Dynamic Programming

Example:

nth Fibonacci number

$F_0 = 0$
$F_1 = 1$
$F_n = F_{n-1} + F_{n-2}$

Recursive algorithm:

Fib(n): $\mathbb{N}$
  if n=0 ret 0
  if n=1 ret 1
    else
      ret Fib(n-1)+Fib(n-2)

DP algorithm:

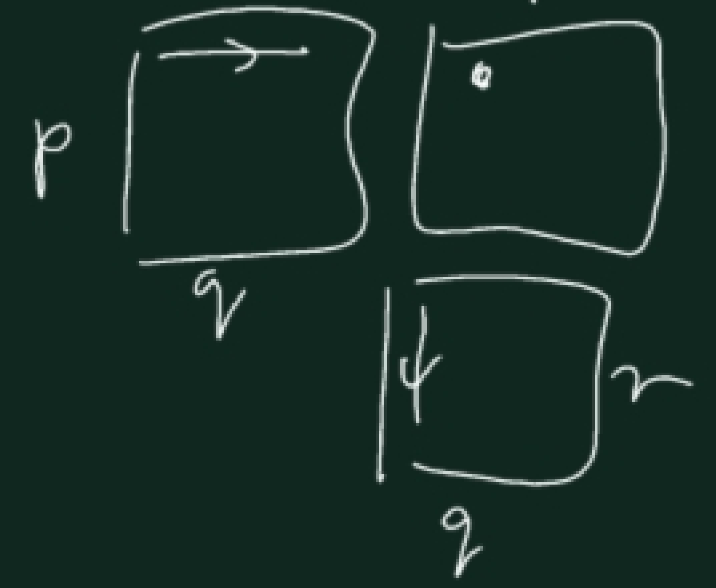
Fib(n): A: $\mathbb{N}[3]$
  $A[0] := 0 \quad A[1] := 1 \quad A[2] := 1$

  for i = 3 ... n
    $A[0] := A[1], \; A[1] := A[2]$
    $A[2] := A[0] + A[1]$

Matrix chain multiplication problem

$$A_1 \times A_2 \times A_3 \times A_4$$

$5\times 10 \quad 10\times 2 \quad 2\times 30 \quad 30\times 4$



The cost of multiplication of a mtxes of size $p\times q$ and $q\times r$ is $\boxed{p\cdot q\cdot r}$

$$((A_1\times A_2)\times A_3)\times A_4 \longleftarrow \overset{100}{5\times 10\times 2} + \overset{300}{5\times 2\times 30} + \overset{600}{5\times 30\times 4}$$

$$(A_1\times A_2)\times(A_3\times A_4) \longleftarrow \overset{100}{5\times 10\times 2} + \overset{240}{2\times 30\times 4} + \overset{40}{5\times 2\times 4}$$

$\boxed{1.}$ Subproblems

$1 \leq i \leq j \leq n$

$A_1 \times A_2 \times A_3 \times \dots \times A_n$

$R[i,j] :=$ is the problem of optimal parenthesization of $A_i \times A_{i+1} \times A_{i+2} \times \dots \times A_j$

