§ 8.4　多元复合函数的求导法则

一元复合函数　$y = f(u), u = \varphi(x)$

求导法则　$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

微分法则　$dy = f'(u)du = f'(u)\varphi'(x)dx$

一、复合函数的中间变量都是一元函数

定理　设$u=u(x),v=v(x)$在点x处可导，

函数$z=f(u,v)$在对应点(u,v)具有连续的偏导数，

则复合函数$z=f\left[u(x),v(x)\right]$在点$x$处的可导，

且有链式法则：

$$\boxed{\frac{dz}{dx}=\frac{\partial f}{\partial u}\frac{du}{dx}+\frac{\partial f}{\partial v}\frac{dv}{dx}.}$$

记住此公式

证 $$\lim_{\Delta x\to 0}\frac{f\left[u(x+\Delta x),v(x+\Delta x)\right]-f\left[u(x),v(x)\right]}{\Delta x}$$

如果上式极限存在，该极限值就是z在点(x,y)的导数$\frac{dz}{dx}$.

因$z=f(u,v)$在点(u,v)可微，则

$$f(u+\Delta u,v+\Delta v)-f(u,v)=\frac{\partial f}{\partial u}\Delta u+\frac{\partial f}{\partial v}\Delta v+o(\rho).$$

其中$\rho=\sqrt{(\Delta u)^2+(\Delta v)^2}$.

而 $$\Delta u=u(x+\Delta x)-u(x),$$

$$\Delta v=v(x+\Delta x)-v(x),$$

$$\frac{f\left[u(x+\Delta x),v(x+\Delta x)\right]-f\left[u(x),v(x)\right]}{\Delta x}$$

$$=\frac{\partial f}{\partial u}\cdot\frac{\Delta u}{\Delta x}+\frac{\partial f}{\partial u}\cdot\frac{\Delta v}{\Delta x}+\frac{o(\rho)}{\Delta x}$$

$$\begin{tikzcd}[arrows=dash]
& z \arrow[dl] \arrow[dr] & \\
u \arrow[d] & & v \arrow[d] \\
t & & t
\end{tikzcd}$$

其中： $\lim\limits_{\Delta x\to 0}\left|\frac{o(\rho)}{\Delta x}\right|=\lim\limits_{\Delta x\to 0}\frac{|o(\rho)|}{\rho}\cdot\frac{\rho}{|\Delta x|}=0$

从而得 $$\boxed{\frac{dz}{dx}=\frac{\partial f}{\partial u}\frac{du}{dx}+\frac{\partial f}{\partial v}\frac{dv}{dx}.}$$ （全导数公式）

说 明: 若定理中 $f(u,v)$ 在点 (u,v) **偏导数连续**减弱为

偏导数存在, 则定理结论不一定成立.

例如:

$$z = f(u,v) = \begin{cases} \dfrac{u^2 v}{u^2+v^2}, & u^2+v^2 \neq 0 \\ 0 & u^2+v^2 = 0 \end{cases}$$

$$u = t,\ v = t$$

易知:

$$\left.\frac{\partial z}{\partial u}\right|_{(0,0)} = f_u(0,0) = 0, \quad \left.\frac{\partial z}{\partial v}\right|_{(0,0)} = f_v(0,0) = 0$$

但复合函数

$$z = f(t,t) = \frac{t}{2}$$

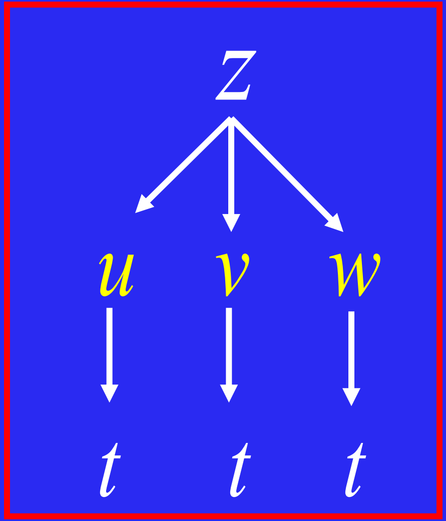
$$\frac{dz}{dt} = \frac{1}{2} \neq \frac{\partial z}{\partial u}\cdot\frac{du}{dt} + \frac{\partial z}{\partial v}\cdot\frac{dv}{dt} = 0\cdot 1 + 0\cdot 1 = 0$$

推广： 设下面所涉及的函数都可微 .

