590 W3

Tuesday, January 26, 2021 10:35 PM

subspace and basis

idea: induced metric on a subspace from the original space

Def Subspace topology / induced topology: (X, τ), $A \subseteq X$. we can topologize A by:

$\tau_A = \{ U \cap A \mid U \text{ is open in } (X,\tau) \}$. (A, τ_A)

Eg 1: $[0,1) \subseteq (\mathbb{R}, \text{usual})$ — (a,b) $0 \le a < b \le 1$ and $[0,c)$ → "generates" topology

2. $\mathbb{Z} \subseteq (\mathbb{R}, \text{usual})$ — lattice. what's the induced topology on $\mathbb{Z}$? discrete ⟸ each point is open

3. $\mathbb{Z} \times \mathbb{Z} \subseteq (\mathbb{R}^2, \text{dictionary topology})$ — induced topology on $\mathbb{Z}\times\mathbb{Z}$? $\boxed{\text{discrete}}$

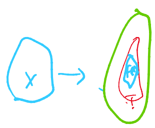
properties of subspace topology: Lemma: $(X, \boxed{\tau_X})$, (Y, τ_Y); $\cup \mid$ subspace topology $\cup \mid$; $(S, \boxed{\tau_S})$, (T, τ_T),

then:

① If $S \in \tau_X$, then $\tau_S \subseteq \tau_X$.

② $C \subseteq S$ is closed iff $\exists$ $A \subseteq X$ closed, st $C = A \cap S$.

③ $f : (X, \tau_X) \to (Y, \tau_Y)$ is continuous, then $f|_S : (S, \tau_S) \to (Y, \tau_Y)$ is also continuous. $\boxed{f|_S^{-1}(A) = f^{-1}(A) \cap S}$



④ $f: (X, \tau_X) \to (Y, \tau_Y)$ is continuous, and $f(X) \subseteq T$, then $g: (X, \tau_X) \to (T, \tau_T)$ defined by $g(x) = f(x)$ is continuous.

Basis:

Idea: In a metric space, $\{B_\varepsilon(P)\}$ serves as a basis.

Def: (X, τ), $\beta \subseteq \tau$ is a basis of X st any $S \in \tau$ is a union of elements in β.

Eg. 1: $\{\mathbb{R}, \text{usual}\}$, then $\{(a,b)\}_{a,b \in \mathbb{R}}$ is a basis.

2. $\{\mathbb{R}, \text{half-open}\}$ then $\{[a,b)\}$ is a basis. indeed we defined half open using such a basis!

3. $\{\mathbb{R}^2, \text{dictionary topology}\}$, $\{$all vertical line segments$\}$ is a basis.

Q: Is a basis necessarily minimal? $\exists \beta'$? $\beta' \subseteq \beta$ — $\boxed{\text{HW question}}$

No! $\{\mathbb{R}, \text{usual}\}$ $\{(c,d) \mid c,d \in \mathbb{Q}\}$ is a basis.

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properties: Lemma 1: $(X, \tau_X) \xrightarrow{f} (Y, \tau_Y)$ f is continuous; $\beta_X \to$ basis $\leftarrow \beta_Y$,

iff $\forall B_Y \in \beta_Y$, $f^{-1}(B_Y)$ is open in X.

f is open iff $\forall B_X \in \beta_X$, $f(B_X)$ is open in Y.

more important

1. ... (X, τ) ... $W \subseteq X$ open?

T is "open" iff $\forall B_X \in \beta_X$, $T(B_X)$ is open ... .

more important

Lemma 2. (X, τ_X), β_X basis, $W \subseteq X$, (open?)

then $W \in \tau_X$ $\overset{\text{iff}}{\Longleftrightarrow}$ $\forall p \in W$ $\exists$ $p \in B_p \in \beta_X$, s.t. $B_p \subseteq W$.

Pf: $\overset{\text{by def.}}{\Longrightarrow}$ W is open, then $W = \bigcup_{i \in I} B_i$, then $\forall p$, $\exists B_p$. ✓

$\Longleftarrow$ $\bigcup_{p \in W} B_p = W$ claim. $\bigcup_{p \in W} B_p \supseteq W$.

