1 Metric Spaces

Definition 1.1. (Metric; Metric space.) Let X be a set. A *metric* on X is a function

$$d : X \times X \longrightarrow \mathbb{R}$$

satisfying the following conditions.

(M1) **(Positivity).** $d(x, y) \geq 0$ for all $x, y \in X$, and $d(x, y) = 0$ if and only if $x = y$.

(M2) **(Symmetry).** $d(x, y) = d(y, x)$ for all $x, y \in X$.

(M3) **(Triangle inequality).** $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$.

The value $d(x, y)$ is sometimes called the *distance from x to y*.

1. Determine whether the following functions define metrics on the corresponding sets. Rigorously justify your answers!
   (a) Let $X = \mathbb{R}$. Define
   $$\begin{aligned} d : \mathbb{R} \times \mathbb{R} &\longrightarrow \mathbb{R} \\ d(x, y) &= (x - y)^2. \end{aligned}$$
   (b) Let $X = \mathbb{R}^2$. Define the texi-cab metric
   $$\begin{aligned} d : \mathbb{R}^2 \times \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ d(\overline{x}, \overline{y}) &= |x_1 - y_1| + |x_2 - y_2|. \end{aligned}$$
   (c) Let X be any set. Define
   $$\begin{aligned} d : X \times X &\longrightarrow \mathbb{R} \\ d(x, y) &= \begin{cases} 0 & x = y \\ 1 & x \neq y. \end{cases} \end{aligned}$$
2. Let (X, d) be a metric space, and let $Y \subseteq X$ be a subset. Show that the restriction $d|_{Y \times Y}$ of d to $Y \times Y \subseteq X \times X$ defines a metric on Y. Conclude that any subset of a metric space inherits a metric space structure.
3. Let $a < b \in \mathbb{R}$. Let $\mathcal{C}(a, b)$ denote the set of continuous functions from the closed interval $[a, b]$ to $\mathbb{R}$. Verify whether each of the following functions defines a metric on the set $\mathcal{C}(a, b)$. Be sure to clearly state which properties of continuous functions and integration you are using!
   (a)
   $$\begin{aligned} d_1 : \mathcal{C}(a, b) \times \mathcal{C}(a, b) &\longrightarrow \mathbb{R} \\ d(f, g) &= \int_a^b |f(x) - g(x)|\, dx \end{aligned}$$
   (b)
   $$\begin{aligned} d_\infty : \mathcal{C}(a, b) \times \mathcal{C}(a, b) &\longrightarrow \mathbb{R} \\ d(f, g) &= \sup_{x \in [a,b]} |f(x) - g(x)| \end{aligned}$$
4. Consider $\mathbb{R}$ with the Euclidean metric. Find an example of a subset of $\mathbb{R}$ that is ...

(a) open and not closed,

(b) closed and not open,

(c) both open and closed,

(d) neither open nor closed.

5. Let $X = \mathbb{R}^2$. Sketch the balls $B_1(0,0)$ and $B_2(0,0)$ for each of the following metrics on $\mathbb{R}^2$. Denote $\overline{x} = (x_1, x_2)$ and $\overline{y} = (y_1, y_2)$.
   (a) $d(\overline{x}, \overline{y}) = ||\overline{x} - \overline{y}|| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$
   (b) $d(\overline{x}, \overline{y}) = |x_1 - y_1| + |x_2 - y_2|$
   (c) $d(\overline{x}, \overline{y}) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$
   (d) $d(\overline{x}, \overline{y}) = \begin{cases} 0, & \overline{x} = \overline{y} \\ 1, & \overline{x} \neq \overline{y} \end{cases}$

6. Let $\{U_i\}_{i \in I}$ denote a collection of open sets in a metric space (X, d).
   (a) Prove that the union $\bigcup_{i \in I} U_i$ is an open set. Do not assume that I is necessarily finite, or countable!
   (b) Show by example that the intersection $\bigcap_{i \in I} U_i$ may not be open. (This means, give an example of a metric space (X, d) and a collection of open sets $U_i \subseteq X$, and prove that $\bigcap_{i \in I} U_i$ is not open).
   (c) Now assume we have a **finite** collection $\{U_i\}_{i=1}^n$ of open sets in a metric space. Prove that the intersection $\bigcap_{i=1}^n U_i$ is open.

7. (a) Rigorously verify that the sets $\{1\}$, $[1, \infty)$, and $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots\} \cup \{0\}$ are all closed subsets of $\mathbb{R}$ (with the Euclidean metric).
   (b) Consider $\mathbb{R}$ with the Euclidean metric. Is the subset $\mathbb{Q} \subseteq \mathbb{R}$ open? Is it closed?
   (c) Recall that $\mathcal{C}(0,2)$ is the set of continuous functions from the closed interval $[0,2]$ to $\mathbb{R}$, and that
   $$d_\infty : \mathcal{C}(0,2) \times \mathcal{C}(0,2) \longrightarrow \mathbb{R}$$
   $$d(f,g) = \sup_{x \in [0,2]} |f(x) - g(x)|$$
   defines a metric on $\mathcal{C}(0,2)$. Determine whether the subset $\{f(x) \in \mathcal{C}(0,2) \mid f(1) = 0\}$ is closed, open, neither, or both.

8. Let X be a **finite** set, and let d be any metric on X. What can you say about which subsets of X are open? Which subsets of X are closed?

9. Let (X, d) be a metric space, and let $x \in X$. Prove that the singleton set $\{x\}$ is closed.

2 Topology on metric spaces

Definition 1. *An **open set** is a subset $U \subseteq X$ that satisfies one of the following equivalent statements:*

1. *every point of U is an interior point.*
2. *U is equal to its interior.*
3. *U can be realized as a union of open balls.*

Definition 2. *A **closed set** is a subset $F \subseteq X$ that satisfies one of the following equivalent statements:*

1. *F contains all of his limit points.*
2. *U is equal to its closure.*

Definition 3. *A function between metric spaces*

$$f : (X, d_X) \to (Y, d_Y)$$

is **continuous** *if for any $x_0 \in X$, for any ϵ, there is a radius r (depending on both x_0 and ϵ) such that;*

$$f(B_r(x_0)) \subseteq B_\epsilon(f(x_0)).$$

1. Show the equivalence of the above three statements in Definition 1.
2. Show that the complement of an open set is closed and vice versa.
3. What are limit points for subsets in a space with the discrete metric?
4. Prove that a set is open/closed in the euclidean metric iff it is open/closed in the taxi-cab metric.
5. What are open and closed sets in a space with the discrete metric?
6. Assume X is a metric space. Prove the following:
   - ϕ, X are open/closed.
   - the arbitrary union of open sets is open.
   - the finite intersection of open sets is open.
   - the arbitrary intersection of closed sets is closed.
   - the finite union of closed sets is closed.
7. Prove the following theorem:

   f is continuous if and only if, for any open set $U \subseteq Y$ the preimage $f^{-1}(U)$ is an open set in X.

   Prove that the same statement holds if you replace open by closed.