MA314

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Writing Project

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Different Models in Hyperbolic Plane and An Elementary Introduction to the Hyperbolic Geometry

**Part1: Introduction to the Hyperbolic Plane**

1) The origin of hyperbolic plane

The discovery of hyperbolic plane can be attributed to people’s consistent interest in proving the Parallel Postulate, given by the great ancient Greek mathematician Euclid, which states that “if a line segment intersects two straight lines forming two interior angles on the same side that sum to less than to right angles, then the two lines, if extend indefinitely, meet on that side on which the angles sum to less than two right angles.” Comparing to the other four axioms, this so-called “Parallel Postulate” seems more like a theorem but an axiom to people, so many mathematicians, throughout history, have been devoting their lives to prove this “postulate” by only using other axioms. However, no one actually succeeded. But a far more exciting result was concluded when people finally realized that this axiom defines certain “geometry”. Moreover, mathematicians like Lobachevski and Bolyai even showed that if we altered the statement of the “Parallel Postulate”, then we can achieve another completely independent but reasonable geometry. At that time, people first ever realized that Euclidean geometry may not be the only existing one, and other newly defined geometries like hyperbolic geometry and spherical geometry has begun their performances on stage. Since then, a new era of geometry and mathematics has started.

2) Why do people want to learn about the hyperbolic geometry?

After people began to accept the idea of hyperbolic geometry, a reasonable question then comes out: Why would we care about this? It turns out that hyperbolic geometry has various applications. It connects to different disciplines that people have never imagined, from general relativity to crystallization, from understanding biological features to enhancing computer science algorithm. However, to understand the hyperbolic geometry means a lot more to mathematics than any of its other applications not only because it touches upon fields like Complex Analysis, Abstract Algebra, Differential Geometry, Number Theory, and Topology, but also because it presents people a brand new angle of mathematics.

**Part2: How do we interpret the hyperbolic plane?**

As people want to start their investigations to the hyperbolic geometry, in particular, the two-dimensional hyperbolic plane here, they have to face a natural question that is how to think of the hyperbolic plane. What does it look like? How can people visualize it in a way that they are familiar with? In order to achieve these goals and help understand the hyperbolic plane, many people create their own models of hyperbolic plane, which are our main focuses in this part.

ℂ The first model we are going to introduce, the Upper-Half Plane Model, also called Poincare Half-Plane Model, was named after the French mathematician, Henri Poincare, but was actually first used by another Italian mathematician Eugenio Beltrami to show the equiconsistency of hyperbolic geometry and Euclidean geometry.

To understand this model, we have to first define its underlying space, which is the upper half-plane in the complex plane :

(Def 1).

[graph 3]

In this model, there are two seemingly different hyperbolic lines, both of which are defined in terms of Euclidean geometry in the complex plane. One type of the hyperbolic line is the intersection of H with a Euclidean line in complex plane perpendicular to the real axis R in C. The other type of the hyperbolic line in this model is the intersection of H with a Euclidean circle centered on the real axis R. However, it may seem a little weird that these “lines” look curved, but they are exactly the shortest paths joining any two points in hyperbolic plane, which is exactly the way we define the term “lines” in Euclidean geometry. The official terminology of this “shortest path” stuff is called “geodesics”, where in Euclidean geometry we just call them lines, so I just copy the terminology “line”, and hope using “lines” can give a better sense to people how the hyperbolic plane behaves differently from the Euclidean plane.

The following graph shows some examples of hyperbolic lines in H:

[graph 4]

Notice that, similar to the case in Euclidean geometry that two distinct points define a unique Euclidean line, each pair of distinct points defines a unique hyperbolic line in the hyperbolic plane. That is, either a Euclidean line passing through the two points when the two points have same real value but different imaginary value, or a Euclidean half-disk passing through the two points centered on the real axis of complex plane if they don’t have same real parts.