$\Downarrow$ by def. so W is open. $B_p \subseteq W \Longrightarrow \bigcup_{p \in W} B_p \subseteq W$. $\square$

(X, τ), $\beta \subseteq \tau$ s.t. $\forall U \in \tau$, $U = \bigcup_{i \in I} B_i$, $B_i \in \beta$.

β is a basis of (X, τ).

Make new space from old. : quotient space, product space.

equivalence relation : S set, $\sim$ relation is a equ relation if $\boxed{\begin{matrix} S \\ s_1 \sim s_2 \end{matrix}}$

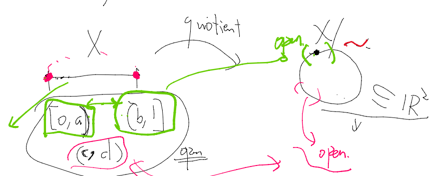
1. reflexive $x \sim x$
2. symmetric $x \sim y \Leftrightarrow y \sim x$
3. transitive $x \sim y$, $y \sim z$. then $x \sim z$.

Any. isomorphism is an equ relation.

For a set X with an equ relation $\sim$, $\forall a \in X$ we can define equivalence class, by $[a] := \{x \in X, x \sim a\}$,

define the $\boxed{\text{quotient}}$ $X/\sim$ by $X/\sim := \{[x] \mid x \in X\}$.

Ex: $[0,1]/0 \sim 1$



Remark : $\pi : X \to X/\sim$, $x \mapsto [x]$ projection map..

1. π is surjective,
2. π is injective iff $X = X/\sim$.

We want $\boxed{\text{the strongest topology}}$

What's a good topology on $X/\sim$? A: π is continuous

Def: (X, τ_X), $\sim$ is an equ. relation on X, we define

$$\tau_\sim := \{U \subseteq X/\sim \mid \pi^{-1}(U) \subseteq X \text{ open}\}.$$

$(X/\sim, \tau_\sim)$ is the quotient space of X by $\sim$. (w.r.t.)

Eg: $\boxed{[0,1]/0 \sim 1}$ $\overset{\text{homeo}}{\sim}$ $\bigcirc \subseteq \mathbb{R}^2$.

Remark: $(X, \tau) \xrightarrow{\pi} (X/\sim, \tau_\sim)$ is not necessarily open.

old space. $[0,1]$ X X

quotient

new space $[0,1]/0 \sim 1 \cong$ circle $\bigcirc$

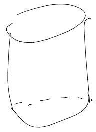
More example

$= [0,1] \times [0,1] \subseteq \mathbb{R}^2 = (x, y)$

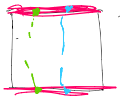
new space $[0,1]/_{0\sim 1} \simeq$

More example

1. cylinder

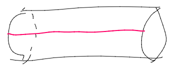


$X =$



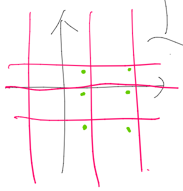
$= [0,1]\times[0,1] \subseteq \mathbb{R}^2 = (x,y)$

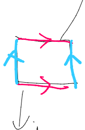
$X/\sim$: $(x,y) \sim (\bar{x},\bar{y})$ iff $\boxed{x=\bar{x},\ y=\bar{y}}$ or $x=\bar{x}$, $y=0$ and $\bar{y}=1$



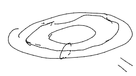
$\subseteq \mathbb{R}^3$

2. Torus $= \boxed{\mathbb{R}^2/\mathbb{Z}^2}$ lattice $=$ □ ; $X = \mathbb{R}^2$ ← usual





$X/\sim$: $(x,y)\sim(\bar{x},\bar{y})$ iff $x = p+\bar{x}$, $y = q+\bar{y}$, $p,q\in\mathbb{Z}$



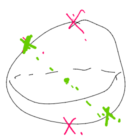
$\subseteq \mathbb{R}^3$

$= \boxed{S^1\times S^1}$

Remark : Fundamental domain : Smallest region. s.t. the gluing is well defined.

