ECE 712: Mini Assignment 2

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Problem

Consider any tall matrix $A \in \mathbb{R}^{m \times n}$, which may be rank deficient, and a corresponding vector $b \in \mathbb{R}^m$.

- Under what condition(s) does an exact solution x exist for the system of equations $Ax = b$?
- Develop a complete description for the solution x.

Solution

We are asked to analyze the linear system

$$Ax = b, \quad A \in \mathbb{R}^{m \times n},\ b \in \mathbb{R}^m,$$

where A is a tall matrix ($m \geq n$) which may be rank-deficient.

Existence of a solution

The system $Ax = b$ has a solution if and only if b lies in the column space of A, i.e.

$$b \in R(A).$$

If $b \notin R(A)$, then no exact solution exists.

Description of the solution

Assume $b \in R(A)$. Then there exists at least one solution x_p such that

$$Ax_p = b.$$

The set of all solutions can be described using the null space $N(A)$. If $z \in N(A)$, then

$$A(x_p + z) = Ax_p + Az = b + 0 = b.$$

Therefore, every solution can be written as

$$x = x_p + z, \quad z \in N(A).$$

Final Answer

The system has a solution if and only if $b \in R(A)$. If a solution exists, then the general solution is

$$x = x_p + N(A),$$

where x_p is any particular solution and $N(A)$ is the null space of A.
If A has full column rank ($\text{rank}(A) = n$), then $N(A) = \{0\}$ and the solution is unique.
If A is rank-deficient, then $N(A)$ has nonzero dimension and the solution set forms an affine subspace.