Another thing that is worth noticing is the definition of the parallel condition in the hyperbolic plane. As we have already constructed our definition of the hyperbolic lines, we have the definition of parallel as follows:

(Def 2)

1. **Poincare Disk Model**

The second model we are introducing is called Poincare Disk Model, named after the same French Mathematician, Henri Poincare. But interestingly, this name is also a misnomer: while the model was named after Poincare, it was first proposed by Eugenio Beltrami, the same Italian mathematics that also invented the Upper-Half Plane Model.

In this model, the underlying space of the hyperbolic plane is the open unit disc D in the complex plane C, where D is defined as:

[Def 3]

The hyperbolic lines in the Disk Model can also be sorted into two types as well. The first type of the hyperbolic plane line is the diameters of the disk, and the second type of the hyperbolic line is the intersection of D and the Euclidean circles that are orthogonal to the boundary of the disk. The graph below shows some hyperbolic lines in the Disk Model.

[graph 5]

As you may notice, there’re lots of similarities between this model and the one we introduced first. In later article, we will discuss in more detail the relationship between the Upper-Half Plane Model and the Poincare Disk Model. Now let’s look at some other models.

**3. The Projective Model**

The third model we are going to introduce is called The Projective Model, or Beltrami-Klein Model, in memory of Eugenio Beltrami and Felix Klein. This model, also first proposed by Beltrami, is similar to the Poincare Disk Model in the sense that the underlying space of it is also the open disk in the Euclidean space. However, the Projective Model serves as a better tool to visualize the hyperbolic lines because they are just correspondent to the open chords of the open disk, which are just straight lines.

[graph 6]

Therefore, when we look at the Projective Model, we don’t need to deal with arcs anymore, but only to deal with Euclidean straight lines, which are much easier. But the disadvantage of this model has also been stated by its name. The word “projective” by itself means there will be a distortion when projecting something onto something else. In this model, the distortion of projection happens to the angles and circles, where angles are distorted and some of the circles in hyperbolic plane will no longer be circular under this model. In fact, circles in the model become increasingly flattened ellipses as they approach to the boundary. This type of projection is sometimes called “non-conformal”.

[graph 7]

1. **The Hyperboloid Model**

The last model we are going to introduce is called the Hyperboloid Model, which is also sometimes referred as Minkowski Model or Lorentz Model, in memory of Hermann Minkowski and Hendrik Lorentz. This model is different from all models we have introduced above because it is a non-planar model of hyperbolic plane and it actually uses a three-dimensional Euclidean geometry to model a two-dimensional hyperbolic geometry. The construction of the Hyperboloid Model can be done by sending the Projective Model onto the upper-half of the hyperboloid given by [Def 3], but the detailed work is not shown here, as it is not our main focus.

[graph 8]

The graph above gives us an intuition what this model looks like, and it clearly shows the relationship between this model and the Poincare Disk Model, where the geodesic (simply the shortest path between two points) in the Hyperboloid Model is projected to the geodesic in the disk, i.e. the hyperbolic lines in the Disk Model.

To directly investigate hyperbolic space by using the hyperboloid model is somehow difficult because it requires enough knowledge of the hyperboloid and high-dimensional geometry. Thus, when we are dealing with low-dimensional hyperbolic space, we prefer to use the models above. But clearly, this model has a great advantage that none of the previous model has, that is it explicitly presents a visualization of a two-dimensional hyperbolic space without huge amount of projections. And you can also produce the other models from this model and use it to see the connection to general relativity.

**Part3: Further understanding of different Models**

**1. Why do people invent so many different models to model the hyperbolic plane?**

As we have seen above, mathematicians have invented many different kinds of models to represent the hyperbolic plane. But why would they do that? The reason might be that none of the models shown above, or ever created, works very well for all cases when studying the hyperbolic geometry. For example, the Projective Model gives us geodesics as straight lines, but at the same time, it distorts the angles, which leads to difficulties when we have to take angles or circles into account.

Therefore, in the following few pages, to understand more deeply the core of these models, we will do some comparisons between models we discussed above and see how are they different and why they have to be different.

**2. Geometries under different models**

As this idea of “geometry” was originated from people’s interests in measuring distances, it is not random that we choose to start our investigations of the geometries with talking about different distance measurements in different models.

