SYMPLECTOMORPHISM GROUP OF RATIONAL SURFACE

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ABSTRACT. We completely determine SMCG and $\pi_1(Ham)$ of a rational surface, which is diffeomorphic to n-point blow up of $\mathbb{C}P^2, n \geq 6$, with a given symplectic form ω of type $\mathbb{A}$ and $\mathbb{D}$ in the K-nef symplectic cone.

Contents

1. Introduction

A symplectic manifold (X, ω) is an even-dimensional manifold X with a closed, nondegenerate two form ω. The symplectomorphism group of (X, ω), denoted by $Symp(X, \omega)$, is the group of diffeomorphisms ϕ of M which preserves ω. When the manifold X is simply connected, $Symp(X, \omega)$ can be given the C^∞-topology and becomes an infinite-dimensional Fréchet Lie group.

For a closed 4−dimensional symplectic manifold (X, ω), since Gromov's work [20], the homotopy type of $Symp(X, \omega)$ has attracted much interest over the past 30 years. For the particular case of some monotone 4−manifolds, the (rational) homotopy of $Symp(X, \omega)$ was fully computed in [20, 3, 17]. However, for an arbitrary symplectic 4 manifold, the complication grows drastically: for $S^2 \times S^2$, see [1, 3]; and [25, 8] for other instances.

Since then there are two parallel stories for simply connected (X^4, ω): one is on the symplectic mapping class group (SMC), which is the discrete group $Symp(X,\omega)/Ham(X,\omega)$; the other is on the (weak) homotopy type of Ham (X,ω). Next we summarize the main idea and facts in the past, together with our new idea and results.

1.1. **Symplectic mapping class group.** Based on the idea of Arnold and Donaldson, Paul Seidel in his thesis [41] shows that there are symplectomorphisms called (generalized) Dehn twists along Lagrangian spheres, which is diffeomorphically but not symplectically isotopic to identity. This initiated the study of SMC. Later on, lots of results about the SMC of an exact symplectic manifold is obtained, see [23],[42], [22], where each of them is related to braid groups on disks.

For the closed symplectic manifolds, the Symplectic mapping class group (SMC), $\pi_0(Symp(X,\omega))$, fits into the short exact sequence

$$1 \to \Gamma(X,\omega) \to \pi_0(Symp(X,\omega)) \to \pi_0(Symp_h(X,\omega)) \to 1, \tag{1}$$

where $\pi_0(Symp_h(X,\omega))$ is the homological trivial part of SMC, called the Torelli Symplectic mapping class group (TSMC). When it comes to the rational manifolds, following form [20, 1, 3, 25, 8] and summarized in [29], it is shown that when the Euler number is no larger than 7, SMC is always equal to the homological action. In other word, TSMC is trivial.

When the Euler number is larger than 7, it becomes more interesting. Jonny Evans in [17], following ideas in [40], showed that the TSMC the monotone 5-point blow up is $\pi_0\mathrm{Diff}^+(S^2,5)$, which is the quotient of 5-strand pure braid group on S^2 quotient its center $\mathbb{Z}_2$. For 6,7,8-point blow up, [40] shows that when endowed with the monotone form, any Dehn twist along a Lagrangian sphere is not symplectically isotopic to identity. And [44] shows that any Dehn twist along a Lagrangian sphere has infinite order in SMC, for monotone 6-point blow up. For more than 8-point blowups, to the author's best knowledge, there's no current known result.

On the other hand, it is conjectured that for a generic form, TSMC is trivial. For no more-than-5-point blow up this is true, see [29] and [28]. And we can prove this type of conjecture for any rational 4-manifold.

First, let's recall a reduced symplectic form

Definition 1.1 (Reduced Symplectic form, cf.[30])**.** *On $\mathbb{C}P^2\#n\overline{\mathbb{C}P^2}, n\geq 3$ with basis $\{H, E_1,\cdots,E_n\}$ of $H_2(X,\mathbb{Z})$, a symplectic form ω is called* **reduced** *if it can be normalized to have area $(1|c_1,\cdots,c_n)$ on the basis such that*

$$1 > c_1 \geq c_2 \geq \cdots \geq c_n > 0 \quad \textit{and} \quad 1 \geq c_i + c_j + c_k. \tag{2}$$

Notice that ω being a symplectic form means it also has positive square, i.e.

$$1 > c_1^2 + c_2^2 + \cdots + c_n^2 \tag{3}$$

The reason why it suffices to consider reduced symplectic form is the following: 1) any ω on a rational 4-manifold is diffeomorphic to a reduced one. 2)Diffeomorphic symplectic forms have homeomorphic $Symp(X,\omega)$.

And throughout this paper we further require that that the symplectic form pairs positive with $c_1(\omega)$. Notice that for a reduced form, $c_1(\omega) = -K = 3H - E_1 - \cdots - E_n$. We'll call this condition $K-nef$,

$$3 > c_1 + c_2 + \cdots + c_n. \tag{4}$$

And we will call the region in the normalized $H_2(X,\mathbb{R})$ the **K-nef reduced symplectic cone**, defined by equations (2), (3), and (4). According to the Dynkin diagram of the root system formed by Lagrangian spheres, we also classify any reduced form into the follwoing 3 types:

- Type $\mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8$ (or simply type $\mathbb{E}$), characterized by the pattern $[\omega] = (1|\underbrace{\frac{1}{3}, \frac{1}{3}, \cdots \frac{1}{3},}_{k}, c_{k+1} \cdots)$, where $c_{k+1} < \frac{1}{3}, k = 6, 7, 8$, respectively.
- Type $\mathbb{D}_m$ (or simply type $\mathbb{D}$), if $[\omega] = (1|a, \underbrace{\frac{1-a}{2}, \frac{1-a}{2}, \frac{1-a}{2}, \cdots \frac{1-a}{2},}_{m}, c_{m+2} \cdots)$, where $\frac{1-a}{2} > c_{m+2} \frac{1}{3} < a < 1 \& m \geqslant 4$, or $a = \frac{1}{3} \& m = 4$.
- Type $\mathbb{A}$, the rest cases.

Notice that a generic symplectic form is of type $\mathbb{A}$, and type $\mathbb{D}, \mathbb{E}$ has higher codimension in the symplectic cone. See section 2 (Proposition 2.1 and 2.2) for more details.

Theorem 1.2 (=Theorem 4.3 and Theorem 6.1)**.** *In the K-nef reduced symplectic cone, the Torelli symplectic mapping class group is completely determined for any type $\mathbb{A}$ or $\mathbb{D}$ symplectic forms.*

- *Any type $\mathbb{A}$ forms has $\pi_0(Symp_h(X, \omega)) = \{1\}$;*
- *Any type $\mathbb{D}_m$ forms has $\pi_0(Symp_h(X, \omega)) = Pb_m(S^2)$, which is the pure braid group of m strands on the sphere.*

In particular, K-nef type $\mathbb{A}$ forms covers all the toric case, and we have

Corollary 1.3 (=Proposition 4.7)**.** *For any toric symplectic 4-manifold, if a symplectomorphism acts trivially on homology, then it is symplectically isotopic to identity.*

1.2. **The idea and Homotopy type of $Symp(X, \omega)$.** The homotopy type of the (compactly supported) Hamiltonian diffeomorphism group is computed for some closed symplectic manifolds or stein domains.

Gromov first computed the full (weak) homotopy type of the Hamiltonian diffeomorphism groups of monotone $\mathbb{C}P^2$, one point blow up, and $S^2 \times S^2$ He showed that $Ham(X, \omega)$ is (weakly) homotopic to the Kähler isometry group for the above cases. The idea is to regard the Hamiltonian group as a composition of 3 factors: Stein domain U, 2-dimensional (configuration of) J-holomorphic curve C, the open stratum of the space of (tamed) almost complex structures $\mathcal{J}_C$. The key input by Gromov is to prove that the $Symp_c$ which is the compactly supported symplectomorphism (Hamiltonian diffeomorphism) of Stein domain $\mathbb{C} \times \mathbb{C}$ is weakly contractible. Later on, progress using this idea has been made by [1, 3, 25, 8] for the non-monotone form, and [25, 17] for the monotone 3,4,5 blowup. We continue to probe the rational symplectic 4-manifolds and stein domains along this line, and prove that

Theorem 1.4. *$Symp_c[(\mathbb{C} - n$ points$) \times \mathbb{C}]$ is weakly contractible. The case $\mathbb{C}^* \times \mathbb{C}$ is due to Evans.*

And using this fact together with a properly chosen configuration which compactifies the Stein domain into a rational 4-manifold, we can obtain the SMC result mentioned in Theorem 1.2 and 6.1. And for the non-monotone form, as noticed in [1, 3] etc, the homotopy type of $Ham(X, \omega)$ involves the complication of the topology of $\mathcal{J}_C$. One approach to deal with this is Donaldson's point of view of infinite-dimensional GIT (see [16]), which is also closely related to the SMC.

1.3. **Discussion: Infinite Dimensional Group Action and Moment Map.** The classical Kempf-Ness theorem and Atiyah-Bott localization theorem state the following about GIT quotient and symplectic reduction:

Let (M, Ω, J) be a Kähler manifold, a Lie group G acts by Kähler isometry, whose Lie algebra is $\mathcal{G}$ and $\mathcal{G}^*$ its dual, with a natural inner product on G invariant under the adjoint action.

- Symplectic Geometry: The action of G on (M, Ω) admits a moment map $\mu : M \to \mathcal{G}^*$. Further, the critical point of the $G-$invariant function $|\mu|^2 := \langle \mu, \mu \rangle : M \to \mathbb{R}$ give the information equivalent cohomology $H^*_G(M) := H^*(X \times_G E_G)$. The gradient flow of $-|\mu|^2$ gives an invariant stratification $M = X_0 \coprod X_1 \coprod \cdots$, where each X_i is the stable manifold of some critical set C_i of $|\mu|^2$.

- Complex Geometry: The action of G on (M, J) could be complexified into $G^{\mathbb{C}}$ on (M, J), whose Lie algebra is $\mathcal{G} + i\mathcal{G}$. And each X_i is invariant under the $G^{\mathbb{C}}$ action. Indeed the $H^*_G(M)$ can be built from $H^*_G(X_i)$.

Then we have the following homeomorphism:

$$\mu^{-1}(0)/G = M^{ss} /\!\!/ G^{\mathbb{C}};$$

and

$$\mu^{-1}(Orb_i)/G = X_i /\!\!/ G^{\mathbb{C}},$$

where Orb_i is the coadjoint orbit corresponding to a preimage of some critical point of $|\mu|^2$.

And Donaldson in [16] considered the following analog of the above for the infinite-dimensional setting. We address the case of the Hamiltonian diffeomorphism group acting on compatible (almost) complex structure here. Let (X, ω) be a symplectic manifold, $\mathcal{J}_\omega$ the space of compatible almost complex structure. there is a natural Kähler structure on $\mathcal{J}_\omega$, by regarding it as the space of section of a bundle over X with fiber as the Siegel upper half-space (a contractible Kähler manifold).

And the Hamiltonian diffeomorphism group $Ham(X, \omega)$ acts on $\mathcal{J}_\omega$ by Kähler isometry, the infinitesimal action of a Hamiltonian vector field V_h w.r.t. h sends $J \in \mathcal{J}_\omega$ to the pull back of the Lie derivative $\mathcal{L}_{V_h} J$. This is the symplectic side analogue. And on the complex side, there is an analogue of the complexification of $Ham(X, \omega)$ which acting on space of (**integrable**) complex structures $\mathcal{J}^{int}_\omega$, by defining the action of ih as $J\mathcal{L}_{V_h} J = \mathcal{L}_{JV_h} J$ such that $V_{ih} := JV_h$. For details, see [16] and [43].

And we have

- Symplectic side: The action of $Ham(X, \omega)$ on $\mathcal{J}_\omega$ admits a moment map $\mu : \mathcal{J}_\omega \to Ham^* = C^\infty_0(X)$, where $C^\infty_0(X)$ is the space of Hamiltonian functions with integral 0 over X. Further, the moment map $\mu(J)$ is indeed the Hermitian scalar curvature $S(J) - \bar{S}$ of the metric g_J, where $\bar{S}$ is the total curvature per unit volume depending on X. Note that for integrable J, $S(J)$ is the usual scalar curvature.
- Complex side: The norm of the the Hermitian scalar curvature is often called the Calabi functional $|\mu|^2 : \mathcal{J}^{int}_\omega \to \mathbb{R}$, $|\mu|^2(J) = \int_X (S(J) - \bar{S})^2 \frac{\omega^n}{n!}$. Its critical points give the information of the equivalent cohomology $H^*_{Ham}(\mathcal{J}^{int}_\omega)$. The critical points of this Calabi functional is the extremal Kähler metric, which is conjectured in [13] to be unique up to the action of Ham. The gradient flow of $-|\mu|^2$ is called the Calabi flow. There is a stratification $\mathcal{J}^{int}_\omega = X_0 \coprod X_1 \coprod \cdots$, where each X_i is diffeomorphic to some critical set C_i of $|\mu|^2$. But we only know the convergence of the Calabi flow near an extremal Kähler metric in [15], no global (in the sense further from the cscK metric) result known, due to the difficulty of the 4th order PDE.

And two related remarks with our results:

Question 1.5. *We can still build the $H^*_{Ham}(\mathcal{J}^{int}_\omega)$ from each stratum of the stratification $\mathcal{J}^{int}_\omega = X_0 \coprod X_1 \coprod \cdots$. And a question suggested by Nitu Kitchloo is the following: is it possible to write down the whole $Ham(X, \omega)$ as a homotopy limit of a collection of finite-dimensional Lie groups (Kähler isometries of extremal metrics)?*

The simplest cases are done by [4] where the $Ham(S^2 \times S^2, \omega)$ is a Kac-Moody group.

Remark 1.6. *One can find a subset $\mathcal{J}_f$ of $\mathcal{J}_\omega$ such that $Symp_h$ acting freely on it. The subset $\mathcal{J}_f / Symp_h$ is the classifying space of $Symp_h$ and hence it topology determines the topology of $Symp_h$. We used this technique in Theorem 6.1. In the meanwhile, this is related to the fact pointed out in [2] equation (4), where*

$BSymp(M,\omega) = \int_{[J\in Z]} BAut(J)$, where Z is the coarse moduli space of complex structures compatible with ω.

1.4. **Organization of the paper.** Section 2 gives the general approach by introducing a new family of divisors.
Section 3 setup inflation and the existence of curves along rays pointing the vertex-A. This is mainly used to deal with type $\mathbb{D}$ forms in section 6.
Section 4 uses a different strategy of inflation plus ball packing arguments to deal with all type $\mathbb{A}$ forms.
Section 5 and 6 complete all type $\mathbb{D}$ cases and the rest paper computes $\pi_1(Ham)$ as well as applying those results on isotopy of symplectic curves.
The last section is some application of rank of $\pi_1(Ham)$ and isotopy of symplectic curves.

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2. Symplectic cone and the strategy

There are 3 key ingredients in our strategy: the inflation process, the ball packing theorems, and a family of the special configuration of the divisor (we call it a filling divisor) with related moduli spaces.

In the inflation process, the ball packing theorems will be discussed in detail in sections 3.3 and 4.

First, we recall some facts about the normalized symplectic cone. Recall from [28] section 2, there is a combinatorial and a Lie theoretic approach for the reduced symplectic cone. And for $X_k = \mathbb{C}P^2\# k\overline{\mathbb{C}P^2}, 3 \leq k \leq 8$, the reduced symplectic cone is a polyhedron with the monotone form as a vertex, while the edge of the polyhedron passing through the monotone vertex 1-1 corresponds to simple roots. There is a generic choice of simple roots given in this way: $MO = H - E_1 - E_2 - E_3$, $MA = E_1 - E_2$, $MB = E_2 - E_3, \cdots$, $ML_{k-1} = E_{k-1} - E_k, \cdots$, $MG = E_7 - E_8$.

When $n \geq 9$, the reduced cone is no longer a Polyhedron. We start from the positive cone $\mathcal{P} = \{e \in H^2(X;\mathbb{R}) | e\cdot e > 0\}$. This is a subset of $\mathbb{R}^{n+1}$, which is called the positive cone. A corollary of [26] is that the action of Diff(X) on $H^2(X,\mathbb{Z})$ is transitive on the positive cone $\mathcal{P}$. When $\chi(X) < 12$, reflection -1 classes is finite, denoting D_{-1}, and the fundamental domain of this action is called the P-Cell. For the basis and canonical class K, when Euler number is less than 12, the **P-Cell** is $PC = \{e \in \mathcal{P} | K\cdot e > 0\}$. When $n \geq 9$, the P-Cell needs to be cut by one more quadratic equation and a K-positive equation a form $\omega = (1|c_1, \cdots, c_n)$:

$$\omega(K) = 3 - \sum_i c_i > 0 \tag{5}$$

and

$$A = PD[\omega], A^2 = 1 - \sum c_i^2 > 0 \tag{6}$$

Further, the reflection group along -1 classes is infinite, guided by Kac-Moody algebra, see [47]. We call this cone the **normalized K-nef symplectic cone**, denoted by P_K^k for k-point blowup of $\mathbb{C}P^2$. Notice that the reduced symplectic cone does not require equation (5). And when $k \leq 9$, the two cones coincide.

We'll summarize these facts as the following proposition:

Proposition 2.1. *Let M be a rational surface. Then*

(i) A convex combination of (normalized) reduced classes is (normalized) reduced. (i') The K-nef condition (5) is linear and hence preserved under convex combinations.

(ii) A reduced class is symplectic if and only it has a positive square.

(iii) For a reduced symplectic class u, its canonical class K_u is

$$K_0 := -3H + \sum_{i=1}^{k} E_i$$

if $M = M_k$, and it is $K_0 := -2F_1 - 2F_2$ if $M = \tilde{M}_1$.

(iv) Every class in $\mathcal{C}_M$ is equivalent to a unique reduced symplectic class under the action of $\mathrm{Diff}^+(\mathrm{M})$. *In other words, up to scaling, the normalized reduced symplectic cone $P(M)$ is a fundamental domain of on $\mathcal{C}_M$ under the action of* $\mathrm{Diff}^+(\mathrm{M})$.

Proposition 2.2. *For $M = M_k, k \geq 3$, consider the convex cone RED in $\mathbb{R}^{\chi-3} = \mathbb{R}^k$ with vertex $-\frac{1}{3}K_0$ and the polygone base in the $c_1c_2\cdots c_{k-1}$ (i.e. $c_k = 0$) hyperplane generated by the following k points G_i:*

$$G_1 = (0, \cdots, 0), G_2 = (1, 0, \cdots, 0), G_3 = (\frac{1}{2}, \frac{1}{2}, 0, \cdots, 0),$$
$$G_4 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, \cdots, 0), \cdots, G_k = (\frac{1}{3}, \cdots, \frac{1}{3}, 0).$$

Then the normalized reduced cone $P(M)$ is the region in RED cut by the quadratic condition having positive square, $\sum_{i=1}^{k} c_i^2 < 1$.

Let

$$l_1 = H - E_1 - E_2 - E_3, \quad l_2 = E_1 - E_2, \quad \cdots \quad , \quad l_k = E_{k-1} - E_k. \tag{7}$$

Then the symplectic classes on each edge T_kG_i are characterized by the property of pairing trivially with l_j for any $j \neq i$ and positively on l_i. Consequently, the reduced symplectic classes are characterized as the symplectic classes which are positive on each E_i and non-negative on each l_i.

Proof. All the statements can essentially be found in Proposition 2.21 and Section 2.2.5 in [27].

What is not explicitly covered in [27] is when $M_k, k \geq 9$.

First, rewrite the normalized reduced condition as

$$1 \geq c_1 + c_2 + c_3, \quad c_1 \geq c_2, \quad c_2 \geq c_3, \quad \cdots, \quad c_{k-1} \geq c_k, \quad c_k > 0. \tag{8}$$

Let Ψ be the translation moving $T_k = (\frac{1}{3}, \cdots, \frac{1}{3})$ to 0. Under this linear translation, $(1|c_1, \cdots, c_k)$ is moved to $x = (x_1, \cdots, x_k) = (c_1 - \frac{1}{3}, \cdots, c_k - \frac{1}{3})$, and the normalized reduced condition (8) can be written as the k homogeneous conditions:

$$0 \geq x_1 + x_2 + x_3, \quad x_1 - x_2 \geq 0, \quad x_2 - x_3 \geq 0, \quad \cdots, \quad x_{k-1} - x_k \geq 0, \quad x_k > -\frac{1}{3}.$$

Clearly, $\Psi(P(M_k))$ has only one vertex at the origin and its opposite face is open and at the hyperplane $x_k = -\frac{1}{3}$. There are k inequalities of the form $\geq$ in (8). Setting $c_k = 0$ and all of the k inequality $\geq$ to be equality except the i-th one, we obtain the k points G_i in the $c_1 \cdots c_{k-1}$ hyperplane. The rays T_kG_i are clearly extremal rays.

Notice that T_k pairs trivially with each l_j, and G_i pairs trivially with each l_j for each $j \neq i$. It follows that T_kG_i pairs trivially with each l_j except for $j = i$. □

A quick remark here is that:

- Type $\mathbb{A}$ is charicterized by $c_1 + c_2 + c_3 < 1$. The reason is that $H - E_1 - E_2 - E_3$ is the only vertex that possibly has 3 intersections with other simple roots in the Dynkin diagram.
- Type $\mathbb{D}_k$ is characterized by $c_1 + c_2 + c_3 = 1$, and $c_1 > c_2 = \cdots = c_k > c_{k+1}$.
- Type E_k is charicterized by $c_1 = c_2 = \cdots = c_k = \frac{1}{3} > c_{k+1}$.

