→ Higher order DE with Variable coeff

① Cauchy Euler eq
② Legendre's eq

} → reduce Constant coeff by using substitution
↓
$y_{sample} = y_{cf} + y_{PS}$

→ $x^n \frac{d^n y}{dx^n} + x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots \; x\frac{dy}{dx} + y = Q(x)$

Cauchy Euler eq

$x = e^t$

$\log x = t$ $\quad \boxed{\frac{dt}{dx} = \frac{1}{x}}$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt}$$

$$\boxed{x\frac{dy}{dx} = \frac{dy}{dt} = Dy} \qquad D = \frac{d}{dt}$$

→ $$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right)$$

$$= \frac{-1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dt}\right)$$

$$= \frac{-1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d}{dt}\left(\frac{dy}{dt}\right)\frac{dt}{dx}$$

$D = \frac{d}{dt}$

$$= \frac{-1}{x^2}\frac{dy}{dt} + \frac{1}{x^2}\frac{d^2 y}{dt^2}$$

$$= \frac{1}{x^2}\left[-D + D^2\right]y$$

$$x^2\frac{d^2 y}{dx^2} = D(D-1)y$$

$$x^3 \frac{d^3y}{dx^3} = D(D-1)(D-2)y$$

$\xrightarrow{ex}$

$$x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = \log x$$

$x = e^t$

$x\frac{dy}{dx} = Dy \qquad D = \frac{d}{dt} \qquad x^2\frac{d^2y}{dx^2} = D(D-1)y$

$$D(D-1)y - Dy + y = t$$

$$(D^2 - D - D + 1)y = t$$

$$(D^2 - 2D + 1)y = t$$

y_{cf} :-

$$D^2 - 2D + 1 = 0$$

$$D = 1, 1$$

$$y_{cf} = (c_1 + c_2 t)e^t$$

To find y_{PI}

undetermined coeff

$y_{trial} = c_0 + c_1 t$

$$(D^2 - 2D + 1)y = t$$

$$(D-1)^2 y = t$$

$$y_{PI} = \frac{1}{(D-1)^2} t = (1-D)^{-2} t$$

$$= (1 + 2D + \cdots\cdots)t$$

$\frac{dt}{dt}$

$$= t + 2(1) = \underline{\underline{t+2}}$$

$x = e^t$

$t = \log x$

$$y_{complete} = y_{cf} + y_{PI} = (c_1 + c_2 t)e^t + t + 2 = \underline{(c_1 + c_2 \ln x)x + \ln x + 2}$$

II $\rightarrow$ Legendre's eq

$$(ax+b)^n \frac{d^n y}{dx^n} + k_1 (ax+b)^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \cdots \quad k_n y = Q(x)$$

$$ax+b = e^t$$

$$\log(ax+b) = t$$

$$\boxed{\frac{dt}{dx} = \frac{1}{ax+b}(a) = \frac{a}{ax+b}}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{a}{ax+b} \frac{dy}{dt}$$

$$(ax+b) \frac{dy}{dx} = a \frac{dy}{dt} \qquad D = \frac{d}{dt}$$

$$\boxed{(ax+b) \frac{dy}{dx} = a\,Dy} \qquad x\frac{dy}{dx} = Dy$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{a}{ax+b} \frac{dy}{dt}\right)$$

$$= \frac{-a^2}{(ax+b)^2} \frac{dy}{dt} + \frac{a}{ax+b} \frac{d}{dx}\left(\frac{dy}{dt}\right)$$

$$= \frac{-a^2}{(ax+b)^2} \frac{dy}{dt} + \frac{a}{ax+b} \frac{d}{dt}\left(\frac{dy}{dt}\right) \frac{dt}{dx}$$

$$a/ax+b$$

$$= \frac{-a^2}{(ax+b)^2} \frac{dy}{dt} + \frac{a^2}{ax+b} \frac{d^2y}{dt^2}$$

$$x^2\frac{d^2y}{dx^2} = D(D-1)y$$

$$(ax+b)^2 \frac{d^2y}{dx^2} = -a^2 Dy + a^2 D^2 y = a^2(D^2 - D)y$$

$$= a^2 D(D-1)y$$

$$(ax+b)^3\frac{d^3y}{dx^3} = a^3D(D-1)(D-2)y$$

ex:-

$$(1+x)^2\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 2\sin(\log(1+x))$$

