CIRCLE ACTIONS AND ISOTOPY ON SPACE OF POLYGONS

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ABSTRACT. Using circle actions on moduli space of polygons in $\mathbb{R}^3$, we construct some non-convex domains with interesting compactly supported symplectomorphism groups. Further, we prove the uniqueness isotopy of projective twists in polygon spaces, and provide some speculations for general toric symplectic manifolds.

CONTENTS

1. INTRODUCTION

In [LLW3 LLWep], it is proved that

Theorem 1.1. *For any symplectic toric (M^4, ω), if $f \in Symp(M, \omega)$ acts trivially on $H_2(M^4, \mathbb{Z})$, then it is symplectically isotopic to the identity.*

Hence it is reasonable to speculate the following very general conjecture:

toricn **Conjecture 1.2.** *In dimension $2k > 4$, any toric manifold M^{2k} has connected $Symp_h$.*

This conjecture is very difficult to prove, since very little is known even in the dim=6 cases. We can start with investigating some approachable examples in higher dimensions, which is the Moduli of space polygons $M(r)$ as a symplectic manifold:

Definition 1.3. *Let $r = (r_1, \cdots, r_n)$ be a tuple of positive numbers. Let $S_{r_i} \to \mathbb{R}^3$ denote the sphere of radius r_i . Let $\mu : S_{r_1} \times \cdots \times S_{r_n} \to \mathbb{R}^3$ denote the addition map, so $\mu(e_1, \cdots, e_n) = e_1 + \cdots + e_n$. $\mu^{-1}(0)$ is invariant under $SO(3)$. The quotient $M(r, n) = \mu^{-1}(0)/SO(3)$ is a manifold of dimension $2n - 6$.*

This manifold $M(r)$ is called the moduli space of spatial polygons, or polygons in 3-space R^3. The polygon should be thought of as the edge vectors $(e_1, \cdots, e_n) \in \Pi S_{r_i}$, placed in three space from tip to tail, up to orientation preserving rigid motions of $\mathbb{R}^3$. Note that the polygons may intersect themselves.

There are natural symplectic structure, torus and circle actions on $M(r)$ induced from $S_{r_1} \times \cdots \times S_{r_n}$.

2. A toy example in dim=4

In dim=2, $M(r,4)$ are all S^2's; and in dimension 4, possible topology of $M(r,4)$ can be $\mathbb{C}P^2$, S^2S^2, and $\mathbb{C}P^2\#n\mathbb{C}P^2, n \leqslant 4$. The symplectic forms heavily depend on the size, and we'll discuss later. Notice that all those manifolds admit Hamiltonian torus actions, but this is no longer true in dimension 6 or higher.

Here we will discuss some isotopy problems on those spaces (mostly covered by [Sei08]), and use this as models for higher dimensional results and compare it with domains.

Let's first we briefly recall the definition of Dehn twists along Lagrangian spheres. The construction of four-dimensional Dehn twists is standard [Ad95; Seixi], but we will need the details as a basis for further discussion. Consider T^*S^2 with its standard symplectic form ω, in coordinates

$$T^*S^2 = \{(u,v) \in \mathbb{R}^3 \times \mathbb{R}^3 \quad \text{s.t.} \quad u,v = 0, ||v|| = 1\}, \quad \omega = du \wedge dv.$$

This carries the $O(3)$-action induced from that on S^2. Maybe less obviously, the function $h(u,v) = ||u||$ induces a Hamiltonian circle action σ on $T^*S^2 \setminus S^2$,

$$\sigma_t(u,v) = \Big(\cos(t)u - \sin(t)||u||v, \cos(t)v + \sin(t)\frac{u}{||u||}\Big).$$

