A study of the spherical coordinates parameterization

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Abstract

This report provides the results of using the spherical coordinates to resolve the constrained problems.

Part I
Introduction

For the resource allocation problem, suppose resource p_i is allocated to i where i is the index of the object which get the resource, p_i is the allocation ratio to the total available resource. Then we can get:

$$\sum_{i=1}^{n} p_i = 1 \tag{1}$$

We can rewrite the equation 1 as bellow:

$$\sum_{i=1}^{n} (r_i)^2 = 1 \tag{2}$$

Any feasible allocation vector $r = (r_1, ..., r_n)$ on the unit ball can be described through $n-1$ angels denoted by $\theta_i, 1 \leq i \leq n-1$, in the following way. Indeed, the spherical coordinated parameterization of r via $\theta^T = (\theta_1, ..., \theta_{n-1})$ is given by:

$$r_i(\theta) = \begin{cases} cos(\theta_1), & i = 1; \\ cos(\theta_i)\prod_{k=1}^{i-1} sin(\theta_k), & 2 \leq i \leq n-1; \\ sin(\theta_{n-1})\prod_{k=1}^{n-2} sin(\theta_k), & i = n. \end{cases} \tag{3}$$

Penalty Methods

In order to adress the constrained optimization problem, several methods are provided. In this work, the penalty method is studied to compare the performance of the spherical coordinated parameterization method.
Given the constrained problem as described in equation 4

$$\begin{aligned} &\min_{x\in\mathbb{R}^n} J(x), \\ &\quad \text{s.t.} \quad 0 \leq x_i \leq 1, i = 1, ..., n \\ &\qquad\qquad \sum_{i=1}^{n} x_i = 1 \end{aligned} \tag{4}$$

THe penalty methods modify the original performance function to penalize the extent to which the constraints are not satisfied. Recall that $|.|$ denotes the Euclidean norm, then the penalized function is defined:

$$\begin{aligned} J_\alpha(x) = J(x) + \frac{\alpha}{2}(|g(x)_+|^2 + |g(x)_\text{-}|^2), \\ where \quad g(x)_+ = (g_1(x)_+, ...g_j(x)_+)^T, and \quad g_i(x)_+ = max(1, g_i(\theta)) - 1 \\ where \quad g(x)_\text{-} = (g_1(x)_\text{-}, ...g_j(x)_\text{-})^T, and \quad g_i(x)_\text{-} = min(0, g_i(\theta)) \end{aligned} \tag{5}$$

Then two-time scale algorithm is used to get the optimal value of x by using the formulars shown in equation 6, equation 7 and equation 8.

$$x_{n+1} = x_n - \epsilon_n \nabla J\alpha_n(x_n)^T \tag{6}$$

$$\alpha_{n+1} = \alpha_n + \delta_n \mathbb{1}_{\{n\in T_i\}} \tag{7}$$

where $\sum \delta_n = +\infty$, and

$$\nabla J\alpha_n(x_n) = \nabla_x J(x_n) + \alpha_n(g(x_n)^T \nabla g(x_n) \mathbb{1}_{\{|g(x_n)|<0\}} + g(x_n)^T \nabla g(x_n) \mathbb{1}_{\{|g(x_n)|>1\}}) \tag{8}$$

Part II
Experiments

Cost function is linear and without noise

In this experiment, the cost function is a linear function. The optimization problem is described as below:

$$
\begin{aligned}
\min_{x_1, x_2 \in \mathbb{R}} \quad & J(x_1, x_2) = -2x_1 - x_2 \\
\text{s.t.} \quad & 0 \leq x_1 \leq 1 \\
& 0 \leq x_2 \leq 1 \\
& \sum_{i=1}^{2} x_i = 1
\end{aligned} \tag{9}
$$

As $\sum_{i=1}^{2} x_i = 1$, we can change equation 9 to:

$$
\begin{aligned}
\min_{x \in \mathbb{R}} \quad & J(x) = -x - 1 \\
\text{s.t.} \quad & 0 \leq x \leq 1
\end{aligned} \tag{10}
$$