$\downarrow$

$$(ax+b)^2\frac{d^2y}{dx^2} + (ax+b)\frac{dy}{dx} + y = 2\sin\log(ax+b)$$

$a=1$
$b=1$

$1+x = e^t$

$\ln(1+x) = t$

$$(1+x)\frac{dy}{dx} = aDy = Dy$$

$$(1+x)^2 d^2y/dx^2 = a^2D(D-1)y = D(D-1)y$$

$$D(D-1)y + Dy + y = 2\sin t$$

$$(D^2 - \cancel{D} + \cancel{D} + 1)y = 2\sin t$$

$$(D^2+1)y = 2\sin t$$

y_{cf}

$$(D^2+1) = 0$$

$$D = \pm i$$

$1+x = e^t$

$$y_{cf} = c_1\cos t + c_2\sin t$$

$$= c_1\cos(\log x) + c_2\sin(\log x)$$

y_{pI}

$t\frac{1}{D}\sin t$

$t\cdot\frac{\sin t}{D}$

$t\int\sin t = -t\cos t$

$$(D^2+1)y = 2\sin t \qquad \sin(ax+b)$$

$$y_{pI} = \frac{1}{D^2+1}2\sin t \xrightarrow{D^2=-a^2=-1} \frac{2\sin t}{-1+1} \rightarrow \times$$

$$-t\cos t = \frac{t\cdot\cancel{2}\sin t}{\cancel{2}D\times D} = t\frac{D\sin t}{D^2} = \frac{t}{-1}\frac{d}{dt}\sin t$$

$$y_{complete} = c_1 \cos(\ln(1+x)) + c_2 \sin(\ln(1+x)) - t \cos t$$

$$= c_1 \cos(\ln(1+x)) + c_2 \sin(\ln(1+x)) - \ln(1+x) \cos(\ln(1+x))$$

$e^t = 1+x$

$t = \ln(1+x)$

Partial differential equation

ODE → single independent
PDE → more than 2

dependent variable, independent variable

its derivatives

$y = f(x, z)$

$$F\left(y, x, t, z \ldots \frac{\partial y}{\partial x}, \frac{\partial y}{\partial t}, \frac{\partial y}{\partial z} \ldots\right) = 0$$

order → Highest partial derivative involved in the eq

$$\frac{\partial y}{\partial x} + \frac{\partial y}{\partial t} + y = 0 \longrightarrow \text{order} = 1$$

$\frac{\partial^2 y}{\partial x^2}, \frac{\partial^2 y}{\partial x \partial z}$

$\frac{\partial^2 y}{\partial z \partial x}, \frac{\partial^2 y}{\partial z^2}$

$$\frac{\partial^2 y}{\partial x \partial z} + \frac{\partial y}{\partial z} + y = 0 \longrightarrow \text{order} = 2$$

degree → Highest power of highest partial derivative involved in the eq such that eq is independent of fractions and radicals

homogeneous / Non homogeneous → every term in your eq should have dependent variable

$$\frac{\partial y}{\partial t} + \frac{\partial y}{\partial x} + y = 2 \longrightarrow \text{Non hom}$$

$$\frac{\partial y}{\partial t} + \frac{\partial^2 y}{\partial x \partial t} + 2y = 0 \longrightarrow \text{homogen}$$

→ ① Linear →

Quasilinear

Semilinear

② Non linear

Linear PDE :- dependent variable, derivatives have degree 1 and also they are not multiplied together

$$\frac{\partial y}{\partial x} + \frac{\partial^2 y}{\partial x^2} + y = \sin x \quad \text{linear}$$

degree 2

$$\left(\frac{\partial y}{\partial x}\right)^2 + \frac{\partial y}{\partial t} = 2 \rightarrow \text{not linear}$$

Quasi linear → only highest derivative in linear and lower order derivative can be non linear

④

$$\frac{\partial^2 u}{\partial x^2} + u\frac{\partial u}{\partial y} + u = 0 \qquad \frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial y}\right)^2 + u = 2$$

Semilinear → only highest derivative is linear and lower derivatives can be non linear

coefficient of derivative only x, y

degree

$$\text{degree} \; \frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial y} + u^2 = 0 \rightarrow \text{degree } ②$$

Semilinear

① $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2 \rightarrow$ linear

→ ② $x\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} + u^2 = 2 \rightarrow$ Semilinear

③ $\frac{\partial^2 u}{\partial x^2} + u\frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial y}\right)^2 = 2 \rightarrow$ Quasilinear

degree

④ $\left(\frac{\partial u}{\partial x}\right)^3 + u = 0 \rightarrow$ Non linear