σ_π is the antipodal map $A(u,v) = (-u,-v)$, while for $t \in (0;\pi)$, σ_t does not extend continuously over the zero-section. Geometrically with respect to the round metric on S^2, σ is the *normalized geodesic flow*, transporting each tangent vector at unit speed (irrespective of its length) along the geodesic emanating from it. Thus, the existence of σ is based on the fact that all geodesics on S^2 are closed. Now take a function $r : \mathbb{R} \to \mathbb{R}$ satisfying $r(t) = 0$ for $t \gg 0$ and $r(-t) = r(t) - t$. The Hamiltonian flow of $H = r(h)$ is $\phi_t(u,v) = \sigma_{t\, r'(||u||)}(u,v)$, and since $r'(0) = 1/2$, the time 2π map can be extended continuously over the zero-section as the antipodal map. The resulting compactly supported symplectic automorphism of T^*S^2,

$$\tau(u,v) = \begin{cases} \sigma_{2\pi\, r'(||u||)}(u,v) & u \neq 0, \\ (0,-v) & u = 0 \end{cases}$$

is called a model Dehn twist. To implant this local model into a given geometric situation, suppose that $L \subset M$ is a Lagrangian sphere in a closed symplectic four-manifold, and choose an identification $i_0 : S^2 \to L$. The Lagrangian tubular neighbourhood theorem tell us that i_0 extends to a symplectic embedding

$$i : T^*_{\leqslant\lambda}S^2 \longrightarrow L$$

of the space $T^*_{\leqslant\lambda}S^2 \subset T^*S^2$ of cotangent vectors of length $\leqslant \lambda$, for some small $\lambda > 0$. By choosing $r(t) = 0$ for $t \geqslant \lambda/2$, one gets a model Dehn twist τ supported inside that subspace, and then one defines the Dehn twist τ_L to be

$$\tau_L(x) = \begin{cases} i\tau i^{-1}(x) & x \in im(i), \\ x & \text{otherwise.} \end{cases}$$

The construction is not strictly unique, but it is unique up to symplectic isotopy.

th:local-fragility

There is a smooth family (ϕ^s) of compactly supported diffeomorphisms of T^*S^2, with the following properties: (1) ϕ^s is symplectic for ω^s; (2) for all $s \neq 0$, ϕ^s is isotopic to the identity by an isotopy in $Aut^c(T^*S^2, \omega^s)$; (3) ϕ^0 is the square τ^2 of a model Dehn twist.

We begin with an elementary general fact. For concreteness, we will identify

th:so3 $\mathfrak{so}_3^*\mathfrak{so}_3\mathbb{R}^3$ by using the cross-product and the standard invariant pairing.

Let M be a symplectic manifold, carrying a Hamiltonian $SO(3)$-action ρ with moment map μ. Then $h = ||\mu||$ is the Hamiltonian of a circle action on $M \setminus \mu^{-1}(0)$.

Proof. h Poisson-commutes with all components of μ (since this is true for the Poisson bracket on $\mathfrak{so}_3^*$, a well-known fact from mechanics), so its flow maps each level set $\mu^{-1}(w)$ to itself. The associated vector field X satisfies

$$X|\mu^{-1}(w) = K_{w/||w||}|\mu^{-1}(w)$$

where K are the Killing vector fields, which is clearly a circle action (the quotient $\mu^{-1}(w)/S^1$ can be identified with the symplectic quotient $M//SO(3)$ with respect to the coadjoint orbit of w). $\square$

¡++¿

The moment map for the $SO(3)$-action on T^*S^2 is $\mu(u,v) = -u \times v$, so the induced circle action is just σ. With respect to the deformed symplectic structures ω^s, the $SO(3)$-action remains Hamiltonian but the moment map is $\mu^s(u,v) = -sv - u \times v$, which is nowhere zero and hence gives rise to a circle action σ^s on the whole cotangent space. As $r \to 0$, σ^s converges on compact subsets of $T^*S^2 \setminus S^2$ to σ. For simplicity, assume that our model Dehn twist τ is defined using a function h which satisfies $h'(t) = 1/2$ for small t. Then

$$\phi^s(u,v) = \sigma^s_{4\pi h'(||v||)}(u,v)$$

for $s \neq 0$ defines a family of compactly supported ω_s-symplectic automorphisms. These are all equal to the identity in a neighbourhood of the zero section, hence they match up smoothly with $\phi^0 = \tau_L^2$. By replacing σ with σ^s in (eq:isotopy ??) one finds ω^s-symplectic isotopies between each ϕ^s, $s \neq 0$, and the identity. This concludes the proof of Proposition th:local-fragility 2. It is no problem to graft this local construction into any Dehn twist, which yields:

th:fragility **Corollary 2.1.** *For any Lagrangian sphere L in a closed symplectic four-manifold M, the square τ_L^2 of the Dehn twist is potentially fragile.* $\square$

First of all, there is an elementary construction based directly on the circle action

th:extend-action σ used in the definition of the Dehn twist.

