- Problem 1.

Let V denote the vector space of univariate polynomials of degree at most 5 and having real coefficients.
Let us consider the subset $W := \{p : p \in V \text{ and } p(\pi) = 0\}$.
a.) Prove that W is a subspace of V.
b.) Find a basis for W, then determine $\dim(W)$.

Solution.
a.) Suppose that $p,\ q \in W$, and $\lambda,\ \mu \in \mathbb{R}$. Then $\lambda\, p + \mu\, q \in W$, because $\deg(\lambda\, p + \mu\, q) \leq 5$, the polynomial $\lambda\, p + \mu\, q$ has real coefficients, and $(\lambda\, p + \mu\, q)\,(\pi) = \lambda\, p(\pi) + \mu\, q(\pi) = 0$.

b.) We claim that, for example, $B := \{(x-\pi),\ (x-\pi)^2,\ \ldots,\ (x-\pi)^5\}$ is a basis for W. This will imply that $\dim(W) = 5$.
Clearly, B is a subset of W.

Now we verify that the elements of the set B are linearly independent. Indeed, suppose that we have, with some real constants $\alpha,\ \beta,\ \gamma,\ \delta$ and ϵ, that
$p(x) := \alpha\,(x-\pi) + \beta\,(x-\pi)^2 + \gamma\,(x-\pi)^3 + \delta\,(x-\pi)^4 + \epsilon\,(x-\pi)^5 = 0$ for all $x \in \mathbb{R}$.
Then, by taking successive derivatives of p, we get for all $x \in \mathbb{R}$ that

$$
\begin{aligned}
0 &= p^{(5)}(x) = 120\,\epsilon,\\
0 &= p^{(4)}(x) = 24\,\delta + 120\,(x-\pi)\,\epsilon,\\
0 &= p'''(x) = 6\,\gamma + 24\,(x-\pi)\,\delta + 60\,(x-\pi)^2\,\epsilon,\\
0 &= p''(x) = 2\,\beta + 6\,(x-\pi)\,\gamma + 12\,(x-\pi)^2\,\delta + 20\,(x-\pi)^3\,\epsilon,\\
0 &= p'(x) = \alpha + 2\,(x-\pi)\,\beta + 3\,(x-\pi)^2\,\gamma + 4\,(x-\pi)^3\,\delta + 5\,(x-\pi)^4\,\epsilon.
\end{aligned}
$$

From this we trivially get $\epsilon = 0$, $\delta = 0$, $\gamma = 0$, $\beta = 0$ and $\alpha = 0$.

Now we verify that $\operatorname{span}(B) = W$. Suppose that $p \in W$ is given arbitrarily. We want to find some real coefficients $\alpha,\ \beta,\ \gamma,\ \delta$ and ϵ such that
$p(x) = \alpha\,(x-\pi) + \beta\,(x-\pi)^2 + \gamma\,(x-\pi)^3 + \delta\,(x-\pi)^4 + \epsilon\,(x-\pi)^5$.
Since π is a root of p, the factorization theorem in algebra tells us that $p(x) = (x-\pi)\,g(x)$ should hold with some polynomial g and $\deg(g) \leq 4$. Suppose g has the form $g_4\,x^4 + g_3\,x^3 + g_2\,x^2 + g_1\,x^1 + g_0$.

So, after a division by the factor $(x-\pi)$, we see that it is enough to find $\alpha,\ \beta,\ \gamma,\ \delta$ and ϵ such that
$\alpha+\beta(x-\pi)+\gamma(x-\pi)^2+\delta(x-\pi)^3+\epsilon(x-\pi)^4=g_4x^4+g_3x^3+g_2x^2+g_1x^1+g_0$,
where $g_j\in\mathbb{R}$ are arbitrary.
By rearranging the left-hand side, we see that it is equal to
$\epsilon x^4+(\delta-4\pi\epsilon)x^3+(\gamma-3\pi\delta+6\pi^2\epsilon)x^2+$
$(\beta-2\pi\gamma+3\pi^2\delta-4\pi^3\epsilon)x+(\alpha-\pi\beta+\pi^2\gamma-\pi^3\delta+\pi^4\epsilon)$.
It is trivial that the (unique) solution of the (triangular) system

$$\begin{aligned}&\epsilon=g_4,\\&\delta-4\pi\epsilon=g_3,\\&\gamma-3\pi\delta+6\pi^2\epsilon=g_2,\\&\beta-2\pi\gamma+3\pi^2\delta-4\pi^3\epsilon=g_1,\\&\alpha-\pi\beta+\pi^2\gamma-\pi^3\delta+\pi^4\epsilon=g_0\end{aligned}$$

is

$$\begin{aligned}&\epsilon=g_4,\\&\delta=g_3+4\pi\epsilon,\\&\gamma=g_2+3\pi\delta-6\pi^2\epsilon,\\&\beta=g_1+2\pi\gamma-3\pi^2\delta+4\pi^3\epsilon,\\&\alpha=g_0+\pi\beta-\pi^2\gamma+\pi^3\delta-\pi^4\epsilon.\end{aligned}$$

- Problem 2.

Let $V:=\{a_1x^2+a_2y^2+a_3xy+a_4x+a_5y+a_6:\ a_i\in\mathbb{R}\}$ denote the vector space of bivariate polynomials of degree at most 2 with real coefficients. Let us consider the Laplacian operator $\Delta=\partial_{x,x}+\partial_{y,y}$ as an endomorphism of V.
a.) Find a basis for $\ker(\Delta)$, and determine $\dim(\ker(\Delta))$.
b.) Determine $\operatorname{rank}(\Delta)$.

Solution.

Let us consider an arbitrary element of
V, $p(x, y) := a_1 x^2 + a_2 y^2 + a_3 x y + a_4 x + a_5 y + a_6$.
Clearly, $\Delta p(x, y) = 2 a_1 + 2 a_2 \in \mathbb{R}$, and in general $2 a_1 + 2 a_2 \neq 0$. Since $\dim(\mathbb{R}) = 1$, and $\mathrm{im}(\Delta)$ is a subspace, we have that $\mathrm{rank}(\Delta) = \dim(\mathrm{im}(\Delta)) = 1$.
Clearly, we also have $\dim(V) = 6$.
So the rank-nullity theorem implies that $\dim(\ker(\Delta)) = \dim(V) - \dim(\mathrm{im}(\Delta)) = 5$.
From the above we also see that $p \in \ker(\Delta)$ if and only if $a_2 = -a_1$. Thus a basis for $\ker(\Delta)$ is, e.g., $\{1, x, y, x y, x^2 - y^2\}$.

■ Problem 3.

Let $V := C^\infty(\mathbb{R})$ denote the space of real functions that are infinitely many times differentiable, and let $A : V \to V$ be the derivative, $A(f) := f'$ for any $f \in V$.
Find the real eigenvalues and eigenfunctions (=eigenvectors) of A.

Solution.
For a given $\lambda \in \mathbb{R}$, the general solution of the ODE $f' = \lambda f$ is $f(x) = c\, e^{\lambda x}$ where $c \in \mathbb{R}$ is arbitrary. In other words, $A(f) = \lambda f$ holds if and only if $f(x) = c\, e^{\lambda x}$.
By choosing for example $c := 1$, the function $f(x) = e^{\lambda x}$ is different from the constant zero function (that is, it is not equal to the additive identity of the vector space V), and clearly, $f \in V$.
Hence any $\lambda \in \mathbb{R}$ is an eigenvalue of the operator A.
It also follows from the above that for a given $\lambda \in \mathbb{R}$, the only eigenfunction (up to a multiplicative a constant) is $e^{\lambda x}$.

■ Problem 4.

Let $S := \{(x_n)_{n \in \mathbb{N}} : x_j \in \mathbb{C} \text{ for all } j \in \mathbb{N}\}$ denote the vector space of complex sequences.

Let $L : S \to S$ denote the left-shift operator, and $R : S \to S$ the right-shift operator, defined as follows.
For any $x = (x_0, x_1, x_2, x_3, \ldots)$ we let $L(x) := (x_1, x_2, x_3, \ldots)$ and $R(x) := (0, x_0, x_1, x_2, \ldots)$.
a.) Determine the complex (including real) eigenvalues of the linear map L.
b.) Determine the complex (including real) eigenvalues of the linear map R.

Solution. The linearity of the maps L and R is trivial.

The constant $\lambda \in \mathbb{C}$ is an eigenvalue of L if and only if there is a sequence $x \in S$, not identically zero, such that $L(x) = \lambda x$, that is, if $L(x) - \lambda x = 0$.
If $x = (x_0, x_1, x_2, x_3, \ldots)$, then $L(x) - \lambda x = (x_1 - \lambda x_0, x_2 - \lambda x_1, x_3 - \lambda x_2, \ldots)$ is the zero sequence if and only if $x_{n+1} = \lambda x_n$ $(n = 0, 1, 2, \ldots)$, that is, if $x_n = \lambda^n x_0$ (here we define $0^0 := 1$). Thus, by setting, for example $x_0 := 1$, we see that any $\lambda \in \mathbb{C}$ is an eigenvalue , because then $x = (1, \lambda, \lambda^2, \lambda^3, \ldots)$ is a non-trivial eigenvector. This eigenvector is unique up to a multiplicative non-zero constant $x_0 \neq 0$.

The constant $\lambda \in \mathbb{C}$ is an eigenvalue of R if and only if there is a sequence $x \in S$, not identically zero, such that $R(x) = \lambda x$, that is, if $R(x) - \lambda x = 0$.
If $x = (x_0, x_1, x_2, x_3, \ldots)$, then $R(x) - \lambda x = (0 - \lambda x_0, x_0 - \lambda x_1, x_1 - \lambda x_2, \ldots)$.
Case 1.) If $\lambda = 0$, then $R(x) - \lambda x$ is zero if and only if $x_n = 0$ for all $n \geq 0$. Therefore there is no non-trivial eigenvector, hence $\lambda = 0$ is not an eigenvalue.
Case 2.) If $\lambda \neq 0$, then the 1st component of $R(x) - \lambda x$ is zero if and only if $x_0 = 0$. So the second component of $R(x) - \lambda x$ is zero if and only if $x_1 = 0$. Proceeding recursively we see that $x_n = 0$ for all $n \geq 0$. Hence any $\lambda \neq 0$ number is not an eigenvalue either.

Problem 5.

Let $V := \{a x^2 + b x + c : a, b, c \in \mathbb{R}\}$ denote the vector space of real univariate polynomials of degree at most two, and let $D : V \to V$ denote the derivative, $D(p) := p'$ for any $p \in V$.
a.) Find a basis for $\ker(D)$, $\operatorname{im}(D)$, and determine $\dim(\ker(D))$, $\dim(\operatorname{im}(D))$ and $\operatorname{rank}(D)$.
b.) By using the basis $B := \{1, x+1, (x+2)(x+1)\}$ for V, find the matrix representation of D.
c.) Find the complex (including real) eigenvalues and eigenvectors of this matrix.

