Excercise 10.6

Q1. Prove that the line of centres of two intersecting circles subtends equal angles at the two points of intersection.

Solution

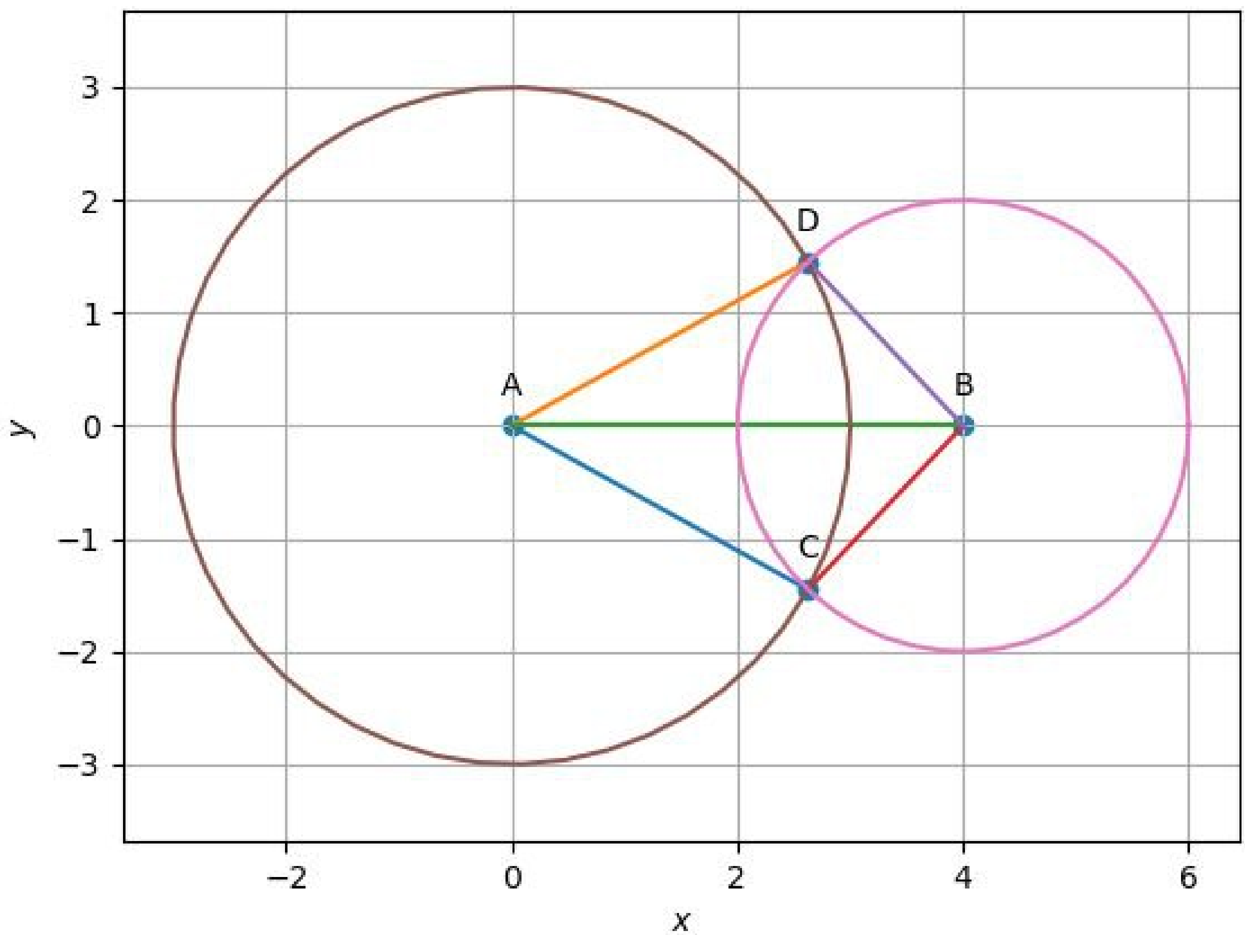

Figure 1:

Construction

Symbol	Values	Description
$\mathbf{A}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	Center of circle 1
r_1	3 units	Radius of the circle 1
$\mathbf{B}$	$\begin{pmatrix} 4 \\ 0 \end{pmatrix}$	Center of circle 2
r_2	2 units	Radius of circle 2
$\mathbf{e}_1$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	Standard basis vector 1
$\mathbf{e}_2$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	Standard basis vector 2

Verification:

The two circle equations are given by:

$$\|x\|^2 - 9 = 0 \tag{1}$$

$$\|x\|^2 - 8\mathbf{e}_1 + 12 = 0 \tag{2}$$

Equation of two conics is given by:

$$\mathbf{x}^\top \mathbf{V}_i \mathbf{x} + 2\mathbf{u}_i^\top \mathbf{x} + f_i = 0, \quad i = 1, 2 \tag{3}$$

Represent the two circles in conic form:

$$\mathbf{x}^\top \mathbf{x} - 9 = 0 \tag{4}$$

$$\mathbf{x}^\top \mathbf{x} + 2\begin{pmatrix} -4 & 0 \end{pmatrix} + 12 = 0 \tag{5}$$

On comparing above two equations with (3), we get:

$$\mathbf{V}_1 = \mathbf{I}, \mathbf{u}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, f_1 = -9 \tag{6}$$

$$\mathbf{V}_2 = \mathbf{I}, \mathbf{u}_2 = \begin{pmatrix} -4 \\ 0 \end{pmatrix}, f_2 = 12 \tag{7}$$

The intersection of the given conics is obtained as

$$\mathbf{x}^\top(\mathbf{V}_1 + \mu\mathbf{V}_2)\mathbf{x} + 2(\mathbf{u}_1 + \mu\mathbf{u}_2)^\top \mathbf{x} + (f_1 + \mu f_2) = 0 \tag{8}$$

$$\mathbf{V}_1 + \mu\mathbf{V}_2 = \begin{pmatrix} \mu + 1 & 0 \\ 0 & \mu + 1 \end{pmatrix} \tag{9}$$

$$\mathbf{u}_1 + \mu\mathbf{u}_2 = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{10}$$

$$f_1 + \mu f_2 = -21 \tag{11}$$

This conic is a single straight line if and only if,

$$\begin{vmatrix} \mathbf{V}_1 + \mu\mathbf{V}_2 & \mathbf{u}_1 + \mu\mathbf{u}_2 \\ (\mathbf{u}_1 + \mu\mathbf{u}_2)^\top & f_1 + \mu f_2 \end{vmatrix} = 0 \tag{12}$$

Substituting equation (9),(10) and (11) in equation (12) We get,

$$\implies \begin{vmatrix} 1+\mu & 0 & -4\mu \\ 0 & 1+\mu & 0 \\ -4\mu & 0 & -9+12\mu \end{vmatrix} = 0 \tag{13}$$

Solving the above equation we get,

$$\mu = -1 \tag{14}$$

Thus, the parameters for a straight line can be expressed as

$$\mathbf{V} = \mathbf{V}_1 + \mu\mathbf{V}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{15}$$

$$\mathbf{u} = \mathbf{u}_1 + \mu\mathbf{u}_2 = \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \tag{16}$$

$$f = f_1 + \mu f_2 = -21 \tag{17}$$

By substituting (15),(16) and (17) in conic equation, we get:

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + 2\begin{pmatrix} 4 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - 21 = 0 \tag{18}$$

