STABILITY OF THE SYMPLECTOMORPHISM GROUP OF RATIONAL SURFACES

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Abstract. We apply Zhang's almost Kähler Nakai-Moishezon theorem and Li-Zhang's comparision of J-symplectic cones to establish a stability result for the symplectomorphism group of a rational surface with Euler number up to 12.

Contents

1. Introduction

Let M be a closed, oriented 4-manifold and ω a symplectic form on M. Let $\mathcal{S}_\omega$ denote the set of homology classes of embedded ω-symplectic spheres and K_ω the symplectic canonical class. For any $A \in \mathcal{S}_\omega$, by the adjunction formula,

$$K_\omega \cdot A = -A \cdot A - 2. \tag{1}$$

Note that we'll need the following subsets of S_ω: let

$$\mathcal{S}_\omega^{\geq n}, \quad \mathcal{S}_\omega^{>n}, \quad \mathcal{S}_\omega^{n}, \quad \mathcal{S}_\omega^{\leq n}, \quad \mathcal{S}_\omega^{<n}$$

be the subsets of symplectic spherical classes with square $\geq n, > n, = n, \leq n, < n$ respectively. The set S_ω^{-2} turns out to be very useful in the study of π_0 and π_1 of $Symp(X_k, \omega)$ where X_k is a rational 4-manifold diffeomorphic to $\mathbb{C}P^2$ blow up at k points, see [LLci] [LLWxi] for details. And this note is a continuation of this idea to investigate how $S_\omega^{\leq -n}, n > 2$, will be related to higher π_i's.

We use the framework of McDuff in [McD01]. Let $G_\omega = Symp(M, \omega) \cap \text{Diff}_0(M)$, where $\text{Diff}_0(M)$ is the identity component of the diffeomorphism group. Here are the main results of this note:

Theorem 1.1. *Let M be a rational surface with $\chi \leq 12$. Let ω and ω' be two symplectic forms on M with $\mathcal{S}_\omega = \mathcal{S}_{\omega'}$. Then*

$$\pi_i(Symp(M, \omega)) = \pi_i(Symp(M, \omega'))$$

for all $i > 0$, and $\pi_0(G_\omega) = \pi_0(G_{\omega'})$.

A stronger version on certain degree homotopy groups is:

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Theorem 1.2. *Let M be a rational surface with $\chi \leq 12$. If $\mathcal{S}_{\omega}^{\geq -n} = \mathcal{S}_{\omega'}^{\geq -n}$, then the groups π_i of G_w and $G_{w'}$ are the same for $1 \leq i \leq 2n-3$.*

Note that the above results generalize the stability results in [AM00; LP04; ALP09; AP13; AE19], etc. Our proof uses the inflation strategy in [McD01], and the cone Theorem in [Zha17] greatly simplified the inflation process. Also, the comparison of J-symplectic cones in [LZ09] plays an important role, see the discussion in section 3.2 for details.

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2. The spaces $\mathcal{T}_u$ and $\mathcal{A}_u$

2.1. **The strategy.**

We start by recalling the strategy in [McD01].

Let $u = [\omega]$ and $\mathcal{T}_u$ be the space of symplectic forms in the class of w. For a rational surface, $\mathcal{T}_u$ is path connected ([LM96; LL01]). In this case, if $\omega, \omega' \in \mathcal{T}_u$ then they are isotopic by Moser's method, and hence the symplectomorphism groups $Symp(M, \omega)$ and $Symp(M, \omega')$ are homeomorphic as topological groups.

Moser's method also grants that $\text{Diff}_0(M)$ acts transitively on $\mathcal{T}_u$ in this case, hence we have the fibration

$$G_\omega \to \text{Diff}_0(M) \to \mathcal{T}_u, \tag{2}$$

Let $\mathcal{A}_u$ be the space of almost complex structures that are compatible with some $\omega \in \mathcal{T}_u$. Consider the space P_u of pairs

$$P_u = \{(\omega, J) | \mathcal{T}_u \times \mathcal{A}_u : \omega \text{ is compatible with } J\},$$

Since the projection $P_u \to \mathcal{A}_u$ is a fibration with the fiber at J being the convex set of J-compatible symplectic forms, the projection is a homotopy equivalence. The projection $P_u \to \mathcal{T}_u$ is also a homotopy equivalence since it is a fibration with the fiber at ω being the contractible set of ω-compatible almost complex structures. Hence T_u and $\mathcal{A}_u$ are homotopy equivalent.

