Topology 590

Friday, January 22, 2021 12:01 PM

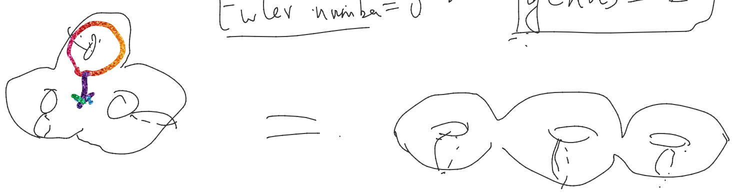
Geometry $\longmapsto$ rigid

Topology (set, topology) $\longmapsto$ soft.

$\hookrightarrow$ invariance property under continuous deformations.

Riemann Surface.

Euler number = 0 $\Leftrightarrow$ genus = 1



Metric space: (S, d)

$d: S \times S \longmapsto \mathbb{R}$. s.t.

① $d(x,y) = 0$ iff $x = y$

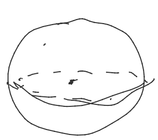
② $d(x,y) = d(y,x) \quad \forall x, y \in S$

③ $d(x,z) \le d(x,y) + d(y,z)$. triangle ineq

Q: what's a continuous function for (S,d)?

$$\lim_{x \to x_0} f(x) = f(x_0) \qquad \boxed{f: S \to \ }$$

"Open ball" (Euclidean space, std metric)



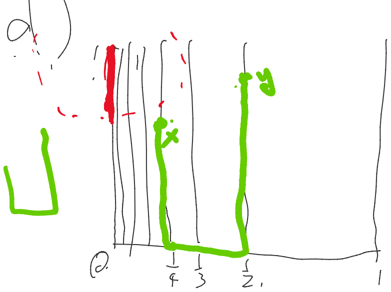
$B_r(p)$ radius = r. center = p

$= \{x \in S \mid d(x,p) < r\}$

(Comb space: d)

$\bigcap$ $\mathbb{R}^2$

$d(x,y) = \sqcup$



$X_0 = \{0\} \times [0,1]$

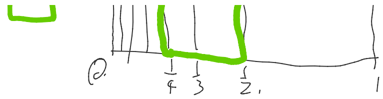
$\cup$

$Y_0 = [0,1] \times \{0\}$

$\cup$

$X_N = \{\frac{1}{N}\} \times [0,1]$

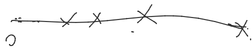
$N \in \mathbb{N}^+$



$X_N = \{\frac{1}{N}\} \times [0,1]$

$N \in \mathbb{N}^+$

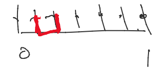
Ex 1: $\{(\frac{1}{N}, 0)\}$



convergent? Y

where? $(0,0)$.

Ex 2. $\{(\frac{1}{N}, 1)\}$



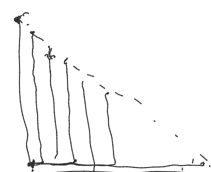
convergent? No

where? Not a Cauchy sequence

$\boxed{d(x_i, x_j) \geq 0.2}$

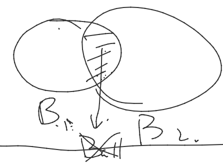
Ex 3: $B_{\frac{1}{3}}(0,1)$ line segment:

Ex 4: $B_{\frac{2}{3}}(0,1)$



change: Open ball → open sets

balls are not closed under $\boxed{\text{operations}}$



Recall: $\boxed{\text{Thm 1}}$: In a metric space (S, d):

1. S, $\emptyset$ is open.
2. $U, V \subseteq S$ open then $U \cap V$ is open.
3. $U_i \subseteq S$, $i \in I$, $\bigcup_{i \in I} U_i$ is open.

open set := unions of open balls

Thm 2: (S, d_1), (T, d_2). $S \xrightarrow{f} T$

f is continuous $\iff$ Any $\boxed{\text{open set}}$ $U \subseteq T$ has

$\lim_{x \to x_0} f(x) = f(x_0)$ a open preimage $f^{-1}(U) \subseteq S$.

Def: $(X, \boxed{\text{topology}})$. A topology τ is a collection of subsets of X.

s.t

1. $X, \emptyset \in \tau$.
2. If $X \supseteq U, V \in \tau$, then $U \cap V \in \tau$
3. If $X \supseteq U_i \in \tau$, $i \in I$, $\bigcup_{i \in I} U_i \in \tau$

Also, any element in τ is called an open subset of X.

