→ Partial diff equation

Classification

1. Linear
2. Quasi linear →
3. Semi linear →
4. Non linear →

$u = f(x, y)$

$\left(\frac{\partial u}{\partial x}\right)^2 + 4\frac{\partial u}{\partial y} = 2$

Every semilinear is Quasi linear

→ Solving

1. Direct Method

$\frac{\partial z}{\partial x \partial y} = \int \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) dx$

$\frac{\partial^2 z}{\partial x^2} = xy^2$

Integrate w.r.t. x keeping y as constant

$f(y)$ $\int \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) dx = \int xy^2\, dx$

$\frac{\partial z}{\partial x} = y^2 \frac{x^2}{2} + c_1 f(y)$

integrate w.r.t x treat y as const

$\int \frac{\partial z}{\partial x} dx = \int \frac{x^2}{2} y^2 dx + \int c_1 f(y)\, dx$

$$\boxed{z = \frac{x^3}{6}y^2 + c_1 x f(y) + c_2 g(y)}$$

$\frac{\partial z}{\partial x \partial y} + xy = 0$

$\int \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) dx = \int xy\, dx$

$\frac{\partial z}{\partial y} = -\frac{x^2}{2}y + c_1 f(y)$

Integrate w.r.t y

$\int \frac{\partial z}{\partial y} dy = \int -\frac{x^2}{2} y\, dy + \int c_1 f_1(y)\, dy + g(x)$

$z = -\frac{x^2 y^2}{4} + h(y) + g(x)$

→ First order Quasi linear equation

$$P(x,y,z)\frac{\partial z}{\partial x} + Q(x,y,z)\frac{\partial z}{\partial y} = R(x,y,z) \longrightarrow \text{Lagrange's equation}$$

Auxillary eq

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Solve simultaneous u_1, v_1

Sol $\boxed{F(u_1, v_1) = 0}$

① $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 3z$

A.E $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{3z}$

① ② ③

① & ②

$$\int \frac{dx}{x} = \int \frac{dy}{y}$$

$$\ln x = \ln y + c$$

$$x = e^{\ln y} \cdot e^{c} = y c_1$$

1st sol $\boxed{\frac{x}{y} = c_1}$

② & ③

$$\int \frac{dy}{y} = \int \frac{dz}{3z}$$

$$\ln y = \frac{1}{3}\ln z + c_2$$

$$3\ln y = \ln z + c_2$$

$$y^3 = z \cdot c_2$$

2nd sol $\boxed{\frac{y^3}{z} = c_2}$

$$\boxed{F\left(\frac{x}{y}, \frac{y^3}{z}\right) = 0}$$

(2) $x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = xyz$

A.E $\frac{dx}{x} = \frac{dy}{-y} = \frac{dz}{xyz}$

(1) (2) (3)

(1) & (2)

$$\int \frac{dx}{x} = \int \frac{dy}{-y}$$

$$\ln x = -\ln y + c$$

$$\ln x + \ln y = c$$

$$xy = e^c = c_1$$

$$\boxed{xy = c_1}$$

(2) & (3)

$$\frac{dy}{-y} = \frac{dz}{xyz} \longrightarrow$$ from no. 1 replace $\boxed{xy = c_1}$

$$\frac{dy}{-y} = \frac{dz}{c_1 z}$$

$$\int \frac{dy}{-y} = \int \frac{dz}{c_1 z}$$

$$-\ln y = \frac{1}{c_1}\ln z + c_2$$

$$\ln y + \frac{1}{c_1}\ln z = -c_2$$

$$y \cdot z^{1/c_1} = c$$

$$\boxed{y\, z^{1/xy} = c}$$

Sol: $F\left(xy,\ y z^{1/xy}\right) = 0$

(3) $y^2 + z^2 \frac{\partial z}{\partial x} - xy\frac{\partial z}{\partial y} + zx = 0$

$$\frac{dx}{y^2+z^2} = \frac{dy}{-xy} = \frac{dz}{-zx} \longrightarrow$$ unable to get 2 soln of variable separable

(2) (3)

→ Method of Multiplier :- x, y, z

l, m, n → $x\ y\ z$

$$x(y^2+z^2) + y(-xy) + z(-zx) = xy^2 + xz^2 - xy^2 - z^2x = 0$$

$$P\frac{\partial z}{\partial x} + Q\frac{\partial z}{\partial y} = R$$

A.E $\qquad \dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R} = \dfrac{P_1\,dx + q_1\,dy + R_1\,dz}{PP_1 + q_1Q + Rr_1}$

0

Multipliers

P_1, q_1, r_1.

Choose multipliers P_1, q_1, r_1 such that

$$\boxed{PP_1 + q_1Q + r_1R = 0}$$

Once you choose multiplier satisfying above condition.

