Wiener index, Harary index and graph properties ☆

Lihua Feng[a], Xiaomin Zhu[a], Weijun Liu[a,b]

[a]*School of Mathematics and Statistics, Central South University, Changsha, Hunan, 410083, China.*
[b]*School of Science, Nantong University, Nantong, Jiangsu, 226019, China.*

Abstract

Finding sufficient conditions for some properties of graphs in light of quantitative methods is an important problem. In this paper, in terms of the Wiener index or Harary index, we present several sufficient conditions for a graph to be k-connected, β-deficient, k-hamiltonian, k-path-coverable or k-edge-hamiltonian.

Keywords: Wiener index, Harary index, Degree sequence, Graph properties.
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1. Introduction

Let $G = (V(G), E(G))$ be a connected graph with $|V(G)| = n$ and $|E(G)| = m$. We use $d(v)$ to denote the degree of a vertex v in G, $\delta(G)$ to denote the minimum degree of G. Let $d(u, v)$ denote the distance between two vertices u and v in G, i.e., the length of a shortest path connecting u and v in G.

The topological indices (also known as the molecular descriptors) had been received much attention in the past decades, and they have been found to be useful in structure-activity relationships (SAR) and pharmaceutical drug design in organic chemistry (see [15, 16, 32]). Many researchers also were devoted to study their graphical properties. Indeed, the topological index of a graph G can be viewed as a graph invariant under the isomorphism of graphs, that is, for some topological index TI, $TI(G) = TI(H)$ if $G \cong H$. Therefore, the results in this paper can also be seen as a topic in extremal graph theory.

One of the most thoroughly studied topological indices was the Wiener index which was proposed by Wiener in 1947 [34]. This index has been shown to possess close relation with the graph distance, which is an important concept in pure graph theory. It is also well correlated with many physical and chemical properties of a variety of classes of chemical compounds. Up to now, there is a tremendous amount of literature on the study of Wiener index and its modifications, see for example [7, 10, 11, 20, 23, 24, 31, 29]. x The *Wiener index* of a graph G, denoted by $W(G)$, is defined as

$$W(G) = \sum_{\{u,v\} \subseteq V(G)} d_G(u, v).$$

If we write $D(v) = \sum_{u \in V(G)} d(u, v)$, then the Wiener index can be rewritten as

$$W(G) = \frac{1}{2} \sum_{v \in V(G)} D(v).$$

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From the above expression, it can be easily verified that

$$D(v) \geq d(v) + 2(n-1-d(v)).$$

In 1993, Plavšić et al. [15, 30] introduced the reciprocal version of the Wiener index, which is now called the Harary index. The Harary index is also extensively studied for various graph classes, such as graphs with fixed matching number [9], graphs with fixed connectivity [25], trees with various parameters [14, 33], bicyclic graphs [36], graphs with minimal Harary index [8]. For other related results, see [28, 30]. For a connected graph G, its *Harary index*, denoted by $H(G)$, is defined as

$$H(G) = \sum_{\{u,v\} \subseteq V(G)} \frac{1}{d(u,v)}.$$

If we write $\widehat{D}(v) = \sum_{u \in V(G)} \frac{1}{d(u,v)}$, then we can rewrite $H(G)$ as

$$H(G) = \frac{1}{2} \sum_{v \in V(G)} \widehat{D}(v).$$

From this expression, it can be easily verified that

$$\widehat{D}(v) \leq d(v) + \frac{1}{2}(n-1-d(v)).$$

For many graph invariants, their computational complexity is usually NP-complete. Thus, finding sufficient conditions for graphs possessing certain properties becomes meaningful in graph theory. For such topic, there are dozens of existing results, the most famous one is the so called the Dirac-type condition. For example, it is known that [2, page 4], if G is a simple graph of order $n \geq k+1$, and if its minimum degree $\delta(G) \geq \frac{1}{2}(n+k-2)$, then G is k-connected. In [5], it is obtained that for a graph G, if $\delta(G) \geq \frac{n+k}{2}$, then G is k-hamiltonian. In [4, page 15], it states that for a connected graph G, if $\delta(G) \geq n-\beta-1$, then G contains a cycle of length at least $n-\beta$, and hence G has a matching of size at least $\frac{n-\beta}{2}$. However, there are only few such conditions in terms of the topological indices. As the Wiener and Harary indices are the two most popular and concise indices, several scholars tried to use them to do so. There are two reasons for using these two indices from mathematical and application-oriented aspects from my point of view. One reason is that investigating "distances in graphs" plays an important role in graph theory. The other reason is that these two indices have many applications in other disciplines such as organic chemistry and biology. A famous example is the gene networks, where the genes exchange information based on the shortest path connecting the two genes. Indeed, this is an example but triggers the hypothesis that distance-based quantities could be crucial for investigating those system. Along these lines, in [12], Hua and Wang gave a sufficient condition for a graph to be traceable by using the Harary index. In [35], Yang presented a sufficient condition for a graph to be traceable by using the Wiener index. The above results are further extended by Liu et al. [26, 27]. Li [21, 22] presented sufficient conditions in terms of the Harary index and Wiener index for a graph to be hamiltonian or hamilton-connected.

Our goal in this paper is, by utilizing the Wiener index, Harary index and the degree conditions, to derive some sufficient conditions for a wide variety of graph properties including k-connected, β-deficient, k-hamiltonian, k-path-coverable or k-edge-hamiltonian. These graph properties are also the concerns of plenty of graph theorists. So our results may be considered as new viewpoints for existing results.

2. Preliminaries

We first present some graph notations and terminologies.

Let K_n, S_n, P_n be the complete graph, the star and the path on n vertices, respectively. For two vertex-disjoint graphs G and H, we use $G \vee H$ to denote the join of G and H; $G \cup H$ to denote their union.

A connected graph G is said to be *k-connected* (or k-vertex connected) if it has more than k vertices and remains connected whenever fewer than k vertices are removed.

The *deficiency* of a graph G, denoted by def(G), is the number of vertices unmatched under a maximum matching in G. In particular, G has a 1-factor if and only if def(G)=0. We call G β-deficient if def$(G) \leq \beta$. Thus a β-deficient graph G of order n has matching number $\frac{n-\beta}{2}$.

A graph G is *k-hamiltonian* if for all $|X| \leq k$, the subgraph induced by $V(G) \setminus X$ is hamiltonian. Thus 0-hamiltonian is the same as *hamiltonian*.

A graph is *traceable* if it contains a hamiltonian path. More generally, G is *k-path-coverable* if $V(G)$ can be covered by k or fewer vertex-disjoint paths. In particular, 1-path-coverable is the same as *traceable*.