Def: continuous function:

$(X, \tau_1) \xrightarrow{f} (Y, \tau_2)$, f is continuous if

$\forall U \in \tau_2$, $f^{-1}(U) \in \tau_1$.

Example:

$\tau = \{\emptyset, X\}$ $\qquad$ τ_1 $\qquad$ τ_2 $\qquad$ $\tau = \{$ any subset of $X\}$

coarse $\longleftrightarrow$ fine

$\tau_2 \geq \tau_1$

stronger $\quad$ weaker

coarse = weak = small $\qquad$ fine = ~~strong~~ = big

Ex: $\mathbb{R}$, $U \subseteq \mathbb{R}$ is open iff U is the union of $[a, b)$ ...

half-open topology.

Is this ~~finer~~ / $\boxed{\text{Stronger}}$ or weaker than the Euclidean?

---

topologies on $\mathbb{R}$

$(\mathbb{R}$, Euclidean metric$)$ $\rightarrow$ usual topology

open set: every point is an interior point.

U is open, if $\forall x \in U$, $\exists \varepsilon$, s.t. $B_\varepsilon(x) \subseteq U$.

open intervals (a, b)

$(\mathbb{R}$, discrete metric$)$ $\qquad$ Any subset is an open set.

$d(x,x) = 0$
$d(x,y) = 1$, $x \neq y$

$(\mathbb{R}$, indescrete topology$)$ $\qquad$ $\{\emptyset, \mathbb{R}\}$ $\rightarrow$ collection of all open sets.

concrete topology

generally not metrizable

Only when (point indescrete)

metrizable

Only when (point, indescrete topology)

Recall: half open topology $(\mathbb{R}, \tau)$

τ is generated by $[a,b)$ → finite ∩, $\boxed{\text{arbitrary } \cup}$

Q: is this one stronger or weaker than the usual τ on $\mathbb{R}$?

A: stronger $\Leftarrow$ open $(a,b) \in \tau$.

want (a,b) as arbitrary $\cup$ of half open

$$(a,b) = \bigcup_{n=1}^{\infty} [a+\tfrac{1}{n}, b) \qquad \blacksquare$$

$\boxed{\text{Dictionary}}$ topology on $\mathbb{R}^2$

ordering

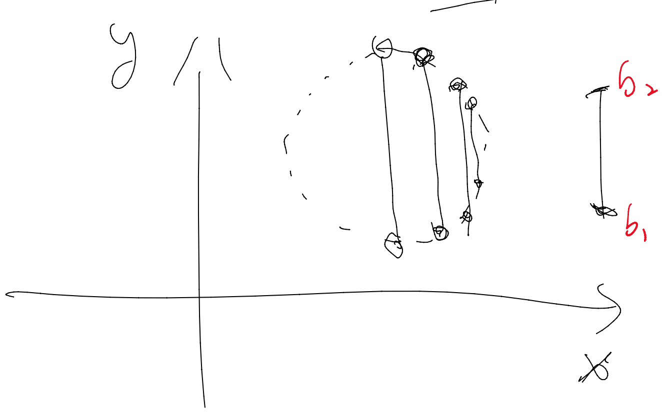
$a < b$

We say $(a,b) < (c,d)$ if either $a<c$ or $a=c$ and $b<d$.

Define D-T- as following:

τ, $U \subseteq \mathbb{R}^2$ is in τ iff U is a union of "open intervals" of such form:

$$U = \{ (x,y) \mid \underset{a,\ b_1}{(a,b)} < \underset{a}{(x,y)} < \underset{a,\ b_2}{(c,d)} \}$$



Q1: is a vertical line open? yes

Q2: is a horizontal line open? No

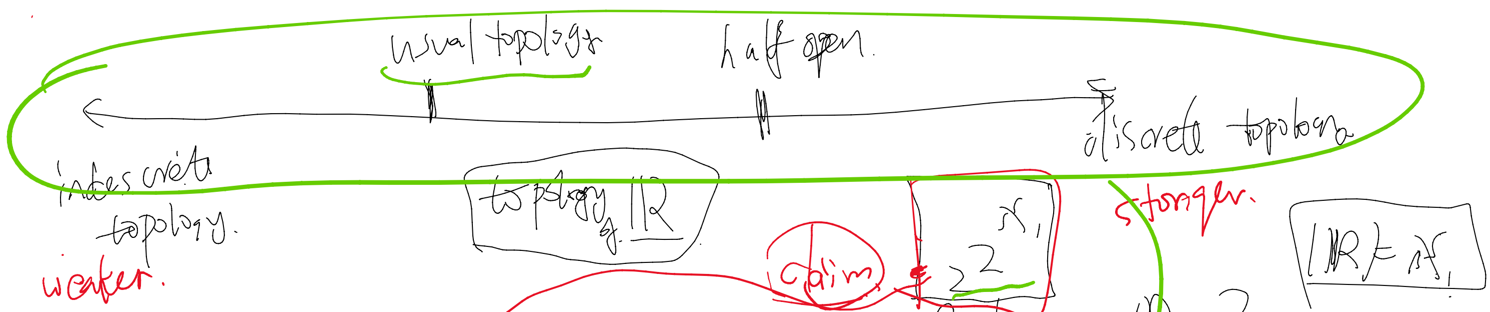
Q3: is D-T- ~~stronger~~ weaker than usual?

