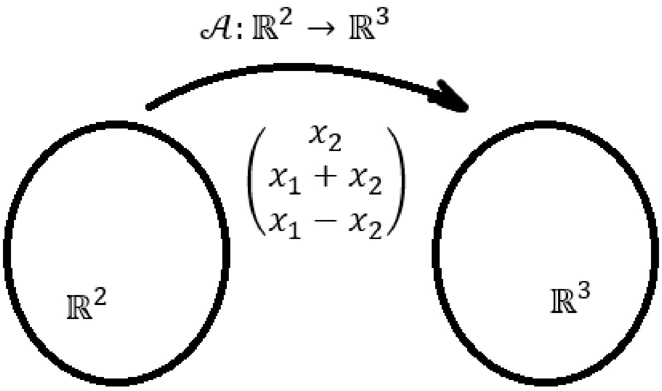
Practice 2
(Supplemental notes)

1. Linear Transformation and Examples

Def: Linear maps are transformations from one vector space to another that have the property of preserving vector addition and scalar multiplication:

$$\mathcal{A}: V \to W$$

Example 1:



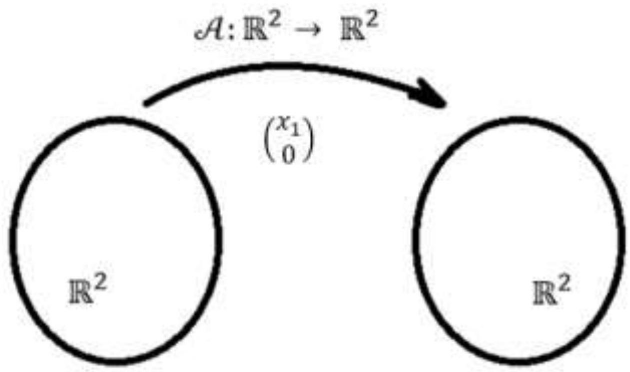
Let's check, is it really linear map?

1. Pick two vectors, $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$. Then $u + v = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix} \begin{matrix} -x_1 \\ -x_2 \end{matrix}$. And $cv = \begin{pmatrix} cv_1 \\ cv_2 \end{pmatrix}$.
   Applying the linear map:
   $$\mathcal{A}(u+v) = \begin{pmatrix} u_2 + v_2 \\ u_1 + v_1 + u_2 + v_2 \\ u_1 + v_1 - u_2 - v_2 \end{pmatrix} = \begin{pmatrix} u_2 \\ u_1 + u_2 \\ u_1 - u_2 \end{pmatrix} + \begin{pmatrix} v_2 \\ v_1 + v_2 \\ v_1 - v_2 \end{pmatrix} = \mathcal{A}(u) + \mathcal{A}(v).$$

2. Pick the vector and the scalar $c \in \mathbb{R}$:
   Applying the linear map:
   $$\mathcal{A}(cv) = \begin{pmatrix} cv_2 \\ c(v_1 + v_2) \\ c(v_1 - v_2) \end{pmatrix} = c \begin{pmatrix} v_2 \\ v_1 + v_2 \\ v_1 - v_2 \end{pmatrix} = c\mathcal{A}(v).$$

These two properties help us to qualify transformation as linear. So, we will use them as the two requirements that must be satisfied in order to be a linear map.

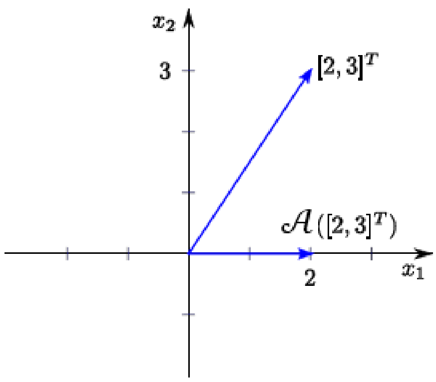
Example 2: Let $\mathcal{A}: \mathbb{R}^2 \to \mathbb{R}^2$ be projection onto the x_1-axis:



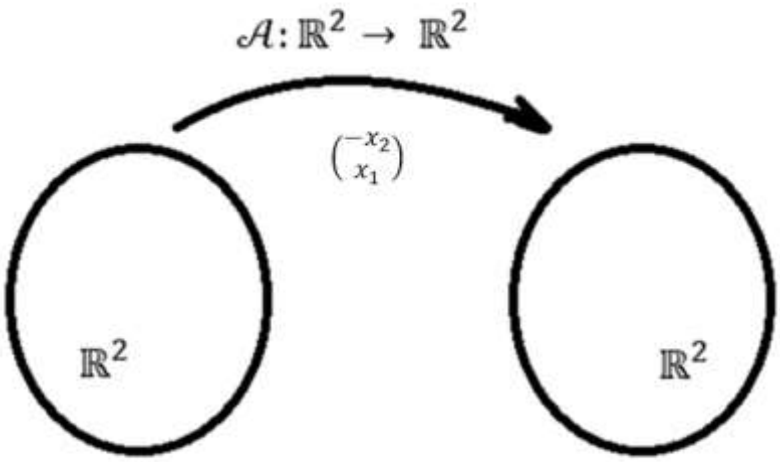
Let's check, is it really linear map?

