Assignment 1

ECE 712: Matrix Computations for Signal Processing

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1. Question 1: PCA Compression and Reconstruction Error

1.1 Method

In this question, we are given a 1000×5 data matrix X (from `X.mat`). First, each column of X is mean-centered to get $X_c = X - \bar{X}$. Then we compute the covariance matrix:

$$C = \frac{1}{N-1} X_c^\top X_c,$$

where $N = 1000$.

Next, we perform eigen-decomposition of C:

$$C = V \Lambda V^\top,$$

where Λ contains the eigenvalues and V the eigenvectors. For a given number of components r, we take the first r columns of V as V_r and project the data:

$$Y = X_c V_r.$$

We can then reconstruct the data (in the centered form) by

$$\hat{X}_r = Y V_r^\top = X_c V_r V_r^\top.$$

The reconstruction error is measured using the Frobenius norm:

$$E(r) = \|X_c - \hat{X}_r\|_F^2.$$

We also compute the cumulative variance explained by the first r components:

$$\text{Cumulative Variance}(r) = \frac{\sum_{i=1}^{r} \lambda_i}{\sum_{i=1}^{5} \lambda_i}.$$

1.2 Results

The PCA results from the dataset are summarized below.

- Eigenvalues (sorted): $\{5.0849,\ 0.0914,\ 0.0272,\ 0.0152,\ 0.0103\}$.
- Cumulative variance explained (%): $\{97.25,\ 98.99,\ 99.51,\ 99.80,\ 100.0\}$.
- Reconstruction error $E(r)$: $\{143.91,\ 52.59,\ 25.45,\ 10.30,\ \approx 0\}$.

Table 1: Summary of PCA results for $r = 1$ to 5.

r	Eigenvalue λ_r	Cumulative Var. (%)	$E(r)$
1	5.0849	97.25	143.91
2	0.0914	98.99	52.59
3	0.0272	99.51	25.45
4	0.0152	99.80	10.30
5	0.0103	100.0	≈ 0

The figures below show the reconstruction error, cumulative variance, and eigenvalue distribution.

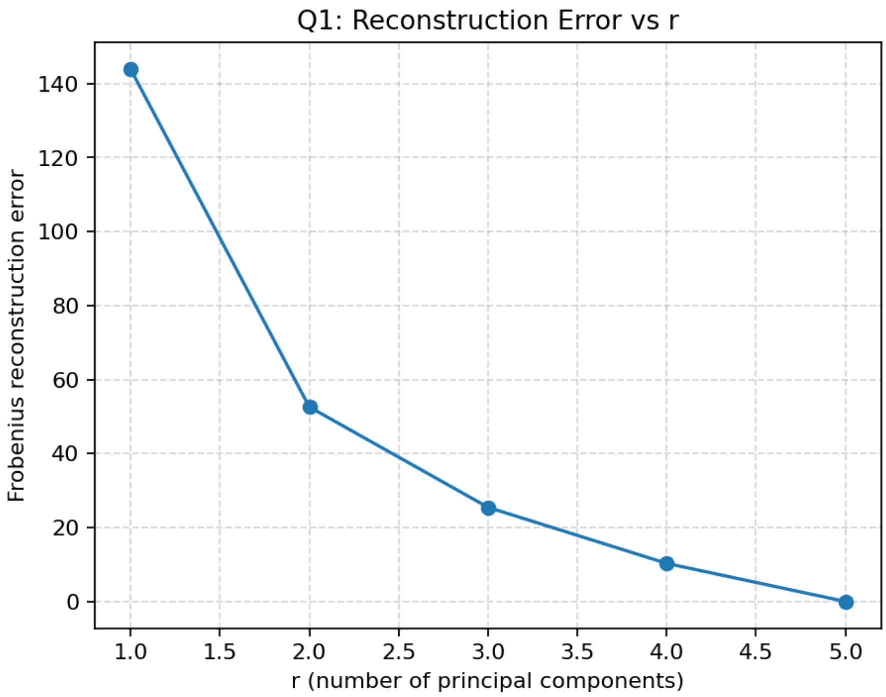


Figure 1: Reconstruction error vs. number of components r.

