3D Point Cloud processing and analysis
Model Fitting
Plane Fitting

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Principle component analysis

Dimensionality reduction method used to reduce the dimensions of large data set by transforming the latter and projecting it into a smaller subspace but still containing most information.

Principle component analysis

- Standardization:
  we need to standardize our data. The aim here is to avoid variable dominance and each variable can contribute equally to the analysis
- Covariance matrix computation:
  get an understanding how the variables of the input data set are varying from each other
- Eigenvectors and Eigenvalues of the covariance matrix:
  Determining the principal components. The values and vectors pointing to the maximum components with largest variance.

Principle component analysis

$$\frac{1}{n}\sum_{i=1}^{n}(\vec{x_i}\cdot\vec{w})\vec{w} = \left(\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)\cdot\vec{w}\right)\vec{w}$$

$$\begin{aligned}
\|\vec{x_i}-(\vec{w}\cdot\vec{x_i})\vec{w}\|^2 &= (\vec{x_i}-(\vec{w}\cdot\vec{x_i})\vec{w})\cdot(\vec{x_i}-(\vec{w}\cdot\vec{x_i})\vec{w}) \\
&= \vec{x_i}\cdot\vec{x_i}-\vec{x_i}\cdot(\vec{w}\cdot\vec{x_i})\vec{w} \\
&\quad -(\vec{w}\cdot\vec{x_i})\vec{w}\cdot\vec{x_i}+(\vec{w}\cdot\vec{x_i})\vec{w}\cdot(\vec{w}\cdot\vec{x_i})\vec{w} \\
&= \|\vec{x_i}\|^2-2(\vec{w}\cdot\vec{x_i})^2+(\vec{w}\cdot\vec{x_i})^2\vec{w}\cdot\vec{w} \\
&= \vec{x_i}\cdot\vec{x_i}-(\vec{w}\cdot\vec{x_i})^2
\end{aligned}$$

$$\begin{aligned}
MSE(\vec{w}) &= \frac{1}{n}\sum_{i=1}^{n}\|\vec{x_i}\|^2-(\vec{w}\cdot\vec{x_i})^2 \\
&= \frac{1}{n}\left(\sum_{i=1}^{n}\|\vec{x_i}\|^2-\sum_{i=1}^{n}(\vec{w}\cdot\vec{x_i})^2\right)
\end{aligned}$$

$$\frac{1}{n}\sum_{i=1}^{n}(\vec{w}\cdot\vec{x_i})^2 = \left(\frac{1}{n}\sum_{i=1}^{n}\vec{x_i}\cdot\vec{w}\right)^2 + \mathrm{Var}\left[\vec{w}\cdot\vec{x_i}\right]$$

Principle component analysis

$$
\begin{aligned}
\sigma_{\vec{w}}^2 &= \frac{1}{n}\sum_i \left(\vec{x_i} \cdot \vec{w}\right)^2 \\
&= \frac{1}{n}(\mathbf{xw})^T(\mathbf{xw}) \\
&= \frac{1}{n}\mathbf{w}^T\mathbf{x}^T\mathbf{xw} \\
&= \mathbf{w}^T\frac{\mathbf{x}^T\mathbf{x}}{n}\mathbf{w} \\
&= \mathbf{w}^T\mathbf{vw}
\end{aligned}
$$

Model Fitting

- Least square fitting
- L1-norm fitting
- Hough transform
- Random sampling consensus (RANSAC)

Plane Model

$Plane\ equation:$

$$n.(P - P_0) = 0; n = [a, b, c]; P(x, y, z); P_0(x_0, y_0, z_0)$$

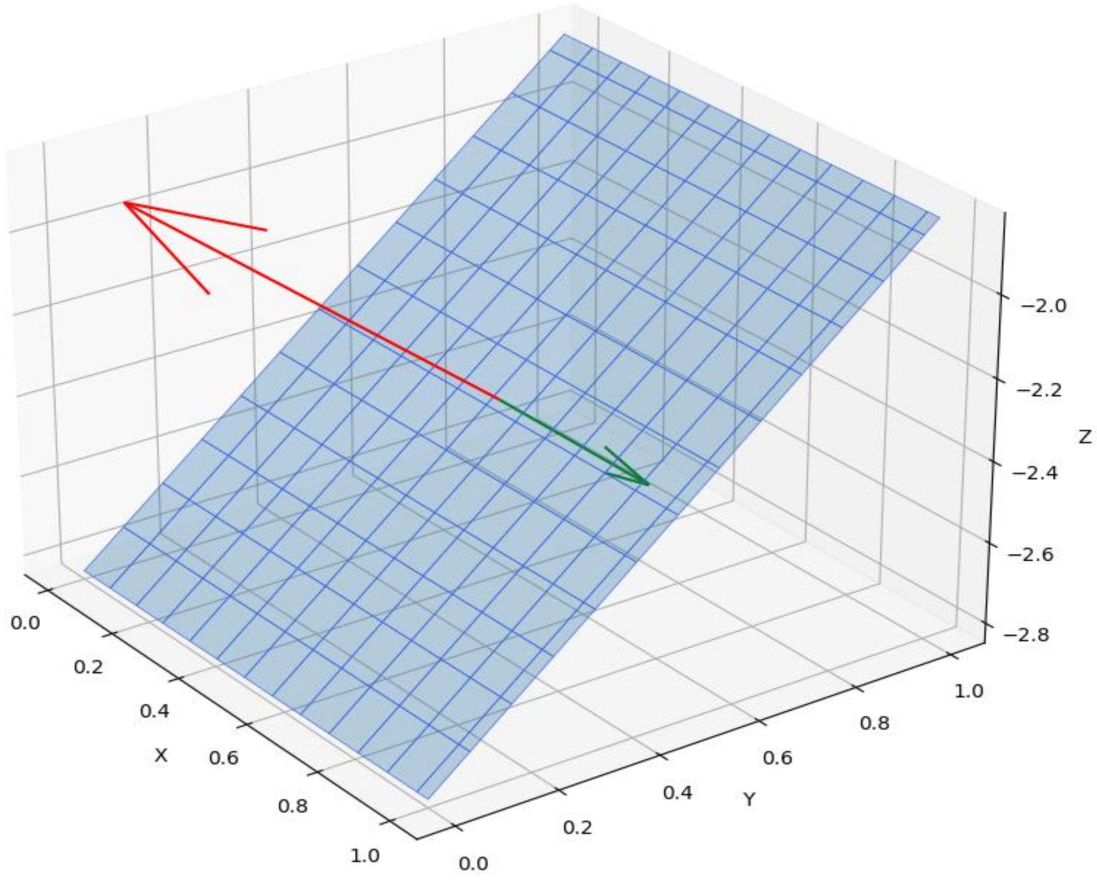
$$\begin{bmatrix} a & b & c \end{bmatrix} . \begin{bmatrix} (x - x_0) \\ (y - y_0) \\ (z - z_0) \end{bmatrix} = 0$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$
$$ax + by + cz - ax_0 - by_0 - cz_0 = 0$$