Solution.
a.) Notice that $p \in \ker(D)$ if and only if $D(p) = p' = 0$, that is, if p is a constant polynomial.
Thus, a basis for $\ker(D)$ is $\{1\}$, for example. Clearly, $\ker(D)$ is a one-dimensional subspace of V ($\ker(D)$ is isomorphic to $\mathbb{R}$), and $\dim(\ker(D)) = 1$.
From the rank-nullity theorem we get that
$\operatorname{rank}(D) = \dim(\operatorname{im}(D)) = \dim(V) - \dim(\ker(D)) = 3 - 1 = 2$. To find a basis for $\operatorname{im}(D)$, we consider a polynomial $p \in V$ with $p(x) = a x^2 + b x + c$. Then $D(p)(x) = 2 a x + b$ is a linear polynomial, hence for example $\{1, x\}$ is a basis for $\operatorname{im}(D)$. This also shows that $\dim(\operatorname{im}(D)) = 2$.

b.) Since D is an endomorphism of V with $\dim(V) = 3$, the linear operator D can be represented by a $3 x 3$ matrix in the given basis B.
We need to express the images of the basis vectors, $D(1)$, $D(x+1)$ and $D((x+2)(x+1))$, as a linear combination of the (same) basis vectors $1, \; x+1, \; (x+2)(x+1)$.
Clearly, one has $D(1) = 0 = 0 \cdot 1 + 0(x+1) + 0(x+1)(x+2)$, so the first column of the matrix of D is $(0, 0, 0)$.
Then $D(x+1) = 1 = 1 \cdot 1 + 0(x+1) + 0(x+1)(x+2)$, so the second column of the matrix of D is $(1, 0, 0)$.
Finally $D((x+2)(x+1)) = 2x + 3 = 1 \cdot 1 + 2(x+1) + 0(x+1)(x+2)$, so the third

column of the matrix of D is $(1, 2, 0)$.

Hence D has the following strictly upper-triangular matrix: $\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$.

c.) The eigenvalues $\lambda_{1,2,3}$ are solutions of the equation

$0 = \det(D - \lambda I) = \det\begin{pmatrix} -\lambda & 1 & 1 \\ 0 & -\lambda & 2 \\ 0 & 0 & -\lambda \end{pmatrix}$. This determinant is equal to $(-\lambda)(-\lambda)(-\lambda)$, because the matrix is strictly upper-triangular.

Thus, $\lambda_{1,2,3} = 0$ is a triple eigenvalue (i.e. an eigenvalue with algebraic multiplicity 3).

To find an eigenvector, we need to find a non-trivial solution $p \in V$ to the equation $D(p) = \lambda\, p$ with $\lambda = \lambda_1 = 0$ (and separately, with $\lambda = \lambda_2$ and $\lambda = \lambda_3$ in general, but now these eigenvalues are all the same). If

$p(x) = a\,(x+2)\,(x+1) + b\,(x+1) + c \in V$ is an arbitrary quadratic polynomial in the given basis B, then we need to solve $D(p) = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} c \\ b \\ a \end{pmatrix} = 0 \begin{pmatrix} c \\ b \\ a \end{pmatrix} = \lambda_1\, p$, that is,

$\begin{pmatrix} b+a \\ 2a \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. These imply that $a = b = 0$, and (to get a non-trivial eigenvector) $0 \neq c = 1$, for example.

So the vector $(1, 0, 0)$ (or any non-zero constant multiple of it) is an eigenvector, representing the "constant 1" polynomial.

Remark. Another way of finding the eigenvalues/eigenvectors: if the derivative of a non-zero polynomial is equal to a constant multiple of the polynomial (that is, if $D(p) = \lambda\, p$ holds), then, by taking into account the degrees on both sides, we see that p must be a non-zero constant and λ=0.

Remark. As we know, the eigenvalues of a matrix do not depend on the chosen basis, but the eigenvectors in general depend on the particular basis used.

Problem 6.

Let $P : \mathbb{R}^3 \to \mathbb{R}^3$ denote the projection onto the x-z-plane.

Let $R : \mathbb{R}^3 \to \mathbb{R}^3$ denote the rotation about the x-axis by $45\,°$ such that the image of both the y-axis and the z-axis lie in the upper half space $z > 0$.
a.) By using the standard basis, find the matrix representation, the real eigenvalues, the real eigenvectors, the trace, and the determinant of P.
b.) By using the standard basis, find the matrix representation, the real eigenvalues, the real eigenvectors, the trace, and the determinant of R.
c.) By using the standard basis, find the matrix representation, the real eigenvalues, the real eigenvectors, the trace, and the determinant of the composition $P \circ R$.
d.) By using the standard basis, find the matrix representation, the real eigenvalues, the real eigenvectors, the trace, and the determinant of the composition $R \circ P$.

Solution.
a.) Let $e_1 := (1, 0, 0)$, $e_2 := (0, 1, 0)$ and $e_3 := (0, 0, 1)$. Then
$P(e_1) = e_1 = 1\,e_1 + 0\,e_2 + 0\,e_3$, $P(e_3) = e_3 = 0\,e_1 + 0\,e_2 + 1\,e_3$ and $P(e_2) = 0 = 0\,e_1 + 0\,e_2 + 0\,e_3$. So P in this basis has the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Since this is a diagonal matrix, the eigenvalues are the diagonal elements $\lambda_{1,2} = 1$, and $\lambda_3 = 0$.

Case 1.) The vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is a non-trivial eigenvector corresponding to $\lambda_{1,2} = 1$ if and only if $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 1 \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, that is, if $b = 0$ and $a, c \in \mathbb{R}$. So we have two, linearly independent non-trivial eigenvectors, e.g., $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. In other words, the eigenspace corresponding to the double eigenvalue $\lambda_{1,2} = 1$ now has dimension 2. Thus, in this example (but not in general!), the algebraic multiplicity of the eigenvalue 1 is the same as its geometric multiplicity.

Case 2.) The vector $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is a non-trivial eigenvector corresponding to $\lambda_3 = 0$ if and only if $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, that is, if $a = c = 0$ and $b \in \mathbb{R}$. So we have one non-trivial eigenvector, e.g., $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ (or any non-zero constant multiple of it). In other words, the eigenspace corresponding to the single eigenvalue $\lambda_3 = 0$ now has dimension 1.

The eigenvalues of a matrix are basis-independent, but the eigenvectors in general depend on the chosen basis.

The trace of $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is 2, while the determinant is 0. The trace and determinant are also basis-independent quantities, because the trace is the sum of the eigenvalues, and the determinant is the product of the eigenvalues.

b.) It is easily seen by using elementary trigonometry, for example, that $R(e_1) = e_1$ and $R(e_2) = \frac{1}{\sqrt{2}}\, e_2 + \frac{1}{\sqrt{2}}\, e_3$ and $R(e_3) = -\frac{1}{\sqrt{2}}\, e_2 + \frac{1}{\sqrt{2}}\, e_3$, so R in this basis has the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$. Its characteristic polynomial is $(\lambda - 1)\left(\lambda^2 - \sqrt{2}\,\lambda + 1\right)$, having only one real root, $\lambda_1 = 1$. The corresponding eigenspace is one-dimensional, and is spanned by $\begin{pmatrix}1\\0\\0\end{pmatrix}$ for example. The trace of R is $1+\sqrt{2}$. The determinant is 1. (By the way, we easily check that $\lambda_{2,3}=(1\pm i)/\sqrt{2}$, and $\lambda_1+\lambda_2+\lambda_3=1+\sqrt{2}$, and $\lambda_1\,\lambda_2\,\lambda_3=1$.)

c.) The matrix representation of the composition $P\circ R$ is the matrix product

$$\begin{pmatrix}1&0&0\\0&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{2}}\\0&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{pmatrix}=\begin{pmatrix}1&0&0\\0&0&0\\0&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{pmatrix}$$

. The real eigenvalues are $\left\{1,\frac{1}{\sqrt{2}},0\right\}$ with corresponding eigenvectors $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}0\\-1\\1\end{pmatrix}\right\}$, respectively, for example.

Can you see this geometrically?
The trace of $P\circ R$ is $1+\frac{1}{\sqrt{2}}$.

We also know that $\det(P\circ R)=\det(P)\det(R)=0$.

d.) The matrix representation of the composition $R\circ P$ is the matrix product

$$\begin{pmatrix}1&0&0\\0&\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{2}}\\0&\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}1&0&0\\0&0&0\\0&0&1\end{pmatrix}=\begin{pmatrix}1&0&0\\0&0&-\frac{1}{\sqrt{2}}\\0&0&\frac{1}{\sqrt{2}}\end{pmatrix}$$

. The real eigenvalues are $\left\{1,\frac{1}{\sqrt{2}},0\right\}$ with corresponding eigenvectors $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\-1\\1\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$, respectively, for example.

Can you see this geometrically?
The trace of $R\circ P$ is $1+\frac{1}{\sqrt{2}}$.

We also know that $\det(R\circ P)=\det(R)\det(P)=0$.

Problem 7.

Let V denote a vector space of dimension two, and let

$A : V \to V$ be a linear map.
Suppose that the map A is represented by the matrix $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ in the basis $\{v_1, v_2\} := \left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$.
Show that if we change the basis to $\{w_1, w_2\} := \left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix}\right\}$, then the trace of A remains the same in the new basis.
Hint: first express $\{v_1, v_2\}$ in the new basis $\{w_1, w_2\}$.

Solution.
By definition, $A(v_1) = 1\, v_1 + 3\, v_2$, and $A(v_2) = 2\, v_1 + 4\, v_2$. To get the matrix of A in the new basis, we need to express both $A(w_1)$ and $A(w_2)$ as a linear combination of the vectors w_1 and w_2.

By solving a simple 2x2 linear system, we express v_1 and v_2 in the new basis as $v_1 = \frac{2}{3}\, w_1 - \frac{1}{3}\, w_2$ and $v_2 = \frac{1}{3}\, w_1 + \frac{1}{3}\, w_2$.
So $A(v_1) = 1\, v_1 + 3\, v_2 = 1\left(\frac{2}{3}\, w_1 - \frac{1}{3}\, w_2\right) + 3\left(\frac{1}{3}\, w_1 + \frac{1}{3}\, w_2\right) = \frac{5}{3}\, w_1 + \frac{2}{3}\, w_2$, and from linearity we get $A(v_1) = A\left(\frac{2}{3}\, w_1 - \frac{1}{3}\, w_2\right) = \frac{2}{3}\, A(w_1) - \frac{1}{3}\, A(w_2)$.
Similarly, $A(v_2) = 2\, v_1 + 4\, v_2 = 2\left(\frac{2}{3}\, w_1 - \frac{1}{3}\, w_2\right) + 4\left(\frac{1}{3}\, w_1 + \frac{1}{3}\, w_2\right) = \frac{8}{3}\, w_1 + \frac{2}{3}\, w_2$, and from linearity we get $A(v_2) = A\left(\frac{1}{3}\, w_1 + \frac{1}{3}\, w_2\right) = \frac{1}{3}\, A(w_1) + \frac{1}{3}\, A(w_2)$.

By comparing the two expressions for $A(v_1)$ we get $\frac{2}{3}\, A(w_1) - \frac{1}{3}\, A(w_2) = \frac{5}{3}\, w_1 + \frac{2}{3}\, w_2$, and similarly, by comparing the two expressions for $A(v_2)$ we get $\frac{1}{3}\, A(w_1) + \frac{1}{3}\, A(w_2) = \frac{8}{3}\, w_1 + \frac{2}{3}\, w_2$. The solution of this system is $A(w_1) = \frac{13}{3}\, w_1 + \frac{4}{3}\, w_2$ and $A(w_2) = \frac{11}{3}\, w_1 + \frac{2}{3}\, w_2$,

so the map A in the new basis $\{w_1, w_2\}$ has the matrix $\begin{pmatrix} \frac{13}{3} & \frac{11}{3} \\ \frac{4}{3} & \frac{2}{3} \end{pmatrix}$.

Its trace is $\frac{13}{3} + \frac{2}{3} = 5$, which is equal to the original trace 1+4=5.

Problem 8.

Let V denote the set of univariate polynomials with real coefficients of degree at most 3, and W denote the set of univariate polynomials with real coefficients of degree at most 4.

Let $A : V \to W$ denote the operator $A(f)(x) := \int_1^x f(t)\, dt$, that is, $A(f)$ denotes the antiderivative of f vanishing at 1. Find the matrix representation of A by using the basis $\{1, x, x^2, x^3\}$ in V, and the basis $\{1, x-1, x^2-x, x^3-x^2, x^4-x^3\}$ in W.

Solution.

