Curves in Surfaces and Mapping Class Group

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Introduction

Goal

To understand constructions of psuedo-Anosov maps and possibly produce a novel construction.

Pseudo-Anosov maps are a particular type of homeomorphisms of topological spaces known as *surfaces.* Homeomorphisms are of utmost interest to topologists, and pseudo-Anosov maps are some of the more complicated ones.

Definition. *A surface is a 2-manifold, i.e. a topological space S with the following properties:*

(a) S is locally Euclidean, i.e. for all $p \in S$ there exist an open neighborhood $U \subset S$ of p and an open subset $V \subset \mathbb{R}^2$ such that U and V are homeomorphic.

(b) S is Hausdorff

(c) S is second-countable.

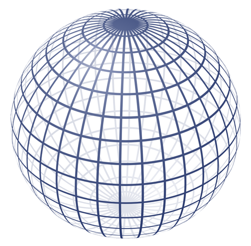


Figure 1: *A sphere*

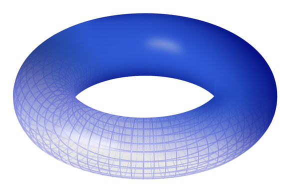


Figure 1: *A torus*

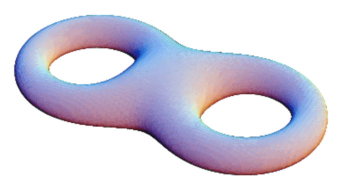


Figure 1: *A genus two torus*

Figure 1: *Examples of surfaces*

Preliminary and Background

Given a compact orientable surface S, the set of all homeomorphisms from S to itself that preserves the orientation of S forms a group, which is denoted as $\mathrm{Homeo}^+(S)$. We introduce the homotopy relation as an equivalence relation on this group. Specifically, if $h \in \mathrm{Homeo}^+(S)$, we let $[h]$ denote the set of all homeomorphisms homotopic to h, and we call $[h]$ the mapping class of h. Then, the collection of all mapping classes of S still forms a group, which we call the *mapping class group* of S, denoted by $\mathrm{Mod}(S)$.

Theorem 1: (Classification of Mapping Class Group)

Given a compact, orientable surface S, let h be a homeomorphism from S to itself. Then, exactly one of the following is true:

- h is periodic, i.e. some power of h is the identity.
- h preserves some finite union of disjoint simple closed curves on S. (If this is the case, we call h *reducible*).
- h is pseudo-Anosov, i.e. no power of h fixes any curve on S.

All constructions of pseudo-Anosov maps that we have studied are constructed as a composition of Dehn twists.

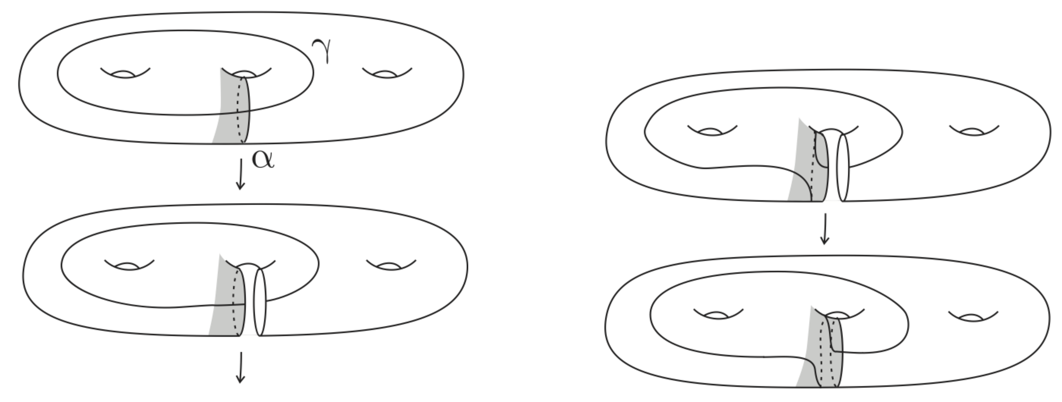


Figure 2: *Demonstration of Dehn Twist*

It is widely known in the literature that there is a standard construction of pseudo-Anosov maps called Penner's construction.

