⇒ First order First degree DE

power of highest derivative

$$F\left(x, y, \frac{dy}{dx}\right)$$

① Variable separable → $f(x)\,dx = f(y)\,dy$

by integrating on both sides

→ $$\frac{dy}{dx} = e^{3x} e^{-2y} + x^2 e^{-2y}$$

$$\frac{dy}{dx} = e^{-2y}\left(e^{3x} + x^2\right)$$

Not a ratio

$$\frac{dy}{e^{-2y}} = \left(e^{3x} + x^2\right) dx$$

$$\int e^{2y}\,dy = \int e^{3x} + x^2 \,dx$$

$$\boxed{\frac{e^{2y}}{2} = \frac{e^{3x}}{3} + \frac{x^3}{3} + C} \rightarrow \text{General solution}$$

$y(0) = 1$ → initial

at $x = 0$ $y = 1$.

$$\frac{e^2}{2} = \frac{e^0}{3} + 0 + c$$

$$\frac{e^2}{2} = 1 + c$$

$$\boxed{c = \frac{e^2}{2} - 1}$$

$$\boxed{\frac{e^{2y}}{2} = \frac{e^{3x}}{3} + \frac{x^3}{3} + \frac{e^2}{2} - 1} \rightarrow \text{Particular solution}$$

II Reduction to Variable Separation

Substitution $\longrightarrow$ Reduce to V.S. $f(x)dx = g(y)dy$

$$\frac{dy}{dx} = (4x+y+1)^2$$

Substitute $4x+y+1 = t \longrightarrow$

diff w.r.t x

$$4 + \frac{dy}{dx} + 0 = \frac{dt}{dx}$$

$$\boxed{\frac{dy}{dx} = \frac{dt}{dx} - 4}$$

$$\frac{dy}{dx} = (\underline{4x+y+1})^2$$

$\downarrow$

$$\frac{dt}{dx} - 4 = t^2$$

$$\frac{dt}{dx} = t^2 + 4$$

$$\frac{dt}{t^2+4} = dx \longrightarrow \text{Variable Separation}$$

$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}(x/a)$ $\quad \int \frac{dt}{t^2+4} = \int dx$

$$\frac{1}{t^2+2^2}$$

$$\Rightarrow \frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right) = x + c$$

$$\Rightarrow \boxed{\frac{1}{2}\tan^{-1}\left(\frac{4x+y+1}{2}\right) = x + c} \longrightarrow \text{General sol}$$

$y(0) = 1 \longrightarrow$

at $x=0$ $y=1$

$$\frac{1}{2}\tan^{-1}\left(\frac{0+1+1}{2}\right) = 0 + c$$

$$\frac{1}{2}\tan^{-1}(1) = c$$

$$\frac{1}{2}\left(\frac{\pi}{4}\right) = c$$

$$\boxed{c = \pi/8}$$

$$\boxed{\frac{1}{2}\tan^{-1}\left(\frac{4x+y+1}{2}\right) = x + \pi/8} \longrightarrow \text{Particular sol}$$

III Homogeneous DE

$$\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$$ are homogeneous functions of same degree

$$f(tx, ty) = t^a f(x,y)$$

$f(x,y) = x+y \longrightarrow f(tx,ty) = tx+ty = t(x+y)$

$= t f(x,y)$

$f(x,y) = x+y+2$

$f(tx,ty) = tx+ty+2 \neq t^a f(x,y)$

Not homogeneous

$$\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)} \xrightarrow{\text{substitution}} y = vx \longrightarrow \text{reduces to Variable separable}$$

v fn dependent on x

ex:- $(x^2-y^2)\,dx = xy\,dy$

$$\frac{dy}{dx} = \frac{x^2-y^2}{xy}$$

$x^2-y^2 \to (tx)^2-(ty)^2 = t^2(x^2-y^2)$

$xy \to (tx)(ty) = t^2xy$

$$= \frac{\cancel{t^2}(x^2-y^2)}{\cancel{t^2}xy}$$

This is homogeneous eq

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$y = vx$$

diff w.r.t x

$$\frac{dy}{dx} = v(1) + x\frac{dv}{dx}$$

$$\boxed{\frac{dy}{dx} = v + x\frac{dv}{dx}}$$

$$\frac{dy}{dx} = \frac{x^2 - y^2}{xy}$$

$$v + x\frac{dv}{dx} = \frac{x^2 - (vx)^2}{x(vx)} = \frac{x^2 - v^2x^2}{vx^2} = \frac{\cancel{x^2}(1-v^2)}{v\cancel{x^2}}$$

$$v + x\frac{dv}{dx} = \frac{1-v^2}{v} \longrightarrow$$ Variable separable eq

$$x\frac{dv}{dx} = \frac{1-v^2}{v} - v = \frac{1-2v^2}{v}$$

$$x\frac{dv}{dx} = \frac{1-2v^2}{v}$$

$$\boxed{\int \frac{f'(x)}{f(x)}\,dx = \log f(x) + c}$$

$$-\frac{1}{4}\int \frac{-4v}{1-2v^2}\,dv = \int \frac{dx}{x}$$

$$-\frac{1}{4}\int \frac{-4v}{1-2v^2}\,dv = \int \frac{dx}{x}$$

$1-2v^2$

diff = $-4v$

$$-\frac{1}{4}\log(1-2v^2) = \log x + c$$

$$-\frac{1}{4}\log(1-2v^2) = \log x + c$$

$$\log(1-2v^2) + 4\log x + 4c = 0$$

$$\log(1-2v^2)x^4 = -4c$$

$y = vx$

$\boxed{v = y/x}$

$$(1-2v^2)x^4 = e^{-4c}$$

$$\left(1-\frac{2y^2}{x^2}\right)x^4 = A$$

→ Non Homogeneous → reducible to homogeneous

$$\frac{dy}{dx} = \frac{ax+by+c\ldots}{a'x+b'y+c'\ldots}$$

$at x + a t y + c$

Case (i)

