Programming theorems

- Programming theorems are problem-program pairs where the program solves the problem. They are frequently used as patterns to plan algorithms when the task to be solved is similar to the problem of the theorem.
- One of the common properties of the programming theorems is that they process a sequence of elementary values produced by an appropriate function. By expressing a programming theorem this way makes it more universal instead of processing the elements of an array: each array can be interpreted as a function over integer interval.
- In the following, some programming theorems will be given (counting, summation, maximum selection, conditional maximum selection, linear search, binary search).

Counting

Problem

Let β be a logical function defined over integers. Let us count the number of elements in the interval $[m..n]$ for which β holds.

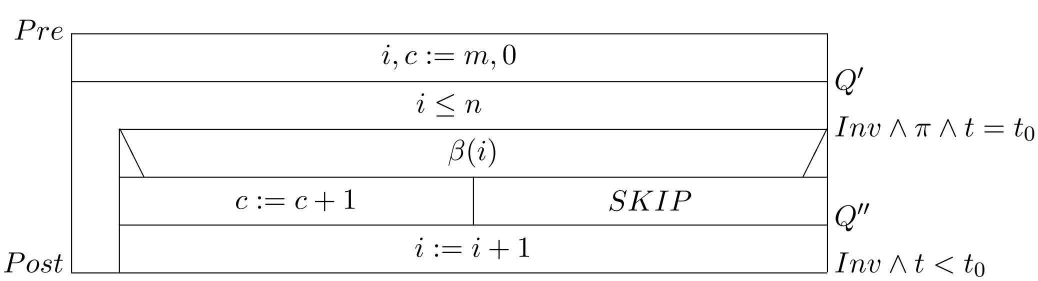
Specification of the problem

$A = (m\colon \mathbb{Z}, n\colon \mathbb{Z}, c\colon \mathbb{N}_0)$

$Pre = (m = m' \land n = n')$

$Post = (Pre \land c = \sum\limits_{i=m}^{n} \chi(\beta(i)))$ where $\chi\colon \mathbb{L} \to \{0,1\}$ and $\chi(true) = 1$ and $\chi(false) = 0$

Program (with annotations)



Assertions needed for proving the correctness of the program

$Q' = (Pre \wedge c = 0 \wedge i = m)$

$Inv = (Pre \wedge i \in [m..n+1] \cup \{m\} \wedge c = \sum\limits_{k=m}^{i-1} \chi(\beta(k)))$

$Q'' = (Inv^{i \leftarrow i+1} \wedge t = t_0)$

variant function $t\colon n - i + 1$

Summation

Problem

Let $\mathcal{H}$ be an arbitrary set where the operation of addition (+) is defined. Suppose that there exists a neutral element of the addition in H. Let the function $f\colon \mathbb{Z} \to \mathcal{H}$ be given. Let us calculate the sum of the values of f over the interval $[m..n]$.

Specification of the problem

$A = (m\colon \mathbb{Z}, n\colon \mathbb{Z}, s\colon \mathcal{H})$

$Pre = (m = m' \wedge n = n')$

$Post = (Pre \wedge s = \sum\limits_{i=m}^{n} f(i))$

Program (with annotations)

Pre	$i, s := m, 0$	
		Q'
	$i \leq n$	
		$Inv \wedge \pi \wedge t = t_0$
	$s := s + f(i)$	
		Q''
$Post$	$i := i + 1$	$Inv \wedge t < t_0$

Assertions needed for proving the correctness of the program

$Q' = (Pre \wedge s = 0 \wedge i = m)$

$Inv = (Pre \wedge i \in [m..n+1] \cup \{m\} \wedge s = \sum\limits_{k=m}^{i-1} f(k))$

$Q'' = (Inv^{i \leftarrow i+1} \wedge t = t_0)$

variant function $t\colon n - i + 1$

Maximum selection

Problem

Consider a non-empty integer interval and a function $f\colon \mathbb{Z} \to \mathcal{H}$ where $\mathcal{H}$ is a totally ordered set. Let us seek the greatest value of the function f and an argument where the function takes its maximum value.

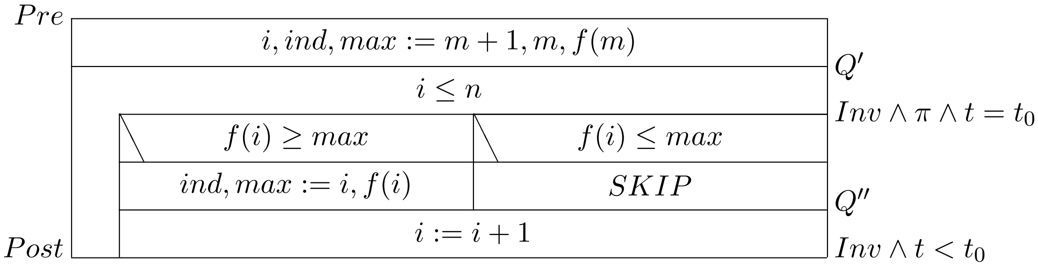
Specification of the problem

$A = (m\colon \mathbb{Z}, n\colon \mathbb{Z}, max\colon \mathcal{H}, ind\colon \mathbb{Z})$

$Pre = (m = m' \wedge n = n' \wedge m \leq n)$

$Post = (Pre \wedge ind \in [m..n] \wedge max = f(ind) \wedge \forall j \in [m..n] : f(j) \leq max)$

Program (with annotations)



