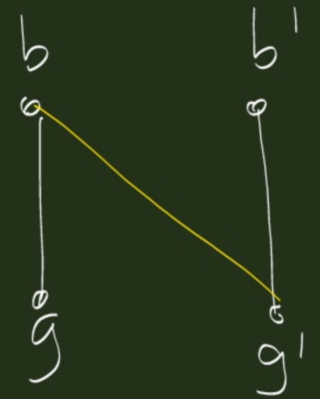
Stable marriage problem

We are given n boys and n girls and all their "preference lists".

We would like to find a matching that is stable = it has no rogue couples.

rogue couple: in a matching M

the couple (b, g) is a rogue couple, if they are not matched to each other by M, but they would prefer each other over their current partner



b likes g' more than g

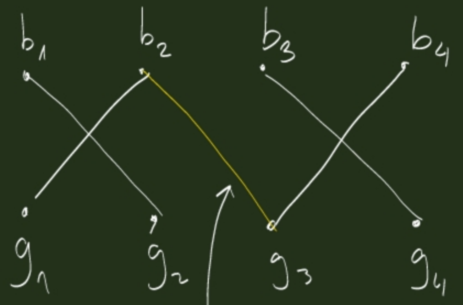
b' likes g more than g'

$b_1, b_2, b_3, b_4 \leftarrow$ boys

$g_1, g_2, g_3, g_4 \leftarrow$ girls

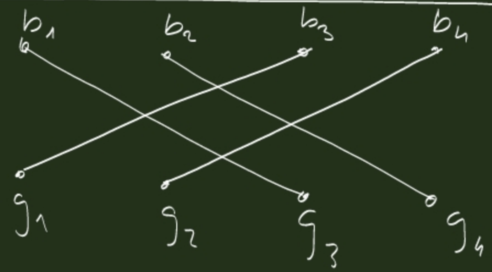
b_1:	g_4	g_1	g_2	g_3	g_1:	b_3	b_4	b_2	b_1
b_2:	g_3	g_2	g_1	g_4	g_2:	b_4	b_3	b_1	b_2
b_3:	g_3	g_2	g_1	g_4	g_3:	b_1	b_3	b_2	b_4
b_4:	g_4	g_1	g_2	g_3	g_4:	b_2	b_4	b_1	b_3

$M = \{(b_1, g_2), (b_2, g_1), (b_3, g_4), (b_4, g_3)\}$



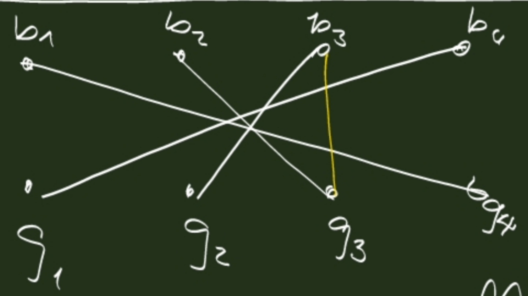
(b_2, g_3) rogue couple

$\Longrightarrow$ M is not stable



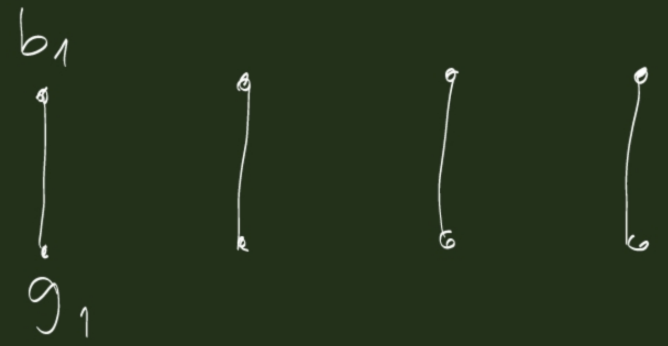
✓

Here every girl is matched to her favorite boy, so it is a stable matching.



(b_3, g_3) rogue c.

not stable



this is stable

There are 6 stable matchings in that setting. Try to find them all!

Is it possible to give preference lists to n boys and n girls such that there exists a stable matching where everybody is matched with the second

first

person on his/her list?

Same question, but "first" instead of "second"

$b_1 : g_3$ ----	$g_1 : b_2$ ----
$b_2 : g_1$ ----	$g_2 : b_5$ ----
$b_3 : g_4$ ----	$g_3 : b_1$ ----
$b_4 : g_5$ ----	$g_4 : b_3$ ----
$b_5 : g_2$ ----	$g_5 : b_4$ ----

yes!

We can write any order on the ---- part.

Everybody with the second ?

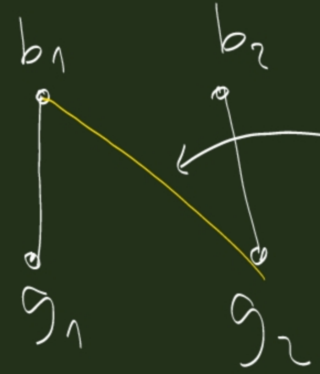
$\underline{\underline{n=2}}$

$$\begin{array}{ll|ll} b_1 : & ?\ ? & g_1 : & ?\ ? \\ b_2 : & ?\ ? & g_2 : & ?\ ? \end{array}$$

Let us assume that b_1 is matched with g_1 and b_2 is matched with g_2 and it is a stable matching, and everybody is matched with the second person on his/her list.

this is not stable

Then:
$$\begin{array}{ll|ll} b_1 : & g_2\ g_1 & g_1 : & b_2\ b_1 \\ b_2 : & g_1\ g_2 & g_2 : & b_1\ b_2 \end{array}$$



this is a rogue couple

$\Longrightarrow$ NOT STABLE

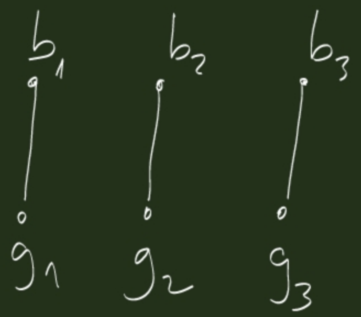
For $n=2$ (2 girls, 2 boys) it is not possible to give preference lists like that.