Remark 2.1. *Note that in [McD01], the $\mathcal{A}$ spaces are almost complex structures* **tamed** *by some ω in the space $\mathcal{T}_u$. Here we are using compatible J's. The proof of the homotopy equivalences between T_u and $\mathcal{A}_u$ are the same, using the fact that the projections have convex fibers.*

Via the homotopy equivalence, we have the following fibration, well defined up to homotopy.

$$G_\omega \to \text{Diff}_0(M) \to \mathcal{A}_u. \tag{3}$$

Let $\mathcal{S}_u$ denote the set of homology classes that are represented by an embedded ω-symplectic sphere for some $\omega \in \mathcal{T}_u$. Notice that $\mathcal{S}_u = \mathcal{S}_\omega$ for any $\omega \in \mathcal{T}_u$.

In the next section, we introduce a decomposition of $\mathcal{A}_u$ via $\mathcal{S}_u$.

2.2. **A fine decomposition of $\mathcal{A}_u$ via symplectic spheres.**

When (X, ω) is a symplectic 4-manifold, we introduced in [LLci] a decomposition of $\mathcal{J}_\omega$ via embedded ω-symplectic spheres of self-intersection at most -2. We introduce the corresponding decomposition for $\mathcal{A}_u$.

For each $A \in \mathcal{S}_\omega^{<0}$ we associate the integer

$$cod_A = 2(-A \cdot A - 1).$$

Definition 2.2. *Let u be a symplectic class. Given a finite subset $\mathcal{C} \subset \mathcal{S}_u^{<0}$,*

$$\mathcal{C} = \{A_1, \cdots, A_i, \cdots, A_n | A_i \cdot A_j \geq 0 \text{ if } i \neq j\},$$

define the codimension of the set $\mathcal{C}$ as $cod(\mathcal{C}) = \sum_{A_i \in \mathcal{C}} cod_{A_i}$.

We call such set $\mathcal{C}$ an **admissible subset of** $S_\omega^{<0}$ *with codimension being $cod(\mathcal{C})$.*

Definition 2.3. *Given $\mathcal{C}$ as above, we define* **prime subsets**

$$\mathcal{A}_{u,\mathcal{C}} = \{J \in \mathcal{A}_u | A \in \mathcal{S}_u \text{ has an embedded } J\text{-hol representative } \textbf{if and only if } A \in \mathcal{C}\}.$$

And we define $cod(\mathcal{A}_{u,\mathcal{C}}) = cod(\mathcal{C})$.

Clearly, we have the decomposition: $\mathcal{A}_u = \amalg_{\mathcal{C}} \mathcal{A}_{u,\mathcal{C}}$. And in Proposition 2.7 we will prove that, under condition 1 below, the $\mathcal{A}_{u,\mathcal{C}}$ are submanifolds of $\mathcal{A}_u$ and the codimension in the definition is the actual codimension.

We choose the following notation:

$$\mathcal{A}_u^{2n} = \amalg_{\mathcal{C}, cod(\mathcal{C}) < 2n} \mathcal{A}_{u,\mathcal{C}}, \quad \text{and} \quad \mathcal{X}_{u,2n} = \amalg_{\mathcal{C}, cod(\mathcal{C}) \geq 2n} \mathcal{A}_{u,\mathcal{C}}.$$

Clearly, $\mathcal{A}_u^{2n} = \mathcal{A}_u \setminus \mathcal{X}_{u,2n}$. Note we have the filtration

$$\mathcal{X}_{u,2n} \subset \cdots \subset \mathcal{X}_{u,2} \subset \mathcal{X}_{u,0} = \mathcal{A}_u,$$

similarly to the one in [LLci] for $\mathcal{J}_\omega$.

It is convenient to also introduce the following notation following the idea in [AP13]:

$$U_{u,\mathcal{C}} = \{J \in \mathcal{A}_u | A \in \mathcal{S}_u \text{ has an embedded } J\text{-hol representative } \textbf{if } A \in \mathcal{C}\}.$$

Clearly, $U_{u,\mathcal{C}} \supset \mathcal{A}_{u,\mathcal{C}}$.

