INFLATION ALONG SINGULAR SURFACES

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ABSTRACT. We prove the stability of $Symp(X, w)$ for one-point blowup of ruled surfaces and study their topological limit. Non-trivial generators of $\pi_0(Symp)$ that differ from Lagrangian Dehn twists are detected.

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1. INTRODUCTION

In this note, we study some topological aspects of sympelctomorphism group irrational ruled surfaces.

More precisely, we now focus on the base being a torus and small number of blowups(up to 4).

In the ruled surface $M_g = \Sigma_g \times S^2$, up to rescaling, any symplectic form is isotopic to $\sigma_{\Sigma_g} \oplus \lambda\sigma_{S^2}$ for some $\lambda > 0$. The same classification result holds also in the blowups $M_g\#\overline{n\mathbb{C}P^2}$: if one picks up ω on $M_g\#\overline{n\mathbb{C}P^2}$, then after rescaling ω has areas $(\lambda, 1, e_1, \cdots, e_n)$ on the homology classes $B, F, E_1, \cdots, E_n$, where $\lambda > 0, 0 < e_i < 1$, $e_1 \geq e_2 \geq \cdots \geq e_n, e_1 < \lambda$; choosing the standard basis $B, F, E_1, \cdots, E_n$ and associate coefficients $(\lambda, 1, e_1, \cdots, e_n)$ to get a cohomology class, then the symplectic forms in this cohomology class are isotopic. cf.

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[12, 11]. After normalization, all possible symplectic form cohomology classes determine a convex region Δ^{n+1} in $\mathbb{R}^{n+1}$, whose boundary walls are n-dimensional convex regions given by . We'll be concerned with symplectic deformations inside this region Δ^{n+1} for the n-points blowups.

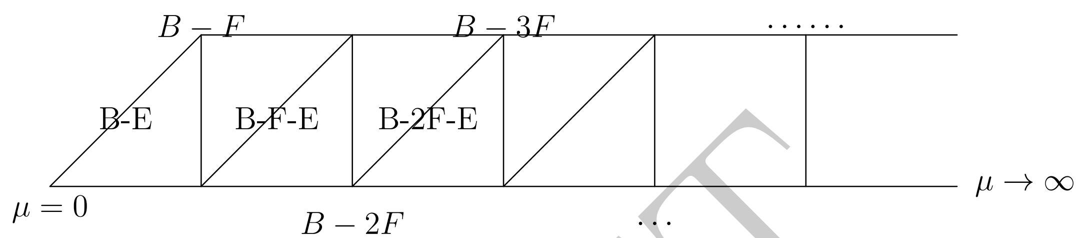

FIGURE 1. (Normalized) Symplectic cone of one-point blowup

We can partition this cone Δ^{n+1} into countably many chambers by linear equations in $\mathbb{R}^{n+1}$ such that each chamber has the same admissible(see section 5 for details) sympelctic embedded curves. Here's a conjecture we make for the topology of $Symp(M,\omega)$:

Conjecture 1.1. *If ω_1 and ω_2 belongs to the same chamber for $\Sigma_g\times S^2$ or its blowup, then $\pi_i(Symp(M,\omega_1))=\pi_i(Symp(M,\omega_2)),\forall i\geq 1$.*

The main concern of this paper is to address the stability of sympelctomorphism group as one of the λ, e_i's changes within the chamber, see section 5 for a precise statement.

Theorem 1.2. *The Conjecture 1.1 holds for $\mathbb{T}^2\times S^2$ blowup at less than or equal to 4 points, where $\omega(\mathbb{T}^2)>2\omega(S^2)$.*

Theorem 1.3. *The Conjecture 1.1 also holds for $\Sigma_g\times S^2$ blowup at one point, $\forall g\geq 1$, with a symplectic form ω such that $\omega(\Sigma_g)>(g-1)\omega(S^2)$.*

Theorem 1.4. *For $\Sigma_g\times S^2$ blowup at one point, $\forall g\geq 1$, the group $Symp_0$ has a topological limit as $\mu\to\infty$, where μ is the ratio of the symplectic area of the base and fiber. Furthermore, this limit has the homotopy type of the group (denoted by $\mathcal{D}_1$) of fiberwise diffeomorphisms of a singular foliation with generic leaves being smooth sphere and exactly one singular leaf being a nodal curve with two sphere component intersecting positively at one point.*

Most techniques here we are using are almost complex structures and J-holomorphic curve they admit. One way to connect the topology of Symplectomorphism group and the almost complex structures is the following fibration

$$Symp(M,\omega) \cap \mathrm{Diff}_0(M) \to \mathrm{Diff}_0(M) \to \mathcal{S}_{[\omega]} \tag{1}$$

by Kronheimer in [9] and later studied in [13] and [2]. Note that many results are obtained by this tech on rational surfaces cf. [1, 3, 10, 4] and [5]. For minimal ruled surface [13, 6]. This current note is the first step toward understanding non-minimal ruled surfaces.

Here $\mathcal{S}_{[\omega]}$ means the space of symplectic forms in the class $[\omega]$ and isotopic to a given form, and $\mathrm{Diff}_0(M)$ is the identity component of the diffeomorphism group. Moser's technique grant a transitive action of $\mathrm{Diff}_0(M)$ on $\mathcal{S}_{[\omega]}$ and hence gives us the fibration 1.

McDuff first observed that there is no direct map between $Symp(M,\omega)$ when deforming ω and one possible way to obtain a map bwtween groups are going through the fibration 1. There's a (weak) homotopy equivalence between $\mathcal{S}_{[\omega]}$ and $\mathcal{A}_\omega$ which is the space of ω-tamed almost complex structures. By the inflation techneique in section 7 one can relate $\mathcal{A}_\omega$'s for ω in different cohomology classes and hence prove stability results of $Symp(M,\omega)$.