中间变量多于两个的情形.

例如, $z = f(u, v, w)$,

$$u = \phi(t),\ v = \psi(t),\ w = \omega(t)$$



$$\frac{dz}{dt} = \frac{\partial z}{\partial u}\cdot\frac{du}{dt} + \frac{\partial z}{\partial v}\cdot\frac{dv}{dt} + \frac{\partial z}{\partial w}\cdot\frac{dw}{dt}$$

$$= f_1'\phi' + f_2'\psi' + f_3'\omega'$$

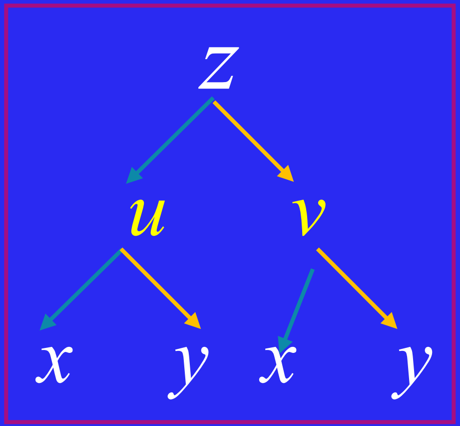
上述两种情况的导数称为全导数

二、复合函数中间变元为多元的情形

定理 设$u=u(x,y),v=v(x,y)$在点(x,y)处的偏导数均存在, 函数$z=f(u,v)$在对应点(u,v)可微, 则复合函数$z=f\left[u(x,y),v(x,y)\right]$在点$(x,y)$处的偏导数$\dfrac{\partial z}{\partial x},\dfrac{\partial z}{\partial y}$均存在, 且有链式法则:

$$\boxed{\begin{aligned}\frac{\partial z}{\partial x}&=\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial x},\\ \frac{\partial z}{\partial y}&=\frac{\partial f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial y}.\end{aligned}}$$

记住此公式



推广, 设$u=u(x,y,s),v=v(x,y,s)$在点(x,y,s)处的偏导数均存在, 函数$z=f(u,v)$在对应点(u,v)可微, 则复合函数$z=f[u(x,y,s),v(x,y,s)]$在点(x,y,s)处的偏导数$\frac{\partial z}{\partial x},\frac{\partial z}{\partial y},\frac{\partial z}{\partial s}$均存在, 且有链式法则:

$$\frac{\partial z}{\partial x}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial x},$$

$$\frac{\partial z}{\partial y}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial y},$$

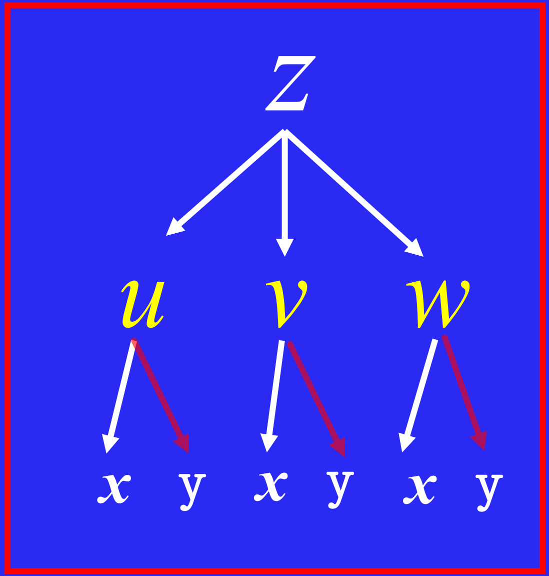
$$\frac{\partial z}{\partial s}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial s}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial s}.$$

