AI Robotics

Learning Objectives

- Understand the purpose and usage of rotation matrices
- Understand the concept of exponential coordinates
- Understand how to rotation is extended into full rigid body motion

Outline

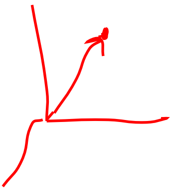
1. Rotations and Angular Velocity

2. Exponential Coordinates

3. Twists, Screws, Wrenches

Vectors

- A free vector is a geometric object described by a length and a direction.
- Any point in a space can be represented by a vector from some reference frame origin to the point



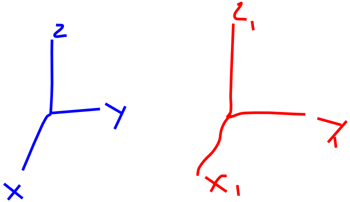
Reference Frames

- Reference frames can be placed at any point and in any orientation
- Reference frames are stationary
- Two fundamental reference frames are used:
  - Fixed/Space Frames denoted $\{s\} = \{\hat{x}_s, \hat{y}_s, \hat{z}_s\}$
  - Body Frame denoted $\{b\} = \{\hat{x}_b, \hat{y}_b, \hat{z}_b\}$
- $\{b\}$ is fixed relative to a specific location on a body
- All reference frames are right handed

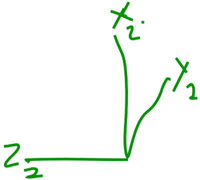
Rotation Matrix

- Rotation matrices can be used for:
  - Represent an orientation
  - Change the reference frame of a vector
  - Rotate a vector or reference frame

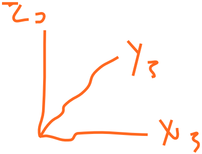
- The rotation matrix R represents a frame



$$R_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$R_2 = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$



$$R_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- In the second view R can provide change of reference frame

$\{a\}, \{b\}, \{c\}$ R_{ab}

R_{bc}

$R_{ac} = R_{a\cancel{b}} R_{\cancel{b}c}$

- The rotation matrix R acts as an operator which rotates a reference frame

$$R = Rot(\hat{\omega}, \theta)$$

$$Rot(\hat{x}, \theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & -s\theta \\ 0 & s\theta & c\theta \end{bmatrix}$$

$$R_x(\theta) R_y(\theta)$$

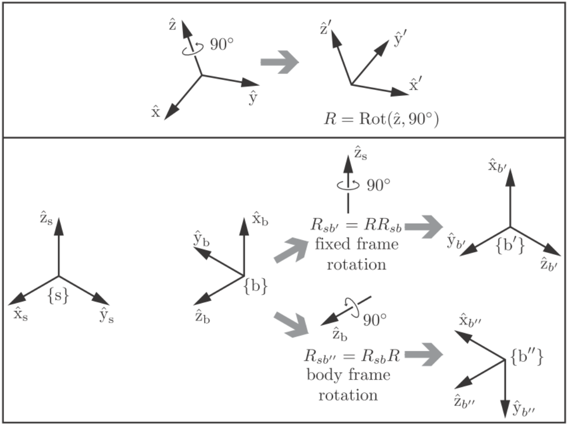
- R is an orthonormal matrix therefore we can write the inner product of the columns of R as

$$r_i^T r_j = \delta_{ij} = \begin{cases} 1 & \text{when } i = j \\ 0 & \text{when } i \neq j \end{cases}$$