Solution : $P_1dx + q_1dy + r_1dz = 0$ — by integrating it you will get

ex:- In this case my multiplier are x, y, z

Confirmation

$$P_1 = x, \quad q_1 = y, \quad r_1 = z$$

$$P_1dx + q_1dy + r_1dz = 0$$

$$\int x\,dx + y\,dy + z\,dz = 0$$

$$\boxed{\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c}$$

(2) & (3)

$$\frac{dy}{-xy} = \frac{dz}{-z\cancel{x}}$$

$$\int\frac{dy}{-y} = \int\frac{dz}{-z}$$

$$-\ln y = -\ln z + c_1$$

$$-\ln y + \ln z = c_1$$

$$\boxed{z/y = c}$$

Sol:- $F\left(\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2}, z/y\right) = 0$

9) $(z-y)\frac{\partial z}{\partial x} + (x-z)\frac{\partial z}{\partial y} = y-x$

$$\frac{dx}{z-y} = \frac{dy}{x-z} = \frac{dz}{y-x}$$

1st set multiplier 1, 1, 1

$dx+dy+dz=0$

2nd set Multipliers x, y, z

$pp_1+qq_1+rr_1=0$

$$1(z-y)+1(x-z)+1(y-x) = z-y+x-z+y-x=0$$

$$\int p_1\,dx + q_1\,dy + r_1\,dz = \int 0$$

$$\int dx + \int dy + \int dz = 0$$

$$\boxed{x+y+z=c}$$

$pp_1+qq_1+rr_1=0$

$$x(z-y)+y(x-z)+z(y-x) = xz-yx+xy-zy+zy-zx$$
$$=0$$

$p_1=x$

$q_1=y$

$r_1=z$

$$\int x\,dx + \int y\,dy + \int z\,dz = \int 0$$

$$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=c$$

$$\boxed{F\left(x+y+z,\ \frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}\right)=0}$$

Method of Separable Variables

$$A\frac{\partial^2 z}{\partial x^2} + B\frac{\partial^2 z}{\partial x \partial y} + C\frac{\partial^2 z}{\partial y^2} = F\left(x,y,z,\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}\right)$$

hyperbolic

parabolic

elliptic

$B^2-4AC>0 \rightarrow$ Hyperbolic PDE

$B^2-4AC=0 \rightarrow$ Parabolic PDE

$B^2-4AC<0 \rightarrow$ elliptic PDE

$$\boxed{Z = f(x, y)}$$

Assume any sol

$$Z = \underline{X(x)}\ \underline{Y(y)}$$

$$\to \quad \frac{\partial^2 z}{\partial x^2} + 4\frac{\partial^2 z}{\partial y^2} = 0$$

$A = 1$

$B = 0$

$C = 4$

$$B^2 - 4AC$$

$$0 - 4(1)(4) < 0$$

<u>elliptic PDE</u>

Assume

$$Z = X(x)\,Y(y)$$

$$\frac{\partial z}{\partial x} = X'(x)\,Y(y)$$

$$\frac{\partial^2 z}{\partial x^2} = X''(x)\,Y(y)$$

$$\frac{\partial z}{\partial y} = X(x)\,Y'(y)$$

$$\frac{\partial^2 z}{\partial y^2} = X(x)\,Y''(y)$$

$$X''Y + 4XY'' = 0$$

$$X''Y = -4XY''$$

$$\frac{X''}{X} = \frac{-4Y''}{Y} = K$$

fun x

fun y

$$\frac{X''}{X} = K$$

$$X'' = KX$$

$$X'' - KX = 0$$

D

$$(D^2 - K)X = 0$$

$$\frac{-4Y''}{Y} = K$$

$$-4Y'' - KY = 0$$

$$(-4D^2 - K)Y = 0$$

①

$$(D^2 - k)x = 0$$

$$D^2 = k$$

$$D = \pm\sqrt{k}$$

$$X(x) = a_1 e^{\sqrt{k}x} + a_2 e^{-\sqrt{k}x}$$

$$(-4D^2 - k)Y = 0$$

$$4D^2 = -k$$

$$D = \frac{\pm i\sqrt{k}}{2}$$

$$Y(y) = b_1 \cos\frac{\sqrt{k}}{2}y + b_2 \sin\frac{\sqrt{k}}{2}y$$

$$Z = X(x)\,Y(y)$$

The complete solution will be

$$Z = \sum X_k Y_k$$

$$= \sum \left(a_k \cos\frac{\sqrt{k}}{2}y + b_k \sin\frac{\sqrt{k}}{2}y\right) e^{\sqrt{k}x}$$

$$+ \; e^{-\sqrt{k}x}\left(a_k \cos\frac{\sqrt{k}}{2}y + b_k \sin\frac{\sqrt{k}}{2}y\right)$$

→ Initial value problem $Z(x,y)$.

$$Z(x,0) = 0$$

$$Z'(x,0) = 0$$

Initial conditions given

→ Boundary value problem

$$Z(0,t) = 0$$

$$Z(l,t) = k$$

find your constants