$$ax + by + cz + d = 0$$

$$\begin{bmatrix} x_1 & y_1 & z_1 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ x_n & y_n & z_n & 1 \end{bmatrix} . \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0$$

$$Ax = 0$$



Plane Fitting approaches

$$f(x, y) = z = \hat{a}x + \hat{b}y + \hat{c}$$

$$E = \sum^{N} \left(z - \left(\hat{a}x + \hat{b}y + \hat{c}\right)\right)^2 \rightarrow \dot{E} = 0$$

$$\frac{\partial E}{\partial \hat{a}} = -2 \sum^{N} x \left(z - \left(\hat{a}x + \hat{b}y + \hat{c}\right)\right) = 0$$

$$\frac{\partial E}{\partial \hat{b}} = -2 \sum^{N} y \left(z - \left(\hat{a}x + \hat{b}y + \hat{c}\right)\right) = 0$$

$$\frac{\partial E}{\partial \hat{c}} = -2 \sum^{N} \left(z - \left(\hat{a}x + \hat{b}y + \hat{c}\right)\right) = 0$$

Plane Fitting approaches

$$\sum^N x\left(z-\left(\hat{a}x+\hat{b}y+\hat{c}\right)\right)=0$$

$$\sum^N y\left(z-\left(\hat{a}x+\hat{b}y+\hat{c}\right)\right)=0$$

$$\sum^N \left(z-\left(\hat{a}x+\hat{b}y+\hat{c}\right)\right)=0$$

$$\sum^N xz-\hat{a}\sum^N x^2-\hat{b}\sum^N yx-\hat{c}\sum^N x=0$$

$$\sum^N yz-\hat{a}\sum^N xy-\hat{b}\sum^N y^2-\hat{c}\sum^N y=0$$

$$\sum^N z-\hat{a}\sum^N x-\hat{b}\sum^N y-\hat{c}\left(\sum^N 1\rightarrow N\right)=0$$

Plane Fitting approaches

$$\hat{a}\sum^N x^2 + \hat{b}\sum^N yx + \hat{c}\sum^N x = \sum^N zx$$
$$\hat{a}\sum^N xy + \hat{b}\sum^N y^2 + \hat{c}\sum^N y = \sum^N zy$$
$$\hat{a}\sum^N x + \hat{b}\sum^N y + \hat{c}N = \sum^N z$$

$$\begin{bmatrix} \sum^N x^2 & \sum^N yx & \sum^N x \\ \sum^N xy & \sum^N y^2 & \sum^N y \\ \sum^N x & \sum^N y & N \end{bmatrix} \cdot \begin{bmatrix} \hat{a} \\ \hat{b} \\ \hat{c} \end{bmatrix} = \begin{bmatrix} \sum^N zx \\ \sum^N zy \\ \sum^N z \end{bmatrix}$$

$$Ax = b$$
$$x = (A^T A)^{-1} A^T b$$

Plane Fitting approaches

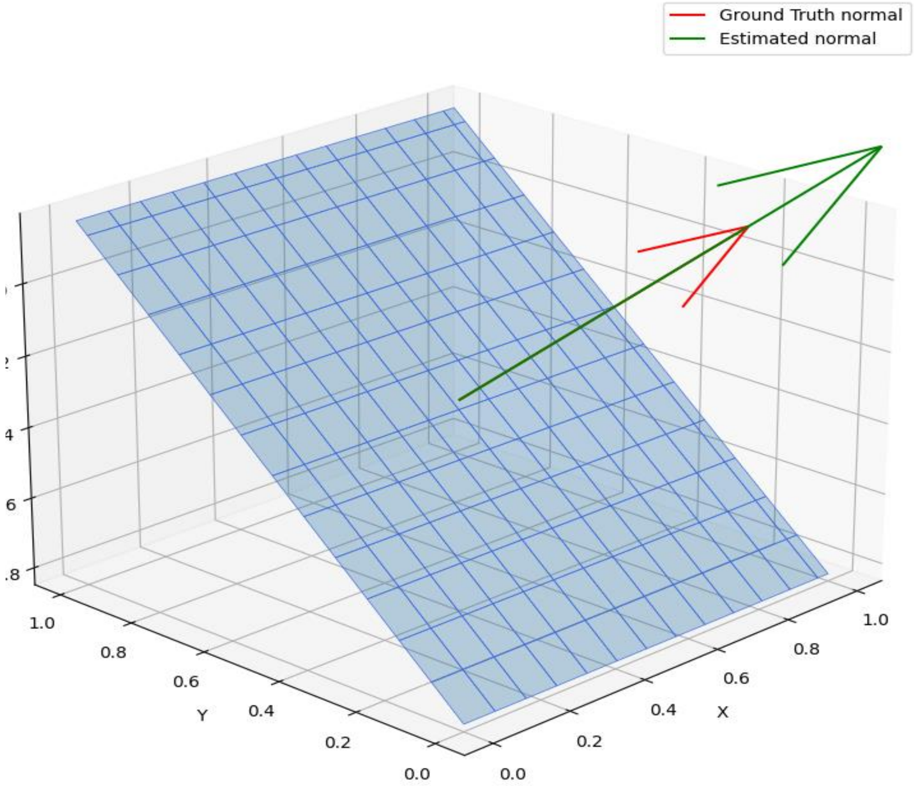
- To avoid ill-conditioned linear system, we can center the data around the mean.

