$\rightarrow \quad (D+2)(D-1)^2 y = e^{-2x} + 2\sinh x$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad e^{x} - e^{-x}$

<u>To find y_{cf}</u>

$(D+2)(D-1)^2 = 0$

roots

$= -2, 1, 1$

$$\boxed{y_{cf} = c_1 e^{-2x} + (c_2 + c_3 x) e^{x}}$$

$$(D+2)(D-1)^2 y = e^{-2x} + e^{x} - e^{-x}$$

$a = -2$

$$y_{PI} = \frac{1}{(D+2)(D-1)^2} e^{-2x} + \frac{e^{x}}{(D+2)(D-1)^2} - \frac{e^{-x}}{(D+2)(D-1)^2}$$

$a = -2 \qquad a = 1 \qquad a = -1$

$P(-2) = 0$

$p'(D) =$

$(D-1)^2 + (D+2)\,2(D-1)$

$p'(-2) = (-3)^2 + 0$

$= 9 \neq 0$

$D = a$

$= -2$

$D = 1$

$$= \frac{x e^{-2x}}{P'(-2)} + \frac{x^2 e^{x}}{p''(1)} - \frac{e^{-x}}{P(-1)}$$

$$= \frac{x e^{-2x}}{9} + \frac{x^2 e^{x}}{6} - \frac{e^{-x}}{(1)(4)}$$

$$= \frac{x e^{-2x}}{9} + \frac{x^2 e^{x}}{6} - \frac{e^{-x}}{4}$$

$P(D) = (D+2)(D-1)^2$

$P(+1) = 0$

$P'(D) = (D-1)^2 + (D+2)2(D-1)$

$P'(1) = 0$

$P''(D) = 2(D-1) + (D+2)(2)$

$\quad + 2(D-1)(1)$

$P''(1) = 0 + 6 + 0$

$P''(1) = 6 \neq 0$

$$y_{complete} = y_{cf} + y_{PI}$$

$=$

Rules for finding Particular Integral

$$(D^n + D^{n-1} + \cdots D)y = Q(x)$$

$$P(D)y = Q(x)$$

an $$y_{PI} = \frac{1}{P(D)} Q(x)$$

1. Case I :- $Q(x) = e^{ax}$

$$P(D)y = e^{ax}$$

$$y_{PI} = \frac{1}{P(D)} e^{ax}$$

$$P(D)\, e^{ax} = P(a)\, e^{ax}$$

$$\frac{1}{P(D)} P(D)\, e^{ax} = \frac{1}{P(D)} P(a)\, e^{ax}$$

$$e^{ax} = P(a) \frac{1}{P(D)} e^{ax}$$

$$y_{PI} = \frac{1}{P(D)} e^{ax} = \frac{e^{ax}}{P(a)} \qquad P(a) \neq 0$$

If $P(a) = 0$ $\qquad y_{PI} = \frac{x e^{ax}}{P'(a)} \qquad P'(a) \neq 0$

If $P'(a) = 0$

$$y_{PI} = \frac{x^2 e^{ax}}{P''(a)} \qquad P''(a) \neq 0$$

Case (ii) :- If $Q(x) = \sin(ax+b)$ (or) $\cos(ax+b)$

$$P(D)y = \sin(ax+b) \text{ (or) } \cos(ax+b)$$

$$P(D)y = \sin(ax+b)$$

$$y_{PI} = \frac{1}{P(D)} \sin(ax+b)$$

$$D \sin(ax+b) = \frac{d}{dx} \sin(ax+b) = a\cos(ax+b)$$

$P(-a^2) \neq 0$

$$D^2 \sin(ax+b) = -a^2 \sin(ax+b)$$

$$D^3 \sin(ax+b) = -a^3 \cos(ax+b)$$

$$D^4 \sin(ax+b) = a^4 \sin(ax+b)$$

$$P(D^2) \sin(ax+b) = P(-a^2) \sin(ax+b)$$

$D^2 = -a^2$

$$\frac{1}{P(D^2)} \sin(ax+b) = \frac{1}{P(-a^2)} \sin(ax+b)$$

$P(-a^2) \neq 0$

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{i(ax+b)} = e^{ib} e^{iax} = e^{ib}(\cos ax + i \sin ax)$$

$$= \text{Img part of } e^{i(ax+b)}$$

Img $\frac{1}{P(D)} e^{iax+b} = e^{ib} \frac{1}{P(D^2)} e^{iax} \xrightarrow{D = ia} e^{ib} \frac{e^{iax}}{P((ia)^2)} = \frac{e^{ib} e^{iax}}{P(-a^2)}$