中间变量多于两个时的情况

设$z=f(u,v,w),u=u(x,y),v=v(x,y),w=w(x,y)$, 则

$$\frac{\partial z}{\partial x}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial x}+\frac{\partial f}{\partial w}\frac{\partial w}{\partial x},$$

$$\frac{\partial z}{\partial y}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial y}+\frac{\partial f}{\partial w}\frac{\partial w}{\partial y}$$



此定理也适用于三元及三元以上的函数

三、复合函数的中间变元既有一元又有多元的情形

定理 设$u=u(x,y)$,$v=v(y)$在点(x,y)处的偏导数，均存在，函数$z=f(u,v)$在对应点(u,v)可微，则复合函数$z=f[u(x,y),v(y)]$在点(x,y)处的偏导数$\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}$均存在，且有链式法则：

$$\boxed{\frac{\partial z}{\partial x}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial x},}$$

$$\boxed{\frac{\partial z}{\partial y}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\frac{dv}{dy}.}$$

由此可推广

2.设$z=f\left(u(x),v(x,y)\right)$则

$$\frac{\partial z}{\partial x}=\frac{\partial f}{\partial u}\frac{du}{dx}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial x}.$$

$$\frac{\partial z}{\partial y}=\frac{\partial f}{\partial v}\frac{\partial v}{\partial y},$$

3.设 $z=f\left(u(x,y),x,y\right)$则

$$\frac{\partial z}{\partial x}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial x}.$$

$$\frac{\partial z}{\partial y}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial y},$$

注: $\frac{\partial z}{\partial x}$和$\frac{\partial f}{\partial x}$不同，

$\frac{\partial z}{\partial x}$是固定$y$,对$x$求导。

$\frac{\partial f}{\partial x}$是固定u,对$x$求导.

口诀:

分段用乘， 分叉用加,

单路全导, 叉路偏导

例1 设$z = e^{x-2y}$，$x = \sin t, y = t^3$，求$\dfrac{dz}{dt}$.

解 $$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$= e^{x-2y} \cos t - 2e^{x-2y} 3t^2$$

例2 设 $z = e^u \sin v$, $u = xy$, $v = x + y$, 求 $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$.

解 $$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$= e^u \sin v \cdot y + e^u \cos v \cdot 1 = e^u (y \sin v + \cos v),$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y} = e^u (x \sin v + \cos v),$$

例3 设 $z=(x+y)^{(x+y)}$，求 $\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}$.

解

令 $u=x+y,\ v=x+y,\ z=u^{v}$

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}=vu^{v-1}\cdot 1+u^{v}\ln u\cdot 1$$

$$=(x+y)^{(x+y)}(1+\ln(x+y))$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}=(x+y)^{(x+y)}(1+\ln(x+y))$$

例4 设$z=u^2+v^2+w, u=\sin x, v=e^{(x+y)}, w=\cos y$,求$\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$.

解

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{du}{dx}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$=2u\cos x+2ve^{(x+y)}$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}+\frac{\partial z}{\partial w}\cdot\frac{dw}{dy}=2ve^{(x+y)}+(-\sin y)$$

例 5 $u=f(x,y,z)=e^{x^2+y^2+z^2}, z=x^2\sin y$,求 $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}$

解:

$$\frac{\partial u}{\partial x}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial x}$$

$$=2x\mathrm{e}^{x^2+y^2+z^2}+2z\mathrm{e}^{x^2+y^2+z^2}\cdot 2x\sin y$$

$$=2x(1+2x^2\sin^2 y)\mathrm{e}^{x^2+y^2+x^4\sin^2 y}$$

例 5 $u = f(x,y,z) = e^{x^2+y^2+z^2}$, $z = x^2 \sin y$, 求 $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}$

$$\frac{\partial u}{\partial y} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial y}$$

$$= 2y\mathrm{e}^{x^2+y^2+z^2} + 2z\mathrm{e}^{x^2+y^2+z^2} \cdot x^2 \cos y$$

$$= 2(y + x^4 \sin y \cos y)\mathrm{e}^{x^2+y^2+x^4 \sin^2 y}$$

例6. 设$z=uv+\sin t, u=e^t, v=\cos t$,求全导数$\dfrac{dz}{dt}$.