Remark 2.3. *We often denote G_1 by O since it is the origin. G_2 is often denoted by A, and it will appear frequently in the rest of the paper. We'll further denote $G_3, G_4, \cdots$ by $B, C, \cdots$ accordingly.*

¡++¿

K-face	Γ_L	ω area
$M_6 A\widehat{B \cdots G_5 \hat{O}} G_6 \cdots$	$\mathbb{E}_6 \times \mathbb{A}$	$\lambda = 1; c_1 = c_2 = \cdots = c_6 = \frac{1}{3} > c_7 \geq \cdots$
$M_7 A\widehat{B \cdots G_6 \hat{O}} G_7 \cdots$	$\mathbb{E}_7 \times \mathbb{A}$	$\lambda = 1; c_1 = c_2 = \cdots = c_7 = \frac{1}{3} > c_8 \geq \cdots$
$M_8 A\widehat{B \cdots G_7 \hat{O}} G_8 \cdots$	$\mathbb{E}_8 \times \mathbb{A}$	$\lambda = 1; c_1 = c_2 = \cdots = c_8 = \frac{1}{3} > c_9 \geq \cdots$
MA	$\mathbb{D}_{n-1}$	$\lambda = 1; c_1 > c_2 = \cdots = c_n$
MAG_n	$\mathbb{D}_{n-2}$	$\lambda = 1; c_1 > c_2 = \cdots = c_{n-1} > c_n$
MAG_{n-1}	$\mathbb{D}_{n-3} \times \mathbb{A}_1$	$\lambda = 1; c_1 > c_2 = \cdots = c_{n-2} > c_{n-1} = c_n$
$MAG_{n-1}G_n$	$\mathbb{D}_{n-3}$	$\lambda = 1; c_1 > c_2 = \cdots = c_{n-2} > c_{n-1} > c_n$
2^2 walls containing $MAB\widehat{\cdots G_{n-3}\hat{O}} G_{n-2}$	$\mathbb{D}_{n-4} \times \mathbb{A}$	inequalities correspond to vertices removed
2^p walls containing $MAB\widehat{\cdots G_{n-p-1}\hat{O}} G_{n-p}$	$\mathbb{D}_{n-p-2} \times \mathbb{A}$	inequalities correspond to vertices removed
2^{n-6} walls containing $M\widehat{BCD\hat{O}}E$	$\mathbb{D}_4 \times \mathbb{A}$	inequalities correspond to vertices removed
Other walls	type $\mathbb{A}$	inequalities correspond to vertices removed

TABLE 1. Regions of possible reduced forms for $\mathbb{C}P^2 \# n\overline{\mathbb{C}P^2}$

We highlight two facts about the general symplectic cone.

1) We can always project the cone on to the coordinate subspace $c_1 \cdots c_{n-m} \subset \mathbb{R}^n, m < n$. Geometrically, this means blow down the last m exceptional curves.

2) The point A is always an open vertex in any cone. It is special in the sense that under base change (38) we can regard any form in the cone as a blowup form from $S^2 \times S^2$ with area ratio at least 1 between the two factors, and point A is the limit point where the ration goes to ∞.

The above two facts indeed illustrate our strategy for π_0:
For type $\mathbb{A}$ forms we use fact 1), and by talking limit of sizes on the last m exceptional curves to be 0, we can use inflation and ball packing facts to compare the SMCG between the blowup and blowdown manifolds.
For type $\mathbb{D}$ forms, motivated by fact 2), we first consider type $\mathbb{D}$ forms that are close to the limit vertex A, then inflation process allows us to extend this information back along the rays through A. Notice that along those rays, as a blowup form from $S^2 \times S^2$ with the fixed area in the fiber class, the area ratio decrease, and the blowup sizes remain the same.

We call the equalities and inequalities between $1, \lambda$ and among $c_i's$ the **size pattern**. Notice that if the size pattern is preserved, we'll stay at the same wall/chamber of the symplectic cone. In particular, preserving the type pattern means that type $\mathbb{A}$, $\mathbb{D}$, or $\mathbb{E}$ of the symplectic form gets preserved.

2.1. **Combinatorics on P-Cells, exceptional classes.** Let K_0 be $-3H + E_1 + \cdots + E_n$. Define the canonical class K_J for a given J. Later if clear in the context, we will simply denote K for the canonical class. We recall from [18]:

Suppose M is an oriented closed manifold with odd intersection form, $b^+ = 1$, $b^- = n$ and no torsion in $H^2(M;\mathbb{Z})$. A basis $(x, \alpha_1, \cdots, \alpha_n)$ for $H^2(M;\mathbb{Z})$ is called standard if $x^2 = 1$, and $\alpha_i^2 = -1$ for each $i = 1, \cdots, n$. Let

$$\begin{aligned} \mathcal{P} &= \{e \in H^2(M;\mathbb{R}) | e \cdot e > 0\} \\ \mathcal{B} &= \{e \in H^2(M;\mathbb{R}) | e \cdot e = 0\} \\ \mathcal{E}_K &= \{e \in H^2(M;\mathbb{R}) | e \cdot e = -1, K \cdot e = -1\}. \end{aligned}$$

We also recall the definition of K-symplectic cone P_K and the $K-$sphere cone S_K^+ here from [31]: $\overline{P}_K$, the closure of P_K, is a $P-$cell and $\kappa(P_K)=-K$.

Definition 2.4. *For a tamed almost complex structure J, the open convex cone $\mathcal{S}_{K_J}^+$ is the interior of the convex cone generated by classes in $S_{K_J}^+$. It is called the positive K_J-sphere cone.*

Let's recall the following from [31]:

Lemma 2.5 (=Lemma 5.24 in [31])**.** *1. P_K is an open convex polytope in $\mathcal{P}$. Each wall of P_K is either a wall of a class in $\mathcal{E}_{M,K}$, or the wall of K if $k\geq 9$.*

2. The face F_{E_k} of $P_{M_k,K}$ corresponding to E_k is naturally identified with $P_{M_{k-1},K}$.

Proposition 2.6 (=Proposition 5.21 in [31])**.** *For any J on $\mathbb{CP}^2\#k\overline{\mathbb{CP}^2}$, the positive $K-$sphere cone $\mathcal{S}_{K_J}^+$ coincides with P_{K_J}.*

Furthermore, the proof of Proposition of 2.6 in [31] is done inductively: when $k\leq 8$, any class in P_K^k is a finite positive combination of exceptional classes in $\mathcal{E}_K$. For $k+1$, by induction, take a wall which is parallel to $-K\cdot p=0, p\in P_K^{k+1}$. This wall is going to intersect P_K^{k+1} at two facets, each of them can be identified with P_K^k by Lemma 2.5. Hence we have the following conclusion:

Lemma 2.7. *Any class in the K-nef symplectic cone P_K can be written as a finite positive combination of exceptional classes in $\mathcal{E}_K$. In particular, any reduced symplectic classes in the K-nef cone can be written as a finite positive combination of exceptional classes $\mathcal{E}_K$.*

3. Inflation and stability for $Symp_h$

First we fix the notation: Let X be $\mathbb{C}P^2\#n\overline{\mathbb{C}P^2}$ with a reduced symplectic form ω, and its class $[\omega]$ is represented using a vector $(1|c_1,c_2,\cdots,c_n)$. We are going to work with the K-nef symplectic cone P_K^k, and prove the stability of π_0,π_1 of *Symp* in this cone .The key technique we use inflation argument in [34] and [7].

3.1. Preliminary.

First, let's recall Theorem 0 of [28]

Lemma 3.1. *Any symplectomorphism of $(X_k:=\mathbb{C}P^2\#k\overline{\mathbb{C}P^2},\omega_k)$ is smoothly isotopic to identity if it acts trivially on homology. Namely, $Symp_h(X,\omega)\subset Diff_0(X)$ for any symplectic rational surface (X,ω).*

And recall several facts about the existence of embedded curves in classes H, $H-E_1$ and E_n:

Lemma 2.19 of [27]

Lemma 3.2. *Let $H,E_1,\cdots,E_n$ be the ordered basis of $H_2(X,\mathbb{Z})$, where the form has area $1,c_1,\cdots,c_n$ respectively. The class E_n has the smallest area among all exceptional sphere classes in X, and hence have an embedded J-holomorphic representative for any $J\in\mathcal{J}_\omega$.*

H and $H-E_1$ are J-nef spherical classes where J tames a reduced ω:

Lemma 3.3. *J-nef spherical classes have non-empty irreducible moduli, and hence always has embedded representatives.*

In particular, for any rational surface with $\chi>4$, $\omega\in[1|c_1,\cdots,c_n]$ reduced, then for any J tames ω, $H-E_1$ has an embedded representative; if further assume that $c_1\leq\frac{1}{2}$, then for any J tames such ω, there's embedded curve in class H.

Those Lemmas allows us to inflate ω along certain line segments in the symplectic cone while keeping J tame, and further grants the inclusion of $\mathcal{A}_\omega$ spaces so that we can apply diagram (3.14).

Lemma 3.4. *For a form $u = (1|c_1, \cdots, c_{n-m}, c_{n-m+1}, \cdots, c_n) \in P^k_K$, here P^k_K is the convex polytope containing the (-K)-effective reduced symplectic cone, $c_{n-m} > c_{n-m+1} = \cdots = c_n$, and c_n is the minimal exceptional size. We define 3 types of lines:* **minimal exceptional line***: $\overline{uu_0}$ where $u_0 := (1|c_1, \cdots, c_{n-m}, 0, \cdots, 0)$,* **A-extremal line***: $\overline{uA}$ and* **O-extremal line***: $\overline{uO}$.*

For any $u_t \in L$ where L is the interior of $\overline{uu_0}, \overline{uA}, \overline{uO}$, we always have $\mathcal{A}_u \subset \mathcal{A}_{u_t}$.

Proof. By Lemma 3.2 and 3.3, we can inflate along embedded $\{E_{n-m+1}, \cdots, E_n\}$; $H - E_1$; H respectively for $\overline{u, u_0}, \overline{u, A}, \overline{u, O}$. □

Also, we'll recall a very useful Lemma about J-holomorphic curves:

Lemma 3.5 (Lemma 4.1 in [46], Lemma 3.2 in [14])**.** *Let $M = \mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$. If $S = aH + \sum b_i E_i$ with $a < 0$ is represented by an irreducible curve, then*

- *$S = -nH + (n+1)E_1 - \sum_{k_j \neq 1} E_{k_j}$ up to a Cremona transform.*
- *Or $S = f^* S'$, where f is a Cremona transformation and S' is a class with $a' > 0$.*

In particular, if the form is reduced, and S is embedded with $S^2 < 0$, then $S = -nH + (n+1)E_1 - \sum_{k_j \neq 1} E_{k_j}$ and it has to be a spherical class.

Note that we are going to often write the class of a curve $S = aH + \sum b_i E_i$ and will often refer to **a-coefficient** of the curve as the coefficient of the hyperplane class in $\mathbb{C}P^2$.

3.2. **Stratification of $\mathcal{A}_\omega$.** The way we stratify $\mathcal{A}_\omega$ will be similar to that of [27]. For any finite subset $\mathcal{C}^{top}_\omega \subset \mathcal{E}_{K_0}$, we are going to use the following partition of $\mathcal{A}_\omega$.

First let's recall the following Lemma from [27]:

Lemma 3.6. *Let X be a rational 4-manifold such that $\chi(X) \leq 12$. Given a finite subset $\mathcal{C} \subset \mathcal{S}^{<0}_\omega$,*

$$\mathcal{C} = \{A_1, \cdots, A_i, \cdots, A_n | A_i \cdot A_j \geq 0 \text{ if } i \neq j\},$$

we have the following **prime submanifolds**

$$\mathcal{J}_\mathcal{C} := \{J \in \mathcal{J}_\omega | A \in \mathcal{S}_\omega \text{ admits a smooth embedded } J\text{−hol representative } \textbf{iff } A \in \mathcal{C}\},$$

which is a submanifold of codimension $cod_\mathcal{C} = \sum_{A_i \in \mathcal{C}} cod_{A_i}$ in $\mathcal{J}_\omega$. Also denote $\mathcal{X}_{2n} = \cup_{cod(\mathcal{C}) \geq 2n} \mathcal{J}_\mathcal{C}$.

Lemma 3.7. *$\mathcal{A}_\omega$ is the disjoint union of three parts:*
Cod=0 part: Characterized by the existence of an embedded curve in each class of $\mathcal{C}^{top}_\omega$. We call it $\mathcal{A}^{top}_\omega$. Since we have restriction on the pseudo-holomorphic representatives in any other classes, $\mathcal{A}^{top}_\omega$ is larger than the generic strata.
$Cod = 2$ part: the union of $\mathcal{A}_{C,\omega}$, where C is an embedded square -2 sphere that pair negatively with some exceptional classes in $\mathcal{C}^{top}_\omega$.
$Cod > 2$ part, the complement of the above two sets in $\mathcal{A}_\omega$.

Proof. The Cod=0 and Cod=2 has correct codimension in $\mathcal{A}_\omega$, by Lemma 3.6.

In particular, if J belongs to the Cod=2 part, there's exactly one exceptional curve E breaks into $C + D$ where C is the unique -2 curve and D is another exceptional curve.

We then argue that the complement has codimension larger than 2.

Consider the stable curve in an exceptional class, $E = \sum_i C_i$ where each C_i is possibly multiple covered. Let $g_J(A)$ be the virtual genus of class A, given by $\frac{A \cdot A + K \cdot A}{2} + 1$. Becasue E is J-nef for a generic J, by Theorem 1.4 of [32], $0 = g_J(E) \geq \sum_i g_J(C_i) \geq 0$. Hence $g_J(C_i) = 0$ for each C_i. Then by the connectedness of $\sum_i C_i$, there must be at least one component with self-intersection at most -2. If this component is embedded and the only negative self-intersection -2 class, then it belongs to the Cod=2 part. Otherwise, by the virtual dimension computation (Theorem 1.6.2 of [21] for example) and transversality for the underlying simple representative, the stratum of such J has codimension larger than 2. Here are more details:

If the only curve with square less than -1 is a simple class with self intersection -2, then it has to be of $\{C, D\}$ by computing the square and pairing with K. More precisely, assume $E = C + \sum D_i + \sum P_j$, so that $C^2 = -2, D_i^2 \geq -1, P_j^2 \geq 0$ By $g_J(C) = g_J(D_i) = 0$, we have $K \cdot C = 0, K \cdot D_i = -1$ and $K \cdot P_J < -1$. Also, we have $K \cdot (C + \sum D_i + \sum P_j) = K \cdot E = -1$. Hence there can only be precisely one D_i.

For all other cases, let's first recall that for a simple class A, with a J-holomorphic representative, the index of A is given by $2g - 2 - 2K_J \cdot A$, where K_J

- if there is a irreducible components with square less than -2.

  Let $E = C_1 + \sum_{i>1} C_i$, such that $C_1^2 < -2$. If C_1 has a simple representative, then we are done. Now let's deal with the case C_1 is multiple covered. Let $C_1 = mC_1', m > 1$ such that C_1' has a simple representative. Notice that we immediately know that $(C_1')^2 \leq -1$.

  Then we have $0 = 2g_J(C_1) = 2 + m^2(C_1')^2 + mK_J \cdot C_1'$. This means that
  $$K_J \cdot C_1' = \frac{-2 - m^2(C_1')^2}{m} < 0.$$
  Hence the simple representative has index less than 2. This means that $\mathcal{A}_{\bar{e}}$ has codimension greater than 2 in $\mathcal{A}_\omega$.

- if there are more than one components with square -2.

  Now let's assume that $E = C_1 + C_2 + \sum_{i>2} C_i$, such that $C_1^2 = C_2^2 = -2$. If both of them have simple representatives, then we are done. Now lets assume some of them are multiple covered, i.e. $C_1 = pC_1', C^2 = qC_2', p, q \geq 1$. Now we still have $0 = 2g_J(C_1) = 2 + p^2(C_1')^2 + pK_J \cdot C_1'$. This means that
  $$K_J \cdot C_1' = \frac{-2 - q^2(C_1')^2}{q} < 0.$$
  Similarly, the index of C_2' is also non-positive. Hence both simple representatives has non positive indices. By the transversality of the simple representatives, $\mathcal{A}_{\bar{e}}$ has codimension greater than 2 in $\mathcal{A}_\omega$.

□

We summarize here the proof of the codimension 2 part of Lemma 3.7.

Lemma 3.8. *Let the homology type e be the set of the homology classes of an irreducible component of a stable curve in the class E, and let $\mathcal{J}_e$ be the space of $J \in \mathcal{J}_\omega$ such that there is a stable curve of type e being J-holomorphic. Then*
1) $\mathcal{J}_e$ is an open subset of $\mathcal{J}_\omega$ if and only if $e = \{E\}$.
2) $\mathcal{J}_e$ is of codimension 2 in $\mathcal{J}_\omega$ if and only if $e = \{C, D\}$, where C is an embedded (-2) sphere and $D \in \mathcal{E}$.

We also make the following convention here, which is important for the rest of section 3:

Convention. *For any reduced from ω s.t.$[\omega] \in P_K^k$, by Lemma 2.7, there is a finite subset of $\mathcal{E}_K$ so that $[\omega]$ can be written as a positive linear combination of those. We will use this subset of $\mathcal{E}_K$ as $\mathcal{C}_\omega^{top}$, and $\mathcal{A}_\omega^{top}$ denote the corresponding open stratum given by Lemma 3.7 for $\mathcal{C}_\omega^{top}$. The codimension 2 part of $\mathcal{A}_w$ is defined accordingly, i.e. it is the collection of $\mathcal{J}_e$ for $E \in \mathcal{C}_\omega^{top}$ of all the possible degeneration of type 2) in Lemma 3.8.*

We also present a Lemma here which simplifies the exposition of the proof of the stability results:

Lemma 3.9. *For the exceptional class E, assume for a given J tames (or is compatible with) ω, it has homology type (2) in 3.8, i.e. the stable curve has two irreducible component classes C and D. They each has an embedded representative and intersects each other transversely. For such J, there is an J-tame (or compatible) inflation along the embedded curves in classes C, D such that ω' tames (is compatible with J) and $[\omega'] = [\omega] + tP.D.(E), 0 \leq t \leq \omega(E)$.*

Proof. For this special case, one just performs negative inflation as in [10] and the enhanced version in [7] for t on both curves in classes C and D.

A more detailed discussion on inflation along nodal curves is given in [11]. □

3.3. **3 types of rays and stability of $Symp_h(X, \omega)$.** In this subsection, we will prove the stability of $Symp_h(X, \omega)$ along three types of rays: the A-extremal, O-extremal, and smallest exceptional rays.

3.3.1. *A-extremal line.* . The idea is the following: there's a "topological limit" when regarding the rational manifold as a blowup from $S^2 \times S^2$ where the ratio between the two component. When the ratio is ∞, we have a (**limiting**) reduced symplectic form being $\omega_0 = (1|1, 0, \cdots, 0)$. Then the ray connecting ω_0 and ω_1 is

$$\omega_t = (1|(c_1 - 1)t + 1, tc_2, \cdots tc_n),$$

and that could be extended for $1 \leq t < \frac{1}{c_2+1-c_1}$. This gives parameterization of rays through vertex A in the reduced cone.

Note that for the blowup of $\Sigma_g \times S^2$ at k points, such a topological limit still exists, see [12, 11] for details. We'll first prove the stability of π_0 and π_1 of $Symp$ along such rays.

Note that we have the following fibration, well defined up to homotopy.

$$G_\omega := Symp_h(X, \omega) \to \mathrm{Diff}_0(X) \to \mathcal{A}_w. \tag{9}$$

Here $\mathcal{A}_w$ is the set of compatible (**note that here we'll possibly use the positive inflation and comparison of tame/compatible cone**) J with some ω in the fixed cohomology class isotopic to a given symplectic form.

Proposition 3.10 (Inflation along the A-extremal line)**.** .

For the following deformation of symplectic form, the π_0, π_1 of $Symp(X, \omega)$ is invariant:

$\omega_1 = (1|c_1, \cdots, c_n)$ *and* $\omega_2 = (1|(c_1 - 1)t + 1, tc_2, \cdots tc_n)$, *for* $1 \leq t < \frac{1}{c_2+1-c_1}$. *(note that this also hold true for* $0 < t < 1$*).*

Notice that $t \geq 1$ means we are moving away from vertex A along the ray $A[\omega]$, and $t < 1$ means moving toward A along the A=extremal ray. Also, moving along the rays through vertex A does not change whether a form is of type $\mathbb{A}$ or type $\mathbb{D}$.

Proof. First, we need to prove that there exist curve set $\mathcal{C}^{top}$ and corresponding top strata $\mathcal{A}^{top}_{\omega_2}$ and $\mathcal{A}^{top}_{\omega_1}$ as in Convention 3.2, such that $\mathcal{A}^{top}_{\omega_2} \subset \mathcal{A}^{top}_{\omega_1}$, and $\mathcal{A}^{top}_{\omega_1} \subset \mathcal{A}^{top}_{\omega_2}$, for some open stratum in $\mathcal{A}_{\omega_1}$ and $\mathcal{A}_{\omega_2}$. Also, we want to prove that for the corresponding Codimension=2 strata, $\mathcal{A}_{C,\omega_1} = \mathcal{A}_{C,\omega_2}$, for any embedded (-2) sphere C coming form a nodal degeneration of $E \in \mathcal{C}^{top}$ with homology type $\{C, D\}$where D is another exceptional curve.

For the case $\mathcal{A}^{top}_{\omega_1} \subset \mathcal{A}^{top}_{\omega_2}$, we can apply Lemma 2.7 and denote the set of exceptional classes appearing in the linear combination by $\mathcal{C}^{top}_{\omega_2}$. Note that $\mathcal{A}^{top}_{\omega_1}$ and $\mathcal{A}^{top}_{\omega_2}$ are the corresponding open strata defined by the curve set $\mathcal{C}^{top}_{\omega_2}$.

For the $\mathcal{A}_{C,\omega_1} \subset \mathcal{A}_{C,\omega_2}$, for an embedded (-2) sphere C, we can apply Lemma 3.9. It ensures that any inflation that can be done for $\mathcal{A}^{top}_{\omega_1}$ and $\mathcal{A}^{top}_{\omega_2}$, we can do the same inflation for $\mathcal{A}^{C,\omega_1}$ and $\mathcal{A}_{C,\omega_2}$ to prove the inclusion.