A graph G is *k-edge-hamiltonian* if any collection of vertex-disjoint paths with at most k edges altogether belong to a hamiltonian cycle in G.

We use $\alpha(G)$ to denote the independence number of a graph G.

An integer sequence $\pi = (d_1 \leq d_2 \leq \cdots \leq d_n)$ is called *graphical* if there exists a graph G having π as its vertex degree sequence; in that case, G is called a *realization* of π. If P is a graph property, such as hamiltonian or k-connected, we call a graphical sequence π *forcibly* P if every realization of π has property P.

We next give some lemmas that will be used later.

Lemma 2.1. *[3] Let G be a graph of order $n \geq 4$ with degree sequence $\pi = (d_1 \leq \cdots \leq d_n)$. If*

$$d_i \leq i + k - 2 \Rightarrow d_{n-k+1} \geq n - i, \text{ for } 1 \leq i \leq \frac{1}{2}(n - k + 1),$$

then G is forcibly k-connected.

Lemma 2.2. *[18] Let $\pi = (d_1 \leq \cdots \leq d_n)$ be a graphical sequence, and let $0 \leq \beta \leq n$ with $n \equiv \beta \pmod 2$. If*

$$d_{i+1} \leq i - \beta \Rightarrow d_{n+\beta-i} \geq n - i - 1, \text{ for } 1 \leq i \leq \frac{1}{2}(n + \beta - 2),$$

then π is forcibly β-deficient.

Lemma 2.3. *[6] Let $\pi = (d_1 \leq \cdots \leq d_n)$ be a graphical sequence and $0 \leq k \leq n-3$. If*

$$d_i \leq i + k \Rightarrow d_{n-i-k} \geq n - i, \text{ for } 1 \leq i < \frac{1}{2}(n - k),$$

then π is forcibly k-hamiltonian.

Lemma 2.4. *[3, 19] Let $\pi = (d_1 \leq d_2 \leq \cdots \leq d_n)$ be a graphical sequence and $k \geq 1$. If*

$$d_{i+k} \leq i \Rightarrow d_{n-i} \geq n - i - k, \text{ for } 1 \leq i < \frac{1}{2}(n - k),$$

then π is forcibly k-path-coverable.

Lemma 2.5. *[17] Let $\pi = (d_1 \le d_2 \le \cdots \le d_n)$ be a graphical sequence and $0 \le k \le n-3$. If*

$$d_{i-k} \le i \Rightarrow d_{n-i} \ge n-i+k, \text{ for } k+1 \le i < \frac{1}{2}(n+k),$$

then π is forcibly k-edge-hamiltonian.

Lemma 2.6. *[1] Let $\pi = (d_1 \le d_2 \le \cdots \le d_n)$ be a graphical sequence and $k \ge 1$. If*

$$d_{k+1} \ge n-k,$$

then π is forcibly $\alpha(G) \le k$.

3. Main Results

Theorem 3.1. *Let G be a connected graph of order $n \ge k+1$.*

(1) If $W(G) \le \frac{1}{2}n(n+1)-k$, then G is k-connected, unless $G = K_{k-1} \vee (K_1 \cup K_{n-k})$.

(2) If $H(G) \ge \frac{1}{2}n(n-2)+\frac{1}{2}k$, then G is k-connected, unless $G = K_{k-1} \vee (K_1 \cup K_{n-k})$.

Proof. Suppose that G is not k-connected, then from Lemma 2.1, there exists an integer $1 \le i \le \frac{n-k+1}{2}$ such that $d_i \le i+k-2$ and $d_{n-k+1} \le n-i-1$. Obviously, $1 \le k \le n-1$.

(1) We first consider $W(G)$. From the definition, we have

$$\begin{aligned} W(G) &= \frac{1}{2}\sum_{v\in V(G)} D(v) \ge \frac{1}{2}\sum_{v\in V(G)} (d(v)+2(n-1-d(v))) \\ &= \frac{1}{2}\sum_{v\in V(G)} (2(n-1)-d(v)) = n(n-1)-\frac{1}{2}\sum_{v\in V(G)} d(v) \\ &\ge n(n-1)-\frac{1}{2}\left[i(i+k-2)+(n-k-i+1)(n-i-1)+(k-1)(n-1)\right] \\ &= \frac{1}{2}n(n-1)-\left[i^2-(n-k+1)i\right]. \end{aligned}$$

Suppose $f(x) = x^2-(n-k+1)x$ with $1 \le x \le \frac{n-k+1}{2}$, then $f(x) \le f(1) = k-n$. Thus

$$W(G) \ge \frac{1}{2}n(n-1)-f(1) = \frac{1}{2}n(n+1)-k,$$

so we get the result.

If $W(G) = \frac{1}{2}n(n+1)-k$, then all the inequalities in the proof should be equalities, so $i=1$, and hence $d_1 = k-1, d_2 = \cdots = d_{n-k+1} = n-2, d_{n-k+2} = \cdots = d_n = n-1$. Thus $G = (K_1 \cup K_{n-k}) \vee K_{k-1}$, which is not k-connected as stated in [1].

(2) We consider $H(G)$. From the definition, we have

$$\begin{aligned} H(G) &= \frac{1}{2}\sum_{v\in V(G)} \widehat{D}(v) \le \frac{1}{2}\sum_{v\in V(G)} (d(v)+\frac{1}{2}(n-1-d(v))) \\ &= \frac{1}{4}n(n-1)+\frac{1}{4}\sum_{v\in V(G)} d(v) \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{4}n(n-1) + \frac{1}{4}\left[i(i+k-2) + (n-k-i+1)(n-i-1) + (k-1)(n-1)\right] \\
&= \frac{1}{2}n(n-1) + \frac{1}{2}\left[i^2 - (n-k+1)i\right].
\end{aligned}$$

As before, we obtain $H(G) \leq \frac{1}{2}n(n-1) + \frac{1}{2}(k-n) = \frac{1}{2}n(n-2) + \frac{1}{2}k$, which contradicts to the assumption, so we get the result.

If $H(G) = \frac{1}{2}n(n-2) + \frac{1}{2}k$, then $i = 1$, the remaining is as in the previous proof. ■

Theorem 3.2. *Let G be a connected graph of order n with $n \equiv \beta \pmod 2$, $0 \leq \beta \leq n$ and $n \geq 10$.*

(1) If $W(G) \leq \frac{1}{2}(n-2)(n+5) + 2\beta$, then G is β-deficient, unless $G = K_1 \vee (2K_1 \cup K_{n-3})$.

(2) If $H(G) \geq \frac{1}{2}(n^2 - 3n + 5) - \beta$, then G is β-deficient, unless $G = K_1 \vee (2K_1 \cup K_{n-3})$.