By solving above equation(18), we get point of contact $\mathbf{q}$:

$$\mathbf{q} = \begin{pmatrix} \frac{21}{8} \\ 0 \end{pmatrix} \tag{19}$$

The points of intersection of line is given by:

$$L: \quad \mathbf{x} = \mathbf{q} + \kappa\mathbf{m} \quad \kappa \in \mathbb{R} \tag{20}$$

with the conic section, we have:

$$\mathbf{x}_i = \mathbf{q} + \kappa_i\mathbf{m} \tag{21}$$

where,

$$\kappa_i = \frac{1}{\mathbf{m}^T\mathbf{V}\mathbf{m}} \left(-\mathbf{m}^T\left(\mathbf{V}\mathbf{q}+\mathbf{u}\right) \pm \sqrt{\left[\mathbf{m}^T\left(\mathbf{V}\mathbf{q}+\mathbf{u}\right)\right]^2 - \left(\mathbf{q}^T\mathbf{V}\mathbf{q} + 2\mathbf{u}^T\mathbf{q} + f\right)\left(\mathbf{m}^T\mathbf{V}\mathbf{m}\right)}\right) \tag{22}$$

On substituting the below values in (22)

$$\mathbf{m} = \mathbf{e}_2, \mathbf{V} = \mathbf{I}, \mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{q} = \begin{pmatrix} \frac{13}{5} \\ 0 \end{pmatrix}, f = -9 \tag{23}$$

We get,

$$\kappa_i = -\frac{29}{20}, +\frac{29}{20} \tag{24}$$

The intersecting points $\mathbf{C}$ and $\mathbf{D}$ are given by:

$$\mathbf{C} = \mathbf{q} + \kappa_1\mathbf{m} = \begin{pmatrix} \frac{21}{8} \\ -\frac{29}{20} \end{pmatrix} \tag{25}$$

$$\mathbf{D} = \mathbf{q} + \kappa_2\mathbf{m} = \begin{pmatrix} \frac{21}{8} \\ \frac{29}{20} \end{pmatrix} \tag{26}$$

Check whether the intersection angles ∠ADB and ∠ACB are equal or not:

1. Finding ∠ADB:

$$\mathbf{A}-\mathbf{D}=\begin{pmatrix}-\frac{21}{8}\\-\frac{29}{20}\end{pmatrix},\mathbf{B}-\mathbf{D}=\begin{pmatrix}\frac{11}{8}\\-\frac{21}{8}\end{pmatrix} \tag{27}$$

$$(\mathbf{A}-\mathbf{D})^\top(\mathbf{B}-\mathbf{D})=-\frac{3}{2} \tag{28}$$

$$\|\mathbf{A}-\mathbf{D}\|\,\|\mathbf{C}-\mathbf{D}\|=6 \tag{29}$$

$$\cos(\angle ADB)=\frac{(\mathbf{A}-\mathbf{D})^\top(\mathbf{B}-\mathbf{D})}{\|\mathbf{A}-\mathbf{D}\|\,\|\mathbf{B}-\mathbf{D}\|} \tag{30}$$

$$\angle ADB=104^\circ \tag{31}$$

2. Finding ∠ACB:

$$\mathbf{A}-\mathbf{C}=\begin{pmatrix}-\frac{21}{8}\\\frac{29}{20}\end{pmatrix},\mathbf{B}-\mathbf{C}=\begin{pmatrix}\frac{11}{8}\\\frac{29}{20}\end{pmatrix} \tag{32}$$

$$(\mathbf{A}-\mathbf{C})^\top(\mathbf{B}-\mathbf{C})=-\frac{3}{2} \tag{33}$$

$$\|\mathbf{A}-\mathbf{C}\|\,\|\mathbf{B}-\mathbf{C}\|=6 \tag{34}$$

$$\cos(\angle ACB)=\frac{(\mathbf{A}-\mathbf{C})^\top(\mathbf{B}-\mathbf{C})}{\|\mathbf{A}-\mathbf{C}\|\,\|\mathbf{B}-\mathbf{C}\|} \tag{35}$$

$$\angle ACB=104^\circ \tag{36}$$

Hence, both the intersecting angles are equal to each other, which satisfies the above condition.