Note that $\mathcal{X}_{u,2n} = \cup_{\mathcal{C}, cod(\mathcal{C}) \geq 2n} U_{u,\mathcal{C}}$, and note that this union is not necessarily disjoint.

We have the following result, which can be proved using the same idea as in [Abr98] appendix and [McD01] Lemma 2.6.

Proposition 2.4. *Let X be a 4-manifold, and $u = [\omega]$ is a cohomology class of some symplectic form. Then $U_{u,\mathcal{C}}$ is a co-oriented Fréchet submanifold of $\mathcal{A}_u$ of (real) codimension $cod(\mathcal{C})$.*

In particular, $\mathcal{X}_{u,2n}$ is the union of submanifolds with codimension at least $2n$.

Note that the above result is very similar but still different from [AP13] Proposition B.1. The only difference is that we consider space $\mathcal{A}_u$ instead of $\mathcal{J}_\omega$ here. And the proof is the same, consider the projection from the universal moduli of curves in classes $\mathcal{C}$ onto $\mathcal{A}_u$ and compute the index of the linearized operator. The index computation goes the same as Proposition B.1 of [AP13].

Now we use the argument in [LLci] to conclude the prime sets $\mathcal{A}_{u,\mathcal{C}}$ are submanifolds of correct codimension in $\mathcal{A}_u$ under the following condition.

Condition 1. *If A is a homology class in $H^2(X;\mathbb{Z})$ with negative self-intersection, which is represented by a simple J-holomorphic map $u : \mathbb{C}P^1 \to M$ for some tamed J, then u is an embedding.*

And by [Zha17] Proposition 4.2, we have

Lemma 2.5. *Condition 1 holds true for $S^2 \times S^2$ and $\mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}, 0 \leq k \leq 9$.*

We need to establish the counterpart of Proposition 2.14 in [LLci] and it suffices to prove Lemma 2.16 in the $\mathcal{A}_u$ setting.

Lemma 2.6. *Assume Condition 1. If $\mathcal{C}' \subset \mathcal{C}$ but $\mathcal{C}' \neq \mathcal{C}$, then $\overline{\mathcal{A}}_{u,\mathcal{C}} \cap \mathcal{A}_{u,\mathcal{C}'} = \emptyset$.*

Proof. Let's assume that $J_0 \in \overline{\mathcal{A}}_{u,\mathcal{C}} \cap \mathcal{A}_{u,\mathcal{C}'}$ which is compatible with ω_0. Since $J_0 \in \overline{\mathcal{A}}_{u,\mathcal{C}}$, we know that there is a sequence of $\{J_n\} \subset \mathcal{A}_{u,\mathcal{C}}$ converging to J_0. Recall remark 2.1, we have a projection from the product space $P_u = \{(\omega, J)|\mathcal{T}_u \times \mathcal{A}_u\}$ to $\mathcal{A}_u$ with convex fibers. Hence we can choose a sequence of pairs $(\omega_n, J_n) \in P_u$, converging to (ω_0, J_0). Note that each J_n is compatible with ω_n, which is diffeomorphic (indeed strongly isotopic) to ω_0 through $f_n \in \mathrm{Diff}(M)$. Now pushing each ω_n to ω_0 by the diffeomorphism f_n, one gets a new sequence of pairs (ω_0, J_n') still converging to (ω_0, J_0), where $J_n' = f_n^* J_n$. Then for a fixed ω_0, we have a sequence of J_n' converging to J_0 in the space $\mathcal{J}_{\omega_0}$, which is the space of ω_0-compatible almost complex structures.

Note that J_n''s can make any class in $\mathcal{C}$ holomorphic but J_0 can only make classes in $\mathcal{C}'$ holomorphic. This contradicts Lemma 2.16 in [LLci]. ∎

Hence we have the following:

Proposition 2.7. *Assume Condition 1 holds. For an admissible set $\mathcal{C}$ with nonempty $\mathcal{A}_{u,\mathcal{C}}$, the prime set $\mathcal{A}_{u,\mathcal{C}}$ is a paracompact Hausdorff submanifold of $\mathcal{A}_u$ with $cod(\mathcal{A}_{u,\mathcal{C}}) = cod(\mathcal{C}) = \sum_{A_i \in \mathcal{C}} cod_{A_i}$.*