Proof. Suppose that G is not β-deficient, then from Lemma 2.2, there exists an integer $1 \leq i \leq \frac{1}{2}(n+\beta-2)$ such that $d_{i+1} \leq i - \beta$ and $d_{n+\beta-i} \leq n-i-2$.

(1) We consider $W(G)$, as in Theorem 3.1, we have

$$\begin{aligned}
W(G) &\geq n(n-1) - \frac{1}{2}\sum_{v \in V(G)} d(v) \\
&\geq n(n-1) - \frac{1}{2}\left[(i+1)(i-\beta) + (n+\beta-2i-1)(n-i-2) + (i-\beta)(n-1)\right] \\
&= \frac{(n-1)(n+2)}{2} + \beta - \frac{1}{2}\left[3i^2 - (2n+2\beta-5)i\right].
\end{aligned}$$

Suppose $f(x) = 3x^2 - (2n+2\beta-5)x$ with $1 \leq x \leq \frac{1}{2}(n+\beta-2)$. It is easy to see that $f_{\max}(x) = \max\{f(1), f(\frac{1}{2}(n+\beta-2)\}$. As $f(1) = 8-2n-2\beta$, $f(\frac{1}{2}(n+\beta-2)) = -\frac{1}{4}(n+\beta-4)(n+\beta-2)$, then we have

$$f(\frac{1}{2}(n+\beta-2) - f(1) = -\frac{1}{4}(n+\beta-10)(n+\beta-4) < 0,$$

so it follows that $f_{\max}(x) = f(1)$. Thus

$$W(G) \geq \frac{(n-1)(n+2)}{2} + \beta - \frac{1}{2}(8-2n-2\beta) = \frac{1}{2}(n-2)(n+5) + 2\beta,$$

so we get the result.

If $W(G) = \frac{1}{2}(n-2)(n+5) + 2\beta$, then $i = 1$. Since $i - \beta \geq 0$, so $\beta = 0$ or 1. If $\beta = 0$, then $d_1 = d_2 = 1$, $d_3 = \cdots = d_{n-1} = n-3$, $d_n = n-1$. Thus $G = K_1 \vee (2K_1 \cup K_{n-3})$, which is not β-deficient as stated in [1]. If $\beta = 1$, then there exist vertices with degree 0 and the extremal graph would be disconnected.

(2) We consider $H(G)$, as in Theorem 3.1, we have

$$\begin{aligned}
H(G) &\leq \frac{1}{4}n(n-1) + \frac{1}{4}\left[(i+1)(i-\beta) + (n+\beta-2i-1)(n-i-2) + (i-\beta)(n-1)\right] \\
&= \frac{1}{2}(n-1)^2 - \frac{1}{2}\beta + \frac{1}{4}\left[3i^2 - (2n+2\beta-5)i\right].
\end{aligned}$$

As in the previous case, $3i^2 - (2n+2\beta-5)i \leq 8 - 2n - 2\beta$. Thus $H(G) \leq \frac{1}{2}(n^2-3n+5) - \beta$, which contradicts to the assumption, so we get the result.

If $H(G) = \frac{1}{2}(n^2-3n+5) - \beta$, then $i = 1$, and the remaining is as the previous proof. ■

Theorem 3.3. *Let G be a connected graph of order $n \geq 3$ and $0 \leq k \leq n-3$.*

(1) (i) For $k = n-4$, if $W(G) \leq \frac{1}{2}n(n-1)+2$, then G is k-hamiltonian, unless $G = K_{k+1} \vee (K_1 \cup K_2)$.

(ii) For $k = n-3$ or $n-5$, if $W(G) \leq \frac{1}{2}n(n-1)+\frac{1}{8}(n-k+1)(n-k-1)$, then G is k-hamiltonian, unless $G = K_{k+1} \vee (2K_1)$ (if $n = k+3$) or $G = K_{k+2} \vee (3K_1)$ (if $n = k+5$).

(iii) For $k \leq n-6$, if $W(G) \leq \frac{1}{2}n(n+1)-k-2$, then G is k-hamiltonian, unless $G = K_{k+1} \vee (K_1 \cup K_{n-k-2})$.

(2) (i) For $k = n-4$, if $H(G) \geq \frac{1}{2}n(n-1)-1$, then G is k-hamiltonian, unless $G = K_{k+1} \vee (K_1 \cup K_2)$.

(ii) For $k = n-3$ or $n-5$, if $H(G) \geq \frac{1}{2}n(n-1)-\frac{1}{16}(n-k+1)(n-k-1)$, then G is k-hamiltonian, unless $G = K_{k+1} \vee (2K_1)$ (if $n = k+3$) or $G = K_{k+2} \vee (3K_1)$ (if $n = k+5$).

(iii) For $k \leq n-6$, if $H(G) \geq \frac{1}{2}n(n-2)+\frac{1}{2}k+1$, then G is k-hamiltonian, unless $G = K_{k+1} \vee (K_1 \cup K_{n-k-2})$.

Proof. Suppose that G is not k-hamiltonian, then from Lemma 2.3, there exists an integer $1 \leq i < \frac{1}{2}(n-k)$ such that $d_i \leq i+k$ and $d_{n-i-k} \leq n-i-1$

(1) As in Theorem 3.1, we have

$$\begin{aligned} W(G) &\geq n(n-1) - \frac{1}{2}\sum_{v \in V(G)} d(v) \\ &\geq n(n-1) - \frac{1}{2}\left[i(i+k) + (n-2i-k)(n-i-1) + (i+k)(n-1)\right] \\ &= \frac{1}{2}n(n-1) - \frac{1}{2}\left[3i^2 - (2n-2k-1)i\right]. \end{aligned}$$

Suppose $f(x) = 3x^2 - (2n-2k-1)x$ with $1 \leq x \leq \frac{1}{2}(n-k-1)$, $0 \leq k \leq n-3$. Since x is an integer, then we have to consider $n-k-1$ is odd or even.

Case 1. If $n-k-1$ is odd, then $1 \leq i \leq \frac{1}{2}(n-k-2)$, $f(\frac{1}{2}(n-k-2)) = -\frac{1}{4}(n-k+4)(n-k-2)$, hence

$$f(\frac{1}{2}(n-k-2)) - f(1) = -\frac{1}{4}(n-k-2)(n-k-4).$$

In this case, we consider two cases. Note that $n \geq k+3$ from assumption.