The paper is organized as follows:

2. The symplectic cone and its partition

Let $\mathcal{S}_\omega$ denote the set of homology classes of embedded ω-symplectic curves and K_ω the symplectic canonical class. For any $A \in \mathcal{S}_\omega$, by the adjunction formula,

$$K_\omega \cdot A = -A \cdot A - 2 + 2g(A). \tag{2}$$

For any integer q, let

$$\mathcal{S}_\omega^{\geq q}, \quad \mathcal{S}_\omega^{>q}, \quad \mathcal{S}_\omega^{q}, \quad \mathcal{S}_\omega^{\leq q}, \quad \mathcal{S}_\omega^{<q}$$

be the subsets of classes with square $\geq q, > q, = q, \leq q, < q$ respectively. For each $A \in \mathcal{S}_\omega^{<0}$ we associate the integer

$$\mathrm{cod}_A = 2(-A\cdot A - 1 + g).$$

3. Homotpoy fibration and the stratification of of $\mathcal{A}_\omega$

Then we define the prime subset $\mathcal{J}_{\mathcal{C}}$ labeled by set $\mathcal{C} \subset \mathcal{S}_\omega^{<0}$ as following:

Definition 3.1. *A subset $\mathcal{C} \subset \mathcal{S}_\omega^{<0}$ is called admissible if*

$$\mathcal{C} = \{A_1, \cdots, A_i, \cdots, A_q | \quad A_i \cdot A_j \geq 0, \quad \forall i \neq j\}.$$

Given an admissible subset $\mathcal{C}$, we define the real codimension of the label set $\mathcal{C}$ as

$$cod(\mathcal{C}) = \sum_{A_i \in \mathcal{C}} cod_{A_i} = \sum_{A_i \in \mathcal{C}} 2(-A_i \cdot A_i - 1 + g_i).$$

Define the **prime subset**

$$\mathcal{J}_{\mathcal{C}} := \{J \in \mathcal{J}_\omega | A \in \mathcal{S}_\omega^{<0} \text{ has an embedded } J\text{-hol representative if and only if } A \in \mathcal{C}\}.$$

The prime subset $\mathcal{J}_\emptyset$ is generally denoted by $\mathcal{J}_{open}$. And if $\mathcal{C} = \{A\}$ contains only one class A, we will use $\mathcal{J}_A$ for $\mathcal{J}_{\{A\}}$.

Notice that these prime subsets are disjoint and we have the decomposition $\mathcal{J}_\omega = \amalg_{\mathcal{C}} J_{\mathcal{C}}$. We define a filtration according to the codimension of these prime subsets:

$$\cdots \subset \mathcal{X}_{2n+1} \subset \mathcal{X}_{2n} (= \mathcal{X}_{2n-1}) \subset \mathcal{X}_{2n-2} \ldots \subset \mathcal{X}_2 (= \mathcal{X}_1) \subset \mathcal{X}_0 = \mathcal{J}_\omega,$$

where $\mathcal{X}_j := \amalg_{\text{cod}(\mathcal{C}) \leq j} \mathcal{J}_{\mathcal{C}}$ is the union of all prime subsets having codimension no less than j.

For classes with negative intersection, we have the following fact

Proposition 3.2. *Let (X, ω) be a symplectic 4-manifold. Suppose $U_{\mathcal{C}} \subset \mathcal{J}_\omega$ is a subset characterized by the existence of a configuration of embedded J-holomorphic curves $C_1 \cup C_2 \cup \cdots \cup C_N$ of negative self-intersection with $\{[C_1], [C_2], \cdots, [C_N]\} = \mathcal{C}$. Then $U_{\mathcal{C}}$ is a co-oriented Fréchet suborbifold of $\mathcal{J}_\omega$ of (real) codimension $2N - 2c_1([C_1] + \cdots + [C_N]) = \sum_i K \cdot [C_i] - [C_i]^2$. Note that this covers Lemma 2.6 in [13] as a special case. Because $K \cdot [C] - [C]^2 = [-2B - (2-2g)F + \cdots] \cdot [B - lF] - [B - lF]^2 = 2l - (2-2g) + 2l = 4l - 2 + 2g$, given $C = B - lF$.*

Proof. Firstly, it suffices to show the above result in the Banach setting. In particular, Theorem 2.1.2 in [8] showed that the space of nodal curve in those fixed classes is a finite co-dimensional Banach analytic subset. Then use the argument in Appendix B of [4], one can construct local chart with codimension $2N - 2c_1([C_1] + \cdots + [C_N])$ of $\mathcal{J}_\omega$ at each point of the space of J so that there is an embedded curve in each component. $\square$

Note that here our open strata is the complement of positive codimension strata in $\mathcal{J}_\omega$ and we a priory don't have control of what classes are embedded in $\mathcal{J}_{open}$.

Now let's compute G-W (or S-W) invariants of some classes. Given a homology class C, let $\mathrm{Gr}(C)$ denote the Gromov invariant of C as defined by Taubes. This counts the number (integer, if dim=4) of embedded J-holomorphic curves in class C through $k(C)$ generic points for a fixed generic J, where

$$k(C) = \frac{1}{2}(c_1(C) + C^2) = \frac{1}{2}(-K \cdot C + C^2).$$

[As pointed by McDuff in the foot note of [**Mcdacs**], this statement holds provided that the class C has no representatives by multiply covered tori of zero self-intersection. It is easy to check that this is true in our following computations. Observe also that each curve is equipped with a sign, and that one takes the algebraic sum.]

Therefore, whenever $\mathrm{Gr}(C) \neq 0$ we know that there have to be embedded J-holomorphic curves in class C for generic J. Hence, by Gromov compactness, there is for each J *some* J-holomorphic or stable map

[Here we also follow [**Mcdacs**] to use the word "curve" to denote the image of a connected smooth Riemann surface under a J-holomorphic map, and reserve the word stable map for a connected union of more than one J-holomorphic curve.]

It was proved by Li-Liu [11] that, if $M = \Sigma_g \times S^2$ or its blowup, where $g > 0$ and $C = pB + qF$, then

$$\mathrm{Gr}(C) = (p+1)^g, \quad \text{provided that } \ k(C) \geq 0.$$

Here $K(C)$ means the number of points in the insertion of GW computation.

In particular, $\mathrm{Gr}(C) \neq 0$ provided that $q \geq g - 1$. When $g = 0$, $\mathrm{Gr}(C) = 1$ for all classes C with $p, q \geq 0$ and $p + q > 0$.

In the genus 1 case, $Gr(B) = 2, K(B) = \frac{1}{2}([2F - \cdots] \cdot B + B^2) = 0$. And the B class has codimension being 0, by the formula in Lemma 3.2. And hence in this case the last Corollary is sufficient.

