

CLASS-9
CHAPTER-10
CIRCLES

Exercise 10.6

Q1. Prove that the line of centres of two intersecting circles subtends equal angles at the two points of intersection.

Solution

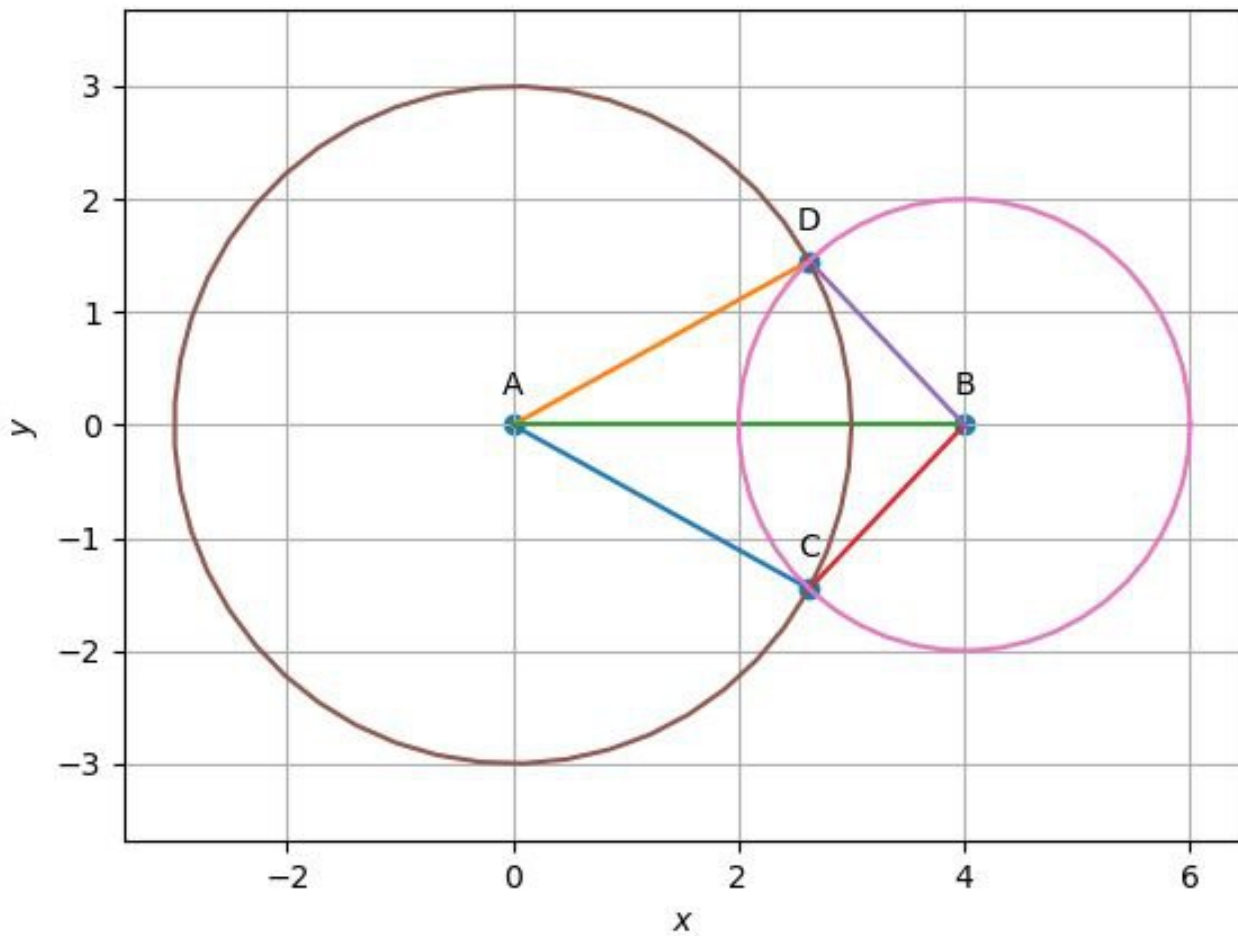


Figure 1:

Construction

Symbol	Values	Description
A	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	Center of circle 1
r_1	3 units	Radius of the circle 1
B	$\begin{pmatrix} 4 \\ 0 \end{pmatrix}$	Center of circle 2
r_2	2 units	Radius of circle 2
\mathbf{e}_1	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	Standard basis vector 1
\mathbf{e}_2	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	Standard basis vector 2

Verification:

The two circle equations are given by:

$$\|x\|^2 - 9 = 0 \quad (1)$$

$$\|x\|^2 - 8\mathbf{e}_1 + 12 = 0 \quad (2)$$

Equation of two conics is given by:

$$\mathbf{x}^\top \mathbf{V}_i \mathbf{x} + 2\mathbf{u}_i^\top \mathbf{x} + f_i = 0, \quad i = 1, 2 \quad (3)$$

Represent the two circles in conic form:

$$\mathbf{x}^\top \mathbf{x} - 9 = 0 \quad (4)$$

$$\mathbf{x}^\top \mathbf{x} + 2 \begin{pmatrix} -4 & 0 \end{pmatrix} \mathbf{x} + 12 = 0 \quad (5)$$

On comparing above two equations with (3), we get:

$$\mathbf{V}_1 = \mathbf{I}, \mathbf{u}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, f_1 = -9 \quad (6)$$

$$\mathbf{V}_2 = \mathbf{I}, \mathbf{u}_2 = \begin{pmatrix} -4 \\ 0 \end{pmatrix}, f_2 = 12 \quad (7)$$