Theorem 2: (Penner's Construction)

Let $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$ be a pair of multicurves on a surface S. Suppose that A and B are in minimal position and the complement of $A \cup B$ is a union of disks and once punctured disks. Then any product of positive Dehn twists about a_j and negative Dehn twists about b_k is pseudo-Anosov, provided that all $n + m$ Dehn twists appear in the product at least once.

This construction is fairly general, so Penner made the following conjecture.

Conjecture (Penner, 1988). *Every pseudo-Anosov mapping class has a power that arises from Penner's construction.*

However, this conjecture has been disproved and it has been shown that there exist pseudo-Anosov maps such that no power of them come from Penner's construction for most surfaces (Shin & Strenner, 2015).

Methods and Results

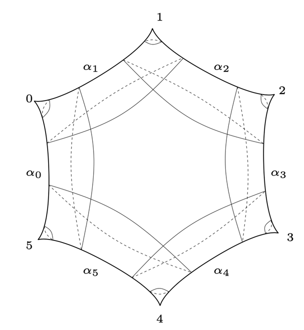
Construction of a pseudo-Anosov Map

Verberne (2019) proposes an method to construct pseudo-Anosov maps, some of which not coming from Penner's Construction.

Theorem (Verberne, 2019). *On $S_{0,6}$, the composition of Dehn twists around the curves $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$*

$$\phi = D_5^2 D_2^2 D_4^2 D_1^2 D_3^2 D_0^2$$

is a pseudo-Anosov map.



A pseudo-Anosov Map Visualized in Polygonal Representation

[[2]]

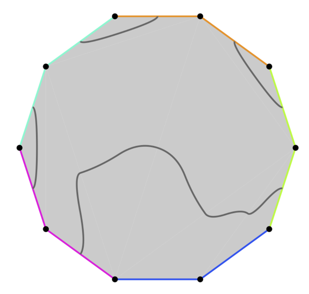


Figure 3: *A curve on the eight-times-punctured sphere.*

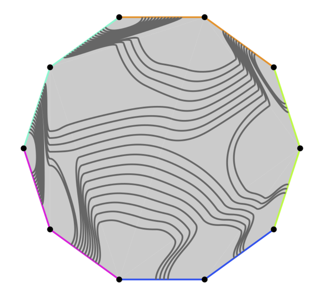


Figure 3: *The image of the curve under ϕ.*

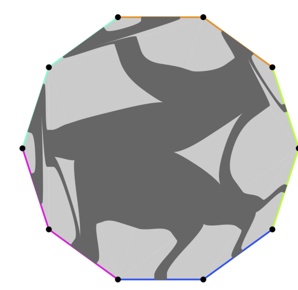


Figure 3: *The image of the curve under ϕ^2.*

In fact, no curve on the surface is preserved under $\phi^n\ \forall n \in \mathbb{N}$. This is a characterization of pseudo-Anosov maps.

Future Directions

These constructions using curves on surfaces contain gaps, i.e. they do not give some power of every pseudo-Anosov map for an arbitrary surface. In particular, there are two questions our team would like answered.

Question 1 Do there exist constructions of pseudo-Anosov maps which are different from current constructions?

Question 2 Using train track methods, does there exist a construction such that every pseudo-Anosov map has a power coming from it?

Answering Question 1. Flipper can guide our intuition. For example, a combination of half-twists gave examples of pseudo-Anosov maps, but we still need to answer if this is different from the other constructions.

Answering Question 2. This will not easily be answered. Answering Question 1 will guide the direction of this question. In particular, if the current constructions are exhaustive, then Question 1, and thus Question 2, will be false.

References

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[5] Shin, Hyunshik, and BalÃ¡zs Strenner. *Pseudo-Anosov mapping classes not arising from Pennerâs construction.* Geometry & Topology 19.6 (2016): 3645-3656.

[6] Verberne, Yvon. *A construction of pseudo-Anosov homeomorphisms using positive twists.* arXiv:1907.05467 (2019).