Linear algebra revision

Vector space:

Vector space: (V, S) pair, where

V: set of "vectors"

S: set of "scalars", e.g.: $\mathbb{R}$ or $\mathbb{C}$ with operations $+$, $-$, $*$, $/$

operations:

addition: $v + w \in V \quad$ if $v, w \in V$

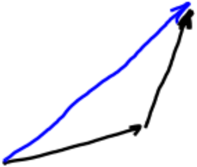
multiplication with scalar: $sv \in V \quad$ if $s \in S, v \in V$

ex: $(\mathbb{R}^n, \mathbb{R})$: $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ list of numbers

ex: space vectors, "arrows"

have direction and magnitude

e.g.: Physics: velocity, force



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Vector space properties:

linear combination: $s_1 v_1 + s_2 v_2 + \ldots + s_n v_n$

linear independence: a set $\{v_1, v_2, \ldots, v_n\}$ is linearly independent
if neither can be composed as a linear combination of the others

$\mathrm{span}(v_1, v_2, \ldots, v_k) = \left\{ \sum_{i=1}^{k} s_i v_i : \; s_i \in S \right\}$ all linear combination of vectors

a basis of V is $\{e_1, e_2, \ldots, e_d\}$
if $e_1, \ldots, e_n$ are linearly independent, and $\mathrm{span}(e_1, \ldots, e_d) = V$

$v = \sum_{i=1}^{d} x_i e_i$, where x_i are "coordinates" (unique for given v)

dimension: d (number of basis vectors, or minimum number of spanning vectors)

ex: $\dim(\mathbb{R}^n) = n$, $\quad e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow i^{\text{th}}$ element

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Linear operations:

$$f: \; V \to W, \quad \text{where } (V,S) \text{ and } (W,S) \text{ are vector spaces}$$

$$f \text{ is linear if:} \quad f(v+w) = f(v) + f(w)$$

$$f(sv) = sf(v)$$

$$\text{therefore} \quad f(s_1 v + s_2 w) = s_1 f(v) + s_2 f(w)$$

ex: $f: \; \mathbb{R}^n \to \mathbb{R}^m$,

$$f\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1 f\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + x_2 f\begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \ldots + x_n f\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

general: $f\left(\sum_{i=1}^{d} x_i e_i\right) = \sum_i x_i f(e_i)$

only need to know the action of f on basis vectors

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In concise form:

$$f\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \underbrace{\begin{bmatrix} & & \\ f(e_1) & \ldots & f(e_n) \\ & & \end{bmatrix}_{m\times n}}_{\underline{\underline{F}} \text{ matrix}} \cdot \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}}_{\underline{x} \text{ vector}} = \sum_i x_i \begin{bmatrix} \\ f(e_i) \\ \\ \end{bmatrix} = \begin{bmatrix} \sum F_{1i}x_i \\ \sum F_{2i}x_i \\ \vdots \\ \sum F_{ni}x_i \end{bmatrix}$$

$$F = \begin{bmatrix} F_{11} & F_{12} & \ldots & F_{1n} \\ F_{21} & F_{22} & \ldots & F_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ F_{m1} & F_{m2} & \ldots & F_{mn} \end{bmatrix}$$

linear operator $+$ basis $\rightarrow$ matrix

$$\left[\underline{\underline{F}} \cdot \underline{x}\right]_j = \sum_{k=1}^{n} F_{jk}x_k = F_{jk}x_k$$

(Einstein convention: sum on repeated indices)

$$\left[\underline{\underline{A}} \cdot \underline{\underline{B}}\right]_{ij} = \sum_k A_{ik}B_{kj}$$

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Matrix operations:

$$f(\cdot) \to \underline{\underline{F}} \quad \text{and} \quad g(\cdot) \to \underline{\underline{G}}, \qquad \text{"is represented by, for a given basis"}, \text{ then}$$

$$\left.\begin{aligned} f + g &\to \underline{\underline{F}} + \underline{\underline{G}} \\ f - g &\to \underline{\underline{F}} - \underline{\underline{G}} \\ sf &\to s\underline{\underline{F}} \end{aligned}\right\} \text{element-wise operations}$$

$$\underbrace{f \circ g}_{f(g(v))} \to \underline{\underline{F}} \cdot \underline{\underline{G}} \text{ matrix product}$$

and

$$\underline{\underline{A}} \cdot (\underline{\underline{B}} \cdot \underline{\underline{C}}) = (\underline{\underline{A}} \cdot \underline{\underline{B}}) \cdot \underline{\underline{C}} \qquad \text{(associative)}$$

$$\underline{\underline{A}} \cdot (\underline{\underline{B}} + \underline{\underline{C}}) = \underline{\underline{A}} \cdot \underline{\underline{B}} + \underline{\underline{A}} \cdot \underline{\underline{C}} \qquad \text{(distributive)}$$

$$s(\underline{\underline{A}} \cdot \underline{\underline{B}}) = (s\underline{\underline{A}}) \cdot B = \underline{\underline{A}} \cdot (s\underline{\underline{B}})$$

$$\underline{\underline{A}} \cdot \underline{\underline{B}} \neq \underline{\underline{B}} \cdot \underline{\underline{A}} \qquad \text{(in general not commutative)}$$

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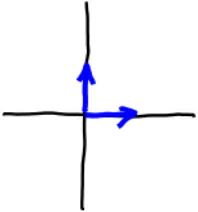


Matrix examples:

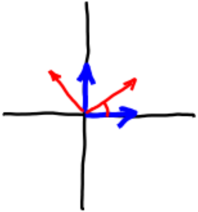
ex: $\mathbb{R}^2 \to \mathbb{R}^2$ geometrical plane, Cartesian basis

identity: $v \mapsto v$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

rescale: $v \mapsto sv$ $\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}$

reflect about x axis: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

rotation by angle ϕ: $\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$



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Matrix properties:

diagonal matrix: $\begin{bmatrix} * & & & 0 \\ & * & & \\ & & * & \\ 0 & & & * \end{bmatrix}$ $A_{ij} = 0$ if $i \neq j$ (off-diagonal elements are zero)

transpose: flip aroud the diagonal

$$\left[\underline{\underline{A}}^T\right]_{ij} = A_{ji}$$

$$\left(\underline{\underline{A}}^T\right)^T = \underline{\underline{A}}, \quad (\underline{\underline{A}} + \underline{\underline{B}})^T = \underline{\underline{A}}^T + \underline{\underline{B}}^T, \quad (\underline{\underline{A}} \cdot \underline{\underline{B}})^T = \underline{\underline{B}}^T \cdot \underline{\underline{A}}^T \quad \text{order!}$$

symmetric: $\underline{\underline{A}} = \underline{\underline{A}}^T$

antisymmetric: $\underline{\underline{A}} = -\underline{\underline{A}}^T$, then diagonal is zero

$$\operatorname{rank}(\underline{\underline{A}}) = \dim(\operatorname{span}(\underbrace{A_{*1}, A_{*2}, \ldots, A_{*n}}_{\text{column vectors}})) = \dim(\operatorname{span}(\underbrace{A_{1*}, A_{2*}, \ldots, A_{m*}}_{\text{row vectors}}))$$

full rank: if $\operatorname{rank}(\underline{\underline{A}}_{m \times n}) = \min(m, n)$

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Block matrices:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot \begin{bmatrix} E & F & G \\ H & I & J \end{bmatrix} = \begin{bmatrix} A \cdot E + B \cdot H & A \cdot F + B \cdot I & A \cdot G + B \cdot J \\ C \cdot E + D \cdot H & C \cdot F + D \cdot I & C \cdot G + D \cdot J \end{bmatrix}$$

block sizes need not be equal, but have to be compatible

ex: spatial transformation (rotation + translation)

$$\begin{bmatrix} \underline{\underline{R}}_1 & \underline{t}_1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \underline{\underline{R}}_2 & \underline{t}_2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \underline{\underline{R}}_1 \cdot \underline{\underline{R}}_2 & \underline{\underline{R}}_1 \cdot \underline{t}_2 + \underline{t}_1 \\ 0 & 1 \end{bmatrix}$$

structure is preserved

sequence of transformations $\rightarrow$ multiplication of matrices

(more details in practicals)

Linear algebra revision

Linear equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned} \qquad \rightarrow \qquad \underline{\underline{A}} \cdot \underline{x} = \underline{b}$$

operations on equations $\Leftrightarrow$ operations on matrix

$$\left[\begin{array}{c|c} \underline{\underline{A}} & \underline{b} \end{array}\right] \quad \begin{array}{c} \text{swap equations} \\ \text{add rows} \\ \text{multiply rows with a scalar} \end{array} \Rightarrow \begin{bmatrix} 1 & 0 & 0 & c_1 \\ 0 & 1 & 0 & c_2 \\ 0 & 0 & 1 & c_3 \end{bmatrix} \Rightarrow \underline{x} = \underline{c}$$

existence of solutions: properties of $\underline{\underline{A}}$ and $\underline{b}$.

ex: $\left.\begin{aligned} x + y &= 2 \\ 2x + 2y &= 5 \end{aligned}\right\}$ no solution

$\underline{\underline{A}} = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$ not full rank

ex: $\left.\begin{aligned} x + y &= 2 \\ 2x + 2y &= 4 \end{aligned}\right\}$ ∞ solutions

Linear algebra revision

Inverse: $n \times n$ matrices

$\underline{\underline{A}}$ is invertible if $\exists \underline{\underline{Z}} : \underline{\underline{A}} \cdot \underline{\underline{Z}} = \underline{\underline{Z}} \cdot \underline{\underline{A}} = \underline{\underline{I}}$

$$\underline{\underline{I}} = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}$$

then: $\underline{\underline{A}}^{-1} = \underline{\underline{Z}}$.

not always intertible: ex: $\underline{\underline{A}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$$

singular = not invertible

properties: if $\underline{\underline{A}}$ and $\underline{\underline{B}}$ are invertible, then

$$\left(\underline{\underline{A}}^{-1}\right)^{-1} = \underline{\underline{A}}$$

$$(\underline{\underline{A}} \cdot \underline{\underline{B}})^{-1} = \underline{\underline{B}}^{-1} \cdot \underline{\underline{A}}^{-1} \quad \text{order!}$$

$$\left(\underline{\underline{A}}^{T}\right)^{-1} = \left(\underline{\underline{A}}^{-1}\right)^{T}$$

use:

$$\underline{\underline{A}} \cdot \underline{x} = \underline{b}$$

$$\underline{\underline{A}}^{-1} \cdot \underline{\underline{A}} \cdot \underline{x} = \underline{\underline{A}}^{-1} \cdot \underline{b}$$

$$\underline{x} = \underline{\underline{A}}^{-1} \cdot \underline{b}$$