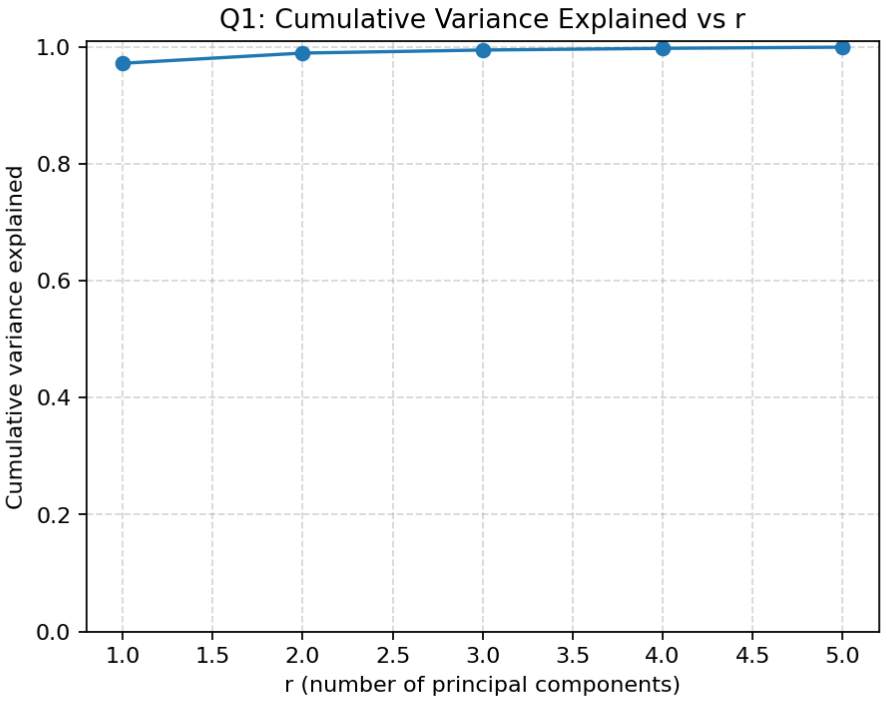


Figure 2: Cumulative variance explained by top r components.

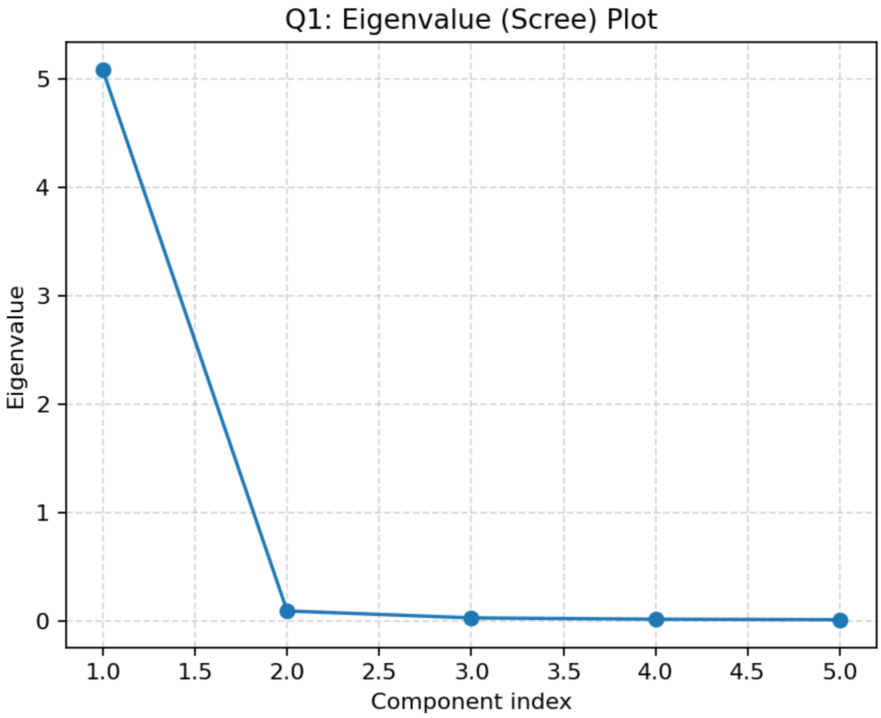


Figure 3: Eigenvalue (scree) plot.

1.3 Discussion

From the results, we can see that most of the variance (around 97%) is already captured by the first component. The remaining components only add a small amount of variance.