In the genus 2 case, $Gr(B) = 4$, however, $K(B) = \frac{1}{2}([2B - 2F - \cdots] \cdot B + B^2) = -1 < 0$. And the B class has codimension being 2, by the formula in Lemma 3.2. This means that the correct class to consider in $\mathcal{J}_{open}$ is $B + F$, which has $Gr(B+F) = 4$, however, $K(B) = \frac{1}{2}([2B - 2F - \cdots] \cdot [B+F] + [B+F]^2) = 2 > 0$. On the other hand, by Lemma 3.2, this class has codimension 0.

Now the question is whether we have the same estimate of $B+F$ in a genus=2?

In general, for genus=$g>2$, we always have $Gr(B+xF)=2^g>0$, and expect to have $K(B+xF)=\frac{1}{2}([2B-2F-\cdots]\cdot[B+xF]+[B+xF]^2)=4x+2-2g\geq 0$. And hence we should have $x\geq g-1$, which agrees the above genus 2 case.

Now in general $B+xF$ might now always be embedded in $\mathcal{J}_{open}$ and hence we are expected to use the nodal inflation/deflation along this class. Note that to enlarge the area of B we want to deflate this class.

4. Existence of curves and invariane of $\mathcal{A}_\omega$

Lemma 4.1. *Suppose $X=\Sigma_g\times S^2\#n\overline{\mathbb{C}P^2}, n\leq 4$, and ω is a reduced symplectic form in the class $\mu B+F-\sum_{i=1}^n a_iE_i'$ as in Lemma* **??**. *If $g=1,2$, and under the above setting, if $A=pB+qF-\sum r_iE_i'\in H_2(X;\mathbb{Z})$ is a class with a simple J-holomorphic representative for some ω-tamed J, then $p\geq 0$.*

And if $p=0$, then $q=0$ or 1. If $p>1$, then $q\geq 1$.

Proof. We start by stating three inequalities: the area inequality (3), the adjunction inequality (5), the r_i integer inequality (6).

The area of the curve class A is positive and hence

$$\omega(A)=p\mu+q-\sum_{i=1}^n a_ir_i>0. \tag{3}$$

Since ω is reduced, its canonical class is

$$K_\omega=-2B+(2g-2)F+E_1'+\cdots+E_n'. \tag{4}$$

(We'll write $2gF-2B-2F+E_1'+\cdots+E_n'$ for convenience of computation.) So we have the following adjunction inequality for simple J-holomorphic curves:

$$0\leq 2g_\omega(A)=A\cdot A+K\cdot A+2=2gq+2(p-1)(q-1)-\sum_{i=1}^n r_i(r_i-1). \tag{5}$$

We now estimate the sum $-\sum_{i=1}^n r_i(r_i-1)$. Since each r_i is an integer, it is easy to see that

$$-\sum_{i=1}^n r_i(r_i-1)\leq 0, \tag{6}$$

and $-\sum_{i=1}^n r_i(r_i-1)=0$ if and only if $r_i=0$ or 1 for each i.

Now let us divide into five cases:

(i) $p \geq 1$, (ii) $p < 0, q \geq 1$, (iii) $p < 0, q \leq 0$, (iv) $p = 0$.

It follows from (6) that $r_i(r_i - 1)$ has to be 0 and hence $r_i \in \{0, 1\}$.

Case (i). $p > 1$ and $q \leq 0$. Then $-\sum_{i=1}^n r_i(r_i - 1) = 2g_\omega(A) - 2(p-1)(q-1) \geq 2 - 2(p-1)(q-1) > 0$. This is impossible. Therefore $q \geq 1$ if $p \geq 1$.

Case (ii). $p < 0$ and $q \geq 1$.

We show that this case is impossible. Because $p \leq -1$, the adjunction inequality (5) implies that

$$0 \geq -2g_\omega(A) \geq gq + 4(q-1) + \sum_{i=1}^n r_i(r_i - 1) \geq (q-1) + \sum_{i=1}^n r_i(r_i - 1).$$

Applying the area equation (3), we have

$$(q-1) + \sum_{i=1}^n r_i(r_i - 1) > (\sum_{i=1}^n a_i r_i - \mu p - 1) + \sum_{i=1}^n r_i(r_i - 1).$$

Since $-\mu p - 1 \geq 0$,

$$(\sum_{i=1}^n a_i r_i - \mu p - 1) + \sum_{i=1}^n r_i(r_i - 1) \geq (\sum_{i=1}^n a_i r_i) + \sum_{i=1}^n r_i(r_i - 1) = \sum_{i=1}^n r_i(r_i - 1 + a_i).$$

For any integer r_i we have $r_i(r_i - 1 + a_i) \geq 0$ due to the reduced condition $1 - a_i \in (0, 1)$. Therefore we would have $-2g_\omega(A) > 0$, which is a contradiction.

Case (iii). $p < 0, q \leq 0$.

We show this case is also impossible. This will follow from the following estimate, under a slightly general assumption:

$$0 \leq 2g_\omega(A) \leq 1 + |p| + |q| - p^2 - q^2, \qquad \text{if } p \leq 0, q \leq 0. \tag{7}$$

Proof of the inequality (7). In order to estimate $-\sum_{i=1}^n r_i(r_i - 1)$ we rewrite the sum

$$\sum_{i=1}^n r_i = \sum_{k=1}^u r_k + \sum_{l=u+1}^n r_l, \tag{8}$$

where each r_k is negative and each r_l is non-negative.