However, before going into the detailed distance formulas of different hyperbolic models, we have to define our meaning of hyperbolic distance.

Informally, we can define that the distance between two points P and Q in the hyperbolic is the length of the shortest curve joining P and Q. The formal definition of the distance needs far more carefulness but we will not talk about it here.

Another thing that we should highlight about the hyperbolic plane before getting into the distance formulas for each of the model is the isometries of the hyperbolic plane. By saying isometry, we have the following definition: (Def 5)

The reason why we have to know about the isometries in hyperbolic plane is that we can take advantage of their properties of preserving distances between two metric spaces, and therefore we can compute the distance formula for a given model by first doing a special case and then deducing the general cases to the special case by applying the hyperbolic isometries. In general, isometries are very powerful tools for a metric space because they can explicitly demonstrate the similarities inside a metric space.

**Isometries of the hyperbolic plane:**

The isometry group in hyperbolic plane, in fact, has very nice expressions and by <Theorem 2.11> in Bonahon, we have that (Def 6).

These two groups of isometries have their own names, where the first one is called the linear fractional maps and the second one is called the antilinear fractional maps. Also, they are sometimes called the Mobius transformation, after August Ferdinand Mobius. The group formed by this transformation is called the Mobius Group, and it follows up with many interesting properties. It would be fun to read some related material if you are interested in this topic.

Then, it’s time to begin our discussion about the different distance formulas for different models.

Let’s see the upper half-plane first. For given two points P and Q in the upper half-plane, we have the distance formula d(P,Q) to be: (Def 7).

In Poincare Disk Model, the distance formula is given by: (Def 8).

And in the Projective Model, the distance formula is given by: (Def 9).

Notice that, the distance formulas in Poincare Disk Model and the Projective Model are exactly the same, and this is due to the fact that both of these two models are projections of a upper hemisphere, but the Poincare Disk Model preserves the angles and circles while the Projective Model is just an orthogonal projection and doesn’t preserve angles.

The last distance formula is given by (Def 10) in the hyperboloid model, where P and Q have become three-dimensional objects.

After having all of these distance formulas, we can do further investigations to other geometric properties. The one we choose to present here is the representation of hyperbolic triangles in different models.

The hyperbolic triangle is defined using on our definition of the hyperbolic lines. Therefore, we define a hyperbolic triangle to be a triangle in the hyperbolic plane that consists of three none-collinear points and the three hyperbolic segments between them. As the definition is given, we know that for sure the representations hyperbolic triangles will be varied due to the different choices of our hyperbolic models. Therefore, let’s see what hyperbolic triangles would look like in different models.

First, let’s look at the upper half-plane model. In this graph shown below, we can see many examples of hyperbolic triangles.

[graph 5]

Because in the upper half-plane model, hyperbolic lines are exactly either half-disks centered on real axis or vertical lines perpendicular to the real axis, the hyperbolic triangle is the area bounded by three intersecting points of the three distinct hyperbolic lines. However, one thing that needs to be known for hyperbolic triangle is that when the intersecting point is on the real axis or at infinity, we call that point an ideal vertex, at where the hyperbolic angle is zero. When a triangle consists of three ideal vertices, we call it an ideal triangle, whose angle sum is zero.

In Poincare Disk Model, the hyperbolic triangle looks similar as the graph below shows.

[graph 6]

To be noticed, when the three vertices are on the boundary of the Disk Model, we say that they form an ideal triangle.

In the Projective Model, the triangles are exactly the Euclidean triangles shaped by the straight lines in the model, and the ideal triangle is formed when its vertices are all on the boundary sphere.

[graph 7]

It is relatively hard to show the hyperbolic triangle in a hyperboloid model the hyperbolic triangle therefore becomes an object in three-dimensional space, so we are not going to include this model when discussing the hyperbolic triangle.

**Relations between some of the models.**

After we have talked about some geometric aspects of the hyperbolic plane, let’s go back to its models and talk about the relationship between some of them, to give us good senses why we need these models, and how did we get to them.

