Revisiting Linear Algebra

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Overview

1. Basics of Linear Algebra

2. Operators in Linear Algebra

3. Few more concepts in Linear Algebra

4. Linear Transformations

5. Hands-on Exercise

What they say about Linear Algebra

"Considering the inconceivable complexity of processes, even in a simple cell, it is little short of a miracle that the simplest possible model - namely, a linear equation between two variables - actually applies in quite a general number of cases "

— Ludwig von Bertalanffy, 1941

What they say about Linear Algebra

"we think basis-free, we write basis-free , but when the chips are down we close the office door and compute with matrices like fury. "

— Irving Kaplansky, 1991

Application domains

- Graphs and Networks, such as analyzing networks
- Fourier Series: Linear Algebra for functions, used widely in signal processing
- Markov Matrices, Population, and Economics, such as population growth
- Linear Algebra for statistics and probability, such as least squares for regression
- Linear Programming, the simplex optimization method
- Computer Graphics, such as the various translation, rescaling and rotation of images

Course Plan

Sunday Sessions on Foundations of AIML

Online Tutorials/Quizzes/Exams

Start-ups and Case Studies

Grading and Attendance Policy

Internship Opportunities - Top Performers

Project Work Component - Promising Students

Basics of Linear Algebra

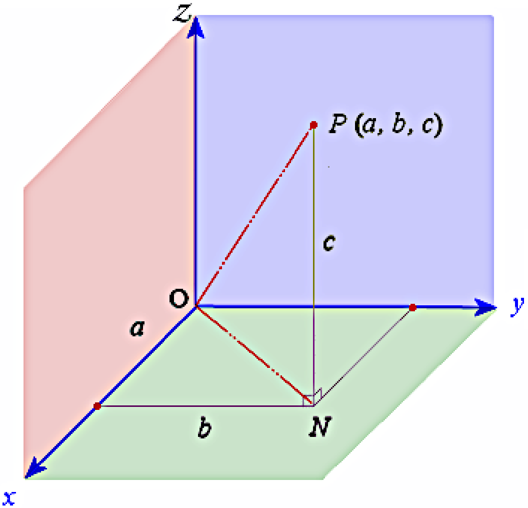
Why Linear Algebra

Simplicity - Occam’s Razon

Well understood - mathematical results, models

Efficiency - speed, memory, power etc

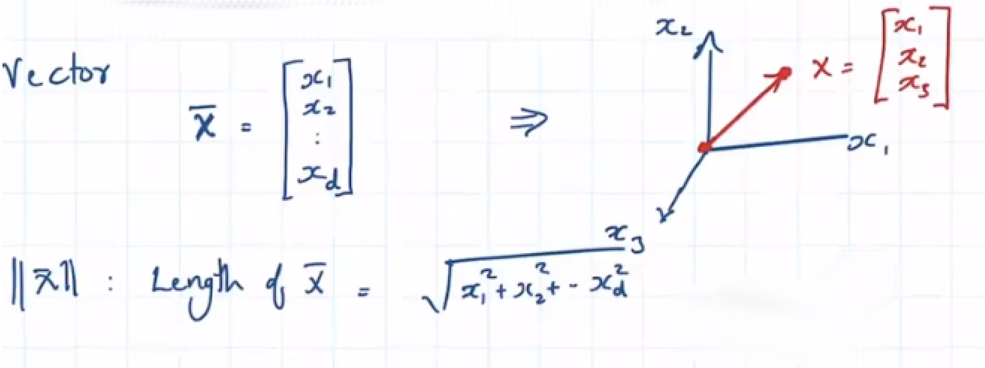
Vector



The distance from $(0, 0, 0)$ to the point $P(a, b, c)$ is given by:

$$\boxed{\text{distance } OP = \sqrt{a^2 + b^2 + c^2}}$$

Vector - Norm or Length



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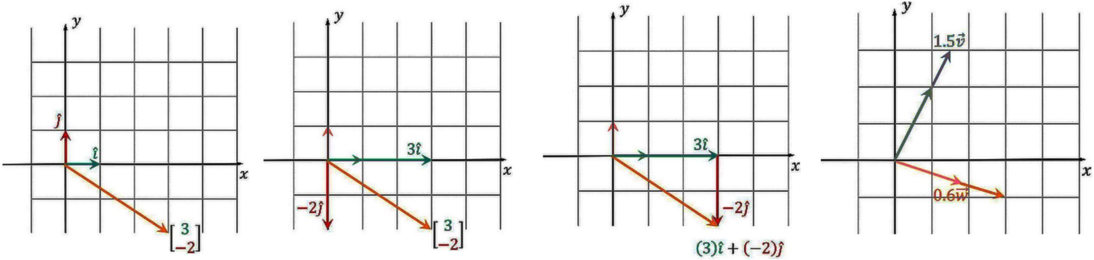
Basis of a vector



Basis vectors - a few more characteristics

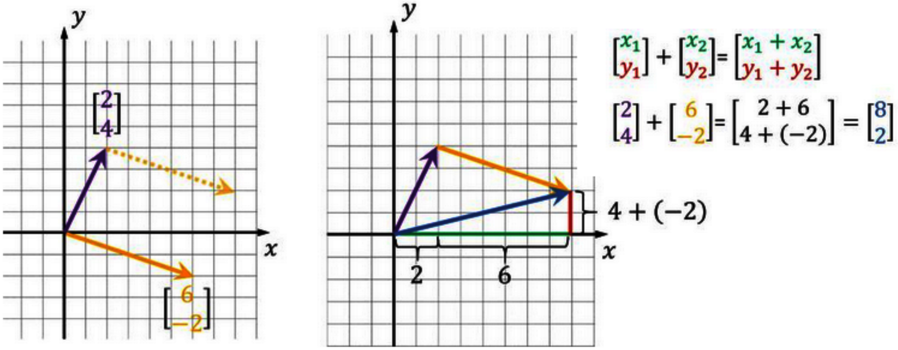
Basis vectors are usually of unit-length

They are linearly independent and orthogonal to each other

The set of basis vectors are not unique

Operators in Linear Algebra

Addition/Subtraction



Addition/Subtraction (python code)

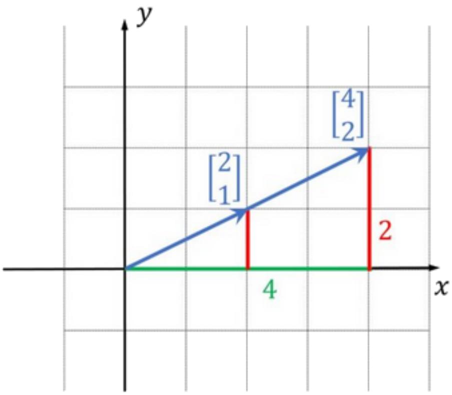
```
In [ ]: # Import NumPy
        import numpy as np
```

```
In [ ]: # Vector addition

        vec1 = np.array([[0],[7],[3]])
        vec2 = np.array([[1],[2],[0]])

        res = vec1 + vec2
        print(res)
```

Scalar Multiplication



$$2 \cdot \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}2x\\2y\end{bmatrix}$$

$$2 \cdot \begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}4\\2\end{bmatrix}$$

This process of “scaling”(stretching) a vector with a "scalar"

Scalar Multiplication

```
In [10]: # Vector-Scalar Multiplication

         # The result will be calculated in the following way
         # res = s*vec1[0] + s*vec1[1] + s*vec1[2]

         s = 3
         vec1 = np.array([5,1,3])

         res = s * vec1

         print(res)
         print(type(res))

         [15  3  9]
         <class 'numpy.ndarray'>
```

Dot Product

$$\bar{X}.\bar{Y} = (x_1y_1 + x_2y_2 = ... + x_dy_d)$$

$$\bar{X}.\bar{Y} = ||X||||Y||Cos\theta$$

Note: Dot product would be zero, if vectors X and Y are orthogonal, or if any of the norm of vectors is zero.