Since $p \leq 0, q \leq 0$, the area inequality (3) takes the following form:

$$-\sum a_i r_i > (|p| + |q|) \geq 1 + (|p| + |q|). \tag{9}$$

Note that there exists at least one negative r_i term, i.e. $u \geq 1$ in (8). An important consequence is

$$\sum_{k=1}^u a_k r_k \leq \sum_{i=1}^n a_i r_i < 0, \quad (\sum_{k=1}^u a_k r_k)^2 \geq (\sum_{i=1}^n a_i r_i)^2. \tag{10}$$

We first observe that, by the Cauchy-Schwarz inequality and (**??**), we have

$$(\sum_{k=1}^{u} a_k r_k)^2 \leq \sum_{k=1}^{u}(r_k)^2 \times \sum_{k=1}^{u}(a_k)^2 \leq \sum_{k=1}^{u}(r_k)^2. \tag{11}$$

Then we do the estimate:

$$\begin{aligned}
\sum_{i=1}^{n} r_i(r_i-1) &= \sum_{i=1}^{n} r_i^2 - \sum_{i=1}^{n} r_i = \sum_{k=1}^{u} r_k^2 - \sum_{k=1}^{u} r_k + (\sum_{l=u+1}^{n} r_l^2 - \sum_{l=u+1}^{n} r_l) \\
&\geq \sum_{k=1}^{u} r_k^2 - \sum_{k=1}^{u} r_k \quad \text{(since } x^2 - x \geq 0 \text{ for any integer)} \\
&\geq (\sum_{k=1}^{u} a_k r_k)^2 - \sum_{k=1}^{u} a_k r_k \quad \text{(follows from the two inequalities:} \qquad (12) \\
&\qquad -\textstyle\sum_{k=1}^{u} r_k > -\sum_{k=1}^{u} a_k r_k \text{ and } \sum_{k=1}^{u} r_k^2 \geq (\sum_{k=1}^{u} a_k r_k)^2) \\
&\geq (\sum_{i=1}^{n} a_i r_i)^2 - \sum_{i=1}^{n} a_i r_i \quad \text{(this crucial step follows from (10))} \\
&> |p| + |q| + (|p|+|q|)^2.
\end{aligned}$$

Because $\sum_{i=1}^{n} r_i(r_i-1)$ is an integer, we actually have

$$\sum_{i=1}^{n} r_i(r_i-1) \geq 1 + |p| + |q| + (|p|+|q|)^2.$$

Now the inequality (7) follows from the inequalities (12) and (5),

$$0 \leq 2g_\omega(A) = 2pq - 2(p+q) + 2 - \sum_{i=1}^{n} r_i(r_i-1) \leq |p| + |q| + 1 - (p^2+q^2).$$

With the inequality (7) established, we note that a direct consequence is that it is impossible to have $p \leq -2, q \leq 0$, or $p \leq 0, q \leq -2$: If $|p| > 1$, $|p| + |q| + 1 - (p^2+q^2)$ is clearly negative since $q^2 \geq |q|, p^2 > |p| + 1$; it is the same if $|q| > 1$.

So the inequality (7) leaves two cases to analyze: $p = q = -1$, or $p = -1, q = 0$. To deal with these two cases, as in the proof of the inequality (7), we assume that $r_k < 0$ for $1 \leq k \leq u$ and $r_l \geq 0$ for $u+1 \leq l \leq n$. Notice that $\sum_{k=1}^{u} r_k^2 - \sum_{k=1}^{u} r_k \leq \sum_{i=1}^{n} r_i(r_i-1)$ as shown in (12).

• $p = -1$ and $q = 0$.

In this case, we have $2g_\omega(A) = 4 - \sum_{i=1}^{n} r_i(r_i-1)$ so $\sum_{k=1}^{u} r_k^2 - \sum_{k=1}^{u} r_k \leq \sum_{i=1}^{n} r_i(r_i-1) = 4$. By the area inequality (3), we have $\sum_{k=1}^{u} r_k < p + q = -1$, and hence $\sum_{k=1}^{u} r_k \leq -1$. It is easy to see that $\{r_k\} = \{-1\}$ or $\{-1, -1\}$. But these possibilities are excluded by the reduced condition $a_i + a_j \le 1 \le \mu$ for any pair i, j and the area inequality (3).

• $p = q = -1$.

In this case, we have $2g_\omega(A) = 8 - \sum_{i=1}^n r_i(r_i - 1)$ so $\sum_{k=1}^u r_k^2 - \sum_{k=1}^u r_k \le 8$. By the area inequality (3), we have $\sum_{k=1}^u r_k < p + q = -2$, and hence $\sum_{k=1}^u r_k \le -2$. It is easy to see that $\{r_k\} = \{-1, -1, -1\}, \{-1, -1, -1, -1\}$ or $\{-1, -2\}$. Again these possibilities are excluded by the reduced condition $a_i + a_j \le 1 \le \mu$ for any pair i, j and the area inequality (3).

Case (iv). $p = 0$.

In this case, by the adjunction inequality (5) we have $2gq - 2(q - 1) - \sum_{i=1}^n r_i(r_i - 1) \ge 0$. Since $-\sum_{i=1}^n r_i(r_i - 1) \le 0$, we have $q \le 1$. If $p = 0, q \le 0$, then we apply the inequality (7) to conclude that $q = 0$. Hence we must have $q = 0$ or 1 if $p = 0$.

In conclusion, only cases (i),(iv) are possible. Moreover, if $p = 0$, then $q = 0$ or 1, if $p \ge 1$, then $q \ge 1$.

□

Corollary 4.2. *For a ruled surface with base genus no larger than 2, blowing up no more than 4 points, then either the class B is embedded, or there is an embedded curve with self-intersection -2 or more negative*

Proof. First we show that B is not a nodal curve with each component being non-negative. Suppose $B = \sum_i \beta_i B_i$, where $\beta_i > 0$ and B_i are classes with self intersection no less than 0. Then homologically square both sides, $B \cdot B = 0$ on the left, and due to positivity of intersection we have $\sum_{i,j}(\beta_i\beta_j)B_i \cdot B_j > 0$, contradiction.

Now we can assume that

$$B = \sum_\beta p_\beta[C_\beta] + \sum_\gamma q_\gamma[C_\gamma],$$

where p_β, q_γ are positive integers, C_β are embedded J-holomorphic curves with $[C_\beta]^2 = -1$, C_γ are simple J-holomorphic rational curves with $[C_\gamma]^2 \ge 0$. Then we also have a contradiction, see below.

