Linear DE of 1st order

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$\text{If}: \quad e^{\int P(x)\,dx}$$

Sol:- $\boxed{y \,\text{If} = \int \text{If}\, Q(x)\,dx + c}$

$\longrightarrow$ $$\frac{dx}{dy} + P(y)x = Q(y)$$

$$\text{If} = e^{\int P(y)\,dy}$$

Sol:- $\boxed{x \,\text{If} = \int \text{If} \times Q(y)\,dy + c}$

$\longrightarrow$ Non Linear $\Longrightarrow$ reducible to linear form

$$\frac{dy}{dx} + P(x)y = y^n Q(x)$$

$$\frac{1}{y^n}\frac{dy}{dx} + \frac{P(x)}{y^{n-1}} = Q(x)$$

$$\frac{1}{y^{n-1}} = z$$

$$\frac{dz}{dx} + (1-n)P(x)z = Q(x)$$

$$\text{If} = e^{\int (1-n)P(x)\,dx}$$

Sol:- $$z \,\text{If} = \int \text{If}\, Q(x)\,dx + c$$

substitute

ex:-

$$x\frac{dy}{dx} + y = x^3 \boxed{y^6}$$

$$\frac{dy}{dx} + \frac{y}{x} = x^3 y^6$$

$$\frac{1}{y^6}\frac{dy}{dx} + \frac{y}{xy^6} = x^3$$

$$\frac{1}{y^6}\frac{dy}{dx} + \frac{1}{x\,\boxed{y^5}} = x^3$$

$$z = \frac{1}{y^5} = y^{-5}$$

$$\frac{dz}{dx} = -5y^{-6}\frac{dy}{dx}$$

$$-\frac{1}{5}\frac{dz}{dx} = \boxed{\frac{1}{y^6}\frac{dy}{dx}}$$

$$-\frac{1}{5}\frac{dz}{dx} + \frac{z}{x} = x^3$$

$$\frac{dz}{dx} - \frac{5z}{x} = -5x^3$$

$$\frac{dz}{dx} - \frac{5z}{x} = -5x^3$$

$$\text{If} = e^{\int -\frac{5}{x}dx} = e^{-5\ln x} = \frac{1}{x^5}$$

Sol:-

$$z \cdot \frac{1}{x^5} = \int \frac{1}{x^5} \cdot -5x^3\,dx + c$$

$$\frac{z}{x^5} = \int -5x^{-2}\,dx + c$$

$$\frac{z}{x^5} = \frac{-5x^{-1}}{-1} + c$$

$$\boxed{\frac{1}{x^5y^5} = \frac{5}{x} + c}$$

Higher order Linear DE

$y, \frac{dy}{dx}, \frac{d^2y}{dx^2} \ldots$ they are not multiplied

→ **n^{th} order LDE**

$$\frac{d^ny}{dx^n} + P_1(x)\frac{d^{n-1}y}{dx^{n-1}} + P_2(x)\frac{d^{n-2}y}{dx^{n-2}} + \ldots P_n(x)y = Q(x)$$

$P_1(x), P_2(x) \ldots$ are functions of x.

n^{th} order Variable coeff Linear DE

n^{th} order **Linear DE with constant coefficient**

$$\frac{d^ny}{dx^n} + k_1\frac{d^{n-1}y}{dx^{n-1}} + k_2\frac{d^{n-2}y}{dx^{n-2}} + \ldots k_ny = \underline{\underline{Q(x)}}.$$

→ Homogeneous and Non homogeneous

$Q(x) = 0 \Rightarrow$ homogeneous eq

$Q(x) \neq 0 \Rightarrow$ Non homogeneous equation

⇒ **Solution of constant coeff LDE of order I**

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$IF = e^{\int P(x)\,dx}.$$

Sol:-

$$y\,e^{\int P(x)dx} = \int Q(x)\,e^{\int P(x)\,dx} + c$$

$$y\,IF = \int IF\,Q(x) + c$$

$$y = e^{-\int P(x)dx}\int Q(x)\,e^{\int P(x)\,dx} + c\,e^{-\int P(x)\,dx}$$

$$y = \underline{c\, e^{-\int P(x)\,dx}} + \underline{e^{-\int P(x)\,dx} \int e^{\int P(x)\,dx}\, Q(x)\, dx}$$

Complementary function

Solution of Homogeneous equation

(No of constant = order of eq)