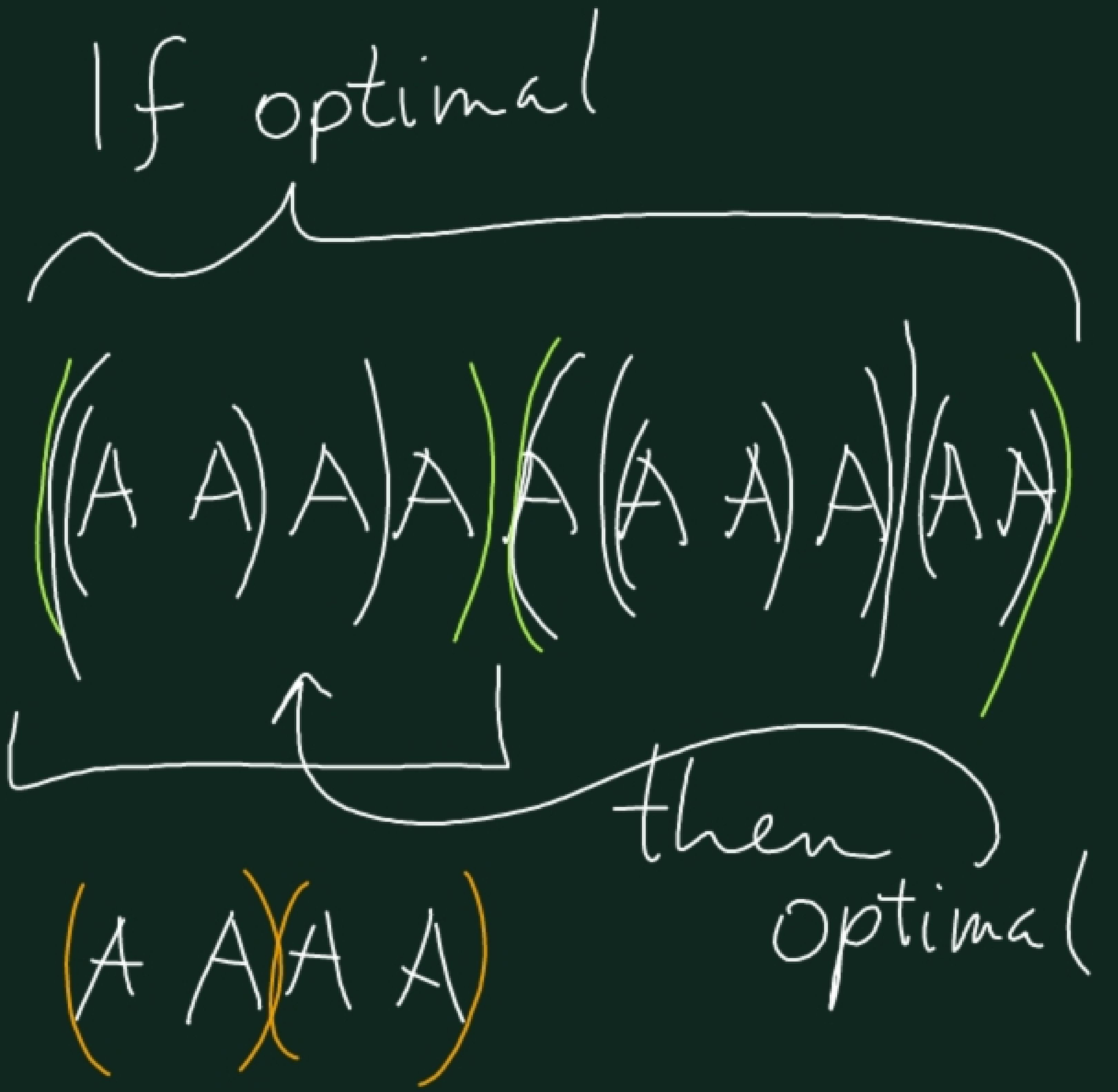
$\rightarrow$ where should we put the brackets so the cost is minimal?

2. Optimal substructure property

If I have an optimal solution for $R[i,j]$, and the last multiplication is between

$$(A_i \times A_{i+1} \times \dots \times A_k) \times (A_{k+1} \times A_{k+2} \times \dots \times A_j)$$

then the other brackets give an optimal solution for $R[i,k]$ and $R[k+1,j]$



$\boxed{3.}$ Recursive formula

$m[i,j] :=$ the cost of the optimal solution for problem $R[i,j]$

from $\boxed{2.}$ it follows:

if in the optimal solution of $R[i,j]$ the last multiplication is after A_k, then we have

$$m[i,j] = \underbrace{m[i,k]}_{A_i \times \dots \times A_k} + \underbrace{m[k+1,j]}_{\times \quad A_{k+1} \times \dots \times A_j} + p_{i-1} \times p_k \times p_j$$