$\mathcal{A}^{top}_{\omega_1} \supset \mathcal{A}^{top}_{\omega_2}$ and $\mathcal{A}_{C,\omega_1} \supset \mathcal{A}_{C,\omega_2}$ follows from Lemma 3.3, since we can always inflate along the embedded curve in class $H - E_1$.

The proof is completed by considering the commutative diagram of the Krohemier fibration regrading ω_1 and ω_2. □

3.3.2. *Inflation and stability along the O-extremal line.*

Lemma 3.11 (the O-extremal line inflation)**.** *For any strata of $\mathcal{A}_\omega$ with codimension 0 or 2, we can always change the cohomology class of ω into ω', such that $\sum c_i^{'} \leq \frac{1}{m}$ for any given integer m, while preserving the (in)equalities of $c_1 \dots c_n$. More concretely, given $(1|c_1, \cdots, c_n)$, we are going to obtain $(1|\frac{c_1}{t}, \cdots, \frac{c_n}{t})$, $\forall t > 1$.*

Proof. First, we need to deal with the top strata:

(1) $\mathcal{A}^{top}_{\omega} \subset \mathcal{A}^{top}_{\omega'}$ is covered by Lemma 3.4.

(2) $\mathcal{A}^{top}_{\omega} \supset \mathcal{A}^{top}_{\omega'}$. This is done by Lemma 2.7.

For those cod=2 strata, $\mathcal{A}_{C,\omega} \subset \mathcal{A}_{C,\omega'}$ is covered by Lemma 3.4 and $\mathcal{A}_{C,\omega} \subset \mathcal{A}_{C,\omega'}$ can be proved using the same inflation for $\mathcal{A}^{top}_{\omega} \supset \mathcal{A}^{top}_{\omega'}$ and Lemma 3.9.

□

3.3.3. *Inflation and stability along the smallest exceptional spheres.* Now we focus on another direction–shrinking the smallest exceptional spheres. And we'll prove that π_0 and π_1 of $Symp_h(X, w)$ are invariant under certain types of moves using inflation (we always use tame-compatible inflation and apply the comparison of cones in [31].)

Lemma 3.12. *For any pair of type $\mathbb{A}$ forms, if the size pattern of $c_1, c_2, \cdots, c_n$ are the same, then the cod 2 part is the same in terms of prime subsets.*

Proof. One only need to check that the curves $E_i - E_j$, $H - E_i - E_j - E_k$ and $2H - \sum E_l$ type of -2 curves.

And they stay the same as long as those (in)equalities of $c_1, c_2, \cdots, c_n$ are the same.

□

Then we'll prepare some inflation Lemma for the proof of Theorem 4.3, which is an inductive process. And the upshot for doing all those steps is that: for a type $\mathbb{A}$ form ω, as long as the (in)equalities of $c_1 \dots c_n$ dose not change, the codim=0 or 2 strata stay the same.

Lemma 3.13. *For a type $\mathbb{A}, \mathbb{D}$ form ω whose class $[\omega] \in P_K$, assume there are m (note that m has to be less than n) exceptional spheres which is smaller than c_1, for $[\omega] = [1, c_1, \cdots, c_n]$. Then for $[\omega_t] = [1, c_1, \cdots, c_{n-m}, t, \cdots, t]$, where $0 < t \leq c_n$, $\mathcal{A}_\omega$ and $\mathcal{A}_{\omega_t}$ has the same strata up to codimension 2. Notice that this allows up to change the smallest blowup size to 0(arbitrary small).*

Proof. We only need to prove the inflation (shrinking area) along the exceptional spheres with smallest sizes. Notice that this shrinking size process does not change $\mathcal{S}^{-2}_\omega$. And we can then prove the full statement by induction.

Now our ω denote the symplectic form we start with and ω_t denote the form after the shrinking size process in the above paragraph. We are going to show that

- $\mathcal{A}^{top}_{\omega} \subset \mathcal{A}^{top}_{\omega_t}$. This can be easily done by inflating along $E_{n-m+1} \cdots E_n$.

- $\mathcal{A}_\omega^{top} \supset \mathcal{A}_{\omega_t}^{top}$. This is granted by Lemma 2.7. The reason is the following: Let's start with
$$[w_0] = [1, c_1, \cdots, c_{n-m}, 0, \cdots, 0].$$
By Lemme 2.7, there're finitely many exceptional curves so that a positive combination of them being $[\omega'] = [1, c_1, \cdots, c_{n-m}, \frac{cn-m+c_n}{2}, \cdots, \frac{cn-m+c_n}{2}]$.
Clearly, there's the following positive linear interpolation
$$[\omega] = \frac{2c_n}{c_{n-m}+c_n}[\omega'] + 1 - \frac{2c_n}{c_{n-m}+c_n}[\omega_0].$$
Hence the proof.

□

We end here by applying the diagram

$$\begin{array}{ccccc} Symp_h(M,\omega) & \longrightarrow & \mathrm{Diff}_0(M) & \longrightarrow & \mathcal{A}_\omega \\ \big\downarrow & & =\big\downarrow & & \hookleftarrow\big\downarrow \\ Symp_h(M,\omega_t) & \longrightarrow & \mathrm{Diff}_0(M) & \longrightarrow & \mathcal{A}_{\omega_t} \end{array} \tag{10}$$

to conclude that

Corollary 3.14. *The pair of symplectic forms related by inflation, either in Lemma 3.13 or 3.11 have the same π_0 and π_1 from $Symp_h$.*

Corollary 3.15. *Take any form ω of type D_k in X_n, $k < n$, s.t. $[\omega]$ is K-nef. We can blow down X_n to $X_{k+1}, \bar{\omega}$ such that $Symp(X_n, \omega)$ and $Symp(X_{k+1}, \bar{\omega})$ has the same π_0 and π_1.*

Proof. Follows from Lemma 3.13 and the same argument form Corollary 3.14. □

3.4. **Preparation for Nakai-Moishozon.** Let $S = aH - \sum b_i E_i$, and denote $c^2 = -S^2$. We have

$$c^2 + a^2 = b_1^2 + \cdots b_n^2, \tag{11}$$

$$-2 + c^2 + 3a \le b_1 + \cdots + b_n. \tag{12}$$

Lemma 3.16. *Let M be an (ordered) n-point blowup of $\mathbb{C}P^2$, with fixed canonical class K. Then for any J tames some ω with $K_\omega = K$, there is a (rational) form of type $\mathbb{D}_{n-1}$ such that all irreducible curves with positive coefficient on H has positive area.*

Proof. We'll fix K_0 as $-3H + E_1 + \cdots + E_n$. It suffices to deal with a reduced symplectic form.

The set-up in [46] is a tamed almost complex structure with $K_J = K_0$. Let $S = aH - \sum b_i E_i$. Note that here we allow b_i to be any integer, and it suffices to only deal with non-negative integer b_i's when $a > 0$.

We now discuss two cases:

Let $c^2 = -S^2$. Recall equation (11) and 12

$$c^2 + a^2 = b_1^2 + \cdots b_n^2, \tag{13}$$

$$-2 + c^2 + 3a \le b_1 + \cdots + b_n. \tag{14}$$

Now we eliminate c^2 by subtracting the first equation by the second inequality.

$$a^2 - 3a + 2 \geq \sum b_i(b_i - 1).$$

Now some clear facts: $b_i(b_i - 1) \geq 0$ since b_i is an integer, and hence $a - 1 \geq b_i$.

Now it is clear that the form $(1, 0, 0, \cdots, 0)$ has positive area on any such classes.

And we want a type $\mathbb{D}_{n-1}$ form, which satisfies our assumptions in Lemma 5.8 and 5.9. And we are going to use the form $(n, n-2, 1, \cdots, 1)$ or $(1, \frac{n-2}{n}, \frac{1}{n}, \cdots, \frac{1}{n})$ after normalization.

We will try to minimize the area of $\omega(C)$ where $\omega = (1, c_1, \frac{1-c_1}{2}, \cdots, \frac{1-c_1}{2})$ and S is our given curve. Here we allow c_1 take value in $[\frac{n-2}{n}, 1]$.

Namely, solve the optimization problem

$$min \quad f(b_1 \cdots, b_n) = a - c_1 b_1 - \frac{1-c_1}{2} b_2 - \cdots - \frac{1-c_1}{2} b_n,$$

subject to the constrain comes from (11), namely, equation

$$g(b_1 \cdots, b_n) = (a-1)(a-2) - \sum (b_i - 1) b_i \geq 0. \tag{15}$$

Note here that we take every number as a real number, except a being an integer. Here note that a and c_1 are fixed constant, and we are looking for solution of $b_1, \cdots, b_n$. By Lagrange multiplier method, max or min appears only if $\nabla f = \lambda \nabla g$, meaning that

$$(c_1, \frac{1-c_1}{2}, \cdots, \frac{1-c_1}{2}) = \lambda(b_1, \cdots b_n), \tag{16}$$

and $(a-1)(a-2) - \sum (b_i - 1) b_i \geq 0$.

Also, clearly, when the last equality hold and (16) is the Cauchy-Schwartz maximal value for $c_1 b_1 + \frac{1-c_1}{2} b_2 + \cdots + \frac{1-c_1}{2} b_n$, which means that this is a minimal solution for the area $\omega(C)$. And we'll prove that when $c_1 = \frac{n-2}{n}$ this min is positive. (Indeed it is positive for any $1 \geq c_1 \geq \frac{n-2}{n}$).

Let's first look at the case where a is $n-1$ for convenience of computation: then note that $(a-1)(a-2) - \sum (b_i - 1) b_i = 0$ has an exact solution subject to (16), namely, $b_1 = n-2, b_2 = \cdots = b_n = 1$. Then we can straightly check $\omega(C) = (n, n-1, 1, \cdots, 1) \cdot (n-1, n-2, 1, \cdots, 1) = n(n-1) - (n-1)(n-2) - (n-1) = n > 0$.

Then note for an arbitrary a, we look at the tuple where $b_1 = a-1, b_2 = \cdots = b_n = \frac{a-1}{n-2}$. Clearly, this tuple satisfies the Cauchy-Schwartz maximal condition. Note that in this case, $\sum (b_i - 1) b_i \geq (b_1 - 1) b_1 = (a-1)(a-2)$, which means the value $\omega(C) = (n, n-1, 1, \cdots, 1) \cdot (a, a-1, \frac{a-1}{n-2}, \cdots, \frac{a-1}{n-2})$ is less than the min value in the allowable domain defined by (15). And this value is $na - (n-2)(a-1) - (n-1)\frac{a-1}{n-2} = 2a + n - 2 - (n-1)\frac{a-1}{n-2} > 0$. This means the minimal solution for $\omega(C)$ is positive. We have completed the proof that when $c_1 = \frac{n-2}{n}$, **this min is positive and is sufficient for our purpose.**

The general conclusion for any $1 \geq c_1 \geq \frac{n-2}{n}$ is similar. We'll still use the testing case with a tuple being $(a-1, \frac{1-c_1}{2c_1}(a-1), \cdots, \frac{1-c_1}{2c_1}(a-1))$. And using the same argument, we can check that the area of this curve is less than the minimal value, but still positive.

$(1, c_1, \frac{1-c_1}{2}, \cdots, \frac{1-c_1}{2}) \cdot (a, a-1, \frac{1-c_1}{2c_1}(a-1), \cdots, \frac{1-c_1}{2c_1}(a-1)) = a - ac_1 - (n-1)\frac{(1-c_1)^2}{4c_1}(a-1) = (1-c_1)(a) + c_1 - (1-c_1)[(n-1)\frac{1-c_1}{4c_1}](a-1)$. Since $c_1 \in [\frac{n-2}{n}, 1]$, we know that $\frac{1-c_1}{4c_1}$ is less than $\frac{1}{n-1}$. Hence that verifies that claim.

□

Lemma 3.17. *Further consider the n-tuple on $\mathbb{C}P^2$ consists n ordered points $p_1, \cdots, p_n$ such that in $p_2, \cdots, p_n$. We call colliding pairs and collinear triples a non-generic condition. Now assume no 2 points in $p_2, \cdots, p_n$ collide, and no pair out of $p_2, \cdots, p_n$ is colinear with the first point (when any point collide with p_1 it count as p_1). Besides those, there is at most one non-generic condition in $p_1, \cdots, p_n$. Then all curves with non-positive coefficient on H has to be $E_1 - E_j$, and hence have positive area.*
As a corollary, the line bundle given by the divisor class which is Poincaré dual to the form in the previous Lemma is ample on the n-point blowup of $\mathbb{C}P^2$, if the n-points satisfy the above assumption.

Proof. Let's further assume the no 2 collide, and no 3 collinear if the triple contains the first point. Note that we always have an integrable blowup complex structure. Then by Castelnuovo's contraction theorem, we can always blow down the ordered exceptional curves $E_n, E_{n-1}, \cdots, E_1$to a $\mathbb{C}P^2$. And any irreducible curve in the blowup either gets contracted in this process (becomes a componentof an exceptional curve of first kind) or projects to an irreducible curve in $\mathbb{C}P^2$, which can only be positive multiple of H. Then all curves with non-positive coefficient on H has to be $E_1 - E_j$. And those all have positive area pairing with the form $(1, \frac{k-2}{k}, \frac{1}{k}, \cdots, \frac{1}{k})$.

By Nakai-Moshezon criteria, the divisor $kH - (k-2)E_1 - E_2 - \cdots - E_n$ gives rise to an ample line bundle.

In particular, it is easy to check that $(1, \frac{k-2}{k}, \frac{1}{k}, \cdots, \frac{1}{k}) \in P_K$.

□

Then we use inflation to prove that any type $\mathbb{A}$ form has connected Torelli $Symp$.

Note that by Lemma 3.1 the Kronhemier fibration now becomes

$$Symp_h(X, \omega_t) \to \mathrm{Diff}_0(X) \to \mathcal{S}_t,$$

and this allows us to study $\pi_0(Symp_h)$ using this framework.

4. Type $\mathbb{A}$ symplectic forms

The purpose of this section is to prove Theorem 4.3. This is a combination of inflation Lemma 3.7 3.11 3.13 and isotopy Lemma 4.1.

4.1. Connections with the space of ball packings.

Firstly let's fix some notation:

(M, ω) is a rational 4-manifold diffeomorphic to $n-m$ points blowup of $\mathbb{C}P^2$, $\vec{c}$ denote the vector with m entries $(c_n, c_n, \cdots, c_n)$ where c_n is smaller than any blowup size in M. Let $(\tilde{M}, \omega_{\vec{c}})$ be the blowup of M with weight $\vec{c}$, and it's clear that this it is diffeomorphic to n point blowup of $\mathbb{C}P^2$, with the form in class $[\omega_{\vec{c}}] = [1, c_1, \cdots, c_{n-m}, c_n, \cdots, c_n]$.

Applying the argument in [25] Theorem 1.6 and Remark 5.8 in [7], we conclude that there's the following fibration

$$Symp(\tilde{M}, \omega_{\vec{c}}) \to Symp(M, \omega) \to Emb_\omega(B^4(\vec{c}), M),$$

where $Emb_\omega(B^4(\vec{c}), M)$ is the space of (ordered) embeddings from the m-disjoint standard balls of size c_n to M.

Also, we follow the idea of Theorem 1.6 in [25]: if we shrink c_n to $t < c_n$ in the pre-image, there is a **restriction map** $Emb_\omega(B^4(\vec{c}), M) \xrightarrow{\phi} Emb_\omega(B^4(\vec{t}), M)$, and if we shrink the size to 0, this map becomes the forgetful map $Emb_\omega(B^4(\vec{c}), M) \to Conf_m^{ord}(M)$, sending balls to their centers.

In particular, we'll consider the following commutative diagram (which is a multi-ball version as in Theorem 1.6 of [25]):

$$
\begin{tikzcd}
Symp(\tilde{M}, \omega_{\vec{c}}) \arrow[r] \arrow[d] & Symp(M,\omega) \arrow[r] \arrow[d, "="'] & Emb_{\omega}(B^4(\vec{c}), M) \arrow[d, "\phi"'] \\
Symp(\tilde{M}, \omega_{\vec{t}}) \arrow[r] & Symp(M,\omega) \arrow[r] & Emb_{\omega}(B^4(\vec{t}), M),
\end{tikzcd}
\tag{17}
$$

where ϕ is the restriction map.

Note that here $Symp(\tilde{M}, \omega_{\vec{c}})$ is suppose to fix all the blowup exceptional divisors and an infinitesimal neighborhood of them. Since our major concern is π_0, we are going abuse notation and regard it as the symplectomorphism group which acting trivially on the homology of $E_{n-m+1}, \cdots, E_n$. In other words, $Symp(\tilde{M}, \omega_{\vec{c}})$ has the same homological action as $Symp(M, \omega)$.

Also,in the proof of Theorem 0 in [28], we indeed showed that

Lemma 4.1. *Take balls $\coprod_i B(c_n), 1 \leq i \leq m$ embedded into M by map ι and we shrink the ball pre-images as in the restriction map above. For any loop of balls $\iota f_{c_n}(t)$(we'll simply denote $\iota_f(t)$) below in $Emb_{\omega}(B^4(\vec{c}), M)$, there is always a positive δ so that the loop of ball $f_{\epsilon}(t)$ is homotopic to identity in $Emb_{\omega}(B^4(\vec{\delta}), M)$.*

Proof. The connectedness is done by [33]. Now we prove the simply connectedness for small enough sizes. This means for a loop of balls $\iota_f(t)$ we want to prove that they are homotopic to identity. The idea is to show that loop of ball has the same π_1 as the loop of its center, which is trivial since the ambient manifold is simply connected. **The following proof is a recap from [28].**

Let x_i be image of the center of $B(c_i)$ under $\iota_f(0)$. $\iota_f(t)(x_i)$ are m disjoint smooth loops which can be isotoped through a family of Hamiltonian diffeomorphism $\gamma(s,t)$ to constant loops at x_i simultaneously:

$$
\begin{cases}
\gamma(0,t) = \gamma(s,0) = id, \\
\gamma(1,t)(x_i) = \iota_f(t)(x_i), \\
\gamma(s,1)(x_i) = x_i
\end{cases}
\tag{18}
$$

For convenience, we may also require $\gamma(s,t)(x_i) \cap \gamma(s,t)(x_j) = \emptyset$ if $i \neq j$.

Take an arbitrary metric g on M and assume $\frac{1}{K} < |\gamma(s,t)(x_i)|_{C^2} < K$ under g. One may choose a sequence of small $\lambda_i > 0$, so that there is a disk centered at x_i of radius r_i (measured by g), denoted as $D_g(x_i, r_i) \subset \mathbb{C}P^2$, which satisfies

$$\iota_f(0)(B(\lambda_i)) \subset D_g(x_i, r_i) \subset \iota_f(0)(B(c_i)).$$

Choose $\delta_i < \lambda_i$ such that $diam_g(\iota_f(t)(B(\delta_i))) < \frac{r_i}{K}$ for all t. Then we have $\gamma(1,t)^{-1}(\iota_f(t)(B(\delta_i))) \subset D(x_i, r_i) \subset \iota_f(0)(B(c_i))$.

Note that $\gamma(s,t)^{-1}(\iota_f(t)(B(\delta_i)))$ does *not* yield a homotopy of loops of embeddings, because we may not require $\gamma(s,1)^{-1}(\iota_f(t)(B(\delta_i)))$ to be independent of s. However, notice that

$$
\gamma(s,1)^{-1}(\iota_f(1)(B(\delta_i))) = \gamma(s,1)^{-1}(\iota_f(0)(B(\delta_i))) \subset \gamma(s,1)^{-1}(D_g(x_i, \frac{r_i}{K})) \subset D_g(x_i, r_i) \subset \iota_f(0)(B(c_i)).
\tag{19}
$$

Therefore, we have a loop of symplectic packing (of $B(\delta_i)$, parametrized by s)

$$\beta(s) := \gamma(s,1)^{-1}(\iota_f(1)(B(\delta_i))) \subset \iota_f(0)(B(c_i)).$$

Now recall Lemma A.2 of [28]:

Lemma 4.2. *The space $Emb(B(c), \delta)$, consisting of symplectic embedded images of an open ball $B(\delta)$ into $B(c)$, is weakly contractible, if $\delta \ll c$. In particular, the loop $\gamma(s,1)^{-1}(\iota_f(t)(B(c_i)))$ inside each $\iota_f(1)(B(\delta_i)))$ can be homotopic to identity.*

We take the concanation of the loop $\gamma(s,1)^{-1}(\iota_f(t)(B(c_i)))$ with the loop in Lemma 4.2, an d this gives an isotopy of a given loop of small balls to id.

□

Lemma 3.2 allows us to simultaneously blow down the minimal exceptional classes E_i and to compare the $Symp$ between the blowup and blow down, in the limit of the inflation process.

4.2. Type $\mathbb{A}$ form and Hamiltonian toric/circle actions.

Theorem 4.3. *Any K-nef type $\mathbb{A}$ form on a rational 4-manifold X has trivial Torelli SMCG.*

Proof. This is a combination of Lemma 3.7 3.11 3.13 and Lemma 4.1. For any ω_0 on X(regarded as $\tilde(M)$), we first use Lemma 3.10 to shrink c_1 no larger than $\frac{1}{2}$.

Then we do such inductive steps:
1) take the number of smallest exceptional spheres, and denote by m
2) apply Lemma 3.11 to reduce $\sum c_i \leq \frac{1}{m}$
3) apply Lemma 3.13 to shrink the smallest exceptional spheres to t very small, so that we can apply Lemma 4.1.