3. The almost Kähler Nakai-Moishezon criterion and J-inflation

3.1. **Nakai-Moishezon and cone theorem in the almost Kähler setting.** Recall the two notions of J-symplectic cones, the J-tame cone and the J-compatible cone:

$$\begin{array}{rcl} \mathcal{K}_J^t & = & \{[\omega] \in H^2(M;\mathbb{R}) | \omega \text{ tames } J\}, \\ \mathcal{K}_J^c & = & \{[\omega] \in H^2(M;\mathbb{R}) | \omega \text{ is compatible with } J\}. \end{array} \tag{4}$$

$\mathcal{K}_J^c$ is also called the almost Kähler cone. Both $\mathcal{K}_J^c$ and $\mathcal{K}_J^t$ are convex cohomology cones contained in the positive cone

$$\mathcal{P} = \{e \in H^2(M;\mathbb{R}) | e \cdot e > 0\}.$$

Clearly, $\mathcal{K}_J^c \subset \mathcal{K}_J^t$. And for an almost Kähler J on a 4-manifold with $b^+ = 1$, they are equal.

Theorem 3.1 (Theorem 1.3 in [LZ09])**.** *Now let M^4 be an almost complex 4-manifold. If $b^+(M^4) = 1$ and $K_J^c \neq \emptyset$, then $\mathcal{K}_J^c = \mathcal{K}_J^t$.*

Let $C_J(M)$ be the curve cone of a compatible almost complex manifold (M, J):

$$C_J(M) = \{\sum a_i [C_i] | a_i > 0,\ C_i \text{ is a irreducible J-holomorphic subvariety } \}.$$

Let $C_J^{\vee,>0}(M)$ be the positive dual of $C_J(M)$ under the homology-cohomology pairing, and set

$$\mathcal{P}_J = C_J^{\vee,>0}(M) \cap \mathcal{P}.$$

Clearly, $\mathcal{K}_J^t \subset C_J^{\vee,>0}(M)$ since the integral of a J-tamed symplectic form over a J-holomorphic subvariety is positive. Motivated by the famous Nakai-Moishezon-Kleiman criterion in algebraic geometry which characterizes the ample cone in terms of the (closure of) curve cone for a projective J, and the recent Kähler version[1] of the Nakai-Moishezon criterion (in dimension 4), which characterizes the Kähler cone in terms of the curve cone for a Kähler J, we ask whether there is a tamed/almost Kähler version of the Nakai-Moishezon criterion.

Here is the tamed Nakai-Moishezon theorem for rational surfaces with $\chi \leq 12$:

Theorem 3.2 (Theorem 1.6 in [Zha17])**.** *Suppose $M = S^2 \times S^2$ or $\mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}, k \leq 9$. For an almost Kähler J on M, the dual cone of the curve cone is the almost Kähler cone, i.e. $C_J^{\vee,>0}(M^4) = K_J^c(M^4)$.*

[1]Established by Buchdahl and Lamari in dimension 4, and by Demailly-Paun in arbitrary dimension.

3.2. **A remark on J-inflations on the 4-manifolds with $b^+ = 1$.** Notice that Theorem 3.2 is stated for an almost Kähler J and the almost Kähler cone, but there is also a version of the tame cone. An important ingredient for Theorem 3.2 is the tamed J-inflation by Lalonde, McDuff [LM96; McD01] and Buse [Bus11]. Note that to prove the almost-Kähler version of Theorem 3.2 (which we are using in this note), one only need a weaker version of Lemma 3.1 in [McD01] and Theorem 1.1 in [Bus11]:

Given a compatible pair (J, ω), one can inflate along a J-holomorphic curve Z, so that there exist a symplectic form ω' taming J such that $[\omega'] = [\omega] + tPD(Z), t \in [0, \lambda)$ where $\lambda = \infty$ if $Z \cdot Z \geq 0$ and $\lambda = \frac{\omega(Z)}{(-Z\cdot Z)}$ if $Z \cdot Z < 0$.

The strategy is **to start with an almost Kähler J and perform the tame inflation** to obtain a J-tame ω' in the correct cohomology class. Then by Theorem 3.1, the almost Kähler cone is the same as the tame cone when $b^+ = 1$, and one can conclude that J is compatible with some symplectic form in the same cohomology class as ω'.

