Practice 1
(Supplemental notes)

1. Conception of Vector Space

Def: A *vector space* is a set V (the elements of which are called vectors) with an addition and a scalar multiplication satisfying the following properties for all $u, v, w \in V$ and $\alpha, \beta \in \mathbb{R}$:
1) $v + w = w + v$,
2) $(u + v) + w = u + (v + w)$,
3) there exists a vector 0 in V such that $v + 0 = v$,
4) for each vector $v \in V$, there exists a vector $-v \in V$ such that $v + (-v) = 0$,
5) $\alpha(v + w) = \alpha v + \alpha w$,
6) $(\alpha + \beta)v = \alpha v + \beta v$,
7) $(\alpha\beta)v = \alpha(\beta v)$,
8) $1v = v$.
So these are simply properties that elements of the vector space must possess.

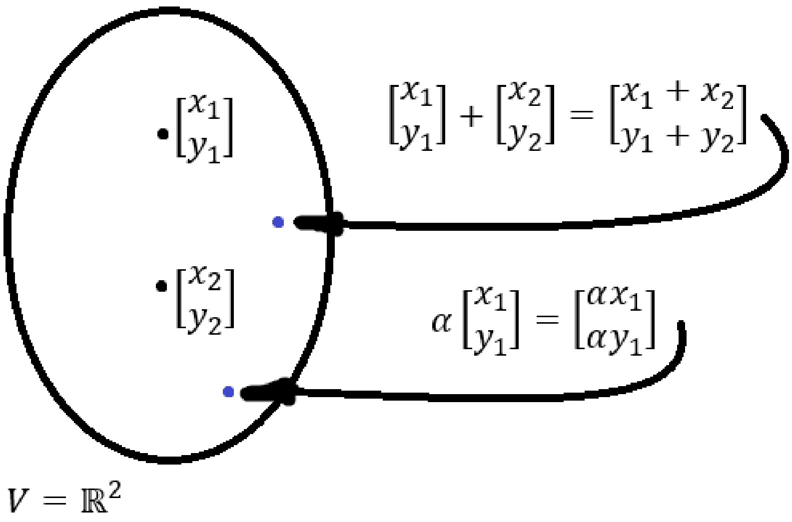
Now, an important property of a vector space is something called *closure*. Closure requires two properties.
- First, for an element a of V, multiplying a by any scalar will give a result that is also within V.
- Second, for any two elements a and b within V, adding two elements will also give a result that is contained within V.

These closure properties are what determine if V is a vector space.

2. Examples of Vector Space

Example 1 (Euclidean space): $\mathbb{R}$ (the set of all real numbers), $\mathbb{R}^n$ (the set of real vectors with a length of some integer n).

<u>For instance, verify that $\mathbb{R}^2$ is the vector space</u>:

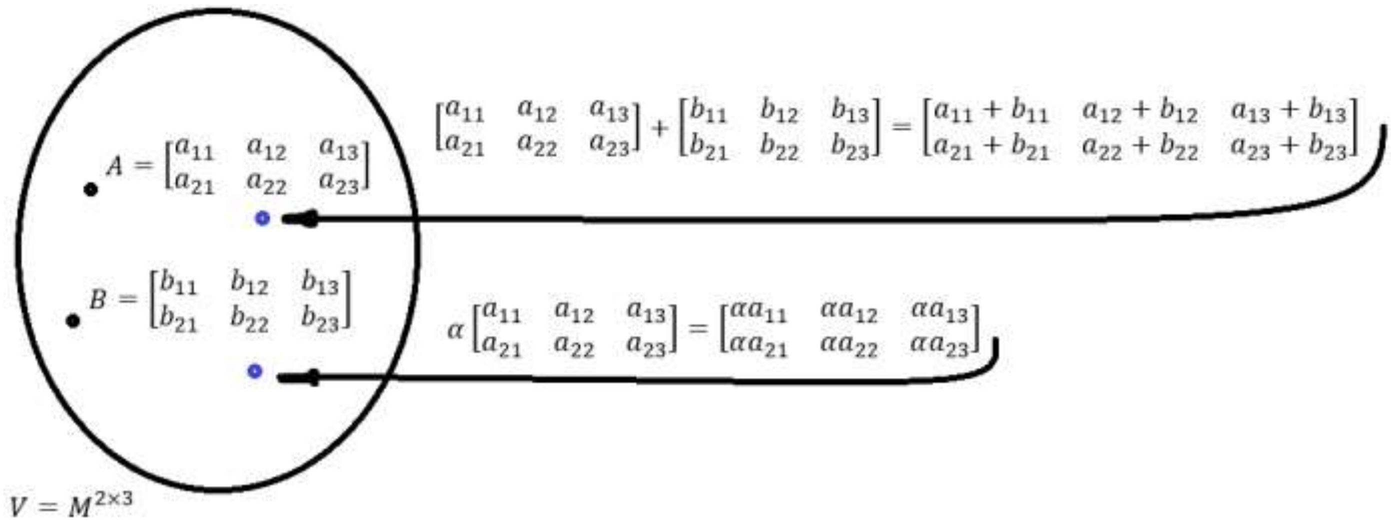


Verifying that $V = \mathbb{R}^2$ satisfies properties (1)-(8) is straightforward.