Suppose that there is a Hamiltonian circle action $\bar\sigma$ on $M \setminus L$ and a Lagrangian tubular neighbourhood $i : T^*_{<\lambda}S^2 \to M$ of L which is equivariant with respect to σ,

th:s2s2 $\bar\sigma$. Then τ_L^2 is isotopic to the identity in $Aut(M)$. $\square$

As in the proof of Proposition th:aut-cotangent ?? take $M = S^2 \times S^2$ with the monotone symplectic form, and $L = \{x_1 + x_2 = 0\}$ the antidiagonal. The diagonal $SO(3)$-action has moment map $\mu(x) = -x_1 - x_2 \in \mathbb{R}^3$, and from Lemma th:so3 2 above we know that $\bar h(x) = ||x_1 + x_2||$ is the moment map for a circle action $\bar\sigma$ on $M \setminus L$. This has the desired property with respect to any $SO(3)$-equivariant Lagrangian tubular neighbourhood for L. A slight refinement of Lemma th:extend-action 2 shows that τ_L itself is symplectically isotopic to the involution $(x_1, x_2) \mapsto (x_2, x_1)$. Somewhat less transparently, this could also

th:linkages be derived from Gromov's Theorem th:gromov ??.

$M(r,5)$ is the quotient of the diagonal action of $SO(3)$ on $(S^2)^5$ with moment map $\mu(x) = -(x_1 + \cdots + x_5)$. The symplectic quotient $M = \mu^{-1}(0)/SO(3)$ is the space of quintuples of vectors of unit length in $\mathbb{R}^3$ which add up to zero, up to simultaneous rotation. This is a compact symplectic four-manifold, and it contains a natural Lagrangian sphere

eq:l1-sphere
$$L_1 = \{x_1 + x_2 = 0\}. \tag{1}$$

Similarly, we define L_i as the permuted L_1 by sending x_k to x_{k+i-1}.

Notice that if we consider plane pentagons there is a similar Lagrangian S^2.

$M \backslash L_1$ carries a Hamiltonian circle action $\bar{\sigma}_1$, given by rotating x_1 around the axis formed by $x_1 + x_2$ while leaving $x_1 + x_2$, x_3, x_4, x_5 fixed. The relevant moment map is $\bar{h}_1(x) = ||x_1 + x_2||$ as before, which already looks much like our standard circle action on $T^*S^2 \setminus S^2$. Indeed, one can find a tubular neighbourhood of L_1 satisfying the conditions of Lemma 2, so $\tau_{L_1}^2$ is symplectically isotopic to the identity.

It is worth while to identify M more explicitly. Take the maps induced by inclusion j and projection p,

$$H^2((S^2)^5;\mathbb{R}) \xrightarrow{j^*} H^2(\mu^{-1}(0);\mathbb{R}) \xleftarrow{p^*} H^2(M;\mathbb{R}).$$

Our group being $SO(3)$, a look at the standard spectral sequence shows that cohomology and equivariant cohomology coincide in degree two. This implies that p^* is an isomorphism. Now, the pullback of the symplectic form on M via p agrees with the restriction of the symplectic form on $(S^2)^5$ via j, and the same holds for the first Chern classes of their respective tangent bundles. We conclude that M is monotone, so by general classification results [LM96] it must be either $\mathbb{C}P^1 \times \mathbb{C}P^1$ or $\mathbb{C}P^2$ blown up at $0 \leqslant k \leqslant 8$ points.