3. Projective space $\mathbb{RP}^2 = S^2/\sim$ antipodal point

$\mathbb{RP}^n = S^n/\sim$



$X = S^2 := \{\vec{v}\in\mathbb{R}^3 \mid \|\vec{v}\| = 1\}$ ($2 \to n$, $3 \to n+1$)

$X/\sim$ ~ $\vec{v}\sim\pm\vec{v}$

$\boxed{Gr_{\mathbb{R}}(n, n+1)}$ ← Grassmannian

fundamental domain

$D^2/\sim$ antipodal on ∂D^2

$D^2/\{\vec{v}\in\mathbb{R}^2 \mid \|\vec{v}\|=1,\ \vec{v}=\pm\vec{v}\}$ $\cong \mathbb{RP}^2$

2nd approach for quotient space induced by a function

$X \xrightarrow{f} Y$ $\boxed{\text{surjective}}$, then we define "∼" induced by f. → equivalence relation

$\forall\, a,b\in X$. $a\sim b$ iff $f(a) = f(b)$.

$X/\sim$ quotient ; $X/\sim \to$ $f(p)\sim f(q)$ iff $p\sim q$

Def : (X,τ_X) $X\xrightarrow{f} Y$ $\boxed{\text{surjective}}$ we define the quotient topology

quotient : $\tau_f := \{U\subseteq Y \mid f^{-1}(U)\subseteq X \text{ open}\}$

strongest topology s.t. f is continuous.

f is the quotient map.

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product topology (X,τ_X) (Y,τ_Y)

$$\begin{array}{ccc} & X\times Y & \\ {\scriptstyle f_x}\swarrow & \text{topologize} & \searrow{\scriptstyle f_y} \\ X & & Y \end{array}$$

coarsest topology s.t. f_X, f_Y are continuous

More on basis, quotients, and products.

Thm (equivalence def of basis) X set, $\mathcal{B}\subseteq 2^X$ s.t.

① $X = \bigcup_{B\in\mathcal{B}} B$

x

Thm (equivalence def of basis) X set, $\beta \subseteq 2^X$ st.

① $X = \bigcup_{B \in \beta} B$

② $B_1, B_2 \in \beta$, $\forall x \in B_1 \cap B_2$, $\exists\, B_x \subseteq B_1 \cap B_2$ and $B_x \in \beta$.

then $\tau = \{\text{Unions of elements in } \beta\} \cup \{\emptyset\}$ is a topology with $\boxed{\text{basis } \beta}$ on X.

Pf: ⓐ Want X open, $\emptyset$ open.

ⓑ Want $U, V \in \tau$, $U \cap V \in \tau$.

By def of τ, $U = \bigcup_{i \in I} B_i$, $V = \bigcup_{j \in J} B_j$. Now $\boxed{U \cap V} = \left(\bigcup_{i \in I} B_i\right) \cap \left(\bigcup_{j \in J} B_j\right)$

$\boxed{\forall}\ x \in U \cap V$, by def of τ, $\exists\, i_x$ and j_x st $x \in B_{i_x} \cap B_{j_x}$

Then: $\exists\, B_x \in \beta$, st. $B_x \subseteq B_{i_x}$ and $B_x \subseteq B_{j_x}$

claim: $U \cap V = \bigcup_{x \in U \cap V} B_x$ How to prove

$\supseteq$

$\subseteq$

ⓒ $\forall\, k \in K$ index set, $A_k \in \tau$ open,

Want $\bigcup_{k \in K} A_k \in \tau$. By def of τ, $A_k = \bigcup_{i \in I} B_i$

$\bigcup_{k \in K} A_k = \bigcup_{k \in K} \left(\bigcup_{i \in I} B_i\right) = \bigcup$ elements in β, is open. $\square$

Recall. Two defs of quotient spaces

diagram commute

$$\begin{tikzcd}
 & (X, \tau_X) \arrow[dl, "\pi"'] \arrow[dr, "f \to \text{map onto } Y"] & \\
(X/\sim, \tau_\sim) \arrow[rr, bend right, "g"'] & & (Y, \tau_f)
\end{tikzcd}$$

g surjective

Lemma (Important)

$$\begin{tikzcd}
(X, \tau_X) \arrow[rr, "f"] \arrow[dr, "g \circ f"'] & & (Y, \tau_f) \arrow[dl, "g"] \\
 & (Z, \tau_Z) &
\end{tikzcd}$$

$f: X \to Y$ surjective

$\tau_f \to$ quotient topology

$\tau_Z \to$ prescribed topology on Z

claim: g is continuous $\iff$ $g \circ f$ is continuous.