解: $$\frac{\mathrm{d}z}{\mathrm{d}t}=\frac{\partial z}{\partial u}\cdot\frac{\mathrm{d}u}{\mathrm{d}t}+\frac{\partial z}{\partial v}\cdot\frac{\mathrm{d}v}{\mathrm{d}t}+\frac{\partial z}{\partial t}$$

$$=v\,\mathrm{e}^t-u\sin t+\cos t$$

$$=e^t(\cos t-\sin t)+\cos t$$

注意: 多元抽象复合函数求导在偏微分方程变形与验证解的问题中经常遇到.

下列两个例题有助于掌握,这方面问题的求导技巧与常用导数符号.

例 7　设 f 可微，$z = f\left(x, e^x\right)$，求 $\dfrac{dz}{dx}$.

解　引入中间变量 $y = e^x, \Rightarrow z = f\left(x, y\right)$. 则

$$\frac{dz}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

$$= \frac{\partial f}{\partial x} + e^x\frac{\partial f}{\partial y} = f_x + e^x f_y.$$

例 8　设 $z=f\left(x^2y,\dfrac{y}{x}\right)$ 在 R^2 内具有关于 x 和 y 的一阶连续偏导数，求 $\dfrac{\partial z}{\partial x},\dfrac{\partial z}{\partial y}$.

解　设 $u=x^2y,\ v=\dfrac{y}{x}, \Rightarrow z=f(u,v).$

由链式法则：

$$\frac{\partial z}{\partial x}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial x}=2xy\frac{\partial f}{\partial u}-\frac{y}{x^2}\frac{\partial f}{\partial v},$$

$$\frac{\partial z}{\partial y}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial y}=x^2\frac{\partial f}{\partial u}+\frac{1}{x}\frac{\partial f}{\partial v}.$$

例9　设f可微，$z=f\left(x-2y^2\right)$，求$\dfrac{\partial z}{\partial x},\dfrac{\partial z}{\partial y}$.

解　令$u=x-2y^2 \Rightarrow z=f(u)$，则

$$\frac{\partial z}{\partial x}=\frac{df}{du}\frac{\partial u}{\partial x}=f'(u)=f'\left(x-2y^2\right),$$

$$\frac{\partial z}{\partial y}=\frac{df}{du}\frac{\partial u}{\partial y}=-4yf'(u)=-4yf'\left(x-2y^2\right).$$

例10 设$z=f(x,u,v), v=g(x,y,u), u=h(x,y)$,

均有连续偏导数。求 $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$

解:

$$\frac{\partial z}{\partial x}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial u}\cdot\frac{\partial h}{\partial x}+\frac{\partial f}{\partial v}\cdot\frac{\partial g}{\partial x}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial u}\cdot\frac{\partial h}{\partial x}$$

$$\frac{\partial z}{\partial y}=\frac{\partial f}{\partial u}\cdot\frac{\partial h}{\partial y}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial u}\cdot\frac{\partial h}{\partial y}$$

四、复合函数的高阶偏导数

复合函数的高阶偏导数是学习时的一个难点，下面以例题形式进行分析和研究.

例10 设$z=f\left(x^2y,\dfrac{y}{x}\right)$, f具有二阶连续偏导数, 求$\dfrac{\partial z}{\partial x},\dfrac{\partial^2 z}{\partial x\partial y}$.

