Linear algebra revision

Determinant: $n \times n$ matrices

$\det \underline{\underline{A}}$: signed n-dimensional volume of generalized paralellogram spanned by column vectors

$\det \underline{\underline{A}} \neq 0 \quad \Leftrightarrow \quad \underline{\underline{A}}$ invertible

ex: 2×2 matrix: $\det \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

general $n \times n$ matrix: signed combination of product of elements

expand along row i: $\det \underline{\underline{A}} = \sum_j (-1)^{i+j} A_{ij} M_{ij}$

minor M_{ij}: det of submatrix: deleting row i and column j of $\underline{\underline{A}}$.

$\det \underline{\underline{A}}^T = \det \underline{\underline{A}}, \quad \det(\underline{\underline{A}} \cdot \underline{\underline{B}}) = \det(\underline{\underline{A}}) \det(\underline{\underline{B}}), \quad \det(\underline{\underline{A}}^{-1}) = 1/\det \underline{\underline{A}}$

inverse: $(\underline{\underline{A}}^{-1})_{ij} = \frac{1}{\det \underline{\underline{A}}} \left[(-1)^{i+j} M_{ij} \right]$

Linear algebra revision

Eigensystems: $n \times n$ matrices

$$\underline{\underline{A}} \cdot \underline{v} = \lambda \underline{v}$$

$\underline{v}$: eigenvector, $\underline{v} \neq 0$

λ: eigenvalue

if $\underline{v}$ is solution, then $s\underline{v}$ is also solution

$$\underbrace{(\underline{\underline{A}} - \lambda \underline{\underline{I}})}_{\text{must be singular}} \cdot \underline{v} = 0$$

characteristic equation: $\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = 0$ $\quad$ n-degree polynomial

ex: generic 2×2 matrix: $\underline{\underline{A}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\det \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} = (a - \lambda)(d - \lambda) - bc = 0$$

Linear algebra revision

Eigensystems

triangular matrices: eg. $\underline{\underline{A}} = \begin{bmatrix} a & * & * \\ 0 & b & * \\ 0 & 0 & c \end{bmatrix}$ upper triangular matrix

$$\det \begin{bmatrix} a-\lambda & * & * \\ 0 & b-\lambda & * \\ 0 & 0 & c-\lambda \end{bmatrix} = (a-\lambda)(b-\lambda)(c-\lambda)(-1)^* = 0$$

eigenvalues are on the diagonal

similarity: suppose $\underline{\underline{P}}$ is invertible

then $\underline{\underline{A}}$ and $\underline{\underline{P}} \cdot \underline{\underline{A}} \cdot \underline{\underline{P}}^{-1}$ are called similar

they have the same eigenvalues (but not necessarily same eigenvectors)

Linear algebra revision

Diagonalization

for $\underline{\underline{A}}$: can find non-singular $\underline{\underline{P}}$ and diagonal $\underline{\underline{D}}$: $\underline{\underline{A}} = \underline{\underline{P}} \cdot \underline{\underline{D}} \cdot \underline{\underline{P}}^{-1}$

$\underline{\underline{A}}$ is similar to a diagonal matrix $\Leftrightarrow$ has n lin. indep. eigenvectors

ex: $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\quad \det = \begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix} = (1-\lambda)^2 = 0 \quad \Rightarrow \quad \lambda = 1$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+y \\ y \end{bmatrix} = 1 \begin{bmatrix} x \\ y \end{bmatrix} \quad \Rightarrow y = 0$$

so eigenvector: $v = \begin{bmatrix} x \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ only one lin.indep. eigenvector

Linear algebra revision

Diagonalization

$$\text{ex: } A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \qquad \det = \begin{bmatrix} 2-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix} = (2-\lambda)^2 = 0 \qquad \Rightarrow \qquad \lambda = 2$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix} = 2 \begin{bmatrix} x \\ y \end{bmatrix} \quad \Rightarrow \quad \text{any } x, y$$

so eigenvector: $v = \begin{bmatrix} x \\ y \end{bmatrix}$, or linearly independent: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Scalar / dot / inner product of vectors

$$\underline{u} \cdot \underline{v} = \underline{u}^T \underline{v} = \sum_i u_i v_i$$

length: $|\underline{v}| = \sqrt{\underline{v} \cdot \underline{v}}$

angle: $\underline{u} \cdot \underline{v} = |\underline{u}||\underline{v}| \cos\theta$

orthogonal: $\underline{u} \cdot \underline{v} = 0$

orthogonal set: pairwise orthogonal

orthonormal set: orthogonal, and normalized: $\underline{u}^{(i)} \cdot \underline{u}^{(j)} = \delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}$

Linear algebra revision

Orthogonal matrix

$$\underline{\underline{Q}} = \begin{bmatrix} \Bigg| & & \Bigg| & & \Bigg| \\ u^{(1)} & \Bigg| & \dots & \Bigg| & u^{(n)} \\ \Bigg| & & \Bigg| & & \Bigg| \end{bmatrix} \qquad \underline{u}^{(i)} \cdot \underline{u}^{(j)} = \delta_{ij} \quad \Leftrightarrow \quad \underline{\underline{Q}}^T \cdot \underline{\underline{Q}} = \underline{\underline{I}}$$

column vectors are orthonormal set

invertible: $\underline{\underline{Q}}^{-1} = \underline{\underline{Q}}^T$

$\det \underline{\underline{Q}} = \pm 1$

preserve dot product:

$$(\underline{\underline{Q}} \cdot \underline{u}) \cdot (\underline{\underline{Q}} \cdot \underline{v}) = Q_{ij} u_j Q_{ik} v_k = u_j \underbrace{Q^T_{ji} Q_{ik}}_{\delta_{jk}} v_k = u_j v_j = \underline{u} \cdot \underline{v}$$

Linear algebra revision

Symmetric diagonalizable matrices $n \times n$

- have n real eigenvalues (maybe with multiplicities)
- eigenvectors for different eigenvalues are orthogonal
- eigenvectors can be chosen all orthogonal: $\underline{\underline{A}} = \underline{\underline{Q}} \cdot \underline{\underline{D}} \cdot \underline{\underline{Q}}^T$

Matrix functions $n \times n$

power: $\underline{\underline{A}}^k = \underbrace{\underline{\underline{A}} \cdot \underline{\underline{A}} \cdot \ldots \cdot \underline{\underline{A}}}_{k}$

polynomial: linear combination of powers

Taylor expansion: ex: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \Rightarrow \quad \exp(\underline{\underline{A}}) = \sum_{n=0}^{\infty} \frac{\underline{\underline{A}}^n}{n!}$