

⇒ First order First degree DE
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 power of highest derivative

$$F(x, y, \frac{dy}{dx})$$

① Variable separable → $f(x)dx = f(y)dy$
 by integrating on both sides

$$\rightarrow \frac{dy}{dx} = e^{3x} e^{-2y} + x^2 e^{-2y}$$

N.B. = ratio

$$\left(\frac{dy}{dx} \right) \cdot e^{2y} = e^{2y} (e^{3x} + x^2)$$

$$\frac{dy}{e^{2y}} = (e^{3x} + x^2) dx$$

$$\int e^{2y} dy = \int (e^{3x} + x^2) dx$$

$$\frac{e^{2y}}{2} = \frac{e^{3x}}{3} + \frac{x^3}{3} + C$$

General solution

$$y(0) = 1 \rightarrow \underline{\underline{\text{initial}}}$$

$$\text{at } x=0 \quad y=1$$

$$\frac{e^2}{2} = \frac{e^0}{3} + 0 + C$$

$$\frac{e^2}{2} = 1 + C$$

$$C = \frac{e^2}{2} - 1$$

$$\frac{e^{2y}}{2} = \frac{e^{3x}}{3} + \frac{x^3}{3} + \frac{e^2}{2} - 1$$

→ Particular solution

II Reducible to Variable Separable

Substitution \longrightarrow Reduce to V.S. $f(x)dx = g(y)dy$

$$\frac{dy}{dx} = (4x+y+1)^2$$

Substitute
diff w.r.t x

$$4x+y+1 = t \longrightarrow$$
$$4 + \frac{dy}{dx} + 0 = \frac{dt}{dx}$$

$$\boxed{\frac{dy}{dx} = \frac{dt}{dx} - 4}$$

$$\frac{dy}{dx} = (4x+y+1)^2$$

\downarrow

$$\frac{dt}{dx} - 4 = t^2$$

$$\frac{dt}{dx} = t^2 + 4$$

$$\frac{dt}{t^2+4} = dx \longrightarrow \text{Variable Separable}$$

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) \quad \int \frac{dL}{t^2+4} = \int dx$$

$$\Rightarrow \frac{1}{2} \tan^{-1}\left(\frac{t}{2}\right) = x + c$$

$$\Rightarrow \boxed{\frac{1}{2} \tan^{-1}\left(\frac{4x+y+1}{2}\right) = x + c} \longrightarrow \text{General sol}$$

$$y(0) = 1 \longrightarrow$$

$$\text{at } x=0 \Rightarrow y=1$$

$$\frac{1}{2} \tan^{-1}\left(\frac{0+1+1}{2}\right) = 0 + c$$

$$\frac{1}{2} \tan^{-1}(1) = C$$

$$\frac{1}{2} \left(\frac{\pi}{4} \right) = C$$

$$C = \pi/8$$

$$\frac{1}{2} \tan^{-1} \left(\frac{4x+y+1}{2} \right) = x + \pi/8 \rightarrow \text{Particular sol}$$

III

Homogeneous DE

$$\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$$

arch homogeneous functions of same degree

$$f(tx, ty) = t^a f(x, y)$$

$$\underline{f(x, y)} = x + y$$

$$f(x, y) = x + y + 2$$

$$f(tx, ty) = tx + ty = t(x + y) = t f(x, y)$$

$$f(tx, ty) = \underline{tx + ty + 2} \neq t^a f(x, y)$$

Not homogeneous

$$\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)} \xrightarrow{\text{substitution}} y = vx \xrightarrow{\text{for derivative}} \text{reduces to variable separation}$$

ex:-

$$(x^2 - y^2) dx = xy dy$$

$$\frac{dy}{dx} = \frac{x^2 - y^2}{xy}$$

$$= \frac{t^2(x^2 - y^2)}{t^2 xy}$$

$$x^2 - y^2 \rightarrow (tx)^2 - (ty)^2 = t^2(x^2 - y^2)$$

$$xy \rightarrow (tx)(ty) = t^2 xy$$

This is homogeneous eq

$$\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$$

diff with $x \rightarrow y = vx$

$$\frac{dy}{dx} = v(1) + x \frac{dv}{dx}$$

$$\boxed{\frac{dy}{dx} = v + x \frac{dv}{dx}}$$

$$\frac{dy}{dx} = \frac{x^2 - y^2}{xy}$$

$$v + x \frac{dv}{dx} = \frac{x^2 - (vx)^2}{x(vx)} = \frac{x^2 - v^2 x^2}{vx^2} = \frac{\cancel{x^2}(1-v^2)}{v\cancel{x^2}}$$

$$v + x \frac{dv}{dx} = \frac{1-v^2}{v} \rightarrow \text{Variable separable eq}$$

$$x \frac{dv}{dx} = \frac{1-v^2}{v} - v = \frac{1-2v^2}{v}$$

$$\frac{x dv}{dx} = \frac{1-2v^2}{v}$$

$$\boxed{\int \frac{f'(x) dx}{f(x)} = \log f(x) + C}$$

$$\left[\frac{-1}{4} \right] \frac{-4v}{1-2v^2} dv = \int \frac{dx}{x}$$

diff $\frac{1-2v^2}{-4v}$

$$\frac{-1}{4} \int \frac{-4v}{1-2v^2} dv = \int \frac{dx}{x}$$

$$\frac{-1}{4} \log(1-2v^2) = \log x + C$$

$$-\frac{1}{4} \log(1-2v^4) = \log x + c$$

$$\log(1-2v^4) + 4\log x + 4c = 0$$

$$\log(1-2v^4) x^4 = -4c$$

$$y = vx$$

$$v = y/x$$

$$(1-2v^4) x^4 = e^{-4c}$$

$$\left(1 - \frac{2y^2}{x^2}\right) x^4 = A$$

→ Non Homogeneous → reducible to homogeneous

$$\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c'}$$

$$ax+ay+c$$

Case (i)