解 设$u=x^2y,\quad v=\dfrac{y}{x},\Rightarrow\quad z=f(u,v)$.

$$z=f(u,v),\quad u=x^2y,\quad v=\frac{y}{x},$$

由链式法则：

$$\frac{\partial z}{\partial x}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial x}=2xy\frac{\partial f}{\partial u}-\frac{y}{x^2}\frac{\partial f}{\partial v},$$

再一次运用链式法则，有

$$\frac{\partial z}{\partial x}=2xy\frac{\partial f}{\partial u}-\frac{y}{x^2}\frac{\partial f}{\partial v}=2xy\cdot f_u(u,v)-\frac{y}{x^2}\cdot f_v(u,v)$$

$$\frac{\partial^2 z}{\partial x \partial y} = 2x\frac{\partial f}{\partial u} + 2xy\frac{\partial f_u}{\partial y} - \frac{1}{x^2}\frac{\partial f}{\partial v} - \frac{y}{x^2}\frac{\partial f_v}{\partial y}$$

$$= 2x\frac{\partial f}{\partial u} + 2xy\left(\frac{\partial^2 f}{\partial u^2}\frac{\partial u}{\partial y} + \frac{\partial^2 f}{\partial u \partial v}\frac{\partial v}{\partial y}\right) - \frac{1}{x^2}\frac{\partial f}{\partial v}$$

$$-\frac{y}{x^2}\left(\frac{\partial^2 f}{\partial v \partial u}\frac{\partial u}{\partial y} + \frac{\partial^2 f}{\partial v^2}\frac{\partial v}{\partial y}\right)$$

$$= 2x^3 y\frac{\partial^2 f}{\partial u^2} + y\frac{\partial^2 f}{\partial u \partial v} - \frac{y}{x^3}\frac{\partial^2 f}{\partial v^2} + 2x\frac{\partial f}{\partial u} - \frac{1}{x^2}\frac{\partial f}{\partial v}.$$

为书写简便，引进记号:

$$f_1=\frac{\partial f}{\partial u}=f_u \Rightarrow$$ 表示函数f对第一个中间变量u求偏导数；

$$f_2=\frac{\partial f}{\partial v}=f_v \Rightarrow$$ 表示函数f对第二个中间变量v求偏导数；

$$f_{12}=\frac{\partial^2 f}{\partial u\partial v}=f_{uv} \Rightarrow$$ 表示函数f先对第一个中间变量u求偏导数, 再对第二个中间变量v求偏导数, 以此类推.

于是，此例的结果可以写为：

$$\frac{\partial z}{\partial x}=\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial x}=2xy\frac{\partial f}{\partial u}-\frac{y}{x^2}\frac{\partial f}{\partial v},$$

$$=2xy\,f_1-\frac{y}{x^2}f_2,$$

$$\frac{\partial^2 z}{\partial x\partial y}=2x^3y\frac{\partial^2 f}{\partial u^2}+y\frac{\partial^2 f}{\partial u\partial v}-\frac{y}{x^3}\frac{\partial^2 f}{\partial v^2}+2x\frac{\partial f}{\partial u}-\frac{1}{x^2}\frac{\partial f}{\partial v}.$$

$$=2x^3yf_{11}+yf_{12}-\frac{y}{x^3}f_{22}+2xf_1-\frac{1}{x^2}f_2.$$

练习

设$z=f(xy,\frac{x}{y})+g(\frac{y}{x})$，其中$f$具有二阶连续的偏导数，

g具有二阶连续的导数，求$\frac{\partial^2 z}{\partial x\partial y}$.

$$\frac{\partial z}{\partial x}=yf_1+\frac{1}{y}f_2-\frac{y}{x^2}g'$$

$$\frac{\partial^2 z}{\partial x\partial y}=f_1-\frac{1}{y^2}f_2+xyf_{11}-\frac{x}{y^3}f_{22}-\frac{1}{x^2}g'-\frac{y}{x^3}g''$$

四、一阶全微分形式的不变性

设$z=f(x,y)$可微, 如果x,y都是自变量, 则

$$\boxed{dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy.}$$

如果x,y不是自变量而是中间变量$x=x(u,v),y=y(u,v)$仍有

$$\boxed{dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy.}$$

由定义 $$dz=\frac{\partial z}{\partial u}du+\frac{\partial z}{\partial v}dv$$

其中 $$\frac{\partial z}{\partial u}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial u},\qquad \frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial v},$$

$$dz=\left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial u}\right)du\quad+\left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial v}\right)dv$$