The first relationship we are going to look at is the relationship between the Upper-Half Plane Model and the Poincare Disk Model. As we might have noticed from the graph or formula presented above, these two models share some similarities and it seems like that the Disk Model is a bending version of the Upper-Half Plane model. In fact, we really can construct a strong relationship between these two models.

Define a linear fractional transformation to be: (Def 12).

We have this map induce an isometry from the Upper-Half Plane to the unit Disk Model. Therefore, we have that: (Def 13)

Knowing this isometry, we now have a bridge connecting these two models, and anything on one of each model can be easily transformed into another correspondent point on the other model.

The second relationship we are going to talk about is the relationship between the Poincare Disk Model and the Projective Model. As we may have noticed from above, these two models share a same underlying space, which is a unit disk in Euclidean plane. More than this, these two models are actually related through projections from the hemisphere model. The graph shown below gives us a good sense what the projections are.

[graph 8]

In this graph, the yellow lines represent the hyperbolic lines projected to the Projective Model from the upper hemisphere, and the red arcs represent the hyperbolic lines projected to the Poincare Disk Model from the same hemisphere. People have given these two kinds of projections different names. The Projective Model, due to the fact that the lines are orthogonally projected, is often called an orthogonal projection to the hemisphere, while the Poincare Disk Model, due to the fact that it preserves angles, is often called stereographic projection to the hemisphere. Actually, this kind of projection (stereographic projection) is often used when people are creating angle-preserve maps.

Then, we are going to show the relationship between the points in both models. (Def 14)

And the graph below shows a nice relationship of the two models, where P,Q are the points in the Projective Plane and P’,Q’ are points after mapping to the Poincare Disk Model.

[graph 7]

**Conclusion**

Although we have spent many pages talking about the idea of different hyperbolic models, there’re still lots of amazing stuffs that we haven’t been able to touch upon. If this “new world” seems interesting to you, there’re still a lot to be working on.

Since two thousands years ago, the ancient Greeks first ever created the Euclidean geometry, this subject has been continuously growing. Now, it’s the era for the hyperbolic geometry.

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**The origin of hyperbolic plane**

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**Part2: How do we interpret the hyperbolic plane?**

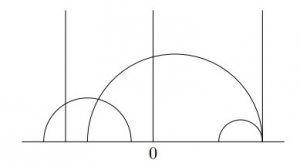
As people want to start their investigations to the hyperbolic geometry, in particular, the two-dimensional hyperbolic plane here, they have to face a natural question that is how to think of the hyperbolic plane. What does it look like? How can people visualize it in a way that they are familiar with? In order to achieve these goals and help understand the hyperbolic plane, many people create their own models of hyperbolic plane, which are our main focuses in this part.

**Upper-Half Plane Model**

The first model we are going to introduce, the Upper-Half Plane Model, also called Poincare Half-Plane Model, was named after the French mathematician, Henri Poincare, but was actually first used by another Italian mathematician Eugenio Beltrami to show the equiconsistency of hyperbolic geometry and Euclidean geometry.

To understand this model, we have to first define its underlying space, which is the upper half-plane H in the complex plane :

(Def 1).

[](http://web.colby.edu/thegeometricviewpoint/files/2016/12/upper-half-plane-1.png)

In this model, there are two seemingly different hyperbolic lines, both of which are defined in terms of Euclidean geometry in the complex plane. One type of the hyperbolic line is the intersection of H with a Euclidean line in complex plane perpendicular to the real axis R in C. The other type of the hyperbolic line in this model is the intersection of H with a Euclidean circle centered on the real axis R. However, it may seem a little weird that these “lines” look curved, but they are exactly the shortest paths joining any two points in hyperbolic plane, which is exactly the way we define the term “lines” in Euclidean geometry. The official terminology of this “shortest path” stuff is called “geodesics”, where in Euclidean geometry we just call them lines, so I just copy the terminology “line”, and hope using “lines” can give a better sense to people how the hyperbolic plane behaves differently from the Euclidean plane.