Subcase 1.1. If $n-4 \leq k \leq n-3$, i.e., $n-k-1 = 3$, then $f_{\max}(x) = f(\frac{1}{2}(n-k-2))$. Thus

$$\begin{aligned} W(G) &\geq \frac{1}{2}n(n-1) - \frac{1}{2}\left[-\frac{1}{4}(n-k+4)(n-k-2)\right] \\ &= \frac{1}{2}n(n-1) + \frac{1}{8}(n-k+4)(n-k-2) \\ &= \frac{1}{2}n(n-1) + 2, \end{aligned}$$

so we get the result.

If $W(G) = \frac{1}{2}n(n-1)+2$, then $i=1$, $d_1 = k+1$, $d_2 = d_3 = n-2$, $d_4 = d_5 = \cdots = d_n = n-1$, thus $G = K_{k+1} \vee (K_1 \cup K_2)$, which is not k-hamiltonian as stated in [1].

Subcase 1.2. If $k < n-5$, then $f_{\max}(x) = f(1)$. Thus

$$W(G) \geq \frac{1}{2}n(n-1) - \frac{1}{2}(-2n+2k+4) = \frac{1}{2}n(n+1) - k - 2,$$

so we get the result.

If $W(G) = \frac{1}{2}n(n+1) - k - 2$, then $i = 1$, $d_1 = 1+k$, $d_2 = \cdots = d_{n-k-1} = n-2$, $d_{n-k} = \cdots = d_n = n-1$, thus $G = K_{k+1} \vee (K_1 \cup K_{n-k-2})$, which is not k-hamiltonian as stated in [1].

Case 2. If $n-k-1$ is even, then $f(1) = -2n+2k+4$, $f(\frac{1}{2}(n-k-1)) = -\frac{1}{4}(n-k+1)(n-k-1)$, $f(\frac{1}{2}(n-k-1)) - f(1) = -\frac{1}{4}(n-k-3)(n-k-5)$. We also have two cases.

Subcase 2.1. If $n-5 \leq k \leq n-3$, i.e., $n-k-1 = 2$ or 4, then $f_{\max}(x) = f(\frac{1}{2}(n-k-1)) = f(1)$. Thus

$$\begin{aligned} W(G) &\geq \frac{1}{2}n(n-1) - \frac{1}{2}\left[-\frac{1}{4}(n-k+1)(n-k-1)\right] \\ &= \frac{1}{2}n(n-1) + \frac{1}{8}(n-k+1)(n-k-1), \end{aligned}$$

so we get the result.

If $W(G) = \frac{1}{2}n(n-1) + \frac{1}{8}(n-k+1)(n-k-1)$, then $i = \frac{1}{2}(n-k-1)$, $d_1 = d_2 = \cdots = d_{\frac{1}{2}(n-k-1)} = \frac{1}{2}(n+k-1)$, $d_{\frac{1}{2}(n-k+1)} = \frac{1}{2}(n+k-1)$, $d_{\frac{1}{2}(n-k+3)} = \cdots = d_n = n-1$, thus $G = K_{\frac{1}{2}(n+k-1)} \vee (\overline{K_{\frac{1}{2}(n-k-1)}} \cup K_1)$, which is $G = K_{k+1} \vee (2K_1)$ (if $n = k+3$) or $G = K_{k+2} \vee (3K_1)$ (if $n = k+5$). They are not k-hamiltonian as stated in [1].

Subcase 2.2. If $k < n-6$, then $f_{\max}(x) = f(1)$. Thus

$$W(G) \geq \frac{1}{2}n(n-1) - \frac{1}{2}(-2n+2k+4) = \frac{1}{2}n(n+1) - k - 2,$$

so we get the result.

If $W(G) = \frac{1}{2}n(n+1) - k - 2$, then $i = 1$, $d_1 = 1+k$, $d_2 = \cdots = d_{n-k-1} = n-2$, $d_{n-k} = \cdots = d_n = n-1$, thus $G = K_{k+1} \vee (K_1 \cup K_{n-k-2})$, which is not k-hamiltonian as stated in [1].

(2) As in Theorem 3.1, we have

$$\begin{aligned} H(G) &\leq \frac{1}{4}n(n-1) + \frac{1}{4}\sum_{v \in V(G)} d(v) \\ &\leq \frac{1}{4}n(n-1) + \frac{1}{4}\left[i(i+k) + (n-2i-k)(n-i-1) + (i+k)(n-1)\right] \\ &= \frac{1}{2}n(n-1) + \frac{1}{4}\left[3i^2 - (2n-2k-1)i\right]. \end{aligned}$$

As in the proof of (1), we distinguish into two cases.

Case 1. If $n-k-1$ is odd, then we have two cases.

Subcase 1.1. If $n-4 \leq k \leq n-3$, i.e., $n-k-1 = 3$, thus

$$H(G) \leq \frac{1}{2}n(n-1) + \frac{1}{4}\left[-\frac{1}{4}(n-k+4)(n-k-2)\right] = \frac{1}{2}n(n-1) - 1,$$

so we get the result.

If $H(G)=\frac{1}{2}n(n-1)-1$, then $i=1$, and the remaining is as the Subcase 1.1 of (1).

Subcase 1.2. If $k<n-4$, then

$$H(G)\leq\frac{1}{2}n(n-1)+\frac{1}{4}(-2n+2k+4)=\frac{1}{2}n(n-2)+\frac{1}{2}k+1.$$

If $H(G)=\frac{1}{2}n(n-2)+\frac{1}{2}k+1$, then $i=1$, the remaining is as the Subcase 1.2 of (1).

Case 2. If $n-k-1$ is even, then we have two cases.

Subcase 2.1. If $n-5\leq k\leq n-3$,

$$H(G)\leq\frac{1}{2}n(n-1)+\frac{1}{4}\left[-\frac{1}{4}(n-k+1)(n-k-1)\right]=\frac{1}{2}n(n-1)-\frac{1}{16}(n-k+1)(n-k-1),$$

so we get the result.

If $H(G)=\frac{1}{2}n(n-1)-\frac{1}{16}(n-k+1)(n-k-1)$, then $i=\frac{1}{2}(n-k-1)$, and the remaining is as the Subcase 2.1 of (1).

Subcase 2.2. If $k<n-5$, then

$$H(G)\leq\frac{1}{2}n(n-1)+\frac{1}{4}(-2n+2k+4)=\frac{1}{2}n(n-2)+\frac{1}{2}k+1,$$

so we get the result.

If $H(G)=\frac{1}{2}n(n-2)+\frac{1}{2}k+1$, then $i=1$, the remaining is as the Subcase 2.2 of (1). ■

If $k=0$, then we immediately obtain the following corollary

Corollary 3.4. *[21, 22] Let G be a connected graph of order $n\geq 6$.*

(i) If $W(G)\leq\frac{1}{2}n(n+1)-2$, then G is hamiltonian, unless $G=K_1\vee(K_1\cup K_{n-2})$.

