CLASS-9
CHAPTER-10
CIRCLES

Excercise 10.6

Q1. Prove that the line of centres of two intersecting circles subtends equal angles at the two points of intersection.

Solution

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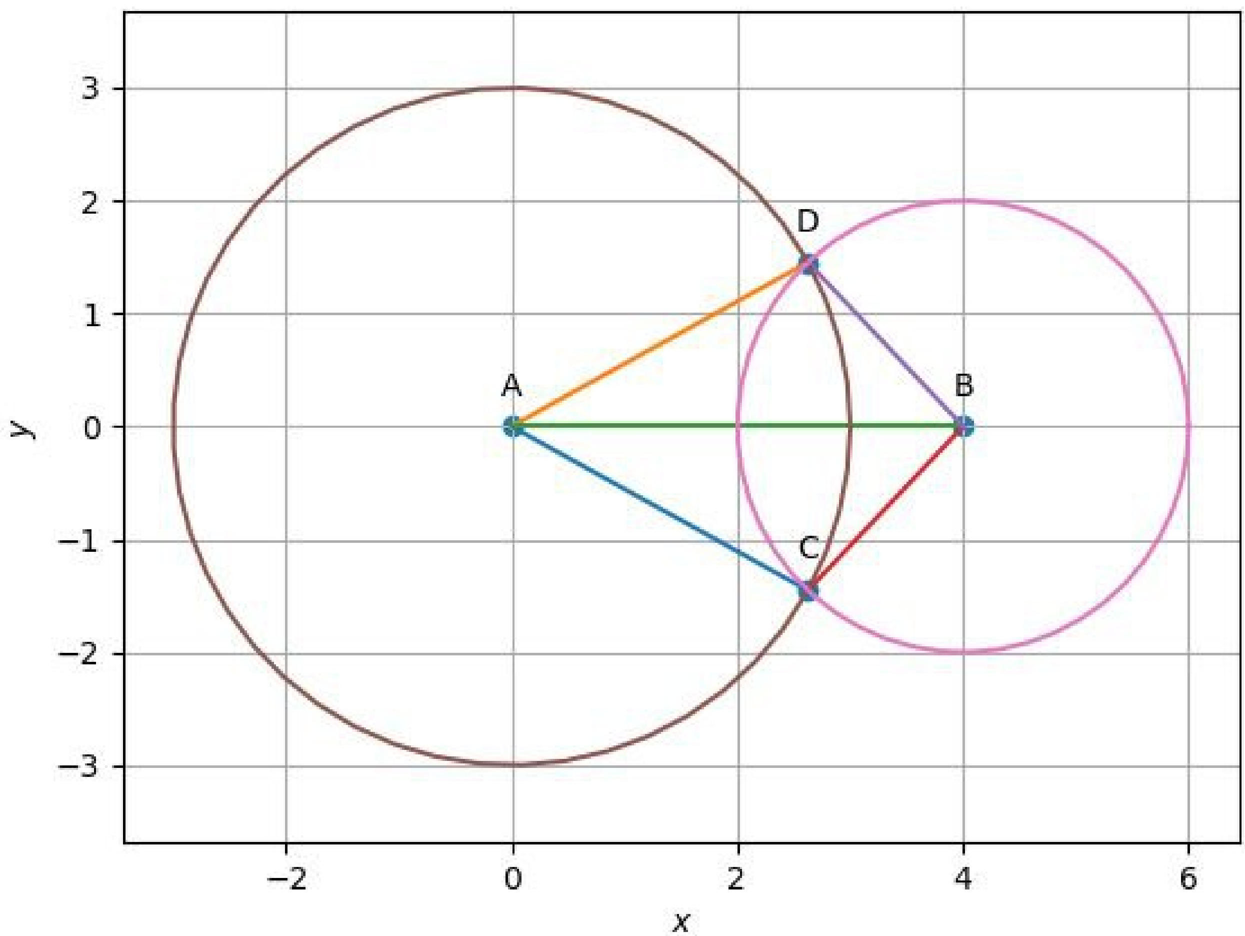

Figure 1:

Construction

:

Symbol	Values	Description
$\mathbf{A}$	$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	Center of circle 1
r_1	3 units	Radius of the circle 1
$\mathbf{B}$	$\begin{pmatrix} 4 \\ 0 \end{pmatrix}$	Center of circle 2
r_2	2 units	Radius of circle 2
$\mathbf{e}_1$	$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$	Standard basis vector 1
$\mathbf{e}_2$	$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$	Standard basis vector 2

Verification:

The two circle equations are given by:

$$\|x\|^2 - 9 = 0 \tag{1}$$
$$\|x\|^2 - 8\mathbf{e}_1 + 12 = 0 \tag{2}$$

Equation of two conics is given by:

$$\mathbf{x}^\top \mathbf{V}_i \mathbf{x} + 2\mathbf{u}_i^\top \mathbf{x} + f_i = 0, \quad i = 1, 2 \tag{3}$$

Represent the two circles in conic form:

$$\mathbf{x}^\top \mathbf{x} - 9 = 0 \tag{4}$$
$$\mathbf{x}^\top \mathbf{x} + 2\begin{pmatrix} -4 & 0 \end{pmatrix} + 12 = 0 \tag{5}$$

On comparing above two equations with (3), we get:

$$\mathbf{V}_1 = \mathbf{I}, \mathbf{u}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, f_1 = -9 \tag{6}$$
$$\mathbf{V}_2 = \mathbf{I}, \mathbf{u}_2 = \begin{pmatrix} -4 \\ 0 \end{pmatrix}, f_2 = 12 \tag{7}$$

The intersection of the given conics is obtained as

$$\mathbf{V}_1 + \mu \mathbf{V}_2 = \begin{pmatrix} \mu + 1 & 0 \\ 0 & \mu + 1 \end{pmatrix} \tag{8}$$
$$\mathbf{u}_1 + \mu \mathbf{u}_2 = \begin{pmatrix} 4 \\ 0 \end{pmatrix} \tag{9}$$
$$f_1 + \mu f_2 = -21 \tag{10}$$

This conic is a single straight line if and only if,

$$\begin{vmatrix} \mathbf{V}_1 + \mu \mathbf{V}_2 & \mathbf{u}_1 + \mu \mathbf{u}_2 \\ (\mathbf{u}_1 + \mu \mathbf{u}_2)^\top & f_1 + \mu f_2 \end{vmatrix} = 0 \tag{11}$$

Substituting equation (8),(9) and (10) in equation (11):

$$\implies \begin{vmatrix} 1+\mu & 0 & -4\mu \\ 0 & 1+\mu & 0 \\ -4\mu & 0 & -9+12\mu \end{vmatrix} = 0 \tag{12}$$

Solving the above equation we get,

$$\mu = -1 \tag{13}$$

Thus, the parameters for a straight line can be expressed as

$$\mathbf{V} = \mathbf{V}_1 + \mu\mathbf{V}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{14}$$

$$\mathbf{u} = \mathbf{u}_1 + \mu\mathbf{u}_2 = \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \tag{15}$$

$$f = f_1 + \mu f_2 = -21 \tag{16}$$

The conic equation is given by:

$$\mathbf{x}^\top \mathbf{V}\mathbf{x} + 2\mathbf{u}^\top \mathbf{x} + f = 0, \tag{17}$$

By substituting (14),(15) and (16) in conic equation (17):

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + 2\begin{pmatrix} 4 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - 21 = 0 \tag{18}$$

$$\implies 8x = 21 \tag{19}$$

$$\implies x = \frac{21}{8} \tag{20}$$

From equation (20), the staright line between the intersection of two circles is given by:

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \mathbf{x} = \frac{21}{8} \tag{21}$$

$$\mathbf{x} = \begin{pmatrix} \frac{21}{8} \\ \lambda \end{pmatrix} \tag{22}$$

$$\mathbf{x} = \begin{pmatrix} \frac{21}{8} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{23}$$