Pf: $\Rightarrow$ easy: g is continuous, f is the quotient map, hence continuous, $g \circ f$ is continuous.

$\Leftarrow$: Know $g \circ f$ continuous $\iff$ $\forall A \in \tau_Z$, $(g \circ f)^{-1} \in \tau_X$. Know: $\forall B \in \tau_f$, $f^{-1}B \in \tau_X$

want $\forall A \in \tau_Z$, $g^{-1}(A) \in \tau_f \Leftarrow \forall A \in \tau_Z$, $f^{-1}(\boxed{g^{-1}(A)}) \in \tau_X$

Thm (Functorality of quotient maps)

$$\begin{tikzcd}
(\mathbb{R} - \{1\}, \text{usual topology}) \arrow[rr, "f"] \arrow[dr, "g \circ f"'] & & (\mathbb{R}_f, \text{usual}) \arrow[dl, "g \text{ continuous}"] \\
 & (\mathbb{R}, \text{usual}) &
\end{tikzcd}$$

claim: ① $f := \begin{cases} x^2, & \forall x < 1 \\ \dfrac{1}{x-1} & \forall x > 1 \end{cases}$ f is surjective, f is continuous

$(\mathbb{R}, \text{usual}) \to (\mathbb{R}, \text{usual})$ ② $g := \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$ ~~not continuous~~

③ $g \circ f = \begin{cases} 1 & x < 1 \\ -1 & x > 1 \end{cases}$ continuous

Thm (functoriality of quotient maps)

$$(X, \tau_X) \xrightarrow[f]{\text{homeo}} (Y, \tau_Y)$$

$$\sim_X \overset{\text{"same"}}{\longleftrightarrow} \sim_Y$$

$x_1 \sim x_2$ iff $f(x_1) \sim f(x_2)$

$g \circ f := \{$

$\boxed{\text{continuous}}$

claim: $(X/\sim_X, \tau_{\sim_X}) \overset{\text{homeo}}{\simeq} (Y/\sim_Y, \tau_{\sim_Y})$

Pf:

$\exists\, g : X/\sim_X \to Y/\sim_Y$

g is a homeo, and ...

$$\begin{tikzcd}
(X, \tau_X) \arrow[r, "f"] \arrow[d, "\pi_X"'] & (Y, \tau_Y) \arrow[d, "\pi_Y"] \\
(X/\sim_X, \tau_{\sim_X}) \arrow[r, "g"] \arrow[u, bend left, "\pi^{-1}"] & (Y/\sim_Y, \tau_{\sim_Y}) \arrow[l, bend right, "g^{-1}"']
\end{tikzcd}$$

$g \circ \pi_X$

g : continuous

g^{-1} : continuous

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product spaces :

weakest topology s.t.

$$\begin{tikzcd}
& X \times Y \arrow[dl, "\pi_X"'] \arrow[dr] & \\
X & & Y
\end{tikzcd}$$

How ?

$\neq \{\text{product of open sets}\}$

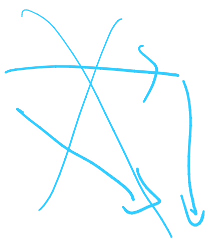
must be

$X/{\sim_X}$ $\quad$ $Y/{\sim_Y}$

...n't g: $\quad [x] \longmapsto [f(x)]$

bijective as maps btw sets.

...n.