Note that this weaker version of inflation indeed covers all the stability of $Symp(M, w)$ results given $b^+(M) = 1$.

4. Stability of $Symp(X, \omega)$, proof of main results

Recall that $\mathcal{S}_u$ is the set of homology classes that are represented by an embedded ω-symplectic sphere for some $\omega \in \mathcal{T}_u$, and $\mathcal{S}_u = \mathcal{S}_\omega$ for any $\omega \in \mathcal{T}_u$. Similarly, we'll also use the notation $\mathcal{S}_u$ for ω' symplectic spherical classes. Let S_J be the set of embedded J-holomorphic rational curve classes. Then clearly $S_J \subset \mathcal{S}_\omega$ for any ω taming J.

Firstly, the inclusion of the curve sets will tell us much about the almost complex structure s-pace.

Lemma 4.1. *Let M be a rational surface with $\chi \leq 12$. If $\mathcal{S}_u \subset \mathcal{S}_{u'}$, then $\mathcal{A}_u \subset \mathcal{A}_{u'}$.*

Proof. Suppose $J \in \mathcal{A}_u$. Then J is compatible with some $\omega \in \mathcal{T}_u$. Note that $S_J \subset \mathcal{S}_\omega$. Since w' is positive on $\mathcal{S}_{\omega'} \supset \mathcal{S}_u$, we have w' is positive on S_J. Then, by Theorem 3.2 and Lemma 2.5 we conclude that w' is in the almost Kähler cone of J. In other words, $J \in \mathcal{A}_{u'}$. We have shown $\mathcal{A}_u \subset \mathcal{A}_{u'}$ □

The following lemma is the case when we have inclusion from two directions.

Lemma 4.2. *If $\mathcal{S}_u^{\geq -n} = \mathcal{S}_{u'}^{\geq -n}$, then $\mathcal{A}_u^{2n} = \mathcal{A}_{u'}^{2n}$.*

Proof. Since $\mathcal{S}_u^{\geq -n} = \mathcal{S}_{u'}^{\geq -n}$, the decompositions of $\mathcal{A}_u^{2n}$ and $\mathcal{A}_{u'}^{2n}$ are indexed by the same set of admissible subsets of $S_u^{<n}$ with codimension less than 2n.

We take any admissible subset $\mathcal{C} \subset S_u^{<0}$ with codimension less than 2n, and for dimension reasons $\mathcal{C}$ has to be a subset of $S_u^{<n}$. Suppose $J \in \mathcal{A}_{u,\mathcal{C}}$, then J is compatible with some $w \in \mathcal{T}_u$ and the **only** J-holomorphic curves are in the classes of $\mathcal{C}$. Since u' is positive on all of the classes in $\mathcal{C}$ and hence by Theorem 3.2, u' is in the almost Kähler cone of J. In other words, $J \in \mathcal{A}_{u',\mathcal{C}}$. The same strategy applies to prove the converse. This means that any pair of corresponding prime subsets of $\mathcal{A}_u$ and $\mathcal{A}_{u'}$ of codimension less than 2n are the same. Then we know that $\mathcal{A}_u^{2n}$ and $\mathcal{A}_{u'}^{2n}$ are the same. □

To be more explicit, let us look at the case $n = 3$. Suppose $J \in \mathcal{A}_{u,A}$ where $A \in \mathcal{S}_u^{-3}$. Then J is compatible with some $w \in \mathcal{T}_u$ and the only J-holomorphic curves with self-intersection at most -3 are in the class A. Since $A \in \mathcal{S}_u^{-3} = \mathcal{S}_{u'}^{-3}$, u' is positive on A, and hence by Theorem 3.2, u' is in the almost Kähler cone of J. In other words, $J \in \mathcal{A}_{u',A}$. Converse inclusion is proved in the same way.