$A_1 : p_0 \times p_1$

$A_2 : p_1 \times p_2$

$\vdots$

$A_i : p_{i-1} \times p_i$

$\vdots$

$A_n : p_{n-1} \times p_n$

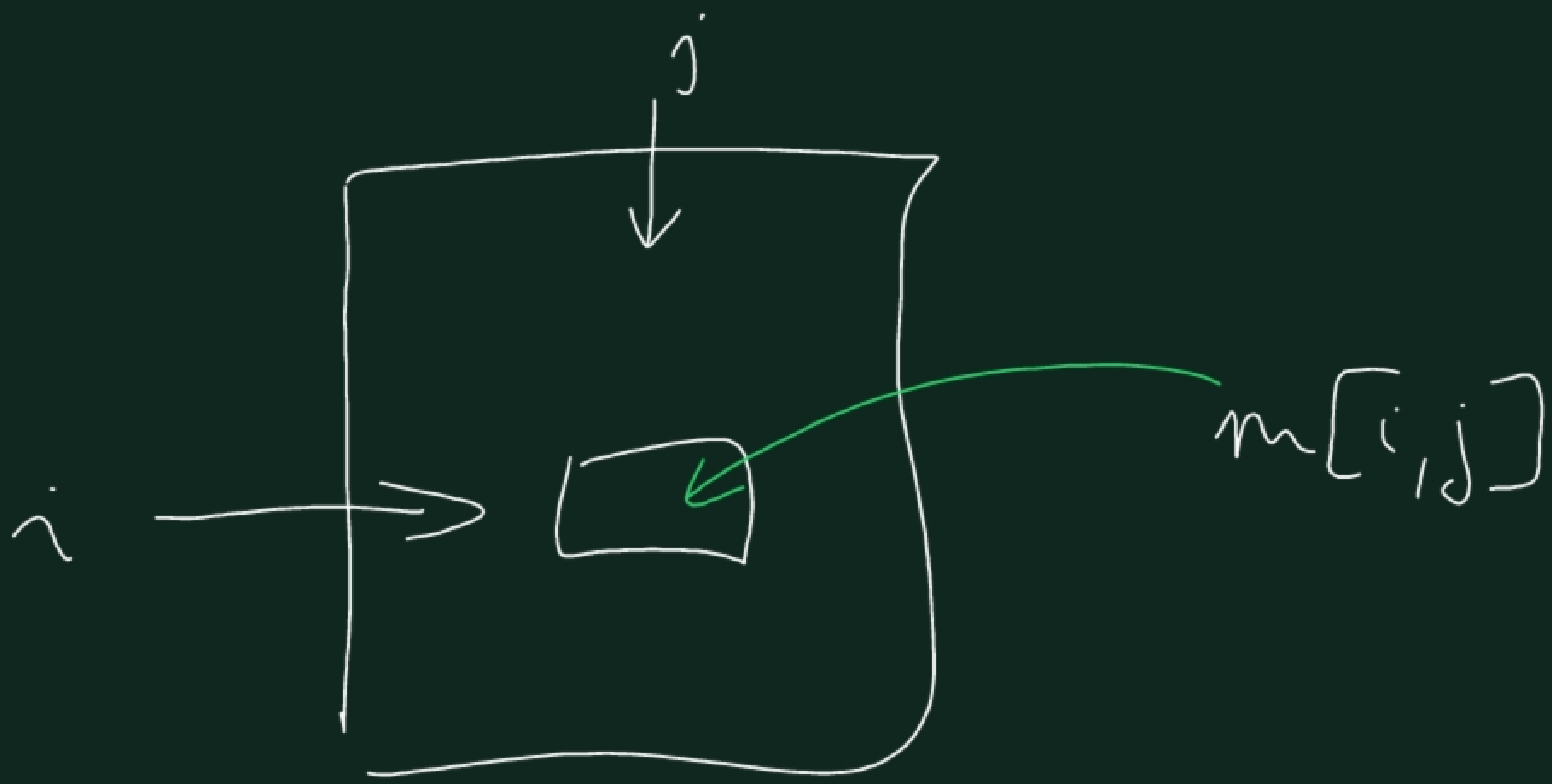
We don't know which "k" this is, so calculate all, and take the minimal one.