Particular integral

Solution of Non homogeneous part

(Not any constants)

Q(x)

Variation of Parameters

Method of undetermined coeff

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$Q(x) = 0$

$$\frac{dy}{dx} + P(x)y = 0$$

$$\frac{dy}{dx} = -P(x)y$$

$$\int \frac{dy}{y} = \int -P(x)\,dx$$

$$\ln y = \int -P(x)\,dx + c$$

$$y = e^{-\int P(x)\,dx + c} = e^{c} \cdot e^{-\int P(x)\,dx} = c_1 e^{-\int P(x)\,dx}$$

$$\boxed{y = c_1 e^{-\int P(x)\,dx}}$$

Complementary function 1st order ODE

$\Rightarrow$ D → Operator form

$$D = \frac{d}{dx} \quad , \quad \frac{d^2}{dx^2} = D^2$$

$$\frac{dy}{dx} = Dy$$

$$\rightarrow \quad \frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots \quad k_n y = Q(x)$$

$$D^n y + k_1 D^{n-1} y + \cdots \quad k_n y = Q(x)$$

$$(D^n + k_1 D^{n-1} + \cdots \quad k_n) y = Q(x)$$

ex. $(D^2 + 5D + 6)y = Q(x)$

Polynomial of number of roots 'n'

$D^2 + 5D + 6$

root $-3, -2$

$(D+3)(D+2)$

Fundamental form

$y_1 = u_1 \qquad y_2 = u_2$

$$y = c_1 u_1 + c_2 u_2$$

$$\rightarrow \quad (D^n + D^{n-1} k_1 + \cdots \quad k_n) y = Q(x)$$

To find complementary function $Q(x)$

$$(D^n + D^{n-1} k_1 + \cdots \quad k_n) y = 0$$

$$D^n + D^{n-1} k_1 + \cdots \quad k_n = 0$$

Auxillary equation

(or) characteristic equation

① Real and distinct
let roots be $m_1, m_2 \ldots \; m_n$ be n distinct root.

roots $= m_1$ $\quad (D-m_1)(D-m_2)\ldots\ldots(D-m_n)y=0$

$D - m_1 = 0$

$\left(\frac{d}{dx} - m_1\right)y = 0$

$\frac{dy}{dx} - m_1 y = 0$

$\frac{dy}{y} = m_1\,dx.$

$\ln y = m_1 x + c$

$$\boxed{y = e^{m_1 x + c} = c_1 e^{m_1 x}}$$

$y = c_1 e^{m_1 x}$

$D - m_2 = 0$

$y = c_2 e^{m_2 x}$

$y = c_3 e^{m_3 x}.$

$c_1 e^{m_1 x}, \; c_2 e^{m_2 x} \ldots \; c_n e^{m_n x}$

$$\boxed{y_{\text{Complementary}} = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots \; c_n e^{m_n x}}$$

$(D^2 + 5D + 6)y = 0 \longrightarrow \frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$