$P(-a^2) \neq 0$

$$= \frac{x e^{ib} e^{iax}}{P'(-a^2)}$$

$\frac{1}{P(D^2)} \sin(ax+b) \underset{D^2 = -a^2}{=} \frac{\sin(ax+b)}{P(-a^2)} \xrightarrow{P(-a^2)=0} \frac{x \sin(ax+b)}{P'(-a^2)}$

$P(-a^2) \neq 0$

$P'(-a^2) \neq 0$

$\boxed{D^2 = -a^2}$

$$\frac{x^2 \sin(ax+b)}{P''(-a^2)}$$

Case 2

If $Q(x) = \sin(ax+b)$ (or) $\cos(ax+b)$

$$\frac{1}{P(D^2)} \sin(ax+b) \text{ (or) } \cos(ax+b) = \frac{\sin(ax+b) \text{ (or) } \cos(ax+b)}{P(-a^2)}$$

$\boxed{D^2 = -a^2}$

replace

If $P(-a^2) = 0$

$$\frac{1}{P(D^2)} \sin(ax+b) \text{ (or) } \cos(ax+b) = \frac{x\left[\sin(ax+b) \text{ (or) } \cos(ax+b)\right]}{P'(-a^2)}$$

$P'(-a^2) \neq 0$

ex:- $(D^3+1)\,y = \cos(2x+1)$

Find Particular integral

$$y_{PI} = \frac{1}{D^3+1}\cos(2x+1) \rightarrow \text{form } \cos(ax+b)$$

$a = 2$

$$D^2 = -a^2 = -2^2 = -4$$

$$= \frac{1}{D^2(D)+1}\cos(2x+1)$$

$$\frac{1}{D-\alpha}Q(x) = e^{\alpha x}\int Q(x)\,e^{-\alpha x}\,dx$$

$$= \frac{1}{(-4)D+1}\cos(2x+1) = \frac{1}{1-4D}\cos(2x+1)$$

$$= \frac{1\,(1+4D)}{(1-4D)(1+4D)}\cos(2x+1)$$

$$= \frac{(1+4D)\cos(2x+1)}{1-16D^2} \xrightarrow{D^2=-4} \frac{(1+4D)\cos(2x+1)}{1-16(-4)}$$

$D = \frac{d}{dx}$ $\quad = \frac{(1+4D)\cos(2x+1)}{65} = \frac{\left(1+4\frac{d}{dx}\right)\cos(2x+1)}{65}$

$$= \frac{1}{65}\left[\cos(2x+1) + 4(2) - \sin(2x+1)\right]$$

$$y_{PS} = \frac{1}{65}\left[\cos(2x+1) - 8\sin(2x+1)\right]$$

(2) $(D^3+4D) = \sin 2x$. Solve

$y_{cf} =$ $D^3 + 4D = 0$

$D(D^2+4) = 0$ roots $= 0, \pm 2i$

$y_{cf} = c_1 e^{0x} + c_2 \cos 2x + c_3 \sin 2x$

$= c_1 + c_2 \cos 2x + c_3 \sin 2x$

To find y_{PI}

$$y_{PI} = \frac{1}{D^3+4D} \sin 2x$$

$a = 2$

$D^2 = -a^2 = -4$

$p(D) = D^3 + 4D$

$p(-4) = 0$

$p'(D) = 3D^2 + 4$

$p'(-a^2) = p'(-4) = 3(-4)+4$
$= -12+4 = -8$

$$\frac{1}{D(D^2+4)} \sin 2x = \frac{x \sin 2x}{p'(-4)}$$

$$\left.\frac{1}{D(-4+4)} \sin 2x\right\} x = \frac{x \sin 2x}{-8}$$

$$= -\frac{x \sin 2x}{8}$$

$\boxed{\text{Case (iii)}}$ If $Q(x) = x^m$ $\quad x^3+x^2+2$

$P(D)y = x^m$ $\quad 1+D \quad D+D^2+$

$$y_{PI} = \frac{1}{P(D)} x^m = [P(D)]^{-1} x^m$$

expand using binomial expansions till D^m term

$(1+x)^{-1} = 1 - x + x^2 + \ldots$

$(1-x)^{-1} = 1 + x + x^2 + \ldots$

$(1+x)^{-2} = 1 - 2x$

$\underline{\underline{ex}}$:- $(D^2+D)y = x^2+2x+4$ $\qquad \underline{\underline{\phi(x) = x^n}}$