Therefore $b_2(M) = 5$ and so $M \setminus (L_1 \cup L_3)$ carries a T^2-action with three fixed points, which means $\chi(M) = 7$.

Hence

$$M \cong \mathbb{C}P^2 \# 4\mathbb{C}P^2.$$

Finally, if we vary this example by considering quintuples of vectors of different lengths, see [HK98; Gol01]. This yields examples of Lagrangian spheres on $\mathbb{CP}^n2\#2\mathbb{CP}^n2$ and $\mathbb{CP}^n2\#3\mathbb{CP}^n2$ with τ^2 symplectically isotopic to the identity, however the relevant symplectic forms are not monotone.

3. Circle actions and symplectic domains in dimension 4

In this section, we will describe a construction of "exotic symplectic $\mathbb{R}^4$" from the space of polygons and exhibit the relation with circle actions.

summary
$$\begin{tikzcd}
Symp_c(U) \arrow[d] & & & \\
Stab^1(C) \arrow[r] & Stab^0(C) \arrow[r] \arrow[d] & Stab(C) \arrow[r] \arrow[d] & Symp_h(X,) \arrow[d] \\
& \mathcal{G}(C) & Symp(C) & \mathcal{C}_0 \simeq \mathcal{J}_{open}
\end{tikzcd} \tag{2}$$

We apply the diagram 2 to the monotone 4-point blow up with the configuration to obtain the following diagram

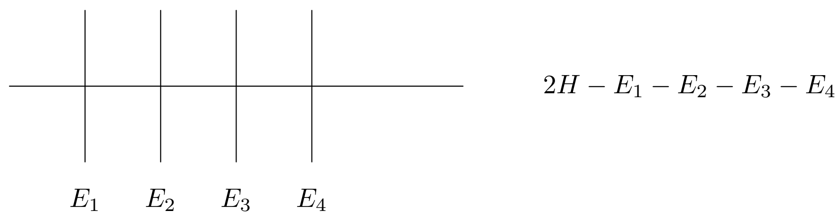


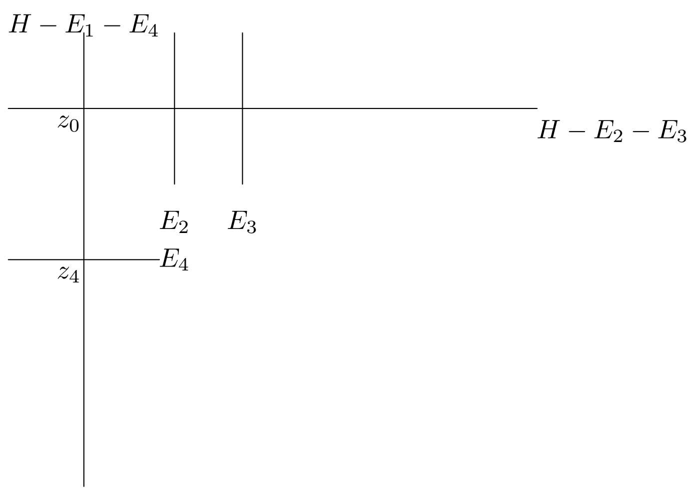
FIGURE 1. Divisor whose complement in X_4 is non-convex 4n

(3)
$$\begin{tikzcd}
\mathbb{Z} = Symp_c(U) = Stab^1(C) \arrow[r] & Stab^0(C) \arrow[r] \arrow[d] & Stab(C) \arrow[r] \arrow[d] & Symp_h(X,) \arrow[d] \\
& \mathbb{Z}^4 & (S^1)^5 \times P_4(S^2)/\mathbb{Z}_2 & \mathcal{S}
\end{tikzcd}$$

Note that $\mathcal{S}$ is homotopic to $\{S^2 - 3 \text{ pts}\}$, and the $\pi_1(S^1)^5$ surjects to $\mathbb{Z} = Symp_c(U)$ (by Lemma 36 in [Eva11] or Lemma 5.5 in [LLWxi]) and $\mathbb{Z}^4 = \mathcal{G}$. Hence we have the equivalence of weak homotopy type $Symp_h(X,) \cong \star$.