Pairing with B, we get

$$0 = B \cdot B = \sum_\beta p_\beta B \cdot [C_\beta] + \sum_\gamma q_\gamma B \cdot [C_\gamma].$$

Note that only one class on the right hand side could have non-zero B coefficient, WOLG we can assume $C_\beta, \beta = 1$ ($\gamma = 1$ could also work here) is that class. Notice that $B \cdot C_\gamma \ge 0$ since they all have non negative F coefficient by Lemma 4.1. Also, $B \cdot C_\beta \ge 0, \beta > 1$. Then this tells us that $B \cdot C_1 < 0$, which means its self intersection is less than -2 (the coefficient on E_i's will increase the negativity). Hence another contradiction.

Now we conclude that any stable curve of B has to have at least one component (possibly multiple covered) whose class is of self-intersection less than -1. In particular, running the same argument in the last paragraph, there must be one component which is simple, whose class has B coefficient being 1 and negative F class. Hence it is an embedded class with self-intersection strictly less than -1.

5. Proof of the theorem

Now let M_μ denote the blowup of $\Sigma_g \times S^2$ at k points with a fixed blowup size, and μ is the area ratio between the base and the fiber. Let $\mathcal{J}_\mu$ denote the space of tamed J in the given class.

Lemma 5.1. *If $g > 0$ then $\mathcal{J}_\mu \subset \mathcal{J}_{\mu+\epsilon}$ for all $1/\mu > g-1, \epsilon > 0$.*

Proof. By [17] Theorem 1.6, we known that for each $J \in \mathcal{J}_\mu$, through each point of M there is a stable J-holomorphic spheres representing the fiber class $F = [pt \times S^2]$.

Now we take a symplectic embedded curve (not necessarily J-holomorphic) in the class $B = [\Sigma \times pt]$. By the above discussion, there is always an embedded curve $\mathcal{S}$ in this class for any $J \in \mathcal{J}_\mu$.

It follows that there is a projection $\pi_J : M \to \mathcal{S}$ onto the leaf space $\mathcal{S}$ Then if σ is any area form on $\mathcal{S}$, its pullback $\pi_J^*(\sigma)$ is J-semi-tame, i.e

$$\pi_J^*(\sigma)(v, Jv) \geq 0, \quad v \in TM.$$

Now we prove the claim, i.e. that if σ is any area form on $\mathcal{S}$ then its pullback $\pi_J^*(\sigma)$ is J-semi-tame. For any a point $p \in \mathcal{S}$ and choose (positively oriented) coordinates (x_1, x_2, y_1, y_2) near p so that the fibers (here we mean the component in the stable curve in class F passing through p) near p have equation $x_i = const$, for $i = 1, 2$, and so that the symplectic orthogonal to the fiber F_p at p

$$Hor_p = \{u : \omega(u, v) = 0, \text{ for all } \ v \in T_p(F_p)\}$$

is tangent at p to the surface $y_1 = y_2 = 0$. Then, at p, the form ω_p may be written

$$\omega_p = adx_1 \wedge dx_2 + bdy_1 \wedge dy_2$$

for some constants $a, b > 0$. Moreover, since J_p preserves the fibers, we can assume that J_p acts on T_pM via the lower triangular matrix

$$J_p = \begin{pmatrix} A & 0 \\ C & J_0 \end{pmatrix}, \quad \text{where} \quad J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

If $u \in Hor_p$, then

$$\omega_p(u, Ju) = adx_1 \wedge dx_2(u, Au) > 0, \quad \text{if} \quad u \neq 0,$$

because ω_p tames J_p. But $\pi_J^*(\sigma)$ is just a positive multiple of $dx_1 \wedge dx_2$ at p. The claim follows.

Given this, if $\omega \in S_\mu$ tames J, then $\omega + \lambda\pi_J^*(\sigma)$ also tames J for all $\lambda > 0$. This means that we can inflate along the class B.

Now note that by adding the class $[B+(g-1)F]$ (which is embedded for the J's in the top stratm) we are enlarging the F class and this works for any $\mu > \frac{1}{1-g}$.

Hence the proof.

□

Remark: Using the singular inflation, we can indeed inflate along the class F and hence remove the assumption on μ.

□

General idea: moving right (toward infinity) is always easy; moving left is harder. And we start with a point on the left, name a target, and do large scale or small scale moves to hit the target.

ONE POINT BLOWUP There are two possibilities:

One: If the target is in the region labeled by $B - kF - E$, then it's always easy.

Two: If the target is in the region labeled by $B - kF$, then we first do a large scale move, then move it right up to the **high point (E has area almost 1)** by inflation on the curve $F - E$, then do large scale move. Note that we can always get to somewhere above our target. The last step is to move down by inflation on E.

Lemma 5.2. *For two symplectic forms ω, ω' such that $PD(\omega) = [\mu, 1, -d_1]$, $PD(\omega') = [\mu', 1, -d_1']$, $1 < \mu' < \mu$. The strata of $\mathcal{J}_\omega$ that ω, ω' have in common are the same.*

Proof. Now let's deal with the inflation when the area of the base getting smaller.

Note that if after inflation, we reach any point in the chamber as in Figure 1, then we can use inflation along both the E curve and $F-E$ curve (both classes have embedded representatives for any J) to reach any point in the same chamber.

- If the target is in the strata labeled by $B-kF-E$. Then we can always inflate along this curve.

  Let's assume that we start with $PD(\omega)=[\mu,1,-d_1]$ and we want $PD(\omega')=[\mu',1,-d_1']$ where $1<\mu'<\mu$. Then we inflate along $B-kF-E$ and there is a family $\omega_t, s.t. PD(\omega_t)=[\mu+t,1+t,-d_1-t]$. And we want $\frac{\mu+t}{1+t}=\mu'$, and this means that $t=\frac{\mu-\mu'}{\mu'-1}$. Then to reach any point in the chamber we can inflate along the E curve and $F-E$ curve. Note that since $\mu-\mu'>0$ we can always achieve this.
- If the target is in the region labeled by $B-kF$, then the above straightforward inflation may not end up in the correct chamber, it may stop at the chamber labelled by $B-kF-E$. There are indeed exactly two cases by doing the above inflation, replacing $B-kF-E$ by $B-kF$: either one stops at the chamber labelled by $B-kF-E$ or the one labelled by $B-kF$. If the latter is the case, then we are done. If we have the former case, then we inflate along $F-E$, then inflate along $B-kF$. Note that we can always get to somewhere above our target. The last step is to move down by inflation on E.