1. Pick two vectors, $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$. Then $u + v = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix}\begin{matrix} -x_1 \\ -x_2 \end{matrix}$. And $cv = \begin{pmatrix} cv_1 \\ cv_2 \end{pmatrix}$.
   Applying the linear map:
   $$\mathcal{A}(u+v) = \begin{pmatrix} u_1 + v_1 \\ 0 \end{pmatrix} = \begin{pmatrix} u_1 \\ 0 \end{pmatrix} + \begin{pmatrix} v_1 \\ 0 \end{pmatrix} = \mathcal{A}(u) + \mathcal{A}(v).$$

2. Pick the vector and the scalar $c \in \mathbb{R}$:
   Applying the linear map:
   $$\mathcal{A}(cv) = \begin{pmatrix} cv_1 \\ c \cdot 0 \end{pmatrix} = c \begin{pmatrix} v_1 \\ 0 \end{pmatrix} = c\mathcal{A}(v).$$



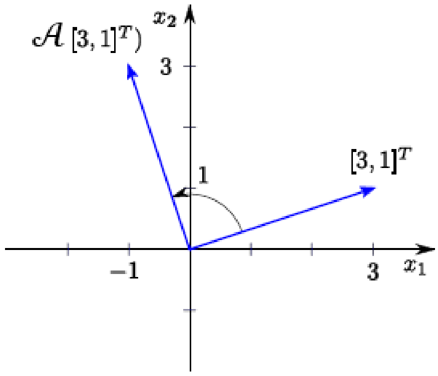
Example 3: Let $\mathcal{A}: \mathbb{R}^2 \to \mathbb{R}^2$ be 90-degree counterclockwise rotation:



Let's check, is it really linear map?

1. Pick two vectors, $u = \begin{pmatrix}u_1\\u_2\end{pmatrix}, v = \begin{pmatrix}v_1\\v_2\end{pmatrix}$. Then $u + v = \begin{pmatrix}u_1+v_1\\u_2+v_2\end{pmatrix}\begin{matrix}-x_1\\-x_2\end{matrix}$. And $cv = \begin{pmatrix}cv_1\\cv_2\end{pmatrix}$.
   Applying the linear map:
$$\mathcal{A}(u+v) = \begin{pmatrix}-u_2-v_2\\u_1+v_1\end{pmatrix} = \begin{pmatrix}-u_2\\u_1\end{pmatrix} + \begin{pmatrix}-v_2\\v_1\end{pmatrix} = \mathcal{A}(u) + \mathcal{A}(v).$$

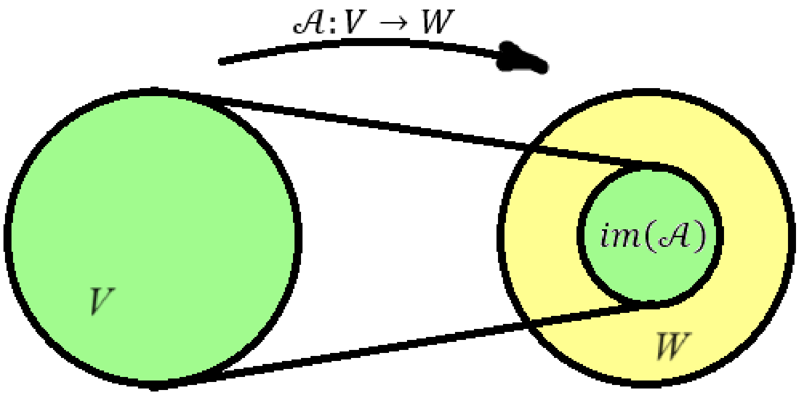
2. Pick the vector and the scalar $c \in \mathbb{R}$:
   Applying the linear map:
$$\mathcal{A}(cv) = \begin{pmatrix}-cv_2\\cv_1\end{pmatrix} = c\begin{pmatrix}-v_2\\v_1\end{pmatrix} = c\mathcal{A}(v).$$



2. Kernel, Range (image), Rank and Nullity

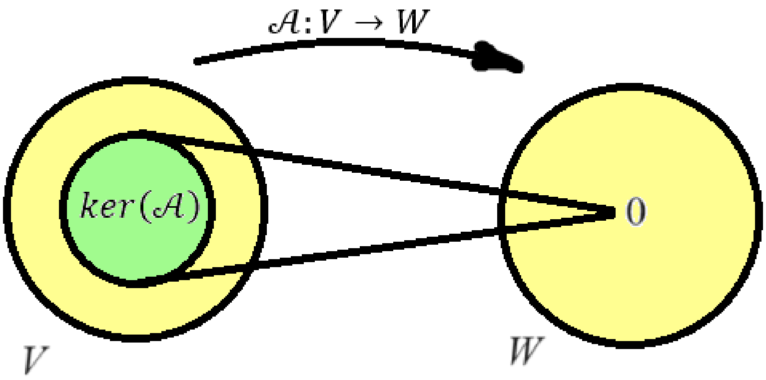
Def: The *range* of $\mathcal{A}$ is defined by: $ran(\mathcal{A}) = im(\mathcal{A}) = \{\mathcal{A}(v): v \in V\}$.
Def: The *kernel* of $\mathcal{A}$ is defined by: $ker(\mathcal{A}) = \{v \in V: \mathcal{A}(v) = 0\}$.
Def: Given a linear map $\mathcal{A}: V \to W$. Then

- dim $(ker(\mathcal{A}))$ is called the *nullity* of $\mathcal{A}$: $null(\mathcal{A})$.
- dim $(ran(\mathcal{A}))$ is called the *rank* of $\mathcal{A}$: $rank(\mathcal{A})$.



Range

Kernel



Problem 2

Def: Let A be an $m \times n$ matrix. The function $\mathcal{A}: V \to W$ defined by
$$\mathcal{A}(x) = Ax$$
is linear.
The function $\mathcal{A}$ is called the linear function corresponding to the matrix A.

Problem 3
Problem 4