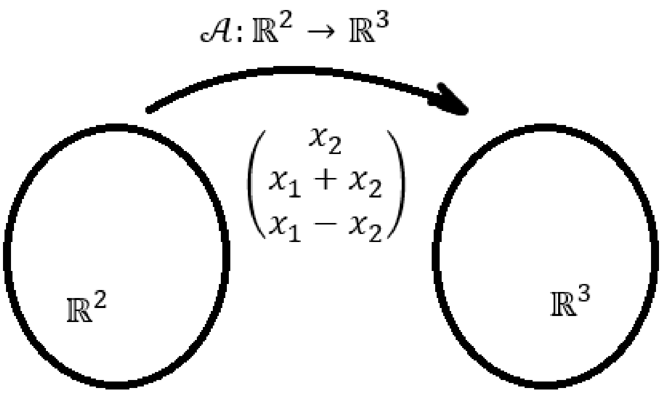
Practice 3
(Supplemental notes)

1. Matrices of Linear Transformations

Example 1: Find a matrix representation of this linear transformation:

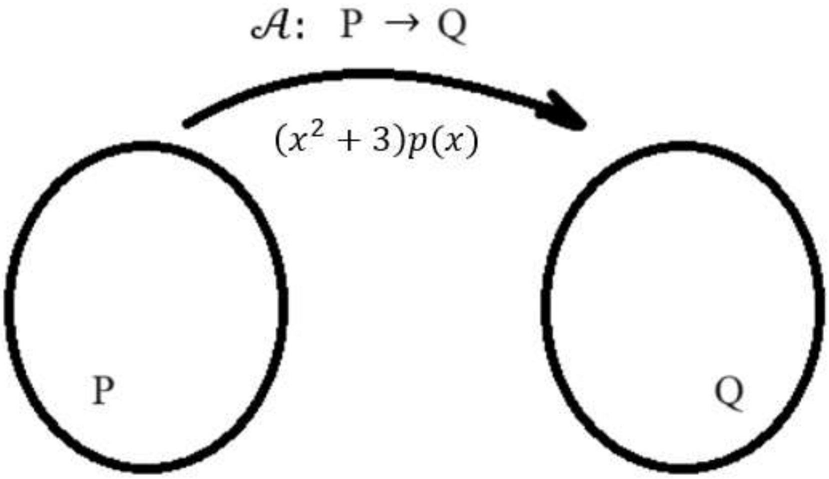


Solution: $\mathbb{R}^2: \left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\} \Rightarrow \mathcal{A}(v) = \mathcal{A}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\1\\1\end{pmatrix}$ *and* $\mathcal{A}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1\\1\\-1\end{pmatrix}$.

Thus, $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}$.

Example 2: Find a matrix representation of the linear transformation that transforms any polynomial of degree at most 2 $p(x)$ into another polynomial $q(x)$ by the rule:

$$(x^2 + 3)p(x)$$



Solution: $B_p = \{1, x, x^2\}, B_q = \{1, x, x^2, x^3, x^4\}$

$$\mathcal{A}(1) = (x^2 + 3) \cdot 1 = x^2 + 3 = 3 \cdot \mathbf{1} + 0 \cdot \boldsymbol{x} + 1 \cdot \boldsymbol{x^2} + 0 \cdot \boldsymbol{x^3} + 0 \cdot \boldsymbol{x^4}$$
$$\mathcal{A}(1) = (x^2 + 3) \cdot x = x^3 + 3x = 0 \cdot \mathbf{1} + 3 \cdot \boldsymbol{x} + 0 \cdot \boldsymbol{x^2} + 1 \cdot \boldsymbol{x^3} + 0 \cdot \boldsymbol{x^4}$$
$$\mathcal{A}(1) = (x^2 + 3) \cdot x^2 = x^4 + 3x^2 = 0 \cdot \mathbf{1} + 0 \cdot \boldsymbol{x} + 3 \cdot \boldsymbol{x^2} + 0 \cdot \boldsymbol{x^3} + 1 \cdot \boldsymbol{x^4}$$