$$=\frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial u}du+\frac{\partial x}{\partial v}dv\right)+\frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial u}du+\frac{\partial y}{\partial v}dv\right)$$

$$=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy.$$

由此可见：不论x, y是自变量还是中间变量，都有

$$\boxed{\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy}$$

这一性质称为**一阶全微分形式不变性.**

全微分运算公式:

$$d(u \pm v) = du \pm dv,$$

$$d(uv) = v\,du + u\,dv,$$

$$d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2} \quad (v \neq 0).$$

例11 已知$e^{-xy}-2z+e^{z}=0$，求$\dfrac{\partial z}{\partial x},\dfrac{\partial z}{\partial y}$和$dz$.

解 $$d(e^{-xy}-2z+e^{z})=0$$

$$e^{-xy}d(-xy)-2dz+e^{z}dz=0$$

$$(e^{z}-2)dz=e^{-xy}(xdy+ydx)$$

$$dz=\frac{ye^{-xy}}{e^{z}-2}dx+\frac{xe^{-xy}}{e^{z}-2}dy$$

$$\frac{\partial z}{\partial x}=\frac{ye^{-xy}}{e^{z}-2},\qquad \frac{\partial z}{\partial y}=\frac{xe^{-xy}}{e^{z}-2}$$

例 12 设 $z=\sqrt[3]{\dfrac{x+y}{x-y}}$，求 dz.

解

$$z=\sqrt[3]{\frac{x+y}{x-y}}\Rightarrow \ln z=\frac{1}{3}[\ln(x+y)-\ln(x-y)].$$

$$\frac{dz}{z}=\frac{1}{3}\left[\frac{dx+dy}{x+y}-\frac{xdy-ydx}{x-y}\right]\Rightarrow dz=\frac{2}{3}\sqrt[3]{\frac{x+y}{x-y}}\cdot\frac{xdy-ydx}{x^2-y^2}$$

$$\Rightarrow \frac{\partial z}{\partial x}=-\frac{2}{3}\cdot\sqrt[3]{\frac{x+y}{x-y}}\cdot\frac{y}{x^2-y^2}$$

$$\frac{\partial z}{\partial y}=\frac{2}{3}\cdot\sqrt[3]{\frac{x+y}{x-y}}\cdot\frac{x}{x^2-y^2}$$

例13 利用全微分形式的不变性求函数 $z=\left(x^2+y^2\right)^{xy}$ 全微分 dz 及其偏导数 $\dfrac{\partial z}{\partial x},\dfrac{\partial z}{\partial y}$.

解 设 $u=x^2+y^2, v=xy, \Rightarrow z=u^v$.

由一阶微分形式的不变性:

$$dz=\frac{\partial z}{\partial u}du+\frac{\partial z}{\partial v}dv=vu^{v-1}du+u^v\ln u\,dv$$

$$=vu^{v-1}d\left(x^2+y^2\right)+u^v\ln u\,d\left(xy\right)$$

$$= vu^{v-1}\left[d\left(x^2\right)+d\left(y^2\right)\right]+u^v \ln u\left(xdy+ydx\right)$$

$$= vu^{v-1}\left[2xdx+2ydy\right]+u^v \ln u\left(xdy+ydx\right)$$

$$=\left[xy\left(x^2+y^2\right)^{xy-1}\cdot 2x+\left(x^2+y^2\right)^{xy}\ln\left(x^2+y^2\right)y\right]dx$$

$$+\left[xy\left(x^2+y^2\right)^{xy-1}\cdot 2y+x\left(x^2+y^2\right)^{xy}\ln\left(x^2+y^2\right)\right]dy.$$

$$\therefore \frac{\partial z}{\partial x}=xy\left(x^2+y^2\right)^{xy-1}\cdot 2x+\left(x^2+y^2\right)^{xy}\ln\left(x^2+y^2\right)y$$

$$\frac{\partial z}{\partial y}=xy\left(x^2+y^2\right)^{xy-1}\cdot 2y+x\left(x^2+y^2\right)^{xy}\ln\left(x^2+y^2\right).$$