$\boxed{B_\varepsilon(p) = \bigcup \text{vertical lines} = \bigcup \text{open sets}}$

$B_\varepsilon(p) \in \tau$

usual topology $\subset \tau$

stronger..



Q1: How many topologies can we find on $\mathbb{R}$?

Q2: Are there linear ordering on those topologies?

No

Q3: Find a topology that's not on.

$\{\emptyset, \boxed{\{1\}}, \mathbb{R}\}$ ← topology neither stronger nor weaker than the usual topology.

$\{\emptyset, [0,1), \mathbb{R}\}$ topology.

---

finite-complement topology:

$\mathbb{R} - \{p\}$ open set.

~~finite~~ $\cap$
arbitrary $\cup$.

Q1: stronger or weaker than usual?

$\mathbb{R} - \{p_1, \cdots, p_n\}$ is open $\Rightarrow$ weaker.

Def: A subset $V \subseteq (X, \tau)$ is closed if $X \setminus V \in \tau$.

closed subset are complement of open subsets.

$\emptyset$: open and closed → $\boxed{\text{clopen sets.}}$

X: open and closed.

X ... nontrivial subsets that

$\emptyset$: open and closed.

Q1: $(\mathbb{R}, \text{usual})$: Are there non-trivial subsets that are both open & closed?

$\boxed{\text{No}}$

Q2: $(\mathbb{R}, \text{half-open})$: Yes;

Q: What's a non-trivial example?

$[a, \infty)$ and $(-\infty, a)$ yes!

Q: $(\mathbb{R}^2, \text{dict topology})$ $\exists$ sets s.t. both closed and open?

$\{(x,y) \in \mathbb{R}^2, (a,b) \leq (x,y) \leq (c,d)\}$

Q: $(\mathbb{R}, \text{finite-complement topology})$ $\exists$ clopen sets?

No: $\boxed{\text{closed sets are finite}}$

Lemma closed sets: (X, τ) the following subset are closed:

① $\emptyset$, X.

② finite union of closed subsets. $\bigcup_{n=1}^{\infty} [\frac{1}{n}, \infty) = (0, \infty)$

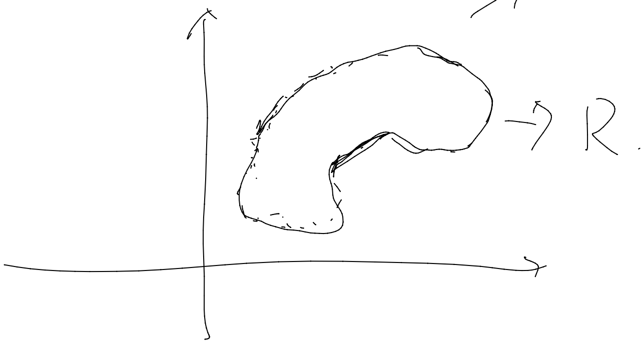
③ arbitrary intersection of closed subsets. □

$(\mathbb{R}^2, \text{usual topology})$

neither open nor closed

interior

maximal open set $\subseteq \mathbb{R}$



① maximal open set $\subseteq R$

② minimal closed set $\supseteq R$ → closure

Def (interior) (X, τ) topological space. $A \subseteq X$, $\{U_i \mid i \in I\}$ be the set of all open sets that are contained in A. Then define

$$\mathring{A} := \bigcup_{i \in I} U_i$$

as the interior of A.

Properties: (X, τ), $A \subseteq X$.

① $\mathring{A} \subseteq A$

② $\mathring{A}$ is open.

③ A is open iff $A = \mathring{A}$

④ $U \subseteq A$ is open, then $U \subseteq \mathring{A}$

Examples:

① $(\mathbb{R}$, half-open topology $[a,b))$ interior $(0,1]$?

