Reducible to homogeneous equation

$$\frac{dy}{dx} = \frac{ax + by + c}{a'x + b'y + c'}$$

Case (i)

$$\frac{a}{a'} \neq \frac{b}{b'}$$

$x = X + h$

$y = Y + k$

$$\frac{dY}{dX} = \frac{aX + bY}{a'X + b'Y} \longrightarrow \boxed{\text{Homogeneous}}$$

$Y = vX$

$\downarrow$

Variable Seperable

Case (ii)

$$\frac{a}{a'} = \frac{b}{b'}$$

$h = \dfrac{\quad}{ab' - ba'}$

Substitution

$t = ax + by$

Case 2

ex $(3y + 2x + 4)dx = (4x + 6y + 5)dy$

$$\frac{dy}{dx} = \frac{2x + 3y + 4}{4x + 6y + 5} = \frac{ax + by + c}{a'x + b'y + c'}$$

$2t + 4$

$a = 2 \quad b = 3$

$a' = 4 \quad b' = 6$

$t = 2x + 3y$

$2t = 4x + 6y$

$$\frac{a}{a'} = \frac{2}{4} = \frac{1}{2}, \quad \frac{b}{b'} = \frac{3}{6} = \frac{1}{2}$$

$$\frac{a}{a'} = \frac{b}{b'} = \frac{1}{2}$$

$t = 2x + 3y$

$$\frac{dt}{dx} = 2 + 3\frac{dy}{dx} \longrightarrow \frac{dy}{dx} = \frac{1}{3}\left[\frac{dt}{dx} - 2\right]$$

$22A + B = 5$

$\frac{44}{7} + B = 5$

$B = 5 - \frac{44}{7} = \frac{-9}{7}$

$\boxed{A = \frac{2}{7} \qquad B = \frac{-9}{7}}$

$A + B/_{7t+22}$

$\frac{2t+5}{7t+22} = \frac{2}{7} - \frac{9}{7(7t+22)}$

$$\int \frac{2}{7} - \frac{9}{7(7t+22)}\, dt = \int dx$$

$\int \frac{1}{7t+22}$

$$\frac{2}{7}t - \frac{9}{7}\frac{\log(7t+22)}{7} = x + c$$

$$\frac{2}{7}t - \frac{9}{49}\log(7t+22) = x + c$$

$$\frac{2}{7}(2x+3y) - \frac{9}{49}\log(7(2x+3y)+22) = x + c$$

→ Linear differential equation

dependent variable y.

all its derivatives $\frac{dy}{dx}, \frac{d^2y}{dx^2} \ldots$ appear in degree 1

and also if they are not multiplied together.

Nonlinear (i) $y + y\frac{dy}{dx} + x = 2$ (iii) $x\frac{dy}{dx} + y = 0$ linear equation

Nonlinear (ii) $y^2 + x\, dy/dx + x = 0$

$$\frac{dy}{dx} = \frac{2x+3y+4}{4x+6y+5}$$

$$\frac{1}{3}\left[\frac{dt}{dx} - 2\right] = \frac{t+4}{2t+5}$$

$$\frac{dt}{dx} = \frac{3(t+4)}{2t+5} + 2$$

$$\frac{dt}{dx} = \frac{3t+12+4t+10}{2t+5} = \frac{7t+22}{2t+5}$$

$$\frac{dt}{dx} = \frac{7t+22}{2t+5}$$

$$\frac{2t+5}{7t+22}\,dt = dx \longrightarrow$$ Variable separable.

$$\int \frac{f'(t)}{f(t)}\,dt \leftarrow \int \frac{2t+5}{7t+22}\,dt = \int dx$$

Partial fraction

$$\frac{2t+5}{7t+22} = A + \frac{B}{7t+22}$$

$$\frac{2t+5}{7t+22} = \frac{A(7t+22)+B}{7t+22}$$

$$A(7t+22) + B = 2t+5$$

let $t=0$ $\quad 22A + B = 5 \longrightarrow ①$

let $t=1$ $\quad 29A + B = 7 \longrightarrow ②$

① − ②

$$-7A = -2$$

$$\boxed{A = 2/7}$$

First order linear DE (Leibnitz equation)

It is in the form of

$$\frac{dy}{dx} + P(x)y = Q(x)$$

P, Q are functions of x.