$$y_{PI} = \frac{1}{(D^2+D)} x^2+2x+4$$

$$= \frac{1}{D(1+D)} x^2+2x+4$$

$$= \frac{1}{D}(1+D)^{-1} x^2+2x+4$$

$$= \frac{1}{D}\left[1-D+D^2- \ldots\ldots\right](x^2+2x+4)$$

$$= \frac{1}{D}\left[(x^2+2x+4) - \frac{d}{dx}(x^2+2x+4) + \frac{d^2}{dx^2}(x^2+2x+4)\right]$$

$$= \frac{1}{D}\left[x^2+\cancel{2x}+4-(\cancel{2x}+2)+(2)\right]$$

$D = \frac{d}{dx}$

$\frac{1}{D-\alpha} = e^{\alpha x}\int e^{-\alpha x}Q(x)$

$= \int Q(x)$

$$= \frac{1}{D}\left[x^2+4\right] = \int x^2+4\,dx = \frac{x^3}{3}+4x$$

$$y_{complete} = y_{cf} + y_{PI}$$

$$= c_1 + c_2 e^{-x} + \frac{x^3}{3}+4x$$

Case iv :- If $Q(x) = e^{ax} V(x)$

$$P(D)\,y = e^{ax} V(x)$$

$$y_{PS} = \frac{1}{P(D)} e^{ax} V(x)$$

let take $u(x)$

$Q(x) = e^{ax} u(x)$

$$D\, e^{ax} u(x) = \frac{d}{dx}\left(e^{ax} u(x)\right) = a e^{ax} u(x) + e^{ax} \frac{d\,u(x)}{dx}$$

$$= a e^{ax} u(x) + e^{ax} D u(x)$$

$$D\, e^{ax} u(x) = e^{ax} (D+a) u(x)$$

* exponential shifting : $P(D)\, e^{ax} u(x) = e^{ax} P(D+a) u(x)$

$$P(D)\, e^{ax} u(x) = e^{ax} P(D+a) u(x)$$

$$\therefore P(D-a)\, e^{ax} u(x) = e^{ax} P(D) u(x)$$

$$P(D)\, e^{ax} u(x) = e^{ax} \boxed{P(D+a)\, u(x)}$$

take = $P(D+a)\, u(x) = V(x)$

$$u(x) = \frac{1}{P(D+a)} V(x)$$

$\frac{1}{P(D)}$

$$\frac{1}{P(D)} P(D)\, e^{ax} u(x) = \frac{1}{P(D)}\left[e^{ax} P(D+a)\, u(x)\right]$$

$$e^{ax} \frac{1}{P(D+a)} V(x) = \frac{1}{P(D)}\left[e^{ax} P(D+a) \cdot \frac{1}{P(D+a)} V(x)\right]$$

$$\boxed{\frac{1}{P(D)} e^{ax} V(x) = e^{ax} \frac{1}{P(D+a)} V(x)}$$ **

ex:- $(D^2 - 2D + 4)y = e^x \cos x$

Find particular integral

$$y_{PI} = \frac{1}{D^2 - 2D + 4} e^x \cos x \rightarrow e^{ax} V(x)$$

$a = 1$

$P(D) \longrightarrow P(D+1)$

$$e^x \cdot \frac{1}{P(D+1)} \cos x = e^x \frac{1}{(D+1)^2 - 2(D+1) + 4} \cos x$$

$$= e^x \cdot \frac{1}{D^2 + 1 + \cancel{2D} - \cancel{2D} - 2 + 4} \cos x = e^x \cdot \frac{1}{D^2 + 3} \cos x \rightarrow \cos(ax+b)$$