$$\bar{x} = \frac{\sum^N x}{N},\ \bar{y} = \frac{\sum^N y}{N},\ \bar{z} = \frac{\sum^N z}{N}$$

$$\begin{bmatrix} \sum^N (x-\bar{x})^2 & \sum^N (x-\bar{x})(y-\bar{y}) & 0 \\ \sum^N (x-\bar{x})(y-\bar{y}) & \sum^N (y-\bar{y})^2 & 0 \\ 0 & 0 & N \end{bmatrix} \cdot \begin{bmatrix} \bar{\hat{a}} \\ \bar{\hat{b}} \\ \bar{\hat{c}} \end{bmatrix} =$$

$$\begin{bmatrix} \sum^N (z-\bar{z})(x-\bar{x}) \\ \sum^N (z-\bar{z})(y-\bar{y}) \\ 0 \end{bmatrix}$$

$$Ax = b$$
$$x = (A^T A)^{-1} A^T b$$



Plane Fitting approaches

$$ax + by + cz + d = 0$$

$$E = \sum^{N} (ax + by + cz + d)^2 \rightarrow \dot{E} = 0$$

$$\frac{\partial E}{\partial \hat{a}} = \sum^{N} 2x\,(ax + by + cz + d) = 0$$

$$\frac{\partial E}{\partial \hat{b}} = \sum^{N} 2y\,(ax + by + cz + d) = 0$$

$$\frac{\partial E}{\partial \hat{c}} = \sum^{N} 2z\,(ax + by + cz + d) = 0$$

$$\frac{\partial E}{\partial \hat{d}} = \sum^{N} 2\,(ax + by + cz + d) = 0$$

Plane Fitting approaches

$$\hat{a}\sum^{N} x^2 + \hat{b}\sum^{N} yx + \hat{c}\sum^{N} xz + \hat{d}\sum^{N} x = 0$$

$$\hat{a}\sum^{N} xy + \hat{b}\sum^{N} y^2 + \hat{c}\sum^{N} yz + \hat{d}\sum^{N} y = 0$$

$$\hat{a}\sum^{N} xz + \hat{b}\sum^{N} yz + \hat{c}\sum^{N} z^2 + \hat{d}\sum^{N} z = 0$$

$$\hat{a}\sum^{N} x \quad + \hat{b}\sum^{N} y \; + \hat{c}\sum^{N} z \quad + \hat{d}N = 0$$

Plane Fitting approaches

$$\begin{bmatrix} \sum^N x^2 & \sum^N yx & \sum^N xz & \sum^N x \\ \sum^N xy & \sum^N y^2 & \sum^N yz & \sum^N y \\ \sum^N xz & \sum^N yz & \sum^N z^2 & \sum^N z \\ \sum^N x & \sum^N y & \sum^N z & N \end{bmatrix} \cdot \begin{bmatrix} \hat{a} \\ \hat{b} \\ \hat{c} \\ \hat{d} \end{bmatrix} = 0$$