We need to express the images of the basis vectors $A(1)$, $A(x)$, $A(x^2)$ and $A(x^3)$ in the basis $\{1, x-1, x^2-x, x^3-x^2, x^4-x^3\}$.

Clearly we have $A(1) = x-1$, $A(x) = \frac{x^2}{2} - \frac{1}{2}$, $A(x^2) = \frac{x^3}{3} - \frac{1}{3}$ and $A(x^3) = \frac{x^4}{4} - \frac{1}{4}$.

It is also easy to see that the following identities hold:

$A(1) = 0 + 1\,(x-1) + 0\,(x^2-x) + 0\,(x^3-x^2) + 0\,(x^4-x^3)$,

$A(x) = 0 + \frac{1}{2}\,(x-1) + \frac{1}{2}\,(x^2-x) + 0\,(x^3-x^2) + 0\,(x^4-x^3)$,

$A(x^2) = 0 + \frac{1}{3}\,(x-1) + \frac{1}{3}\,(x^2-x) + \frac{1}{3}\,(x^3-x^2) + 0\,(x^4-x^3)$,

$A(x^3) = 0 + \frac{1}{4}\,(x-1) + \frac{1}{4}\,(x^2-x) + \frac{1}{4}\,(x^3-x^2) + \frac{1}{4}\,(x^4-x^3)$,

so A has the following matrix representation: $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ 0 & 0 & \frac{1}{3} & \frac{1}{4} \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix}$.

Problem 9.

Let $V := C^\infty(\mathbb{R})$ denote the set of functions $f : \mathbb{R} \to \mathbb{R}$ that are infinitely many times differentiable.
Let $A : V \to V$ denote the operator $A(f)(x) := \int_0^x f(t)\, dt$, that is, $A(f)$ denotes the antiderivative of f vanishing at the origin. Find the real eigenvalues of A, then the corresponding eigenfunctions.

Solution.
For a fixed $\lambda \in \mathbb{R}$ we have $A(f) = \lambda f$ if and only if $\int_0^x f(t)\, dt = \lambda f(x)$ (for any $x \in \mathbb{R}$).
Suppose that f is a non-trivial solution of this integral equation (non-trivial here means that f is not the zero function).
Clearly, both sides are differentiable. By taking the derivative at x we get the necessary (but not sufficient!) condition $f(x) = \lambda f'(x)$ (referred to as the "modified problem").

Case 1.) If $\lambda \neq 0$, then the modified problem is the same as $f'(x) = \frac{1}{\lambda} f(x)$, and the general solution to this ODE is $f(x) = c \exp(x / \lambda)$, where $c \in \mathbb{R} \setminus \{0\}$ is arbitrary.
Here we need $c \neq 0$, otherwise f is trivial. Hence if (λ, f) is an eigenpair for the original problem, then $\lambda \neq 0$ and $f(x) = c \exp(x / \lambda)$ with $c \neq 0$ must hold.
However, if we substitute this into the left-hand side of the original equation, we get that $\int_0^x f(t)\, dt = \int_0^x c \exp(t / \lambda)\, dt = c \lambda e^{x/\lambda} - c \lambda$.
So the original equation holds if and only if $c \lambda e^{x/\lambda} - c \lambda = \lambda c \exp(x / \lambda)$, that is, if $c \lambda = 0$, which is a contradiction. So $\lambda \neq 0$ is *not* an eigenvalue of the operator A.

Case 2.) If $\lambda = 0$, then $f(x) = \lambda f'(x) = 0$ for any $x \in \mathbb{R}$, hence f is the zero function, so $\lambda = 0$ is not an eigenvalue of the modified problem, hence it is not an eigenvalue of A.

The operator A has no real eigenvalues. The above proof also shows that A has no complex eigenvalues either.

Problem 10.

Find two matrices $A, B \in \mathbb{R}^{n \times n}$ such that
$e^A e^B \neq e^{A+B} \neq e^B e^A \neq e^A e^B$.

Solution.

For example, $A := \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ and $B := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ will work. Then $A + B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, and, by using Hermite–Lagrange interpolation for example, we have that $e^A = \begin{pmatrix} e & 0 \\ e-1 & 1 \end{pmatrix}$, $e^B = \begin{pmatrix} 1 & 0 \\ 0 & e \end{pmatrix}$, $e^{A+B} = \begin{pmatrix} e & 0 \\ e & e \end{pmatrix}$, and $e^A e^B \neq e^{A+B} \neq e^B e^A \neq e^A e^B$.

Problem 11.

Compute the matrix e^A for $A := \begin{pmatrix} 0 & -3 & 2 \\ 0 & 0 & 10 \\ 0 & 0 & 0 \end{pmatrix}$.

1st solution.

Since $A^2 = \begin{pmatrix} 0 & 0 & -30 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $A^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, we have

$$e^A = \mathcal{I} + \sum_{k=1}^{\infty} \frac{A^k}{k!} = \mathcal{I} + \sum_{k=1}^{2} \frac{A^k}{k!} = \mathcal{I} + A + \frac{1}{2} A^2 = \begin{pmatrix} 1 & -3 & -13 \\ 0 & 1 & 10 \\ 0 & 0 & 1 \end{pmatrix}.$$

2nd solution.

The eigenvalues of A are clearly $\lambda_{1,2,3} = 0$, so
$n = 3,\ \deg(p) \le 2,\ k = 1,\ j = 1,\ m_1 = 3,\ i \in \{0, 1, 2\},\ p(z) = \alpha z^2 + \beta z + \gamma$, therefore

In[∘]:=
```
(α z^2 + β z + γ /. z → 0) == Exp[0]
```
Out[∘]=
```
γ == 1
```

In[∘]:=
```
(D[α z^2 + β z + γ, z] /. z → 0) == Exp[0]
```
Out[∘]=
```
β == 1
```

In[∘]:=
```
(D[α z^2 + β z + γ, z, z] /. z → 0) == Exp[0]
```
Out[∘]=
```
2 α == 1
```

hence $e^A = p(A) = \alpha A^2 + \beta A + \gamma \mathcal{I} =$

In[∘]:=
$$\frac{1}{2}\begin{pmatrix} 0 & -3 & 2 \\ 0 & 0 & 10 \\ 0 & 0 & 0 \end{pmatrix} . \begin{pmatrix} 0 & -3 & 2 \\ 0 & 0 & 10 \\ 0 & 0 & 0 \end{pmatrix} + 1 \begin{pmatrix} 0 & -3 & 2 \\ 0 & 0 & 10 \\ 0 & 0 & 0 \end{pmatrix} + 1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ // MatrixForm}$$

Out[∘]//MatrixForm=
$$\begin{pmatrix} 1 & -3 & -13 \\ 0 & 1 & 10 \\ 0 & 0 & 1 \end{pmatrix}$$

■ Problem 12.

Compute the matrix e^A for $A := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

1st solution.

We observe that the first few powers of A follow a simple pattern:

In[∘]:=
$$\text{Table}\left[\text{MatrixForm@MatrixPower}\left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, n\right], \{n, 0, 8\}\right]$$

Out[∘]=
$$\left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\}$$

hence it is worth separating the even and odd indices to get $e^A = \mathcal{I} + \sum_{k=1}^{\infty} \frac{A^k}{k!} =$

$$\left[\mathcal{I}-\begin{pmatrix}\frac{1}{2!}&0\\0&\frac{1}{2!}\end{pmatrix}+\begin{pmatrix}\frac{1}{4!}&0\\0&\frac{1}{4!}\end{pmatrix}-\begin{pmatrix}\frac{1}{6!}&0\\0&\frac{1}{6!}\end{pmatrix}\pm\ldots\right]+\left[\begin{pmatrix}0&\frac{1}{1!}\\-\frac{1}{1!}&0\end{pmatrix}+\begin{pmatrix}0&-\frac{1}{3!}\\\frac{1}{3!}&0\end{pmatrix}+\begin{pmatrix}0&\frac{1}{5!}\\-\frac{1}{5!}&0\end{pmatrix}+\ldots\right]$$
$$=\begin{pmatrix}\cos(1)&0\\0&\cos(1)\end{pmatrix}+\begin{pmatrix}0&\sin(1)\\-\sin(1)&0\end{pmatrix}=\begin{pmatrix}\cos(1)&\sin(1)\\-\sin(1)&\cos(1)\end{pmatrix}.$$

2^{nd} solution.

The eigenvalues of A are $\lambda_1 = i$ and $\lambda_2 = -i$, so $n = 2$, $\deg(p) \le 1$, $k = 2$, $j \in \{1, 2\}$, $m_1 = 1$, $m_2 = 1$, $i = 0$, $p(z) = \alpha z + \beta$. Here i is an index, and i is the imaginary unit. So

In[∘]:=
```
(α z + β /. z → ⅈ) == Exp[ⅈ]
```
Out[∘]=
```
ⅈ α + β == ⅇ^ⅈ
```

In[∘]:=
```
(α z + β /. z → -ⅈ) == Exp[-ⅈ]
```
Out[∘]=
```
-ⅈ α + β == ⅇ^-ⅈ
```

In[∘]:=
```
Solve[ⅈ α + β == ⅇ^ⅈ ∧ -ⅈ α + β == ⅇ^-ⅈ, {α, β}] // Expand
```
Out[∘]=

$$\left\{\left\{\alpha \to \frac{i\, e^{-i}}{2} - \frac{i\, e^{i}}{2},\ \beta \to \frac{e^{-i}}{2} + \frac{e^{i}}{2}\right\}\right\}$$

By using Euler's identity $e^{i\phi} = \cos(\phi) + i \sin(\phi)$ with $\phi = 1$, we now have

In[∘]:=

$$\left\{\left\{\alpha \to \frac{i\, e^{-i}}{2} - \frac{i\, e^{i}}{2},\ \beta \to \frac{e^{-i}}{2} + \frac{e^{i}}{2}\right\}\right\} \text{ // ComplexExpand}$$

Out[∘]=
```
{{α → Sin[1], β → Cos[1]}}
```

Therefore $e^A = p(A) = \alpha A + \beta \mathcal{I} =$

In[∘]:=

$$\text{Sin[1]}\begin{pmatrix}0&1\\-1&0\end{pmatrix}+\text{Cos[1]}\begin{pmatrix}1&0\\0&1\end{pmatrix}\text{ // MatrixForm}$$

Out[∘]//MatrixForm=

$$\begin{pmatrix}\text{Cos[1]}&\text{Sin[1]}\\-\text{Sin[1]}&\text{Cos[1]}\end{pmatrix}$$

Problem 13.

Compute the matrix A^{2019} for $A := \begin{pmatrix} 2 & 2 \\ 3 & -3 \end{pmatrix}$.

Solution.
The eigenvalues are $\lambda_1 = -4$ and $\lambda_2 = 3$, so $n = 2, k = 2, m_1 = 1,\ m_2 = 1$. We let $f(z) := z^{2019}$. Then $A^{2019} = f(A) = p(A)$, where $\deg(p) \le n - 1 = 1$, hence $p(z) = \alpha z + \beta$. The system $p^{(i)}(\lambda_j) = f^{(i)}(\lambda_j)$ with $j = 1, \ldots, k$ and $i = 0, \ldots, m_j - 1$ now reads as $p(-4) = f(-4)$ and $p(3) = f(3)$, that is, $(-4)\alpha + \beta = (-4)^{2019}$ and $3\alpha + \beta = 3^{2019}$, from which we have $\alpha = \frac{1}{7}(3^{2019} - (-4)^{2019})$, $\beta = \frac{1}{7}(4 \times 3^{2019} + 3(-4)^{2019})$.