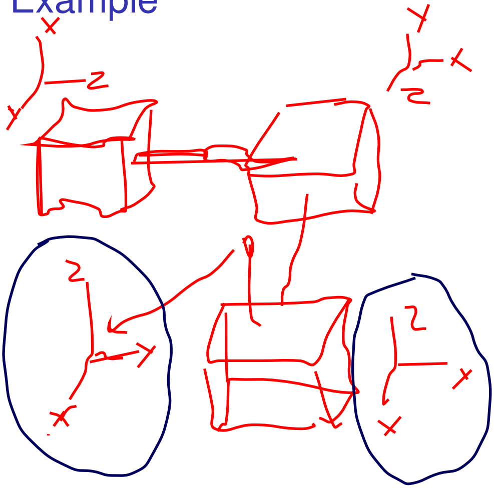
- From this two key properties of Rotation matrices can be determined

$$RR^T = R^T R = I$$
$$det(R) = \pm 1$$

- The order in which the rotation matrix is applied matters
- Pre-multiplying R results in a rotation about an axis $\hat{\omega}$ in the fixed frame
- Post-multiplying R results in a rotation about $\hat{\omega}$ in the body frame



Example



$$R_{ab} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$R_{bc} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$R_{cd} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_{ad} = R_{ab} R_{bc} R_{cd}$$

SO(3)

- Rotation Matrices are part of the Special Orthogonal Group (SO)
- Members of SO(n) satisfy the following properties:
  - $RR^T = \mathbb{I}$
  - $det(R) = 1$
- In general a group G has the following axioms:
  - Closure
  - Associativity
  - Identity Element
  - Inverse Element

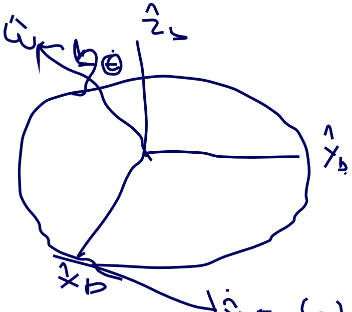
- R preserves distance $||Rq - Rp|| = ||q - p|| \; \forall q, p \in \mathbb{R}^3$
- R preserves orientation $R(v \times w) = Rv \times Rw \; \forall v, w \in \mathbb{R}^3$

$$
\begin{aligned}
||Rq - Rp||^2 &= (R(q-p))^T(R(q-p)) = (q-p)^T R^T R(q-p) \\
&= (q-p)^T(q-p) = ||q-p||^2
\end{aligned}
$$

Angular Velocities

- Given a reference frame attached to a rotating body the change in its orientation is described by a rotation angle θ about some axis $\hat{\omega}$
- Angular velocity $\omega = \hat{\omega}\dot{\theta}$
- The velocities of the individual components $\hat{x}, \hat{y}, \hat{z}$ are given by

$$\dot{\hat{x}}_b = \omega \times \hat{x}_b$$
$$\dot{\hat{y}}_b = \omega \times \hat{y}_b$$
$$\dot{\hat{z}}_b = \omega \times \hat{z}_b$$



$\dot{\hat{x}} = \omega \times \hat{x}_b$

Angular Velocities

- The above equations can be rewritten in the space frame coordinates as

$$\begin{aligned}
\dot{r}_i &= \omega_s \times r_i \\
\dot{R} &= \begin{bmatrix} \omega_s \times r_1 & \omega_s \times r_2 & \omega_s \times r_3 \end{bmatrix} = \omega_s \times R \\
&= [\omega_s] R
\end{aligned}$$

- For any vector $\mathbf{x} \in \mathbb{R}^3$ there is a matrix $[\mathbf{x}] \in \mathbb{R}^{3x3}$ such that $\mathbf{x} \times p = [\mathbf{x}]p$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$[\mathbf{x}] = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix}$$

- An alternative way of looking at rotation is by the rotation around an axis ω for a rotation angle θ
- All rotations are elements of $SO(3)$ it follows that we can parameterise a rotation matrix by a rotation axis ω and a rotation angle θ
- This mapping is done with the use of exponential coordinates