The above graph shows some examples of hyperbolic lines in H:

Notice that, similar to the case in Euclidean geometry that two distinct points define a unique Euclidean line, each pair of distinct points defines a unique hyperbolic line in the hyperbolic plane. That is, either a Euclidean line passing through the two points when the two points have same real value but different imaginary value, or a Euclidean half-disk passing through the two points centered on the real axis of complex plane if they don’t have same real parts.

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(Def 2)

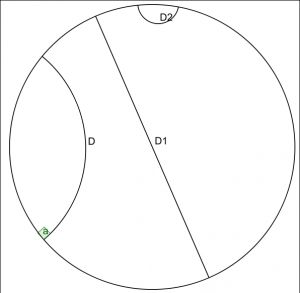
**Poincare Disk Model**

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In this model, the underlying space of the hyperbolic plane is the open unit disc D in the complex plane C, where D is defined as:

[Def 3]

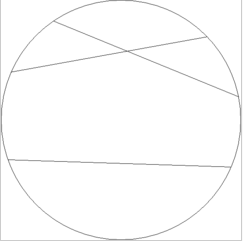
The hyperbolic lines in the Disk Model can also be sorted into two types. The first type of the hyperbolic plane line is the diameters of the disk (D1), and the second type of the hyperbolic line is the intersection of D and the Euclidean circles that are orthogonal to the boundary of the disk (D&D2), as the graph below shows.

[](http://web.colby.edu/thegeometricviewpoint/files/2016/12/disk-mode.png)

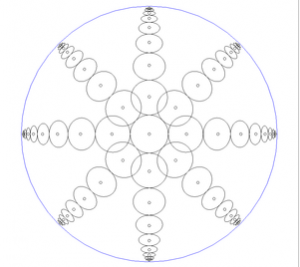
As you may notice, there’re lots of similarities between this model and the one we introduced first. In later article, we will discuss in more detail the relationship between the Upper-Half Plane Model and the Poincare Disk Model. Now let’s look at some other models.

**The Projective Model**

The third model we are going to introduce is called The Projective Model, or Beltrami-Klein Model, in memory of Eugenio Beltrami and Felix Klein. This model, also first proposed by Beltrami, is similar to the Poincare Disk Model in the sense that the underlying space of it is also the open disk in the Euclidean space. However, the Projective Model serves as a better tool to visualize the hyperbolic lines because they are just correspondent to the open chords of the open disk, which are just straight lines.

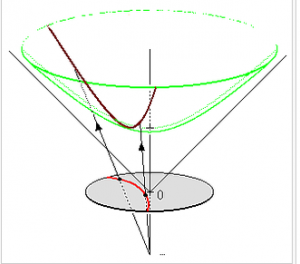
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Therefore, when we look at the Projective Model, we don’t need to deal with arcs anymore, but only to deal with Euclidean straight lines, which are much easier. But the disadvantage of this model has also been stated by its name. The word “projective” by itself means there will be a distortion when projecting something onto something else. In this model, the distortion of projection happens to the angles and circles, where angles are distorted and some of the circles in hyperbolic plane will no longer be circular under this model. In fact, circles in the model become increasingly flattened ellipses as they approach to the boundary. This type of projection is sometimes called “non-conformal”.

[](http://web.colby.edu/thegeometricviewpoint/files/2016/12/projective-model_2.png)

**The Hyperboloid Model**

The last model we are going to introduce is called the Hyperboloid Model, which is also sometimes referred as Minkowski Model or Lorentz Model, in memory of Hermann Minkowski and Hendrik Lorentz. This model is different from all models we have introduced above because it is a non-planar model of hyperbolic plane and it actually uses a three-dimensional Euclidean geometry to model a two-dimensional hyperbolic geometry. The construction of the Hyperboloid Model can be done by sending the Projective Model onto the upper-half of the hyperboloid given by [Def 3], but the detailed work is not shown here, as it is not our main focus.

[](http://web.colby.edu/thegeometricviewpoint/files/2016/12/hyperboloid-model.png)

The graph above gives us an intuition what this model looks like, and it clearly shows the relationship between this model and the Poincare Disk Model, where the geodesic (simply the shortest path between two points) in the Hyperboloid Model is projected to the geodesic in the disk, i.e. the hyperbolic lines in the Disk Model.