Therefore $A^{2019} = \alpha A + \beta \mathcal{I} = \alpha \begin{pmatrix} 2 & 2 \\ 3 & -3 \end{pmatrix} + \beta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2\alpha + \beta & 2\alpha \\ 3\alpha & -3\alpha + \beta \end{pmatrix} =$
$\begin{pmatrix} \frac{1}{7}(6 \times 3^{2019} + (-4)^{2019}) & \frac{2}{7}(3^{2019} - (-4)^{2019}) \\ \frac{3}{7}(3^{2019} - (-4)^{2019}) & \frac{1}{7}(3^{2019} + 6(-4)^{2019}) \end{pmatrix}$.

Problem 14.

Let $A := \begin{pmatrix} 0 & -2 \\ -1 & -1 \end{pmatrix}$.

a.) Compute $\cos(A)$ and $\sin(A)$.

b.) Verify that $\cos^2(A) + \sin^2(A) = \mathcal{I}$, where $f^2(A) := f(A)\, f(A)$ and $\mathcal{I}$ is the identity matrix.

Solution.
a.) The eigenvalues of A are $\{-2,\ 1\}$. The eigenvalues are distinct, each eigenvalue has multiplicity 1. Since A is a 2x2 matrix, the interpolation polynomial has degree at most 1, so $p(z) = \alpha z + \beta$.

To find $\cos(A)$, we solve $\begin{cases} p(-2) = \cos(-2), \\ p(1) = \cos(1). \end{cases}$ The solution of this linear system is

$\alpha = \frac{1}{3}(\cos(1) - \cos(2))$ and $\beta = \frac{1}{3}(2\cos(1) + \cos(2))$.

Hence

$$\cos(A) = p(A) = \alpha A + \beta \mathcal{I} = \begin{pmatrix} \frac{1}{3}(2\cos(1) + \cos(2)) & -\frac{2}{3}(\cos(1) - \cos(2)) \\ \frac{1}{3}(-\cos(1) + \cos(2)) & \frac{1}{3}(\cos(1) + 2\cos(2)) \end{pmatrix}.$$

To find $\sin(A)$, we solve $\begin{cases} p(-2) = \sin(-2), \\ p(1) = \sin(1). \end{cases}$ The solution of this linear system is

$\alpha = \frac{1}{3}(\sin(1) + \sin(2))$ and $\beta = \frac{1}{3}(2\sin(1) - \sin(2))$. Hence

$$\sin(A) = p(A) = \alpha A + \beta \mathcal{I} = \begin{pmatrix} \frac{1}{3}(2\sin(1) - \sin(2)) & -\frac{2}{3}(\sin(1) + \sin(2)) \\ \frac{1}{3}(-\sin(1) - \sin(2)) & \frac{1}{3}(\sin(1) - 2\sin(2)) \end{pmatrix}.$$

b.) From this we have $\cos(A)\cos(A) + \sin(A)\sin(A) =$

$$\begin{pmatrix} \frac{2\sin^2(1)}{3} + \frac{\sin^2(2)}{3} + \frac{2\cos^2(1)}{3} + \frac{\cos^2(2)}{3} & -\frac{2\sin^2(1)}{3} + \frac{2\sin^2(2)}{3} - \frac{2}{3}\cos^2(1) + \frac{2\cos^2(2)}{3} \\ -\frac{\sin^2(1)}{3} + \frac{\sin^2(2)}{3} - \frac{1}{3}\cos^2(1) + \frac{\cos^2(2)}{3} & \frac{\sin^2(1)}{3} + \frac{2\sin^2(2)}{3} + \frac{\cos^2(1)}{3} + \frac{2\cos^2(2)}{3} \end{pmatrix} = \mathcal{I}.$$

Problem 15.

By using Hermite–Lagrange interpolation, compute the inverse of the matrix $A := \begin{pmatrix} 3 & -3 & 2 \\ -1 & 5 & -2 \\ -1 & 3 & 0 \end{pmatrix}$.

The characteristic polynomial of A is

In[∘]:=
```
Det[λ (1 0 0; 0 1 0; 0 0 1) - (3 -3 2; -1 5 -2; -1 3 0)]
```

Out[∘]=

$-16 + 20\lambda - 8\lambda^2 + \lambda^3$

which can be factorized to get $(\lambda - 2)^2(\lambda - 4)$. Now $n = 3$, $k = 2$, $\lambda_1 = 2$, $\lambda_2 = 4$, $m_1 = 2$, $m_2 = 1$. Hence the index pairs (j, i) are $(1, 0)$, $(1, 1)$, $(2, 0)$.

Let $f(z) := z^{-1}$, then $A^{-1} = f(A) = p(A)$ with $\deg(p) \leq 2$, that is, $p(z) = \alpha z^2 + \beta z + \gamma$.

We use $p^{(i)}(\lambda_j) = f^{(i)}(\lambda_j)$, where $j = 1, \ldots, k$ and $i = 0, \ldots, m_j - 1$.

In other words, $p(2) = f(2) = 1/2$, $p'(2) = f'(2) = -1/4$, $p(4) = f(4) = 1/4$, from which we obtain

In[∘]:=
```
Solve[(α z^2 + β z + γ == 1/z /. z → 2) ∧
  (D[α z^2 + β z + γ, z] == D[1/z, z] /. z → 2) ∧ (α z^2 + β z + γ == 1/z /. z → 4)]
```

Out[∘]=

$$\left\{\left\{\alpha \to \frac{1}{16}, \beta \to -\frac{1}{2}, \gamma \to \frac{5}{4}\right\}\right\}$$

Therefore

$$A^{-1} = p(A) = \alpha A^2 + \beta A + \gamma \mathcal{I} = \frac{1}{16}\begin{pmatrix} 10 & -18 & 12 \\ -6 & 22 & -12 \\ -6 & 18 & -8 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 3 & -3 & 2 \\ -1 & 5 & -2 \\ -1 & 3 & 0 \end{pmatrix} + \frac{5}{4}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{8} & \frac{3}{8} & -\frac{1}{4} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{4} \\ \frac{1}{8} & -\frac{3}{8} & \frac{3}{4} \end{pmatrix}.$$

And indeed:

In[∘]:=

$$\begin{pmatrix} \frac{3}{8} & \frac{3}{8} & -\frac{1}{4} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{4} \\ \frac{1}{8} & -\frac{3}{8} & \frac{3}{4} \end{pmatrix} . \begin{pmatrix} 3 & -3 & 2 \\ -1 & 5 & -2 \\ -1 & 3 & 0 \end{pmatrix} \text{// MatrixForm}$$

Out[∘]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

■ Problem 16.

By using Hermite–Lagrange interpolation, solve the following system of differential equations

$$\begin{cases} x'(t) = 8\,x(t) - y(t) \\ y'(t) = 6\,x(t) + 3\,y(t) \end{cases}$$ with initial conditions

$x(0) = 2$, $y(0) = 3$. The final answer can contain only real numbers.

Solution.

By introducing the notation $X(t) := \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ we know from the theory of differential equations that $X(t) = e^{A\,t}\,X(0) = e^{A\,t}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ with $A := \begin{pmatrix} 8 & -1 \\ 6 & 3 \end{pmatrix}$.

The eigenvalues of A are $\lambda_1 = 5$ and $\lambda_2 = 6$, hence $n = 2$, $k = 2$, $m_1 = 1$, $m_2 = 1$, so $(j, i) = (1, 0)$ and $(j, i) = (2, 0)$.

We let $f(z) := \exp(z\,t)$. Then the (parametric) interpolation polynomial satisfies $\deg(p) \leq 1$, so $p(z) = \alpha\,z + \beta$, and $p^{(i)}(\lambda_j) = f^{(i)}(\lambda_j) = t^i\,e^{\lambda_j\,t}$.

In other words, $p(5) = e^{5\,t}$ and $p(6) = e^{6\,t}$, that is, $5\,\alpha + \beta = e^{5\,t}$ and $6\,\alpha + \beta = e^{6\,t}$, hence $\alpha = e^{6\,t} - e^{5\,t}$, $\beta = 6\,e^{5\,t} - 5\,e^{6\,t}$. So $e^{A\,t} = \alpha\,A + \beta\,\mathcal{I} = \alpha\begin{pmatrix} 8 & -1 \\ 6 & 3 \end{pmatrix} + \beta\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3\,e^{6\,t} - 2\,e^{5\,t} & e^{5\,t} - e^{6\,t} \\ 6\,e^{6\,t} - 6\,e^{5\,t} & 3\,e^{5\,t} - 2\,e^{6\,t} \end{pmatrix}$, therefore the solution of the system of differential equations is $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{A\,t}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3\,e^{6\,t} - e^{5\,t} \\ 6\,e^{6\,t} - 3\,e^{5\,t} \end{pmatrix}$.

And indeed:

In[ ]:=
```
DSolve[{x'[t] == 8 x[t] - y[t], y'[t] == 6 x[t] + 3 y[t], x[0] == 2, y[0] == 3},
  {x[t], y[t]}, t] // Expand
```

Out[ ]=
$\{\{x[t] \to -e^{5\,t} + 3\,e^{6\,t},\ y[t] \to -3\,e^{5\,t} + 6\,e^{6\,t}\}\}$

Problem 17.

By using Hermite–Lagrange interpolation, solve the following system of differential equations

$\begin{cases} x'(t) = 8\,x(t) + y(t) \\ y'(t) = -4\,x(t) + 12\,y(t) \end{cases}$ with initial conditions $x(0) = 2,\ y(0) = 3$. The final answer can contain only real numbers.

Solution.

The eigenvalues of $A := \begin{pmatrix} 8 & 1 \\ -4 & 12 \end{pmatrix}$ are $\lambda_{1,2} = 10$, so $n = 2$, $k = 1$, $m_1 = 2$, hence $(j, i) = (1, 0)$ and $(j, i) = (1, 1)$.

Proceeding similarly as before, we have $p(10) = e^{10\,t}$ and $p'(10) = t\,e^{10\,t}$, that is, $10\,\alpha + \beta = e^{10\,t}$ and $\alpha = t\,e^{10\,t}$, from which we obtain $\alpha = t\,e^{10\,t}$, $\beta = e^{10\,t} - 10\,e^{10\,t}\,t$.

Hence $e^{A\,t} = \alpha\,A + \beta\,\mathcal{I} = \alpha \begin{pmatrix} 8 & 1 \\ -4 & 12 \end{pmatrix} + \beta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{10\,t}\,(1 - 2\,t) & e^{10\,t}\,t \\ -4\,e^{10\,t}\,t & e^{10\,t}\,(1 + 2\,t) \end{pmatrix}$,

therefore the solution is $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{A\,t} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2\,e^{10\,t} - e^{10\,t}\,t \\ 3\,e^{10\,t} - 2\,e^{10\,t}\,t \end{pmatrix}$.

Problem 18.

By using Hermite–Lagrange interpolation, solve the following system of differential equations

$\begin{cases} x'(t) = 8\,x(t) + 2\,y(t) \\ y'(t) = -4\,x(t) + 12\,y(t) \end{cases}$ with initial conditions $x(0) = 2,\ y(0) = 3$. The final answer can contain only real numbers.

Solution.

