# ECEN 649 Pattern Recognition – Spring 2018

## Problem Set 1

Due on: Feb 8

**Problems:**

1. This problem demonstrates nicely subtle issues regarding partial information and prediction. A certain show host has placed a case with US$1,000,000 behind one of three identical doors. Behind each of the other two doors he placed a donkey. The host asks the contestant to pick one door but not to open it. The host then opens one of the other two doors to reveal a donkey. He then asks the contestant if he wants to stay with his door or switch to the other unopened door. Assume that the host is honest and that if the contestant initially picked the correct door, the host randomly picks one the two donkey doors to open. Which of the following strategies is rationally justifiable:

   (a) The contestant should never switch to the other door.

   (b) The contestant should always switch to the other door.

   (c) There is not enough information or the choice between (a) and (b) is indifferent.

   To get full credit, you must argue this by correctly computing the probabilities of success.

2. The random experiment consists of throwing two fair dice. Let us define the events:

   $D = \{\text{the sum of the dice equals } 6\}$

   $E = \{\text{the sum of the dice equals } 7\}$

   $F = \{\text{the first die lands } 4\}$

   $G = \{\text{the second die lands } 3\}$

   Show the following, both by arguing and by computing probabilities:

   (a) $D$ is not independent of $F$ and $D$ is not independent of $G$.

   (b) $E$ is independent of $F$ and $E$ is independent of $G$.

   (c) $E$ is not independent of $(F, G)$, in fact, $E$ is completely determined by $(F, G)$. (Here is an example where an event is independent of each of two other events but is not independent of the joint occurrence of these events.)

3. Suppose that a typist monkey is typing randomly, but that each time he types the "wrong character," it is discarded from the output. Assume also that the monkey types 24-7 at the rate of one character per second, and that each character can be one of 27 symbols (the alphabet without punctuation plus space). Given that *Hamlet* has about 130,000 characters, what is the average number of days that it would take the typist monkey to compose the famous play?

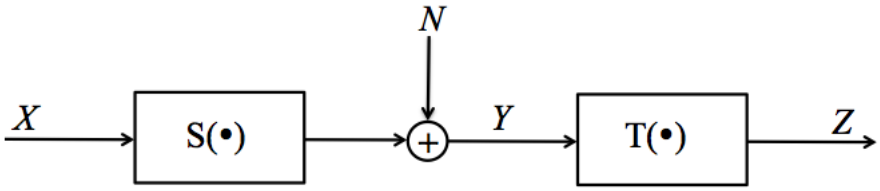
4. Suppose that 3 balls are selected without replacement from an urn containing 4 white balls, 6 red balls, and 2 black balls. Let $X_i = 1$ if the $i$-th ball selected is white, and let $X_i = 0$ otherwise, for $i = 1, 2, 3$. Give the joint PMF of

   (a) $X_1$, $X_2$

   (b) $X_1$, $X_2$, $X_3$

5. Consider 12 independent rolls of a 6-sided die. Let $X$ be the number of 1's and let $Y$ be the number of 2's obtained. Compute $E[X]$, $E[Y]$, $\mathrm{Var}(X)$, $\mathrm{Var}(Y)$, $E[X+Y]$, $\mathrm{Var}(X+Y)$, $\mathrm{Cov}(X,Y)$, and $\rho(X,Y)$. (Hint: You may want to compute these in the order given.)

6. Consider the system represented by the block diagram below.




   The functionals are given by $S(X) = aX + b$, and $T(Y) = Y^2$. The additive noise is $N \sim N(0, \sigma_N^2)$. Assuming that the input signal is $X \sim N(\mu_X, \sigma_X^2)$:

   (a) Find the pdf of $Y$.

   (b) Find the pdf of $Z$.

   (c) Compute the probability that the output is bounded by a constant $k > 0$, i.e., find $P(Z \leq k)$.

7. (Bi-variate Gaussian Distribution) Suppose $(X, Y)$ are jointly Gaussian.

   (a) Show that the joint pdf is given by:

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \times \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 - 2\rho\frac{(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right]\right\}$$

   where $E[X] = \mu_X$, $\mathrm{Var}(X) = \sigma_X^2$, $E[Y] = \mu_Y$, $\mathrm{Var}(Y) = \sigma_Y^2$, and $\rho$ is the correlation coefficient between $X$ and $Y$.

   (b) Show that the conditional pdf of $Y$, given $X = x$, is a univariate Gaussian density with parameters:

$$\mu_{Y|X} = \mu_y + \rho\frac{\sigma_y}{\sigma_x}(x - \mu_x) \quad \text{and} \quad \sigma_{Y|X}^2 = \sigma_y^2(1-\rho^2)$$

(c) Conclude that the conditional expectation $E[Y|X]$ (which can be shown to be the "best" predictor of $Y$ given $X$), is in the Gaussian case a linear function of $X$. This is the foundation of optimal linear filtering in Signal Processing. Plot the regression line for the case $\sigma_x = \sigma_y$, $\mu_x = 0$, fixed $\mu_y$ and a few values of $\rho$. What do you observe as the correlation $\rho$ changes? What happens for the case $\rho = 0$?

8. Consider the example of a random sequence $X(n)$ of 0-1 binary r.v.'s given in class:

   - Set X(0) = 1
   - From the next 2 points, pick one randomly and set to 1, the other to zero.
   - From the next 3 points, pick one randomly and set to 1, the rest to zero.
   - From the next 4 points, pick one randomly and set to 1, the rest to zero.
   - ...

   Show that $X(n)$:

   (a) converges to 0 in probability

   (b) converges to 0 in the mean-square sense

   (c) does not converge to 0 with probability 1. In fact, show that

$$P\left(\lim_{n\to\infty} X(n) = 0\right) = 0$$