Suppose $J \in \mathcal{A}_{u,\{A_1,A_2\}}$ where $A_i \in \mathcal{S}_u^{-2}$. Then J is compatible with some $w \in \mathcal{T}_u$ and the **only** J-holomorphic curves with self-intersection at most -2 are in the class A_i, $i = 1, 2$. Since $A_i \in \mathcal{S}_u^{-2} = \mathcal{S}_{u'}^{-2}$, u' is positive on A_i (as well as on $\mathcal{S}_u^{-1}$) and hence by Theorem 3.2, u' is in the almost Kähler cone of J. In other words, $J \in \mathcal{A}_{u',\{A_1,A_2\}}$. Converse inclusion is proved in the same way. The case that the admissible set $\mathcal{C} = \{A\}$ where A is of square -2 also follows from the same argument.

The following is essentially due to McDuff's beautiful observation in [McD01].

Lemma 4.3. *Suppose there is an inclusion map $\mathcal{A}_u \subset \mathcal{A}_{u'}$ and $\mathcal{A}_u^{2n} = \mathcal{A}_{u'}^{2n}$. Then the groups π_i of G_w and $G_{w'}$ are the same for $1 \le i \le 2n-3$, where $[\omega] = u$ and $[\omega'] = u'$.*

Proof. Suppose there is an inclusion map $\mathcal{A}_u \subset \mathcal{A}_{u'}$, then there is an induced map $G_w \to G_{w'}$ well defined up to homotopy and makes the diagram (3) for w and w' homotopy commute.

$$\begin{array}{ccccc} G_\omega & \longrightarrow & \mathrm{Diff}_0(M) & \longrightarrow & \mathcal{A}_u \\ \downarrow & & \downarrow & & \downarrow \\ G_{\omega'} & \longrightarrow & \mathrm{Diff}_0(M) & \longrightarrow & \mathcal{A}_{u'}. \end{array} \tag{5}$$

The complement of $\mathcal{A}_u \subset \mathcal{A}_{u'}$ has codimension $2n$, since by 4.2 they have the same prime sets up to codimension $2n-2$, i.e $\mathcal{A}_u^{2n} = \mathcal{A}_{u'}^{2n}$, and $\mathcal{X}_{u,2n}$, $\mathcal{X}_{u',2n}$ each has codimension 2n. Then the inclusion induce an isomorphism $\pi_i(\mathcal{A}_u) \to \pi_i(\mathcal{A}_{u'})$ for $i \le 2n-2$. Therefore, from the homotopy commuting diagram and the associated diagram of long exact homotopy sequences of homotopy groups, the induced homomorphisms $\pi_i(G_w) \to \pi_i(G_{w'})$ are isomorphisms for $1 \le i \le 2n-3$.

□

Remark 4.4. *Note that we indeed only need the fact that $U_{u,\mathcal{C}}$ are submanifolds of correct codimension. Note that $\mathcal{A}_u^{2n} = \mathcal{A}_u \setminus \mathcal{X}_{u,2n} = \mathcal{A}_{u'} \setminus \mathcal{X}_{u',2n}$. The point is to argue that both $\mathcal{X}_{u,2n}$ and $\mathcal{X}_{u',2n}$ has codimension $2n$ or higher in $\mathcal{A}_{u'}$. Then either $\mathcal{X}_{u,2n} = \amalg_{\mathcal{C}} \mathcal{A}_{u,\mathcal{C}}, cod(\mathcal{C}) \ge 2n$ or $\mathcal{X}_{u,2n} = \cup_{\mathcal{C}} U_{u,\mathcal{C}}, cod(\mathcal{C}) \ge 2n$ would work. Because we know that by Proposition 2.4, each $U_{u,\mathcal{C}}, cod(\mathcal{C}) \ge 2n$ is a submanifolds of codimension no less than 2n; or we can use Proposition 2.7 (which depends on Proposition 2.4) to conclude that each $\mathcal{A}_{u,\mathcal{C}}, cod(\mathcal{C}) \ge 2n$ is a submanifolds of codimension no less than 2n.*

Hence we only need 2.4 to run the long exact homotopy sequences of homotopy groups. But note that to use the Proposition 3.2 we still need the decomposition of $\mathcal{A}_u^{2n}$ into a disjoint union of prime sets, but those prime sets don't necessarily need to be submanifolds.

This is to say that Lemma 2.5 and 2.6 are indeed not necessary in the proof of the above Lemmas. But it's still worth mentioning the manifold structures about the prime subsets.