cc4 **Proposition 3.1.** *There is a non-standard symplectic form on $\mathbb{C} \times \mathbb{C}$, which is non-convex and has $Symp_c$ homotopic to $\mathbb{Z}$.*

Proof. Choose the following configuration in the equal size 4-point blowup



And the diagram 2 becomes

(4)
$$\begin{tikzcd}
Symp_c(U) = Stab^1(C) \arrow[r] & Stab^0(C) \arrow[r] \arrow[d] & Stab(C) \arrow[r] \arrow[d] & Symp_h(X,) \arrow[d] \\
& \mathbb{Z}^3 & (S^1)^4 & \mathcal{S}
\end{tikzcd}$$

We know from [LLWxi] that $Symp_h(X,)$ is weakly homotopic to $\mathcal{J}_{open}$.

The complement is biholomorphic to $\mathbb{C}\times\mathbb{C}$. When $c_1 > c_4$, the complement is convex, and when $c_1 = c_4$ it is not. This is because there should be positive solution $\{d_0, d_1, d_2, d_3, d_4 \in RR^+\}$ for the following equation,

$$\begin{aligned} PD([l]) = aH - c_1E_1 - c_2E_2 - c_3E_3 - c_4E_4 \\ = d_0(H - E_1 - E_4) \\ +d_1(H - E_2 - E_3) \tag{5} \\ +d_2E_2 + d_3E_3 + d_4E_4. \end{aligned}$$

(6)

And we immediately see that $d_0 = c_1$ and $d_0 = c_4 + d_4$. Hence $c_1 = c_4 + d_4$ If there's a positive solution, then we must have $c_1 > c_4$. And it is easy to check that if $c_1 > c_4$, we can always find a positive solution.

And $\mathcal{S}$ is the filling of the space $\mathcal{J}_{open}$ by the prime submanifold $E_1 - E_4$. In this case, the $Symp_h(X,)$ has the same homotopy group as $\mathcal{S}$ for rank larger than 1. For the fundamental group, $\pi_1 Symp_h(X,) = \pi_1(\mathcal{S})\mathbb{Z}$ by the homotopy exact sequence. This means the prime submanifold $E_1 - E_4$ is contractible.

Otherwise it is non-convex as a symplectic manifold. And $\mathcal{S}$ is the same as the space $\mathcal{J}_{open}$ (the prime submanifold $E_1 - E_4$ does not exist because $c_1 = c_4$.) In this case, $Symp_h(X,)$ is weakly homotopic to $\mathcal{S}$, the LES yields that $Stab(C)$ is weakly contractible, which means $Symp_c(U) \cong \mathbb{Z}$.

“ abc ” ”

□

The scheme to directly prove the above theorem is the following:

Proof. Firstly we fix notation: the complement of divisor of figure 4n 1 in X_4 is denoted by W_4.

And the the complement of divisor of the following figure figure 4n 1 in X_4 is denoted by U_4.

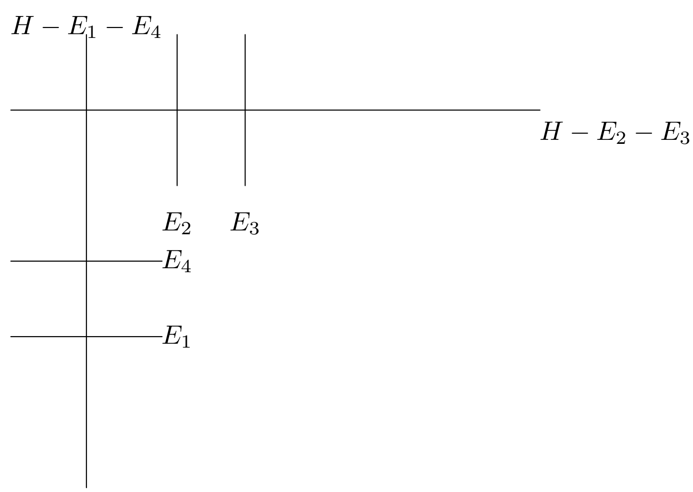


FIGURE 2. Divisor whose complement in X_4 is non-convex 4c

Note that U_4 is biholomorphic to $\mathbb{C} \times \mathbb{C}^*$.

