Topology 590

Friday, January 22, 2021 12:01 PM

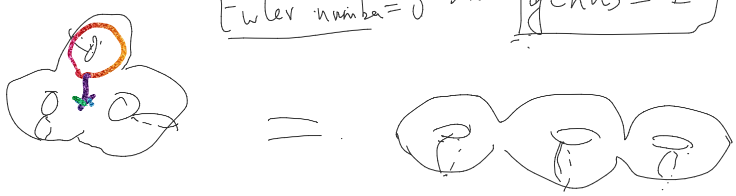
Geometry $\longmapsto$ rigid

Topology (set, topology) $\longmapsto$ soft.

$\hookrightarrow$ invariance property under continuous deformations.

Riemann Surface.

Euler number = 0 $\Leftrightarrow$ $\boxed{\text{genus} = 1}$



Metric space: (S, d)

$d: S \times S \longmapsto \mathbb{R}$. s.t.

① $d(x,y) = 0$ iff $x = y$

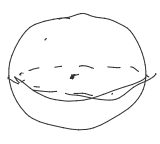
② $d(x,y) = d(y,x)$ $\forall x, y \in S$

③ $d(x,z) \le d(x,y) + d(y,z)$. triangle ineq

Q: what's a continuous function for (S,d)?

$$\lim_{x \to x_0} f(x) = f(x_0) \qquad \boxed{f: S \to}$$

"Open ball" (Euclidean space, std metric)

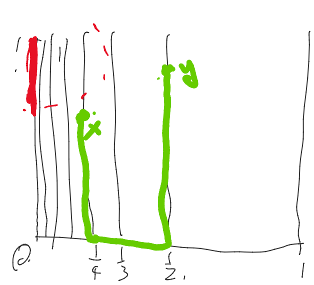


$B_r(p)$ radius = r. center = p

$= \{x \in S \mid d(x,p) < r\}$

(Comb space: d) $\subset \mathbb{R}^2$

$d(x,y) =$ 



$X_0 = \{0\} \times [0,1]$

$\cup$

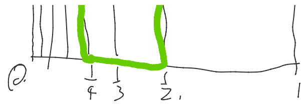
$Y_0 = [0,1] \times \{0\}$

$\cup$

$X_N = \{\frac{1}{N}\} \times [0,1]$

$N \in \mathbb{N}^+$

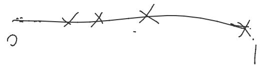
$u_i(x, 0) = \square$



$X_N =: \{\frac{1}{N}\} \times [0, r]$

$N \in \mathbb{N}^+$

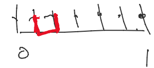
Ex 1: $\{(\frac{1}{N}, 0)\}$



convergent? Y

where? $(0,0)$.

Ex 2. $\{(\frac{1}{N}, \mathbb{Q})\}$



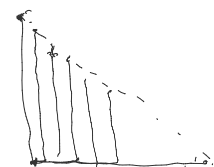
convergent? No

where? Not a Cauchy sequence

$d(x_i, x_j) \geq 0.2$

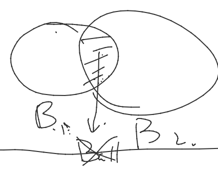
Ex 3: $B_{\frac{r}{3}}(0,1)$ line segment:

Ex 4: $B_{\frac{2}{7}}(0,1)$



change Open ball

balls are not closed under operations



open sets

Recall: Thm 1: In a metric space (S, d):

① $S, \emptyset$ is open.

② $U, V \subseteq S$ open then $U \cap V$ is open.

③ $U_i \subseteq S$, $i \in I$, $\bigcup_{i \in I} U_i$ is open.

open set := unions of open balls

Thm 2: (S, d_1), (T, d_2). $S \xrightarrow{f} T$

f is continuous $\Longleftrightarrow$ Any open set $U \subseteq T$ has a open preimage $f^{-1}(U) \subseteq S$.

$\lim_{x \to x_0} f(x) = f(x_0)$

Def: $(X, \text{topology})$ τ. A topology τ is a collection of subsets of X s.t.

① $X, \emptyset \in \tau$.

② If $X \supseteq U, V \in \tau$, then $U \cap V \in \tau$

③ If $X \supseteq U_i \in \tau$, $i \in I$, $\bigcup_{i \in I} U_i \in \tau$

Also, any element in τ is called an open subset of X.

Def: continuous function:

$(X, \tau_1) \xrightarrow{f} (Y, \tau_2)$, f is continuous if

$\forall U \in \tau_2, \quad f^{-1}(U) \in \tau_1$.