The eigenvalues of $A := \begin{pmatrix} 8 & 2 \\ -4 & 12 \end{pmatrix}$ are $\lambda_1 = 10 + 2\,i$, $\lambda_2 = 10 - 2\,i$, so $n = 2$, $k = 2$, $m_1 = 1$, $m_2 = 1$, hence $(j, i) = (1, 0)$ and $(j, i) = (2, 0)$. Here i is an index, and i is

the imaginary unit.
Proceeding similarly as before, we have $p(\lambda_1) = e^{\lambda_1 t}$ and $p(\lambda_2) = e^{\lambda_2 t}$, that is, $\lambda_1 \alpha + \beta = e^{\lambda_1 t}$ and $\lambda_2 \alpha + \beta = e^{\lambda_2 t}$, from which we get
$\alpha = \frac{e^{t \lambda_1}}{\lambda_1 - \lambda_2} - \frac{e^{t \lambda_2}}{\lambda_1 - \lambda_2}$, $\beta = \frac{e^{t \lambda_2} \lambda_1}{\lambda_1 - \lambda_2} - \frac{e^{t \lambda_1} \lambda_2}{\lambda_1 - \lambda_2}$.
We now use Euler's formula to get rid of all complex numbers:
$\alpha = \frac{e^{t \lambda_1}}{\lambda_1 - \lambda_2} - \frac{e^{t \lambda_2}}{\lambda_1 - \lambda_2} = \frac{1}{4 i} (e^{10 t} e^{2 i t} - e^{10 t} e^{-2 i t}) =$
$\frac{e^{10 t}}{4 i} (\cos(2 t) + i \sin(2 t) - (\cos(-2 t) + i \sin(-2 t))) = \frac{e^{10 t}}{4 i} 2 i \sin(2 t) = \frac{1}{2} e^{10 t} \sin(2 t)$,
and, similarly,
$\beta = \frac{1}{4 i} (e^{10 t} e^{-2 i t} (10 + 2 i) - e^{10 t} e^{2 i t} (10 - 2 i)) = \frac{e^{10 t}}{4 i} (4 i \cos(2 t) - 20 i \sin(2 t)) =$
$e^{10 t} (\cos(2 t) - 5 \sin(2 t))$.
Hence $e^{A t} = \alpha A + \beta \mathcal{I} = \alpha \begin{pmatrix} 8 & 2 \\ -4 & 12 \end{pmatrix} + \beta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} =$
$\begin{pmatrix} e^{10 t} (\cos(2 t) - \sin(2 t)) & e^{10 t} \sin(2 t) \\ -2 e^{10 t} \sin(2 t) & e^{10 t} (\cos(2 t) + \sin(2 t)) \end{pmatrix}$, therefore
$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{A t} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} e^{10 t} (2 \cos(2 t) + \sin(2 t)) \\ e^{10 t} (3 \cos(2 t) - \sin(2 t)) \end{pmatrix}$.

Problem 19.

Let $A := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2018 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

a.) By evaluating the infinite sum explicitly, compute the matrix $\sum_{k=0}^{\infty} \frac{A^k}{k!}$, where $A^0 := \mathcal{I}$.

b.) Compute $\exp(A)$, that is e^A, via interpolation.

Solution.

a.) Observe that $A^k = \begin{pmatrix} \frac{1}{2}-\frac{1}{2}(-1)^{1+k} & 0 & \frac{1}{2}+\frac{1}{2}(-1)^{1+k} \\ 0 & 2018^k & 0 \\ \frac{1}{2}-\frac{(-1)^k}{2} & 0 & \frac{1}{2}+\frac{(-1)^k}{2} \end{pmatrix}$, hence

$\sum_{k=0}^{\infty}\frac{A^k}{k!} = \begin{pmatrix} \frac{1}{2e}+\frac{e}{2} & 0 & -\frac{1}{2e}+\frac{e}{2} \\ 0 & e^{2018} & 0 \\ -\frac{1}{2e}+\frac{e}{2} & 0 & \frac{1}{2e}+\frac{e}{2} \end{pmatrix} = \begin{pmatrix} \cosh(1) & 0 & \sinh(1) \\ 0 & e^{2018} & 0 \\ \sinh(1) & 0 & \cosh(1) \end{pmatrix}$.

b.) To compute $\exp(A)$ via Hermite–Lagrange interpolation, notice that the eigenvalues are $\{-1, 1, 2018\}$. The quadratic interpolation polynomial is

$p(z) = -\frac{-2017+2019\,e^2-2\,e^{2019}}{8\,144\,646\,e}\,z^2 - \frac{1-e^2}{2\,e}\,z - \frac{-2\,035\,153-2\,037\,171\,e^2+e^{2019}}{4\,072\,323\,e}$,
so $p(A) = -\frac{-2017+2019\,e^2-2\,e^{2019}}{8\,144\,646\,e}\,A^2 - \frac{1-e^2}{2\,e}\,A - \frac{-2\,035\,153-2\,037\,171\,e^2+e^{2019}}{4\,072\,323\,e}\,\mathcal{I} =$
$\begin{pmatrix} \frac{1}{2e}+\frac{e}{2} & 0 & -\frac{1}{2e}+\frac{e}{2} \\ 0 & e^{2018} & 0 \\ -\frac{1}{2e}+\frac{e}{2} & 0 & \frac{1}{2e}+\frac{e}{2} \end{pmatrix}$.

Problem 20.

Let $A := \begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix}$. Prove that
$(\exp(A\,t))' = A\exp(A\,t) = \exp(A\,t)\,A$, where the derivative of a matrix function $B(t)$ is understood componentwise.

Solution.
The characteristic polynomial of A is $\lambda(\lambda-3)(\lambda+4)=0$, so $\lambda_1=0$, $\lambda_2=3$, $\lambda_3=-4$.
By using interpolation we get that $\exp(A\,t) =$
$\begin{pmatrix} \frac{1}{84}\left(-7+27\,e^{-4t}+64\,e^{3t}\right) & \frac{2}{7}\,e^{-4t}\left(-1+e^{7t}\right) & \frac{1}{84}\left(-7-9\,e^{-4t}+16\,e^{3t}\right) \\ \frac{1}{14}\left(-7-9\,e^{-4t}+16\,e^{3t}\right) & \frac{1}{7}\,e^{-4t}\left(4+3\,e^{7t}\right) & \frac{1}{14}\left(-7+3\,e^{-4t}+4\,e^{3t}\right) \\ \frac{1}{84}\left(91-27\,e^{-4t}-64\,e^{3t}\right) & -\frac{2}{7}\,e^{-4t}\left(-1+e^{7t}\right) & \frac{1}{84}\left(91+9\,e^{-4t}-16\,e^{3t}\right) \end{pmatrix}$,

so $(\exp(A\,t))'$ =

$$\begin{pmatrix} \frac{1}{7}e^{-4t}(-9+16e^{7t}) & \frac{2}{7}e^{-4t}(4+3e^{7t}) & \frac{1}{7}e^{-4t}(3+4e^{7t}) \\ \frac{6}{7}e^{-4t}(3+4e^{7t}) & \frac{1}{7}e^{-4t}(-16+9e^{7t}) & \frac{6}{7}e^{-4t}(-1+e^{7t}) \\ \frac{1}{7}e^{-4t}(9-16e^{7t}) & -\frac{2}{7}e^{-4t}(4+3e^{7t}) & -\frac{1}{7}e^{-4t}(3+4e^{7t}) \end{pmatrix}.$$

One directly checks that this last matrix is equal to $A\exp(A\,t)$ or to $\exp(A\,t)\,A$.

Problem 21.

Let $A := \begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix}$. Prove that $\exp(i\,A) = \cos(A) + i\sin(A)$, where i denotes the imaginary unit.

Solution. For this exercise you may want to use a computer.
By interpolation we have $\cos(A)$ =

$$\begin{pmatrix} 1+\frac{1}{84}(-91+64\cos(3)+27\cos(4)) & \frac{2}{7}(\cos(3)-\cos(4)) & \frac{1}{84}(-7+16\cos(3)-9\cos(4)) \\ \frac{1}{14}(-7+16\cos(3)-9\cos(4)) & 1+\frac{1}{7}(-7+3\cos(3)+4\cos(4)) & \frac{1}{14}(-7+4\cos(3)+3\cos(4)) \\ \frac{1}{84}(91-64\cos(3)-27\cos(4)) & -\frac{2}{7}(\cos(3)-\cos(4)) & 1+\frac{1}{84}(7-16\cos(3)+9\cos(4)) \end{pmatrix},$$

and $\sin(A)$ =

$$\begin{pmatrix} \frac{1}{84}(64\sin(3)-27\sin(4)) & \frac{2}{7}(\sin(3)+\sin(4)) & \frac{1}{84}(16\sin(3)+9\sin(4)) \\ \frac{1}{14}(16\sin(3)+9\sin(4)) & \frac{1}{7}(3\sin(3)-4\sin(4)) & \frac{1}{14}(4\sin(3)-3\sin(4)) \\ \frac{1}{84}(-64\sin(3)+27\sin(4)) & -\frac{2}{7}(\sin(3)+\sin(4)) & \frac{1}{84}(-16\sin(3)-9\sin(4)) \end{pmatrix},$$

and $\exp(i\,A)$ =

$$\begin{pmatrix} -\frac{1}{12}+\frac{16\cos(3)}{21}+\frac{9\cos(4)}{28}+i\left(\frac{16\sin(3)}{21}-\frac{9\sin(4)}{28}\right) & \frac{2\cos(3)}{7}-\frac{2\cos(4)}{7}+i\left(\frac{2\sin(3)}{7}+\frac{2\sin(4)}{7}\right) & -\frac{1}{12}+\frac{4\cos(3)}{21}-\frac{3\cos(4)}{28}+i\left(\frac{4\sin(3)}{21}+\frac{3\sin(4)}{28}\right) \\ -\frac{1}{2}+\frac{8\cos(3)}{7}-\frac{9\cos(4)}{14}+i\left(\frac{8\sin(3)}{7}+\frac{9\sin(4)}{14}\right) & \frac{3\cos(3)}{7}+\frac{4\cos(4)}{7}+i\left(\frac{3\sin(3)}{7}-\frac{4\sin(4)}{7}\right) & -\frac{1}{2}+\frac{2\cos(3)}{7}+\frac{3\cos(4)}{14}+i\left(\frac{2\sin(3)}{7}-\frac{3\sin(4)}{14}\right) \\ \frac{13}{12}-\frac{16\cos(3)}{21}-\frac{9\cos(4)}{28}+i\left(-\frac{16\sin(3)}{21}+\frac{9\sin(4)}{28}\right) & -\frac{2\cos(3)}{7}+\frac{2\cos(4)}{7}+i\left(-\frac{2\sin(3)}{7}-\frac{2\sin(4)}{7}\right) & \frac{13}{12}-\frac{4\cos(3)}{21}+\frac{3\cos(4)}{28}+i\left(-\frac{4\sin(3)}{21}-\frac{3\sin(4)}{28}\right) \end{pmatrix},$$

from which we verify the identity.

Problem 22. [Diagonalization of non-symmetric matrices]

Show that the matrix $A := \begin{pmatrix} -19 & 8 & 96 & 48 \\ 64 & 49 & 16 & 8 \\ 32 & 18 & 21 & 4 \\ -44 & -28 & -24 & 1 \end{pmatrix}$ is diagonalizable, then find its decomposition $A = K\,D\,K^{-1}$ with a suitable diagonal matrix D.

Solution.

```
Eigenvalues[{{-19, 8, 96, 48}, {64, 49, 16, 8}, {32, 18, 21, 4}, {-44, -28, -24, 1}}]
```

```
{65, -39, 13, 13}
```

```
Eigenvectors[{{-19, 8, 96, 48}, {64, 49, 16, 8}, {32, 18, 21, 4}, {-44, -28, -24, 1}}]
```

```
{{-4, -18, -9, 14}, {20, -14, -7, 8}, {1, -2, 0, 1}, {2, -4, 1, 0}}
```

These eigenvectors are linearly independent (because the determinant is non-zero):

```
Det[{{-4, -18, -9, 14}, {20, -14, -7, 8}, {1, -2, 0, 1}, {2, -4, 1, 0}}]
```

```
-676
```

The matrix K consists of the eigenvectors as column vectors, so

$$K = \begin{pmatrix} -4 & 20 & 1 & 2 \\ -18 & -14 & -2 & -4 \\ -9 & -7 & 0 & 1 \\ 14 & 8 & 1 & 0 \end{pmatrix}.$$

Then $K^{-1} A K = D$ is indeed diagonal, containing the eigenvalues:

```
Inverse[(-4 20 1 2; -18 -14 -2 -4; -9 -7 0 1; 14 8 1 0)].(-19 8 96 48; 64 49 16 8; 32 18 21 4; -44 -28 -24 1).(-4 20 1 2; -18 -14 -2 -4; -9 -7 0 1; 14 8 1 0) // MatrixForm
```

$$\begin{pmatrix} 65 & 0 & 0 & 0 \\ 0 & -39 & 0 & 0 \\ 0 & 0 & 13 & 0 \\ 0 & 0 & 0 & 13 \end{pmatrix}$$

■ Problem 23. [Diagonalization of non-symmetric matrices]

Show that the matrix $A := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is *not* diagonalizable.