Notice that after all those steps, we are going toward a type $\mathbb{A}$ symplectic form on a rational 4-manifold with smaller Euler number (reduced by m), but we can never achieve that. However, we can apply the following argument:

In the LES of commutative diagram (17), we apply corollary 3.14,

(20)

$$\begin{array}{ccccccc}
\pi_1[Emb_\omega(B^4(\vec{c}), M)] & \longrightarrow & \pi_0 Symp(\tilde{M}, \omega_{\vec{c}}) & \longrightarrow & \pi_0 Symp(M,\omega) & \longrightarrow & 1 = \pi_0[Emb_\omega(B^4(\vec{c}), M)] \\
\phi_* \downarrow & & (= by 3.14) \downarrow & & = \downarrow & & \downarrow \\
\pi_1[Emb_\omega(B^4(\vec{t}), M)] & \longrightarrow & \pi_0 Symp(\tilde{M}, \omega_{\vec{t}}) & \longrightarrow & \pi_0 Symp(M,\omega) & \longrightarrow & 1 = \pi_0[Emb_\omega(B^4(\vec{t}), M)]
\end{array}$$

Here $Symp(\tilde{M}, \omega_{\vec{c}})$ is allowed to permute the homology classes in M, but is required to fix the exceptional classes by blowing up $\vec{c}$ since $Emb(B_i, M) is ordered.$

Note that at this stage we have no information yet on $\pi_0 Symp(\tilde{M}, \omega_{\vec{c}})$. To conclude it's the same as $Symp_h(M,\omega)$, we do the following argument:

Assume for contradiction there's a nontrival loop $f(t)$ in $\pi_1[Emb_\omega(B^4(\vec{c}), M)]$ which maps non-trivially into $\pi_0 Symp(\tilde{M}, \omega_{\vec{c}})$. Then by Lemma 4.1, there exist δ small so that $f_\delta(t)$ is the image of $f(t)$ under ϕ into $\pi_1[Emb_\omega(B^4(\vec{\delta}), M)]$, such that $f_\delta(t)$ is trivial. Then chase the commutative diagram and letting $t=\delta$ in (17), the image of $f(t)$ in $\pi_0 Symp(\tilde{M}, \omega_{\vec{\delta}})$ is non-trivial, a contridication against Corollary 3.14.

Now we proved that $\pi_0 Symp(\tilde{M}, \omega_{\vec{c}})$ is the same as the blow-down $Symp(M,\omega)$. Notice that (M,ω) is again a rational 4-manifold with a type $\mathbb{A}$ form.

Then we repeat the above 3 step, until we obtain a less than 5 point blowup.

□

Remark 4.4. *Even if positive coefficient H class changes, they are covered by other -1 curves. Compare to Anjos [6].*

Remark 4.5. *Type $\mathbb{D}$ doesn't work, because blowing down means go to point A. Or change -2 curve $H - E_1 - E_2 - E_3$.*

Lemma 4.6. *Any reduced symplectic form that's not in P_K^k does not admit any torus action. And they even does not support any Hamiltonian circle action.*

Proposition 4.7. *A 4-dimensional toric symplectic manifold has connected Torelli symplectomorphism group and hence has finite symplectic mapping class group.*

Proof. Toric form has to be of type $\mathbb{A}$, since a blowdown of a toric form is also toric. Then we must have the first 4-point blowup with E_1, E_2, E_3, E_4 being of type $\mathbb{A}$. Then the proposition is covered by Theorem 4.3. $\square$

Remark 4.8. *Note that Lemma 4.7 does not hold for circle actions. For example, on 5-point blowup of $\mathbb{C}P^2$, the form $(1|\frac{2}{3}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$ admits a circle action, but its Torelli SMCG is infinite. This is noticed by [28] and [5].*

Corollary 4.9. *Let (M, ω) be a toric 4-manifold. Then the set of Hamiltonian conjugacy classes of maximal 2-tori in $Ham(M, \omega)$ is finite.*

Proof. This is directly corollary of Proposition 4.7 and Theorem 1.3 in [39]. $\square$

5. Toric packing forms and a filling divisor

We then focus on the divisor configuration and related fibrations to deal with the type $\mathbb{D}$ forms. First let's recall the sequence below which appeared in [29] and [28].

$$\begin{tikzcd} Symp_c(U) = Stab^1(C) \arrow[r] & Stab^0(C) \arrow[r] \arrow[d] & Stab(C) \arrow[r] \arrow[d] & Symp_h(X,\omega) \arrow[d] \\ & \mathcal{G}(C) & Symp(C) & \mathcal{C}_0 \simeq \mathcal{J}_C \end{tikzcd} \tag{21}$$

We choose the configuration C for $X = \mathbb{C}P^2 \# n\overline{\mathbb{C}P^2}$ to be the following diagram of homology classes, and $\mathcal{J}_C$ is the subspace of $\mathcal{J}_\omega$ such that each homology class has an embedded J-holomorphic representative.

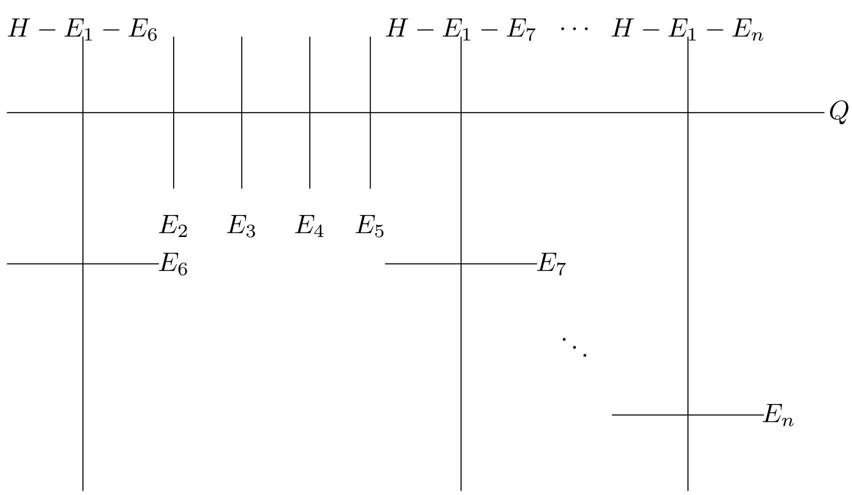


Figure 1. A filling divisor in X_n

To establish the fibration (21), we need the following lemma similar to [29].

Lemma 5.1. *$Stab(C) \to Symp(C)$ is surjective, and hence diagram 21 is a fibration.*

Proof. We adapt the proof of Proposition 5.4 in [28].

Denote the spheres in class $H-E_1-E_i$ as S_i, the ones in class E_i as e_i, and the one in class $2H-E_1-\cdots-E_5$ as Q. Let $p_i = S_i \cap Q$ or $e_i \cap Q$ for $i \geq 6$ and $2 \leq i \leq 5$, respectively, and $q_i = S_i \cap e_i$ for $i \geq 6$. Recall from [17] and [29], $Symp(C) = \Pi_{i=1}^n Symp(S_i; p_i, q_i) \times Symp(Q, n-1) \times Symp(e_i, q_i)$, where $Symp(S_i; p_i, q_i)$ is the symplectomorphism group of the sphere in class $H - E_1 - E_i$ fixing the intersection point p_i, q_i, and $Symp(Q, n-1)$ is the symplectomorphism group of Q fixing $(n-1)$ intersection points. Since f is a group homomorphism, we only need to show the projection to each factor is surjective. This is clear for $Symp(e_i, q_i)$ and $Symp(S_i; p_i, q_i)$ factors.

The only thing we need to prove is the restriction map of f is surjective on the factor $Symp(Q, n-1)$. Note this means for any given $h^{(2)} \in Symp(Q, n-1)$ we need to find a symplectomorphism $h^{(4)} \in Stab(C)$ which fixes the whole configuration C as a set, whose restriction on Q is $h^{(2)}$. To achieve this, we can blow down the exceptional spheres $S_1 \cdots S_n$, and obtain a $\mathbb{C}P^2 \setminus \coprod_{i=1}^n B(i)$ with a conic S^2 in homology class $2H$ and five disjoint balls $\coprod_{i=1}^{n-1} B(i)$ each centered on this conic and the intersections are 5 disjoint disks on this S^2, along with five symplectic rational curves D_i which are proper transform of E_i, $i \geq 6$.

Note that by the identification in Lemma 5.3 in [28], this blow down process sends $h^{(2)}$ in $Symp(Q, n-1)$ to a unique $\overline{h^{(2)}}$ in $Symp(S^2, \coprod_{i=1}^{n-1} D_i)$. It suffice to find a symplectomorphism $\overline{h^{(4)}}$ whose restriction is $\overline{h^{(2)}}$, and fixing the image of balls $\coprod_{i=1}^{n-1} B(i)$ and D_i. Because blowing the balls $\coprod_{i=1}^{n-1} B(i)$ up and by Definition 5.1 in [28], we obtain a symplectomorphism $h^{(4)} \in Stab(C)$ whose restriction is the given $h^{(2)} \in Symp(Q, n-1)$.

Now for a given $h^{(2)} \in Symp(Q, n-1)$, we will first consider its counterpart $\overline{h^{(2)}}$ in $Symp(S^2, \coprod_{i=1}^{n-1} D_i)$. One can always find $f^{(4)} \in Symp(\mathbb{C}P^2, \omega)$ whose restriction on S^2 is $\overline{h^{(2)}}$ in $Symp(S^2, \coprod_{i=1}^{n-1} D_i)$. We can construct $f^{(4)}$ using the method as in Lemma 2.5 in [29]: $\overline{h^{(2)}}$ in $Symp(S^2, \coprod_{i=1}^{n-1} D_i)$ is a Hamiltonian diffeomorphism on S^2 because S^2 is simply connected, therefore, the Hamiltonian function can be extended to a neighborhood so that the induced Hamiltonian diffeomorphism $f^{(4)} \in Symp(\mathbb{C}P^2 \# n\overline{\mathbb{C}P^2}, \omega)$ equals $\overline{h}^{(2)}$ when restricted to Q. $f^{(4)}$ clearly fixes the $(n-1)$ intersection disks $\coprod_{i=1}^{n-1} D_i$.

Then we need another symplectomorphism $g^{(4)} \in Symp(\mathbb{C}P^2, \omega)$ so that $g^{(4)}$ move the the five symplectic balls back to their original position in $\mathbb{C}P^2$. This can be done by connectedness of ball packing relative a divisor (the conic in class $2H$ in our case). Namely, by Lemma 4.3 and Lemma 4.4 in [45], there exists a symplectomorphism $g^{(4)} \in Symp(\mathbb{C}P^2, \omega)$ such that the composition $\overline{F^{(4)}} = g^{(4)} \circ f^{(4)}$ is a symplectomorphism fixing the five balls. After this, the isotopy from $\overline{F^{(4)}}(D_i)$ back to D_i can be achieved by the usual J-holomorphic technique: pick an ω-compatible almost complex structure J_0 so that $\overline{F^{(4)}}(D_i)$ is J_0-holomorphic, and there is a path of ω-compatible $J_t, t \in [0,1]$ so that D_i is J_1-holomorphic, and for all J_t, the J_t-holomorphic curves that passes through $D_i \cap Q$ are embedded. This can be achieved by a virtual dimension count that bubbling has always codimension 2 in our semi-positive situation. The resulting isotopy of rational curves can therefore be extended to a Hamiltonian isotopy and hence be composted with $\overline{F^{(4)}}(D_i)$.

The end result of the above discussion is a symplectomorphism $\overline{h^{(4)}}$ of $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$, which fixes the $(n-1)$ balls and D_i. Upon blowing up the these balls we obtain an element $h^{(4)}$ in $Stab(C)$, which is a ball swapping symplectomorphism whose restriction on $Symp(C)$ creates the group $Symp(Q, n-1)$. Hence this restriction map $Stab(C) \to Symp(C)$ is surjective.

It is clear that the action of $Stab(C)$ on $Symp(C)$ is transitive and by Theorem A in [**Pai60**] $Stab(C) \to Symp(C)$ is a fibration. The rest parts of the diagram being a fibration is the same as the arguments in [29].

□

5.1. **An interesting corner of the symplectic cone.** Next we show that there is a corner with $A \in P_k$ as a vertex, so that the symplectic form is reduced, K-nef, and the complement (U, ω) induced form X_n is Stein when the form class is in $H^2(X_n, \mathbb{Q})$.

Further, we'll show that this region also support a Hirzbruch toric model in the next section.

Let $[\omega] = [1|c_1 \cdots c_n], n \geq 6$. we will focus on the following corner of RED as in proposition 2.2:

$$c_1 > \frac{n-6}{n-3}. \tag{22}$$

Lemma 5.2. *The region defined by condition (22) and the reduced condition is a subset of the normalized cone for any n. Further, any reduced form ω satisfies (22) automatically satisfies (5), meaning that $[\omega] \in P_K$.*

Proof. Recall the normalized cone from Proposition 2.2. When $n = 6, 7, 8, 9$ this is obvious because the normalized cone is the polyhedron P^n.

For $n > 9$ we need to check some quadratic condition. Now choose form $\omega = (1|c_1, \cdots, c_n)$ such that $c_k = (1-c_1)/2, \forall k \geq 2$. And we need to check equation 5 and 6:

- Equation (5) now is $c_1 + (n-1)\frac{1-c_1}{2} < 3$, which is to say $c_1 > \frac{n-7}{n-3}$. And hence (22) implies (5).
- Equation (6) is to say
$$c_1^2 + (n-1)(\frac{1-c_1}{2})^2 < 1.$$
This is always true if $c_k = (1-c_1)/2, \forall k \geq 2$. For a reduced form, we have $c_i \leq (1-c_1)/2, i > 1$. Hence '(22) implies (6).

□

Lemma 5.3. *Let X_n be $\mathbb{C}P^2 \# n\overline{\mathbb{C}P^2}, n > 5$, endowed with a reduced $\omega = (1, c_1, \cdots, c_n)$ with rational period (i.e. all $c_i \in \mathbb{Q}$), and assume the quantitative conditions*

Then $(U = X_n - C, \omega)$ is Stein, where C is the configuration as in 21 and ω is the induced form from (X_n, ω). In particular, U is symplectomorphic to $\mathbb{P}^1 - \{p_1, \cdots p_{n-5}\} \times D^2$ as a Stein domain, where $\mathbb{P}^1$ and D^2 is endowed with a symplectic form with appropriate areas.

Proof. Consider the Stein domain given by the complement of the third Hirzebruch surface $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(3))$ with the zero section (of self-intersection)

From Proposition 3.3 in [29], we only need to check the following to show that U is Stein: Up to rescaling by a positive integer l, we can write $PD([l\omega]) = aH - b_1E_1 - b_2E_2 - b_3E_3 - \cdots - b_nE_n$ with $a, b_i \in \mathbb{Z}^{\geq 0}$. Further, we assume $b_1 \geq \cdots \geq b_n$. Then we can represent $PD([l\omega])$ as a positive integral combination of all elements in the set $\{2H - E_1 - E_2 - E_3 - E_4 - E_5, H - E_1 - E_6, E_2, E_3, E_4, E_5, E_6; H - E_1 - E_j, E_j, \}$, where $7 \leq i \leq n$ which is the homology type of C.

This is a direct computation:

$$\begin{aligned} PD([l\omega]) = aH - b_1E_1 - b_2E_2 - b_3E_3 - \cdots - b_nE_n \\ = d_0(2H - E_1 - E_2 - E_3 - E_4 - E_5) \\ +d_1(H - E_1 - E_6) \\ +d_2E_2 + \cdots d_6E_6 \\ +\sum_{i=7}^n d_iE_i + \sum_{i=7}^n f_i(H - E_1 - E_i) \end{aligned} \tag{23}$$

Compare the coefficient we have $2d_0 + d_1 + \sum_{i=7}^n f_i = a$ and $d_0 + d_1 + \sum_{i=7}^n f_i = b_1$. Hence we have $d_0 = a - b_1$ and $d_1 = a - 2d_0 = 2b_1 - a - \sum_{i=7}^n f_i$. Meanwhile to make sure the above equation have a solution, we also need $d_1 - d_i = b_i > 0$, which means $2b_1 - a - \sum_{i=7}^n f_i - a > b_n$. This is equivalent to $(k-1)c_1 - c_{2k} - c_{2k+1} > (k-2)$, when $n = 2k+1$, $(k-1)c_1 - c_{2k} > (k-2)$, when $n = 2k$.

For the last statement, note that

Lemma 5.4. *With the numerical condition given in (22), the above complement U admit finite type Stein structure. And hence $Symp_c(U)$ is weakly homotopic to $Symp_c((\mathbb{C}-\{p_1,p_2,\cdots,p_{n-6}\})\times\mathbb{C})$.*

Proof. The proof is the same as Proposition 3.3 in [29], checked case by case below.

□

□

Lemma 5.5. *$U=(\mathbb{C}-\{p_1,p_2,\cdots,p_{n-6}\})\times\mathbb{C}$, $Symp_c(U,\omega_{std})$ is weakly contractible. In particular, it is connected.*

Proof. Denote $U_m=(\mathbb{C}-\{p_1,p_2,\cdots,p_m\})\times\mathbb{C}$ where $m=n-6$.

There is a partial compactification of U_m into $M_c=S^2\times\mathbb{C}$ with homology class $F\in H_2(M_c,\mathbb{Z})$ by adding the m disks as fibers $F_1,\cdots F_m$ and an infinity section D.

In the proof of Gromov's theorem (cf. [37] proof of Theorem 9.5.1), $Symp_c(M_c;D)$ (fixing D point-wise), which can be identified with $Symp(S^2\times S^2;\{B,F\})$ (fixing one fiber and the section at infinity point-wise), is indeed $Symp_c(\mathbb{C}^2)$ and is shown to be contractible. And we consider the space of configuration C of pairwise disjoint symplectic spheres $C=\{C_1,\cdots C_m\}$ in homology class F, such that $C_i\cap D=F_i\cap D$. The space $\mathcal{C}$ of $C=\{C_1,\cdots C_m\}$ is contractible. This is because the space of each C_i passing through $F_i\cap D$ is contractible, and the point set $\{p_1,p_2,\cdots,p_m\}$ is fixed because it is a subset of D which is fixed point-wisely.

$Symp_c(M_c;D)$ acts transitively on $\mathcal{C}$. And we have the following fibrations:

$$\begin{array}{ccccc} Symp_c(U_m)=Stab^0(C) & \longrightarrow & Stab(C) & \longrightarrow & Symp_c(S^2\times\mathbb{C}) \\ & & \downarrow & & \downarrow \\ & & \Pi_{i=1}^m Symp(C_i) & & \mathcal{C} \end{array} \tag{24}$$

Then it follows from LES

$$0=\pi_{n+1}(\mathcal{C})\to\pi_n(Symp_c(U_m))\to\pi_n(Symp_c(\mathbb{C}^2))\to\pi_n(\mathcal{C})=0$$

that the $Symp_c(U_m)$ is weakly contractible.

□

Also we have

Lemma 5.6. *If condition (22) holds, the complement U has a connected compactly supported symplectomorphism group.*

Proof. Complement U is Stein by Lemma 5.2. The result follows from Lemma 5.5 and Proposition 2.1 in [17]. □

Hence for a reduced form ω with condition (22), Diagram 21 is a fibration.

$$\begin{array}{ccccccc} Symp_c(U_{n-6}) & \longrightarrow & Stab^0(C) & \longrightarrow & Stab(C) & \longrightarrow & Symp_h(X,\omega) \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \mathbb{Z}^{2n-7} & & (S^1)^{2n-6}\times \mathrm{Diff}^+(S^2,n-1) & & \mathcal{C}_0\simeq\mathcal{J}_C \end{array} \tag{25}$$

Note: $\mathcal{J}_C$ is not contractible but still connected.

And we also have connecting map from $\pi_1(S^1)^{2n-6}$ to $\mathbb{Z}^{2n-7}$ is always surjective, by Lemma 2.9 in [29]. $Symp_c(U) \simeq \star$ when condition (22) holds. Hence we know $Stab(C) \simeq S^1 \times \mathrm{Diff}^+(S^2, n-1)$. And we have the right end of LES of the fibration being

$$\cdots \to \pi_1(\mathcal{C}_0) \to \pi_0 \mathrm{Diff}^+(S^2, n-1) \to \pi_0(Symp(X, \omega)) \to 1. \tag{26}$$

5.2. **Toric packing forms.** Now we prove the following Lemma:

Lemma 5.7. *After a series of base change, the existence of the toric packing in a Hirzeburch surface, such that the vertical curves in Figure 1 is a ball when blown down.*

Proof. We explicit give the base change here:

For $n = 2k+2$,

$$\begin{aligned} B = (k-1)H - (k-2)E_1 - \sum_j E_j, 6 \leq j \leq n = 2k+2,\\ F = H - E_1,\\ E_i' = E_{i+1}, 1 \leq i \leq 4\\ E_i' = H - E_{i+1} - E_1, \forall 5 \leq i \leq n, \end{aligned} \tag{27}$$

For $n = 2k+3$,

$$\begin{aligned} h = (k)H - (k-1)E_1 - \sum_j E_j, 6 \leq j \leq n = 2k+3,\\ E_i' = E_{i+1}, 1 \leq i \leq 4\\ E_i' = H - E_{i+1} - E_1, \forall 5 \leq i \leq n, \end{aligned} \tag{28}$$

Notice that the existence of the toric model is equivalent to the negative section of the Hirzeburch surface having positive area.

Under this base change, the negative section is $-H + 2E_1 - \sum_{i=6}^n E_i$ in the original H, E_i basis. It has positive symplectic area if and only if we can find such a packing of balls in the Hirzebruch model. This means $2E_1 - H - \sum_{i=6}^n E_i > 0$.

Notice that if we assume (22), i.e. $\omega(E_1)/\omega(H) > \frac{n-6}{n-3}$, (or larger $\frac{n-4}{n-2}$ for example,) then there's always such a toric model.

□

Notice that condition (22) also guarantees that the complement is convex as a symplectic domain.

Here the region that the above theorem deals with is the condition (22).