Example:

$\tau = \{\phi, X\}$ τ_1 τ_2 $\tau = \{$ any subset of $X\}$

coarse $\longleftrightarrow$ fine

$\parallel$ $\parallel$

weak $\tau_2 \geq \tau_1$ ~~strong~~ strong

$\parallel$ stronger weaker $\parallel$

small big

Ex: $\mathbb{R}$, $U \subseteq \mathbb{R}$ is open iff U is the union of $[a, b)$ ...

half-open topology.

Is this ~~finer~~ stronger or weaker than the Euclidean?

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topologies on $\mathbb{R}$

$(\mathbb{R}, \text{Euclidean metric}) \to$ usual topology

open set — every point is an interior point.

U is open, if $\forall x \in U$, $\exists \varepsilon$, s.t $B_\varepsilon(x) \subseteq U$.

open intervals

(a, b)

$(\mathbb{R}, \text{discrete metric})$ Any subset is an open set.

$d(x, x) = 0$

$d(x, y) = 1, x \neq y$

$(\mathbb{R}, \text{indescrete topology})$ concrete topology

$\{\phi, \mathbb{R}\} \to$ collection of all open sets.

generally not metrizable

Only when (point indescrete)

∪ metrizable

Only when (point, indescrete topology)

Recall: half open topology $(\mathbb{R}, \tau)$

τ is generated by $[a,b)$ → finite ∩, $\boxed{\text{arbitrary } \cup}$

Q: is this one stronger or weaker than the usual τ on $\mathbb{R}$?

A: stronger $\Leftarrow$ open $(a,b) \in \tau$.

want (a,b) as arbitrary $\cup$ of half open

$$(a,b) = \bigcup_{\boxed{n=1}}^{\infty} [a+\tfrac{1}{n}, b) \qquad \square$$

Dictionary topology on $\mathbb{R}^2$

We say $(a,b) < (c,d)$ if either $a<c$ or $a=c$ and $b<d$.

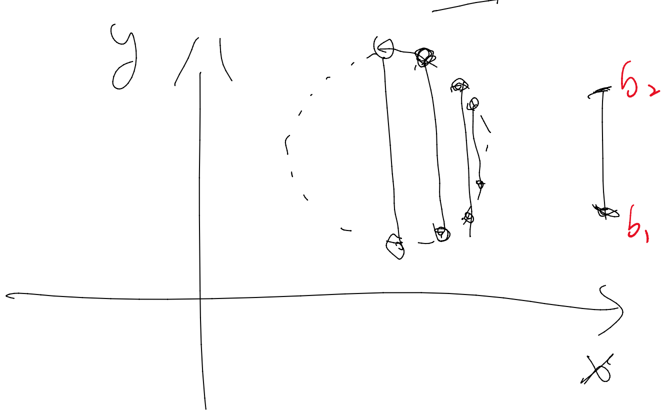
ordering. $a < b$

Define D-T- as following:

τ, $U \subseteq \mathbb{R}^2$ is in τ iff U is a union of "open intervals" of such form:

$$U = \{(x,y) \mid (a,b) < (x,y) < (c,d)\}$$

a_1 b_1 a a b_2



Q1: is a vertical line open? yes

Q2: is a horizontal line open? No

Q3: is D-T- ~~stronger~~ weaker than usual?

$\boxed{B_\varepsilon(p) = \bigcup \text{ vertical lines} = \bigcup \text{ open sets}}$

$B_\varepsilon(p) \in \tau$

usual topology $\subseteq \tau$ stronger..

usual topology   half open

indiscrete topology   discrete topology

weaker   stronger

topology of $\mathbb{R}$

claim $\aleph_1$ $\leq 2^{2^{\aleph_1}}$

$|\mathbb{R}| = \aleph_1$

$\boxed{Q1}$: How many topologies can we find on $\mathbb{R}$?

Q2: Are there linear ordering on those topologies?

No

Q3: Find a topology that's not on.

$\{\emptyset, \boxed{[-1, 1]}, \mathbb{R}\}$ ← topology neither stronger nor weaker than the usual topology.

$\{\emptyset, [0,1), \mathbb{R}\}$ topology.

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finite-complement topology:

$\mathbb{R} - \{p\}$ open set.

finite $\cap$

arbitrary $\cup$.

Q1: stronger or weaker than usual?

$\mathbb{R} - \{p_1 \cdots p_n\}$ is open $\Rightarrow$ weaker.

Def: A subset $V \subseteq (X, \tau)$ is closed if $X \setminus V \in \tau$.

closed subset are complement of open subsets.

$\emptyset$: open and closed → clopen sets.

X: open and closed.

X ... subsets that

$\emptyset$: open and closed.

Q1: $(\mathbb{R}, \text{usual})$: Are there non-trivial subsets that are both open & closed?

No

Q2: $(\mathbb{R}, \text{half-open})$: Yes!

Q: What's a non-trivial example?

$[a, \infty)$ and $(-\infty, a)$ yes!