As r increases, the reconstruction error keeps getting smaller, which makes sense since we are keeping more information.

1.4 Conclusion

In this dataset, using just the first principal component already keeps most of the data variation. If we use more components, the error decreases but the improvement becomes smaller. A reasonable choice is $r = 1$ or $r = 2$, depending on how much accuracy we need.

2. Question 2: Proof of Minimum Reconstruction Error of PCA

2.1 Objective

Let $X \in \mathbb{R}^{m\times d}$ be a mean-centered data matrix. For any r-dimensional orthonormal basis $U_r \in \mathbb{R}^{d\times r}$ with $U_r^\top U_r = I_r$, the orthogonal projection of the data onto $\mathrm{span}(U_r)$ and its reconstruction are

$$\hat{X}_r = XU_rU_r^\top.$$

The reconstruction error is defined as

$$E(U_r) = \|X - \hat{X}_r\|_F^2 = \|X - XU_rU_r^\top\|_F^2.$$

Our goal is to show that the PCA basis U_r gives the smallest possible reconstruction error among all orthonormal bases.

2.2 Proof (Projection ⇒ Trace Maximization ⇒ PCA)

Because $P_{U_r} = U_rU_r^\top$ is an orthogonal projector, we have

$$\|X\|_F^2 = \|XP_{U_r}\|_F^2 + \|X(I - P_{U_r})\|_F^2 \quad \Rightarrow \quad E(U_r) = \|X\|_F^2 - \|XU_rU_r^\top\|_F^2.$$

Thus, minimizing $E(U_r)$ is the same as maximizing $\|XU_r\|_F^2$. Since

$$\|XU_r\|_F^2 = \mathrm{tr}(U_r^\top X^\top XU_r),$$

we want to find U_r that makes this trace as large as possible.

Let $X^\top X = V\Lambda V^\top$ be the eigen-decomposition with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d$. From the properties of symmetric matrices, the maximum trace is obtained when U_r is made of the first r eigenvectors of $X^\top X$:

$$U_r = [v_1, \ldots, v_r].$$

In that case,

$$\mathrm{tr}(U_r^\top X^\top XU_r) = \sum_{i=1}^{r} \lambda_i.$$

Therefore, projecting onto these eigenvectors (the PCA subspace) gives the smallest possible reconstruction error.

2.3 Conclusion

PCA provides the orthogonal basis that minimizes the reconstruction error, and the minimum possible error is

$$E_{\min}(r) = \sum_{i=r+1}^{d} \sigma_i^2,$$

where σ_i are the singular values of X. This result matches what we observed in Question 1, where the reconstruction error decreases as r increases.

3. Question 3: Minimum-Variance FIR Filter Design

3.1 Objective

In this question, we are given a mean-centered coloured random sequence $x[n]$ (from `xb.mat`). The goal is to design a finite impulse response (FIR) filter $h[n]$ with length N and unit norm $\|h\|_2 = 1$ so that the output $y[n] = (h * x)[n]$ has the smallest possible variance.

3.2 Method

First, we calculate the sample autocorrelation $r_x[\ell]$ of $x[n]$. Then we form a Toeplitz covariance matrix

$$R_x = \text{Toeplitz}(r_x[0], r_x[1], \ldots, r_x[N-1]).$$

The output variance can be written as

$$\sigma_y^2 = h^\top R_x h, \quad \text{subject to } \|h\|_2 = 1.$$

The minimum variance happens when h is the eigenvector of R_x that corresponds to its smallest eigenvalue. In other words, we just find the smallest eigenvalue and take the corresponding eigenvector as the filter.

3.3 Implementation

The Python script `q3_minvar_fir_final.py` loads the signal, computes $r_x[\ell]$, builds R_x, and finds the minimum-variance filter for $N = 8, 12, 16, 24$. It also saves the plots:

- `q3_autocorr.png`: autocorrelation of $x[n]$
- `q3_var_vs_N.png`: output variance vs. N
- `q3_impulse_N{N}.png`: impulse response of $h[n]$
- `q3_spectrum_N{N}.png`: magnitude spectrum $|H(e^{j\omega})|$

3.4 Results

The input signal clearly has correlation (it is not white noise), which can be seen in Fig. 4. As the filter length N increases, the output variance becomes smaller (see Fig. 5). The quantitative results are listed in Table 2.