$$\frac{a}{a'} \neq \frac{b}{b'}$$

✓ $x = X + h$

$dx = dX$

✓ $y = Y + k$

$dy = dY$

$$\frac{dy}{dx} = \frac{dY}{dX}$$

h, k are constants

Case (ii) $\frac{a}{a'} = \frac{b}{b'}$

$ab' = a'b$

$\boxed{ab' - a'b = 0}$

Substitute

$\boxed{ax+by = t}$

$$\frac{dy}{dx} = \frac{a(X+h)+b(Y+k)+c}{a'(X+h)+b'(Y+k)+c'} = \frac{aX+ah+bY+bk+c}{a'X+a'h+b'Y+b'k+c'} = \frac{aX+bY+(ah+bk+c)}{a'X+b'Y+(a'h+b'k+c')}$$

① $ah + bk + c = 0$

② $a'h + b'k + c' = 0$

Solve for h, k such ①, ② satisfies

$$\frac{dy}{dx} = \frac{ax+by}{a'x+b'y}$$

$\downarrow$

$y = vx$

$\downarrow$ ~~Variable~~ separable

$$\boxed{h = \frac{bc' - b'c}{ab' - a'b} \qquad k = \frac{ca' - ac'}{ab' - a'b}}$$

ex:- ① $\dfrac{dy}{dx} = \dfrac{x+y-2}{-x+y-4}$

$\dfrac{a}{a'} = \dfrac{b}{b'}$?? $\quad a = 1, b = 1$

$a' = -1 \quad b' = 1$

$\dfrac{a}{a'} \neq \dfrac{b}{b'}$

Case i $\quad -1 \neq 1$

② $\dfrac{dy}{dx} = \dfrac{3y+2x+4}{6y+4x+5}$

$a = 2 \qquad b = 3$

$a' = 4 \qquad b' = 6$

$\dfrac{2}{4} = \dfrac{3}{6}$

$\dfrac{1}{2} = \dfrac{1}{2}$

Case (ii)

$$\boxed{\frac{a}{a'} = \frac{b}{b'}}$$

$x = X + h$

$y = Y + k$

$$\frac{dy}{dx} = \frac{x+y-2}{-x+y-4}$$

$$\frac{dy}{dx} = \frac{X+h+Y+k-2}{-(X+h)+Y+k-4} = \frac{X+Y+(h+k-2)}{-X+Y+(-h+k-4)}$$

$h + k - 2 = 0$

$-h + k - 4 = 0$

$2k - 6 = 0$

$\boxed{k = 3}$

$h = -1$

$h + k - 2 = 0$

$h + 3 - 2 = 0$

$\boxed{h = -1}$

$$\boxed{\begin{aligned} x &= X - 1 \\ y &= Y + 3 \end{aligned}}$$

$$\frac{dy}{dx} = \frac{X + Y}{-X + Y} \longrightarrow \text{This is homogeneous eq}$$

Substitute $Y = vX$

$$\frac{dY}{dX} = v + X\frac{dv}{dX}$$

$$v + X\frac{dv}{dX} = \frac{X + vX}{-X + vX} = \frac{X(1+v)}{X(-1+v)}$$

$$v + X\frac{dv}{dX} = \frac{1+v}{v-1}$$

$$X\frac{dv}{dX} = \frac{1+v}{v-1} - v = \frac{1 + v - v^2 + v}{v-1}$$

$$X\frac{dv}{dX} = \frac{1 + 2v - v^2}{v-1}$$

$$\frac{-1}{2}\int \frac{-2(v-1)}{1+2v-v^2}\,dv = \int \frac{dX}{X}$$

$2 - 2v$

$2(1-v)$

$$\frac{-1}{2}\int \frac{2(1-v)}{1+2v-v^2}\,dv = \int \frac{dX}{X}$$

$$-\frac{1}{2}\log(1+2v-v^2) = \log x + c$$

$$\log(1+2v-v^2) + 2\log x + 2c = 0$$

$$\log(1+2v-v^2)x^2 = -2c$$

$$(1+2v-v^2)\ x^2 = e^{-2c}$$

$y = vx$

$v = \frac{y}{x}$

$$\left(1+\frac{2y}{x}-\frac{y^2}{x^2}\right)x^2 = A$$

$x = X-1 \Rightarrow X = x+1$

$y = Y+3$

$Y = y-3$

$$\left(1+\frac{2(y-3)}{x+1}-\frac{(y-3)^2}{(x+1)^2}\right)(x+1)^2 = A$$

$$(y-3)^2 - 2(y-3)(x+1) - (x+1)^2 = A$$