We can more explicitly write down the toric ball swapping model for the $2k$ and $2k+1$ points blow up correspondingly, and note that Lemma 5.8 and 5.9 are special cases of Lemma 5.7:

Lemma 5.8. *Let $X_{2k} = \mathbb{C}P^2 \# 2k\overline{\mathbb{C}P^2}$, with a reduced form ω such that (22) holds, Then there is a toric ball packing model with $2k-1$ balls packed on the curve $B + (k-1)F$ in the Hirzebruch surface $H(2k-2)$, and the ball swapping of two balls with different size is isotopic to identity.*

Proof. We will we will blow down $2k-1$ spheres on $Q \subset C$ such that it is a sphere in class $B+(k-1)F$ on $S^2 \times S^2$.

Do $n-5=2k-5$ compositions of base changes:

The first pair being

$$\begin{aligned} B &= H - E_{2k}, \\ F &= H - E_{2k-1}, \\ E'_{2k-1} &= H - E_{2k} - E_{2k-1}, \\ E'_i &= E_i, \forall 1 \le i \le 2k-2, \end{aligned} \tag{29}$$

and

$$\begin{aligned} h &= B + F - E'_1, \\ e_1 &= B - E'_1 = H - E_1 - E_{2k}, \\ E_2 &= F - E'_1 = H - E_1 - E_{2k-1}, \\ e_3 &= H - E_{2k-1} - E_{2k}, \\ E_j &= E'_{j-2}, \forall 3 \le j \le 2k. \end{aligned} \tag{30}$$

And the $i-th$ pair being

$$\begin{aligned} B &= H - E_{2k}, \\ F = H - E_1 - E2k - 2i + 2 &- \cdots - E_{2k}, \\ E'_i &= H - E_{2k} - E_{2k-i}, \\ E'_{2k-j} &= E_{2k-j-3}, \forall 1 \le j \le i, \end{aligned} \tag{31}$$

and

$$\begin{aligned} h &= B + F - E'_1, \\ e_1 &= B - E'_1 = H - E_1 - E_{2k}, \\ E_2 &= F - E'_1 = H - E_1 - E_{2k-1}, \\ e_3 &= H - E_{2k-1} - E_{2k}, \\ &\ddots \\ E_j &= E'_{j-2}, \forall 3 \le j \le 2k. \end{aligned} \tag{32}$$

And so on

$\ddots$

Finally

$$
\begin{aligned}
b &= H - E_1,\\
f = h - e_{2k} &= kH - (k-1)E_1,\\
e'_1 &= H - E_1 - E_6,\\
e'_2 &= H - E_1 - E_7,\\
&\ddots\\
e'_{2k-5} &= H - E_1 - E_{2k},\\
e'_{2k-j+1} &= E_j, \forall 2 \le j \le 5,
\end{aligned} \tag{33}
$$

Then existence of $H(n-2)$ is equivalent to the curve $B - (k-1)F$ having positive symplectic area. And it is exactly the condition (22) for the even n case.

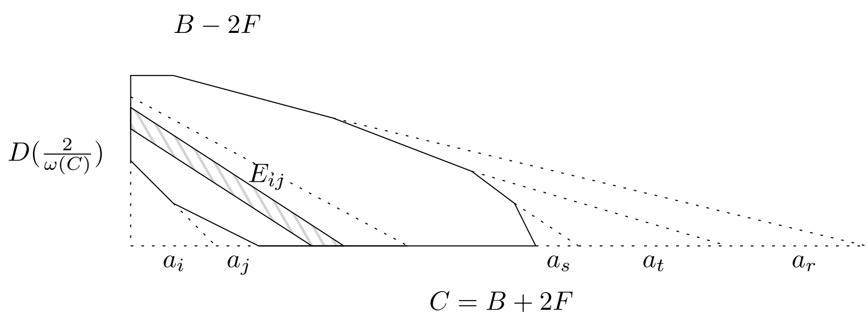


FIGURE 2. Standard toric packing and ball swapping in $H(4)$

□

Lemma 5.9. *Let $X_{2k+1} = \mathbb{C}P^2 \# (2k+1)\overline{\mathbb{C}P^2}$, with a reduced form ω such that (22) holds. Then there is a toric ball packing model with $2k$ balls packed on the curve $kh - (k-1)e_1$ in the Hirzebruch surface $H(2k-1)$, and the ball swapping of two balls with different size is isotopic to identity.*

Proof. First do base change

$$
\begin{aligned}
B &= H - E_{2k+1},\\
F &= H - E_{2k},\\
E'_{2k} &= H - E_{2k} - E_{2k-1},\\
E'_i &= E_i, \forall 1 \le i \le 2k-1,
\end{aligned} \tag{34}
$$

And then do another base change:

$$
\begin{aligned}
h &= B + F - E'_1,\\
e_1 &= B - E'_1,\\
E_2 &= F - E'_1,\\
E_j &= E'_{j-1}, \forall 3 \le j \le 2k+1.
\end{aligned} \tag{35}
$$

Then do another $n-7=2k-6$ compositions of base changes, which is basically the first $2k-6$ base changes we did in Lemma 5.8.

Then existence of $H(n-2)$ is equivalent to the curve $ke_1-(k-1)h$ having positive symplectic area. And it is exactly the condition (22) for the odd n case.

Picture? □

Theorem 5.10. *For any rational surface $\mathbb{C}P^2\#k\overline{\mathbb{C}P^2}, k>5$:*

There's an open region of the reduced cone where ω is of type $\mathbb{A}$, s.t. $1\to 1\to \pi_0(Symp(X,\omega))\to W(\Gamma_L)\to 1$.

There's a family of symplectic form of type $\mathbb{D}$, s.t. $1\to \pi_0(Diff^+(S^2,n))\to \pi_0(Symp(X,\omega))\to W(\Gamma_L)\to 1, 3<n\le \chi(X)-3$.

Proof will be given in each cases.

Lemma 5.11. *For n-point blowup, $n>5$, when $\omega=(1|c_1,\cdots,c_n)$ is a reduced form. If it satisfies*

- *1)condition (22)*
- *2) there are at least $n-3$ distinct values in $\{c_2,\cdots,c_n\}$*

then $TSMC$ is trivial.

Proof. Let $G=\{1\}$ the trivial group, denote TSMC by H. One only need to show

$G\twoheadrightarrow H$:
From LES (26), TSMC is a quotient of $P_{n-1}(S^2)/\mathbb{Z}_2$ by $Im\phi$.

Use the corresponding toric ball-swapping model 5.8 or 5.9. If there are at least $n-3$ distinct values $\{c_2,\cdots,c_n\}$, then by Lemma 5.2 in [28] there is a generating set in the $Im\phi$. Hence we have $G\twoheadrightarrow H$.

And this means H is trivial.

□

6. Type $\mathbb{D}$ forms

Notice that because of Lemma 3.13, we only need to deal with the full type D cases, namely, type $\mathbb{D}_{n-1}$ forms on X_n. We'll prove Theorem 6.1 via moduli of polarized Kähler structures.

Theorem 6.1. *For any K-nef type $\mathbb{D}_k$ symplectic form $\omega_{\mathbb{D}_k}$ on $X_n, n>k$, $\pi_0(Symp(X_n,\omega_{\mathbb{D}_k}))=PB_k(S^2)$.*

6.1. Construction of the moduli space.

We first define a partial compactification of a subset in Conf$(\mathbb{C}P^2,n)$, the configuration space of n *ordered* points $p_1,\cdots,p_n$ on $\mathbb{C}P^2$. Let $\mathcal{U}$ be the subset of $(\mathbb{C}P^2)^n$ where

- p_1 is not collinear with any two of p_i, $i\ge 2$,
- there are no collinear relations of more than 3 points,
- no more than 6 points lie on the same conic,
- $p_i\ne p_j$ for $i,j\ge 2$.

We emphasize that there are more discriminant locus of codimension 2 than those listed above, but those will not concern us and won't be removed.

Finally, we blow up the locus $p_1=p_i$ for $i\ge 2$, and denote the resulting space as $\mathcal{Q}_n$, see discussion before Definiton 6.18 for more details.

Definition 6.2. *We define $\mathcal{Q}_n$, the space of admissible ordered n-tuple $p_1, \cdots, p_n$ in $\mathbb{C}P^2$ to be the ordered n points where each of $p_2, \cdots, p_n$ is allowed to collide with p_1, and any 3 of $p_2, \cdots, p_n$ is allowed to be collinear, and there's no conic passing through $p_1, \cdots, p_5$ and a sixth point. Besides these non-generic relations, any two points do not collide and any 3 points are not collinear.*

We also call colliding pairs and collinear triples a non-generic condition. And beside the above assumpsions, at most one non-generic condition is allowed in $p_1, \cdots, p_n$.

We consider the moduli space of almost Kähler structures which compatible with the given ω. The construction is given by [19] Corollary 3.2 (for Sobolev H^k completion), where $\mathcal{M}^k_{a,\omega}$ is given by the quotient $\mathcal{C}^k_{a,\omega}/Symp^{k+1}(M,\omega)$. For the moduli space of smooth almost complex structures, one can take the inverse limit on k, and endow the inverse limit topology of the moduli space $\mathcal{M}_{a,\omega} = \mathcal{J}_\omega/Symp$. We can restrict the group action to make a moduli space $\mathcal{J}_\omega/Symp_h$ up to the action of $Symp_h$.

Further, since the action of $Symp$ or $Symp_h$ on $\mathcal{J}_\omega$ fixes each strata, we can consider the subspace $(\mathcal{J}_\omega - \mathcal{J}_4)/Symp_h \subset \mathcal{J}_\omega/Symp_h$, by restricting the action to a certain strata $(\mathcal{J}_\omega - \mathcal{J}_4)$, where

Definition 6.3. $(\mathcal{J}_\omega - \mathcal{J}_4)$ *is defined as the subspace of $\mathcal{J}_\omega$ such that the following classes $E_1, E_2, \cdots, E_n, H - E_1 - E_6, \cdots, H - E_1 - E_n$, and $2H - E_1 - \cdots E_5$*
- *either each has an embedded representative,*
- *or at most one class has an pseudo-holomorphic representative with at most 2 irreducible components, and each components has negativity at least -2.*

Lemma 6.4. *By blowing up one element in $\mathcal{Q}_n$ (as in definition 6.18) in $\mathbb{C}P^2$, one obtain a Kähler rational surface whose complex structure belongs to $\mathcal{J}_\omega - \mathcal{J}_4$.*

Proof. The Kähler property comes from Lemma 3.17.

Now we only need to check that curves in class $E_1, E_2, \cdots, E_n, H - E_1 - E_6, \cdots, H - E_1 - E_n$, and $2H - E_1 - \cdots E_5$ are all embedded or at most bubble a (-2) negative sphrical component.

Note that $E_1, E_2, \cdots, E_n, H - E_1 - E_6, \cdots, H - E_1 - E_n$, are already covered by Lemma Lemma 3.17. We only need to take care of $2H - E_1 - \cdots E_5$. By definition 6.18, there's no conic passing through $p_1, \cdots, p_5$ and a sixth point. By Lemma 3.17 all the rational curves has non-negative H-coefficient. Hence they only possible pseudoholomorphic representitive for the class $2H - E_1 - \cdots E_5$ is either an embedded curve or a nodal curve with 2 embedded component being $H - E_1 - E_i$, $H - E_j - E_k - E_l$, where $\{i, j, k, l\} = \{2, 3, 4, 5\}$.

Also, note that in $\mathcal{Q}_n$, at most one non-generic condition is allowed. This means that there's at most one curve with genativity less than or equal to -2, among $E_i - \sum E_j, H - \sum E_k$. Hence the complex structure we obtained by blowing up $\mathcal{Q}_n$ is in $\mathcal{J}_\omega - \mathcal{J}_4$.

□

Then we highlight some important facts about the space $(\mathcal{J}_\omega - \mathcal{J}_4)$:

Lemma 6.5. *For a given form $\omega \in MA$ on a rational surface X with $\chi(X) > 8$, the action of $Symp_h$ on $\mathcal{J}_\omega - \mathcal{J}_4$ is free. And hence $\pi_i(Symp_h) = \pi_{i+1}(\mathcal{J}_\omega - \mathcal{J}_4)/Symp_h$ for $i = 0, 1$.*

Proof. By Lemma 3.8 in [28], there is an action of $Symp_h$ on $\mathcal{J}_\omega - \mathcal{J}_4$.

Consider the configuration of homology classes as in 3. We now show that if $Symp_h(X,\omega)$ fix some J in $\mathcal{J}_\omega - \mathcal{J}_4$, then it fix the tangent space of the intersecting point of two minimal area exceptional spheres, since E_5 and $H - E_1 - E_5$ are both of minimal area among exceptional spheres.

Then we consider the intersection point of embedded $J-$holomorphic spheres on the sphere in homology class E_5 or $H - E_1 - E_5$:

- $J \in \mathcal{J}_C$, there are 5 intersection points on each sphere in homology class E_5 or $H - E_1 - E_5$.
- $J \in \mathcal{J}_{\mathcal{C}}$, where $\mathcal{C}$ is the following:

element in $\mathcal{C}$	# of g.i.p. on E_5	# of g.i.p. on $H-E_1-E_5$
$H-E_2-E_3-E_5$	3	5
$H-E_2-E_4-E_5$	3	5
$H-E_3-E_4-E_5$	3	5
E_1-E_5	3	5
$H-E_2-E_3-E_4$	5	3
E_1-E_2	5	3
E_1-E_3	5	3
E_1-E_4	5	3

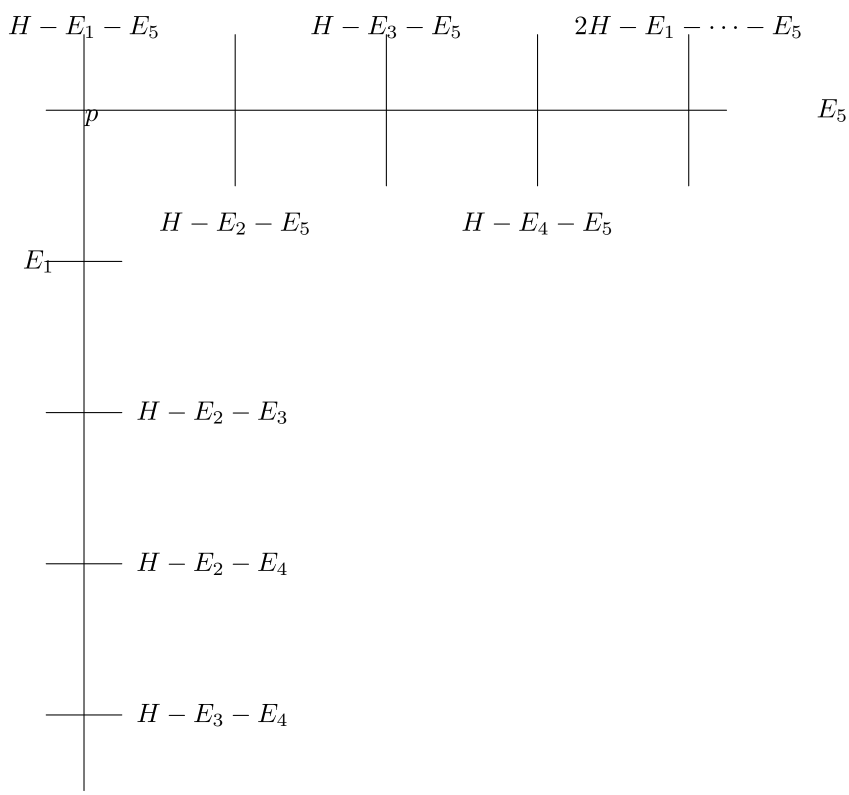


FIGURE 3. Configuration of two minimal area exceptional classes for $\omega \in MA$.

There are always more than 3 intersection points on the sphere in class E_5 and $H-E_1-E_5$. Hence if the action fix J, it is an isometry fixing both spheres. This means the action fixes the tangent space at p, and hence the identity map.

□

Lemma 6.6. *$\mathcal{J}_\omega$ is a contractible Fréchet manifold, which means its topology is induced from a complete metric, but the metric is not necessarily induced from a norm.*
$\mathcal{J}_4$ is a union of (possibly infinitely many) submanifolds of $\mathcal{J}_\omega$, where each of them has real codimension no

less than 4.
$(\mathcal{J}_\omega - \mathcal{J}_4)$ has trivial fundamental group.

Proof. The first statement is know to Gromov, by convexity. The second statement follows from the fact that the removed strata are degeneration of the $E_1, E_2, \cdots, E_n, H - E_1 - E_6, \cdots, H - E_1 - E_n$, and $2H - E_1 - \cdots E_5$ which either has more than one (-2) curve or has embedded sphere whose square is less than -2. The 3rd statement immediately follows from the second one. □

Then we have two nice properties on the group action $Symp_h$:

Lemma 6.7. *The action of $Symp$ on $(\mathcal{J}_\omega - \mathcal{J}_4)$ is free and proper.*

Proof. For the free action, we have Lemma 6.5 dealing with the general case.

By Theorem 3.3 in [19] (after taking inverse limit this holds for $\mathcal{C}_{a,\omega}/Symp^{k+1}(M,\omega)$), this action of $Symp$ on $\mathcal{J}_\omega$ is proper.

Note that our definition here and is the map $G \times X \to X \times X : (g, x) \to (gx, x)$ being a proper map, i.e., inverses of compact sets are compact.

Since the group $Symp/Symp_h$ is discrete, then we know the action of $Symp$ on $\mathcal{J}_\omega$ is also proper, and the restriction on $(\mathcal{J}_\omega - \mathcal{J}_4)$ is proper.

□

Lemma 6.8. *$\mathcal{J}_4$ is closed in $\mathcal{J}_\omega$. And hence $(\mathcal{J}_\omega - \mathcal{J}_4)$ is a Fréchet manifold.*

Proof. We know $\mathcal{J}_4$ is the union of many submanifolds of codimension no less than 4. Then we prove that $\mathcal{J}_4$ is closed in $\mathcal{J}_\omega$, i.e. the closure of each submanifold does not intersect $(\mathcal{J}_\omega - \mathcal{J}_4)$.

Firstly, note that we can use a finite set of negative sphere classes to label the submanifolds in $\mathcal{J}_4$. This finite subset consist those of negative coefficients on H and those coefficients are 0,1,2. Then if we have a sequence of $\mathcal{J}_4$ that converges to $(\mathcal{J}_\omega - \mathcal{J}_4)$, there must be a subsequence all located in the same submanifoldlabelled by a finite set of negative curves.

Firstly we prove that there cannot be the following case: $\mathcal{J}_{D_1,\cdots D_n}$ converges to $\mathcal{J}_{D_0}$ or $\mathcal{J}_{top}$. The reason is that if

For a sphere class B such that B is an irreducible component of the Gromov limit of some -1 sphere classes listed in definition 6.3 with $B^2 < -2$. First recall that for any such B, there exist some class E listed in definition 6.3 such that $E^2 = -1$ and $B \cdot E \le -1$.(We just consider the class E containing B as an irreducible component, either $B \cdot H \le 0$ or $B \cdot H > 0$ we have $B \cdot E \le -1$.) Assume for contradiction that some strata in $\mathcal{J}_4$ has a limit point in $(\mathcal{J}_\omega - \mathcal{J}_4)$. Then have $B = \sum_i A_i$, where each A_i has an embedded representative for some $J \in \mathcal{J}_\omega - \mathcal{J}_4$.

Then we consider the intersection pairing with E of the above equation $E \cdot B = E \cdot \sum_i A_i$. There are two possibilities: 1) $E = A_i$ for some i. Then we have $-1 \ge -1 + 1 + \sum_{j \ne i} E \cdot A_j$, contradiction since each $E \cdot A_j \ge 0$. 2) $E \ne A_i$. Then we have $-1 \ge \sum_i E \cdot A_i$, contradiction. This means that $\mathcal{J}_4$ is closed in $\mathcal{J}_\omega$.

Immediately following from the last statement, we have $(\mathcal{J}_\omega - \mathcal{J}_4)$ is a Fréchet manifold.

□

Then combine the above with the slice theorem (Theorem 5.6, cf. Corollary 5.3 in [19]). we have

Lemma 6.9. *The orbit space $(\mathcal{J}_\omega - \mathcal{J}_4)/Symp_h$ is Hausdorff and locally modelled by Fréchet spaces. The orbit projection of the free proper action $Symp_h$ on $(\mathcal{J}_\omega - \mathcal{J}_4)$ is a fibration with fiber $Symp_h$.*

Proof. The Hausdorff property follows from Proposition 6.1 in [19]. And the local Fréchet follows from Corollary 5.7 in [19], since the slice is a submanifold of the Fréchet manifold $\mathcal{J}_\omega$, and it's homeomorphic to the open set of $(\mathcal{J}_\omega - \mathcal{J}_4)/Symp_h$.

The orbit projection $\pi : (\mathcal{J}_\omega - \mathcal{J}_4) \to \mathcal{B} = (\mathcal{J}_\omega - \mathcal{J}_4)/Symp_h$ is clearly surjective. And we now show it's a submersion. For any given point p in $\mathcal{B}$, consider a tangent vector $\vec{v} \in T_p\mathcal{B}$ represented by a path γ_t. On the local chart of $(\mathcal{J}_\omega - \mathcal{J}_4)/Symp_h$ containing p, by the slice theorem (IHL version of Theorem 5.6 in [19]), we can lift γ_t into a path Γ_t s.t. $P = \Gamma_0 \in \pi^{-1}(p)$ in the slice S_P which is a subset of $(\mathcal{J}_\omega - \mathcal{J}_4)$. Denote the tangent vector of Γ_t at $t = 0$ by $\vec{u}$ Then $d\pi_{(P)}(\vec{u}) = \vec{v}$. This means the projection is a submersion in the differential geometric sense. It is a homotopic submersion.