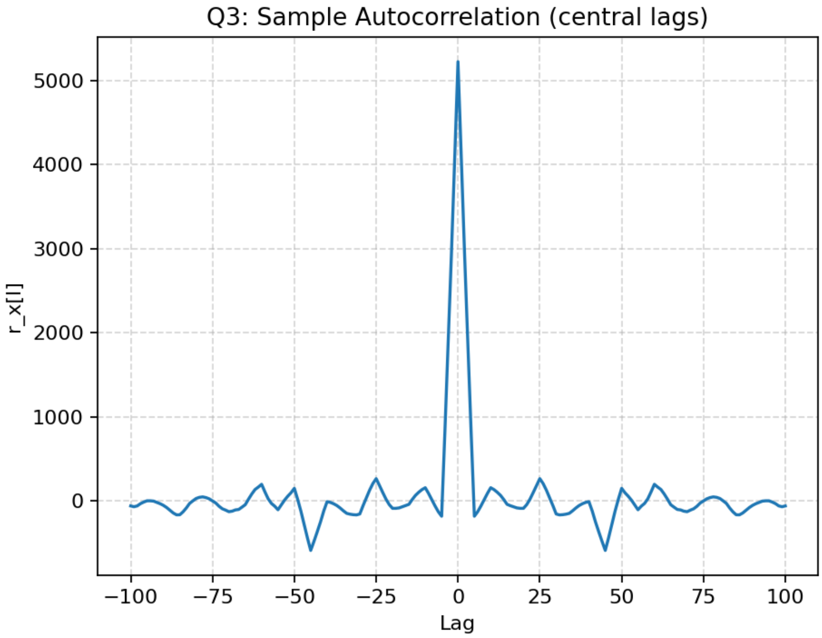


Figure 4: Autocorrelation of $x[n]$.

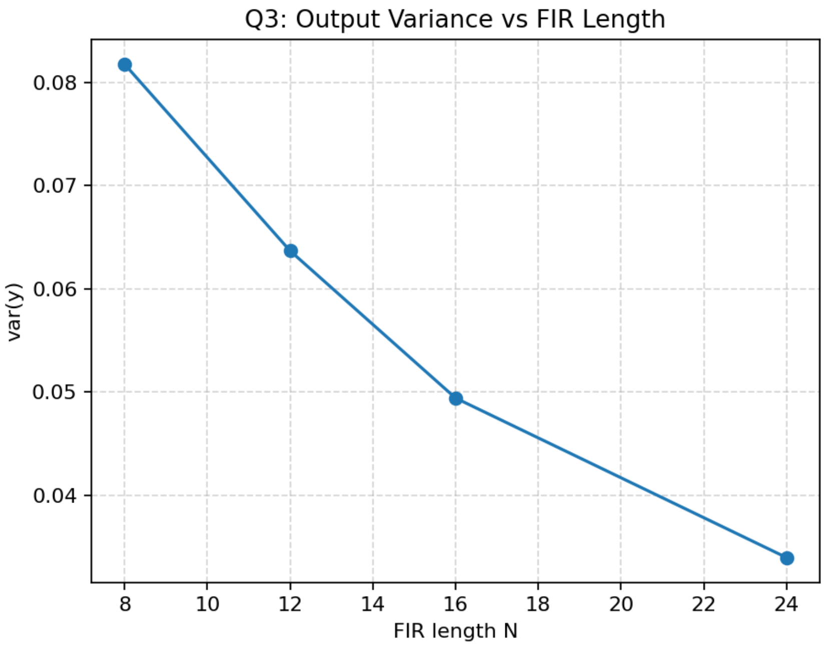
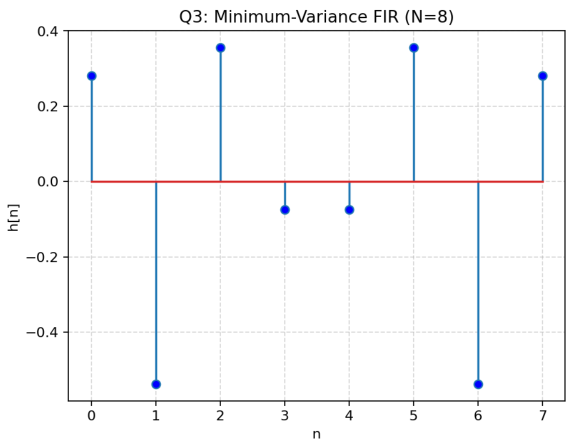


Figure 5: Output variance σ_y^2 vs. filter length N.

