AI Robotics

Learning Objectives

- Apply the Jacobian to solve the velocity kinematics problem
- Understand how the Jacobian relates wrenches to joint torques
- Understand how kinematic singularities effect robots and relate to the form of the Jacobian

Outline

1. Jacobians
   - Space Jacobian
   - Body Jacobian

2. Statics
   - Joint torques
   - Singularities

Velocity Kinematics

- Previously we saw the forward kinematics which is the determination of the position and orientation of the end-effector based on the configuration of its joint variables.
- Velocity kinematics is the determination of the end-effector based on the velocities of the joint variables.

Jacobian Matrices

- Jacobian for a function $f(\theta)$ is defined as

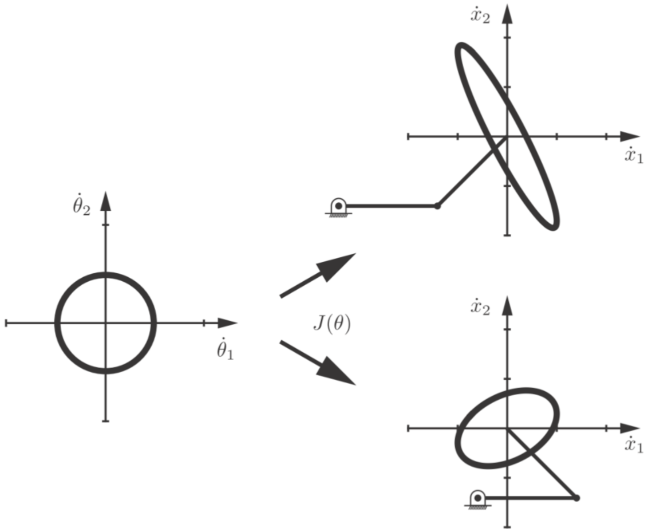
$$J(\theta) = \left[\frac{\partial f}{\partial \theta}\right]$$

- If the f is a function of time, the Jacobian relates the velocities of f and θ

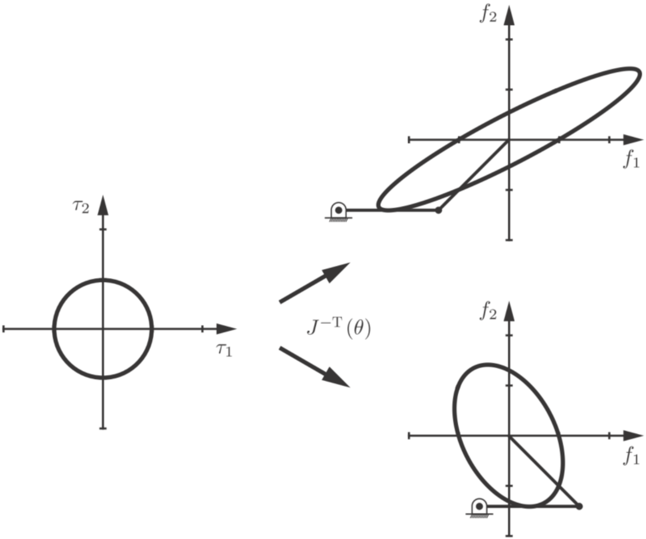
$$\begin{aligned}\frac{df}{dt} &= \frac{\partial f(\theta)}{\partial \theta}\frac{\mathrm{d}\theta(t)}{\mathrm{d}t} = \frac{\partial f(\theta)}{\partial \theta}\dot{\theta} \\ &= J(\theta)\dot{\theta}\end{aligned}$$

- The Jacobian gives the sensitivity of the end-effector velocity to the joint velocity

Manipulability Ellipsoid



Force Ellipsoid



Outline

Space Jacobian

- Recall that we solve the foward kinematics problem with the Product of Exponentials

$$T_{sb}(\theta_0, \ldots, \theta_n) = e^{[S_1]\theta_1} \ldots e^{[S_n]\theta_n} M$$

- The space Jacobian relates the joint velocity $\dot{\theta}$ to the spatial twist $\mathcal{V}_s = (\omega_s, \nu_s)$

$$\mathcal{V}_s = J_s(\theta)\dot{\theta} = J_{s1}(\theta)\dot{\theta}_1 + \cdots + J_{sn}(\theta)\dot{\theta}_n$$

Space Jacobian Derivation

- The spatial twist $\mathcal{V}_s$ is $[\mathcal{V}_S] = \dot{T}T^{-1}$

$$\begin{aligned}\dot{T} &= \left(\frac{d}{dt}e^{[\mathcal{S}_1]\theta_1}\right)\cdots e^{[\mathcal{S}_n]\theta_n}M + e^{[\mathcal{S}_1]\theta_1}\left(\frac{d}{dt}e^{[\mathcal{S}_2]\theta_2}\right)\cdots e^{[\mathcal{S}_n]\theta_n}M + \cdots \\ &= [\mathcal{S}_1]\,\dot{\theta}_1 e^{[\mathcal{S}_1]\theta_1}\cdots e^{[\mathcal{S}_n]\theta_n}M + e^{[\mathcal{S}_1]\theta_1}[\mathcal{S}_2]\,\dot{\theta}_2 e^{[\mathcal{S}_2]\theta_2}\cdots e^{[\mathcal{S}_n]\theta_n}M + \cdots\end{aligned}$$