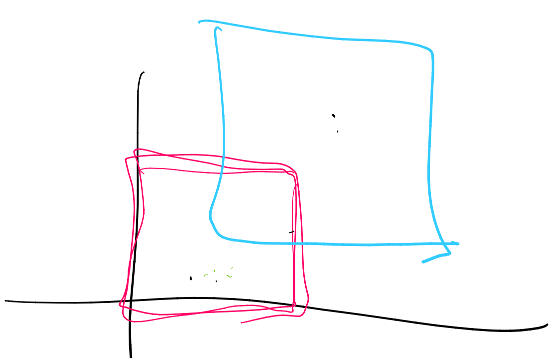
Lemma

...mios)

Lemma

$\boxed{\ldots \pi_Y}$ $\quad$ π_X and π_Y continuous

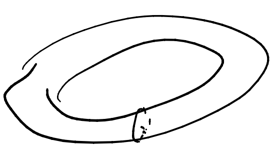
Y



Def: $\mathcal{B} := \{ \text{product of open sets} \}$

Eg.

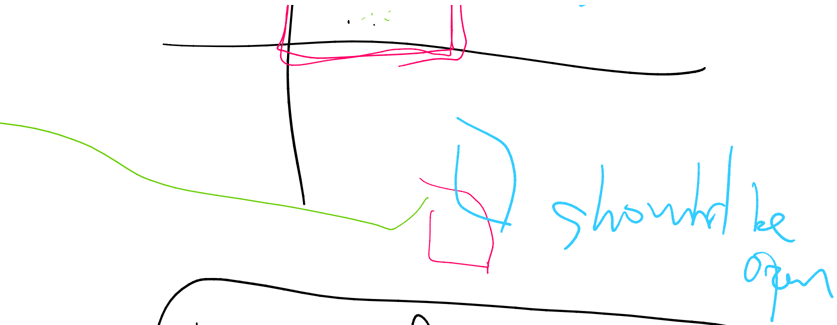
1. $\mathbb{R}^3 \supseteq$  $\overset{\text{homeo}}{\simeq}$ $\underset{\uparrow}{I} \times I \Big/ (x,0)\sim(x,1)$, with $[0,1]$ above the arrow; homeo $\simeq$

2. $\mathbb{R}^3 \supseteq$  $\overset{\text{homeo}}{\simeq}$ $\boxed{I\times I \Big/ \begin{matrix}(x,0)\sim(x,1)\\(0,y)\sim(1,y)\end{matrix}} \nearrow \mathbb{R}^2/\mathbb{Z}^2$; homeo $\simeq$

3. $\mathbb{R}^n/\mathbb{Z}^n \overset{\text{homeo}}{\simeq} T^n \overset{\text{homeo}}{\simeq} \prod_{i=}^{n} S^1_i$

Properties of product spaces

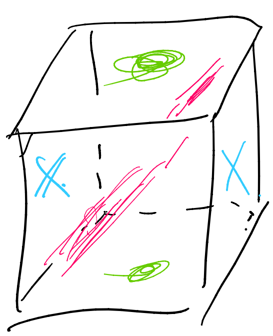
Fa



$\longrightarrow \boxed{\text{basis for } \tau_{(X\times Y)}}$

$S^1 \times I$

$S^1 \times S^1$

$T^3 \simeq$ 

$(X, Y, \tau_{X\times Y}) \xrightarrow{\pi_X} (X, \tau_X)$

$$\begin{tikzcd} & X \times Y \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ X & & Y \end{tikzcd}$$

$\pi_X : (x, y) \to x$

$\pi_Y : (x, y) \to y$

Lemma: π_X and π_Y are open.

products of open $\subseteq X$ and Y

pf: π_X sends basic open sets to open

$B_i \in \mathcal{B}_{X \times Y}$

$$\pi_X \left(\bigcup_i B_i \right) = \bigcup_i \pi_X (B$$

Lemma (Important)

$$\begin{tikzcd} & (X \times Y, \tau_{X \times Y}) \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ (X, \tau_X) & & (Y, \tau_Y) \\ & (A, \tau_A) \arrow[ul, "f"] \arrow[ur, "g"'] \arrow[uu, bend right, "h"'] & \end{tikzcd}$$

Pf: $\Longleftarrow$ done, because π_X, π_Y are continuous

$\Longrightarrow$ We assume f, g are continuous.

Want: h^{-1}(every

$(X, Y, \tau_{X \times Y}) \to (X, \tau_X)$

then π_X and π_Y are continuous

② Same setting, then

π_X and π_Y are **open**!

sets,

quotient maps are not always open!!