- Recall the linear differential equation $\dot{x}(t) = ax(t)$

It has the solution $x(t) = e^{at}x_0$
Where e^{at} has the series expansion

$$e^{at} = 1 + at + \frac{(at)^2}{2!} + \frac{(at)^3}{3!} + \cdots$$

We can represent the same problem in a matrix form
$\dot{x}(t) = Ax(t)$ which has the solution

$$x(t) = e^{At}x_0$$

Like the previous case e^{At} can be expressed in a series expansion

$$e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots$$

- The motion of a point p about an axis $\hat{\omega}$ is given by

$$\dot{p} = \hat{\omega} \times p(t)$$

- Using the 3×3 skew-symmetric representation

$$\dot{p} = [\hat{\omega}]p(t)$$
$$p = e^{[\hat{\omega}]\theta} p(0)$$

Which has a closed form solution

$$Rot(\omega, \theta) = e^{[\hat{\omega}]\theta} = \boxed{I + sin\theta[\hat{\omega}] + (1 - cos\theta)[\hat{\omega}]^2}$$

- The components of $\omega\theta \in \mathbb{R}^3$ are the exponential coordinates for the rotation matrix R

- Treated rotation and translations separately, how to represent them in a single representation
- Define the homogeneous transformation matrix

$$T = \begin{bmatrix} R & p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_1 \\ r_{21} & r_{22} & r_{23} & p_2 \\ r_{31} & r_{32} & r_{33} & p_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T^{-1} = \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix}$$

- T can no longer be applied to a point $p \in \mathbb{R}^3$ so we introduce the homogeneous representation of a point

$$p = \begin{bmatrix} p_x \\ p_y \\ p_z \\ 1 \end{bmatrix}$$

- A transformation is a rigid body transformation if it satisfies he following:
  - Transformation preserves length
  - Transformation preserves the cross product
- These properties have the following results:
  - Inner products are preserved
  - Angles are preserved
  - Orthogonal vectors remain orthogonal
  - Right hand reference frames remain right handed

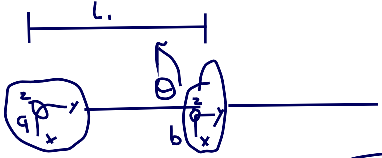
- Transformation matrices are elements of $SE(3)$ and have the following properties
  - The inverse of a transformation matrix is also a transformation matrix
  - The product of two transformation matrices is also a transformation matrix
  - Multiplication is associative
  - Multiplication is not generally commutative
  - Distances between points is preserved
  - Orientation between vectors is preserved

Use of Transformation matrices

- Represents a rigid body configuration
- Change of reference frame
- As an operator that displaces a vector
- As an operator that displaces a frame

Example

- Consider the rotation of the ~~second~~ First link in a 2R planar robot



$$R_{ab} = \begin{bmatrix} c\theta & -s\theta & 0 \\ s\theta & c\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_{ab} = \begin{bmatrix} R_{ab} & p_{ab} \\ 0 & 1 \end{bmatrix}$$

$$p_{ab} = \begin{bmatrix} 0 \\ L_1 \\ 0 \end{bmatrix}$$

- Screw theory is a mathematical framework of rigid body mechanics
- Rigid body motion is defined as a rotation about an axis and a translation about the same axis
- Three objects make up Screw theory:
  - Screw: A 6 vector which is made up of a pair of 3 vectors such as the linear velocity and angular velocity
  - Twist: Represents the velocity of a rigid body
  - Wrench: Represents the forces and torques applied to a body

- Linear and Angular velocities can be combined using T
- For rotations we could go from $SO(3)$ to $so(3)$ the skew-symmetric representation of rotations by $\dot{R}R^{-1}$
- The same can be applied to transformation matrices

$$
\begin{aligned}
T^{-1}\dot{T} &= \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{R} & \dot{p} \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} R^T\dot{R} & R^T\dot{p} \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} [\omega_b] & \nu_b \\ 0 & 0 \end{bmatrix}
\end{aligned}
$$

- The representation in the space frame $\{s\}$ can be found by evaluating $\dot{T}T^{-1}$