Then one have the following fibration

$$Stab(E_1) \to Symp_c(W) \to Orb(E_1).$$

Here $Orb(E_1)$ is the space of embedded pseudo-holomorphic sphere in homology class E_1.

Now we introduce the notation of the space $\mathcal{C}_4^0$ of a standard test sphere in X_4 :

Definition 3.2. *A standard test sphere is an embedded symplectic sphere S in homology class E_1 in X_4 with a fixed configuration as in figure 1, such that*
1)S is disjoint from the spheres in $H - E_2 - E_3, E_2, E_3, E_4$, and hence the intersection point of S with the sphere in $H - E_1 - E_4$ is disjoint from z_0 and z_4.
2) there exist a $J \in J_\omega$ such that the spheres are simultaneously J-holomorphic;
3) the intersection points between any pair of spheres are w-orthogonal.

Then the space of $\mathcal{C}_4^0$ is contractible. **Note: if the form is reduced, this is only possible when $\omega(E_1) = \cdots = \omega(E_4)$.**

And we can see that $\pi_4 : \mathcal{C}_4^0 \to \{S^2 - \{z_0, z_4\}\}$, by sending the sphere S to its intersection point with the sphere in class $H - E_1 - E_4$, is a fibration. (same proof as Lemma 43 in [Eva11]). And we have the following fibration:

$$\mathcal{F} \to \mathcal{C}_4^0 \to \{S^2 - \{z_0, z_4\}\}.$$

Hence $\mathcal{F} \sim \pi_0(\mathcal{F}) = \pi_1(S^2 - \{z_0, z_4\}) = \mathbb{Z}$.

Now the same argument in Proposition 45 of [Eva11] gives that $\pi_0(Symp_c(W_4)) \hookrightarrow \mathbb{Z}$.

And use the same straitly in [Wu14], there is a the ball swapping subgroup that injects into $\pi_0(Symp_c(W_4))$ (this needs the free action on the fiber $\mathcal{F}$ and the last Lemma in [Eva11]), that is isomorphic to $\mathbb{Z}$. Hence $\pi_0(Symp_c(W_4)) \simeq \mathbb{Z}$.

□

Remark 3.3. *Notice that such generator of $\mathbb{Z}$ indeed can be realized as a circle action of the closed manifold X_4.*

4. Isotopy on polygon spaces

Now we come back to Dehn twists. The first hope is a special case of Conjecture 1.2:

toricpol **Conjecture 4.1.** *If $(M(r,n),\omega)$ admit a Hamiltonian torus action, then it has connected $Symp_h$.*

As a first step, we consider the moduli space of plane polygons, and they are natural Lagrangian submanifolds in $M(r)$. The topology of those submanifolds can vary, but in certain special cases, they are projective spaces $\mathbb{C}P^m$. And we can define Dehn twist along those Lagrangian submanifolds. Because the natural circle action on $M(r,n)$, we are able to prove that

projpol **Theorem 4.2.** *Dehn twists along projective Lagrangian submanifolds in $M(r,n)$ are isotopic to identity.*

The above conjecture can be regarded as a generalization of Theorems on $\pi_0(Symp)$ of small rational manifolds to higher dimensions. The remaining is to show that (projective) Dehn twists are the only possible generator of SMCG for $(M(r,n),\omega)$, then Theorem 4.2 would imply Conjecture 4.1.

deftwist

4.1. **Projective Twists.** Given a closed Riemannian manifold (L,g) with $H^1(L;\mathbb{R})=0$ and admitting a periodic (co-)geodesic flow $\Phi_L^t\colon T^*L\to T^*L$, Seidel ([Sei00]) constructs a symplectomorphism in $\mathrm{Symp}_c(T^*L)$. We review the construction (in the notation of [MW]) of this class of symplectomorphisms, which we call twists. More specifically, for $L\cong S^{2n+1}$ this is the well-known symplectic *Dehn twist*, and in the cases $L\in\{\mathbb{RP}^n,\mathbb{CP}^n,\mathbb{HP}^n\}$ the construction yields what we call a *projective twist*.