$$\frac{a}{a'} \neq \frac{b}{b'}$$

$$\checkmark x = X+h$$

$$\checkmark y = Y+k$$

$$dy = dY$$

$$\frac{dy}{dx} = \frac{dY}{dX}$$

h, k are constants

Case (ii)

$$\frac{a}{a'} = \frac{b}{b'}$$

$$ab' = a'b$$

$$ab' - a'b = 0$$

Substitute

$$ax+by = t$$

$$\frac{dy}{dx} = \frac{a(x+h) + b(y+k) + c}{a'(x+h) + b'(y+k) + c'}$$

$$= \frac{ax + \check{a}h + by + \check{b}k + \check{c}}{a'x + a'h + b'y + b'k + c'} = \frac{ax + by + (ah+bk+c)}{dx + b'y + (a'h+b'k+c')}$$

$$\begin{aligned} \textcircled{1} - ah + bk + c &= 0 \\ \textcircled{2} - a'h + b'k + c' &= 0 \end{aligned} \quad \left. \begin{array}{l} \text{Solve for } h, k \\ \text{such } \textcircled{1}, \textcircled{2} \text{ satisfies} \end{array} \right\}$$

$$\frac{dy}{dx} = \frac{ax+by}{a'x+b'y}$$

$$\downarrow$$

$$y = vx$$

\downarrow ~~Variable~~ reducible

$$\boxed{\begin{aligned} h &= \frac{bc' - b'c}{ab' - a'b} \\ k &= \frac{ca' - ac}{ab' - a'b} \end{aligned}}$$

ex:- $\textcircled{1} \frac{dy}{dx} = \frac{x+y-2}{-x+y-4}$

$$\frac{a}{a'} = \frac{b}{b'} \quad ?! \quad a=1, b=1$$

$$a'=-1, b'=1$$

$$\frac{a}{a'} \neq \frac{b}{b'}$$

Case i $-1 \neq 1$

$$x = x + h$$

$$y = y + k$$

$$\frac{dy}{dx} = \frac{x+y-2}{-x+y-4}$$

$$\frac{dy}{dx} = \frac{x+h+y+k-2}{-(x+h)+y+k-4} = \frac{x+y+(h+k-2)}{-x+y+(-h+k-4)}$$

$$h = -1$$

$$\textcircled{2} \frac{dy}{dx} = \frac{3y+2x+9}{6y+4x+5}$$

$$a=2 \quad b=3$$

$$a'=4 \quad b'=6$$

$$\frac{2}{4} = \frac{3}{6}$$

$$\frac{1}{2} = \frac{1}{2}$$

Case (ii)

$$\boxed{\frac{a}{a'} = \frac{b}{b'}}$$

$$\begin{aligned} h+k-2 &= 0 \\ -h+k-4 &= 0 \end{aligned}$$

$$\underline{\hspace{10em}}$$

$$2k-6=0$$

$$\boxed{k=3}$$

$$h+k-2=0$$

$$h+3-2=0$$

$$h = -1$$

$$x = x-1$$

$$y = y+3$$

$$\frac{dy}{dx} = \frac{x+y}{-x+y} \rightarrow \text{This is homogeneous eq}$$

$$\text{Substitute } y = vx$$

$$\frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$v + x \frac{dv}{dx} = \frac{x+vx}{-x+vx} = \frac{x(1+v)}{x(-1+v)}$$

$$v + x \frac{dv}{dx} = \frac{1+v}{v-1}$$

$$\frac{x dv}{dx} = \frac{1+v}{v-1} - v = \frac{1+v-v^2+v}{v-1}$$

$$\frac{x dv}{dx} = \frac{1+2v-v^2}{v-1}$$

$$\frac{-1}{2} \int \frac{-2(v-1)}{1+2v-v^2} dv = \int \frac{dx}{x}$$

$2-2v$
 $2(1-v)$

$$\frac{-1}{2} \int \frac{2(1-v)}{1+2v-v^2} dv = \int \frac{dx}{x}$$

$$-\frac{1}{2} \log(1+2v-v^2) = \log x + C$$

$$\log(1+2v-v^2) + 2\log x + 2C = 0$$

$$\log(1+2v-v^2) x^2 = -2C$$

$$(1+2v-v^2) x^2 = e^{-2C}$$

$$\left(1 + \frac{2y}{x} - \frac{y^2}{x^2}\right) x^2 = A$$

$$y = vx$$

$$v = \frac{y}{x}$$

$$x = x-1 \Rightarrow x = x+1$$

$$y = y+3$$

$$\underline{\underline{y = y-3}}$$

$$\left(1 + \frac{2(y-3)}{x+1} - \frac{(y-3)^2}{(x+1)^2}\right) (x+1)^2 = A$$

$$(y-3)^2 - 2(y-3)(x+1) - (x+1)^2 = A$$