Assertions needed for proving the correctness of the program

$Q' = (Pre \wedge max = f(m) \wedge ind = m \wedge i = m+1)$
$Inv = (Pre \wedge i \in [m..n+1] \wedge max = f(ind) \wedge \forall j \in [m..i-1] : f(j) \leq max)$
$Q'' = (Inv^{i \leftarrow i+1} \wedge t = t_0)$
variant function $t\colon n - i + 1$

Conditional maximum selection

Problem

Let $f\colon \mathbb{Z} \to \mathcal{H}$ and $\beta\colon \mathbb{Z} \to \mathbb{L}$ be functions defined over integers where $\mathcal{H}$ is a totally ordered set. Let us find the maximum value of the function f over the set $[m..n] \bigcap \lceil \beta \rceil$, and if exists, an argument in $[m..n] \bigcap \lceil \beta \rceil$ where the function takes its maximum value.

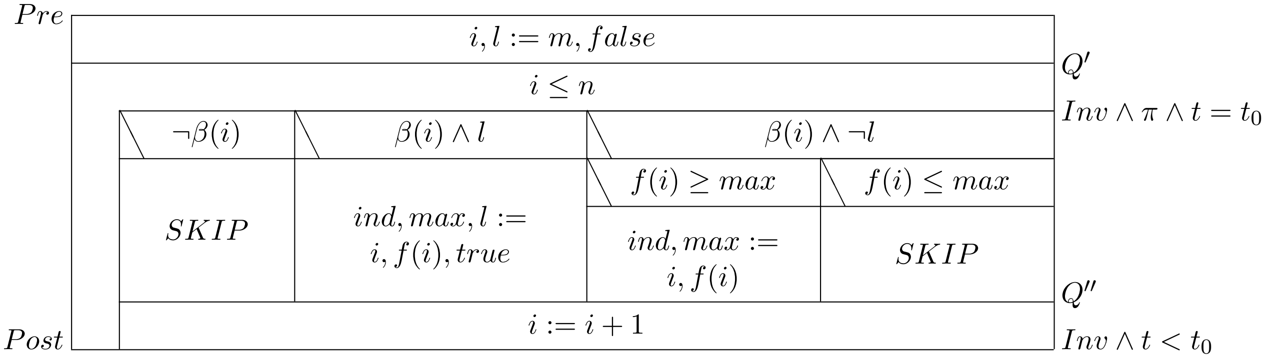
Specification of the problem

$A = (m\colon \mathbb{Z}, n\colon \mathbb{Z}, h\colon \mathcal{H}, ind\colon \mathbb{Z}, l\colon \mathbb{L})$

$Pre = (m = m' \wedge n = n')$

$Post = (Pre \wedge l = (\exists k \in [m..n] : \beta(k)) \wedge (l \implies (ind \in [m..n] \wedge max = f(ind) \wedge \beta(ind) \wedge \forall j \in [m..n] \cap \lceil \beta \rceil : f(j) \leq max))$

Program (with annotations)



Assertions needed for proving the correctness of the program

$Q' = (Pre \wedge l = false \wedge i = m)$
$Inv = (Pre \wedge i \in [m..n+1] \cup \{m\} \wedge l = (\exists k \in [m..i] : \beta(k)) \wedge (l \implies (ind \in [m..i] \wedge max = f(ind) \wedge \beta(ind) \wedge \forall j \in [m..i] \cap \lceil\beta\rceil : f(j) \leq max))$ $Q'' = (Inv^{i \leftarrow i+1} \wedge t = t_0)$
variant function t: $n - i + 1$

Counting

Problem

Let β be a logical function defined over integers. Let us count the number of elements in the interval $[m..n]$ for which β holds.

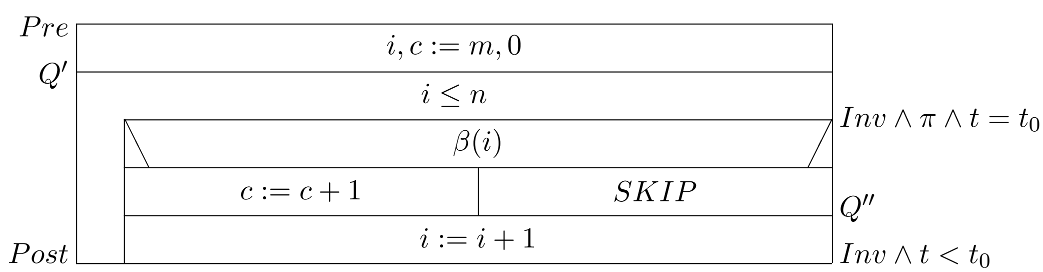
Specification of the problem

$A = (m\colon \mathbb{Z}, n\colon \mathbb{Z}, c\colon \mathbb{N}_0)$

$Pre = (m = m' \wedge n = n')$

$Post = (Pre \wedge c = \sum\limits_{i=m}^{n} \chi()\beta(i)))$ where $\chi\colon \mathbb{L} \to \{0,1\}$ and $\chi(true) = 1$ and $\chi(false) = 0$

Program (with annotations)



Assertions needed for proving the correctness of the program

$Q' = (Pre \land c = 0 \land i = m)$
$Inv = (Pre \land i \in [m..n+1] \cup \{m\} \land c = \sum_{k=m}^{i-1} \chi\beta(k))$
$Q'' = (Inv^{i \leftarrow i+1} \land t = t_0)$
variant function t: $n - i + 1$