Matrices - Solving Equations

x y z

Unknown variables

$3x - 5y + 4z = 7$

$x - 2y + 7z = 2$

$6x - 8y + z = 0$

Coefficients **Variables**

$$\begin{aligned} 3x - 5y + 4z &= 7 \\ x - 2y + 7z &= 2 \\ 6x - 8y + 1z &= 0 \end{aligned} \longrightarrow \begin{bmatrix} 3 & -5 & 4 \\ 1 & -2 & 7 \\ 6 & -8 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ 0 \end{bmatrix}$$

Matrices - Solving Equations

```
In [1]: import numpy as np
        import matplotlib.pyplot as plt

        %matplotlib inline
```

```
In [2]: a = np.array([[3,-5,4],[1,-2,7],[6,-8,1]])
        b = np.array([[7],[2],[0]])
```

```
In [3]: x = np.linalg.solve(a, b)
        print(x)

        [[-12.        ]
         [ -9.07407407]
         [ -0.59259259]]
```

Matrices - Solving Equations

$$\underset{A}{\begin{bmatrix}2 & 2\\1 & 3\end{bmatrix}}\underset{\vec{x}}{\begin{bmatrix}x\\y\end{bmatrix}}=\underset{\vec{b}}{\begin{bmatrix}-4\\-1\end{bmatrix}}$$

$$\underbrace{A^{-1}A}_{\text{The “do nothing” matrix}}\vec{x} = A^{-1}\vec{b}$$

Few more concepts in Linear Algebra

Rank, Determinant, Linear Dependency

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then

$$\boxed{A^{-1} = \frac{1}{\underbrace{ad - bc}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$$

Inverse of A

Determinant of A

Adjoint of A

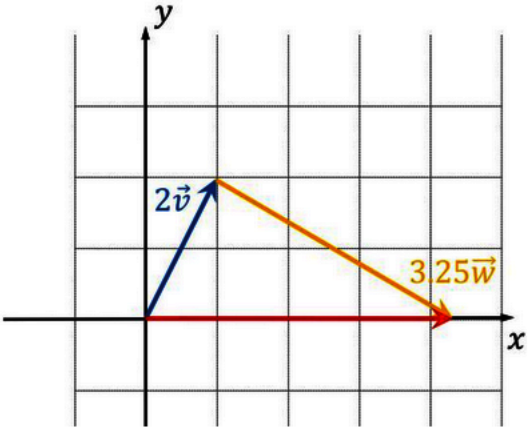
Note: A^{-1} exists only when $ad - bc \neq 0$

Span of Vectors

The "span" of $\vec{v}$ and $\vec{w}$ is the set of all their Linear combinations

$$a \cdot \vec{v} + b \cdot \vec{w}$$

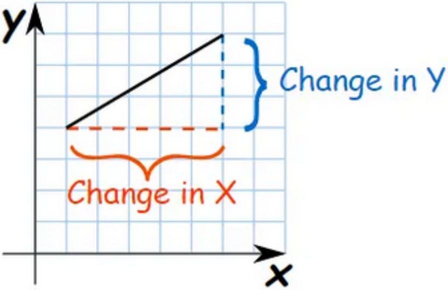
Let a and b vary over all real numbers



Conventional Equation for a Line

```
y = mx + c (in 2d)
ax + by + c = 0(General Equation)
```

