ECE 712: Mini Assignment 3

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Problem

Let the singular value decomposition (SVD) of a matrix $A \in \mathbb{R}^{m \times n}$ be

$$A = U\Sigma V^T,$$

where

$$U = \begin{bmatrix} U_1 & U_2 \end{bmatrix}, \quad V = \begin{bmatrix} V_1 & V_2 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \Sigma_r & 0 \\ 0 & \Sigma_2 \end{bmatrix},$$

and $\Sigma_r = \operatorname{diag}(\sigma_1, \ldots, \sigma_r)$ contains the r largest singular values. The truncated rank-r approximation of A is defined as

$$A_r = U_1 \Sigma_r V_1^T.$$

- Show that $A_r = P_c A$, where $P_c = U_1 U_1^T$ is a projection matrix, and identify the corresponding subspace.
- Show that $A_r^T = P_r A^T$, where $P_r = V_1 V_1^T$ is a projection matrix, and identify the corresponding subspace.

Solution

SVD and truncated approximation

From the SVD, we expand A as

$$A = U\Sigma V^T = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_r & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}.$$

The truncated rank-r approximation is

$$A_r = U_1 \Sigma_r V_1^T.$$

1) Proof that $A_r = P_c A$

Define the projection matrix

$$P_c = U_1 U_1^T.$$

Then

$$\begin{aligned} P_c A &= U_1 U_1^T\, U \Sigma V^T \\ &= U_1 \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \Sigma_r & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} \\ &= U_1 \Sigma_r V_1^T \\ &= A_r. \end{aligned}$$

Thus, A_r is obtained by projecting A onto the subspace

$$\mathcal{S}_c = \operatorname{span}(U_1),$$

which is called **the r-dimensional subspace defined by U_1, i.e. a subspace of R(A)**. If $r = \operatorname{rank}(A)$, then $\mathcal{S}_c = \mathcal{R}(A)$, the column space of A.

2) Proof that $A_r^T = P_r A^T$

Define the projection matrix

$$P_r = V_1 V_1^T.$$

Using transpose,

$$A_r^T = (U_1 \Sigma_r V_1^T)^T = V_1 \Sigma_r U_1^T.$$

Now compute:

$$\begin{aligned} P_r A^T &= V_1 V_1^T (V \Sigma^T U^T) \\ &= V_1 \begin{bmatrix} I & 0 \end{bmatrix} \Sigma_r \begin{bmatrix} U_1^T \\ U_2^T \end{bmatrix} \\ &= V_1 \Sigma_r U_1^T \\ &= A_r^T. \end{aligned}$$

Thus, A_r^T is obtained by projecting A^T onto the subspace

$$\mathcal{S}_r = \operatorname{span}(V_1),$$

which is called **the r-dimensional subspace defined by U_1, i.e. a subspace of R(A)**. If $r = \operatorname{rank}(A)$, then $\mathcal{S}_r = \mathcal{R}(A^T) = \mathcal{N}(A)^\perp$.