Counterexample 1: A collection of all vectors of the form: $\begin{bmatrix} x \\ 2 \end{bmatrix}$.

Example 2 (Matrix space): The set $V = M^{m \times n}$ of $m \times n$ matrices is a vector space with usual matrix addition and scalar multiplication.

For instance, verify that $M^{2\times 3}$ is the vector space:

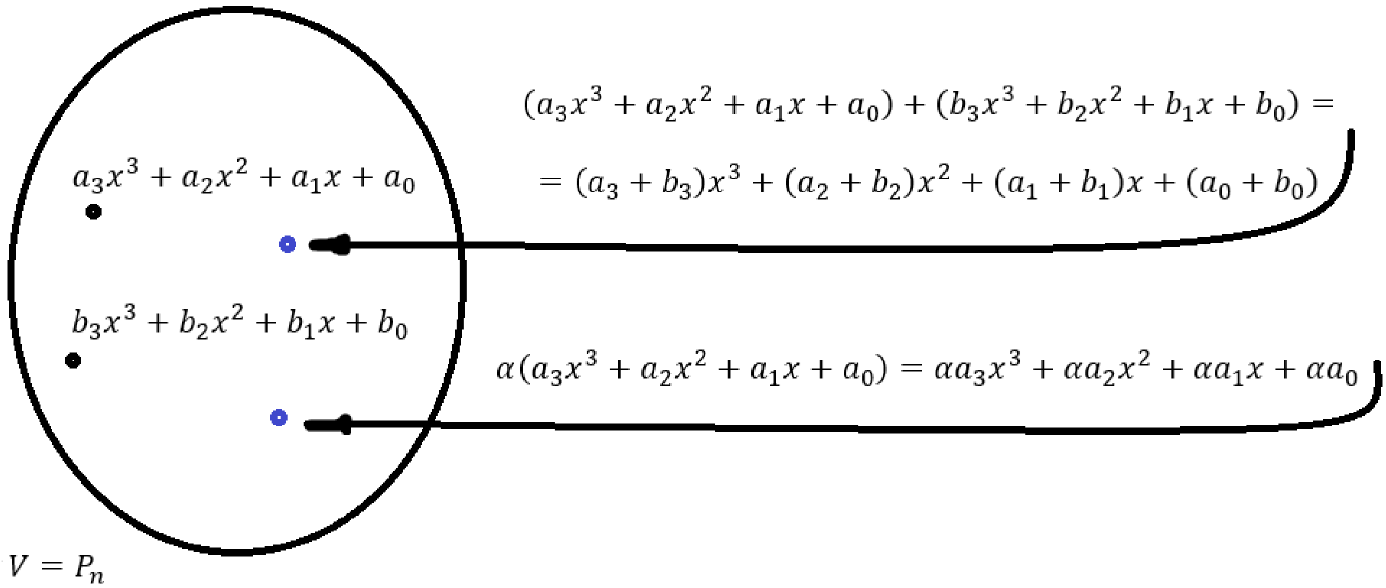


Verifying that $V = M^{2\times 3}$ satisfies properties (1)-(8) is straightforward.

Counterexample 2: The set of all invertible 2×2 matrices does not form a vector space.

Example 3 (Polynomial space): The set $V = P_n$ of all polynomials of degree less than n with real coefficients is a vector space.

For instance, verify that P_3 is the vector space:



Verifying that $V = P_3$ satisfies properties (1)-(8) is straightforward.

Counterexample 3: The set of polynomials of degree exactly n does not form a vector space.

Example 4 (Function space): Let $V = C[a,b]$ be the set of all continuous functions $f: \mathbb{R} \to \mathbb{R}$, let the field of scalars be $\mathbb{R}$, and define vector addition and scalar multiplication by

- $f + g$ is the continuous function defined by $(f+g)(x) = f(x) + g(x)$;
- αf is the continuous function defined by $(\alpha f)(x) = \alpha \cdot f(x)$.

With this addition and scalar multiplication, this set is a vector space.

Counterexample 4: The set of functions satisfying $f(0) = 1$ is not a vector space.

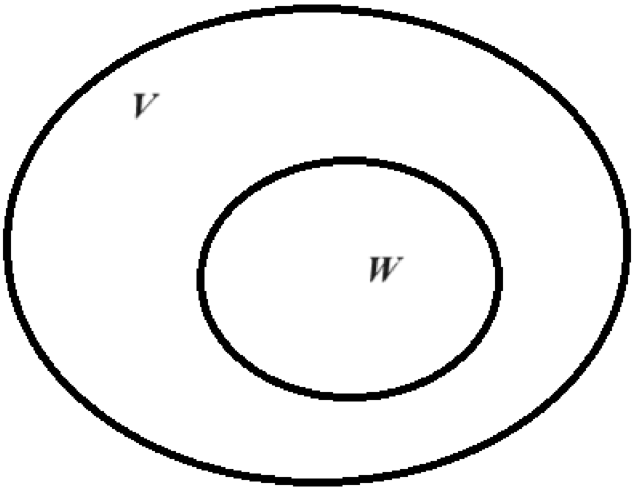
Example 5 (Infinite sequences): Let V be the set of infinite sequences of real numbers $(x_1, x_2, x_3, \dots)$, let the field of scalars be $\mathbb{R}$, and define vector addition and scalar multiplication by