$D^2 + 5D + 6 = 0$

$D^2 + 3D + 2D + 6 = 0$

$(D+3)(D+2) = 0$

roots $= \underset{\to m_1}{-3}, \; \underset{\to m_2}{-2}$

$y_{cf} = c_1 e^{-3x} + c_2 e^{-2x}$

II Real and repeated roots — nth order

let $m, m, \dots m$ repeating n times

$$y_{cf} = c_1 e^{mx} + c_2 e^{mx} + \dots c_n e^{mx}$$

$$= (c_1 + c_2 + \dots c_n) e^{mx}$$

$$y_{cf} = c \cdot e^{mx} \quad \times$$

$\neq$ # order of eq

$\Rightarrow$

Ex:-

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$$

$$D^2y + 4Dy + 4y = 0$$

$$D^2 + 4D + 4 = 0$$

roots $-2, -2$

$$y \neq c_1 e^{-2x} + c_2 e^{-2x} = c e^{-2x} \quad \times$$

$$y = v(x) e^{-2x}$$

$$y'(x) = v'(x) e^{-2x} + -2e^{-2x} v(x) = (v' - 2v) e^{-2x}$$

$$y''(x) = (v'' - 2v') e^{-2x} + (v' - 2v) e^{-2x} (-2)$$

$$y''(x) = (v'' - 4v' + 4v) e^{-2x}$$

$$D^2y + 4Dy + 4y = 0$$

$$\left[v'' - 4v' + 4v\right]e^{-2x} + 4(v' - 2v)e^{-2x} + 4ve^{-2x} = 0$$

$$v'' e^{-2x} = 0$$

$$\boxed{v'' = 0}$$

$$\int \frac{d^2v}{dx^2} = \int 0 \Rightarrow \int v''\,dv = \int 0\,dx$$

$$\int v' = 0 + c$$

$$\int \frac{dv}{dx} = 0 + \int c_1$$

$$\int v'\,dv = \int c_1\,dx.$$

$v(x) = c_1x + c_2$

$$\boxed{v = c_1x + c_2}$$

$$y = (c_1x + c_2)e^{-2x}$$ $-2, -2$

$$= (c_1 + c_2x)e^{-2x}$$

$$y = (c_1 + c_2x + c_3x^2)e^{-2x}$$ $-2, -2, -2$

n repeated roots

$$\boxed{y_{cf} = (c_1 + c_2x + c_3x^2 + \dots c_nx^{n-1})e^{mx}}$$ → n repeated roots

→ III Imaginary roots (pair)

$$\frac{d^2y}{dx^2}+y=0 \quad (D^2+1)y=0$$

$$D^2+1=0$$

$$\text{roots} = \pm i$$

$$\alpha+i\beta, \quad \alpha-i\beta$$

Sol:- $y = c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha-i\beta)x}$ $\qquad e^{i\theta} = \cos\theta + i\sin\theta$

$$= c_1 e^{\alpha x} e^{i\beta x} + c_2 e^{\alpha x}\cdot e^{-i\beta x}$$

$$= e^{\alpha x}\left[c_1 e^{i\beta x} + c_2 e^{-i\beta x}\right]$$

$$= e^{\alpha x}\left[\left[c_1\cos\beta x + ic_1\sin\beta x\right] + c_2\cos\beta x - ic_2\sin\beta x\right]$$

$$= e^{\alpha x}\left[(c_1+c_2)\cos\beta x + (ic_1 - ic_2)\sin\beta x\right.$$

$\alpha \pm i\beta$ $\qquad y = e^{\alpha x}\left[A\cos\beta x + B\sin\beta x\right]$

$$\boxed{y = e^{\alpha x}\left[c_1\cos\beta x + c_2\sin\beta x\right]}$$ → Imaginary

→ repeated Imaginary $\alpha \pm i\beta, \alpha \pm i\beta$ $\qquad y = e^{\alpha x}\left[(c_1+c_2x)\cos\beta x + (c_3+c_4x)\sin\beta x\right]$

① real and distinct roots n roots

$m_1, m_2 \cdots\cdots m_n$

$$\boxed{y_{cf} = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots c_n e^{m_n x}}$$

② Real and repeated roots $m, m \ldots m$ n times

$$y_{cf} = \left[c_1 + c_2 x + c_3 x^2 + \cdots c_n x^{n-1}\right] e^{mx}$$

③ Imaginary roots (pair) $\underline{\underline{\alpha \pm i\beta}}$

$$\underline{\underline{y_{cf} = e^{\alpha x}\left[c_1 \cos\beta x + c_2 \sin\beta x\right]}}$$

$\underline{\underline{\text{ex:-}}}$ roots $= \underline{\underline{1, 1}}, 2, -3$

$$y_{cf} = c_1 e^{2x} + c_2 e^{-3x} + \left(c_3 + c_4 x\right) e^{x}$$

→ roots $\underline{\underline{1 \pm i}}$, $\underline{\underline{1, 1}}$, 2, 3

$\alpha = 1$
$\beta = 1$

$$y_{cf} = c_1 e^{2x} + c_2 e^{3x} + (c_3 + c_4 x) e^{x} + e^{x}(c_5 \cos x + c_6 \sin x)$$

→ $\underline{\underline{1, 1 \text{ roots}}}$

$(D-1)(D-1)$

$D^2 - 1$

$(D^2 - 1) y = 0 \longrightarrow \frac{d^2 y}{dx^2} - y = 0$

1. $\frac{d^2y}{dx^2} + \frac{6\,dy}{dx} + 9y = 0$

2. $(D^3 + D^2 + 4D + 4)\,y = 0$

3. $(D^4 - 4D^2 + 4)\,y = 0$

4. $(D^2 + 1)^3\,y = 0$

5. $(D^4 + 4)\,y = 0$