Thus, $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

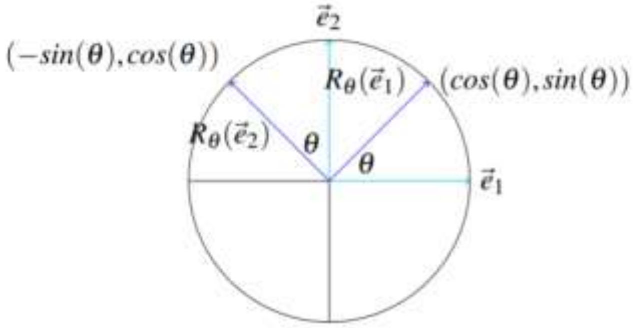
The same linear transformation in different bases generally has DIFFERENT matrices.

Problem 5

2. Some linear transformations on $\mathbb{R}^2$

Example 1 (Rotation):

- Counter-clockwise rotation by an angle θ about the origin



Then the matrix A of $\mathcal{A}$ is given by

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

- Clockwise rotation by an angle θ about the origin:

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Problem 6

Example 2 (Reflection):

- Reflection against the x-axis:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

- Reflection against the y-axis

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example 3 (Projection):

- Projection on the x-axis:

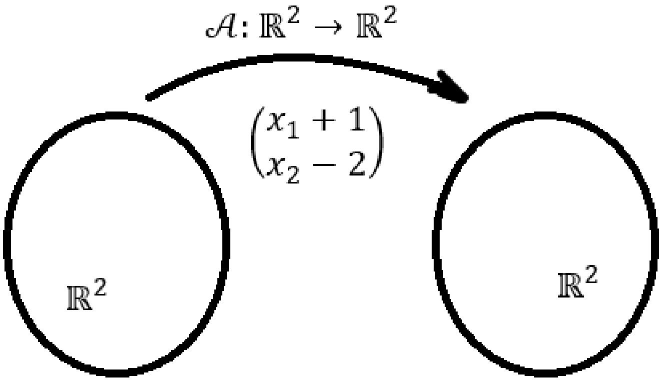
$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

- Projection on the y-axis

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

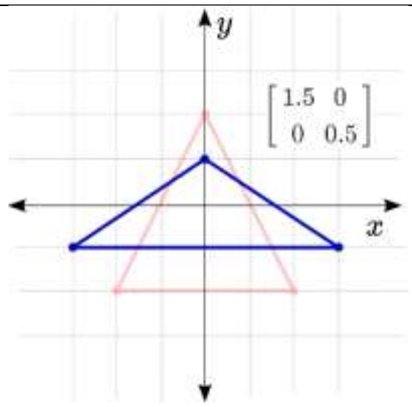
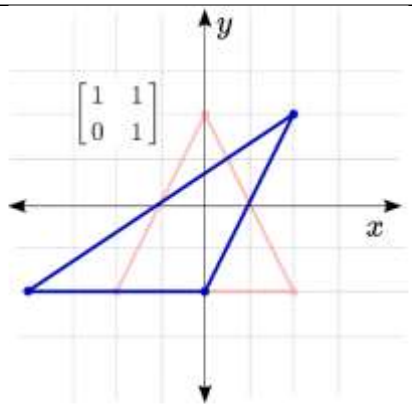
Not every function between vector spaces is a linear transformation.

Example :



