1. (a) set $C = \{ x \in R ; f(x) \le k \}$.

$\forall x_1, x_2 \in C$. $f(x_1) \le K$. $f(x_2) \le k$.

then if $f(x_1, x_2) \in C$. and $\forall \theta \in (0,1)$. $f(x,y)$ is convex function.

$f(\theta x_1 + (1-\theta) x_2) \le \theta f(x_1) + (1-\theta) f(x_2) \le \theta k + (1-\theta) k = k$.

$\therefore f(\theta x_1 + (1-\theta) x_2) \in C$.

then set $C = \{ x \in R: f(x) \le k \}$ is convex set.

(b). $f(x)$ $g(x)$ are convex functions. then:

$f(x_1) \ge f(x_0) + \nabla f(x_0)^T [x_1 - x_0]$ $\forall x_1, x_2 \in C$.

$g(x_1) \ge g(x_0) + \nabla g(x_0)^T [x_1 - x_0]$ $\forall x_1\ x_2 \in C$.

then $f(x_1) + g(x_1) \ge f(x_0) + g(x_0) + (\nabla f(x_0)^T + \nabla g(x_0)^T) \cdot [x_1 - x_0]$. --- ①

Let $h(x) = f(x) + g(x)$. to show $h(x)$ is convex. function

$h(x_1) = f(x_1) + g(x_1)$ $h(x_0) = g(x_0) + f(x_0)$

$\nabla h(x) = \nabla f(x) + \nabla g(x)$

$\therefore$ $h(x_1) \ge h(x_0) + \nabla h(x_0)^T (x_1 - x_0]$

$\therefore$ the combination of $f(x)$, $g(x)$ is also convex.

2. min $f(x,y) = 2x + 3y$. st: $x + y = B$ $\overset{C_1}{x \ge 0}$ $\overset{C_2}{y \ge 0}$.

$\nabla f = \binom{2}{3}$. Let lagrange factor be λ. and KKT factor μ_1 μ_2.

$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} \mu_1 + \mu_2 \\ \mu_1 + \mu_3 \end{bmatrix}$

$\begin{cases} \mu_1 (x + y - B) = 0. \\ \mu_2 x = 0. \\ \mu_3 y = 0 \end{cases}$

case (i): both C_1 and C_2 inactive.

then $\mu_2 = \mu_3 = 0$. $\begin{cases} \mu_1 + \lambda = 2 \\ \mu_1 + \lambda = 3 \end{cases}$ not exist.

case (ii): C_2 active and C_3 inactive. $\mu_2 \ne 0$ $\mu_3 = 0$.

then $\begin{cases} \mu_1 + \mu_2 + \lambda = 2 \\ \mu_1 + \lambda = 3 \end{cases} \Rightarrow \begin{cases} \mu_2 = 1 \\ x^* = 0 \\ y^* = B \end{cases}$ $f(x^*, y^*) = 3B$.

C_2 inactive and C_3 active $\mu_2 = 0$, $\mu_3 \ne 0$.

then $\begin{cases} \mu_1 + \lambda = 2 \\ \mu_1 + \lambda + \mu_3 = 3 \end{cases} \Rightarrow \begin{cases} \mu_3 = 1 \\ x^* = B \\ y^* = 0 \end{cases}$ $f(x^*, y^*) = 2B$.

$\therefore$ min $f(x,y) = 2B$.

case (ii) Both active. $\mu_2 \neq 0$, $\mu_3 \neq 0$.

$$\begin{cases} \mu_1+\mu_2+\lambda=2 \\ \mu_1+\mu_3+\lambda=3 \end{cases} \Rightarrow \begin{cases} x^*=0 \\ y^*=0. \end{cases}$$ not satisfied the constrain.

$\therefore$ As show above. $x^*=B$. $y^*=0$. make $f(x,y)$ min.

3. cones means for $\forall x \in C$. $\forall \alpha \geq 0$, $\Rightarrow \alpha x \in C$.

and we can judge from the graph. and formula

(a) (b) (d) are cones.

(c) (e) (f) are not cones.

(a) for $X \cdot (x_1, x_2)$ $\begin{cases} x_2 \geq 3|x_1| \\ -3x_2 \geq |x_1| \end{cases} \Rightarrow \alpha X\ (\alpha x_1, \alpha x_2)$ $\begin{cases} \alpha x_2 \geq 3\alpha|x_1| \\ -3\alpha x_2 \geq \alpha|x_1| \end{cases}$

αX is still in the original set C.

(b) for $X\ (x_1, x_2)$ $\begin{cases} 2x_1 \geq x_1 \\ 2x_1 \geq x_2 \end{cases} \Rightarrow \alpha X\ (\alpha x_1, \alpha x_2)$ $\begin{cases} 2\alpha x_1 \geq x_1 \\ 2\alpha x_1 \geq x_2. \end{cases}$

αX is still in the original set C.

(d) for $X\ (x_1, x_2)$ $x_1 \geq -2x_2 \Rightarrow \alpha X\ (\alpha x_1, \alpha x_2)$ $\alpha x_1 \geq -2\alpha x_2$.

αX is still in the original set C.

(c) for $X(x_1, x_2)$ $\begin{cases} 2x_2 \geq x_1+1 \\ 2x_2 \geq x_1+1 \end{cases} \Rightarrow \begin{cases} 2\alpha x_2 \geq \alpha x_1+\alpha. \\ 2\alpha x_2 \geq \alpha x_1+\alpha. \end{cases}$ not in set C. for all $\alpha \geq 0$.

(e) for $X(x_1, x_2)$ $x_1^2 \leq x_2 \Rightarrow \alpha^2 x_1^2 \leq \alpha x_2$ not in set C.

(f) for $X\ (x_1, x_2)$ $x_1^2 \geq x_2 \Rightarrow \alpha^2 x_1^2 \geq \alpha x_2$ not in set C.

4. to show $\min_{x \cdot y} f(x,y) = x \cdot y$ $2(x+y) \leq 2$ $x>0$ $y>0$

$2(x+y) \leq 2 \Rightarrow x+y \leq 1$.

$\nabla f = \begin{bmatrix} y \\ x \end{bmatrix}$ let KKT factor be μ_1 μ_2. μ_3

then $\begin{cases} \mu_1(x+y-1)=0. & \cdots ① \\ \mu_2 x=0 & \cdots ② \\ \mu_3 y=0 & \cdots ③ \end{cases}$ $\begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} \mu_1+\mu_2 \\ \mu_1+\mu_3 \end{bmatrix} \Rightarrow \begin{matrix} y=\mu_1+\mu_2 \\ x=\mu_1+\mu_3. \end{matrix}$

from ②, ③. for $x \neq 0$, $y \neq 0$. then $\mu_2=\mu_3=0$.

$\therefore x=\mu_1=y=\mu_1$ $\therefore x=y=\frac{1}{2}$

constrain $2(x+y) \leq 2$ is active and $x>0$, $y>0$ is inactive.

5. (a) NO

$f'(x) = \frac{2}{\ln x}$ $\quad f''(x) = -\frac{1}{x \ln^2 x}$

$x > 0$ and $\ln^2 x > 0$.

$\therefore f''(x) < 0$

$\therefore$ it is not convex function.

(b) YES

$f'(x) = a \quad f''(x) = 0. \geq 0.$

$\therefore$ it is a convex function.

(c). YES.

$$\nabla f(x) = \begin{bmatrix} 4x^3 - 4y \\ 4y^3 - 4x \end{bmatrix}.$$

$$H = \begin{bmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{bmatrix}$$

Obviously. H is positive definite matrix.

So, it's a convex function.