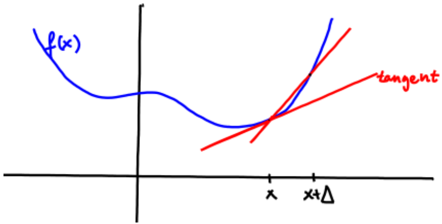
Calculus revision



Differentiation:

derivative of f: "local slope"

$$\underbrace{f'(x) = \frac{df}{dx}}_{\text{notation}} = \underbrace{\frac{\Delta f}{\Delta x}}_{\text{intuition}} = \underbrace{\lim_{\Delta \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta}}_{\text{definition}} \qquad \underbrace{df = f'(x)\, dx}_{\text{notation}}$$

use: x=time: growth rate, rate of change; e.g.: velocity $v = \frac{ds}{dt}$

x=space: gradient; e.g.: force $F = \frac{d\text{Energy}}{dx}$

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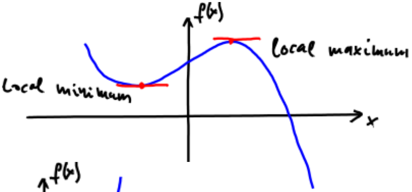
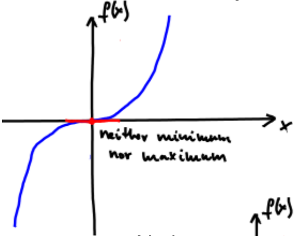
Differentiation and extrema:

(local) extremum $\underbrace{\Rightarrow}_{\text{at inner point if differentiable}}$ $f'(x) = 0$



careful: $\Leftarrow$ not necessarily true:

e.g.: x^3 at $x = 0$



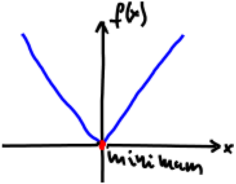
careful: endpoints can be extrema without $f'(x) = 0$

e.g.: $f: [1,2] \to \mathbb{R},\ f(x) = x^2$

careful: if not differentiable

e.g.: $|x|$ at $x = 0$

$$\frac{d|x|}{dx} = \begin{cases} 1 \text{ if } x > 0 \\ -1 \text{ if } x < 0 \\ \text{undefined if} x = 0 \end{cases}$$



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Differentiation rules:

power functions:

$$x' = 1$$
$$\left(x^2\right)' = 2x$$
$$\vdots$$
$$(x^n)' = nx^{n-1}$$

linear: $(f(x) + g(x))' = f'(x) + g'(x)$ or: $(f+g)' = f' + g'$

$(a\,f(x))' = a\,f'(x)$ (a constant)

product: $(f\,g)' = f'\,g + f\,g'$

ratio: $\left(\frac{f}{g}\right)' = \frac{f'\,g - f\,g'}{g^2}$

composition: notation $f \circ g$: $(f \circ g)(x) = f(g(x))$

chain rule: $(f \circ g)' = (f' \circ g) \cdot g'$

ex: $\frac{d\sqrt{2x+1}}{dx} = \frac{1}{2}(2x+1)^{-1/2} \cdot 2 = \frac{1}{\sqrt{2x+1}}$

$f(x) = \sqrt{x} = x^{1/2}$, $f'(x) = \frac{1}{2}x^{-1/2}$; $g(x) = 2x + 1$

Calculus revision

Inverse, exponential and trigonometric:

inverse function: notation: $f^{-1}(x)$, (ambiguous but widely used notation)

$$(f \circ f^{-1})(x) = f(f^{-1}(x)) = x \qquad / \tfrac{d}{dx}$$

$$f'(f^{-1}(x)) \cdot (f^{-1})'(x) = 1 \qquad \text{so} \qquad (f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

exponential: $\exp(x)$: $\frac{d}{dx} e^x = e^x$

log: inverse of exp: $\log'(x) = (\exp^{-1})'(x) = \frac{1}{\exp(\log(x))} = \frac{1}{x}$