□

MANY-POINT BLOWUP There are many possibilities, here we record the general strategy: first do a large scale move, then move it right up to the **high point (depending on the region, see below)** by inflation on some curve (depending on the region), then do large scale move. Finally, inflate $E_i's$.

Now the target is in the region labeled by $B-kF-\sum_{i=1}^n E_i+\sum_k E_{i_k}$, note that in each region there's at most one of this curve.

Then we first do a large scale move, then move it right up to the **high point (E_{i_k}'s all have area almost 1)** by inflation on the curve $F-E_{i_k}$'s more their degeneration's, note this is always possible by positivity of intersection. Note that we can always get to somewhere above our target by another large scale move. The last step is to move down by inflation on E_i's.

6. Singular foliation and topological colimit

In this section we define and calculate the limit G_∞ and then show how to understand the relationship between the different spaces $\mathcal{A}_\mu$. Note that we are fixing the blowup size here and μ means the ratio of the base and fiber.

First let's recall the following Lemma in [17] and also used in [16].

Lemma 6.1. *Let (Z, ω) be a symplectic ruled 4-manifold diffeomorphic to $S^2 \times Y^2 \# \overline{\mathbb{C}P}^2$, and let J be an ω-tamed almost-complex structure. Then Z admits a* **singular foliation** *given by a proper projection $\pi : Z \to Y$ onto Y such that*

i) there is a singular value $y^* \in Y$ such that π is a spherical fiber bundle over $Y - y^*$, and each fiber $\pi^{-1}(y)$, $y \in Y - y^*$, is a J-holomorphic smooth rational curve in the class F;

ii) the fiber $\pi^{-1}(y^*)$ consists of the two exceptional J-holomorphic smooth rational curves in the classes $F - E$ and E.

6.1. **The basic idea.** Recall the fibration

$$G_\mu \to \mathrm{Diff}_0(M) \to \mathcal{A}_\mu.$$

Here $\mathrm{Diff}_0(M)$ is the identity component of the group of diffeomorphisms of M and $\mathcal{A}_\mu$ is the space of all symplectic forms on M in the cohomology class $[\omega_\mu]$ that are isotopic to ω_μ. By Moser's theorem the group $\mathrm{Diff}_0(M)$ acts transitively on $\mathcal{A}_\mu$ via an action that we will write as

$$\phi \cdot \omega = \phi_*(\omega) = (\phi^{-1})^* \omega,$$

so that $\mathcal{A}_\mu$ is simply the homogeneous space $\mathrm{Diff}_0(M)/G_\mu$. Let $\mathcal{A}_\mu$ denote the space of almost complex structures that are **tamed (in the stability proof we use compatible for high genus, but here we already know there's a limit)** by some form in $\mathcal{A}_\mu$.

Lemma 6.2. *$\mathcal{A}_\mu$ is homotopy equivalent to $\mathcal{A}_\mu$.*

Lemma 6.3. *If $g > 0$ then $\mathcal{A}_\mu \subset \mathcal{A}_{\mu+\epsilon}$ for all $\mu, \epsilon > 0$.*

proof:

Use Lemma 6.1, there's a singular foliation. It follows that there is a projection $\pi_J : M \to \Sigma$ onto the leaf space Σ of this foliation. We now show in that if $\underline{\sigma}$ is any area form on Σ then its pullback $\pi_J^*(\underline{\sigma})$ is J-semi-tame, i.e

$$\pi_J^*(\underline{\sigma})(v, Jv) \geq 0, \quad v \in TM.$$

Granted this, if $\omega \in \mathcal{A}_\mu$ tames J, then $\omega + \kappa \pi_J^*(\underline{\sigma})$ also tames J for all $\kappa > 0$. The result is then immediate.

Corollary 6.4. *For any $\mu, \epsilon, \epsilon' > 0$ there are maps $\mathcal{A}_\mu \to \mathcal{A}_{\mu+\epsilon}$ and $G_\mu \to G_{\mu+\epsilon}$ that are well defined up to homotopy and make the following diagrams homotopy commute:*

$$
(a) \qquad \begin{array}{ccccc}
G_\mu & \to & \mathrm{Diff}_0(M) & \to & \mathcal{A}_\mu \\
\downarrow & & \downarrow = & & \downarrow \\
G_{\mu+\epsilon} & \to & \mathrm{Diff}_0(M) & \to & \mathcal{A}_{\mu+\epsilon},
\end{array}
$$

$$
(b) \qquad \begin{array}{ccc}
G_\mu & \to & G_{\mu+\epsilon} \\
 & \searrow & \downarrow \\
 & & G_{\mu+\epsilon+\epsilon'}
\end{array} \qquad .
$$

Proof. The maps $\mathcal{A}_\mu \to \mathcal{A}_{\mu+\epsilon}$ are induced from the inclusions $\mathcal{A}_\mu \subset \mathcal{A}_{\mu+\epsilon}$ using the homotopy equivalences $\mathcal{A}_\mu \simeq \mathcal{A}_\mu$ in Lemma 6.2 above. Since G_μ is the fiber of the map $\mathrm{Diff}_0(M) \to \mathcal{A}_\mu$, there are induced maps $G_\mu \to G_{\mu+\epsilon}$ making diagram (a) homotopy commute. The rest is obvious.

□

This corollary illustrates the essential feature of our approach. Statements that are true only up to homotopy on the level of the groups G_μ are true on the nose for the spaces $\mathcal{A}_\mu$. The discussion below shows that it is easy to understand what happens to the groups G_μ as μ increases. Much of the rest of the paper is devoted to understanding (on the level of the spaces $\mathcal{A}_\mu$) what happens as μ decreases. For this we use the Lalonde–McDuff technique of symplectic inflation.

Proof of Proposition 1.4.