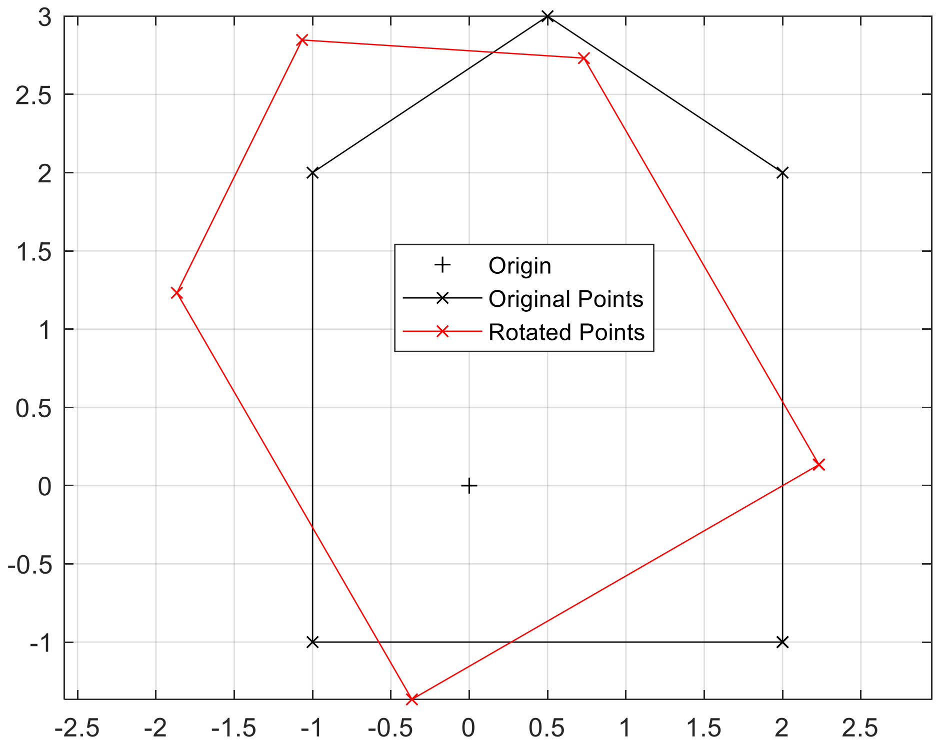
In this case, $\mathcal{A}$ adds $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ to each vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. This type of mapping is called a ***translation***. However, $\mathcal{A}$ is not a linear transformation.

Some other examples of transformation:

Scaling The *a* element will determine how much x_1 affects x_2, and the *d* element will determine how much y_1 affects y_2. If we only change these values, this will result in a scale in the *x* or *y* axis (or both). Reducing both of these to zero is a special case where every single point is shrunk down into the origin!	$A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ $\mathcal{A}(v) = Av$ such that $\mathcal{A}: \mathbb{R}^2 \to \mathbb{R}^2$	
Shearing The two off-diagonal elements control the shear transform. The *b* element determines how much the old *y* affects the new *x*, and the *c* element determines how much the old *x* affects the new *y*. Keeping *a* and *d* at 1 and adjusting these one at a time produces a shear/skew effect.	$A = \begin{bmatrix} 1 & b \\ c & 1 \end{bmatrix}$ $\mathcal{A}(v) = Av$ such that $\mathcal{A}: \mathbb{R}^2 \to \mathbb{R}^2$	

Properties of linear transformations:

1. 0 always maps to 0. There is no way to move the origin.
2. Linear transformations are always odd $\mathcal{A}(-v) = -\mathcal{A}(v)$. This results in a sort of mirroring effect. If you pick any point and see how it moves, the point exactly opposite (through the origin) will move the opposite way, and will continue to be its mirror. You can imagine there is a pin through the origin and everything is stretching and mirroring around it



3. Linear transformations chain through multiplication. If we want to scale some points, then shear them, then rotate them, we just need to multiply all the matrices together:
$$\mathcal{A}_1\left(\mathcal{A}_2\left(\mathcal{A}_3(v)\right)\right) = A_3A_2A_1v.$$
In other words, composition of linear transformations corresponds to the multiplication of their matrices.

Problem 7, problem 8 and problem 9.

In short about eigenvectors, eigenvalues:

$\begin{pmatrix} -1 & -6 \\ 2 & 6 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} -5 \\ 4 \end{pmatrix}$	Average life of vectors ☹
$\begin{pmatrix} -1 & -6 \\ 2 & 6 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix}$	$\begin{pmatrix} 4 \\ -2 \end{pmatrix} = 2\begin{pmatrix} 2 \\ -1 \end{pmatrix}$	$Av = \lambda v$

https://commons.wikimedia.org/wiki/File:Mona_Lisa_eigenvector_grid.png#/media/File:Mona_Lisa_eigenvector_grid.png