trigonometric functions:

$$\sin'(x) = \cos(x)$$

$$\cos'(x) = -\sin(x)$$

$$\tan'(x) = \left(\frac{\sin(x)}{\cos(x)}\right)' = \frac{\cos^2(x) - \sin(x) \cdot (-\sin(x))}{\cos^2(x)} = \frac{1}{\cos^2(x)}$$

inverse trigonometric example:

$$\frac{d}{dx} \sin^{-1}(x) = \frac{1}{\cos(\sin^{-1}(x))} = \frac{1}{\sqrt{1-x^2}}$$

$\sin(\phi) = x, \ \phi = \sin^{-1}(x)$
$\cos(\phi) = \sqrt{1-x^2}$

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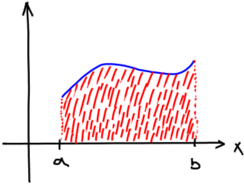
Integration:

antiderivative of $f(x)$ is $F(x)$ if $F'(x) = f(x)$

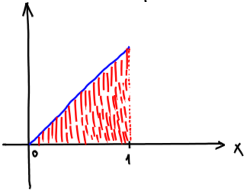
notation: $\int f(x)\,dx = F(x) + C$ **indefinite integral**

ex: $\int x^n dx = \frac{1}{n+1} x^{n+1} + C$ recall: $\frac{d}{dx} x^k = k\,x^{k-1}$

definite integral: $\int\limits_a^b f(x)\,dx = \text{Area} = F(b) - F(a)$



ex: $\int\limits_0^1 x\,dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1^2}{2} - \frac{0^2}{2} = \frac{1}{2}$



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Integration techniques:

change of variables:

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du$$

$$u = g(x), \quad \frac{du}{dx} = g'(x), \quad du = g'(x)\, dx$$

ex: $\int \sin(x^2)\, x\, dx = \int \sin(u) \frac{du}{2} = -\frac{1}{2}\cos(x^2) + C$

$$u = x^2, \quad du = 2x\, dx$$

integration by parts: recall: $(uv)' = u'v + uv'$

$$uv = \int u'v + \int uv' \quad \Rightarrow \quad \int uv' = uv - \int u'v$$

ex: $\int \underbrace{\sin(x)}_{v'} \underbrace{x}_{u} dx = -x\cos(x) + \int 1 \cdot \cos(x)\, dx$

$v = -\cos(x)$

$= \sin(x) - x\cos(x) + C$

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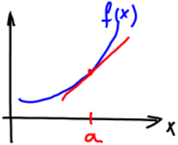

Power series:

$$f(x) \approx f(a) + f'(a)\cdot(x-a)$$
$$= c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots + c_n(x-a)^n$$

plug in $x = a$: $\quad f(a) = c_0$

$\dfrac{d}{dx}$, then $x = a$: $\quad f'(a) = c_1$

$$f''(a) = 2c_2 \qquad c_2\,2(x-a) \to 2c_2$$

$$f'''(a) = 3\cdot 2\cdot c_3 \qquad c_3 3(x-a)^2 \to c_3 3{\cdot}2(x-a) \to c_3 3{\cdot}2$$

$$f^{(n)}(a) = \underbrace{n(n-1)\ldots 3\cdot 2\cdot 1}_{n!\ \text{factorial}}\cdot c_n$$

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}}{k!}(x-a)^k + R_n$$

$$R_n = \frac{f^{(n+1)}(a^*)}{(n+1)!}(x-a)^{n+1} \qquad \text{where } a^* \in [a, x]$$