If $L\cong S^n$, for $\epsilon>0$, define an auxiliary function $r_\epsilon\in C^\infty([0,1],\mathbb{R})$ such that $0<r_\epsilon(t)<\pi$ for all $t<\epsilon$ and

sphereconv

$$r_\epsilon(t)=\begin{cases}\frac{1}{2}-t & t\ll\epsilon\\ 0 & t\geqslant\epsilon\end{cases}\tag{7}$$

If L is a (real, complex or quaternionic) projective space, and $\epsilon>0$, let $r_\epsilon\in C^\infty([0,1],\mathbb{R})$ such that $0<r_\epsilon(t)<2\pi$ for all $t<\epsilon$ and

projconv

$$r_\epsilon(t)=\begin{cases}1-t & t\ll\epsilon\\ 0 & t\geqslant\epsilon\end{cases}\tag{8}$$

Let $\|\cdot\|_L$ be the norm associated to the given Riemannian metric g. Consider the unit disc bundle D_1T^*L, where $D_sT^*L:=\{v\in T^*L;\|v\|_L\leqslant s\}$, with associated standard symplectic form $\omega\in\Omega^2(D_1T^*L)$ and contact form $\lambda\in\Omega^1(ST^*L)$, where ST^*L is the unit cotangent bundle.

The normalised cogeodesic flow Φ_L^t, which coincides with the Reeb flow for λ, satisfies $\Phi_L^1=Id$ and can be extended to a Hamiltonian S^1-action σ_t on twistlocal $D_1T^*L\setminus L$, with moment map $\mu:D_1T^*L\to\mathbb{R}$, $\mu(v)=\|v\|_L$. Define the model twist map $\tau_L^{loc}\colon D_1T^*L\to D_1T^*L$ as follows.

For L isomorphic to a sphere, set

$$\tau_L^{loc}(\xi)=\begin{cases}\sigma_{r(\|\xi\|_L)}(\xi) & \xi\notin L\\ -\xi & \xi\in L.\end{cases}\tag{9}$$

For L isomorphic to a projective space, let

$$\tau_L^{loc}(\xi)=\begin{cases}\sigma_{r(\|\xi\|_L)}(\xi) & \xi\notin L\\ \xi & \xi\in L.\end{cases}\tag{10}$$

Proof of Theorem 4.2:

Proof. This is a straight forward argument by the constructions of projective twists.

Let's assume the Lagrangian L is projective and its complement has a circle action.

Now let's take the unit disk bundle $T^*_{\leq 1}L$. By the construction of projective twists τ_L, there is a family of smooth maps f_t such that $f_0=id$ and $f_1=\tau_L$. Now we will take the circle action of $M\setminus T^*_{\leq 1}L$, and use a radio Hamiltonian function to cut-off. The upshot is on the set $T^*_{=1}L$, the circle action is the one induced from $M\setminus T^*_{\leq 1}L$; while on each level $T^*_{a,a<1}L$, the cut-off untwists the action τ_L.

□

Here we also list several other notable theorem about projective twist due to Seidel:

Theorem 4.3. *[Sei00, Corollary 4.5] Let (L, g) be a Riemannian manifold admitting a periodic (co-)geodesic flow and satisfying $H^1(L;\mathbb{R}) = 0$. Then the symplectomorphisms τ_L^{loc} have infinite order in $\pi_0(\mathrm{Symp}_c(T^*L))$.*

Theorem 4.4. *[Sei00, Proposition 4.6] For $L \cong \mathbb{CP}^n$, the symplectomorphism τ_L^{loc} of Definition (4.1) is isotopic to the identity in $\mathrm{Diff}_c(T^*\mathbb{CP}^n)$.*