The proof of Theorem 6.9 of [19] confirms that Corollary 5.3 in [19] holds in the IHL(inverse Hilbert limit) setting. Then we have an invariant nbhd at any given point in $(\mathcal{J}_\omega - \mathcal{J}_4)$. This means that for any fiberwise continuous map $[0, 1] \times S^n \to (\mathcal{J}_\omega - \mathcal{J}_4)$, we can make the interval $[0, 1]$ in the normal direction of the fiber. Then consider the invariant nbhd of any preimage of the point 0 in $(\mathcal{J}_\omega - \mathcal{J}_4)/Symp_h$ in $(\mathcal{J}_\omega - \mathcal{J}_4)$, we know that along the path $[0, 1]$ the fiber are identified homeomorphically. Then each S^n must be homotopic in each fiber. This means that all vanishing cycles of all dimensions are trivial, and all emerging cycles are trivial.

Then by Theorem A in [38], the orbit projection $(\mathcal{J}_\omega - \mathcal{J}_4) \to (\mathcal{J}_\omega - \mathcal{J}_4)/Symp_h$ is a fibration with fiber $Symp_h$.

□

Lemma 6.10. *$\pi_1[(\mathcal{J} - \mathcal{J}_4)/Symp_h]$ is the same as $\pi_0(Symp)$.*

Proof. Immediately follows from the LES of the orbit projection fibration $Symp_h \to (\mathcal{J}_\omega - \mathcal{J}_4) \to (\mathcal{J}_\omega - \mathcal{J}_4)/Symp_h$, we have

$$\pi_1(Symp_h) \to \pi_1(\mathcal{J}_\omega - \mathcal{J}_4) \to \pi_1((\mathcal{J}_\omega - \mathcal{J}_4)/Symp_h) \to \pi_0(Symp_h) \to 0.$$

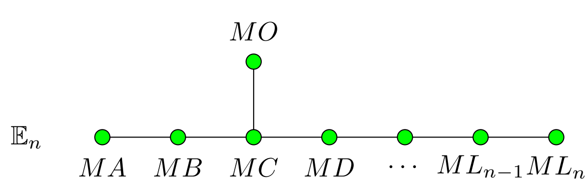
□

6.2. **Reduce to Hirzebruch surface.** We can use a suitable composition of base change to reduce it to $H(n-2)$.

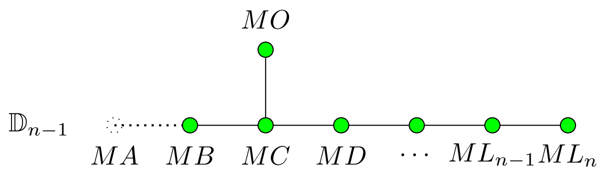
Lemma 6.11. *When $n-2 = 2k$ is even, the base sphere $Q = 2H - E_1 - \cdots - E_5$ becomes $b + kf - e'_1 - e'_2 - \cdots e'_{2k+2}$. Then the (2k+2)-point blowup can be identified with blow up (n-1) points on the sphere $kb + f$ in $S^2 \times S^2$. The condition for existing a toric Hirzebruch surface model is $b - kf$ having positive area. This is to say that $kc_1 - c_{2k+2} > (k-1)$.*

When $n-2 = 2k+1$ is odd, the base sphere $Q = 2H - E_1 - \cdots - E_5$ becomes $(k+1)h - ke_{2k+3} - e_1 - e_2 - \cdots - e_{2k+2}$. Then the (2k+3)-point blowup can be identified with blow up (n-1) points on the sphere $kh - (k-1)e_1$ in one point blowup. The condition for existing a toric Hirzebruch surface model is $(k+1)e_{2k+3} - kh$ having positive area. This is to say that $kc_1 > (k-1) + c_{2k+2} + c_{2k+3}$.

Recall the reduced symplectic cone $\mathcal{R}_n$ is a subset of P^n labeled by the extended $\mathbb{E}_n$ with all simple roots $MO, MA, \cdots, ML_n$:



When deform ω in $\mathcal{R}_n$, one obtain a sublattice of $\mathbb{E}_n$. And note that if we put the assumption in Lemma 6.11, then we obtain a sublattice of $\mathbb{D}_{n-1}$:



And the TSMC of the above sublattice is $\pi_0\text{Diff}^+(S^2, n-1)$, and we have the table of its deformations.

K-face	Γ_L	$TSMC = G_\omega$	ω area
MA	$\mathbb{D}_{n-1}$	$\pi_0\text{Diff}^+(S^2, n-1)$	$\lambda = 1; c_1 > c_2 = \cdots = c_n$
MAL_n	$\mathbb{D}_{n-2}$	$\pi_0\text{Diff}^+(S^2, n-2)$	$\lambda = 1; c_1 > c_2 = \cdots = c_{n-1} > c_n$
MAL_{n-1}	$\mathbb{D}_{n-3} \times \mathbb{A}_1$	$\pi_0\text{Diff}^+(S^2, n-3)$	$\lambda = 1; c_1 > c_2 = \cdots = c_{n-2} > c_{n-1} = c_n$
$MAL_{n-1}L_n$	$\mathbb{D}_{n-3}$	$\pi_0\text{Diff}^+(S^2, n-3)$	$\lambda = 1; c_1 > c_2 = \cdots = c_{n-2} > c_{n-1} > c_n$
4 walls containing $MAB\widehat{\cdots L_{n-3}}\hat{O}L_{n-2}$	$\mathbb{D}_{n-4} \times \mathbb{A}$	$\pi_0\text{Diff}^+(S^2, n-4)$	inequalities correspond to vertices removed
2^{n-k} walls containing $MAB\widehat{\cdots L_{k-1}}\hat{O}L_k$	$\mathbb{D}_{k-2} \times \mathbb{A}$	$\pi_0\text{Diff}^+(S^2, k-1)$	inequalities correspond to vertices removed
2^{n-4} walls containing $M\widehat{BCD}\hat{O}E$	$\mathbb{D}_4 \times \mathbb{A}$	$\pi_0\text{Diff}^+(S^2, 4)$	inequalities correspond to vertices removed
Other walls	type $\mathbb{A}$	trivial	inequalities correspond to vertices removed

TABLE 2. TSMC for $\mathbb{C}P^2 \# n\overline{\mathbb{C}P^2}$ with a type $\mathbb{D}$ symplectic form

Hence we have the following Lemma:

6.3. **Proof in the general MA cases.**

Lemma 6.12. *For n-point blowup, when ω is on edge MA and condition in (22) holds, then $G = \pi_0 \mathit{Diff}(S^2, n-1)$.*

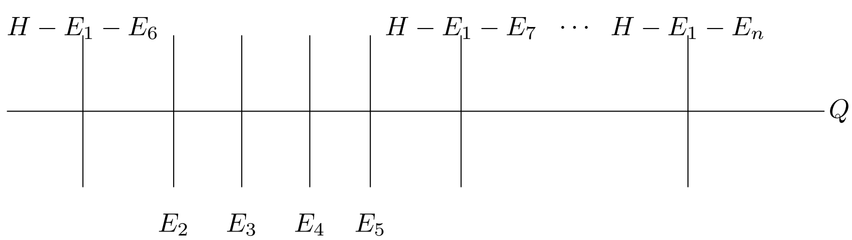
Proof. 1)$G \twoheadrightarrow H$:
This is by the toric $H(n-2)$ model and ball swapping Lemma.

2) $H \twoheadrightarrow G$:

Firstly, there is a well defined space $(\mathcal{J}_\omega - \mathcal{J}_4)/Symp_h$ such that $\pi_1[(\mathcal{J}_\omega - \mathcal{J}_4)/Symp_h]$ is the same as $\pi_0(Symp_h)$, see definition 6.3 and Lemma 6.10.

By Lemma 6.13 6.15 below, there exist continuous maps γ, β, such that the composition of $\beta \circ \alpha \circ \gamma$ in (6.15) is an isomorphism. Then the map on π_1 induced the desired $H \twoheadrightarrow G$.

□



Over $\mathcal{Q}_n$ there is a canonical family of rational surfaces $\mathcal{Y}_n \to \mathcal{Q}_n$. Start with a trivial family of $\mathbb{C}P^2$ over $\mathcal{Q}_n$, one obtains n canonical sections $s_1, \cdots, s_n$ given by π_i. One then first blow up s_1, then the rest of s_i, this yields the family $\mathcal{Y}_n$. The fiber over $q = (p_1, \cdots, p_n)$ is given by the blowup of $\mathbb{C}P^2$ at q. Over the discriminant locus where $p_1 = p_i$ for some i, the rational surface is given by first blowing up p_1, then the rest of p_i. This yields the following lemma

Lemma 6.13. *For a rational point $\omega \in MA$ with $\omega(H) \geq n\omega(E_2)$, there exists a well defined continuous map*

$$\alpha : \mathcal{Q}_n \quad \mapsto (\mathcal{J}^c_\omega - \mathcal{X}^c_4)/Symp_h.$$

Proof. We will first show the existence of a fiberwise symplectic form on $\mathcal{Y}_n$ which is homologous to ω. We denote the family of (integrable) complex structures of X_n obtained by blowing up the $\mathcal{Q}_n$ by $\mathcal{X}$ and a paarticular fiber blowing up the tuple $q \in \mathcal{Q}_n$ by $\mathcal{Y}_q$.

Up to a rescaling, we can write $PD([l\omega]) = aH - b'E_1 - \cdots - bE_{n-1} - bE_n$ with $a, b' > b \in \mathbb{Z}^{>0}$, and $a = b' + 2b$.

Consider the divisor D of $\mathcal{Y}_q$, which is a linear combination of the universal line class (where over each fiber has class H) and the canonical exceptional divisors by the blow-ups of s_i, so that over each fiber $\mathcal{Y}_q$ we have $[D_q] = PD([l\omega]) = aH - b'E_1 - \cdots - bE_{n-1} - bE_n$ for $q \in \mathcal{Q}_n$.

Clearly, we have $D_q \cdot D_q > 0$. Let C_q be any curve in $\mathcal{Y}_q$, and by definition 6.18, either $C_q \cdot H > 0$, or C_q is a negative curve appears in Lemma 6.3.

By Lemma 3.17, $D_q \cdot C_q > 0$ if $C_q \cdot H > 0$. And it's easy to check that if C_q is a negative curve appears in Lemma 6.3, we also have $D_q \cdot C_q > 0$. Hence by Nakai-Moshenzon criteria, D is a relative ample divisor which induces a family of embeddings of $\mathcal{Y}_q$ into $\mathbb{C}P^N$.

Equipping $\mathbb{P}H^0(X; D)$ with a Fubini-Study form, one has a fiberwise symplectic structure ω_q on $\mathcal{Y}_q$ from the pullback, which is diffeomorphic through some $\iota_q : \mathcal{Y} \to X_n$ so that $\omega_q = \iota_q^*\omega$ from [33]. For each fiber $\mathcal{Y}_q$, the embedding pushes forward to a J_q through ι_q, which gives an integrable almost complex structure $(\iota_q)_*(J_q) \in J_\omega$. Two different choices of ι_q differ by a symplectomorphism in $Symp_h(X, \omega)$, and all these almost complex structures satisfy the assumptions of Lemma 6.3. Hence this construction yields a well-defined continuous map

$$\alpha : \mathcal{Q}_n \quad \mapsto (\mathcal{J}^c_\omega - \mathcal{X}^c_4)/Symp_h.$$

□

Lemma 6.14. *There is a well defined section map γ sending $\mathcal{B}_{n-1}$ to $\mathcal{Q}_n$.*

Proof. Consider the zeroth and first Hirzebruch surface $F(0)$ and $F(1)$, and topologically identify them with $S^2 \times S^2$ and $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$, respectively. Fix a holomorphic map $u : \mathbb{C}P^1 \to F(0)$ or $F(1)$ in the class $B + sF \in H^2(S^2 \times S^2)$ when $n - 1 = 2s + 2$ or $sH - (s-1)E_1 \in H^2(\mathbb{C}P^2 \# n\overline{\mathbb{C}P^2})$ when $n - 1 = 2s + 3$ with $\mathbb{C}P^1$. Then the above $(n-1)$-tuple will be sent to a point on the respective minimal rational surface. The blow-up of these points yields a rational surface in $\mathcal{Q}_n$.

□

Lemma 6.15. *There's a well defined continuous map $\beta : (\mathcal{J}_\omega - \mathcal{J}_4)/Symp_h \to Conf_{n-1}(S^2)/PSL(2, \mathbb{C})$ so that the following diagram commutes, and the composition $\beta \circ \alpha \circ \gamma$ is an isomorphism.*

$$\begin{tikzcd}
\mathcal{Q}_n \arrow[r, "\alpha"] & (\mathcal{J}_\omega - \mathcal{J}_4)/Symp_h \arrow[r, "\beta"] & \mathcal{B}_{n-1}, \arrow[ll, bend right, "\gamma"']
\end{tikzcd}$$

Proof. As usual, we start with a reduced basis and a configuration as in Figure 1, but we will need to make some adjustments later. Clearly, if $\mathcal{J}_{open}$ is the open strata of almost complex structures where all curves in Figure 1 have embedded representatives, we can map any element in $\mathcal{J}_{open}/Symp$ to the equivalence class given by the $(n-1)$ geometric intersection points on the Q-class sphere. There are three kinds of strata in $\mathcal{J}_\omega - (\mathcal{J}_4 \cup \mathcal{J}_{open})$.

- *J admits $E_1 - E_i$.* The existence of such -2 spheres does not affect the irreducibility of any curves in Figure 1. Indeed, since all curves in Figure 1 pairs nonnegatively with $E_1 - E_i$, if any of the curves in Figure 1 is reducible, there has to be another curve with self-intersection of -2 or lower, hence J would fall inside $\mathcal{J}_4$. This allows us to define β as in the case of $\mathcal{J}_{open}$.
- *J admits $H - E_i - E_j - E_k$, $2 \leq i < j < k \leq n$.* If $\max(i, j, k) > 5$, one may check that $H - E_i - E_j - E_k$ again pairs non-negatively with all curves in Figure 1, so again the absence of breaking of curves in Figure 1 allows us to define β similarly.

  If $2 \leq i < j < k \leq 5$, we assume without loss of generality that $i = 2, j = 3, k = 4$. Q breaks into two irreducible components of classes $H - E_2 - E_3 - E_4$ and $H - E_1 - E_5$. In below, we will denote the J-holomorphic curve in class A as $J(A)$ when such a curve is unique. Consider any sequence $J_k \in \mathcal{J}_{open}$ that converges to a J_∞ admitting $J_\infty(H-E_2-E_3-E_4)$. Consider the Gromov convergence sequence $\{u_k : \mathbb{C}P^1 \to X_n\}$ to the stable curve $J_\infty(H-E_2-E_3-E_4) \cup J_\infty(H-E_1-E_5)$, and require

$$u_k(p_i^k) = J_k(2H - E_1 - \cdots - E_5) \cap J_k(E_i), 2 \leq i \leq 5, \tag{36}$$

$$u_k(p_i^k) = J_k(2H - E_1 - \cdots - E_5) \cap J_k(H - E_1 - E_j), 6 \leq j \leq n. \tag{37}$$

  We also denote

$$p_j^\infty := J_\infty(H - E_2 - E_3 - E_4 - E_5) \cap J_\infty(E_i), 2 \leq i \leq 4,$$
$$p_j^\infty := J_\infty(H - E_2 - E_3 - E_4 - E_5) \cap J_\infty(H - E_1 - E_j), 5 \leq j \leq n.$$

  Note that all other curves of the form $H - E_1 - E_j$ are disjoint from the component $H - E_1 - E_5$. Therefore, when one requires $p_2^k = 0, p_3^k = 1, p_4^k = \infty$ in the reparametrization process during the bubble forming, there is a small neighborhood $U \subset \mathbb{C}P^1$ that contains all p_5^k for large enough k, so that all $p_j^k \notin U$ for $j \geq 6$ and large k. Since U can be taken arbitrarily small, this shows that the configurations given by $[p_1^k, \cdots, p_n^k] \in \mathrm{Conf}(\mathbb{C}P^1, n)/PSL(2, \mathbb{C})$ converge to $[p_1^\infty, \cdots, p_n^\infty]$.
- *J admits $2H - E_1 - \cdots - E_5 - E_l$ for some $l \geq 6$.* Without loss of generality, we assume $l = 6$. Similar to the above case, we consider a sequence $J_k \in \mathcal{J}_{open}$ that converges to J. Require the corresponding u_k and p_i^k to satisfy the same conditions (36)(37), as well as

$$p_j^\infty := J_\infty(2H - E_1 - E_2 - E_3 - E_4 - E_5 - E_6) \cap J_\infty(E_i), 2 \leq i \leq 6,$$
$$p_j^\infty := J_\infty(2H - E_2 - E_3 - E_4 - E_5 - E_6) \cap J_\infty(H - E_1 - E_j), 7 \leq j \leq n.$$

  From essentially the same argument, one has a small open disk U in $\mathbb{C}P^1$ such that all p_i^k is outside U for large enough k, except $p_6^k \in U$ for the same range of k from the reparametrization process. Therefore, the configuration $[p_i^k] \in \mathrm{Conf}(\mathbb{C}P^1, n)/PSL(2, \mathbb{C})$ converges to $[p_i^\infty]$.

then we have a continuous (one can check this map preserve limit) map $\beta : (\mathcal{J}_\omega - \mathcal{J}_4)/Symp_h \to Conf_{n-1}(S^2)/PSL(2, \mathbb{C})$.

To prove that $\beta \circ \alpha \circ \gamma = id$, we go through the diagram. Given a configuration of points in $\mathcal{B}_{n-1}$, γ gives a rational surface by blowing up a fixed rational curve in the class $kb + f$ of $S^2 \times S^2$ (or $kh - (k-1)e_1$ in $\mathbb{C}P^2\#\overline{\mathbb{C}P}^2$ respectively). We call such new exceptional classes $e_1, \cdots, e_{n-1}$ (or $e_2, \cdots, e_n$, respectively). The map α uses a projective embedding from this rational surface to $\mathbb{C}P^N$, where the pulled back symplectic form is symplectomorphic to our given symplectic form. The image of α is the push-forward of our integrable complex structure to the symplectic manifold we studied, and we denote the images of classes h, b, f, e_i by the same notation. We now perform a base change as in Lemma 5.7, so that $e_i = H - E_1 - E_i$ for $i \geq 5$, $e_i = E_{i+1}$, $kh - (k-1)e_1 - e_2 \cdots - e_n = 2H - E_1 - \cdots - E_5$ for $4 \geq i \geq 1$ when n is odd; and $b + kf - e_1 \cdots - e_{n-1} = 2H - E_1 - \cdots - E_5$ for $4 \geq i \geq 1$ when n is even. After this rename of our classes, the above β bounds to satisfy $\beta \circ \alpha \circ \gamma = id$ tautologically.

□

Remark 6.16. *take the space of above curves with orthogonal intersection at each point, denoted $\mathcal{C}_0$. Then for a fixed smooth X and a fixed sphere Q, choosing $(n-1)$ pairwise distinct points on Q and pick up smooth sphere which can be parallelled transported (connection?) to each other, then there is a $Conf_{n-1}(S^2)$ family of sphere configurations, they are not symplectomorphic (but diffeomorphic).*

There are two cases, corresponding n even or odd.
When n even, as in Lemma 6.11, the n-point blowup can be can be identified with blow up (n-1) points on the sphere $kb+f$ in $S^2 \times S^2$, one can consider a $Conf_{n-1}(S^2)$ family because there are $n-1$ equal-area "vertical" non-intersecting exceptional spheres.

When n odd, as in Lemma 6.11, the n-point blowup can be can be identified with blow up (n-1) points on the sphere $kh-(k-1)e_1$ in one point blowup, one can consider a $Conf_{n-1}(S^2)$ family because there are $n-1$ equal-area "vertical" non-intersecting exceptional spheres.

we can find an embedded sphere in the class $kb+f$ of $S^2 \times S^2$ (or $kh-(k-1)e_1$ in one point blowup respectively,) and by blowing up $(n-1)$ points, we have a $Conf_{n-1}(S^2)$ family of almost complex structures in $(\mathcal{J}-\mathcal{J}_4)/Symp$.

Remark 6.17. *Then consider the subspace $\mathcal{J}_v$ of $\mathcal{J}_\omega$, such that either every curve class in C_n has embedded pseudo-holomorphic representative, or only the vertical classes in the picture below could degenerate into a nodal curve with a single (-2) sphere and a (-1) sphere. (Note that we require Q is always has an embedded representative, and vertical classes at most has two irreducible components, and other negative curves could appear.)*

Similar to Lemma 6.5, we have

By the slice theorem 5.6 [19] (cf. Theorem 6.9 [19]), the orbit projection is a fibration with fiber $Symp_h$.

Then we notice that $\mathcal{J}_\omega - \mathcal{J}_v$ is a union of finitely many codimension 2 submanifolds with higher codimension manifolds. Then we know that $\mathcal{J}_v$ is path connected but it could have non-trivial π_1.

We first define a partial compactification of a subset in $\mathrm{Conf}(\mathbb{C}P^2, n)$, the configuration space of n *ordered* points $p_1, \cdots, p_n$ on $\mathbb{C}P^2$. Let $\mathcal{U}$ be the subset of $(\mathbb{C}P^2)^n$ where

- p_1 is not collinear with any two of p_i, $i \geq 2$,
- there are no collinear relations of more than 3 points,
- no more than 6 points lie on the same conic,
- $p_i \neq p_j$ for $i, j \geq 2$.

We emphasize that there are more discriminant loci of codimension 2 than those listed above, but those will not concern us and won't be removed.

Finally, we blow up the locus $p_1 = p_i$ for $i \geq 2$, and denote the resulting space as $\mathcal{Q}_n$.

Definition 6.18. *We define $\mathcal{Q}_n$, the space of admissible ordered n-tuple $p_1, \cdots, p_n$ in $\mathbb{C}P^2$ to be the ordered n points where each of $p_2, \cdots, p_n$ is allowed to collide with p_1, and any 3 of $p_2, \cdots, p_n$ is allowed to be collinear, and there's no conic passing through $p_1, \cdots, p_5$ and a sixth point. Besides these non-generic relations, any two points do not collide and any 3 points are not collinear.*

We also call colliding pairs and collinear triples a non-generic condition. And besides the above assumptions, at most one non-generic condition is allowed in $p_1, \cdots, p_n$.