To directly investigate hyperbolic space by using the hyperboloid model is somehow difficult because it requires enough knowledge of the hyperboloid and high-dimensional geometry. Thus, when we are dealing with low-dimensional hyperbolic space, we prefer to use the models above. But clearly, this model has a great advantage that none of the previous model has, that is it explicitly presents a visualization of a two-dimensional hyperbolic space without huge amount of projections. And you can also produce the other models from this model and use it to see the connection to general relativity.

**Part3: Further understanding of different Models**

**Why do people invent so many different models to model the hyperbolic plane?**

As we have seen above, mathematicians have invented many different kinds of models to represent the hyperbolic plane. But why would they do that? The reason might be that none of the models shown above, or ever created, works very well for all cases when studying the hyperbolic geometry. For example, the Projective Model gives us geodesics as straight lines, but at the same time, it distorts the angles, which leads to difficulties when we have to take angles or circles into account.

Therefore, in the following few pages, to understand more deeply the core of these models, we will do some comparisons between models we discussed above and see how are they different and why they have to be different.

**Geometries under different models**

As this idea of “geometry” was originated from people’s interests in measuring distances, it is not random that we choose to start our investigations of the geometries with talking about different distance measurements in different models.

However, before going into the detailed distance formulas of different hyperbolic models, we have to define our meaning of hyperbolic distance.

Informally, we can define that the distance between two points P and Q in the hyperbolic is the length of the shortest curve joining P and Q. The formal definition of the distance needs far more carefulness but we will not talk about it here.

Another thing that we should highlight about the hyperbolic plane before getting into the distance formulas for each of the model is the isometries of the hyperbolic plane. By saying isometry, we have the following definition: (Def 5)

The reason why we have to know about the isometries in hyperbolic plane is that we can take advantage of their properties of preserving distances between two metric spaces, and therefore we can compute the distance formula for a given model by first doing a special case and then deducing the general cases to the special case by applying the hyperbolic isometries. In general, isometries are very powerful tools for a metric space because they can explicitly demonstrate the similarities inside a metric space.

The isometry group in hyperbolic plane, in fact, has very nice expressions and by <Theorem 2.11> in Bonahon, we have that (Def 6).

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Then, it’s time to begin our discussion about the different distance formulas for different models.

Let’s see the upper half-plane first. For given two points P and Q in the upper half-plane, we have the distance formula d(P,Q) to be: (Def 7).

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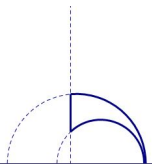
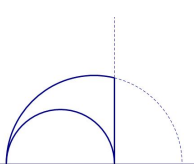
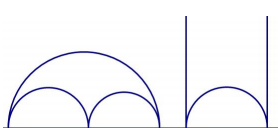
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After having all of these distance formulas, we can do further investigations to other geometric properties. The one we choose to present here is the representation of hyperbolic triangles in different models.

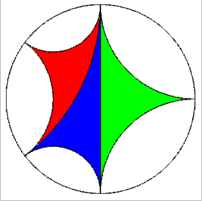
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[](http://web.colby.edu/thegeometricviewpoint/files/2016/12/hyperbolic-triangle_1.png)[](http://web.colby.edu/thegeometricviewpoint/files/2016/12/hyperbolic-triangle_2.png)[](http://web.colby.edu/thegeometricviewpoint/files/2016/12/hyperbolic-triangle_3.png)

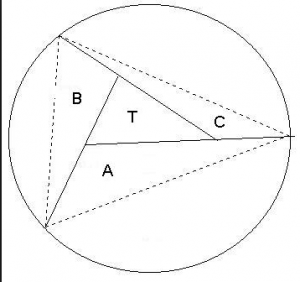
Because in the upper half-plane model, hyperbolic lines are exactly either half-disks centered on real axis or vertical lines perpendicular to the real axis, the hyperbolic triangle is the area bounded by three intersecting points of the three distinct hyperbolic lines. However, one thing that needs to be known for hyperbolic triangle is that when the intersecting point is on the real axis or at infinity, we call that point an ideal vertex, at where the hyperbolic angle is zero. When a triangle consists of three ideal vertices, we call it an ideal triangle, whose angle sum is zero.