The intersection of the given conics is obtained as

$$\mathbf{V}_1 + \mu \mathbf{V}_2 = \begin{pmatrix} \mu + 1 & 0 \\ 0 & \mu + 1 \end{pmatrix} \quad (8)$$

$$\mathbf{u}_1 + \mu \mathbf{u}_2 = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \quad (9)$$

$$f_1 + \mu f_2 = -21 \quad (10)$$

This conic is a single straight line if and only if,

$$\begin{vmatrix} \mathbf{V}_1 + \mu \mathbf{V}_2 & \mathbf{u}_1 + \mu \mathbf{u}_2 \\ (\mathbf{u}_1 + \mu \mathbf{u}_2)^\top & f_1 + \mu f_2 \end{vmatrix} = 0 \quad (11)$$

Substituting equation (8),(9) and (10) in equation (11):

$$\Rightarrow \begin{vmatrix} 1+\mu & 0 & -4\mu \\ 0 & 1+\mu & 0 \\ -4\mu & 0 & -9+12\mu \end{vmatrix} = 0 \quad (12)$$

Solving the above equation we get,

$$\mu = -1 \quad (13)$$

Thus, the parameters for a straight line can be expressed as

$$\mathbf{V} = \mathbf{V}_1 + \mu \mathbf{V}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad (14)$$

$$\mathbf{u} = \mathbf{u}_1 + \mu \mathbf{u}_2 = \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \quad (15)$$

$$f = f_1 + \mu f_2 = -21 \quad (16)$$

The conic equation is given by:

$$\mathbf{x}^\top \mathbf{V} \mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0, \quad (17)$$

By substituting (14),(15) and (16) in conic equation (17), we get:

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + 2 \begin{pmatrix} 4 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - 21 = 0 \quad (18)$$

By solving above equation (18), we get point of contact \mathbf{q} :

$$\mathbf{q} = \begin{pmatrix} \frac{21}{8} \\ 0 \end{pmatrix} \quad (19)$$

The points of intersection of line is given by:

$$L : \quad \mathbf{x} = \mathbf{q} + \kappa \mathbf{m} \quad \kappa \in \mathbb{R} \quad (20)$$

with the conic section, we have:

$$\mathbf{x}_i = \mathbf{q} + \kappa_i \mathbf{m} \quad (21)$$

where,

$$\kappa_i = \frac{1}{\mathbf{m}^\top \mathbf{V} \mathbf{m}} \left(-\mathbf{m}^\top (\mathbf{V} \mathbf{q} + \mathbf{u}) \pm \sqrt{[\mathbf{m}^\top (\mathbf{V} \mathbf{q} + \mathbf{u})]^2 - (\mathbf{q}^\top \mathbf{V} \mathbf{q} + 2\mathbf{u}^\top \mathbf{q} + f) (\mathbf{m}^\top \mathbf{V} \mathbf{m})} \right) \quad (22)$$

On substituting the below values in (22)

$$\mathbf{m} = \mathbf{e}_2, \mathbf{V} = \mathbf{I}, \mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{q} = \begin{pmatrix} \frac{13}{5} \\ 0 \end{pmatrix}, f = -9 \quad (23)$$

We get,

$$\kappa_i = -\frac{29}{20}, +\frac{29}{20} \quad (24)$$

The intersecting points \mathbf{C} and \mathbf{D} are given by:

$$\mathbf{C} = \mathbf{q} + \kappa_1 \mathbf{m} = \begin{pmatrix} \frac{21}{8} \\ -\frac{29}{20} \end{pmatrix} \quad (25)$$

$$\mathbf{D} = \mathbf{q} + \kappa_2 \mathbf{m} = \begin{pmatrix} \frac{21}{8} \\ \frac{29}{20} \end{pmatrix} \quad (26)$$

Check whether the intersection angles $\angle ADB$ and $\angle ACB$ are equal or not:

1. Finding $\angle ADB$:

$$\mathbf{A} - \mathbf{D} = \begin{pmatrix} -\frac{21}{8} \\ -\frac{29}{20} \end{pmatrix}, \mathbf{B} - \mathbf{D} = \begin{pmatrix} \frac{11}{8} \\ -\frac{21}{8} \end{pmatrix} \quad (27)$$

$$(\mathbf{A} - \mathbf{D})^\top (\mathbf{B} - \mathbf{D}) = -\frac{3}{2} \quad (28)$$

$$\|\mathbf{A} - \mathbf{D}\| \|\mathbf{B} - \mathbf{D}\| = 6 \quad (29)$$

$$\cos(\angle ADB) = \frac{(\mathbf{A} - \mathbf{D})^\top (\mathbf{B} - \mathbf{D})}{\|\mathbf{A} - \mathbf{D}\| \|\mathbf{B} - \mathbf{D}\|} \quad (30)$$

$$\angle ADB = 104^\circ \quad (31)$$

2. Finding $\angle ACB$:

$$\mathbf{A} - \mathbf{C} = \begin{pmatrix} -\frac{21}{8} \\ \frac{29}{20} \end{pmatrix}, \mathbf{B} - \mathbf{C} = \begin{pmatrix} \frac{11}{8} \\ \frac{29}{20} \end{pmatrix} \quad (32)$$

$$(\mathbf{A} - \mathbf{C})^\top (\mathbf{B} - \mathbf{C}) = -\frac{3}{2} \quad (33)$$

$$\|\mathbf{A} - \mathbf{C}\| \|\mathbf{B} - \mathbf{C}\| = 6 \quad (34)$$

$$\cos(\angle ACB) = \frac{(\mathbf{A} - \mathbf{C})^\top (\mathbf{B} - \mathbf{C})}{\|\mathbf{A} - \mathbf{C}\| \|\mathbf{B} - \mathbf{C}\|} \quad (35)$$

$$\angle ACB = 104^\circ \quad (36)$$

Hence, both the intersecting angles are equal to each other, which satisfies the above condition.