What about $\underline{n=3}$?

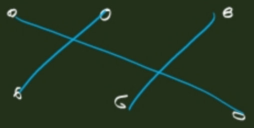
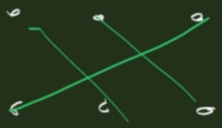
We can assume that b_i is matched with g_i for all $i=1,2,3$, and also that they are all second on each others preference lists.

b_1:	g_2	g_1	g_3	g_1:	b_2	b_1	b_3
b_2:	g_3	g_2	g_1	g_2:	b_3	b_2	b_1
b_3:	g_1	g_3	g_2	g_3:	b_1	b_3	b_2

M:



Can we complete the preference lists in such a way that $M = \{(b_1, g_1), (b_2, g_2), (b_3, g_3)\}$ is stable? YES

Observation:  and  are also STABLE

$b_1: \quad g_2 \ \underline{g_1} \ g_n \ g_{n-1} \ \dots \ g_4 \ g_3$

$b_2: \quad g_3 \ \underline{g_2} \ g_1 \ g_n \ \dots \ g_5 \ g_4$

$b_3: \quad g_4 \ g_3$

$\vdots \quad \vdots$

$b_n:$

$g_1: \quad b_2 \ \underline{b_1} \ b_n \ b_{n-1} \ \dots \ b_4 \ b_3$

$g_2: \quad b_3 \ \underline{b_2} \ b_1 \ b_n \ \dots \ b_5 \ b_4$

$g_3: \quad b_4 \ b_3$

Check if for $n=4$ is it a good example for the question!

"Many" stable matchings here.

n stable matchings at least (every column)

How many stable matchings can we have for n boys and n girls? What is the most we can have?

Exact number not known.

Approximately?

$\Theta(n)$? $\Theta(n^2)$? $\Theta(n^k)$? $\Theta(n!)$?

unlikely

$n!$ matchings all together

$\Theta(a^n)$

For n boys and n girls we can give preference lists such that the number of stable matchings is $\geqslant \sqrt{2}^{\,n}$.

$$\begin{cases} b_1 : g_1 \ g_2 \ - - - - \\ b_2 : g_2 \ g_1 \ - - - - \end{cases} \qquad g_1 : - - - - - - \ b_1$$

$$\hphantom{\begin{cases} b_1 \end{cases}} \qquad g_2 : - - - - \ b_2$$

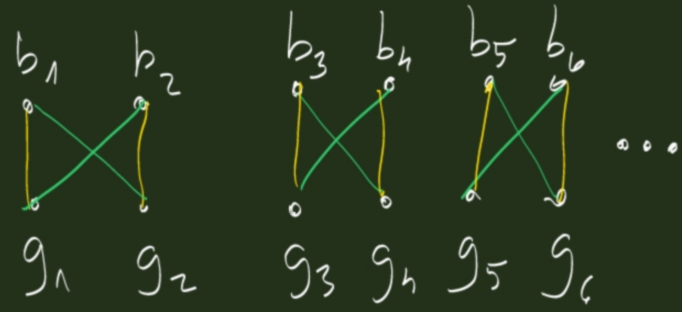
$$\begin{cases} b_3 : g_3 \ g_4 \ - - - - \\ b_4 : g_4 \ g_3 \ - - - - \end{cases} \qquad g_3 : - - - - - \ b_3$$

$$\hphantom{\begin{cases} b_1 \end{cases}} \qquad g_4 : - - - - - \ b_4$$

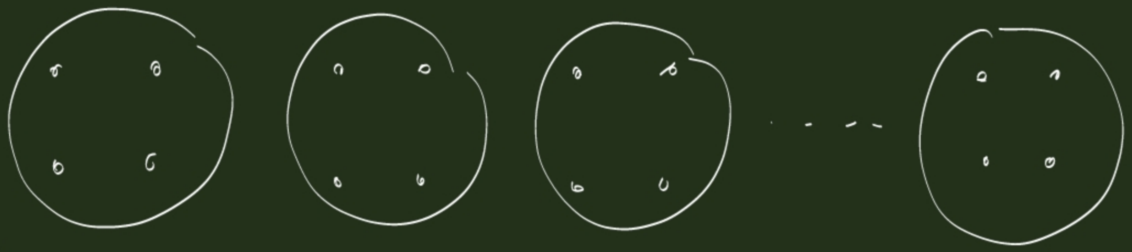
$$\begin{cases} b_5 : g_5 \ g_6 \ - - - - \\ b_6 : g_6 \ g_5 \ - - - - - \end{cases} \qquad g_5 : - - - - - \ b_5$$

$$\hphantom{\begin{cases} b_1 \end{cases}} \qquad g_6 : - - - - \ b_6$$

$$\vdots \qquad\qquad\qquad \vdots$$



in any group of 4 we can either take the green or the yellow matching



in every group we choose || or X

$\frac{n}{2}$ groups

How many matchings do we have?

$$2^{\frac{n}{2}}$$

Are they all stable?

if b_i is matched to g_i $\rightarrow$ b_i cannot be in a rogue couple

if b_i is matched to the other girl $\rightarrow$ — " —

Final question: is it true that for <u>any</u> preference list there is at least one stable matching?