Remark 4.5. *When $n = 1$, $\mathcal{S}_\omega^{-1}$ is the same for any reduced ω.*

Since $\mathcal{S}_u = \mathcal{S}_\omega$ for any $\omega \in \mathcal{T}_u$, the following is equivalent to Theorem 1.1.

Proposition 4.6. *If $\mathcal{S}_u \subset \mathcal{S}_{u'}$ and $\mathcal{S}_u^{\ge -n} = \mathcal{S}_{u'}^{\ge -n}$, then the groups π_i of G_w and $G_{w'}$ are the same for $i \le 2n-3$.*

Consequently, if $\mathcal{S}_u = \mathcal{S}_{u'}$, then π_i of G_w and $G_{w'}$ are the same for all i.

Proof. It follows from the three lemmas above, Lemma 4.3, Lemma 4.1, Lemma 4.2.

□

Finally, there's a stronger version which assures the stability on each level (Theorem 1.2). Note that the previous statements are assuming some inclusion relation $S_u \subset S_{u'}$, and the final theorem we'll drop this assumption.

Theorem 4.7. *Let M be a rational surface with $\chi(M) \leq 12$. Let ω and ω' be two symplectic forms on M whose cohomology classes are u and u' respectively. If $\mathcal{S}_u^{\geq -n} = \mathcal{S}_{u'}^{\geq -n}$ for some $n \geq 2$.*

Then $\pi_i(Symp(M,\omega)) = \pi_i(Symp(M,\omega'))$ for $1 \leq i \leq 2n-3$.

Proof. Firstly let's recall some facts on linear (in)equalities in $\mathbb{R}^n$.

Lemma 4.8. *One linear equation in n variables defines a hyperplane, cutting $\mathbb{R}^n$ into two half-spaces. Any set defined by a system of linear inequalities is convex.*

Also, we will use the following (standard) notation of symmetric difference of two sets A, B:

$$A \triangle B = (A \setminus B) \cup (B \setminus A) .$$

Now let's start with $\mathcal{S}_u$ and $\mathcal{S}_{u'}$, which are the sets of classes of negative spheres that are symplectic w.r.t ω and ω' respectively. We take $\mathcal{S}_u \triangle \mathcal{S}_{u'}$, and because $\mathcal{S}_u^{\geq -n} = \mathcal{S}_{u'}^{\geq -n}$ we know that $\mathcal{S}_u \triangle \mathcal{S}_{u'}$ consists only of classes whose square is less than $-n$. Also, for a fixed symplectic form class on a rational surface with $\chi(M) < 12$, this is a finite set, and we denote its cardinality by k. Note that each of the negative classes defines a linear equation by setting its area to be 0, and hence each of them cut the symplectic cone into two parts.

Now consider the line segment $\mathcal{L}$ connecting u and u' in the reduced symplectic cone. Firstly, this line segment never intersects the hyperplane defined by any class in $\mathcal{S}_u^{\geq -n} = \mathcal{S}_{u'}^{\geq -n}$. This is because of convexity, and we only need to check the endpoints of this line segment. Also, $\mathcal{L}$ intersect each of the hyperplane defined by elements in $\mathcal{S}_u \triangle \mathcal{S}_{u'}$ exactly at one point. This is by definition of $\mathcal{S}_u \triangle \mathcal{S}_{u'}$. Finally and very importantly, $\mathcal{L}$ does not intersect any other hyperplanes defined by a class that's not in $\mathcal{S}_u \triangle \mathcal{S}_{u'}$: the reason is if it does, then this means one of u, u' makes this class positive-area and the other one doesn't, meaning this class is in $\mathcal{S}_u \triangle \mathcal{S}_{u'}$.

Then we call the intersection points p_i for each class $[C_i] \in \mathcal{S}_u \triangle \mathcal{S}_{u'}$. And we call the line segment $\mathcal{L}$ in generic position if those p_i's are distinct.