$$Ax = 0$$

Solution is found by finding the eigenvector of the smallest eigenvalues of A

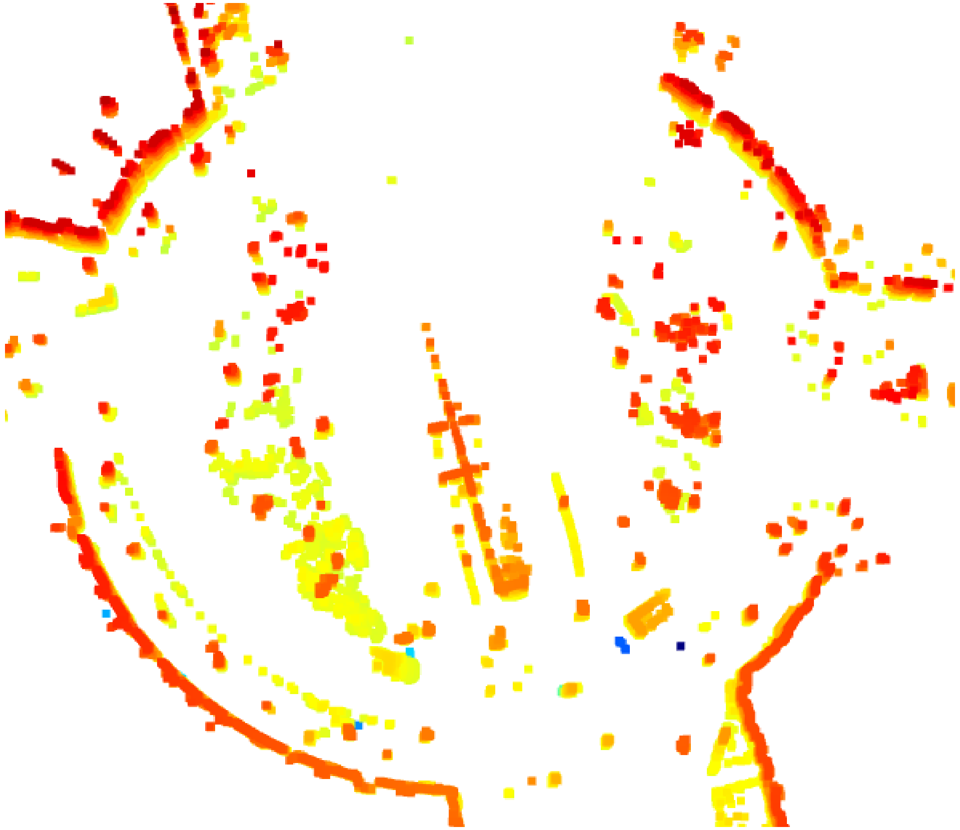
L1_Norm

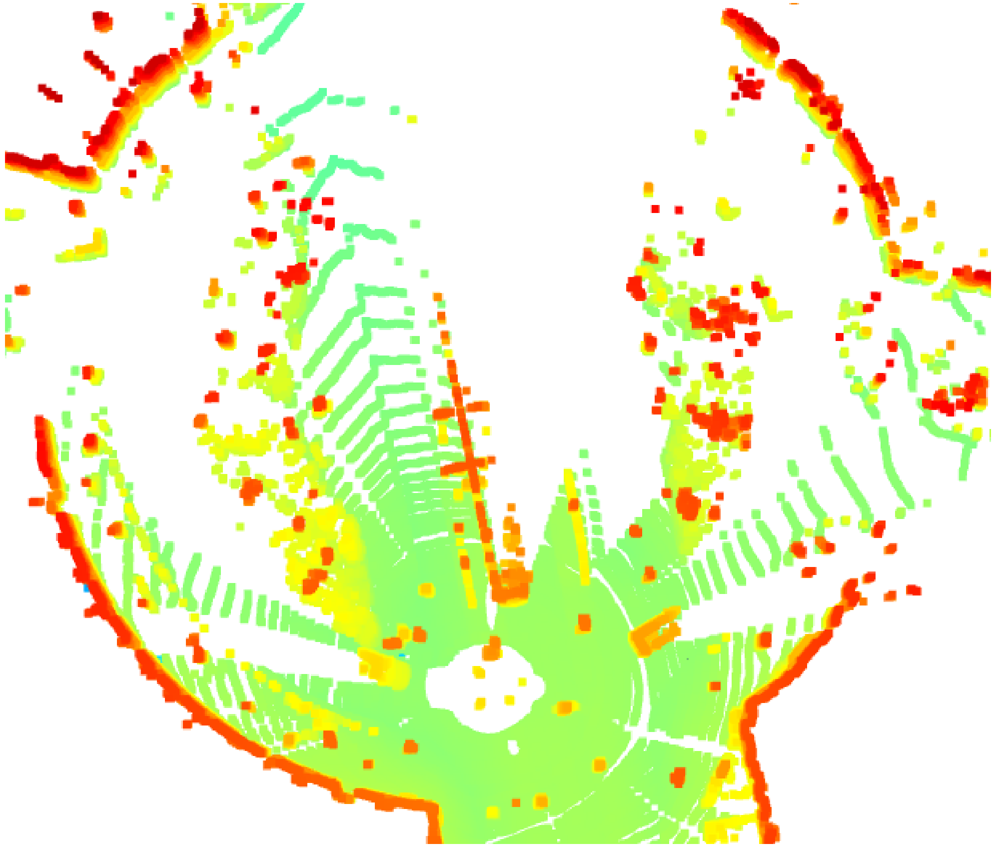
- The main limitation of least squared errors is the sensitivity to outliers
- Outliers pull the fitting in their direction
- Ordinary least squares minimizes the sum of the squares of the residuals, i.e., the L2-norm
- L1-norm can be used as alternative which minimizes the sum of the absolute of the residuals
- Compared to L2-norm, L1-norm regression reduces the influence of the outlier

Random Sample Consensus (RANSAC)

- Estimates the parameters of a mathematical model iteratively from a set of observations
- Randomly select a sample that is a minimal subset of the points required to solve the model
- Solve the model
- Compute the error function for each point
- Count the points that are consistent with the model, i.e., inliers $e < t$
- Repeat the steps for N iterations, the model with most inliers will be chosen

RANSAC





Plane Fitting approaches

- Standardization
- Covariance matrix computation
- Eigenvector of the small Eigenvalue of the covariance matrix

$$ax + by + cz + d = 0$$

$$\begin{bmatrix} x_1 & y_1 & z_1 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ x_n & y_n & z_n & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0$$

$$Ax = 0$$

COV= $$\begin{array}{cc} & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c} x \\ y \\ z \end{array} & \begin{bmatrix} var(x) & cov(x,y) & cov(x,z) \\ cov(x,y) & var(y) & cov(y,z) \\ cov(x,z) & cov(y,z) & var(z) \end{bmatrix} \end{array}$$

Hough Transform

- Feature extraction technique used to isolate features of a particular shape within an image
- Classical Hough transform is most commonly used to detect lines, circles or ellipses
- First maps the data from the image space to the parameter space
- The solution is found through a voting method

Hough Transform

Line example:

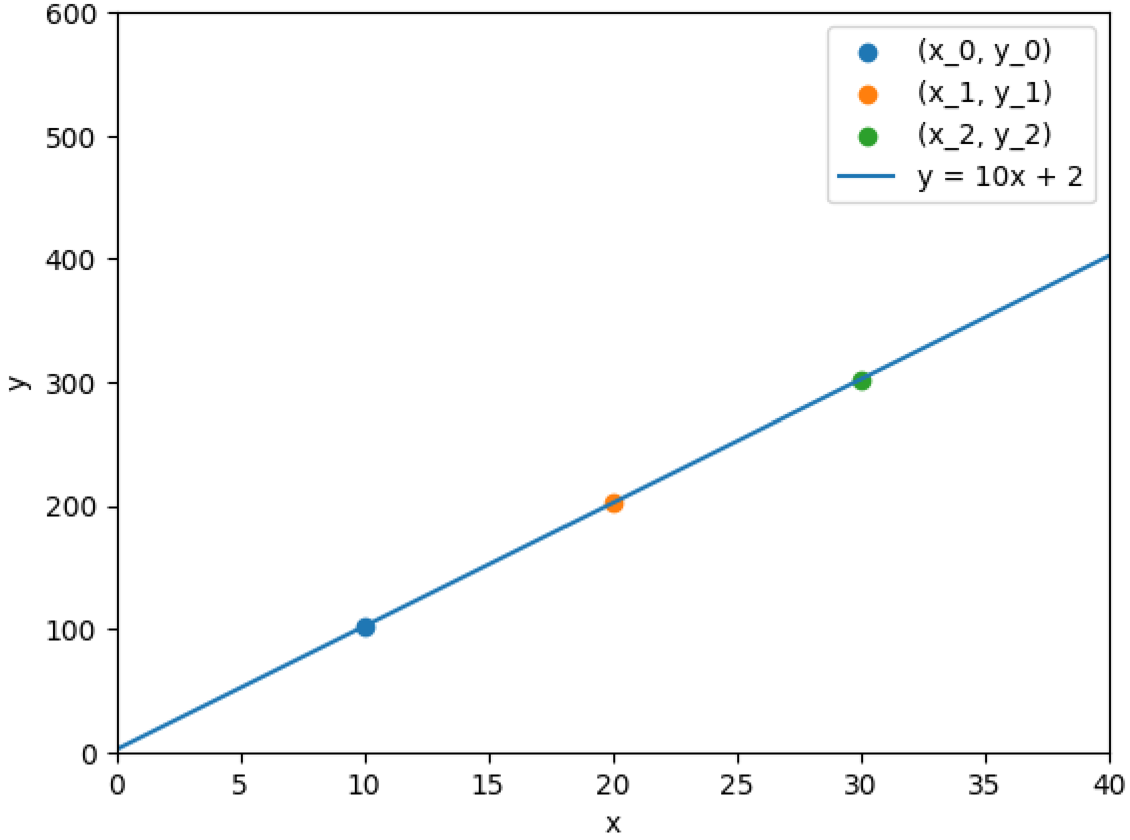
$$y = ax + b$$

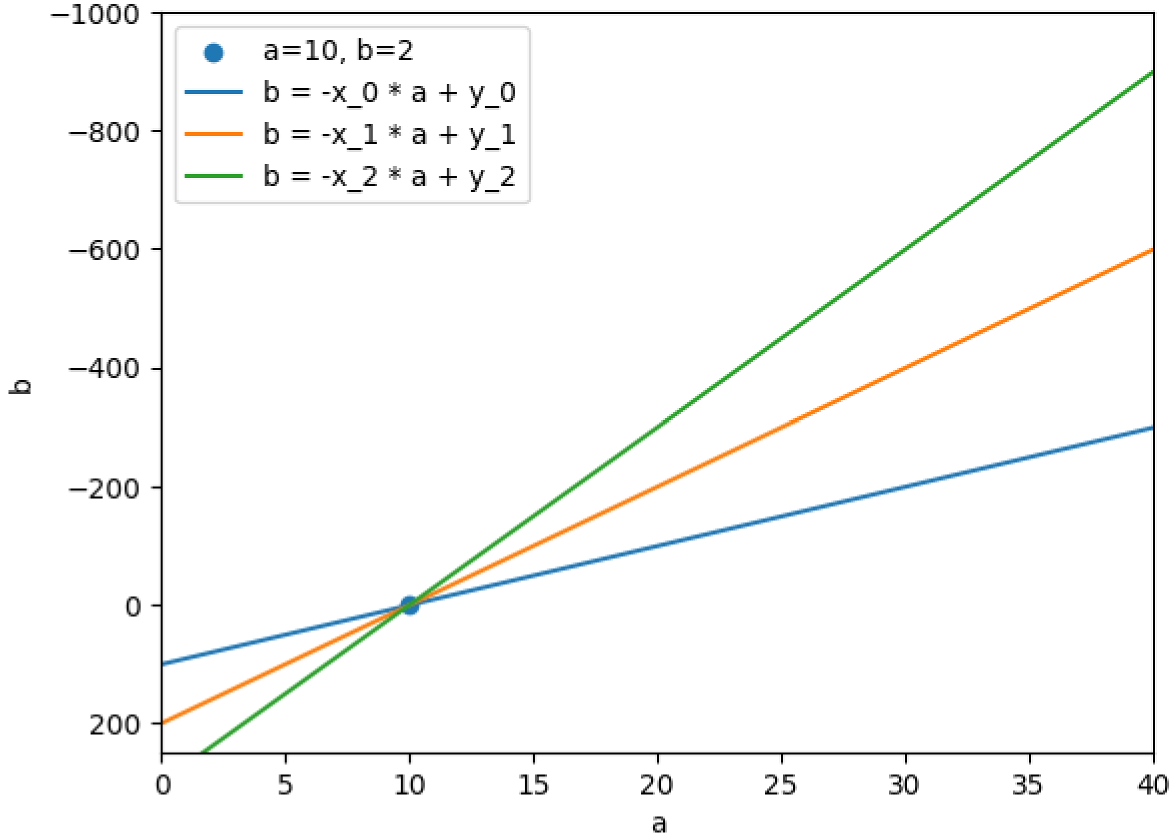
Parameter space:

$$b = -ax + y$$

We can find (a, b) in the parameter space for any points $(x_0, y_0), (x_1, y_1) \dots (x_n, y_n)$ That satisfy the line equation

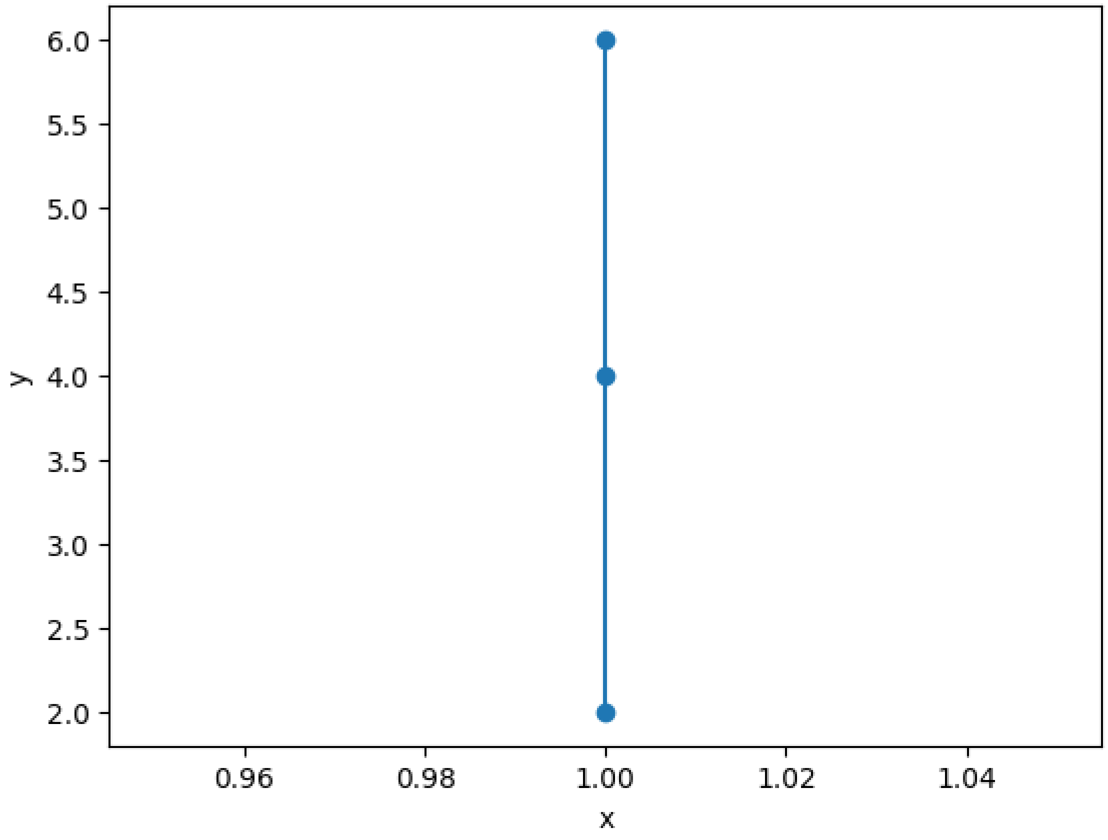
Hough Transform





Hough Transform

- Straight lines pose a problem
- The parameter a rises to infinity
- A better model is adopted, with Polar Coordinates



Hough Transform

- For any point P(x, y) in line there is a normal vector perpendicular that satisfies $\vec{n}\,.\vec{p} = 0$
- Expressing all possible vectors starting from the origin in polar coordinates