Hint.
Assume $A = K\,D\,K^{-1}$ and show that this is impossible: what is D for this matrix?

■ Problem 24. [Diagonalization of symmetric real matrices]

The matrix $A := \begin{pmatrix} 2 & -2\sqrt{2} & 3\sqrt{2} \\ -2\sqrt{2} & 4 & 3 \\ 3\sqrt{2} & 3 & 0 \end{pmatrix}$ is real and symmetric, hence diagonalizable. Find its decomposition $A = Q\,D\,Q^{T}$, where D is diagonal and Q is orthogonal.

Solution.
This time we know that spec(A) must be real:

```
Eigenvalues[{{2, -2 √2, 3 √2}, {-2 √2, 4, 3}, {3 √2, 3, 0}}]
```

$$\left\{6, -3\sqrt{3}, 3\sqrt{3}\right\}$$

In[∘]:=

```
CharacteristicPolynomial[{{2, -2 √2, 3 √2}, {-2 √2, 4, 3}, {3 √2, 3, 0}}, λ]
```

$$-162 + 27\lambda + 6\lambda^2 - \lambda^3 == 0$$

Out[∘]=

```
True
```

In[∘]:=

```
Divisors[162]
```

In[∘]:=

$$\frac{\{1, 2, 3, 6, 9, 18, 27, 54, 81, 162\}}{\{1, -1\}}$$

In[∘]:=

$$-162 + 27\lambda + 6\lambda^2 - \lambda^3 \;/.\; \lambda \to 6$$

Out[∘]=

```
0
```

In[∘]:=

$$\frac{-162 + 27\lambda + 6\lambda^2 - \lambda^3}{\lambda - 6} \text{ // Simplify}$$

Out[∘]=

$$27 - \lambda^2$$

The eigenvectors:

```
Eigenvectors[(2, -2 √2, 3 √2; -2 √2, 4, 3; 3 √2, 3, 0)]
```

$$\text{Eigenvectors}\left[\begin{pmatrix} 2 & -2\sqrt{2} & 3\sqrt{2} \\ -2\sqrt{2} & 4 & 3 \\ 3\sqrt{2} & 3 & 0 \end{pmatrix}\right]$$

$$\left\{\left\{-\frac{1}{\sqrt{2}}, 1, 0\right\}, \left\{-\frac{3\sqrt{2}+2\sqrt{6}}{3\left(2+\sqrt{3}\right)}, -\frac{3+2\sqrt{3}}{3\left(2+\sqrt{3}\right)}, 1\right\}, \left\{-\frac{-3\sqrt{2}+2\sqrt{6}}{3\left(-2+\sqrt{3}\right)}, -\frac{-3+2\sqrt{3}}{3\left(-2+\sqrt{3}\right)}, 1\right\}\right\}$$

The normalized eigenvectors: $\frac{v_j}{\|v_j\|}$

$$\left\{\frac{\left\{-\frac{1}{\sqrt{2}}, 1, 0\right\}}{\text{Norm}\left[\left\{-\frac{1}{\sqrt{2}}, 1, 0\right\}, 2\right]}, \frac{\left\{-\frac{3\sqrt{2}+2\sqrt{6}}{3(2+\sqrt{3})}, -\frac{3+2\sqrt{3}}{3(2+\sqrt{3})}, 1\right\}}{\text{Norm}\left[\left\{-\frac{3\sqrt{2}+2\sqrt{6}}{3(2+\sqrt{3})}, -\frac{3+2\sqrt{3}}{3(2+\sqrt{3})}, 1\right\}, 2\right]}, \frac{\left\{-\frac{-3\sqrt{2}+2\sqrt{6}}{3(-2+\sqrt{3})}, -\frac{-3+2\sqrt{3}}{3(-2+\sqrt{3})}, 1\right\}}{\text{Norm}\left[\left\{-\frac{-3\sqrt{2}+2\sqrt{6}}{3(-2+\sqrt{3})}, -\frac{-3+2\sqrt{3}}{3(-2+\sqrt{3})}, 1\right\}, 2\right]}\right\} \text{ // FullSimplify}$$

$$\left\{\left\{-\frac{1}{\sqrt{3}}, \sqrt{\frac{2}{3}}, 0\right\}, \left\{-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{2}}\right\}, \left\{\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{2}}\right\}\right\}$$

These are the column vectors of the orthogonal matrix Q:

$$\left\{\left\{-\frac{1}{\sqrt{3}}, \sqrt{\frac{2}{3}}, 0\right\}, \left\{-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{2}}\right\}, \left\{\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{2}}\right\}\right\} \text{ // Transpose // MatrixForm}$$

$$\begin{pmatrix} -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

The columns of Q are linearly independent:

$$\text{Det}\left[\begin{pmatrix} -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\right]$$

1

The columns of Q are orthonormal:

$$\left\{-\frac{1}{\sqrt{3}}, \sqrt{\frac{2}{3}}, 0\right\}.\left\{-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{2}}\right\}$$

0

$$\left\{-\frac{1}{\sqrt{3}}, \sqrt{\frac{2}{3}}, 0\right\}.\left\{\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{2}}\right\}$$

0

$$\left\{-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{2}}\right\}.\left\{\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{2}}\right\}$$

0

$$\left\{-\frac{1}{\sqrt{3}}, \sqrt{\frac{2}{3}}, 0\right\}.\left\{-\frac{1}{\sqrt{3}}, \sqrt{\frac{2}{3}}, 0\right\}$$

1

$$\left\{-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{2}}\right\}.\left\{-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{2}}\right\}$$

1

$$\left\{\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{2}}\right\}.\left\{\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{2}}\right\}$$

1

Hence, the columns of Q form an orthonormal basis (ONB, eigenbasis).

We indeed have the decomposition $Q^{\mathrm{T}} A\, Q = D$ with D containing the corresponding eigenvalues:

```
Transpose[ ( -1/√3  -1/√3  1/√3 ; √(2/3)  -1/√6  1/√6 ; 0  1/√2  1/√2 ) ].( 2  -2 √2  3 √2 ; -2 √2  4  3 ; 3 √2  3  0 ).( -1/√3  -1/√3  1/√3 ; √(2/3)  -1/√6  1/√6 ; 0  1/√2  1/√2 ) //
  Simplify // MatrixForm
```

$$\begin{pmatrix} 6 & 0 & 0 \\ 0 & -3\sqrt{3} & 0 \\ 0 & 0 & 3\sqrt{3} \end{pmatrix}$$

Moreover, we have $Q^T Q = Q Q^T = I$:

```
Transpose[ ( -1/√3  -1/√3  1/√3 ; √(2/3)  -1/√6  1/√6 ; 0  1/√2  1/√2 ) ].( -1/√3  -1/√3  1/√3 ; √(2/3)  -1/√6  1/√6 ; 0  1/√2  1/√2 ) // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
( -1/√3  -1/√3  1/√3 ; √(2/3)  -1/√6  1/√6 ; 0  1/√2  1/√2 ).Transpose[ ( -1/√3  -1/√3  1/√3 ; √(2/3)  -1/√6  1/√6 ; 0  1/√2  1/√2 ) ] // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Problem 25. [Application of diagonalization]
  Solve the linear system $A\,x = b$ with

$A := \begin{pmatrix} -19 & 8 & 96 & 48 \\ 64 & 49 & 16 & 8 \\ 32 & 18 & 21 & 4 \\ -44 & -28 & -24 & 1 \end{pmatrix}$ and $b := \begin{pmatrix} 8 \\ -3 \\ 5 \\ 1 \end{pmatrix}$ by using the diagonal form of the non-symmetric matrix A in Problem 22.

Solution.

We saw in Problem 22 that A is regular (because 0 is not an eigenvalue). Moreover, we also constructed a suitable regular matrix K and diagonal matrix D such that $A = K\,D\,K^{-1}$. Clearly, this time $\exists\, D^{-1}$, and D^{-1} is the diagonal matrix consisting of the reciprocals of the eigenvalues of A.

```
Inverse[{{65, 0, 0, 0}, {0, -39, 0, 0}, {0, 0, 13, 0}, {0, 0, 0, 13}}] // MatrixForm
```

$$\begin{pmatrix} \frac{1}{65} & 0 & 0 & 0 \\ 0 & -\frac{1}{39} & 0 & 0 \\ 0 & 0 & \frac{1}{13} & 0 \\ 0 & 0 & 0 & \frac{1}{13} \end{pmatrix}$$

Therefore $A\,x = b$ is equivalent to $K\,D\,K^{-1}\,x = b$, from which we get $x = K\,D^{-1}\,K^{-1}\,b$, so

```
{{-4, 20, 1, 2}, {-18, -14, -2, -4}, {-9, -7, 0, 1}, {14, 8, 1, 0}}.Inverse[{{65, 0, 0, 0}, {0, -39, 0, 0}, {0, 0, 13, 0}, {0, 0, 0, 13}}].
  Inverse[{{-4, 20, 1, 2}, {-18, -14, -2, -4}, {-9, -7, 0, 1}, {14, 8, 1, 0}}].{8, -3, 5, 1}
```

$$\left\{\frac{592}{845}, -\frac{911}{845}, -\frac{33}{845}, \frac{593}{845}\right\}$$

The above result is of course the same as a direct application of

Mathematica's Solve:

```
Solve[{{-19, 8, 96, 48}, {64, 49, 16, 8}, {32, 18, 21, 4}, {-44, -28, -24, 1}}.{x1, x2, x3, x4} == {8, -3, 5, 1}]
```

$$\mathrm{Solve}\left[\begin{pmatrix} -19 & 8 & 96 & 48 \\ 64 & 49 & 16 & 8 \\ 32 & 18 & 21 & 4 \\ -44 & -28 & -24 & 1 \end{pmatrix}.\{x1, x2, x3, x4\} == \{8, -3, 5, 1\}\right]$$

$$\left\{\left\{x1 \to \frac{592}{845},\ x2 \to -\frac{911}{845},\ x3 \to -\frac{33}{845},\ x4 \to \frac{593}{845}\right\}\right\}$$

Problem 26. [Application of diagonalization]

By using diagonalization, compute A^{2019}, where $A := \begin{pmatrix} 2 & 2 \\ 3 & -3 \end{pmatrix}$ is the matrix appearing in Problem 13.

Solution.

This non-symmetric matrix A can be diagonalized as $A = K\,D\,K^{-1}$, where $D = \begin{pmatrix} -4 & 0 \\ 0 & 3 \end{pmatrix}$ and $K = \begin{pmatrix} -1 & 2 \\ 3 & 1 \end{pmatrix}$.

Now $A^{2019} = A\,A\ \ldots A = (K\,D\,K^{-1})(K\,D\,K^{-1})\ldots(K\,D\,K^{-1}) = K\,D^{2019}\,K^{-1}$. But D is diagonal, so we have $D^{2019} = \begin{pmatrix} (-4)^{2019} & 0 \\ 0 & 3^{2019} \end{pmatrix}$,

therefore

$$A^{2019} = \begin{pmatrix} -1 & 2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} (-4)^{2019} & 0 \\ 0 & 3^{2019} \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 3 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} (-4)^{2019} & 0 \\ 0 & 3^{2019} \end{pmatrix} \begin{pmatrix} -\frac{1}{7} & \frac{2}{7} \\ \frac{3}{7} & \frac{1}{7} \end{pmatrix} =$$

$$\begin{pmatrix} \frac{1}{7}\left(6 \times 3^{2019} + (-4)^{2019}\right) & \frac{2}{7}\left(3^{2019} - (-4)^{2019}\right) \\ \frac{3}{7}\left(3^{2019} - (-4)^{2019}\right) & \frac{1}{7}\left(3^{2019} + 6\,(-4)^{2019}\right) \end{pmatrix},$$

which is the same matrix as the one obtained by interpolation in Problem 13.

Problem 27. [Application of diagonalization]

Solve the linear system $A\,x = b$ with

$$A := \begin{pmatrix} 2 & -2\sqrt{2} & 3\sqrt{2} \\ -2\sqrt{2} & 4 & 3 \\ 3\sqrt{2} & 3 & 0 \end{pmatrix} \text{ and } b := \begin{pmatrix} \sqrt{2} \\ 2 \\ 3 \end{pmatrix} \text{ by using}$$

the orthonormal eigenbasis generated by the real symmetric matrix A in Problem 24.

Solution.

Let $(v_1,\ v_2,\ v_3)$ denote the column vectors of Q in Problem 24, forming an orthonormal eigenbasis.
The matrix A is now invertible, since spec(A) does not contain 0, so division by λ_j (any eigenvalue of A) makes sense here.
Let us express the unknown vector x and the given vector b in the orthonormal basis $(v_1,\ v_2,\ v_3)$:
$x = \sum_{j=1}^{3} \xi_j\, v_j$, and $b = \sum_{j=1}^{3} \beta_j\, v_j$. Here, due to orthogonality, we have $\beta_j = \langle b,\, v_j \rangle$.
Then $A\,x = b$ becomes $A\left(\sum_{j=1}^{3} \xi_j\, v_j\right) = \sum_{j=1}^{3} \beta_j\, v_j$, that is, $\sum_{j=1}^{3} \xi_j\, A\, v_j = \sum_{j=1}^{3} \beta_j\, v_j$, or $\sum_{j=1}^{3} \xi_j\, \lambda_j\, v_j = \sum_{j=1}^{3} \beta_j\, v_j$, which can be rearranged as $\sum_{j=1}^{3} \left(\xi_j\, \lambda_j - \beta_j\right) v_j = 0$. Since the vectors v_j are linearly independent, this implies that $\xi_j\, \lambda_j - \beta_j = 0$ for each j.
Hence we have determined the unknown coefficients as $\xi_j = \frac{\beta_j}{\lambda_j}$, which can also be written as $\xi_j = \frac{\langle b, v_j \rangle}{\lambda_j}$. Thus $x = \sum_{j=1}^{3} \frac{\langle b, v_j \rangle}{\lambda_j}\, v_j$.

Therefore, by using the data from Problem 24, we obtain that the solution vector x is

$$\frac{\{\sqrt{2}, 2, 3\}.\{-\frac{1}{\sqrt{3}}, \sqrt{\frac{2}{3}}, 0\}}{6} \{-\frac{1}{\sqrt{3}}, \sqrt{\frac{2}{3}}, 0\} +$$
$$\frac{\{\sqrt{2}, 2, 3\}.\{-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{2}}\}}{-3\sqrt{3}} \{-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{2}}\} +$$
$$\frac{\{\sqrt{2}, 2, 3\}.\{\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{2}}\}}{3\sqrt{3}} \{\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{2}}\} \text{ // Simplify}$$

$$\left\{\frac{5}{9\sqrt{2}}, \frac{4}{9}, \frac{4}{9}\right\}$$

Notice that this is essentially a Fourier expansion in 3D.

The above solution x is of course the same as a direct application of *Mathematica*'s Solve:

$$\text{Solve}\left[\begin{pmatrix} 2 & -2\sqrt{2} & 3\sqrt{2} \\ -2\sqrt{2} & 4 & 3 \\ 3\sqrt{2} & 3 & 0 \end{pmatrix}.\{x1, x2, x3\} == \left\{\sqrt{2}, 2, 3\right\}\right]$$

$$\left\{\left\{x1 \to \frac{5}{9\sqrt{2}}, x2 \to \frac{4}{9}, x3 \to \frac{4}{9}\right\}\right\}$$

■ Problem 28. [Computing the directional derivative (a.k.a. Gateaux derivative) for a functional in an inner-product space]

Suppose $(V, \langle\cdot, \cdot\rangle)$ is an inner-product space, and fix a vector $v \in V$. Let $F : V \to \mathbb{R}$ denote the functional $F(x) := \langle x, x\rangle - \langle v, x\rangle$ (defined for any $x \in V$). Show that for any $x \in V$ and $u \in V$ we have

$$\lim_{t\to 0,\, t\in\mathbb{R}} \frac{F(x+t\,u) - F(x)}{t} = \langle 2\,x - v, u\rangle.$$

Hint: first show that for any $x \in V$, $u \in V$ and $t \in \mathbb{R}$ we have $F(x + t\,u) - F(x) = t^2\langle u, u\rangle + t\langle 2\,x - v, u\rangle$.

■ Problem 29. [Computing the directional derivative (a.k.a. Gateaux derivative) for a functional in an inner-product space]

Let $A \in \mathbb{R}^{n \times n}$ be a *symmetric* matrix and $b \in \mathbb{R}^n$. Let $F: \mathbb{R}^n \to \mathbb{R}$ denote the functional $F(x) := \frac{1}{2}\langle x, A x\rangle - \langle b, x\rangle$ (defined for any $x \in \mathbb{R}^n$). Show that for any $x \in \mathbb{R}^n$, $u \in \mathbb{R}^n$ we have $\lim_{t \to 0,\, t \in \mathbb{R}} \frac{F(x+t u) - F(x)}{t} = \langle A x - b, u\rangle$.

Hint: first show that for any $x \in \mathbb{R}^n$, $u \in \mathbb{R}^n$ and $t \in \mathbb{R}$ we have $F(x + t u) - F(x) = \frac{1}{2} t^2 \langle u, A u\rangle + t\langle A x - b, u\rangle$.

- Problem 30. [The parallelogram law in inner-product spaces]

Suppose $(V, \langle \cdot, \cdot\rangle)$ is an inner-product space, and let $\|\cdot\|$ denote the induced norm, that is $\|x\| := \sqrt{\langle x, x\rangle}$ for $x \in V$. Show that for $\forall\ x \in V$, $\forall\ y \in V$ we have $\|x + y\|^2 + \|x - y\|^2 = 2\,\|x\|^2 + 2\,\|y\|^2$.

Solution: https://en.wikipedia.org/wiki/Parallelogram_law#The_parallelogram_law_in_inner_product_spaces

---

- Problem 31. [CSB in $\mathbb{R}^2$]

Let us consider the vector space $V := \mathbb{R}^2$ with the standard inner product and standard norm. Prove the Cauchy–Schwarz–Bunyakovsky inequality $|\langle x, y\rangle| \le \|x\|\,\|y\|$ (for any $x, y \in V$).

Hint: show that $\left| x_1 y_1 + x_2 y_2 \right| \leq \sqrt{x_1{}^2 + x_2{}^2}\ \sqrt{y_1{}^2 + y_2{}^2}$ for any $x_j, y_j \in \mathbb{R}$.

■ Problem 32. [Gram–Schmidt in $\mathbb{R}^4$]

By considering the standard inner product in $\mathbb{R}^4$, carry out the GS process with linearly independent input vectors $v_1 := \begin{pmatrix} -4 \\ -18 \\ -9 \\ 14 \end{pmatrix}$, $v_2 := \begin{pmatrix} 20 \\ -14 \\ -7 \\ 8 \end{pmatrix}$, $v_3 := \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix}$.

Solution. By using the formulae we discussed during the lecture, we get

$$f_1 = \begin{pmatrix} -4 \\ -18 \\ -9 \\ 14 \end{pmatrix},\ f_2 = \begin{pmatrix} \frac{13728}{617} \\ -\frac{2392}{617} \\ -\frac{1196}{617} \\ \frac{78}{617} \end{pmatrix},\ f_3 = \begin{pmatrix} -\frac{3}{469} \\ -\frac{202}{469} \\ \frac{368}{469} \\ -\frac{24}{469} \end{pmatrix},\ \text{hence } e_1 = \frac{1}{\sqrt{617}} \begin{pmatrix} -4 \\ -18 \\ -9 \\ 14 \end{pmatrix},$$

$$e_2 = \frac{1}{26\sqrt{\frac{469}{617}}} \begin{pmatrix} \frac{13728}{617} \\ -\frac{2392}{617} \\ -\frac{1196}{617} \\ \frac{78}{617} \end{pmatrix},\ e_3 = \frac{1}{\sqrt{\frac{377}{469}}} \begin{pmatrix} -\frac{3}{469} \\ -\frac{202}{469} \\ \frac{368}{469} \\ -\frac{24}{469} \end{pmatrix}.$$

■ Problem 33. [Gram–Schmidt in $C([-1, 1])$]

Let us consider the vector space $V := C([-1, 1])$, that is, the set of continuous functions on the interval $[-1, 1]$, equipped with the inner product

$\langle f, g\rangle := \int_{-1}^{1} f(x)\, g(x)\, dx$. Carry out the GS process with linearly independent input vectors $v_1 := 1$, $v_2 := x$, $v_3 := x^2$.

Solution. By using the formulae we discussed during the lecture, we get $f_1 = 1$, $f_2 = x$, $f_3 = x^2 - \frac{1}{3}$, hence $e_1 = \frac{1}{\sqrt{2}}$, $e_2 = \sqrt{\frac{3}{2}}\, x$, $e_3 = \sqrt{\frac{45}{8}}\, (x^2 - \frac{1}{3})$.

■ Problem 34. [Example of a norm: the induced norm]
Suppose $(V, \langle\cdot, \cdot\rangle)$ is an inner-product space, and let $\|\cdot\| : V \to \mathbb{R}$ denote the function $\|x\| := \sqrt{\langle x, x\rangle}$ (for any $x \in V$). Show that $(V, \|\cdot\|)$ is a normed space.

Hint: to verify the triangle inequality, first square it, then use the CSB inequality.

■ Problem 35. [Equivalent norms]
Let us consider the vector space $V := \mathbb{R}^2$. Show that the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent.

Hint: one needs to show that there are positive constants $0 < m < M$ such that for any $x \in \mathbb{R}^2$ we have $m\, \|x\|_2 \leq \|x\|_1 \leq M\, \|x\|_2$, that is, with $x = (x_1, x_2)$, we have $m\, \sqrt{x_1^2 + x_2^2} \leq |x_1| + |x_2| \leq M \sqrt{x_1^2 + x_2^2}$.
Solution: pick, for example, $m := 1$ and $M := \sqrt{2}$.

Problem 36. [Matrix operator norms induced by the p-norms]
Let $A := \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Compute $\|A\|_1$, $\|A\|_2$ and $\|A\|_\infty$.

Solution: by using the formulae we discussed during the lecture we get $\|A\|_1 = 6$, $\|A\|_2 = \sqrt{15 + \sqrt{221}} \approx 5.465$, $\|A\|_\infty = 7$.

■ Problem 37. [Example of a metric: the induced metric]
Suppose $(V, \|\cdot\|)$ is a normed vector space, and let $|\cdot, \cdot| : V \times V \to \mathbb{R}$ denote the function $|x, y| := \|x - y\|$ (defined for any $x \in V$, $y \in V$). Show that $(V, |\cdot, \cdot|)$ is a metric space.

Hint: verify that this $|\cdot,\cdot|$ satisfies all axioms of a metric.

■ Problem 38. [Example of a metric]
Let $V := \mathbb{R}^+$ denote the set of positive reals. The function $|\cdot, \cdot| : V \times V \to \mathbb{R}$ is defined by $|x, y| := \left|\ln\left(\frac{y}{x}\right)\right|$ (for any $x \in V$, $y \in V$), that is, by the modulus of the logarithm of the ratio of the arguments.
Show that $(V, |\cdot, \cdot|)$ is a metric space.

Hint: verify that this $|\cdot,\cdot|$ satisfies all axioms of a metric.

Problem 39. [Example of a metric]

Find the Hamming distance between the words “00100101” and “10011101”.

Hint: https://en.wikipedia.org/wiki/Hamming_distance

Solution:

```
HammingDistance["00100101", "10011101"]
```

```
4
```

Problem 40. [Example of a metric]

Let us consider the set of positive integers $M := \mathbb{N}^+$ with the function $|\cdot, \cdot| : M \times M \to \mathbb{R}$ defined by

$$|n, m| := \begin{cases} \max\left(\frac{1}{n}, \frac{1}{m}\right), & \text{for } n \neq m \ (n, m \in M), \\ 0, & \text{for } n = m \in M. \end{cases}$$

Show that $|\cdot, \cdot|$ is a metric on M, and the distance (with respect to this metric) of any two points $n, m \in M$ is at most 1.

Problem 41. [Example of a metric]

Show that the taxicab distance (that is, the l_1 distance) in $\mathbb{R}^3$ satisfies the axioms of a metric.

Hint: verify that the function $|x, y| := \|x - y\|_1 := |x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|$ (for any $x = (x_1, x_2, x_3) \in V$, $y = (y_1, y_2, y_3) \in V$) satisfies the axioms of a metric.

- Problem 42. [Example of a metric]
  Let us consider $V := C([0, 1])$, that is, the vector space of continuous functions defined on the interval [0, 1]. We define a function $|\cdot, \cdot| : V \times V \to \mathbb{R}$ by $|f, g| := \int_0^1 |f(x) - g(x)|\, dx$ (for any $f, g \in V$). Show that $|\cdot, \cdot|$ satisfies all axioms of a metric.

- Problem 43. [Example of a metric]
  Let us define a function $|\cdot, \cdot| : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $|x, y| := \frac{|x-y|}{1+|x-y|}$ (for any $x, y \in \mathbb{R}$). Show that $|\cdot, \cdot|$ satisfies all axioms of a metric, and the distance (with respect to this metric) of any two points $x, y \in \mathbb{R}$ is at most 1.

- Problem 44. [Negative definite matrix]
  Check that the matrix $A := \begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{pmatrix}$ is negative

definite, by
a.) computing the eigenvalues directly;
b.) using Sylvester's criterion;
c.) verifying that the corresponding quadratic form is negative for $x \neq 0$.

Hints.
a.) The eigenvalues of A are all negative:
$\lambda_1 = -2 - \sqrt{2} < 0,\ \lambda_2 = -2 < 0,\ \lambda_3 = -2 + \sqrt{2} < 0.$
b.) The sign pattern for the $k \times k$ top left subdeterminants (i.e., leading principal minors) is $(-, +, -)$.
c.) With $x = (x_1, x_2, x_3)$, the corresponding quadratic form $\langle A x, x\rangle = -x_1^2 - 2 x_2^2 + 2 x_1 x_3 - 3 x_3^2 = -(x_1 - x_3)^2 - 2 x_2^2 - 2 x_3^2$ is clearly non-positive, and for $x \neq 0$ it is strictly negative.

Problem 45. [Positive semidefinite matrix]

Check that the matrix $A := \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ is positive semidefinite, by
a.) computing the eigenvalues directly;
b.) using Sylvester's criterion;
c.) verifying that the corresponding quadratic form is non-negative for all $x \in \mathbb{R}^3$.

Hints.
a.) The eigenvalues of A are all nonnegative: $\lambda_1 = 3,\ \lambda_2 = 0,\ \lambda_3 = 0.$
b.) The sign of the 1×1 principal minors is $+$, the sign of the 2×2 principal minors is 0, the sign of the 3×3 principal minor is 0.

c.) With $x = (x_1, x_2, x_3)$, the corresponding quadratic form $\langle A x, x \rangle = x_1^2 + 2 x_1 x_2 + x_2^2 + 2 x_1 x_3 + 2 x_2 x_3 + x_3^2 = (x_1 + x_2 + x_3)^2$ is clearly non-negative, and for some $x \neq 0$ it can be zero, hence A is positive semidefinite but not positive definite.

Problem 46. [Testing definiteness of a matrix]

Determine the definiteness of the symmetric matrix

$$A := \begin{pmatrix} 1 & 4 & 6 \\ 4 & 2 & 1 \\ 6 & 1 & 6 \end{pmatrix}.$$

Solution: the sign pattern for the first two leading principal minors $\begin{pmatrix} 1 & 4 & 6 \\ 4 & 2 & 1 \\ 6 & 1 & 6 \end{pmatrix}$, $\begin{pmatrix} 1 & 4 & 6 \\ 4 & 2 & 1 \\ 6 & 1 & 6 \end{pmatrix}$ is $(+, -)$, hence A is neither positive definite, nor negative definite, nor positive semidefinite. The sign pattern for the first leading principal minor of $-A = \begin{pmatrix} -1 & -4 & -6 \\ -4 & -2 & -1 \\ -6 & -1 & -6 \end{pmatrix}$ is $(-)$, hence $-A$ is not positive semidefinite, so A is not negative semidefinite. Therefore A is indefinite.
Remark: direct computation of the eigenvalues $\{-11.08\ldots, -2.259\ldots, 4.349\ldots\}$ confirms the above.

Problem 47. [SVD]

a.) Find the SVD of the matrix $A := \begin{pmatrix} 3 & 0 & 4 \\ 0 & 5 & 0 \end{pmatrix}$.

b.) Check that the dyadic decomposition holds: $A = \sum_{k=1}^{r} \sigma_k u_k {v_k}^{\mathrm{T}}$.

c.) Compute the Moore–Penrose pseudoinverse A^+.

Hints.

a.)

• As it is known, the non-zero eigenvalues of $A^T A = \begin{pmatrix} 9 & 0 & 12 \\ 0 & 25 & 0 \\ 12 & 0 & 16 \end{pmatrix}$ or $A A^T = \begin{pmatrix} 25 & 0 \\ 0 & 25 \end{pmatrix}$ coincide. They are now {25, 25}, so the (non-zero) singular values of A are $\sigma_1 = \sqrt{25} = 5$, $\sigma_2 = \sqrt{25} = 5$. As usual, we have $\sigma_1 \geq \sigma_2 > 0$. The positive singular values are *uniquely* determined. We also have that the third eigenvalue of $A^T A$ is 0.

• The columns of V are (some) orthonormal eigenvectors of $A^T A$, corresponding to the eigenvalues $\lambda = 25$, $\lambda = 25$ and $\lambda = 0$. So we need to solve $\begin{pmatrix} 9 & 0 & 12 \\ 0 & 25 & 0 \\ 12 & 0 & 16 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, with $\lambda = 0$ and $\lambda = 25$. For $\lambda = 0$, we get $3x + 4z = 0$, $y = 0$. Hence, for example, $\begin{pmatrix} -1 \\ 0 \\ 3/4 \end{pmatrix}$ is an eigenvector.

For $\lambda = 25$, we get $4x = 3z$, $y = y$. Hence, for example, $\begin{pmatrix} 1 \\ 0 \\ 4/3 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ are orthogonal eigenvectors. (Recall: given a set of linearly independent [eigen]vectors, one can always use the Gram–Schmidt algorithm to generate a set of orthogonal/orthonormal [eigen]vectors in the same subspace.) The three eigenvectors $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \\ 4/3 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 0 \\ 3/4 \end{pmatrix}$ are orthogonal but not orthonormal. Simple normalization yields the orthonormal eigenvectors $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 3/5 \\ 0 \\ 4/5 \end{pmatrix}$ and $\begin{pmatrix} -4/5 \\ 0 \\ 3/5 \end{pmatrix}$. So $V = \begin{pmatrix} 0 & \frac{3}{5} & -\frac{4}{5} \\ 1 & 0 & 0 \\ 0 & \frac{4}{5} & \frac{3}{5} \end{pmatrix}$. The matrix V is *not unique* in general, but

$V\,V^{\mathsf{T}} = V^{\mathsf{T}}\,V$ = the appropriate identity matrix.

• To determine the matrix U, we use the relation $A\,v_j = \sigma_j\,u_j$, where $j = 1, 2, \ldots, \min(m, n)$, the dimensions of A are m and n, σ_j is a singular value of A, and u_j and v_j are the j^{th} column of U and V, respectively. From this we get

$$u_1 = \frac{1}{5}\begin{pmatrix} 3 & 0 & 4 \\ 0 & 5 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ and } u_2 = \frac{1}{5}\begin{pmatrix} 3 & 0 & 4 \\ 0 & 5 & 0 \end{pmatrix}\begin{pmatrix} \frac{3}{5} \\ 0 \\ \frac{4}{5} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{ hence } U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Note that the matrix U is *not unique* in general, but $U\,U^{\mathsf{T}} = U^{\mathsf{T}}\,U$ = the appropriate identity matrix.

• Thus we have obtained the SVD for A as $A = U\,\Sigma\,V^{\mathsf{T}}$ with $U = (u_1, u_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$,

$$\Sigma = \operatorname{diag}(\sigma_1, \sigma_2) = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \end{pmatrix}, \quad V = (v_1, v_2, v_3) = \begin{pmatrix} 0 & \frac{3}{5} & -\frac{4}{5} \\ 1 & 0 & 0 \\ 0 & \frac{4}{5} & \frac{3}{5} \end{pmatrix}.$$

• Remark. The columns of U are (some) orthonormal eigenvectors of $A\,A^{\mathsf{T}}$, corresponding to the eigenvalues $\lambda = 25$ and $\lambda = 25$. So we could solve $\begin{pmatrix} 25 & 0 \\ 0 & 25 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 25\begin{pmatrix} x \\ y \end{pmatrix}$, which is an identity, meaning that this time we are free to choose any two *orthonormal* planar vectors. We can pick for example $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, implying $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, which satisfies $A = U\,\Sigma\,V^{\mathsf{T}}$. However, one could also consider another ordering of these column vectors and obtain $U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, which would *not* satisfy the requirement $A = U\,\Sigma\,V^{\mathsf{T}}$. Therefore, we determine the matrix U from the relation $A\,v_j = \sigma_j\,u_j$.

b.)

The rank of A is 2, hence $r = 2$. Consequently,
$\sum_{k=1}^{r} \sigma_k\,u_k\,v_k^{\mathsf{T}} = \sigma_1\,u_1\,v_1^{\mathsf{T}} + \sigma_2\,u_2\,v_2^{\mathsf{T}} =$

$$5\begin{pmatrix} 0 \\ 1 \end{pmatrix}(0, 1, 0) + 5\begin{pmatrix} 1 \\ 0 \end{pmatrix}\left(\frac{3}{5}, 0, \frac{4}{5}\right) = 5\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + 5\begin{pmatrix} \frac{3}{5} & 0 & \frac{4}{5} \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 4 \\ 0 & 5 & 0 \end{pmatrix} = A.$$

c.)

As we know, $A^{+} = V\,\Sigma^{+}\,U^{\mathrm{T}}$. Now $\Sigma^{+} = \begin{pmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{5} \\ 0 & 0 \end{pmatrix}$, hence the generalized inverse is $A^{+} = \begin{pmatrix} \frac{3}{25} & 0 \\ 0 & \frac{1}{5} \\ \frac{4}{25} & 0 \end{pmatrix}$.

- **Problem 48.** [Application of the pseudoinverse: least-squares curve fitting] ***see in a separate file***

- **Problem 49.** [Application of SVD: data compression] ***see in a separate file***

- **Problem 50.** [Application of SVD: PCA] ***see in a separate file***