Method to solve linear DE of 1st order

Integrating factor $\Longrightarrow M(x)$

$$M(x)\left[\frac{dy}{dx} + P(x)y = Q(x)\right]$$

$$M(x)\frac{dy}{dx} + M(x)P(x)y = M(x)Q(x)$$

Consider

$$\frac{d}{dx}\left(M(x)y\right) = M(x)\frac{dy}{dx} + M'(x)y$$

$$M'(x) = M(x)P(x)$$

$$\frac{M'(x)}{M(x)} = P(x)$$

$$\int \frac{M'(x)}{M(x)}\,dx = \int P(x)\,dx$$

$$\ln M(x) = \int P(x)\,dx + c$$

$$M(x) = e^{\int P(x)\,dx}$$

$$\frac{dy}{dx} + \underline{P(x)}\, y = Q(x)$$

$$\boxed{I.F = e^{\int P(x)\,dx}}$$

$$e^{\int P(x)\,dx} \frac{dy}{dx} + e^{\int P(x)\,dx} P(x)\, y = e^{\int P(x)\,dx} Q(x)$$

$$\frac{d}{dx}\left[e^{\int P(x)dx} \cdot y \right] = e^{\int P(x)dx} Q(x)$$

$$\boxed{e^{\int P(x)\,dx}\, y = \int e^{\int P(x)\,dx} Q(x) + c}$$

Sol:-

$$\boxed{If\, y = \int If \cdot Q(x)\, dx + c}$$

(or)

$$\frac{dx}{dy} + P(y)\, x = Q(y)$$

If :- $e^{\int P(y)\,dy}$

Sol.

$$\boxed{If\, x = \int If\, Q(y)\, dy + c}$$

Ex:

$$x\,dy - (y - 3x^2)\,dx = 0$$

$$x\,dy = (y - 3x^2)\,dx \qquad \frac{dy}{dx} = \frac{y - 3x^2}{x}$$

$$x\frac{dy}{dx} - y = -3x^2$$

$$\frac{dy}{dx} - \frac{y}{x} = -3x \longrightarrow \text{First order LDE}$$

$$\frac{dy}{dx} + P(x)\,y = Q(x) \qquad P(x) = \frac{-1}{x}$$

$$Q(x) = -3x$$

$$I.F = e^{\int P(x)\,dx} = e^{\int \frac{-1}{x}dx} = e^{-\ln x} = \frac{1}{x}$$

Sol:-

$$IF\,y = \int IF\,Q(x)\,dx + c$$

$$\frac{1}{x}y = \int \frac{1}{\cancel{x}}(-3\cancel{x})\,dx + c$$

$$\frac{y}{x} = -3x + c$$

$$\boxed{\frac{y}{x} + 3x = c} \longrightarrow \text{General sol}$$

$y(1) = 2$

$x = 1 \quad y = 2$

$2 + 3 = c$

$c = 5$

$$\frac{y}{x} + 3x = 5 \longrightarrow \text{Particular solution}$$

$\Rightarrow$ Non Linear equation but reducible to linear

(Bernoulli's equation

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \longrightarrow y^n \to \text{non linear term}$$

$n \neq 1$

$$\frac{1}{y^n}\frac{dy}{dx} + \frac{P(x)}{y^{n-1}} = Q(x)$$

Substitute $y^{1-n} = z$

$$\frac{1}{y^{n-1}} = z \qquad (1-n)y^{-n}\frac{dy}{dx} = dz/dx$$

$$\frac{(1-n)}{y^n}\frac{dy}{dx} = dz/dx$$

$$\frac{1}{y^n}\frac{dy}{dx} = \frac{1}{1-n}\frac{dz}{dx}$$

$$\frac{1}{1-n}\frac{dz}{dx} + P(x)z = Q(x)$$

$$\frac{dz}{dx} + P(x)z = Q(x) \qquad \frac{dz}{dx} + \boxed{P(x)(1-n)}z = (1-n)Q(x)$$

I.F $\to e^{\int P(x)(1-n)\,dx}$

$z = y^{1-n}$

Sol:- I.F $\cdot z = \int$ I.F. $Q(x)\,dx + c$

I.F $y^{1-n} = \int$ I.F $Q(x)\,dx + c$

ex:-

$$x\frac{dy}{dx} + y = x^3 y^6$$

$$\frac{dy}{dx} + \frac{y}{x} = x^3 y^6$$

$$\frac{1}{y^6}\frac{dy}{dx} + \frac{y}{xy^6} = x^3$$

$$\frac{1}{y^6}\frac{dy}{dx} + \frac{1}{x\boxed{y^5}} = x^3$$

$$z = \frac{1}{y^5} = y^{-5}$$

$$-\frac{1}{5}\frac{dz}{dx} + \frac{z}{x} = x^3$$

$$\frac{dz}{dx} = -5y^{-6}\frac{dy}{dx}$$

$$\frac{dz}{dx} - \frac{5z}{x} = -5x^3$$

$$-\frac{1}{5}\frac{dz}{dx} = \boxed{\frac{1}{y^6}\frac{dy}{dx}}$$

$$\frac{dz}{dx} - \frac{5z}{x} = -5x^3$$

$$\text{If} = e^{\int -\frac{5}{x}dx} = e^{-5\ln x} = \frac{1}{x^5}$$