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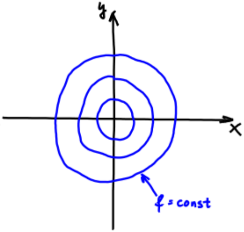
Functions of multiple variables:



ex: $f : \mathbb{R}^2 \to \mathbb{R}$ $\quad f(x,y) = x^2 + y^2$

partial derivatives: vary single variable, keep others fixed

$$\frac{\partial f(x,y)}{\partial x} = \lim_{\Delta \to 0} \frac{f(x+\Delta, y) - f(x,y)}{\Delta} = 2x$$

$$\frac{\partial f(x,y)}{\partial y} = \lim_{\Delta \to 0} \frac{f(x, y+\Delta) - f(x,y)}{\Delta} = 2y$$

gradient: $\nabla f = \begin{bmatrix} \partial f / \partial x \\ \partial f / \partial y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$

multiple derivatives:

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x} 2x = 2$$

$$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x} 2y = 0$$

Calculus revision

Functions of multiple variables:

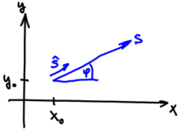
tangent plane: $z = f(x_0, y_0) + \left.\frac{\partial f}{\partial x}\right|_{x0,y0} (x - x_0) + \left.\frac{\partial f}{\partial y}\right|_{x0,y0} (y - y_0)$

total derivative, "infinitesimal change" notation:

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

ex: change in time: $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$ where $\frac{dx}{dt}$, $\frac{dy}{dt}$ are velocity components

directional derivative:



$$dx = \cos(\phi)\ ds$$

$$dy = \sin(\phi)\ ds$$

$$df = \frac{\partial f}{\partial x}\cos(\phi)\ ds + \frac{\partial f}{\partial y}\sin(\phi)\ ds \quad \Rightarrow \quad \frac{df}{ds} = \frac{\partial f}{\partial x}\cos(\phi) + \frac{\partial f}{\partial y}\sin(\phi)$$

$$\frac{df}{ds} = \nabla f \cdot \hat{s}, \qquad \hat{s} = \begin{bmatrix} \cos(\phi) \\ \sin(\phi) \end{bmatrix}$$

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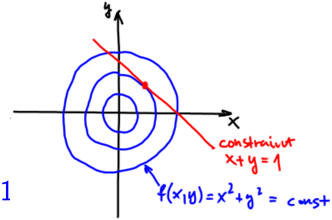
Constrained optimization:

ex: find minimum of $f(x,y) = x^2 + y^2$,

subject to $x + y = 1$



constraint function: $0 = g(x,y) \quad = x + y - 1$

$\nabla g \perp$ constraint curve (hypersurface)

at extremum (x_0, y_0): $\nabla f \perp$ constraint as well.

so at (x_0, y_0): $\left.\begin{array}{l} \nabla f = \lambda \nabla g \\ g(x,y) = 0 \end{array}\right\}$ solve!

ex: $\left.\begin{array}{c} \begin{bmatrix} 2x \\ 2y \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ x + y - 1 = 0 \end{array}\right\}$ $\quad \begin{array}{r} 2x = \lambda \\ 2y = \lambda \\ x + y = \frac{\lambda}{2} + \frac{\lambda}{2} = 1 \end{array} \quad \Rightarrow \quad \begin{array}{c} \lambda = 1 \\ x = y = \frac{1}{2} \end{array}$

equivalent formulation: minimize $L(x,y,\lambda) := f(x,y) - \lambda g(x,y)$

$$\left.\begin{array}{l} \frac{\partial L}{\partial x} = \frac{\partial f}{\partial x} - \lambda \frac{\partial g}{\partial x} = 0 \\ \frac{\partial L}{\partial y} = \frac{\partial f}{\partial y} - \lambda \frac{\partial g}{\partial y} = 0 \end{array}\right\} \Rightarrow \nabla f - \lambda \nabla g = 0$$

$$\frac{\partial L}{\partial \lambda} = g(x,y) = 0$$

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Online help:

https://openstax.org/subjects/math

then choose Calculus 1, Calculus 2, Calculus 3 (parts)

https://www.wolframalpha.com/

for online symbolic algebra, e.g.: check your results

(of course not for midterm / final exam!)