Now let's take a line segment $\mathcal{L}$ in generic position, then there are $k+1$ intervals on $\mathcal{L}$. Let's take $u_0 := u$, $u_k := u'$ from the first and last interval, then take one point from the interior of each of the rest $k-1$ intervals, and denote by $u_i, 1 \leq i < k$. Then for each $0 \leq i < k$, we consider u_i and u_{i+1}. For this pair, we clearly have $\mathcal{S}_{u_i} \subset \mathcal{S}_{u_{i+1}}$ or $\mathcal{S}_{u_{i+1}} \subset \mathcal{S}_{u_i}$. In any case, we have the inclusion relation between $\mathcal{A}_{u_i}$ and $\mathcal{A}_{u_{i+1}}$. Then note that they have the same codimension less then 2n part, and by Lemma 4.3 this means that $\pi_j(Symp(M,\omega_i)) = \pi_j(Symp(M,\omega_{i+1}))$ for $1 \leq j \leq 2n-3$, where $[\omega_i] = u_i$ and $[\omega_{i+1}] = u_{i+1}$. Then doing induction we have our desired result about ω and ω'.

Finally, if the line segment $\mathcal{L}$ is not in the generic position, then we can do a small perturbation of u within its absolute stable region, making $\mathcal{L}$ in generic position. Then reason is that fixing u', there's only a 0-measure choice of u in $\mathbb{R}^n$ making $\mathcal{L}$ not in generic position. Hence we finished the proof.

□

5. Concluding remark and discussions on the space of ball packings

We end bringing up the close relation between the topology of $Symp(M,\omega)$ and the space of embedded symplectic balls. Recall Theorem 2.5 in [LP04] which reveals this relation,

Theorem 5.1. *Let M be a symplectic 4-manifold, and let c be a positive real number. Suppose that (1) the space $Emb(B^4(c), M)$, which is the space of subsets of M that are images of symplectic embeddings of the standard closed ball of capacity c, is nonempty and connected, and*
(2) the exceptional curve that one gets by blowing up an arbitrary element of $Emb(B^4(c), M)$ cannot degenerate in $\tilde{M}$, where $\tilde{M}$ is the blowup of M.
Then there is a homotopy fibration

$$Symp(\tilde{M}, \omega_c) \to Symp(M, \omega) \to Emb_\omega(B^4(c), M),$$

where ω_c is the blow-up symplectic form with the exceptional class having area c.

Remark 5.2. *The theorem also works if we replace $B^4(c)$ by a finite number of disjoint symplectic balls $\coprod B^4(c_i)$ if they satisfy the corresponding versions of conditions (1) and (2) in Theorem 5.1.*

Applying the argument in [LP04] Theorem 1.6 and the above stability result of $Symp(M, \omega)$, we have the following stability result for the space of ball embeddings:

Lemma 5.3. *Notation as above, and let M denote a small rational manifold with $\chi(M) \leq 11$. Then the (weak) homotopy type of $Emb_\omega(B^4(c), M)$ is stable if c does not cross any of the hyperplane in the reduced cone defined by a negative curve class of the blowup $\tilde{M} := M\#\overline{\mathbb{C}P^2}$.*

This is to say that the group of symplectomorphisms $Symp(\tilde{M}, \omega_c)$ is (weakly) homotopy equivalent to the group $Symp_p(M, \omega)$ fixing a point, if c is smaller than the smallest area of the negative curves.

Note that [LW19] conjectured a higher homotopical generalization of Biran's stabilization theorem:

Conjecture 5.4. *Given any symplectic manifold X, consider the forgetful map from the embedding of k symplectic ball to their center*

$$\sigma : Emb_k(c_i) \to Conf_k(X),$$

where $Conf_k(X)$ is the configuration space of k points in X. Then for any $m \in \mathbb{Z}^+$, there exists an $\epsilon(m) > 0$, when $c_i < \frac{\epsilon(m)}{k}$, such that σ is m-connected.

And note that Theorem 1.2 and the multi-ball version Lemma 5.3 implies conjecture is true for $X_0 = \mathbb{C}P^2$ with no more than 9 embedded balls.

Moreover, by [McD08] and [LLWxi], $\pi_1(Symp(X_k, \omega))$ is finitely generated and the (free) rank is bounded by $\frac{k(k+1)}{2}$. We are going to prove in an upcoming work [LLW p] that even though there are infinitely many negative curves when we blow up $\mathbb{C}P^2$ at more than 9 points (symplectic balls), the stability at π_1 level is still true.

Hence it is reasonable to speculate that a version of Theorem 1.2 still holds for $X_n, n \geq 9$ and hence Conjecture 5.4 holds for $X_0 = \mathbb{C}P^2$ with any number of embedded balls.

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