Gale-Shapley algorithm

b_1 : ~~3~~ ~~2~~ 5 1 4

b_2 : ~~1~~ 2 5 3 4

b_3 : 4 3 2 1 5

b_4 : ~~1~~ 3 4 2 5

b_5 : ~~1~~ 2 4 5 3

g_1 : 3 5 2 1 4

g_2 : 5 2 1 4 3

g_3 : 4 3 5 1 2

g_4 : 1 2 3 4 5

g_5 : 2 3 4 1 5

$\boxed{\text{GSA}}$

*	g_1	g_2	g_3	g_4	g_5
1.	~~b_2~~, ~~b_4~~, b_5		b_1	b_3	
2.	b_5	b_2	~~b_1~~, b_4	b_3	
3.	b_5	b_2, ~~b_1~~	b_4	b_3	
4.	b_5	b_2	b_4	b_3	b_1

$\boxed{\text{girl version:}}$

	b_1	b_2	b_3	b_4	b_5
1. *	g_4	g_5	g_1	g_3	g_2

Theorem: The GSA provides a stable matching
for <u>any</u> preference lists.

girl

Definition: A stable matching is a boy
girl
optimal stable matching if every boy
boy her
gets the best girl from his realm
boys
of possibility, which is the set of girls
her
that can be his partner in stable matchings.

Theorem

The original GSA is boy optimal.
The girl version is girl optimal.

Consequence: if the original GSA and the girl version provides the same matching then there is only one stable matching in that scenario.

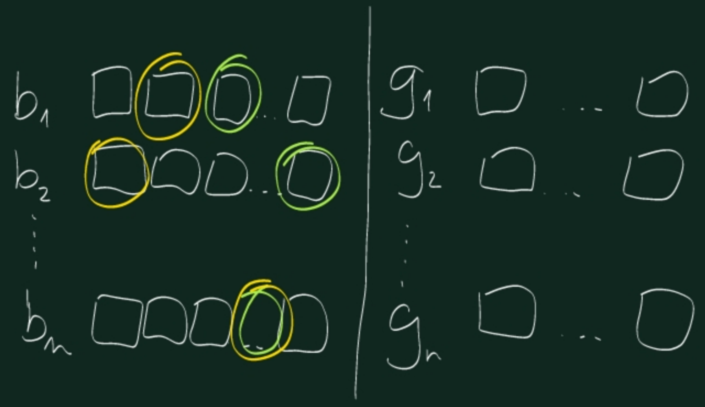
The GSA will stop after $\leq n^2+1$ days

$\longrightarrow$ at least one girl crossed out each day

Exercise:

$$\boxed{\text{Prove that the GSA always stops after } \leq (n-1)^2+1 \text{ days.}}$$

For n girls and n boys you can give preference lists such that the GSA goes on for exactly $(n-1)^2+1$ days.



(1.) There is always at least 1 girl who never <u>rejects</u> any boys, so she is never crossed from the lists.

$\boxed{\text{minus } n \text{ crosses}}$

(2.) No boy crosses his final partner from his list, so we have minus $-n$ crosses $+1$ because for one boy these two girls are the same.

$\boxed{\text{minus } n-1 \text{ crosses}}$

$$n^2+1-n-(n-1)=$$
$$=n^2-2n+2=$$
$$=(n-1)^2+1$$

Exercise:

Construct preference lists for

3 girls 3 boys such that GSA goes for 5 days

4 girls 4 boys 10 days

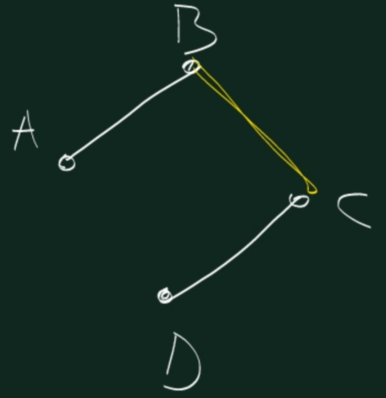
⋮

n girls n boys — " — $(n-1)^2+1$ days

Generalization: Stable roommate problem:

We have n girls with preference lists of each other and we would like to have a matching among them which is stable. $\longrightarrow$

Ann: BCD
Betty: CAD
Carla: ABD
Denise: ACB



AC is <u>not</u> rogue

BC is rogue

{AB}, {CD} is NOT STABLE

There is no algorithm solving the stable roommate problem, because for some preference lists there are no stable matchings at all.

(check all matchings on the last page

(AB) (CD)
(AC) (BD)
(AD) (BC)

None of them are stable)

Now we go back to boys and girls!

Exercise:

Prove that if all girls have the same preference list, then there is only 1 stable matching.

$b_1 : g_1 \ \cdots$

$b_2 : \cancel{g_1}\ g_2$

$\quad\ \ g_2$

$b_3 : \cancel{g_1}$ —— ——

$\quad\ \ \cancel{g_1}\ \cancel{g_2}$

$\quad\ \ g_2\ g_1$

$\vdots$

$b_n :$ ——

$g_1 : b_1\ b_2\ b_3 \ \dots\ b_n$

$g_2 : b_1\ b_2\ b_3 \ \dots\ b_n$

$\vdots$

$g_n : b_1\ b_2\ b_3 \ \dots\ b_n$

b_1	g_1 1 2 3 4 5
b_2	g_2 —‖—
b_3	g_3 —‖—
b_4	g_4 —‖—
b_5	g_5 —‖—

GSA:

If the first girl on b_1's list is g_1, then they will be matched to each other.

Both the GSA and the girl version of the GSA will match b_1 to the first girl on his list. (We will call her g_1.)

So any stable matching will match b_1 to g_1.

If g_1 is the first on b_2's list, then he will get the second girl on his list, but if g_1 is not the first on b_2's list, then he will get the first girl on his list

This is true for both the original GSA and the girl version.

$\longrightarrow$ So this is true for <u>any</u> stable matching.

Inductional step: if we know that b_1 is always matched to g_1, b_2 is always matched to g_2, b_3 is always g_3, ..., b_{k-1} is always m. to g_k then:

For the kth boy:

b_k is matched to the first girl on his list that is different from $g_1, g_2, g_3, \dots, g_{k-1}$ by both versions of the GSA. We will call this girl g_k. So <u>all</u> stable matchings match b_k to g_k.

Exercise:

Prove that if a stable matching is <u>boy optimal</u> then at most <u>one</u> boy can be matched to the last girl on his list.