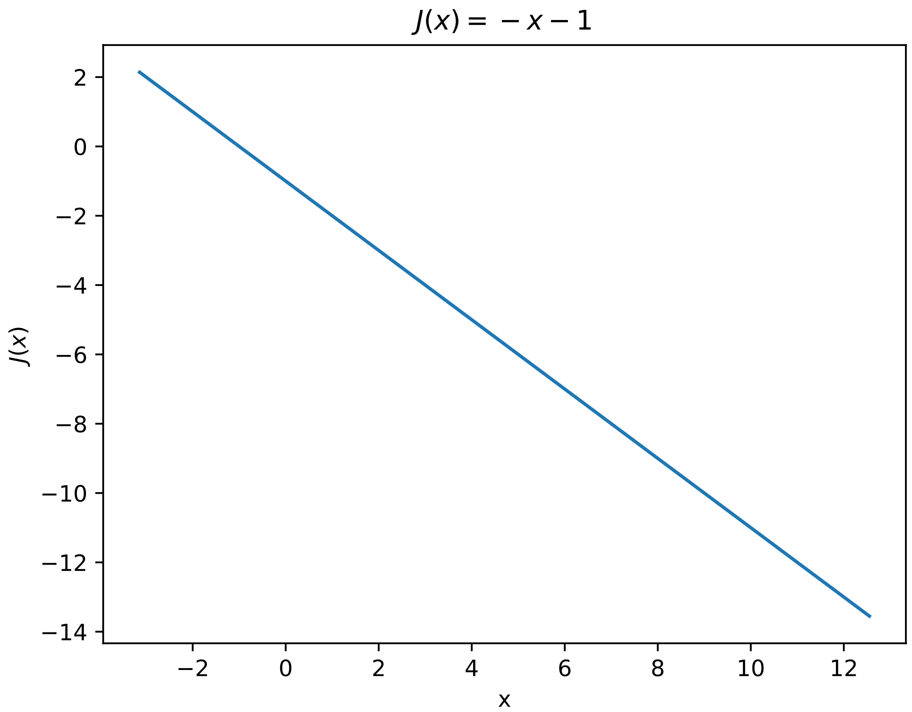

Figure 1: The plot of the cost function: $J(x) = -x - 1$.

$$
\min_{\theta \in \mathbb{R}} \quad J(\theta) = -cos(\theta)^2 - 1 \tag{11}
$$

The plot of the equation 10 is shown in figure 1. By using the equation 3, we can change the constrained optimization probblem to an unconstrained problem by using the spherical coordinates parameterization method. The unconstrained problem by using the spherical coordinates parameterization method is formally described in equation 11. The plot of the equation 11 is shown in figure 2.

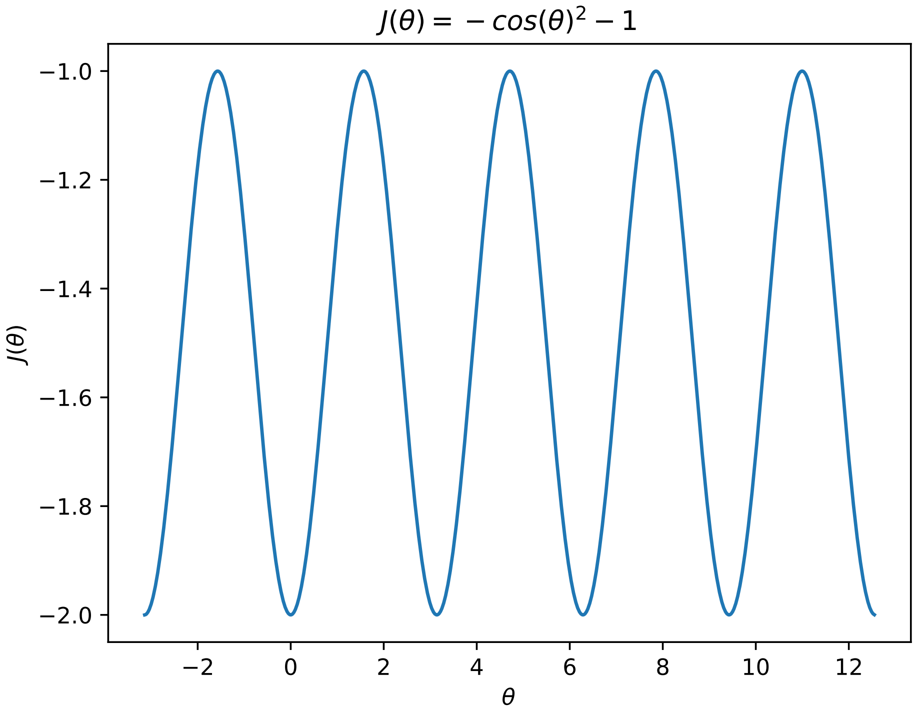

Figure 2: The plot of the cost function: $J(\theta) = -cos(\theta)^2 - 1$.