) open. $\square$

f and g are continuous

f

h is continuous,

Note:

$$\boxed{\begin{aligned} f &= \pi_X \circ h \\ g &= \pi_Y \circ h \end{aligned}}$$

ns.

(base open sets) $\to$ open in A.

$\Longrightarrow$ f, g are continuous.

Want: h^{-1} (every ...

$U \times V$, $U \in \tau_X$ and $V \in \tau_Y$

$h^{-1}\left(\prod_{i \in 2} U_i\right)$

$\boxed{h^{-1}(U \times V)} = \{ a \in A \mid h(a) \in \ldots$

$= \{ a \in A \mid (f(a), g(\ldots$

$= \{ a \in A \mid f(a) \in \ldots$

$= f^{-1}(U) \cap \ldots$

$\pi_i^{-1}(U_i) \to$ not ne...

$i \in I$

$\mathbb{R} \to \mathbb{R}$

$f_1 = x^2$ continuous

$f_2 = x^3 + x$

$f_3 = \sin x$

$\boxed{\mathbb{R} \to \mathbb{R}^3} : x \mapsto (x^2, x^3+x, \sin x)$ is continuous

Infinite product:

$\prod_{i \in I} X_i$ $\qquad I$

choice of topology?

box topology $\boxed{\tau_\square}$: has a...

). base open sets) $\to$ open in A

$^{-1}(U)$ and $g^{-1}(V) \in \mathcal{T}_A$.

$U \times V \}$

$a)) \in U \times V \}$

$\in U$ and $g(a) \in V \}$

$f^{-1}(U)$ $\to$ intersection of opensets in A

$\to$ open in A!

cessarily open.

$\square$

is an index set.

$\{0,1\}^{\mathbb{R}}$

basis $\{$ product of open sets in X_i $\}$

$\rho_\square$

"Important" lemma still holds? **No!**

We have to deal

Correct choice: product topology for ∞-comp

$\boxed{\tau_\Pi}$

$\boxed{\beta_\Pi} \longrightarrow$ basis $= \{$ product

Important lemma still holds.

Compare $\beta_\square$ and β_Π:

$(X, X \times$

① $\beta_\square$ and β_Π are both generalizations of $\beta_{X \times Y}$.

same. $\to$ product topology

② $\beta_\Pi \subseteq \beta_\square$

$\boxed{\text{"weakest"}}$ top

$\boxed{\beta_\Pi}$

with ∞ $\bigcap$.

onents :

of finite open sets in X_i's $\}$.

fixed value

$) \subseteq (\prod_{i \in I} X_i)$

nb space topology of $\beta_{\square}$ or $\beta_{\prod}$

on $(X_1 \times X_2)_\tau$

ology on product spaces

③ Convention : ~~We use~~ When not explic

product topology on $\prod_{i \in I} X_i$ m

Compactness Hausdorff.

Rmk : quotient space $\underline{D_2 \cup_f M}$ ...

How to distinguish spaces ?

$\triangleleft$ Prove $X \overset{homeo}{\cong} Y$, we only need

$\triangleleft$ Prove $X \not\cong Y$, hard.

Cpt, Hausdorff, connectedness, ...

Cpt : Def. (X, τ_X), $A \subseteq X$. A

finiteness

$\forall$ collection $\mathcal{C} \subseteq \tau_X$ if $A \subseteq \bigcup_{i \in \mathcal{C}} U_i$

itting addressed,

eanel B_{Π}.

---

ostrant

$\overline{D_2 \amalg M}$

f

a map $X \xrightarrow{f} Y$ → homeo

invariant property under homeo

$|X|$

t is compact if

, we can find a finite $F \subseteq C$, st.

finiteness

$\bigcup_{i \in C} U_i$

$A \subseteq \bigcup_{i \in F} U_i$.