(ii) If $H(G)\geq\frac{1}{2}n(n-2)+1$, then G is hamiltonian, unless $G=K_1\vee(K_1\cup K_{n-2})$.

Theorem 3.5. *Let G be a connected graph of order n, $1\leq k\leq n-3$.*

(1) (i) For $\frac{n-2}{5}\leq k\leq n-4$ and $n-k-1$ is odd, if $W(G)\leq\frac{1}{8}\left[5n^2+(2k-2)n+k^2+2k-8\right]$, then G is k-path-coverable, unless $G=K_{\frac{n-k-2}{2}}\vee(\overline{K_{\frac{n+k-2}{2}}}\cup K_2)$.

(ii) For $\frac{n-5}{5}\leq k\leq n-3$ and $n-k-1$ is even, if $W(G)\leq\frac{1}{8}\left[5n^2+(2k-4)n+k^2-1\right]$, then G is k-path-coverable, unless $G=K_{\frac{n-k-1}{2}}\vee(\overline{K_{\frac{n+k-1}{2}}}\cup K_1)$.

(iii) For $k<\frac{n-5}{5}$, if $W(G)\leq\frac{1}{2}\left[n^2+(2k+1)n-k^2-5k-4\right]$, then G is k-path-coverable, unless $G=K_1\vee(\overline{K_{k+1}}\cup K_{n-k-2})$.

(2) (i) For $\frac{n-2}{5}\leq k\leq n-4$ and $n-k-1$ is odd, if $H(G)\geq\frac{1}{16}\left[7n^2-(2k+10)n-k^2-2k+8\right]$, then G is k-path-coverable, unless $G=K_{\frac{n-k-2}{2}}\vee(\overline{K_{\frac{n+k-2}{2}}}\cup K_2)$.

(ii) For $\frac{n-5}{5}\leq k\leq n-3$ and $n-k-1$ is even, if $H(G)\geq\frac{1}{16}\left[7n^2-(2k+8)n-k^2+1\right]$, then G is k-path-coverable, unless $G=K_{\frac{n-k-1}{2}}\vee(\overline{K_{\frac{n+k-1}{2}}}\cup K_1)$.

(iii) For $k < \frac{n-5}{5}$, if $H(G) \geq \frac{1}{4}\left[2n^2 - (2k+4)n + k^2 + 5k + 4\right]$, then G is k-path-coverable, unless $G = K_1 \vee (\overline{K_{k+1}} \cup K_{n-k-2})$.

Proof. Suppose that G is not k-path-coverable, then from Lemma 2.4, there exists an integer $1 \leq i \leq \frac{n-k-1}{2}$ such that $d_{i+k} \leq i$ and $d_{n-i} \leq n-i-k-1$.

(1) We consider $W(G)$, as in Theorem 3.1,

$$\begin{aligned} W(G) &\geq n(n-1) - \frac{1}{2}\sum_{v \in V(G)} d(v) \\ &\geq n(n-1) - \frac{1}{2}\left[(i+k)i + (n-2i-k)(n-i-k-1) + i(n-1)\right] \\ &= n(n-1) - \frac{1}{2}(n-k)(n-k-1) - \frac{1}{2}\left[3i^2 - (2n-4k-1)i\right]. \end{aligned}$$

Suppose $f(x) = 3x^2 - (2n-4k-1)x$ with $1 \leq x \leq \frac{1}{2}(n-k-1)$, $1 \leq k \leq n-3$. Since x is an integer, then we have to consider $n-k-1$ is odd or even.

Case 1. If $n-k-1$ is odd, then $1 \leq i \leq \frac{1}{2}(n-k-2)$. Easily, $f(1) = 4+4k-2n$, $f(\frac{1}{2}(n-k-2)) = -\frac{1}{4}(n-k-2)(n-5k+4)$, $f(\frac{1}{2}(n-k-2)) - f(1) = -\frac{1}{4}(n-k-4)(n-5k-2)$. In this case, we consider two cases. Note that $n \geq k+3$ from assumption.

Subcase 1.1. If $\frac{n-2}{5} \leq k \leq n-4$, then $f_{\max}(x) = f(\frac{1}{2}(n-k-2))$. Thus

$$\begin{aligned} W(G) &\geq n(n-1) - \frac{1}{2}(n-k)(n-k-1) - \frac{1}{2}\left[-\frac{1}{4}(n-k-2)(n-5k+4)\right] \\ &= \frac{1}{8}\left[5n^2 + (2k-2)n + k^2 + 2k - 8\right], \end{aligned}$$

so we get the result.

If $W(G) = \frac{1}{8}\left[5n^2 + (2k-2)n + k^2 + 2k - 8\right]$, then from the above proof, $i = \frac{1}{2}(n-k-2)$, and hence $d_1 = d_2 = \cdots = d_{\frac{n+k-2}{2}} = \frac{n-k-2}{2}$, $d_{\frac{n+k}{2}} = d_{\frac{n+k+2}{2}} = \frac{n-k}{2}$, $d_{\frac{n+k+4}{2}} = \cdots = d_n = n-1$. Thus $G = K_{\frac{n-k-2}{2}} \vee (\overline{K_{\frac{n+k-2}{2}}} \cup K_2)$, which is not k-path-coverable as stated in [1].

Subcase 1.2. If $k < \frac{n-2}{5}$, then $f_{\max}(x) = f(1)$. Thus

$$\begin{aligned} W(G) &\geq n(n-1) - \frac{1}{2}(n-k)(n-k-1) - \frac{1}{2}(4+4k-2n) \\ &= \frac{1}{2}\left[n^2 + (2k+1)n - k^2 - 5k - 4\right]. \end{aligned}$$

If $W(G) = \frac{1}{2}\left[n^2 + (2k+1)n - k^2 - 5k - 4\right]$, then $i = 1$, and hence $d_1 = d_2 = \cdots = d_{1+k} = 1$, $d_{k+2} = \cdots = d_{n-1} = n-k-2$, $d_n = n-1$. Thus $G = K_1 \vee (\overline{K_{k+1}} \cup K_{n-k-2})$, which is not k-path-coverable as stated in [1].

Case 2. If $n-k-1$ is even, $f(1) = 4+4k-2n$, $f(\frac{n-k-1}{2}) = -\frac{1}{4}(n-k-1)(n-5k+1)$, $f(\frac{n-k-1}{2}) - f(1) = -\frac{1}{4}(n-k-3)(n-5k-5)$. We also have two cases.