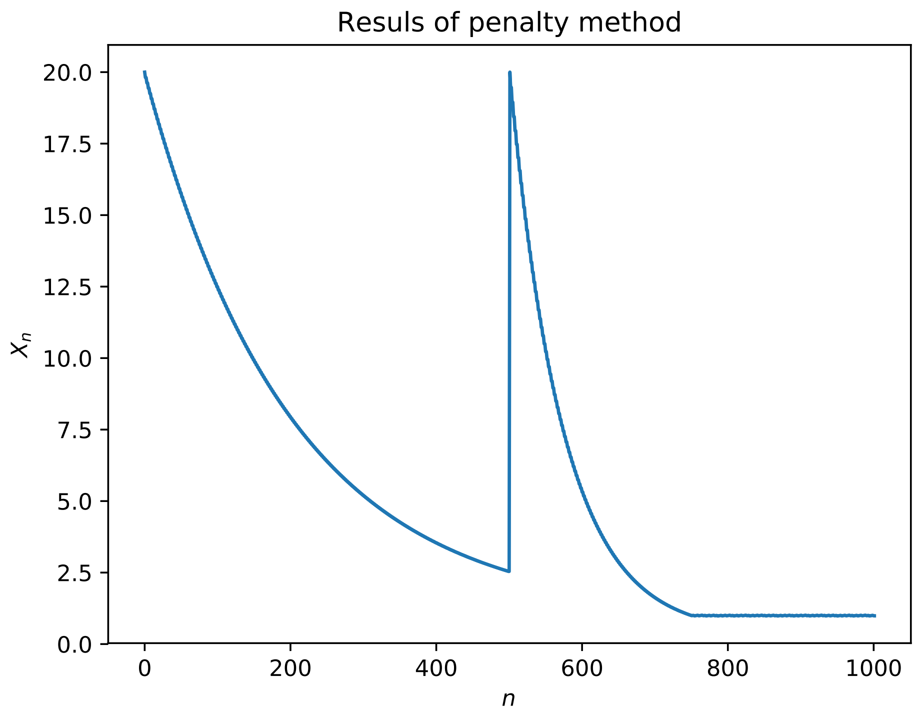

Figure 3: The estimation process of the optimal x when the penalty method is used.Here the initial value of x is set to 20. The total iteration for T_i is 500. The $\alpha_1 = 1$ and $\alpha_{n+1} = \alpha_n + \delta_n$, where $\delta_n = e^n$.

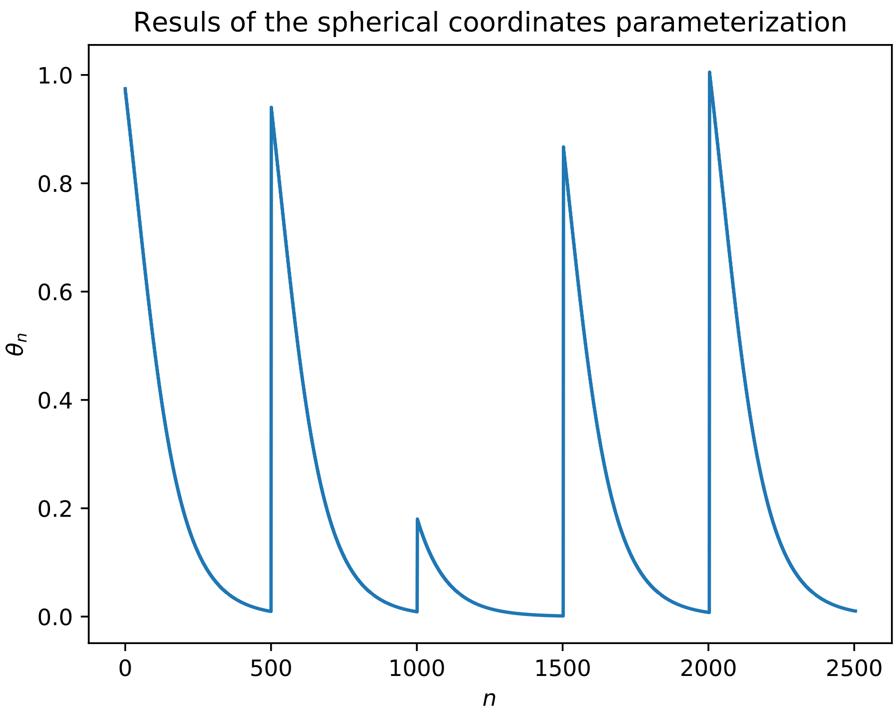

Figure 4: The estimation process of the optimal θ when the spherical coordinated parameterization is used. In order to make sure the global optimal value is collected, 5 times experiments with random initialized value of θ is done and the global minimum vlaue is collected from those 5 experiments.

Optimal x	method	estimated x	execution time(ms)
1.0	penalty method	0.9947	1.09
1.0	the spherical coordinates parameterization	1.0000	14.16

Table 1: The results of the penalty method and spherical coordinate parameterization.