- Linear and Angular velocities can be combined in a single six vector called the Spatial Velocity or Twist

$$\mathcal{V}_b = \begin{bmatrix} \omega_b \\ \nu_b \end{bmatrix} \in \mathbb{R}^6$$

$$T^{-1}\dot{T} = [\mathcal{V}_b] = \begin{bmatrix} [\omega_b] & \nu_b \\ 0 & 0 \end{bmatrix}$$

- The representation in the space frame $\{s\}$ can be found by evaluating $\dot{T}T^{-1}$

$$\dot{T}T^{-1} = \begin{bmatrix} \dot{R} & \dot{p} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R^T & -R^T p \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \dot{R}R^T & \dot{p} - \dot{R}R^T \\ 0 & 0 \end{bmatrix}$$

$$[\mathcal{V}_s] = \begin{bmatrix} [\omega_s] & \nu_s \\ 0 & 0 \end{bmatrix}$$

- To go from the twist in one frame to that of a new frame we cannot use the subscript cancellation rule with the Transformation matrix T as T is 4×4 and $\mathcal{V}$ is 6×1
- To provide change of reference frame we need to define the adjoint representation of T

$$[Ad_T] = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix}$$

- The combination of linear and angular motion can be described with the motion of a screw.
- Consider a rigid body transformation involving a rotation about an axis for an angle of θ which is followed with a linear translation along the same axis for a distance d.
- The pitch of a screw is given as $h = \frac{d}{\theta}$ making the total linear motion $h\theta$

- Every rigid-body displacement can be expressed as a displacement along a screw axis $\mathcal{S}$
- Similar to the exponential coordinate for rotation we can define the exponential coordinate for a homogeneous transform as $\mathcal{S}\theta$ where $\mathcal{S}$ is the screw axis and θ the distance to travel along $\mathcal{S}$

$$\mathcal{S} = \begin{bmatrix} \omega \\ \nu \end{bmatrix} \in \mathbb{R}^6$$

$$[\mathcal{S}] = \begin{bmatrix} [\omega] & \nu \\ 0 & 0 \end{bmatrix}$$

- We can define the six dimension exponential coordinates of a transformation as $\mathcal{S}\theta$ analagous to the case for rotation $\hat{\omega}\theta$
- Two cases to consider
  - If the pitch is finite then θ is the angle of rotation about the screw axis
  - If the pitch is infinite then θ is the linear distance traveled along the screw axis

For the case of finite pitch

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} e^{[\omega]\theta} & (I\theta + (1 - \cos\theta)[\omega] + (\theta - \sin\theta[\omega]^2)\nu \\ 0 & 1 \end{bmatrix}$$

For infinite pitch

$$e^{[\mathcal{S}]\theta} = \begin{bmatrix} I & \nu\theta \\ 0 & 1 \end{bmatrix}$$

- We can define the six dimension representation of a force as $\mathcal{F}$ this is called the wrench

$$\mathcal{F} = \begin{bmatrix} m_a \\ f_a \end{bmatrix} \in \mathbb{R}^6$$

- Here f_a describes the 3 spatial components of an applied force
- m_a describes the torque/moment generated by the force applied at a point r from the origin and is given as $m_a = r_a \times f_a$
- If multiple wrenches act on a body then the total wrench is just the vector sum of all wrenches
- We can use the Adjoint representation of the transformation matrix to go from one frame to another for example:

Given a wrench $\mathcal{F}_a$,

$$\mathcal{F}_b = [Ad_{T_{ab}}]^T \mathcal{F}_a$$

Summary

- Rotation matrices describe reference frames, allow to convert from one frame to another, and allow to transform a reference frame over some angle θ
- Homogeneous transformation matrices describe rigid-body motion.
- The Twist $\mathcal{V}$ describes the velocity of the rigid-body
- Wrenches $\mathcal{F}$ describe forces and torques applied to rigid bodies

- Next Lecture
  - Using the elements describe in this lecture to solve inverse kinematics problems in robots