Subcase 2.1. If $\frac{n-5}{5} \leq k \leq n-3$, then $f_{\max}(x) = f(\frac{n-k-1}{2})$. Thus

$$\begin{aligned} W(G) &\geq n(n-1) - \frac{1}{2}(n-k)(n-k-1) - \frac{1}{2}\left[-\frac{1}{4}(n-k-1)(n-5k+1)\right] \\ &= \frac{1}{8}\left[5n^2 + (2k-4)n + k^2 - 1\right], \end{aligned}$$

so we get the result.

If $W(G) = \frac{1}{8}\left[5n^2 + (2k-4)n + k^2 - 1\right]$, then $i = \frac{n-k-1}{2}$, and hence $d_1 = d_2 = \cdots = d_{\frac{n+k-1}{2}} = \frac{n-k-1}{2}$, $d_{\frac{n+k+1}{2}} = \frac{n-k-1}{2}$, $d_{\frac{n+k+3}{2}} = \cdots = d_n = n-1$. Thus $G = K_{\frac{n-k-1}{2}} \vee (\overline{K_{\frac{n+k-1}{2}}} \cup K_1)$, which is not k-path-coverable as stated in [1].

Subcase 2.2. If $k < \frac{n-5}{5}$, then $f_{\max}(x) = f(1)$. Thus

$$\begin{aligned} W(G) &\geq n(n-1) - \frac{1}{2}(n-k)(n-k-1) - \frac{1}{2}(4+4k-2n) \\ &= n^2 - \frac{1}{2}(n-k)(n-k-1) - 2k - 2. \end{aligned}$$

If $W(G) = n^2 - \frac{1}{2}(n-k)(n-k-1) - 2k - 2$, then $i = 1$, and hence $d_1 = d_2 = \cdots = d_{1+k} = 1$, $d_{k+2} = \cdots = d_{n-1} = n-k-2$, $d_n = n-1$. Thus $G = K_1 \vee (\overline{K_{k+1}} \cup K_{n-k-2})$, which is not k-path-coverable as stated in [1].

(2) We consider $H(G)$, as in Theorem 3.1, we have

$$\begin{aligned} H(G) &\leq \frac{1}{4}n(n-1) + \frac{1}{4}\sum_{v \in V(G)} d(v) \\ &\leq \frac{1}{4}n(n-1) + \frac{1}{4}\left[(i+k)i + (n-2i-k)(n-i-k-1) + i(n-1)\right] \\ &= \frac{1}{4}n(n-1) + \frac{1}{4}(n-k)(n-k-1) + \frac{1}{4}\left[3i^2 - (2n-4k-1)i\right]. \end{aligned}$$

The proof is similar to the problem on $W(G)$.

Case 1. If $n-k-1$ is odd, we have two cases.

Subcase 1.1. If $\frac{n-2}{5} \leq k \leq n-4$, then

$$\begin{aligned} H(G) &\leq \frac{1}{4}n(n-1) + \frac{1}{4}(n-k)(n-k-1) + \frac{1}{4}\left[-\frac{1}{4}(n-k-2)(n-5k+4)\right] \\ &= \frac{1}{16}\left[7n^2 - (2k+10)n - k^2 - 2k + 8\right]. \end{aligned}$$

If $H(G) = \frac{1}{16}\left[7n^2 - (2k+10)n - k^2 - 2k + 8\right]$, then $i = \frac{n-k-2}{2}$, and the remaining is as the Subcase 1.1 of (1).

Subcase 1.2. If $k < \frac{n-2}{5}$, then

$$\begin{aligned} H(G) &\leq \frac{1}{4}n(n-1) + \frac{1}{4}(n-k)(n-k-1) + \frac{1}{4}(4+4k-2n) \\ &= \frac{1}{4}\left[2n^2 - (2k+4)n + k^2 + 5k + 4\right]. \end{aligned}$$

If $H(G) = \frac{1}{4}\left[2n^2 - (2k+4)n + k^2 + 5k + 4\right]$, then $i = 1$, and the remaining is as the Subcase 1.2 of (1).

Case 2. If $n-k-1$ is even, we also have two cases.

Subcase 2.1. If $\frac{n-5}{5} \leq k \leq n-3$, then

$$H(G) \leq \frac{1}{4}n(n-1) + \frac{1}{4}(n-k)(n-k-1) + \frac{1}{4}\left[-\frac{1}{4}(n-k-1)(n-5k+1)\right]$$

$$= \frac{1}{16}\left[7n^2-(2k+8)n-k^2+1\right].$$

If $H(G)=\frac{1}{16}\left[7n^2-(2k+8)n-k^2+1\right]$, then $i=\frac{n-k-1}{2}$, and the remaining is as the Subcase 2.1 of (1).

Subcase 2.2. If $k<\frac{n-5}{5}$, then $H(G)\le \frac{1}{4}\left[2n^2-(2k+4)n+k^2+5k+4\right]$.

If $H(G)=\frac{1}{4}\left[2n^2-(2k+4)n+k^2+5k+4\right]$, then $i=1$, and the remaining is as the Subcase 2.2 of (1). ■

If $k=1$, then we obtain the following corollary regarding traceable.

Corollary 3.6. *[35, 12] Let G be a connected graph of order $n\ge 11$.*

(i) If $W(G)\le \frac{1}{2}\left[n^2+3n-10\right]$, then G is traceable, unless $G=K_1\vee(\overline{K_2}\cup K_{n-3})$.

(ii) If $H(G)\ge \frac{1}{2}\left[n^2-3n+5\right]$, then G is traceable, unless $G=K_1\vee(\overline{K_2}\cup K_{n-3})$.

Theorem 3.7. *Let G be a connected graph of order n with $0\le k\le n-3$.*

(1) (i) For $k=n-4$, if $W(G)\le\frac{1}{8}\left[5n^2-(2k+2)n+k^2-2k-8\right]$, then G is k-edge-hamiltonian, unless $G=K_{k+1}\vee(K_1\cup K_2)$.

(ii) For $k=n-3$ or $n-5$, if $W(G)\le\frac{1}{8}\left[5n^2-(2k+4)n+k^2-1\right]$, then G is k-edge-hamiltonian, unless $G=K_{k+1}\vee(2K_1)$ (if $k=n-3$) or $K_{k+2}\vee(3K_1)$ (if $k=n-5$).

(iii) For $k<n-5$, if $W(G)\le\frac{1}{2}n(n+1)-k-2$, then G is k-edge-hamiltonian, unless $G=K_{k+1}\vee(K_1\cup K_{n-k-2})$.