Proof. We first show that we can understand the limit $G_\infty = \lim G_\mu$ by studying the space $\cup_\mu \mathcal{A}_\mu$. Let J_{split} be the standard blowup of the product almost complex structure on M. Because J_{split} is tamed by ω_μ the map $\mathrm{Diff}_0(M) \to \mathcal{A}_\mu$ lifts to

$$\mathrm{Diff}_0(M) \to \mathcal{A}_\mu : \quad \phi \mapsto (\phi_*(\omega_\mu), \phi_*(J_{split})).$$

Composing with the projection to $\mathcal{A}_\mu$ we get a map

$$\mathrm{Diff}_0(M) \to \mathcal{A}_\mu : \quad \phi \mapsto \phi_*(J_{split})$$

that is not a fibration but has homotopy fiber G_μ. Since $\mathcal{A}_\mu$ is an open subset of $\mathcal{A}_{\mu+\epsilon}$ for all $\epsilon > 0$, the homotopy limit $\lim_\mu \mathcal{A}_\mu$ of the spaces $\mathcal{A}_\mu$ is homotopy equivalent to the union $\mathcal{A}_\infty = \cup_\mu \mathcal{A}_\mu$. Hence G_∞, which is defined to be the homotopy limit of the G_μ, is homotopy equivalent to the homotopy fiber of the map $\mathrm{Diff}_0(M) \to \mathcal{A}_\infty$.

To understand $\mathcal{A}_\infty$ we proceed as follows. Let Folbe the space of all singular foliations of $\Sigma \times S^2$ by spheres in the fiber class F as in Lemma 6.1. Since S^2 is compact and simply connected, each generic leaf of this foliation has trivial holonomy and hence has a neighborhood that is diffeomorphic to the product $D^2 \times S^2$ equipped with the trivial foliation with leaves $pt \times S^2$. It follows that $\mathrm{Diff}(M)$ acts transitively on Fol via the map $\phi \mapsto \phi(\mathcal{F}_{split})$, where $\mathcal{F}_{split}$ is the flat foliation by the spheres $pt \times S^2$. Similarly, $\mathrm{Diff}_0(M)$ acts transitively on the connected component Fol_0 of Fol that contains $\mathcal{F}_{split}$. Hence there is a fibration sequence

$$\mathcal{D} \cap \mathrm{Diff}_0(M) \to \mathrm{Diff}_0(M) \to \mathrm{Fol}_0.$$

It is not hard to see that the group $\mathcal{D} \cap \mathrm{Diff}_0(M)$ is connected, and so equal to $\mathcal{D}_1$.

Next, observe that there is a map $\mathcal{A}_\infty \to \mathrm{Fol}_0$ given by taking J to the singular foliation of M by J-spheres in class F or $F - E$. Standard arguments in [14] Ch 2.5 show that this map is a fibration with contractible fibers. Hence it is a homotopy equivalence. Moreover, it fits into the commutative diagram:

$$\begin{array}{ccc} \mathrm{Diff}_0(M) & \to & \mathcal{A}_\infty \\ \downarrow & & \downarrow \\ \mathrm{Diff}_0(M) & \to & \mathrm{Fol}_0, \end{array}$$

where the map $\mathrm{Diff}_0(M) \to \mathcal{A}_\infty$ is given as above by the action $\phi \mapsto \phi_*(J_{split})$. Hence there is an induced homotopy equivalence from the homotopy fiber G_∞ of the top row to the fiber $\mathcal{D}_1$ of the second.

□

Remark 6.5. Implicit in the above argument is the following description of the map $G_\infty \to \mathcal{D}_1$. Let $\mathcal{J}_\mu$ denote the space of all almost complex structures tamed by ω_μ. Since the image of the group G_μ under the map $\mathrm{Diff}_0(M) \to \mathcal{A}_\mu$ is contained in $\mathcal{J}_\mu$ there is a commutative diagram

$$\begin{array}{ccc} & & \mathcal{D}_1 \\ & & \downarrow \\ G_\mu & \xrightarrow{\iota} & \mathrm{Diff}_0(M) \\ \downarrow & & \downarrow \\ \mathcal{J}_\mu & \longrightarrow & \mathrm{Fol}. \end{array}$$

Because $\mathcal{J}_\mu$ is contractible, the inclusion $\iota : G_\mu \to \mathrm{Diff}_0(M)$ lifts to a map $\tilde{\iota} : G_\mu \to \mathcal{D}_1$. Now take the limit to get $G_\infty \to \mathcal{D}_1$.

7. Inflation Theorem

Theorem 7.1. *For a symplectic four manifold (M^4, J, τ_0) such that J is a τ_0-tame almost complex structure. Assume that M admits embedded J-holomorphic curves $u_i : (\Sigma_i, j_i) \to (M^4, J), i = 1, 2$ in homology classes Z_1 and Z_2 with $Z_i^2 = -m_i$ and $Z_1 \cdot Z_2 = 1$. For* *all ϵ there exist a family of symplectic forms $\tau_{\mu,\eta}$ all taming J and $[\tau_{\mu,\eta}] = [\tau_0] + \mu a Z_1 + \eta a Z_2$ for all $0 \leq \mu \frac{\tau_0(Z_1)}{m_1} - \epsilon$ and $0 \leq \eta \frac{\tau_0(Z_2)}{m_2} - \epsilon$. Here aZ_i is the Poincaré dual of Z_i.*

A new nbhd theorem is in Guadagni, Symplectic neighborhood of crossing divisors arxiv.1611.02363 and is definitely helpful (which allows cycles beyond chain types) to improve the singular inflation theorem.

Proof: Note this is a compatible to tame inflation proof

Given symplectic (M, ω) Suppose we have a divisor $D \subset M$, normal crossing, no cycle in its augmented graph (Γ, a) (each component being a node and each intersection being an edge).

Now, if we have an ω'-orthogonal divisor (D', ω') with augmented graph (Γ, a), which is the same as that of the neighborhood triple (X, ω, D), then there exist neighborhood N' of D' symplectomorphic to a neighborhood of D and sending D' to D (See [15] and [7]).

Now let's focus on the case when D only has two components intersecting transversely, i.e. the augmented graph being

Suppose the two components are Z_1 and Z_2 and take $N(Z_1)$ ($N(Z_2)$ respectively) a neighborhood of Z_1 consisting of the unit disk bundle over the curve in class Z_1. Denote by r_1 and r_1 the radial coordinate of $N(Z_1)$ ($N(Z_2)$ respectively). We assume $\tau_0(Z_1) = 1$ and $\tau_0(Z_2) = b$. Denote by σZ_1 and σZ_2 the area form on Z_1 and Z_2 such that $\int_{Z_1} \sigma Z_1 = 1$ and $\int_{Z_b} \sigma Z_2 = b$. We then can choose a connections on the disk bundles such that the connection one-forms α and β on the bundle over Z_1 and Z_2 obeys $d\alpha = m_1 \pi * (\sigma Z_1)$ and $d\beta = m_2 \pi * (\sigma Z_2)$ where π being the bundle projections (there's no confusion so we do not distinguish) respectively.