Remark 4.5. *With the conventions (7) and (8), the isomorphisms $S^1 \cong \mathbb{RP}^1$, $S^2 \cong \mathbb{P}^1$ and $S^4 \cong \mathbb{P}^1$ induce identifications $\tau^2_{S^1} \simeq \tau_{\mathbb{RP}^1}$, $\tau^2_{S^2} \simeq \tau_{\mathbb{P}^1}$ and $\tau^2_{S^4} \simeq \tau_{\mathbb{P}^1}$ respectively (see [Sei00]).*

Now suppose $L \subset M$ is a Lagrangian embedding of a Riemannian manifold L as above into a symplectic manifold (M, ω). By the Weinstein neighbourhood theorem, a neighbourhood of $L \subset M$ can be identified with a neighbourhood of $L \subset T^*L$, a disc bundle $D_{\leqslant s}T^*L$. If $s > \epsilon$ (and ϵ is as in (7)), this identification can be used to implant the model twist map into M, by symplectically extending the embedding $\iota\colon L \hookrightarrow M$ to $D_{\leqslant s}T^*L \longrightarrow M$.

Let $K \in \{S^n, \mathbb{RP}^n, \mathbb{CP}^n, \mathbb{HP}^n\}$, or in general a Riemannian manifold as above. A *framed exact Lagrangian* is an exact Lagrangian submanifold $L \subset M$ together with an equivalence class $[f]$ of diffeomorphisms $f\colon K \to L$; $f_1 \sim f_2$ iff $f_2^{-1}f_1$ is isotopic, in $\mathrm{Diff}(K)$, to an element of the isometry group $\mathrm{Iso}(K, g)$. An equivalence class $[f]$ as above is called a *framing*.

Let $L \subset M$ be a framed exact Lagrangian submanifold and τ_L^{loc} a model twist supported in the interior of $D^*_{\leqslant s}L$. Consider the symplectomorphism defined as

$$\tau_L \cong \begin{cases} \iota \circ \tau_L^{loc} \circ \iota^{-1} & \text{on } \mathrm{Im}(\iota) \\ \mathrm{Id} & \text{elsewhere} \end{cases}$$

In the case where L is a sphere, the map τ_L is the standard symplectic *Dehn* twist. When L is a projective space, the resulting map is called *projective twist*. In this paper, the appellation *Dehn* is exclusively reserved for twists that are constructed from a Lagrangian sphere.

Remark 4.6. *We refer to [Sei00, Section 4.b] for the choices involved in this construction (in particular the auxiliary functions r_ϵ).*

Remark 4.7. *Theorem 4.4 implies that given a symplectic manifold (M, ω), any Lagrangian $L \cong \mathbb{CP}^n \subset M$ will define an element τ_L that is isotopic to the identity in $\mathrm{Diff}_c(M)$.*

5. Remarks and Conjectures

Moreover, if we remove certain divisors from $M(r, n)$, some interesting symplectic domains can be obtained. This will provide us plenty of examples that have surprising dynamics behavior.

Conjecture 5.1. *There are non-convex symplectic forms ω in contractible domains D in any dimension$\geqslant 4$, so that the compactly supported symplectomorphism group $Symp_c(D, \omega)$ has infinitely many connected components.*

In dimension 4, we can construct "exotic symplectic $\mathbb{R}^4$" from $\mathbb{C}P^2\#4\mathbb{C}P^2$, and in higher dimension from $M(r,n)$. Recall in dim=4, we choose a filling divisor C and consider its complement. The complement can be non-Stein and even does not have an outpointing Liouville vector field. There are only some constructions of non-convex $\mathbb{R}^4$ by Gromov in [Gro85], by Bates-Peschke in [BP90] and later by Roger Caslas [Casxi]. Very little symplectic dynamics about this is known. Even less is known for higher dimensional SMCG. Notice that in dimension 4, there is an interesting relationship with the boundary twist in [HPW16], and the Gompf's Surgery in [Gom98]. This is also related to CR structures and overtwistedness of stable Hamiltonian structures on the boundary of the domain.

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