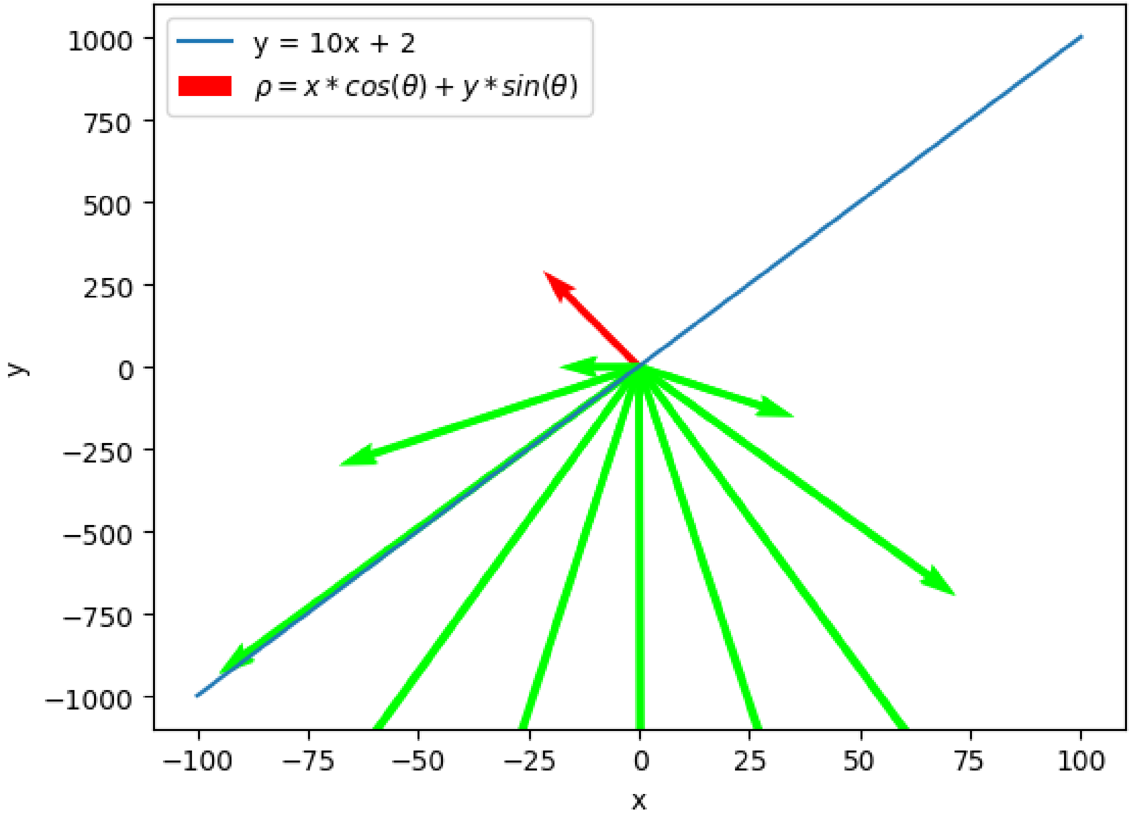
$$\vec{p}(rcos(\theta), rsin(\theta))$$

- For any point $p_0(x, y)$ we have

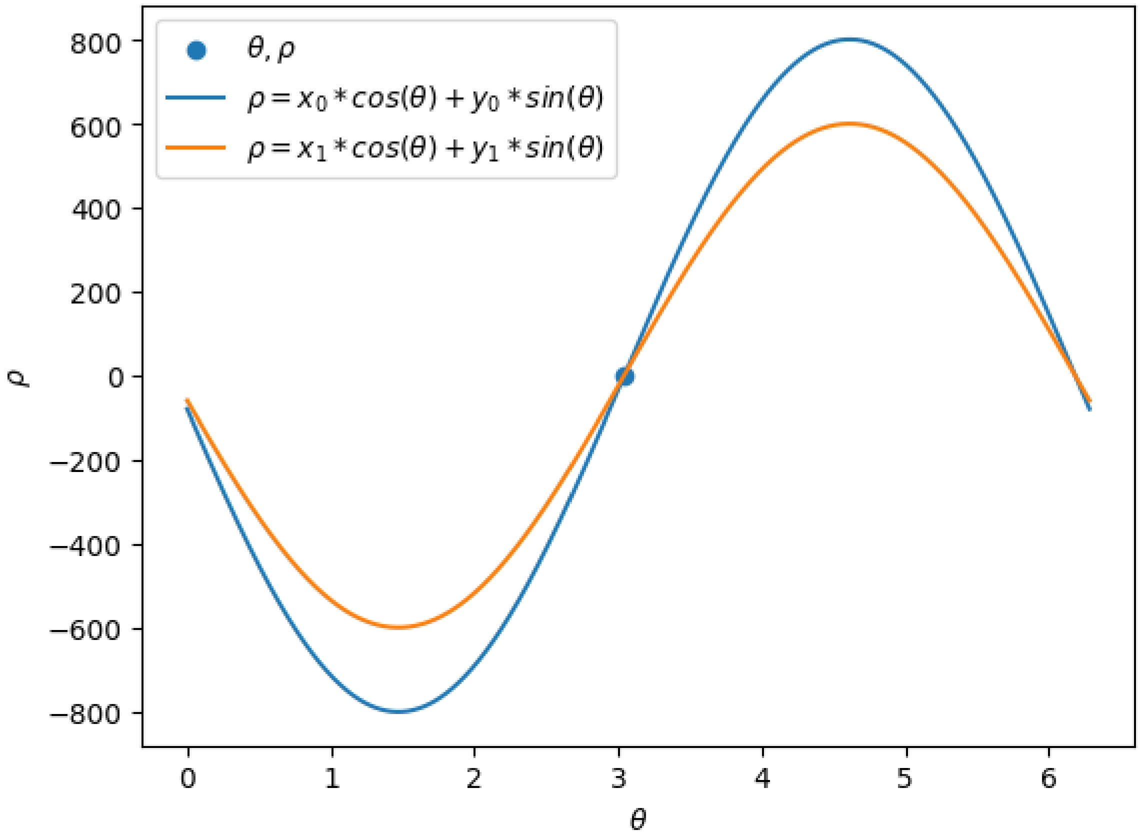
$$(\overrightarrow{p_0} - \vec{p}).\vec{p} = 0 \rightarrow \overrightarrow{p_0}.\vec{p} - \vec{p}.\vec{p} = 0$$