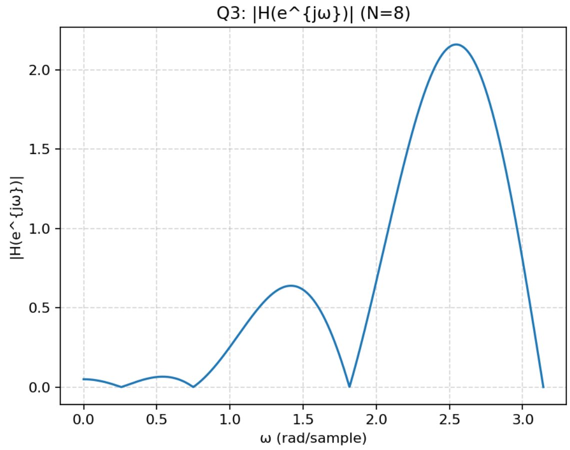
Table 2: Result summary for different FIR lengths.

N	σ_y^2 (variance)	$\lambda_{\min}(R_x)$
8	0.0817	409.53
12	0.0637	318.93
16	0.0494	247.38
24	0.0340	170.04

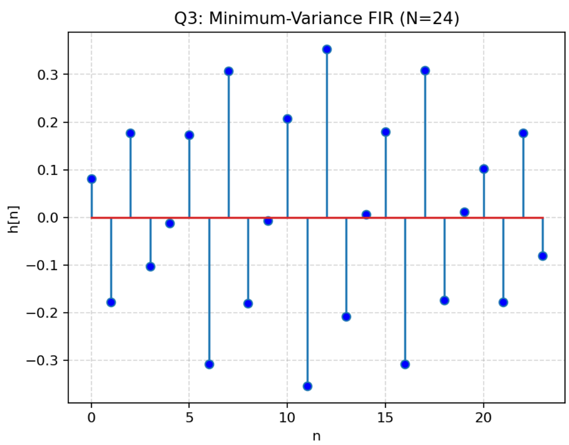
Some example impulse and magnitude responses are shown in Fig. 6. When N is larger, the filter becomes smoother and can reduce more of the correlated part of the signal.



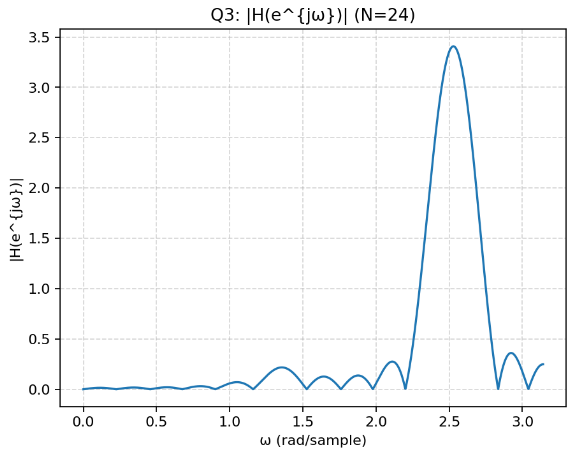
(a) $h[n]$ for $N = 8$



(b) $|H(e^{j\omega})|$ for $N = 8$



(c) $h[n]$ for $N = 24$



(d) $|H(e^{j\omega})|$ for $N = 24$

Figure 6: Impulse and magnitude responses for two filter lengths.

3.5 Discussion and Conclusion

From the table and plots, we can see that as N increases, the filter can make the output smoother and the variance smaller. This matches the idea that a longer filter has more freedom to cancel out the correlation in $x[n]$. All the results follow the theory that the minimum-variance filter corresponds to the smallest eigenvector of R_x. In summary, the experiment confirms that the eigen-decomposition of the covariance matrix can be used to design such filters, and the implementation works as expected.

References

- J. Reilly, *Fundamentals of Linear Algebra for Signal Processing*, Lecture Notes, ECE 712, McMaster University.