1 Metric Spaces

Definition 1.1. (Metric; Metric space.) Let X be a set. A *metric* on X is a function

$$d : X \times X \longrightarrow \mathbb{R}$$

satisfying the following conditions.

(M1) **(Positivity).** $d(x, y) \geq 0$ for all $x, y \in X$, and $d(x, y) = 0$ if and only if $x = y$.

(M2) **(Symmetry).** $d(x, y) = d(y, x)$ for all $x, y \in X$.

(M3) **(Triangle inequality).** $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$.

The value $d(x, y)$ is sometimes called the *distance from x to y*.

A set X endowed with a metric d is called a *metric space*, and is denoted (X, d) (or simply X when the metric is clear from context).

Theorem 1.2. (The Euclidean Metric). *Define*

$$d : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

as follows. For $\overline{x} = (x_1, \ldots, x_n)$ *and* $\overline{y} = (y_1, \ldots, y_n)$*, let*

$$\begin{aligned} d(\overline{x}, \overline{y}) &= ||\overline{x} - \overline{y}|| \\ &= \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}. \end{aligned}$$

Then d is a metric, called the Euclidean metric*, and makes* $(\mathbb{R}^n, d)$ *into a metric space.*

Proof. We need to verify that d satisfies the three conditions that define a metric.

Step 1. Verify that d satisfies condition (M1).

Step 2. Verify that d satisfies condition (M2).

Step 3. Explain why, to verify (M3), it's enough to check that

$$(d(\overline{x}, \overline{y}) + d(\overline{y}, \overline{z}))^2 \geq d(\overline{x}, \overline{z})^2$$

Step 4. Expand $(d(\overline{x}, \overline{y}) + d(\overline{y}, \overline{z}))^2 = (||\overline{x} - \overline{y}|| + ||\overline{y} - \overline{z}||)^2$.

Step 5. Expand

$$\begin{aligned} d(\overline{x}, \overline{z})^2 &= (\overline{x} - \overline{z}) \cdot (\overline{x} - \overline{z}) \\ &= \Big((\overline{x} - \overline{y}) + (\overline{y} - \overline{z}) \Big) \cdot \Big((\overline{x} - \overline{y}) + (\overline{y} - \overline{z}) \Big) \end{aligned}$$

Step 6. Conclude that d satisfies (M3).

$\square$

In-class Exercises

1. Determine whether the following functions define metrics on the corresponding sets. Rigorously justify your answers!

   (a) Let $X = \mathbb{R}$. Define

   $$\begin{aligned} d : \mathbb{R} \times \mathbb{R} &\longrightarrow \mathbb{R} \\ d(x, y) &= (x - y)^2. \end{aligned}$$

   (b) Let $X = \mathbb{R}^2$. Define

   $$\begin{aligned} d : \mathbb{R}^2 \times \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ d(\overline{x}, \overline{y}) &= |x_1 - y_1| + |x_2 - y_2|. \end{aligned}$$

   (c) Let X be any set. Define

   $$\begin{aligned} d : X \times X &\longrightarrow \mathbb{R} \\ d(x, y) &= \begin{cases} 0 & x = y \\ 1 & x \neq y. \end{cases} \end{aligned}$$

2. Let (X, d) be a metric space, and let $Y \subseteq X$ be a subset. Show that the restriction $d|_{Y \times Y}$ of d to $Y \times Y \subseteq X \times X$ defines a metric on Y. Conclude that any subset of a metric space inherits a metric space structure.

3. **(Optional)** Let $a < b \in \mathbb{R}$. Let $\mathcal{C}(a, b)$ denote the set of continuous functions from the closed interval $[a, b]$ to $\mathbb{R}$. Verify whether each of the following functions defines a metric on the set $\mathcal{C}(a, b)$. Be sure to clearly state which properties of continuous functions and integration you are using!

   (a)
   $$\begin{aligned} d_1 : \mathcal{C}(a, b) \times \mathcal{C}(a, b) &\longrightarrow \mathbb{R} \\ d(f, g) &= \int_a^b |f(x) - g(x)| \, dx \end{aligned}$$

   (b)
   $$\begin{aligned} d_\infty : \mathcal{C}(a, b) \times \mathcal{C}(a, b) &\longrightarrow \mathbb{R} \\ d(f, g) &= \sup_{x \in [a,b]} |f(x) - g(x)| \end{aligned}$$

4. **(Optional)** Let (X, d) be a metric space. Which of the following functions $\widetilde{d} : X \times X \to \mathbb{R}$ defines a new metric space structure on X?

   (a) For any $x, y \in X$, $\widetilde{d}(x, y) = c\Big(d(x, y)\Big)$ for $c \in \mathbb{R}, c > 0$.

   (b) For any $x, y \in X$, $\widetilde{d}(x, y) = \Big(d(x, y)\Big)^2$.

   (c) For any $x, y \in X$, $\widetilde{d}(x, y) = \min(d(x, y), 1)$.

   (d) For any $x, y \in X$, $\widetilde{d}(x, y) = \max(d(x, y), 1)$.

   (e) For any $x, y \in X$, $\widetilde{d}(x, y) = \dfrac{d(x, y)}{1 + d(x, y)}$.