$$T^{-1} = M^{-1}e^{-[\mathcal{S}_n]\theta_n}\cdots e^{-[\mathcal{S}_1]\theta_1}$$

$$\begin{aligned}[\mathcal{V}_s] = [\mathcal{S}_1]\,\dot{\theta}_1 &+ e^{[\mathcal{S}_1]\theta_1}[\mathcal{S}_2]\,e^{-[\mathcal{S}_1]\theta_1}\dot{\theta}_2 \\ &+ e^{[\mathcal{S}_1]\theta_1}e^{[\mathcal{S}_2]\theta_2}[\mathcal{S}_3]\,e^{-[\mathcal{S}_2]\theta_2}e^{-[\mathcal{S}_1]\theta_1}\dot{\theta}_3 + \cdots\end{aligned}$$

$$\mathcal{V}_s = \underbrace{\mathcal{S}_1}_{J_{s1}}\dot{\theta}_1 + \underbrace{\mathrm{Ad}_{e^{[\mathcal{S}_1]\theta_1}}(\mathcal{S}_2)}_{J_{s2}}\dot{\theta}_2 + \underbrace{\mathrm{Ad}_{e^{[\mathcal{S}_1]\theta_1}e^{[\mathcal{S}_2]\theta_2}}(\mathcal{S}_3)}_{J_{s3}}\dot{\theta}_3 + \cdots$$

Space Jacobian Overview

- Robot with joint configuration $\theta = (\theta_0, \dots, \theta_n)$
- Find screw axis $\mathcal{S}_1 = [\omega_{s1}, \nu_{s1}]$ when robot is in the home position. $J_{s1} = \mathcal{S}_1$
- Find screw axis $\mathcal{S}_2 = [\omega_{s2}, \nu_{s2}]$ after moving joint 1 from zero to θ_1. $J_{s2} = \mathcal{S}_2$
- Find screw axis $\mathcal{S}_3 = [\omega_{s3}, \nu_{s3}]$ after moving first and second joints from zero to θ_1 and θ_2. $J_{s3} = \mathcal{S}_3$

Example: Space Jacobian

Outline

Body Jacobian

- The same procedure can be applied to determine the relationship between the joint rates $\dot{\theta}$ and the body twist $\mathcal{V}_b$
- We derived the space Jacobian from the relationship $[\mathcal{V}_s] = \dot{T}T^{-1}$ the same process can be undertaken by starting with the relationship $[\mathcal{V}_b] = T^{-1}\dot{T}$

- The body and space Jacobians are related through the Adjoint of the transformation matrix

$$J_s(\theta) = [Ad_{T_{sb}}]J_b(\theta)$$
$$J_b(\theta) = [Ad_{T_{bs}}]J_s(\theta)$$

Outline

Statics of Open Chains

- Suppose that during movement of the end-effector an external force is applied to the tip.
- What are the torques required to be generated by the joint motors?
- Power generated at the joints must be equal to the power generated at the tip
- Denote the force at the tip as f_{tip} and the joint torque as τ

$$f_{tip}^T v_{tip} = \tau^T \dot{\theta}$$

- Previous seen that $v_{tip} = J(\theta)\dot{\theta}$

$$f_{tip}^T J(\theta)\dot{\theta} = \tau^T \dot{\theta}$$
$$\tau = J^T(\theta) f_{tip}$$

- This problem can be recast to use wrenches $\mathcal{F} = [m, f]$ and twists $\mathcal{V}$.

$$\tau^T \dot{\theta} = \mathcal{F}_b^T \mathcal{V}_b$$
$$\tau = J_b^T(\theta)\mathcal{F}_b = J_s^T(\theta)\mathcal{F}_s$$

- If an external wrench $-\mathcal{F}$ is applied to the end-effector the above equation calculates the neccessary joint torques to create an equal and opposing wrench $\mathcal{F}$

Outline

Singularity Analysis

- Kinematic singularities are points in which the end-effector can no longer move in one or more directions
- These points correspond to a Jacobian which is not full-rank $rank(J(\theta)) < min(6, n)$
- In this case not all columns of the Jacobian are linearly independent

Singularity Analysis

- Two or more collinear revoulte joint axes
- Coplanar and parallel revolute joint axes
- Four revolute joint axis which intersect at a point

Summary

- Velocity kinematics is the process of determining the configuration of the end-effector based on the configuration of the joints
- The Jacobian matrix is an essential part of mapping the relationship between joint rates and tip velocity
- The Jacobian matrix can be expressed in either the space frame or the body frame with the transformation given as
- Kinematic singularities can occur for certain configurations. In these configurations the robot loses its ability to move in one or more directions
- The Jacobian is also used to calculate the tip force given the joint motor torques and alternatively the required joint motor torques to create a desired tip force

- Next Lecture
  - Inverse Kinematics