In Poincare Disk Model, the hyperbolic triangle looks similar as the graph below shows.

[](http://web.colby.edu/thegeometricviewpoint/files/2016/12/disk-model-triangle.png)

To be noticed, when the three vertices are on the boundary of the Disk Model, as what it is in this graph above, we say that they form an ideal triangle.

In the Projective Model, the triangles are exactly the Euclidean triangles shaped by the straight lines in the model, and the ideal triangle is formed when its vertices are all on the boundary sphere. The graph below gives a hyperbolic triangle T with vertices A, B, and C in the Projective Model.

[](http://web.colby.edu/thegeometricviewpoint/files/2016/12/triangle-in-projective-mdoel.png)

It is relatively hard to show the hyperbolic triangle in a hyperboloid model the hyperbolic triangle therefore becomes an object in three-dimensional space, so we are not going to include this model when discussing the hyperbolic triangle.

**Relations between some of the models.**

After we have talked about some geometric aspects of the hyperbolic plane, let’s go back to its models and talk about the relationship between some of them, to give us good senses why we need these models, and how did we get to them.

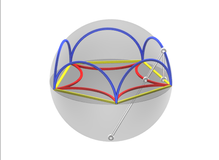
The first relationship we are going to look at is the relationship between the Upper-Half Plane Model and the Poincare Disk Model. As we might have noticed from the graph or formula presented above, these two models share some similarities and it seems like that the Disk Model is a bending version of the Upper-Half Plane model. In fact, we really can construct a strong relationship between these two models.

Define a linear fractional transformation to be: (Def 12).

We have this map induce an isometry from the Upper-Half Plane to the unit Disk Model. Therefore, we have that: (Def 13)

Knowing this isometry, we now have a bridge connecting these two models, and anything on one of each model can be easily transformed into another correspondent point on the other model.

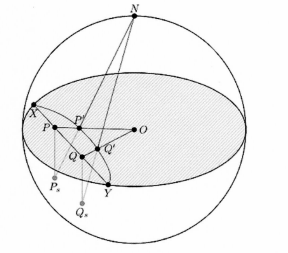
The second relationship we are going to talk about is the relationship between the Poincare Disk Model and the Projective Model. As we may have noticed from above, these two models share a same underlying space, which is a unit disk in Euclidean plane. More than this, these two models are actually related through projections from the hemisphere model. The graph shown below gives us a good sense what the projections are.

[](http://web.colby.edu/thegeometricviewpoint/files/2016/12/relationship.png)

In this graph, the yellow lines represent the hyperbolic lines projected to the Projective Model from the upper hemisphere, and the red arcs represent the hyperbolic lines projected to the Poincare Disk Model from the same hemisphere. People have given these two kinds of projections different names. The Projective Model, due to the fact that the lines are orthogonally projected, is often called an orthogonal projection to the hemisphere, while the Poincare Disk Model, due to the fact that it preserves angles, is often called stereographic projection to the hemisphere. Actually, this kind of projection (stereographic projection) is often used when people are creating angle-preserve maps.

Then, we are going to show the relationship between the points in both models. (Def 14)

And the graph below shows a nice relationship of the two models, where P,Q are the points in the Projective Plane and P’,Q’ are points after mapping to the Poincare Disk Model.

[](http://web.colby.edu/thegeometricviewpoint/files/2016/12/relationship_2.png)

**Conclusion**

Although we have spent many pages talking about the idea of different hyperbolic models, there’re still lots of amazing stuffs that we haven’t been able to touch upon. If this “new world” seems interesting to you, there’re still a lot to be working on.

Since two thousands years ago, the ancient Greeks first ever created the Euclidean geometry, this subject has been continuously growing. Now, it’s the era for the hyperbolic geometry.

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