Equation (23) can be expressed in the form of parametric equation

$$\mathbf{x} = \mathbf{q} + \lambda\mathbf{m} \tag{24}$$

The distance form origin to point $\mathbf{x}$ is given by

$$\|\mathbf{x}\|^2 = d^2 \tag{25}$$

Then substituting (24) in (25) yeilds,

$$\implies (\mathbf{q} + \lambda\mathbf{m})^\top (\mathbf{q} + \lambda\mathbf{m}) = d^2 \tag{26}$$

$$\implies \mathbf{q}^\top \mathbf{q} + (\lambda\mathbf{m})^\top \lambda\mathbf{m} + \mathbf{q}^\top \lambda\mathbf{m} + (\lambda\mathbf{m})^\top \mathbf{q} = d^2 \tag{27}$$

$$\implies \|\mathbf{q}\|^2 + \lambda^2 \|\mathbf{m}\|^2 + 2\lambda\mathbf{q}^\top \mathbf{m} = d^2 \tag{28}$$

$$\implies \lambda^2 \|\mathbf{m}\|^2 + 2\lambda\mathbf{q}^\top \mathbf{m} + \|\mathbf{q}\|^2 = d^2 \tag{29}$$

where

$$\mathbf{q} = \begin{pmatrix} \frac{21}{8} \\ 0 \end{pmatrix}, \mathbf{m} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ and } d = r_1 = 3 \tag{30}$$

substituting the values of (30) in (29) gives

$$\implies \lambda^2(1) + 2\lambda \begin{pmatrix} \frac{21}{8} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{441}{64} = 9 \tag{31}$$

$$\implies \lambda^2 = \frac{135}{64} \tag{32}$$

$$\implies \lambda_i = \pm\frac{3\sqrt{5}}{8} \tag{33}$$

The intersecting points **C** and **D** are given by:

$$\mathbf{C} = \mathbf{q} + \lambda_1 \mathbf{m} = \begin{pmatrix} \frac{21}{8} \\ -\frac{3\sqrt{5}}{8} \end{pmatrix} \tag{34}$$

$$\mathbf{D} = \mathbf{q} + \lambda_2 \mathbf{m} = \begin{pmatrix} \frac{21}{8} \\ \frac{3\sqrt{5}}{8} \end{pmatrix} \tag{35}$$

Check whether the intersection angles ∠ADB and ∠ACB are equal or not:

1. Finding ∠ADB:

$$\mathbf{A} - \mathbf{D} = \begin{pmatrix} -\frac{21}{8} \\ -\frac{3\sqrt{5}}{8} \end{pmatrix}, \mathbf{B} - \mathbf{D} = \begin{pmatrix} \frac{11}{8} \\ -\frac{3\sqrt{5}}{8} \end{pmatrix} \tag{36}$$

$$(\mathbf{A} - \mathbf{D})^\top(\mathbf{B} - \mathbf{D}) = -\frac{3}{2} \tag{37}$$

$$\|\mathbf{A} - \mathbf{D}\| \, \|\mathbf{C} - \mathbf{D}\| = 6 \tag{38}$$

$$\cos(\angle ADB) = \frac{(\mathbf{A} - \mathbf{D})^\top(\mathbf{B} - \mathbf{D})}{\|\mathbf{A} - \mathbf{D}\| \, \|\mathbf{B} - \mathbf{D}\|} \tag{39}$$

$$\angle ADB = 104^\circ \tag{40}$$

2. Finding ∠ACB:

$$\mathbf{A} - \mathbf{C} = \begin{pmatrix} -\frac{21}{8} \\ \frac{3\sqrt{5}}{8} \end{pmatrix}, \mathbf{B} - \mathbf{C} = \begin{pmatrix} \frac{11}{8} \\ \frac{3\sqrt{5}}{8} \end{pmatrix} \tag{41}$$

$$(\mathbf{A} - \mathbf{C})^\top(\mathbf{B} - \mathbf{C}) = -\frac{3}{2} \tag{42}$$

$$\|\mathbf{A} - \mathbf{C}\| \, \|\mathbf{B} - \mathbf{C}\| = 6 \tag{43}$$

$$\cos(\angle ACB) = \frac{(\mathbf{A} - \mathbf{C})^\top(\mathbf{B} - \mathbf{C})}{\|\mathbf{A} - \mathbf{C}\| \, \|\mathbf{B} - \mathbf{C}\|} \tag{44}$$

$$\angle ACB = 104^\circ \tag{45}$$

Hence, both the intersecting angles are equal to each other, which satisfies the above condition.