Thm 1: Any closed subset A of a compact

D_n S_n are compact

Pf: $X = A \cup (X \setminus A)$

Thm 2: (Heine-Borel): $(\mathbb{R}^n$, usual$)$,

Thm 3: (Tychonoff) (X, τ_X) (Y, τ_Y) are cpt. sp

Hausdorff is a seperation property

Def (Hausdorff): (X, τ_X) $\forall$ $x, y \in X$

$\exists$ U and $V \in \tau_X$ s.t. $U \cap V = \phi$ and $x \in U$, $y \in V$

Eg. Non-Hausdorff space: $\{0, 1\} = X$ $\tau_X =$

, we can

space is compact.

X

$\square$

$A \subseteq \mathbb{R}^n$ is cpt iff A is closed and bounded.

nce, then $(X \times Y, \mathcal{T}_{X \times Y})$ is cpt.

finite

$\to$ T_0 T_1 $T_{1\frac{1}{2}}$ T_2 T_3 T_4

Hausdorff

Any reasonable object should be hausdorff

$\subseteq X$

$\neq X$.

$= \{\emptyset, X\}$ We can not seperate $0, 1$.

Any set S with $|S| \geq$

Eg: $\left(\dfrac{\text{continuous map}}{\text{does not preserve } T_2}\right)$:

$\boxed{\begin{array}{l} \{0,1\} \\ \tau = \{\phi, \{0\} \\ \quad \{1\}, \{0,1\}\} \end{array}}$

Hausdorff

Thm: $(X, \tau_X) \xrightarrow{g} (Y, \tau_Y)$, g is a homeomorphi

pf:

Interaction between cpt and Haus

Thm (X, τ_X) Hausdorff, $A \subseteq X$ cpt.

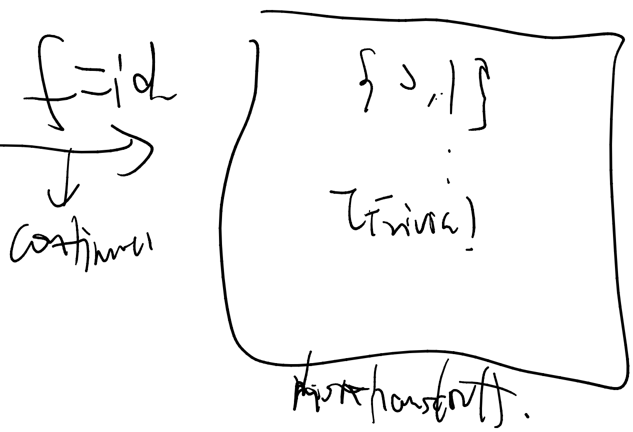
Pf: Next week. $\square$

Rmk: (X, τ_X) is cpt and Hausdorff. $A \subseteq X$

Application: $(X, \tau_X) \xrightarrow{\quad} (Y, \tau_Y)$ continuous surjec

D_n A S_n

2, T(trivial), → non-Hausdorff.



ism, then X is Hausdorff iff Y is hausdorff.

$\square$

---

dorffness.

), A is closed.

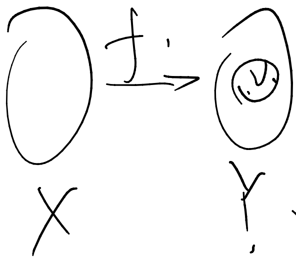
closed $\Leftrightarrow$ cpt.

tion, X and Y are cpt. and Hausdorff,

then f is a quotient map.

Pf: Want: ① $f^{-1}(V)$ is open in

quotient topology ⇐ ② $T_Y = \{ U \subseteq Y \mid$



$Y \subseteq Y$ $\quad f(f^{-1}(V)) = V$ be

~~open~~ closed $\quad$ closed

want f is a close map.

$Z = f^{-1}(V)$ closed in X $\longleftarrow$ f is continuous $\longrightarrow$ $f(Z) = V$ is

This proves ②.

$\overline{D_n} \longrightarrow \mathbb{R}^{n+1}$

$|\vec{x}| \le 1 \subseteq D_n$

$x \longmapsto \left(\frac{2\sqrt{\|x\|(1-\|x\|)}}{\|x\|}, \; 2\|x\|-1 \right.$

$D_2 \cup_f M \longleftarrow$ polygon re

$\| \longrightarrow$ How to prove?

homeo

and Hausdorff,

X if U is open in Y ← f continuous

$f^{-1}(U)$ is open } ← weakest topology

cause f is surjective.