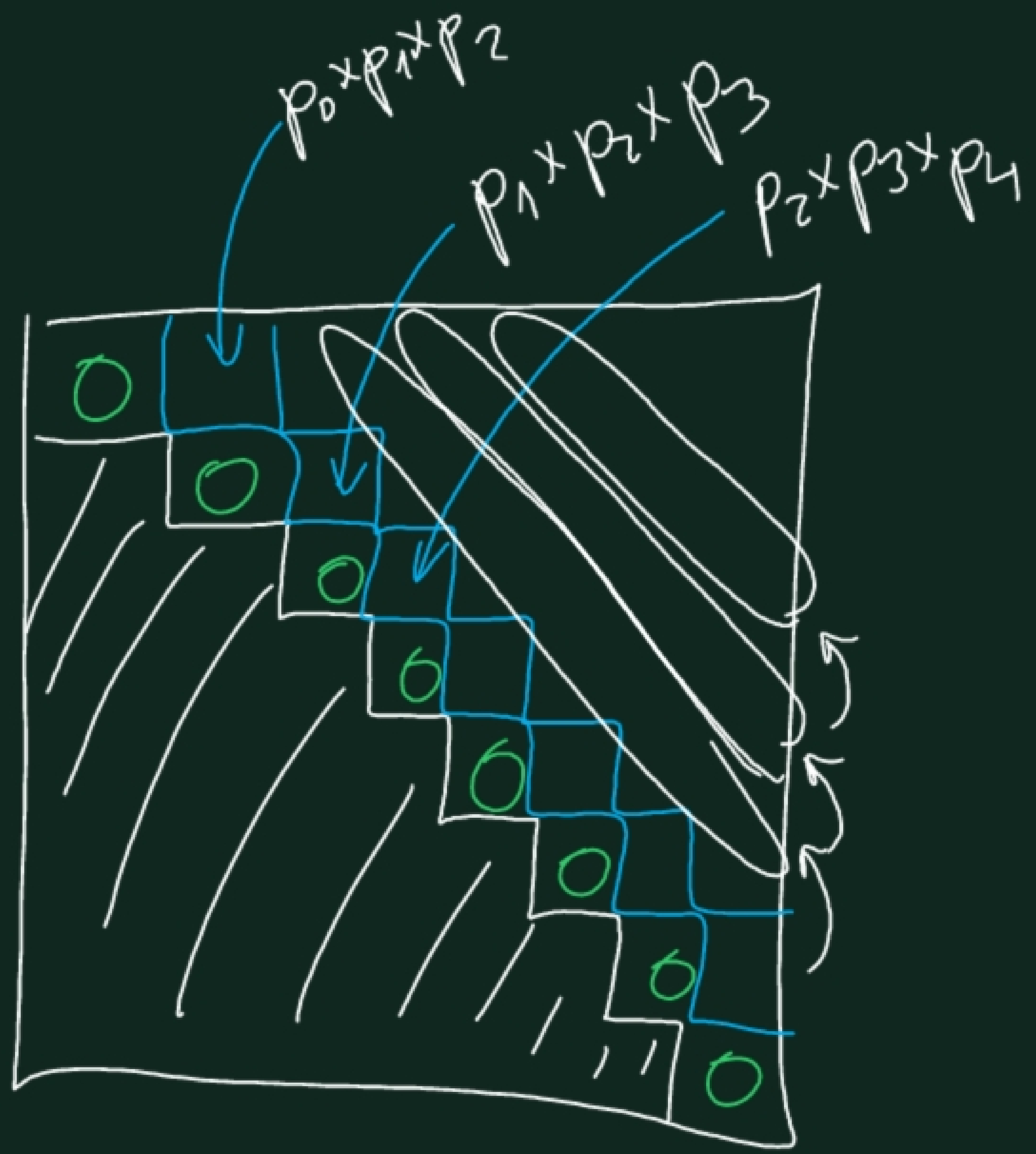
$$\boxed{\begin{aligned} m[i,j] &= \min_{i \le k < j} \left\{ m[i,k] + m[k+1,j] + p_{i-1} \times p_k \times p_j \right\} \\[2em] m[i,i] &:= 0 \\[1em] m[i,i+1] &= p_{i-1} \times p_i \times p_{i+1} \end{aligned}}$$