(2) (i) For $k=n-4$, if $H(G)\ge\frac{1}{16}\left[7n^2+(2k-10)n-k^2+2k+8\right]$, then G is k-edge-hamiltonian, unless $G=K_{k+1}\vee(K_1\cup K_2)$.

(ii) For $k=n-3$ or $n-5$, if $H(G)\ge\frac{1}{16}\left[7n^2+(2k-8)n-k^2+1\right]$, then G is k-edge-hamiltonian, unless $G=K_{k+1}\vee(2K_1)$ (if $k=n-3$) or $K_{k+2}\vee(3K_1)$ (if $k=n-5$).

(iii) For $k<n-5$, if $H(G)\ge\frac{1}{2}n(n-2)+\frac{1}{2}k+1$, then G is k-edge-hamiltonian, unless $G=K_{k+1}\vee(K_1\cup K_{n-k-2})$.

Proof. Suppose that G is not k-edge-hamiltonian, then from Lemma 2.5, there exists an integer $k+1\le i\le\frac{1}{2}(n+k-1)$ such that $d_{i-k}\le i$ and $d_{n-i}\le n-i+k-1$.

(1) As in Theorem 3.1, we have

$$\begin{aligned} W(G) &\ge n(n-1)-\frac{1}{2}\sum_{v\in V(G)}d(v)\\ &\ge n(n-1)-\frac{1}{2}\left[(i-k)i+(n-2i+k)(n-i+k-1)+i(n-1)\right]\\ &= n(n-1)-\frac{1}{2}(n+k)(n+k-1)-\frac{1}{2}\left[3i^2-(2n+4k-1)i\right].\end{aligned}$$

Suppose $f(x)=3x^2-(2n+4k-1)x$, $k+1\le x\le\frac{1}{2}(n+k-1)$, $0\le k\le n-3$. Since x is an integer, we have to consider $n+k-1$ is odd or even.

Case 1. If $n+k-1$ is odd, then $k+1 \le i \le \frac{1}{2}(n+k-2)$ and $n-k-1$ is also odd, Obviously, $f_{\max}(x) = \max\{f(k+1), f(\frac{1}{2}(n+k-2))\}$. Easily,

$$f(\frac{1}{2}(n+k-2)) = -\frac{1}{4}(n+k-2)(n+5k+4), \qquad f(k+1) = -(k+1)(2n+k-4),$$

$$f(\frac{1}{2}(n+k-2)) - f(k+1) = -\frac{1}{4}(n-k-2)(n-k-4).$$

Next, we consider two cases. Note that $n \ge k+3$ from assumption.

Subcase 1.1. If $k+3 \le n \le k+4$, i.e., $n-k-1=3$, then $f_{\max}(x) = f(\frac{1}{2}(n+k-2)) = f(k+1)$. Hence

$$\begin{aligned} W(G) &\ge n(n-1) - \frac{1}{2}(n+k)(n+k-1) - \frac{1}{2}\left[-\frac{1}{4}(n+k-2)(n+5k+4)\right] \\ &= \frac{1}{8}\left[5n^2 - (2k+2)n + k^2 - 2k - 8\right]. \end{aligned}$$

If $W(G) = \frac{1}{8}\left[5n^2 - (2k+2)n + k^2 - 2k - 8\right]$, then we must have $i = \frac{1}{2}(n+k-2)$, and hence $d_1 = d_2 = \cdots = d_{\frac{1}{2}(n-k-2)} = \frac{1}{2}(n+k-2)$, $d_{\frac{1}{2}(n-k)} = d_{\frac{1}{2}(n-k+2)} = \frac{1}{2}(n+k)$, $d_{\frac{1}{2}(n-k+4)} = \cdots = d_n = n-1$, thus $G = K_{\frac{1}{2}(n+k-2)} \vee (\overline{K_{\frac{1}{2}(n-k-2)}} \cup K_2) = K_{k+1} \vee (K_1 \cup K_2)$, which is not k-edge-hamiltonian as stated in [1].

Subcase 1.2. If $n > k+4$, then $f_{\max}(x) = f(k+1)$. Thus

$$\begin{aligned} W(G) &\ge n(n-1) - \frac{1}{2}(n+k)(n+k-1) - \frac{1}{2}\left[-(k+1)(2n+k-4)\right] \\ &= \frac{1}{2}n(n+1) - k - 2. \end{aligned}$$

If $W(G) = \frac{1}{2}n(n+1) - k - 2$, then $i = k+1$. Therefore $d_1 = k+1$, $d_2 = \cdots = d_{n-k-1} = n-2$, $d_{n-k} = \cdots = d_n = n-1$, thus $G = K_{k+1} \vee (K_1 \cup K_{n-k-2})$, which is not k-edge-hamiltonian as stated in [1].

Case 2. If $n+k-1$ is even, then $n-k-1$ is also even, and we easily see

$$f(k+1) = -(k+1)(2n+k-4), \qquad f(\frac{1}{2}(n+k-1)) = -\frac{1}{4}(n+k-1)(n+5k+1),$$

$$f(\frac{1}{2}(n+k-1)) - f(k+1) = -\frac{1}{4}(n-k-3)(n-k-5).$$

We also have the following two cases.

Subcase 2.1. If $k+3 \le n \le k+5$, i.e., $n-k=3$ or $n-k=5$, then $f_{\max}(x) = f(\frac{1}{2}(n+k-1)) = f(k+1)$. Therefore

$$\begin{aligned} W(G) &\ge n(n-1) - \frac{1}{2}(n+k)(n+k-1) - \frac{1}{2}\left[-\frac{1}{4}(n+k-1)(n+5k+1)\right] \\ &= \frac{1}{8}\left[5n^2 - (2k+4)n + k^2 - 1\right]. \end{aligned}$$

If $W(G) = \frac{1}{8}\left[5n^2 - (2k+4)n + k^2 - 1\right]$, then $i = \frac{1}{2}(n+k-1)$, so $d_1 = d_2 = \cdots = d_{\frac{1}{2}(n-k-1)} = \frac{1}{2}(n+k-1)$, $d_{\frac{1}{2}(n-k+1)} = \frac{1}{2}(n+k-1)$, $d_{\frac{1}{2}(n-k+3)} = \cdots = d_n = n-1$, thus $G = K_{\frac{1}{2}(n+k-1)} \vee (\overline{K_{\frac{1}{2}(n-k-1)}} \cup K_1)$ (so $G = K_{k+1} \vee (2K_1)$ for $k = n-3$ and $K_{k+2} \vee (3K_1)$ for $k = n-5$), which is not k-edge-hamiltonian as stated in [1].