X is cpt $\Longrightarrow$ Z is cpt in X

cpt. in Y $\xRightarrow{Y \text{ is Hausdorff}}$ V is closed.

$\square$

continuous

Image is S_n.

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representation

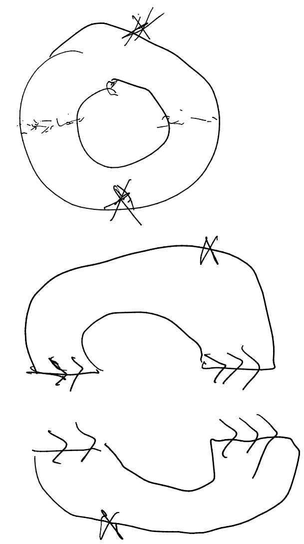
$\| \longrightarrow$ How to prove?

$\mathbb{RP}^2$

$\|$

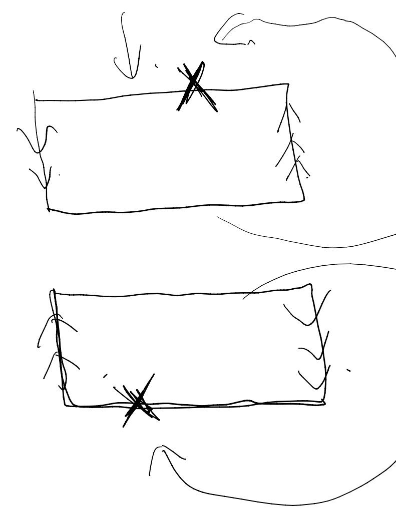
$\bar{D}_2$

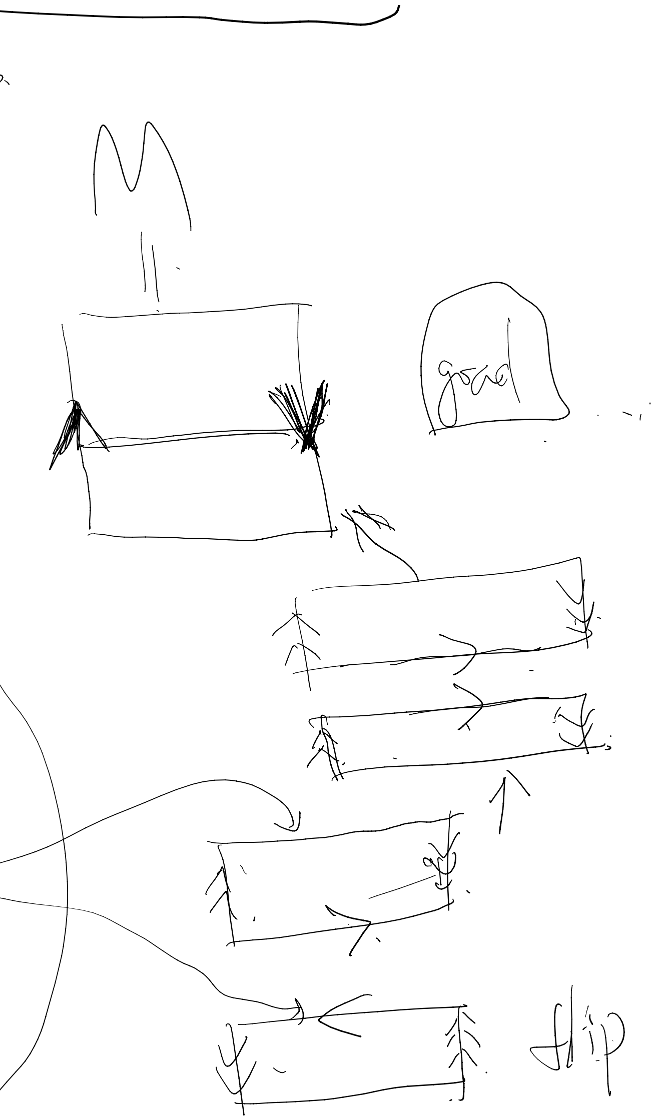
Show:



home…

$\cong$





flip horizontally