$$A_{i-1} \times \underbrace{A_i \times A_{i+1}} \times A_{i+3} \times \dots$$

$$\underbrace{\left(A_i \times A_{i+1} \times \dots A_k\right)}_{\text{matrix of size } p_{i-1} \times p_k} \times \underbrace{\left(A_{k+1} \times \dots \times A_j\right)}_{p_k \times p_j}$$

4. Table to fill





Example: A_1 A_2 A_3 A_4 A_5 A_6

25×30 30×10 10×5 5×10 10×15 15×20

	1	2	3	4	5	6
1	0	7500	5250			
2		0	1500	3000		
3			0	500	1500	
4				0	750	2250
5					0	3000
6						0

$25\times 30\times 10$

$30\times 10\times 5$

$10\times 5\times 10$

$5\times 10\times 15$

$10\times 15\times 20$

$$m[1,3]=\min\begin{cases}\overset{0}{m[1,1]}+\overset{1500}{m[2,3]}+\overset{3750}{25\times 30\times 5}\\ \overset{7500}{m[1,2]}+\overset{0}{m[3,3]}+\overset{1250}{25\times 10\times 5}\end{cases} 5250$$

$$m[2,4]=\min\begin{cases}\overset{0}{m[2,2]}+\overset{500}{m[3,4]}+\overset{3000}{30\times 10\times 10}\\ \overset{1500}{m[2,3]}+\overset{0}{m[4,4]}+\overset{1500}{30\times 5\times 10}\end{cases}$$

$$m[3,5]=\min\begin{cases}\overset{0}{m[3,3]}+\overset{750}{m[4,5]}+\overset{750}{10\times 5\times 15}\\ \overset{500}{m[3,4]}+\overset{0}{m[5,5]}+\overset{1500}{10\times 10\times 15}\end{cases}$$

$$m[4,6]=\min\begin{cases}\overset{0}{m[4,4]}+\overset{3000}{m[5,6]}+\overset{1000}{5\times 10\times 20}\\ \overset{750}{m[4,5]}+\overset{0}{m[6,6]}+\overset{1500}{5\times 15\times 20}\end{cases}$$

25 30 10 5 10 15 20

$$m[1,4] = \min \begin{cases} m[1,1] + m[2,4] + 25 \times 30 \times 10 \\ m[1,2] + m[3,4] + 25 \times 10 \times 10 \\ m[1,3] + m[4,4] + 25 \times 5 \times 10 \end{cases}$$

$$m[2,5] = \min \begin{cases} m[2,2] + m[3,5] + 30 \times 10 \times 15 \\ m[2,3] + \\ m[2,4] + \end{cases}$$

$$m[3,6] = \min \begin{cases} m[3,3] + m[4,6] + 10 \times 5 \times 20 \\ m[3,4] + m[5,6] + 10 \times 10 \times 20 \\ m[3,5] + m[6,6] + 10 \times 15 \times 20 \end{cases}$$

$m[1,5] = \ldots$

$m[2,6] = \ldots$

$m[1,6] = \ldots$

		5250	6500	7875	10000
			3000	4500	6750
				1500	3250
					2250

5. Giving an optimal solution:

In a new table keep track of the "k" values, where the minimum is taken.

k where the minimum is taken

	1	2	3			
1	0	1	1			3
2		0	2	3		
3			0	3	3	
4				0	4	5
					0	5
						0

$$\big((A_1 (A_2 A_3))((A_4 A_5) A_6)\big)$$