$a = 1$

$D^2 = -a^2 = -1$

$$= e^x \cdot \frac{1}{(-1) + 3} \cos x$$

$$\boxed{y_{PI} = \frac{e^x \cos x}{2}}$$

Rules of finding Particular integral

$Q(x)$

1. $Q(x) = e^{ax}$

2. $Q(x) = \sin(ax+b)$ (or) $\cos(ax+b)$

3. $Q(x) = x^m$

4. $Q(x) = e^{ax} V(x)$

Case 5 :- $\boxed{\text{If} \quad Q(x) = x V(x)}$

$p(D) y = x v(x)$

$$\boxed{y_{PI} = \frac{1}{P(D)} x v(x)}$$

$Q(x) = x u(x)$

$$D(x u(x)) = \frac{d}{dx}(x u) = u + x D u$$

$$D^2(x u) = D(u + x D u) = Du + x D^2 u + Du$$

$\vdots$

$$\left\{ D^n(uv) = (D^n u) v + {}^nc_1 (D^{n-1} u)(Dv) + {}^nc_2 (D^{n-2} u)(D^2 v) + \cdots {}^nc_n (u) D^n v \right.$$

by leibnitz theorem of successive differentiation

$D^n = P(D)$

$D^n = P(D)$

$n D^{n-1} = P'(D)$

$$D^n(u x) = (D^n u) x + {}^nc_1 (D^{n-1} u)(1) + 0 + \cdots 0$$

$n D^{n-1}$

$$\boxed{\begin{array}{l} D^n = P(D) \\ P'(D) = n D^{n-1} \end{array}}$$

$$\downarrow$$

$$P(D)(u x) = x P(D) u + P'(D) u$$

$n D^{n-1}$

$$\downarrow$$

let $P(D) u = V(x) \Rightarrow \boxed{u = \frac{1}{P(D)} V(x)}$

$$P(D)\left[\frac{1}{P(D)} x V(x)\right] = x P(D) \cdot \frac{1}{P(D)} V(x) + P'(D) \cdot \frac{1}{P(D)} V(x)$$

$$\frac{1}{P(D)} x V(x) = \left(x - \frac{1}{P(D)} P'(D)\right) \frac{1}{P(D)} V(x) \quad //$$

$$\boxed{\frac{1}{P(D)}\left(x\,V(x)\right) = \left[x - \frac{1}{P(D)}P'(D)\right]\frac{1}{P(D)}V}$$

ex:- $(D+1)^2 y = x\cos x$ Solve

To find y_{cf}

$$(D+1)^2 y = 0$$

$$(D+1)^2 = 0 \Rightarrow \text{roots} = -1, -1$$

$$y_{cf} = (C_1 + C_2 x)e^{-x}$$

To find particular integral

$$(D+1)^2 y = x\cos x$$

$$y_{PI} = \frac{1}{(D+1)^2}x\cos x \qquad P(D) = (D+1)^2$$

$$y_{PI} = \left[x - \frac{1}{P(D)}P'(D)\right]\frac{1}{P(D)}V(x)$$

$$= \left[x - \frac{1}{(D+1)^2}2(D+1)\right]\frac{1}{(D+1)^2}\cos x$$

$$= \left[x - \frac{2}{(D+1)}\right]\frac{1}{D^2+2D+1}\cos x \rightarrow \cos(ax+b)$$

$a = 1$

$D^2 = -a^2 = -1$

$$-1 + 2D + 1 = \frac{1}{2D}$$

$$= \left(x - \frac{2}{D+1}\right) \frac{1}{-1+2D+1} \cos x$$

$$D^2 = -1$$

$$\left(x - \frac{2}{D+1}\right) \frac{1}{2D} \cos x = \left(x - \frac{2}{D+1}\right) \frac{1}{2} \int \cos x \, dx$$

$$= \left(x - \frac{2}{D+1}\right) \frac{1}{2} \sin x$$

$$= \frac{x}{2} \sin x - \frac{1}{D+1} \sin x$$

$$D^2 = -a^2$$

$$\frac{1}{D-\alpha} Q(x) = e^{\alpha x} \int e^{-\alpha x} \sin x \, dx$$

$$= \frac{x}{2} \sin x - \frac{1}{D+1} \times \frac{(D-1)}{(D-1)} \sin x$$

$$= \frac{x}{2} \sin x - (D-1) \frac{1}{D^2-1} \sin x \longrightarrow \begin{matrix} \sin(ax+b) \\ D^2 = -a^2 \end{matrix} \quad a = 1$$

$$= \frac{x}{2} \sin x - (D-1) \frac{1}{-1-1} \sin x$$

$$= \frac{x}{2} \sin x + \frac{1}{2} (D-1) \sin x$$

$$= \frac{x}{2} \sin x + \frac{1}{2} (\cos x - \sin x)$$