We consider the moduli space of almost Kähler structures which compatible with the given ω. The construction is given by [19] Corollary 3.2 (for Sobolev H^k completion), where $\mathcal{M}^k_{a,\omega}$ is given by the quotient $\mathcal{C}^k_{a,\omega}/Symp^{k+1}(M,\omega)$. For the moduli space of smooth almost complex structures, one can take the inverse limit on k, and endow the inverse limit topology of the moduli space $\mathcal{M}_{a,\omega} = \mathcal{J}_\omega/Symp$. We can restrict the group action to make a moduli space $\mathcal{J}_\omega/Symp_h$ up to the action of $Symp_h$.

Further, since the action of $Symp$ or $Symp_h$ on $\mathcal{J}_\omega$ fixes each strata, we can consider the subspace $(\mathcal{J}_\omega - \mathcal{J}_4)/Symp_h \subset \mathcal{J}_\omega/Symp_h$, by restricting the action to a certain strata $(\mathcal{J}_\omega - \mathcal{J}_4)$, where

Definition 6.19. *$(\mathcal{J}_\omega - \mathcal{J}_4)$ is defined as the subspace of $\mathcal{J}_\omega$ such that the following classes $E_1, E_2, \cdots, E_n, H - E_1 - E_6, \cdots, H - E_1 - E_n$, and $2H - E_1 - \cdots E_5$*
- *either each has an embedded representative,*
- *or at most one class has an pseudo-holomorphic representative with at most 2 irreducible components, and each components has negativity at least -2. More explicitly, there's at most one (-2) class such as $E_1 - E_i$, $H - E_p - E_q - E_r, p, q, r \geq 2$ and $2H - E_1 - \cdots - E_5 - E_k, k \geq 6$.*

7. Fundamental Group of the Hamiltonian diffeomorphism Group and applications

There has been a complete understanding of $\pi_1 Ham$ for any symplectic forms on a small rational surface, see tables 5, 4 and 3 in [27]. Also, the understanding of SMCG and $\pi_1 Ham$ helped to prove the isotopy uniqueness of certain symplectic curves, as in [28]. In this section, we extend those results to any rational surface with a symplectic form of type $\mathbb{A}$ or $\mathbb{D}$ in the K-nef symplectic cone.

K-face	Γ_L	N_ω	$\pi_1(Symp_h(X,\omega))$	ω area
Point M	$\mathbb{A}_4$	0	trivial	monotone, $\lambda = 1; c_1 = c_2 = c_3 = c_4$
MO	$\mathbb{A}_3$	4	$\mathbb{Z}^4$	$\lambda < 1; c_1 = c_2 = c_3 = c_4$
MA	$\mathbb{A}_3$	4	$\mathbb{Z}^4$	$\lambda = 1; c_1 > c_2 = c_3 = c_4$
MB	$\mathbb{A}_1 \times \mathbb{A}_2$	6	$\mathbb{Z}^6$	$\lambda = 1; c_1 = c_2 > c_3 = c_4$
MC	$\mathbb{A}_1 \times \mathbb{A}_2$	6	$\mathbb{Z}^6$	$\lambda = 1; c_1 = c_2 = c_3 > c_4$
MOA	$\mathbb{A}_2$	7	$\mathbb{Z}^7$	$\lambda < 1; c_1 > c_2 = c_3 = c_4$
MOB	$\mathbb{A}_1 \times \mathbb{A}_1$	8	$\mathbb{Z}^8$	$\lambda < 1; c_1 = c_2 > c_3 = c_4$
MOC	$\mathbb{A}_2$	7	$\mathbb{Z}^7$	$\lambda < 1; c_1 = c_2 = c_3 > c_4$
MAB	$\mathbb{A}_2$	7	$\mathbb{Z}^7$	$\lambda = 1; c_1 > c_2 > c_3 = c_4$
MAC	$\mathbb{A}_1 \times \mathbb{A}_1$	8	$\mathbb{Z}^7$	$\lambda = 1; c_1 > c_2 = c_3 > c_4$
MBC	$\mathbb{A}_1 \times \mathbb{A}_1$	8	$\mathbb{Z}^7$	$\lambda = 1; c_1 = c_2 > c_3 > c_4$
MOAB	$\mathbb{A}_1$	9	$\mathbb{Z}^8$	$\lambda < 1; c_1 > c_2 > c_3 = c_4$
MOAC	$\mathbb{A}_1$	9	$\mathbb{Z}^9$	$\lambda < 1; c_1 > c_2 = c_3 > c_4$
MOBC	$\mathbb{A}_1$	9	$\mathbb{Z}^9$	$\lambda < 1; c_1 = c_2 > c_3 > c_4$
MABC	$\mathbb{A}_1$	9	$\mathbb{Z}^9$	$\lambda = 1; c_1 > c_2 > c_3 > c_4$
MOABC	trivial	10	$\mathbb{Z}^{10}$	$\lambda < 1; c_1 > c_2 > c_3 > c_4$

Table 3. Γ_L and $\pi_1(Symp_h(X,\omega))$ for $\mathbb{C}P^2 \# 4\overline{\mathbb{C}P^2}$

We give the following change of basis in $H^2(X,\mathbb{Z})$ in preparation for section 3. Note that $X = S^2 \times S^2 \# k\overline{\mathbb{C}P^2}, k \geq 1$ can be naturally identified with $\mathbb{C}P^2 \# (k+1)\overline{\mathbb{C}P^2}$. Denote the basis of H_2 by $B, F, E_1', \cdots, E_k'$ and $H, E_1, \cdots, E_k, E_{k+1}$ respectively. Then the transition on the basis is explicitly given

k-Face	Γ_L	N_ω	$\pi_1(Symp_h(X,\omega))$	$\omega-$area
Point M	$\mathbb{A}_1\times\mathbb{A}_2$	0	$\mathbb{Z}^2$	$(\frac{1}{3},\frac{1}{3},\frac{1}{3})$: monotone
Edge MO:	$\mathbb{A}_2$	1	$\mathbb{Z}^3$	$\lambda<1; c_1=c_2=c_3$
Edge MA:	$\mathbb{A}_1\times\mathbb{A}_1$	2	$\mathbb{Z}^4$	$\lambda=1; c_1>c_2=c_3$
Edge MB:	$\mathbb{A}_1\times\mathbb{A}_1$	2	$\mathbb{Z}^4$	$\lambda=1; c_1=c_2>c_3$
ΔMOA:	$\mathbb{A}_1$	3	$\mathbb{Z}^5$	$\lambda<1; c_1>c_2=c_3$
ΔMOB:	$\mathbb{A}_1$	3	$\mathbb{Z}^5$	$\lambda<1; c_1=c_2>c_3$
ΔMAB:	$\mathbb{A}_1$	3	$\mathbb{Z}^5$	$\lambda=1; c_1>c_2>c_3$
T_{MOAB}:	trivial	4	$\mathbb{Z}^6$	$\lambda<1; c_1>c_2>c_3$

TABLE 4. Γ_L and $\pi_1(Symp_h(X,\omega))$ for $\mathbb{C}P^2\#3\overline{\mathbb{C}P^2}$

k-Face	Γ_L	N_ω	$\pi_1(Symp_h(X,\omega))$	ω area
OB	$\mathbb{A}_1$	0	$\mathbb{Z}^2$	$c_1=c_2$
ΔBOA	trivial	1	$\mathbb{Z}^3$	$c_1\neq c_2$

TABLE 5. Γ_L and $\pi_1(Symp_h(\mathbb{C}P^2\#2\overline{\mathbb{C}P^2}))$

by

$$
\begin{aligned}
B &= H-E_2,\\
F &= H-E_1,\\
E_1' &= H-E_1-E_2,\\
E_i' &= E_{i+1}, \forall i\geq 2,
\end{aligned}
\tag{38}
$$

with the inverse transition given by:

$$
\begin{aligned}
H &= B+F-E_1',\\
E_1 &= B-E_1',\\
E_2 &= F-E_1',\\
E_j &= E_{j-1}', \forall j>2.
\end{aligned}
\tag{39}
$$

$\nu H-c_1E_1-c_2E_2-\cdots-c_kE_k=\mu B+F-a_1E_1'-a_2E_2'-\cdots-a_{k-1}E_{k-1}'$ if and only if

$$\mu=(\nu-c_2)/(\nu-c_1), a_1=(\nu-c_1-c_2)/(\nu-c_1), a_2=c_3/(\nu-c_1),\cdots, a_{k-1}=c_k/(\nu-c_1). \tag{40}$$

In [28] the following results are proved:

Proposition 7.1. *Let (X,ω) be $\mathbb{C}P^2$ or its blow up at several points with a given reduced form, $(\tilde{X}_k,\tilde{\omega}_\epsilon)$ be the blow up of X at k points with different small size(less than any blow up size of X), then the rank of $\pi_1(Symp_h(\tilde{X}_k))$ can exceed $\pi_1(Symp_h(X))$ at most $rk+k(k-1)/2$, where r is the rank of $\pi_2(X)$.*

Lemma 7.2. *Note that if we allow the blow up sizes to be all equal, then counting Hamiltonian bundle gives the following:*

$$Rank[\pi_1(Symp(\tilde{X}_k,\tilde{\omega}_\epsilon)]\leq Rank[\pi_1(Symp(X,\omega))]+rk,$$

where where r is the rank of $\pi_2(X)$, and k is the number of points of blow up of $\tilde{X}_k$ from X.

In particular, for $S^2\times S^2$ with size $(\mu,1),\mu\geq 1$, an equal blow-up of k points, one can easily check that the exceptional sphere E_k in $\tilde{X}_k$ with the smallest blow-up size always has an embedded representative. Hence by counting the Hamiltonian bundle one obtain an upper-bound of the rank of $\pi_1(Symp_h(S^2\times S^2))$ plus $2k$. Note that rank of $\pi_1(Symp_h(S^2\times S^2))$ means the free rank, where for monotone $S^2\times S^2$ is 0 and non-monotone $S^2\times S^2$ is 1.

7.1. **Fundamental Group and Seidel representations.** We use the same strategy as in [28]. We will generalize Lemma 5.33 and Theorem 5.35 in [28], which together give the precise rank of $\pi_1(Symp_h(X,\omega))$ in the cases where we know the SMC:

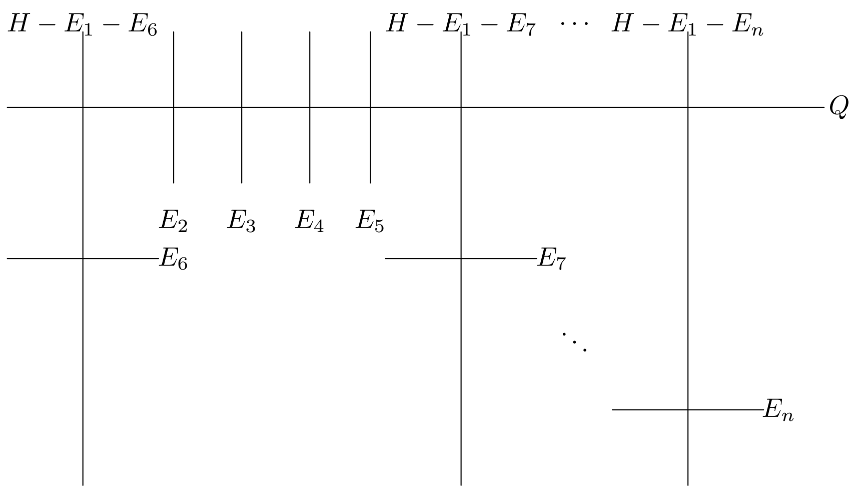
The case of type $\mathbb{A}$ forms on $\mathbb{C}P^2\#n\overline{\mathbb{C}P^2}$

Theorem 7.3. *Suppose (22) holds and the TSMC is connected, then the upper-bound given in Theorem 5.35 equals the lower-bound given in Lemma 5.33.*

In particular, for a generic symplectic form (there's no -2 Lagrangian sphere) then rank of $\pi_1(Symp_h(X,\omega)) = n(n+1)/2$.

Proof. **The upper bound** This is given by [35] and [28]. $1+2+\cdots+n = n(n+1)/2$.

The lower bound



Where $Q = 2H - E_1 - \cdots - E_5$.

Complement $U = (\mathbb{C} - \{p_1, p_2, \cdots, p_{n-6}\}) \times \mathbb{C}$, and by Lemma 5.5, its compactly supported symplectomorphism group is connected.

And for a reduced form with $\omega = (1|c_1, \frac{1-c_1}{2}, \cdots, \frac{1-c_1}{2})$ and $c_1 > \frac{n-7}{n-3}$, Diagram 21 is

$$\begin{array}{ccccccc} Symp_c(U_{n-6}) & \longrightarrow & Stab^0(C) & \longrightarrow & Stab(C) & \longrightarrow & Symp_h(X,\omega) \\ & & \downarrow & & \downarrow & & \downarrow \\ & & \mathbb{Z}^{2n-7} & & (S^1)^{2n-6} \times \mathrm{Diff}^+(S^2, n-1) & & \mathcal{C}_0 \simeq \mathcal{S} \end{array} \tag{41}$$

When TSMC is connected, we consider the following portion of the LES:

$$1 \to \mathbb{Z} \to \pi_1(Symp_h(X,\omega)) \to \pi_1(\mathcal{S}) \xrightarrow{f} \pi_0(Stab(C)) \to 1 \tag{42}$$

Note that we can think the following sequence

$$\pi_1(Symp_h(X,\omega))/\mathbb{Z} \to \pi_1(\mathcal{S}) \xrightarrow{f} \pi_0(Stab(C)) \to 1 \tag{43}$$

and its abelinization gives the following:

$$\mathbb{Z} \to \pi_1(Symp_h(X,\omega)) \to Ab[\pi_1(\mathcal{S})] \xrightarrow{f} Ab[\pi_0(Stab(C))] = \mathbb{Z}^{(n-1)(n-4)/2} \to 1 \tag{44}$$

Now we deal with $\pi_1(\mathcal{S})$, and show that its rank in $(n-1)^2+1$:

We need to find out the -2 curve that pairs at least one of the -1 curve negative in the configuration.

A complete list is the following:

$$6 \quad H-E_i-E_j-E_k, \quad i,j,k \in \{2,\cdots,5\};$$

$$(n-2)(n-1)/2 \quad E_s-E_t, \quad s,t \in \{2,\cdots,n\};$$

$$(n-1)(n-2)/2 \quad H-E_1-E_p-E_q, \quad p,q \in \{6,\cdots,n\}.$$

$$n-5 \quad 2H-E_1-\cdots E_5-E_i, i>5.$$

Hence the total number is $(n-1)(n-2)+n-1=(n-1)^2+1$.

□

The case of type $\mathbb{D}$ forms on $\mathbb{C}\mathbf{P}^2\#\mathbf{n}\overline{\mathbb{C}\mathbf{P}^2}$

For the convenience of computation, we divide all type $\mathbb{D}$ forms into 3 families, one can check table 1 to see this is a complete list.

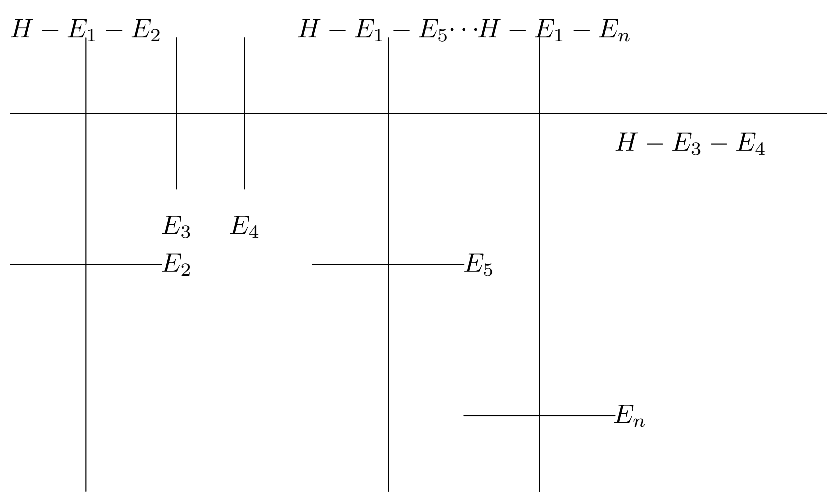
- MA or $MAL_{i_1}\cdots L_{i_k}$.
- MO, MB, MC, MD or $MXL_{i_1}\cdots L_{i_k}$ where $X=O,B,C,D$.
- ME or $MEL_{i_1}\cdots L_{i_k}$.

7.1.1. *MA or $MAL_{i_1}\cdots L_{i_k}$ cases.*

Proposition 7.4. *The rank of $\pi_1(Symp_h(X,\omega))$ for the $MA, MAE, \cdots,$ cases of a $n-$points blow up:*
• if $\omega \in MA$, then $rk(\pi_1)=n$, and in particular, we know it is free abelian, i.e. $\pi_1(Ham)=\mathbb{Z}^n$.
• If ω lies on any other lines, we can always find a lower-bound such that it equals the upper-bound given by [35] and Proposition 3.21 in [28].

Proof. Here we list all the cases with $6 \le n \le 8$:

- MA case of $\mathbb{C}P^2\#n\overline{\mathbb{C}P^2}$:



k-Face	M	rank $\pi_1(Symp_h(X,\omega))$	upper bound
MA	$\mathbb{C}P^2\#6\overline{\mathbb{C}P^2}$	6	11=5+6
MAE	$\mathbb{C}P^2\#6\overline{\mathbb{C}P^2}$	11	11=5+6
MA	$\mathbb{C}P^2\#7\overline{\mathbb{C}P^2}$	7	13=6+7
MAF	$\mathbb{C}P^2\#7\overline{\mathbb{C}P^2}$	13	13=6+7
MAE	$\mathbb{C}P^2\#7\overline{\mathbb{C}P^2}$	17	$17 = 5 + 6 \times 2$
MAEF	$\mathbb{C}P^2\#7\overline{\mathbb{C}P^2}$	18	18=5+6+7
MA	$\mathbb{C}P^2\#8\overline{\mathbb{C}P^2}$	8	15=7+8
MAG	$\mathbb{C}P^2\#8\overline{\mathbb{C}P^2}$	15	15=7+8
MAF	$\mathbb{C}P^2\#8\overline{\mathbb{C}P^2}$	20	$20 = 6 + 7 \times 2$
MAFG	$\mathbb{C}P^2\#8\overline{\mathbb{C}P^2}$	21	$21 = 6 + 7 + 8$
MAE	$\mathbb{C}P^2\#8\overline{\mathbb{C}P^2}$	23	$23 = 5 + 6 \times 3$
MAEG	$\mathbb{C}P^2\#8\overline{\mathbb{C}P^2}$	25	$25 = 17 + 8$
MAEF	$\mathbb{C}P^2\#8\overline{\mathbb{C}P^2}$	27	$25 = 11 + 7 \times 2$
MAEFG	$\mathbb{C}P^2\#8\overline{\mathbb{C}P^2}$	26	$26 = 5 + 6 + 7 + 8$

TABLE 6. rank of $\pi_1(Symp_h(\mathbb{C}P^2\#n\overline{\mathbb{C}P^2}))$

It is easy to check that for a symplectic form s.t. (22) holds, the complement of the above complement is a convex symplectic $(\mathbb{C} - (n-5)pts) \times \mathbb{C}$.

Then the diagram 21 becomes

$$\begin{tikzcd}
Symp_c(U) = Stab^1(C) \arrow[r] & Stab^0(C) \arrow[r] \arrow[d] & Stab(C) \arrow[r] \arrow[d] & Symp_h(X,\omega) \arrow[d] \\
 & \mathbb{Z}^6 & (S^1)^7 \times P_5(S^2)/\mathbb{Z}_2 & \mathcal{S}
\end{tikzcd} \tag{45}$$

We then have the LES with g being an isomorphism:

$$\mathbb{Z} \to \pi_1(Symp_h(X,\omega)) \to \pi_1(\mathcal{S}) \xrightarrow{f} \pi_0(Stab(C)) \xrightarrow{g} \pi_0(Symp_h) \to 1. \tag{46}$$

Lemma 7.5. *1)The only (-2) curves that generate $\pi_1(\mathcal{S})$ are in classes E_1-E_3, E_1-E_4, $H-E_2-E_3-E_4$, $H-E_3-E_4-E_5, \cdots H-E_3-E_4-E_n$. 2) For any symplectic -2 sphere classes listed as above, homologous symplectic -2 spheres are isotopic. 3) Hence the rank of $\pi_1(\mathcal{S})$ is $n-1$.*

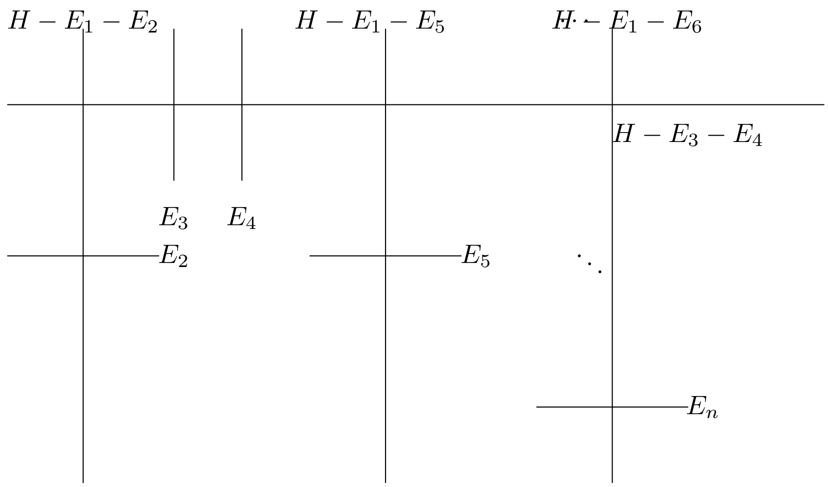
Proof. 1) can be easily checked from the size of the symplectic form. 3) is a corollary of 1) and 2). Now we prove 2)basically use the same argument as in Lemma 3.45 in [28] for this case.