Subcase 2.2. If $n > k+5$, then $f_{\max}(x) = f(k+1)$. Thus

$$\begin{aligned} W(G) &\geq n(n-1) - \frac{1}{2}(n+k)(n+k-1) - \frac{1}{2}\left[-(k+1)(2n+k-4)\right] \\ &= \frac{1}{2}n(n+1) - k - 2, \end{aligned}$$

so we get the result.

If $W(G) = \frac{1}{2}n(n+1) - k - 2$, then $i = k+1$. So $d_1 = k+1$, $d_2 = \cdots = d_{n-k-1} = n-2$, $d_{n-k} = \cdots = d_n = n-1$, thus $G = K_{k+1} \vee (K_1 \cup K_{n-k-2})$, which is not k-edge-hamiltonian as stated in [1].

(2) We now consider $H(G)$. As in Theorem 3.1, we have

$$\begin{aligned} H(G) &\leq \frac{1}{4}n(n-1) + \frac{1}{4}\sum_{v \in V(G)} d(v) \\ &\leq \frac{1}{4}n(n-1) + \frac{1}{4}\left[(i-k)i + (n-2i+k)(n-i+k-1) + i(n-1)\right] \\ &= \frac{1}{4}n(n-1) + \frac{1}{4}(n+k)(n+k-1) + \frac{1}{4}\left[3i^2 - (2n+4k-1)i\right]. \end{aligned}$$

The remaining proof is similar to the proof on $W(G)$.

Case 1. If $n+k-1$ is odd, then we have two cases.

Subcase 1.1. If $n-4 \leq k \leq n-3$, i.e., $n-k-1 = 3$, then

$$\begin{aligned} H(G) &\leq \frac{1}{4}n(n-1) + \frac{1}{4}(n+k)(n+k-1) + \frac{1}{4}\left[-\frac{1}{4}(n+k-2)(n+5k+4)\right] \\ &= \frac{1}{16}\left[7n^2 + (2k-10)n - k^2 + 2k + 8\right]. \end{aligned}$$

If $H(G) = \frac{1}{16}\left[7n^2 + (2k-10)n - k^2 + 2k + 8\right]$, then $i = \frac{1}{2}(n+k-2)$, and the remaining is as Subcase 1.1 of (1).

Subcase 1.2. If $k < n-4$, then

$$\begin{aligned} H(G) &\leq \frac{1}{4}n(n-1) + \frac{1}{4}(n+k)(n+k-1) + \frac{1}{4}\left[-(k+1)(2n+k-4)\right] \\ &= \frac{1}{2}n(n-2) + \frac{1}{2}k + 1. \end{aligned}$$

If $H(G) = \frac{1}{2}n(n-2) + \frac{1}{2}k + 1$, then $i = k+1$, and the remaining is as Subcase 1.2 of (1).

Case 2. If $n+k-1$ is even, then we also have two cases.

Subcase 2.1. If $n-5 \leq k \leq n-3$, i.e., $n-k = 3$ or $n-k = 5$, then

$$\begin{aligned} H(G) &\leq \frac{1}{4}n(n-1) + \frac{1}{4}(n+k)(n+k-1) + \frac{1}{4}\left[-\frac{1}{4}(n+k-1)(n+5k+1)\right] \\ &= \frac{1}{16}\left[7n^2 + (2k-8)n - k^2 + 1\right]. \end{aligned}$$

If $H(G) = \frac{1}{16}\left[7n^2 + (2k-8)n - k^2 + 1\right]$, then $i = \frac{1}{2}(n-k-1)$, and the remaining is as the Subcase 2.1 of (1).

Subcase 2.2. If $k < n-5$, then

$$H(G) \leq \frac{1}{4}n(n-1) + \frac{1}{4}(n+k)(n+k-1) + \frac{1}{4}\left[-(k+1)(2n+k-4)\right]$$

$$= \frac{1}{2}n(n-2)+\frac{1}{2}k+1.$$

If $H(G)=\frac{1}{2}n(n-2)+\frac{1}{2}k+1$, then $i=k+1$, $d_1=k+1$, and the remaining is as the Subcase 2.2 of (1). ■

Theorem 3.8. *Let G be a connected graph of order n.*

(1) If $W(G)\le \frac{1}{2}n(n-1)+\frac{1}{2}k(k+1)$, then G satisfies $\alpha(G)\le k$, unless $G=\overline{K_{k+1}}\vee K_{n-k-1}$.

(2) If $H(G)\ge \frac{1}{2}n(n-1)-\frac{1}{4}k(k+1)$, then G satisfies $\alpha(G)\le k$ unless $G=\overline{K_{k+1}}\vee K_{n-k-1}$.

Proof. Suppose that G does not satisfy $\alpha(G)\le k$, then from Lemma 2.6, $d_{k+1}\le n-k-1$.

(1) As in Theorem 3.1, we have

$$\begin{aligned} W(G) &\ge n(n-1)-\frac{1}{2}\sum_{v\in V(G)}d(v)\\ &\ge n(n-1)-\frac{1}{2}\left[(k+1)(n-k-1)+(n-k-1)(n-1)\right]\\ &= \frac{1}{2}n(n-1)+\frac{1}{2}k(k+1).\end{aligned}$$

This is a contradiction, so we get the result.

If $W(G)=\frac{1}{2}n(n-1)+\frac{1}{2}k(k+1)$, $d_1=d_2=\cdots=d_{k+1}=n-k-1$, $d_{k+2}=\cdots=d_n=n-1$. So $G=\overline{K_{k+1}}\vee K_{n-k-1}$, which does not satisfy $\alpha(G)\le k$.

(2) As in Theorem 3.1, we have

$$\begin{aligned} H(G) &\le \frac{1}{4}n(n-1)+\frac{1}{4}\sum_{v\in V(G)}d(v)\\ &\le \frac{1}{4}n(n-1)+\frac{1}{4}\left[(k+1)(n-k-1)+(n-k-1)(n-1)\right]\\ &= \frac{1}{2}n(n-1)-\frac{1}{4}k(k+1).\end{aligned}$$

Thus $H(G)\le \frac{1}{2}n(n-1)-\frac{1}{4}k(k+1)$, which leads to a contradiction. The equality case is similar to that of (1). ■

4. Concluding Remarks

In this paper, we use the most popular and well known topological indices, namely Wiener index and Harary index, to provide several sufficient conditions for graphs possessing certain properties. There are also many other well studied topological indices such as the degree distance, the eccentric connectivity index, the eccentric distance sum, the connective eccentricity index, graph eigenvalues. From the ideas of this paper, we can try to present sufficient conditions for graphs possessing many other properties. There are also many other properties of graphs such as the toughness and the thickness, can we study them in terms of some topological indices? We will leave them to study in our future work.

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