$\boxed{\text{interior}} \leftarrow \bigcup_{n=1}^{\infty} [\frac{1}{n}, 1) = (0,1)$

sketch: assume $1 \in \bigcup [a, b_i)$

$\boxed{\text{ex?}} \leftarrow$? $\exists N$, s.t. $1 \in [a, b_N)$

$[a, b_N) \not\subseteq (0,1]$

② $(\mathbb{R}$, finite-complement topology$)$ Interior of $(0,1]$?

interior of $(0,1] \Leftarrow \mathbb{R} - \{p_1, \dots, p_n\} \not\subseteq (0,1]$

$= \emptyset$

Def (closure) $A \subseteq (X, \tau)$, $\{V_j \mid j \in J\}$: collection of all closed subset of X ~~that~~ contains A. Then we define

$$\bar{A} := \bigcap_{j \in J} V_j$$ as the closure of A.

properties:

① $\bar{A} \supseteq A$

② $\bar{A}$ is closed.

③ A is closed iff $A = \bar{A}$.

④ $A \subseteq V$, V is closed, then $\bar{A} \subseteq V$

Example: Q1: ($\mathbb{R}$, usual topology) closure of $\{1, \frac{1}{2}, \frac{1}{3}, \dots \frac{1}{n}\}$

$$\| $$

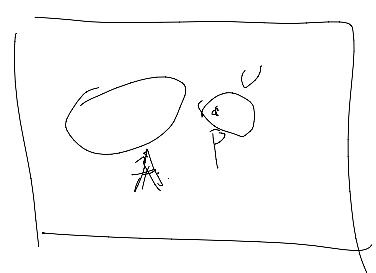
$\{1, \frac{1}{2}, \frac{1}{3} \dots \frac{1}{n} \dots, 0\}$

Lemma: $A \subseteq (X, \tau)$, then $p \in \bar{A}$ $\boxed{\text{iff}}$ for every open set $U \subseteq X$ containing p, $U \cap A \neq \emptyset$.

$\Rightarrow$: $p \in \bar{A}$, and $\exists\, U \subseteq X$ open s.t $p \in U$ and $U \cap A = \emptyset$.

Denote C by the complement of A.

$C = X \setminus A$



$p \in U \Rightarrow p \notin C$

C is closed and $A \subseteq C \Rightarrow \bar{A} \subseteq C$

$\Rightarrow p \in C$ contradiction. Π

and $A \subseteq C$ ...

$\Rightarrow p \in C$. contradiction. $\square$

$(-\infty, 0]$ $(1, \infty)$

<u>Q</u>: ($\mathbb{R}$, half-open topology) closure $(0,1]$?

closure of A $\left((-\infty,0) \cup (1,\infty)\right)^c$

two approaches: ① A^c $\left(\text{interior of } A^c\right)^c$

② $[0,1]$

Continuity:

<u>Recall</u>: HW $(X, d_1) \xrightarrow{f} (Y, d_2)$. f is continuous iff

$\forall\, U \subseteq Y$ <u>open</u>, $f^{-1}(U) \subset X$ is <u>open</u>.

<u>Def</u>: $(X, \tau_1) \xrightarrow{f} (Y, \tau_2)$ f is continuous if

$\forall\, U \subseteq Y$ open, $f^{-1}(U) \subseteq X$ is open

---

<u>Closed under composition</u>: $(X, \tau_1) \xrightarrow[\text{continuous}]{f} (Y, \tau_2) \xrightarrow[\text{continuous}]{g} (Z, \tau_3)$

$g \circ f$ is continuous, because $\forall\, U \subseteq Z$ open,

$g^{-1}(U)$ is open in Y, then $\underbrace{f^{-1} g^{-1}(U)}_{(g\circ f)^{-1}(U)}$ is open in X

Thm: $(X, \tau_1) \xrightarrow{f} (Y, \tau_2)$. f is continuous iff

Thm: $(X,\tau_1) \longrightarrow (Y,\tau_2)$. f is continuous iff

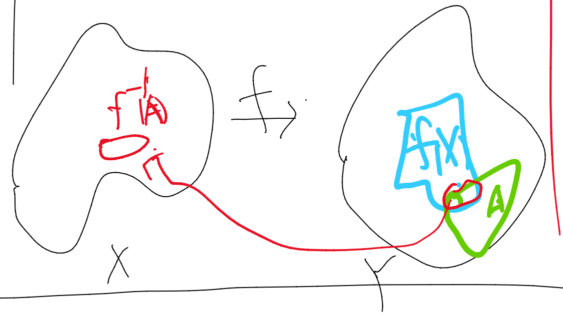
$\forall\, V \subseteq Y$ closed, $f^{-1}(V) \subseteq X$ is also closed.