- $(x_1, x_2, x_3, \dots) + (y_1, y_2, y_3, \dots) = (x_1 + y_1, x_2 + y_2, x_3 + y_3, \dots)$
- $\alpha(x_1, x_2, x_3, \dots) = (\alpha x_1, \alpha x_2, \alpha x_3, \dots)$

With this addition and scalar multiplication, this set is a vector space.

3. Subspace

It is possible for one vector space to be contained within a larger vector space. A *subspace* is a smaller set within a vector space that is itself a vector space:



every element of W is also element of V

Every element of W is already also in our vector space V, so we know *they all obey the properties* we discussed. This means that the only requirement we need to check in order to determine whether W is a vector space, is that it satisfies the properties of *closure*. If W is closed, it will also be a vector space, and because it is completely contained in the larger vector space V, we will then call it a *subspace of* V.

Def: Let V be a vector space. A subset W of V is called a subspace if the following hold:
1) $0 \in W$;
2) $u, v \in W$ implies $u + v \in W$;
3) $v \in W, \alpha \in \mathbb{R}$ implies $\alpha v \in W$.

Example 0: Every vector space has a zero subspace $\{0\}$.
Example 1: A plane in $\mathbb{R}^3$ through the origin is a subspace of $\mathbb{R}^3$.
Example 2: Let W be the subset of $M^{2\times 2}$ consisting of all matrices of the form:

$$\begin{bmatrix} a & b \\ c & -a \end{bmatrix}$$

W is a subspace of $M^{2\times 2}$.

Example 3: Let W be the subset of $C[a, b]$ consisting of all functions $f: \mathbb{R} \to \mathbb{R}$ for which $f(1) = 0$. W is a subspace of $C[a, b]$.

Not subspaces: $\mathbb{R}^2$ is not a subspace of $\mathbb{R}^3$; any straight line in $\mathbb{R}^2$ not passing through the origin is not a vector space; …

4. Basis and dimension of a Vector Space. Examples of standard bases

Def: Let $\{v_1, v_2, \dots, v_n\}$ be the set of the vectors. Then the *span* of these vectors, $Span\{v_1, v_2, \dots, v_n\}$, is said to be the space of all vectors that are a linear combination of this set of vectors.

Def: Let S be a subspace of V. Then the set $\{v_1, v_2, \dots, v_n\}$ is said to be a spanning set for S if $Span\{v_1, v_2, \dots, v_n\} = S$.

The difference between the two is that a span is the set of all possible linear combinations, while a spanning set is a specific subset of vectors that can be used to generate all other vectors in the space.

Def: Let $v_1, v_2, v_3, \dots, v_n$ be vectors in a vector space V. The set $\{v_1, v_2, v_3, \dots, v_n\}$ is a *basis* for V if

1) $v_1, v_2, v_3, \dots, v_n$ are linearly independent;
2) $Span\{v_1, v_2, v_3, \dots, v_n\} = V$.

The first condition says that there aren't more vectors than necessary in the set. The second says there are enough to be able to generate V.

$$\boxed{\text{Linearly Independent Set}} \subseteq \boxed{\text{Basis}} \subseteq \boxed{\text{Spanning Set}}$$

Example 1: Show that $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$ is a basis for VS $\mathbb{R}^2$.

Solution: We check the two properties of basis:

1. $Span\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\} = \mathbb{R}^2$?
2. Are $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$ linearly independent?

Example 2: Show that $\left\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right\}$ is a basis for VS $M^{2\times 2}$.

Solution: We check the two properties of basis:

1. $Span\left\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right\} = M^{2\times 2}$?
2. Are $\left\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right\}$ linearly independent?

Example 3: Show that $\{1, x, x^2\}$ is a basis for VS P_3.
Solution: We check the two properties of basis:
1. $Span\{1, x, x^2\} = P_3$?
2. Are $\{1, x, x^2\}$ linearly independent?

Examples 1 – 3 represent the *standard basis*.

Def: Let V be a vector space. If V has a basis consisting of n vectors, we say that V is *finite dimensional* and has dimension n (written $dimV = n$). If V does not have a basis consisting of finitely many vectors, we say that V is *infinite dimensional*.

In addition, if V is a vector space of dimension n and W is a subspace of V, then $dimW \leq n$.

Example 1: $dim\mathbb{R}^n = n$

Example 2: $dimM^{m\times n} = mn$

Example 3: $dimP_n = n + 1$

Example 4: The space of real-valued functions on $\mathbb{R}$ is infinite dimensional.

Problem 1 from practice notes.