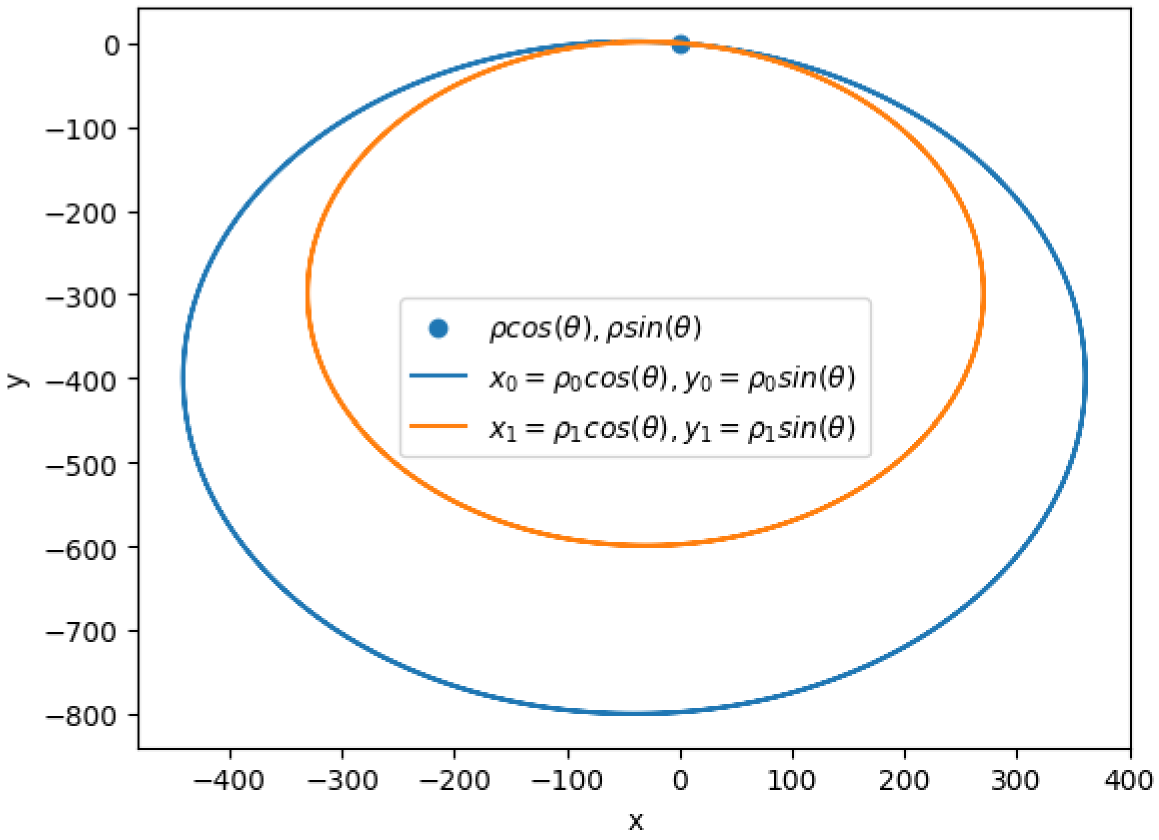
$$rxcos(\theta) + rysin(\theta) = r^2(\cos^2(\theta) + \sin^2(\theta))$$

$$x\cos(\theta) + y\sin(\theta) = r$$


y = 10x + 2
$\rho = x * cos(\theta) + y * sin(\theta)$
1000
750
500
250
0
−250
−500
−750
−1000
y
−100
−75
−50
−25
0
25
50
75
100
x