Now in a very small tubular nbhd of Z_1 and Z_2, but away from the nbhd intersection point (considered as the product of two small disks), we choose the symplectic form τ_0 to be diffeomorphic to the following: near Z_1 and away from Z_2, $\tau_0 \sim (1 + m_1 r_1^2)\pi * (\sigma Z_1) + 2r_1 dr_1 \wedge \alpha$;
near Z_2 and away from Z_1, $\tau_0 \sim (1 + m_2 r_2^2)\pi * (\sigma Z_2) + 2r_2 dr_2 \wedge \beta$;
in the intersection nbhd (product of two disks), $\tau_0 \sim 2r_1 dr_1 \wedge \alpha + 2r_2 dr_2 \wedge \beta$.

Note that by changing the coordinate on Z_1 and Z_2, we did not change the form, only changing the parametrization.

The point is the form on the product of two disks perfectly match the forms on Z_1 nbhd and Z_2 nbhd, when restricted to Z_1 and Z_2. The patch together to be the original σZ_1 and σZ_2.

Then for the inflation form $\tau_{\mu\eta}$, We still do the similar modification as [6]:

near Z_1 and away from Z_2, $\tau_{\mu\eta} = (1 + m_1 r_1^2 - m_1 f(\mu))\pi * (\sigma Z_1) + [2r_1 - f'(\mu)]dr_1 \wedge \alpha$;
near Z_2 and away from Z_1, $\tau_{\mu\eta} = (1 + m_2 r_2^2 - m_2 g(\eta))\pi * (\sigma Z_2) + [2r_2 dr_2 - g'(\eta)] \wedge \beta$;

Those are the same treatment and the same quadratic estimate will work.

Then, in the intersection nbhd (product of two disks), $\tau_{\mu\eta} = F(\mu, \eta, r_1) 2r_1 dr_1 \wedge \alpha + G(\mu, \eta, r_2) 2r_2 dr_2 \wedge \beta$.

Here $F(\mu, , \eta, r_1)$ and $G(\mu, \eta, r_2)$ are non-decreasing functions in r.

And $F(\mu, \eta, r_1)$ is obtained by smoothly (means keeping track of every order derivative, Whitney extension theorem could do this) interpolating the function $1 - f(\mu)$ and $1 - \frac{g'(\eta)}{2r_{01}}$, while $G(\mu, \eta, r_2)$ is by smoothly interpolating the function $1 - g(\eta)$ and $1 - \frac{f'(\mu)}{2r_{01}}$.

Note that we want
1) the form $\tau_{\mu\eta}$ obtained both ways on the boundary disks match each other.
2) on the Z_1 and Z_2, $\tau_{\mu\eta}$ scale τ_0 as desired.
And it is easy to check the above form satisfies both conditions.

Now we do the quadratic estimate in the nbhd of the product of disks $S = D_1 \times D_2$, where $0 \in D_1 \subset Z_1$, $\quad 0 \in D_2 \subset Z_2$: We'll use the splitting of the tangent space $T_p(S) = E_1 \oplus E_2$, where E_i tangents to D_i.

Under this choice of splitting, let's assume that

$$J_p = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where A, B, C, D are 2×2 matrices.

Now let's do the general computation and let

$$\mathcal{J}_0 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

$$\begin{aligned} &\tau_{\mu\eta}((v, w), J_p(v, w)) \\ = &F(\mu, \eta, r_1) 2r_1 dr_1 \wedge \alpha((v, w), J_p(v, w)) + G(\mu, \eta, r_2) 2r_2 dr_2 \wedge \beta((v, w), J_p(v, w)) \end{aligned}$$

$$= F v^\top \mathcal{J}_0^\top A v + F v^\top \mathcal{J}_0^\top B w + G w^\top \mathcal{J}_0^\top C v + G w^\top \mathcal{J}_0^\top D w. \tag{13}$$

We want to prove this is positive at least for the nbhd where both disks have sufficient small radius.

Note that since the curve at $r_1 = 0$ or $r_2 = 0$ is J-holomorphic we can assume that the fiber disks are locally J-holomorphic near $r_1 = 0$ or $r_2 = 0$ and this means $B = C = 0$ when $r_1 = 0$ or $r_2 = 0$.

Also note that by the standalization of the nbhd, we achieved both ω orthogonal and J-orthogonal. Since we started with triple (N, ω, J) where N is the nbhd with two J-holomorphic curve intersecting at one point. Then we use a diffeomorphism supported along N, making two curves ω' orthogonal. This diffeomorphism also pushes forward J, and both curves are still J'-holomorphic. Since near the intersection point, disk on one curve is the base and the disk on the other curve is the fiber; we also know that they are both J'-hol after the push forward. In the above (and below) argument we still use J to denote J'.

To justify $B = C = 0$, we only need to show that J preserves the base and fiber. And this is the J-hol condition. And this means that local J matrix has to be blockwise diagonal.

Then we know that

$$\tau_0((v, w), J_p(v, w)) = v^\top \mathcal{J}_0^\top Av + w^\top \mathcal{J}_0^\top Dw > 0$$

Since

$$J_p = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}^2 = -Id,$$

when $r_1 = 0$ or $r_2 = 0$, we can find a neighborhood $r_1, r_2 < \delta$ and a positive constants K and L depending only on J s.t.

$$||v||^2 \leq Kv^\top \mathcal{J}_0^\top Av, \quad ||w||^2 \leq Kv^\top \mathcal{J}_0^\top Dv,$$

and

$$v^\top \mathcal{J}_0^\top Bw \leq L||v||||w||, \quad w^\top \mathcal{J}_0^\top Cv \leq L||v||||w||.$$

Then the above shows that for sufficient small nbhd, i.e. $r_1, r_2 < \delta$, equation (13) can be made positive, because F, G as functions are uniformly bounded with value greater than 1.

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