Argue by contradiction. Assume there are $k>1$ isotopic classes for one of those classes. Then there are $k>1$ isotopic classes for each of those classes, since those are transitive under the homological action.

Then we show that the lower bound of rank $\pi_1 Ham(X\#\overline{\mathbb{C}P^2},\tilde\omega)$ exceeds the upper bound, for any $k>1$, where $\tilde\omega$ is the blowup form of a ball with very small size. First look at the $k=2$ case, the upper bound of $\pi_1 Ham(X\#\overline{\mathbb{C}P^2},\tilde\omega)$ is $1+2(n-1)+n = 3n-1$, computed from the equal size blowup of a non-monotone $S^2\times S^2$. And the lower bound is given by counting (-2) sphere classes, given by $k(n-1+n-4)$, because there are n-1 (-2) curves given by subtracting E_{n+1} from the vertical classes, and there are n-4 (-2) curves given by $2H-E_1-\cdots-E_5-E_j, j\geq 6$. Clearly, when $k\geq 2, n>5$, $k(2n-5)>3n-1$, a contradiction.

□

Then we can compute the precise rank as follows: in equation (46), the map g is an isomorphism, which follows from the fact that both groups are Hopfian and epimorphism means isomorphism. Then we know that $\pi_1(\mathcal{S})$is abelian since it's a quotient group of an abelian group. Then it is isomorphic to $H^1(\mathcal{S}, \mathbb{Z})$ by generalized Alexander duality. By Lemma 7.5, $H^1(\mathcal{S}, \mathbb{Z}) = \mathbb{Z}^{n-1}$. Hence we trace back equation (46), the extension of a free abelian group by another free abelian group is free abelian. Hence $\pi_1(Symp) = \mathbb{Z}^n$.



For a symplectic form s.t. (22) holds, the complement of the above complement is a convex symplectic $(\mathbb{C} - (n-5)points) \times \mathbb{C}$.

We then have the LES with g being an isomorphism:

$$\mathbb{Z} \to \pi_1(Symp_h(X, \omega)) \to \pi_1(\mathcal{S}) \xrightarrow{f} \pi_0(Stab(C)) \xrightarrow{g} \pi_0(Symp_h) \to 1. \tag{47}$$

And $\pi_1(\mathcal{S})$ is generated by (-2) curves:
for $\mathbb{C}P^2 \# 7\overline{\mathbb{C}P^2}$, by $E_1 - E_3, E_1 - E_4$, $H - E_2 - E_3 - E_4$, $H - E_3 - E_4 - E_5, H - E_3 - E_4 - E_6$, $H - E_3 - E_4 - E_7$;
for $\mathbb{C}P^2 \# 8\overline{\mathbb{C}P^2}$, by $E_1 - E_3, E_1 - E_4$, $H - E_2 - E_3 - E_4$, $H - E_3 - E_4 - E_5, H - E_3 - E_4 - E_6$, $H - E_3 - E_4 - E_7$, $H - E_3 - E_4 - E_8$.

- For MAE of $\mathbb{C}P^2 \# 6\overline{\mathbb{C}P^2}$, the upper bound is provided by the above table. Now we consider the lower-bound.

  We still consider the configuration as in the MA case. Now $\pi_1(\mathcal{S})$ has 8 more generators, by $E_2 - E_6, \cdots E_5 - E_6$; $H - E_1 - E_2 - E_6, \cdots, H - E_1 - E_5 - E_6$. And this time the map g in sequence (46) is no longer an isomorphism, instead, it is the forget strand map $1 \to \mathbb{F}_3 \to P_5/\mathbb{Z}_2 \to P_4/\mathbb{Z}_2 \to 1$.

  Hence we consider the following:

  $$\mathbb{Z} \to \pi_1(Symp_h(X, \omega)) \to \pi_1(\mathcal{S}) \xrightarrow{f} Ker(f) = \mathbb{F}_3 \to 0$$

  And its abelianization $\mathbb{Z} \to \pi_1(Symp_h(X, \omega)) \to Ab[\pi_1(\mathcal{S})] \xrightarrow{f} Ker(f) = \mathbb{Z}^3 \to 0$

  Since the rank of $Ab[\pi_1(\mathcal{S})]$ is $5 + 8 = 13$, we know rank of $\pi_1(Symp_h(X, \omega))$ is no less than $13 - 3 + 1 = 11$, which matches the upper-bound.

- Every other case listed in the table 6 can be covered using the above approach.

- 7-MAF case: $\pi_1(\mathcal{S})$ is generated by 6 curves from 7-MA, 5 curves $E_2 - E_7, \cdots, E_6 - E_7$; and 5 curves $H - E_1 - E_7 - E_2, \cdots, H - E_1 - E_7 - E_6$.

  By the abelianization argument, the lower bound of $\pi_1(Symp_h(X,\omega))$ is $13 = 1+5+5+6-4$, where 4 is the rank of $\mathbb{F}_4$.

- 7-MAEF case: $\pi_1(\mathcal{S})$ is generated by 6 curves form 7-MA, 5 curves $E_2 - E_7, \cdots, E_6 - E_7$; and 5 curves $H - E_1 - E_7 - E_2, \cdots, H - E_1 - E_7 - E_6$; 4 curves $E_2 - E_6, \cdots, E_5 - E_6$, and 4 curves $H - E_1 - E_6 - E_2, \cdots, H - E_1 - E_6 - E_5$.

  By the abelianization argument, the lower bound of $\pi_1(Symp_h(X,\omega))$ is $18 = 1+5+5+6+4+4-7$, where 7 is the rank of $\mathbb{F}_4 \rtimes \mathbb{F}_3$.

- 7-MAE case: $\pi_1(\mathcal{S})$ is generated by the same set as in MAEF, except $E_6 - E_7$. Everything else is the same. Hence the lower bound is 17.

- 8-MAG case: $\pi_1(\mathcal{S})$ is generated by 7 curves from 8-MA, 6 curves $E_2 - E_8, \cdots, E_7 - E_8$; and 6 curves $H - E_1 - E_8 - E_2, \cdots, H - E_1 - E_8 - E_6$.

  By the abelianization argument, the lower bound of $\pi_1(Symp_h(X,\omega))$ is $15 = 1+7+6+6-5$, where 5 is the rank of $\mathbb{F}_5$.

- 8-MAE case:

- 8-MAFG case: $\pi_1(\mathcal{S})$ is generated by 7 curves from 8-MA, 6 curves $E_2 - E_8, \cdots, E_7 - E_8$; and 6 curves $H - E_1 - E_8 - E_2, \cdots, H - E_1 - E_8 - E_6$; 5 of $E_i - E_7$ and 5 of $H - E_1 - E_7 - E_i$, where $2 \leq i \leq 6$.

  By the abelianization argument, the lower bound of $\pi_1(Symp_h(X,\omega))$ is $21 = 1+7+6+6+5+5-9$, where 9 is the rank of $\mathbb{F}_5 \rtimes \mathbb{F}_4$.

- 8-MAF case: one curve $(E_7 - E_8)$ less than 8-MAFG, and the lower bound is 20.

- 8-MAEFG case: besides 8-MAFG case, $\pi_1(\mathcal{S})$ is generated by 4 curves of $E_i - E_6$ and 4 of $H - E_1 - E_6 - E_i$, where $2 \leq i \leq 5$.

  By the abelianization argument, the lower bound of $\pi_1(Symp_h(X,\omega))$ is $21 = 1+7+6+6+5+5+4+4-12$, where 12 is the rank of $\mathbb{F}_5 \rtimes \mathbb{F}_4 \rtimes \mathbb{F}_3$.

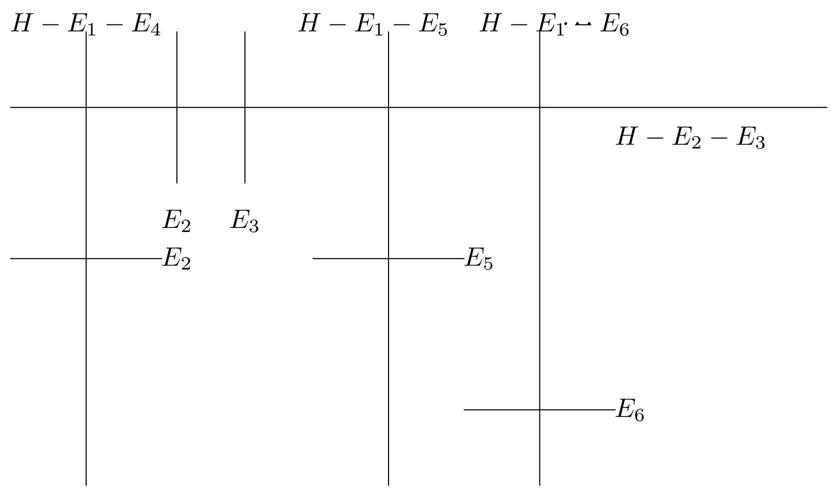
- 8-MAEG case: one curve $(E_6 - E_7)$ less than 8-MAFG, and the lower bound is 25.

- 8-MAEF case: one curve $(E_7 - E_8)$ less than 8-MAFG, and the lower bound is 25.

- 8-MAE case: 3 curve $(E_6 - E_7, E_6 - E_8, E_7 - E_8)$ less than 8-MAFG, and lower bound is 23.

□

7.1.2. *MO,MB, MC, MD or* $MXL_{i_1}\cdots L_{i_k}$ *cases cases.* We start with 6-MO. And we'll use the following configuration as in 6-MA.



But we will do a Cremona transformation by $H - E_3 - E_4 - E_5$.

Now the new form has

$$h = 2H - E_3 - E_4 - E_5, \quad e_4 = E_1, \quad e_5 = E_2 \quad e_6 = E_6,$$
$$e_1 = H - E_4 - E_5, \quad e_2 = H - E_3 - E_5, \quad e_3 = H - E_3 - E_4.$$

We also have $2h - e_1 - \cdots - e_5 = H - E_1 - E_2$.

And it's easy to check that if $\omega(e_1) > \omega(e_4) + \omega(e_5) + \omega(e_6)$, then the complement is convex, and one can show from the diagram 21 that $Symp_h$ is connected.

And now $\pi_1(\mathcal{S})$ is generated by the following 10 curves:

$e_1 - e_4, e_1 - e_5, e_1 - e_6$; and $h - e_2 - e_4 - e_5, h - e_2 - e_4 - e_6, h - e_2 - e_5 - e_6, h - e_3 - e_4 - e_5, h - e_3 - e_4 - e_6, h - e_3 - e_5 - e_6, h - e_4 - e_5 - e_6$.

Then from the sequence

$$\mathbb{Z} \to \pi_1(Symp_h(X,\omega)) \to \pi_1(\mathcal{S}) \xrightarrow{f} \pi_0(Stab(C)) \xrightarrow{g} \pi_0(Symp_h) = 1. \tag{48}$$

we have rank of $\pi_1(Symp_h(X,\omega))$ is no less than 1+10=11, which equals the upper bound given by 7.1.

On the other hand, using the blow-up stability, we can compute 6 ME case here:

First let $(\tilde{M}, \tilde{\omega})$ be a blow up of (M,ω) at a ball with very small size, then we have

$Symp(\tilde{M}, \tilde{\omega}) \cong Symp(M; B^4)^{U(2)} \to Symp(M,\omega) \to M.$

Here "very small" size means that the exceptional curve never degenerates in the blowup.

For the proof, let us first recall the Theorem 2.5 in [25]: Let M be a symplectic 4-manifold, and let c be a positive real number. Suppose that
(1)the space of symplectic embeddings of the standard closed ball of capacity c is nonempty and connected, and
(2) the exceptional curve that one gets by blowing up an arbitrary ball if size c cannot degenerate in $\tilde{M}$.

Then there is a homotopy fibration

$$Symp(\tilde{M}_c) \to Symp(M,\omega) \to Emb_\omega(B_4(c), M).$$

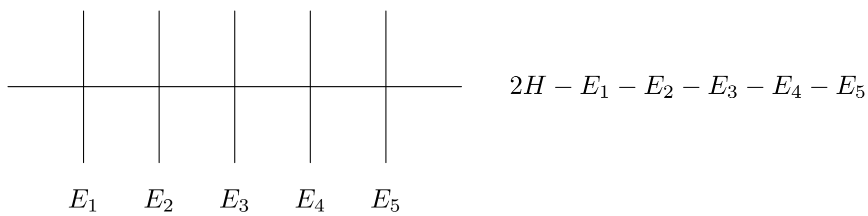
Also, if the exceptional curve never degenerates in the blowup, then $Symp(\tilde{M}_c)$ is homotopy equivalent to $Symp_p(M,\omega)$ which is the group of symplectomorphisms that fix a point p in (M,ω). cf. [25] Proposition 4.21, [8] Proposition 1.2, and [6] Lemma 3.5.

Now the LES is

$$\pi_2(M) \to \pi_1(Symp(\tilde{M},\tilde{\omega})) \to \pi_1(Symp(M,\omega)) \to \pi_1(M) \to \pi_0(Symp(\tilde{M},\tilde{\omega})) \to \pi_0(Symp(M,\omega)) \to 1.$$

Then we have $\pi_0(Symp(\tilde{M},\tilde{\omega})) \cong P_5(S^2)/\mathbb{Z}_2$, and $\pi_1(Symp(\tilde{M},\tilde{\omega})) \cong \pi_2(M) = \mathbb{Z}^6$.

7.1.3. *ME or $MEL_{i_1}\cdots L_{i_k}$ cases.* We start with the ME on a 6-point blowup here. Consider the following configuration together with E_6.



The complement $\mathbb{T}^*\mathbb{R}P^2$ has $Symp_c$ weakly homotopic to $\mathbb{Z}$. The homotopy LES's of the diagram will have an S^1 in the π_1 level of the fiber, and every higher π_i being trivial.

For the space of configuration, its fundamental group is generated by 5 curves $E_1 - E_6, \cdots, E_5 - E_6, 2H - E_1 - \cdots - E_5 - E_6$. Then at least, the fundamental group has rank 6.

On the other hand, by blowing down to the monotone 5-point blowup, the upper bound of the rank of the fundamental group is 6. Hence the rank is precisely 6.

For more points blow up, we will consider the above configuration together with $E_6, E_7, \cdots, E_n$. And a similar computation will give us that the rank is precisely the number of symplectic classes among $E_1 - E_i, \cdots, E_5 - E_i, 2H - E_1 - \cdots - E_5 - E_i, E_i - E_j$, where $i, j \geq 6$.

Combining the above discussion, we have the following theorem:

Proposition 7.6. *For a rational surface with and the Lagrangian system is of type $\mathbb{A}$ or $\mathbb{D}$ in the K-nef symplectic cone,*

$$Q = PR[\pi_0(Symp_h(X,\omega))] + Rank[\pi_1(Symp(X,\omega))] - rank[\pi_0(Symp_h(X,\omega))]$$

is a constant only depending on the topology. In particular, $Q = 1 + 2 + \cdots n$ for the n points blow up of $\mathbb{C}P^2$.

7.2. **Isotopy of Symplectic Spheres.**

Lemma 7.7. *For any symplectic -2 sphere classes, homologous symplectic -2 spheres are isotopic.*

Proof. The proof is similar to that of Lemma 7.5.

First, note that it suffices to prove any (-2) curve which pairs negative with some classes in Figure 1. Because the transitivity of homology action grants that (-2) symplectic sphere classes have the same number of isotopy class.

Argue by contradiction. Assume there are $k > 1$ isotopic classes for one of those classes. We already proved the MA case, and for the other cases, we showed that the upper bound of rank $\pi_1 Ham(X,\omega)$ equals the lower bound, which is given by counting the (-2) symplectic classes.

If such classes have isotopy classes $k > 1$, there will be immediately a contradiction where the lower bound is greater than the upper bound. □

Proposition 7.8. *The symplectic spheres in the class $2H - E_1 - \cdots E_n$ of $\mathbb{C}P^2\#n\overline{\mathbb{C}P^2}$ are isotopic.*

Proof. First we prove $Symp_h$ acts transitively on space of curve in class $2H - E_1 - \cdots - E_n$:

Lemma 7.9. *For a rational manifold $M = S^2 \times S^2\#n\overline{\mathbb{C}P^2}$, the group $Symp_h$ acts transitively on the space of symplectic spheres in class $Q = 2H - E_1 - \cdots - E_n$.*

Proof. Denote a symplectic sphere S_i in the homology class $[S_i] = 2H-E_1-\cdots-E_n$. For each pair (M, S_i), by [36], there is a set $\mathcal{C}_i$ of disjoint (-1) symplectic spheres C_i^l for $l = 1, \cdots, n-1$ such that

$$[C_i^l] = H - E_1 - E_l, \text{for} \quad l = 2, \cdots n.$$

Blowing down the set $\{C_i^1, \cdots C_i^{n-1}\}$ separately, results in $(M_i, \tilde{S}_i, \mathcal{B}_i)$ where M_i is a symplectic $S^2 \times S^2$ (this is because 10 is a even number) with a symplectic form admitting from the original symplectic form of M, $\tilde{S}_i$ a symplectic sphere in M_i(homology class can be computed using the projection formula), and $\mathcal{B}_i = \{B_i^2, \cdots, B_i^{n-1}\}$ is a symplectic ball packing in $M_i \setminus \tilde{S}_i$ corresponding to $\mathcal{C}_i$. For any two pairs, since the symplectic forms are homologous, by [24], there is a symplectomorphism Φ from $(M_1, \tilde{S}_1)$ to $(M_2, \tilde{S}_2)$, such that for fixed l, $Vol(\Phi(B_1^l)) = Vol(B_2^l)$. Then according to [3], we can choose this Φ such that the two symplectic spheres are isotopic, i.e. $\Phi(\tilde{S_1}) = \tilde{S}_2$. Then apply Theorem 1.1 in [9], there is a compactly supported Hamiltonian isotopy ι of $(M_2, \tilde{S}_2)$ such that the symplectic ball packing $\Phi(\mathcal{B}_1))$ and $\mathcal{B}_2$ is connected by ι in $(M_2, \tilde{S}_2)$. Then $\iota \circ \Phi$ is a symplectomorphism between the tuples $(M_i, \tilde{S}_i, \mathcal{B}_i)$ and hence blowing up induces a symplectomorphism $\psi : (M_1, \tilde{S}_1, \mathcal{B}_1) \to (M_2, \tilde{S}_2, \mathcal{B}_2)$. Further note that ψ preserve homology classes and hence $\psi \in Symp_h(M)$. □

Then the proposition follows from the connectedness of $Symp_h$ when the class $2H - E_1 - \cdots - E_n$ is symplectic.

□

Appendix A. A proof that ball swapping= Lagrangian Dehn twists

We show that for type $\mathbb{D}$ forms appears in theorem 1.2, the SMCG is generated by squared Dehn twist along Lagrangian spheres, and there is a canonical way to realize the generators as ball swapping. The strategy is first to deal with small ball sizes so that there is an isotopy between ball swapping and Lagrangian Dehn twists, and then show that the inflation process preserves this isotopy.

Firstly, in Figure 5.8, when the ball sizes are very small (this means we take a point in the symplectic cone that is close enough to the vertex A), two balls have a packing in a larger Euclidean ball(a Darboux chart). Then we have an isotopy between ball swapping and Lagrangian Dehn twists.

Then let's deal with the larger size case. We will denote the symplectic form class in the previous case by u and the latter case by u'.

Notice that by [40], Lagrangian Dehn twists can be realized as an image under a connecting homomorphism of a nontrivial loop in the space of symplectic form on the Weinstein neighborhood of the Lagrangian sphere to $\pi_0 Symp$. Hence entending the family of form to the closed manifold, we have the following fibration sequence, where the connecting map in the LES gives the above interpretation of Dehn twists:

$$Symp(M, \omega) \to \mathrm{Diff}_0(M) \to \mathcal{S}_u.$$

On the other hand, ball swapping can be realized as an image of a nontrivial loop in the space of on the blow-down to $\pi_0 Symp$. Hence we have the following fibration sequence, where the connecting map in the LES gives the above interpretation of ball swapping:

$$Symp(M, \omega) \to Symp(\mathbb{C}P^2, \omega_{FS}) \to \mathcal{B}_u.$$

Joining the LSE of the two fibrations together, we have the following diagram:

$$\begin{tikzcd}
& \pi_1(Symp(\mathbb{C}P^2)) \arrow[r, "t'"] \arrow[dl, "t"'] & \pi_1(\mathcal{B}(u')) \arrow[rr, "t'"] \arrow[dl] \arrow[dd] & & \pi_0(Symp(u')) \arrow[r, "t'"] \arrow[dl] \arrow[dd] & 0 \\
\pi_1(\mathrm{Diff}(X_k)) \arrow[r] & \pi_1(\mathcal{S}(u')) \arrow[rr, "t'"] \arrow[dd] & & \pi_0(Symp(u')) \arrow[r, "t'"] \arrow[dd] & 0 & \\
& \pi_1(Symp(\mathbb{C}P^2)) \arrow[r] \arrow[dl] & \pi_1(\mathcal{B}(u)) \arrow[rr, "t'"] \arrow[dl] & & \pi_0(Symp(u')) \arrow[r, "t'"] \arrow[dl] & 0 \\
\pi_1(\mathrm{Diff}(X_k)) \arrow[r] & \pi_1(\mathcal{S}(u)) \arrow[rr, "t'"] & & \pi_0(Symp(u)) \arrow[r, "t'"] & 0 &
\end{tikzcd}$$

When the ball sizes are small, we have an isomorphism between the right ends of the two corresponding long exact sequences

And after inflation, for u', by 5-Lemma, we still have an isomorphism, hence the isotopy between ball swapping and Lagrangian Dehn twists.

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