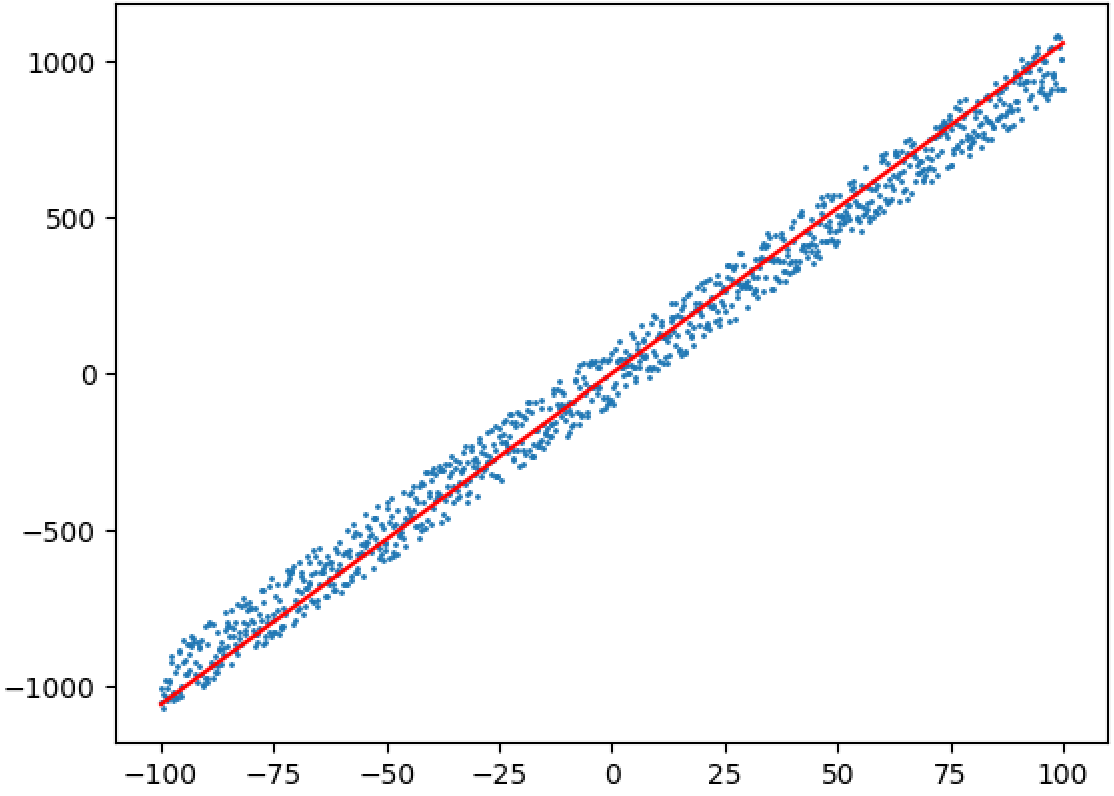
Hough Transform



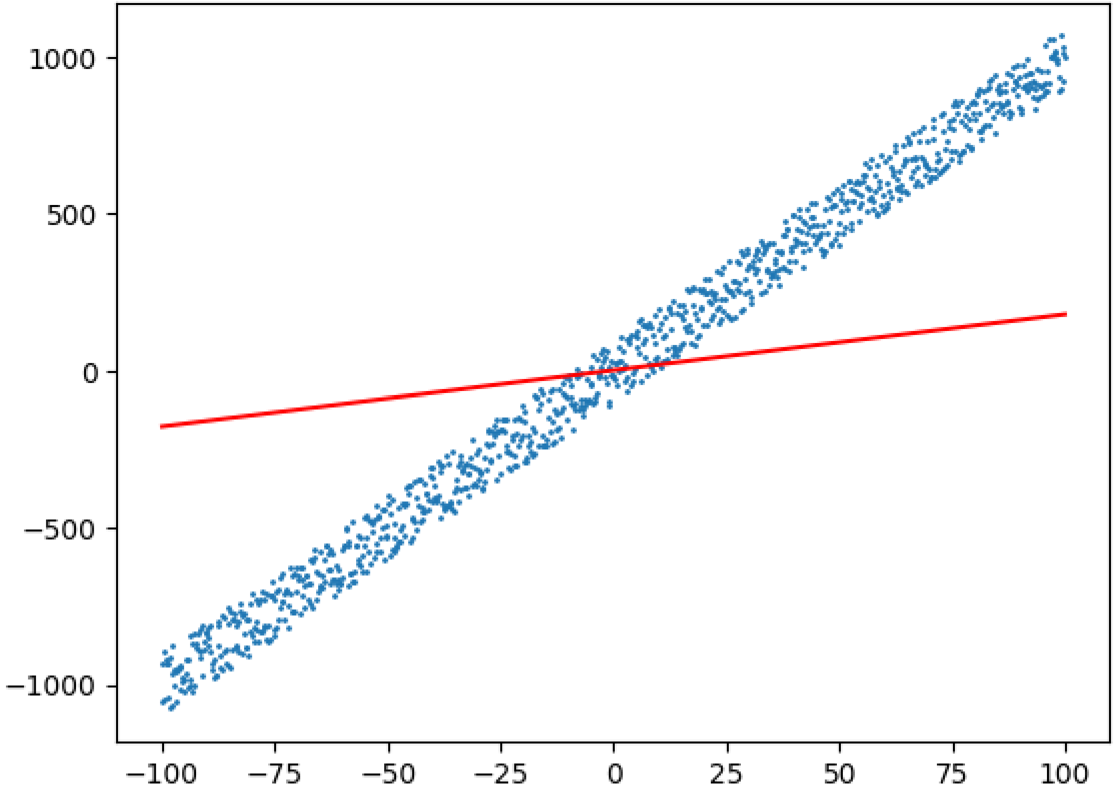


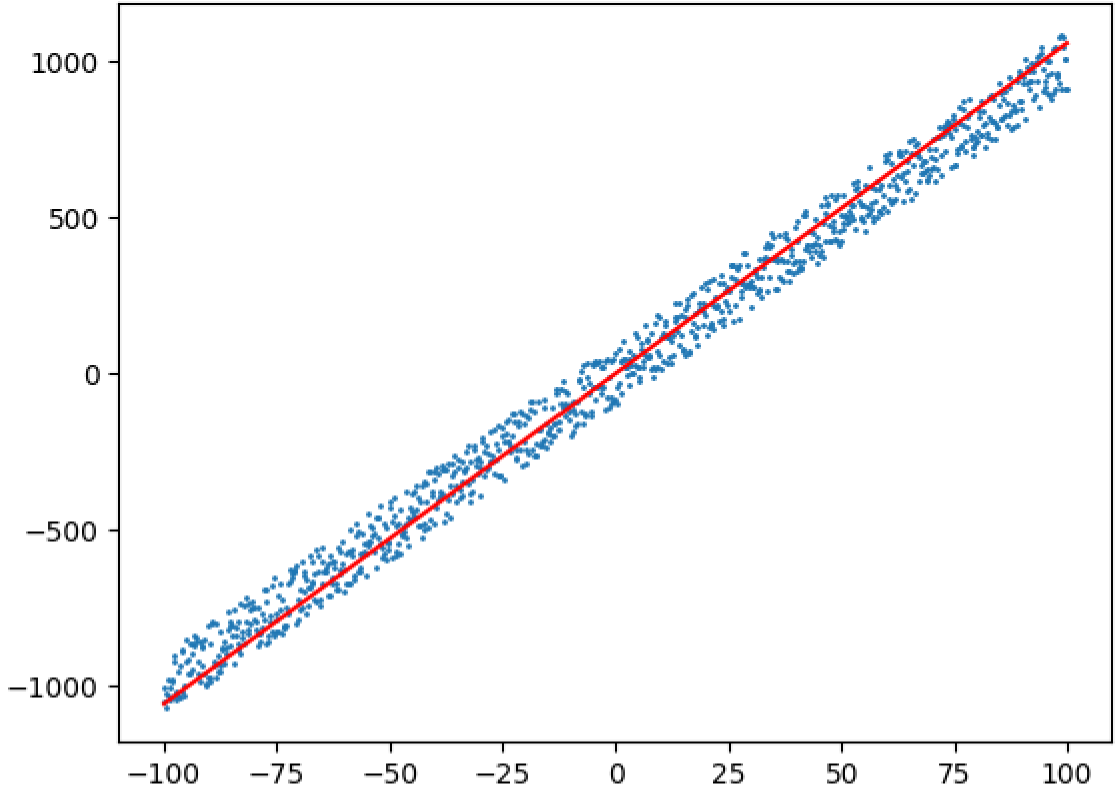
Line fitting with Hough transform

- Generate the θ range $(-\frac{\pi}{2}, \frac{\pi}{2}$, step:$\frac{\pi}{1000})$
- Calculate for each point P^m the set of (θ_i, ρ_i^m)
- Set a voting strategy to chose the line or lines that satisfy one or multiple criteria.



Line fitting with Hough transform





Plane fitting classical Hough transform

- $P\{x, y, z\} \rightarrow \rho = x\cos(\theta)\sin(\phi) + y\sin(\theta)\sin(\phi) + zcos(\theta)$
- Traverse all $P_i \in P$ compute ρ_i^m for all θ_i, ϕ_j
- Accumulate all $\theta_i, \phi_i, \rho_i^m$ in a container
- Increment the number of occurrences where θ_i, ϕ_i, ρ_i for all m points
- Filter the accumulator elements based on the number of occurrences