Pf: Lemma: $\forall\, A \subseteq Y$, $f^{-1}(Y\setminus A) = X \setminus f^{-1}(A)$

pf: $\{x\in X \mid f(x)\in Y\setminus A\} = \{x\in X \mid f(x)\notin A\}$ □

$X - \{x \mid f(x)\in A\}$

f is not necessarily onto.



$f^{-1}(A) := \{x\in X \mid f(x)\in A\}$

$\Rightarrow$ $\forall\, U \subseteq Y$ open, $f^{-1}(U)\subseteq X$ is open $\Rightarrow$ $\forall\, C$ closed,

its complement $U = Y\setminus C$ is open, $f^{-1}(Y\setminus C)$ is open in X,

$X\setminus f^{-1}(C)$ is open $\Rightarrow$ $f^{-1}(C)$ is closed. □

$\Leftarrow$?

Def (open/closed maps) $(X,\tau_1) \xrightarrow{f} (Y,\tau_2)$

f is an open map if $\forall\, U \subseteq X$ open, $f(U)$ is open in Y

f is closed - - - $V\subseteq X$ closed, $f(V)$ — closed —

Example: $(\mathbb{R}, \text{half open}) \xrightarrow{f} (\mathbb{R}, \text{usual})$

$[a,b)$ open; f^{-1}(open); open set $(\ )$

stronger; open & closed; weaker

$x \longrightarrow x$

Q1: Is f continuous? No, Yes ✓

Q2: Is f open? No

Q3: Is f closed? No

$[0,\infty)$ is neither open nor closed

$(0,1) = \bigcup [\frac{1}{n},1)$

Q3: Is f closed? No — not closed

Homeomorphisms → equivalent relation between topological space.

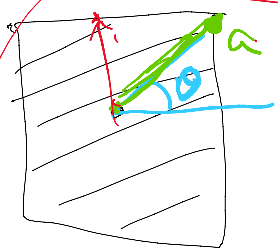
Def: $(X, \tau_1) \xrightarrow{f} (Y, \tau_2)$ f is a homeomorphism [inverse]

if f is bijective, continuous and open. f^{-1} is continuous

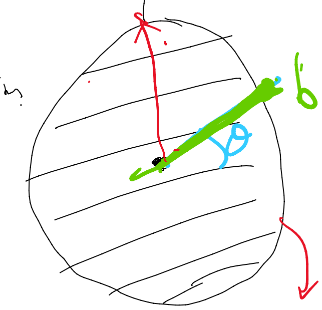
If $\exists$ f homeomorphism between X, Y, then X, Y are homeomorphic.

$(\mathbb{R}^2, \text{Euclidean})$ $[-1,1]\times[-1,1]$ $\{x^2+y^2 \le 1\}$

prove



homeomorphism $\cong$



ball in taxi-cab metric ball in usual metric

topologies are the same. $\Leftarrow$ $c\|\ \|_2 \le \|\ \|_1 \le C\|\ \|_2$

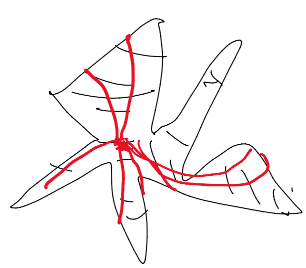
For each ray: f: a → b, end pt → end pt. scale the interior of the ray.

(0, 0) → (0, 0),

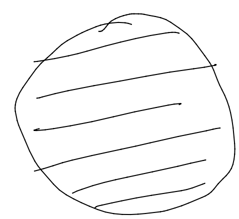
Same → Forevery ray.

claim: f is homeo.

Q:



$\overset{?}{\cong}$ Yes



star shaped

induced from (R, usual)

S^1? No ← harder

induced from $(\mathbb{R}, \text{usual})$

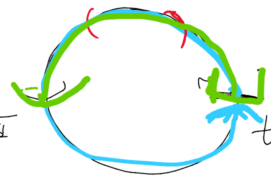
Eg.: $[0, 1)$

(a, b) open

$[0, c)$ open

$t \to (\cos 2\pi t, \sin 2\pi t)$

$[0,1)$



circle $\subseteq \mathbb{R}^2$

"open intervals"

induced topology from $\mathbb{R}^2$.

Q1: Is f bijective?

Yes

Q2: Is f continuous?

Yes!

Q3: Is f open?

No

$[0, \frac{1}{2})$ open $\to$ not open!