$$m = \frac{\text{Change in Y}}{\text{Change in X}}$$



Generic Equations in Linear Algebra

$$w_1x_1 + w_2x_2 + \ldots + +w_nx_n + w_o = 0$$

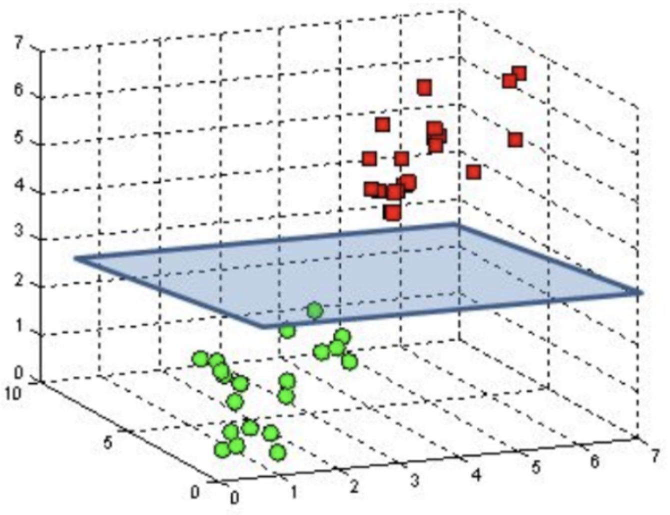
$$\mathbf{w} = [w_0, w_1, w_2, \ldots, w_n]$$

$$\mathbf{x} = [1, x_1, x_2, \ldots, x_n]$$

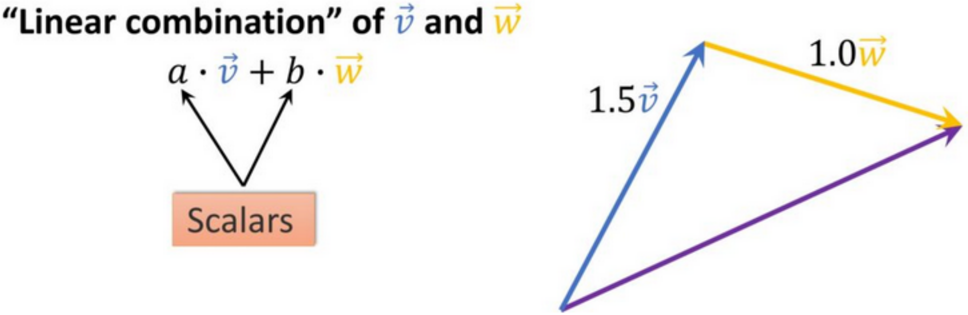
$$\mathbf{w} \cdot \mathbf{x} = \mathbf{w}^{\mathrm{T}}\mathbf{x} = \sum_{i=0}^{n} w_i * x_i$$

The transpose is just to write it in a matrix multiplication form.

Solving Generic Equations



Linear Combination of Vectors



Linear Combination - Python Code

```
In [6]: # Linear combination of vectors

        v = np.array([[3],[1]])
        w = np.array([[-2],[-4]])
```

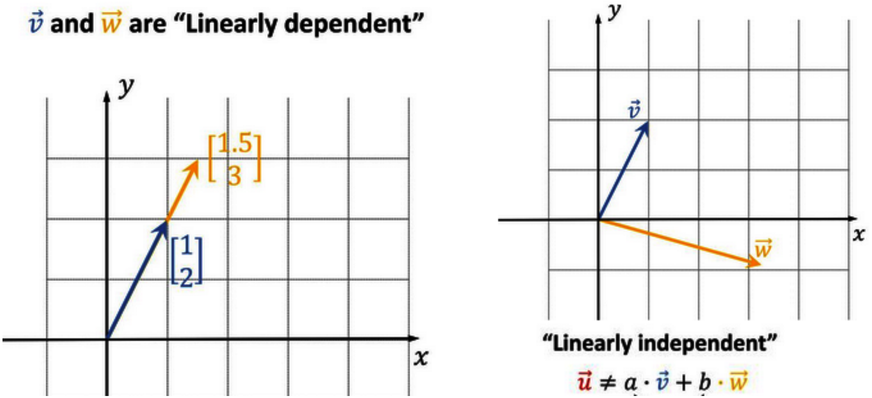
```
In [7]: a = 1.5
        b = 1
```

```
In [8]: vec_new = a*v + b*w

        print(vec_new)

        [[ 2.5]
         [-2.5]]
```

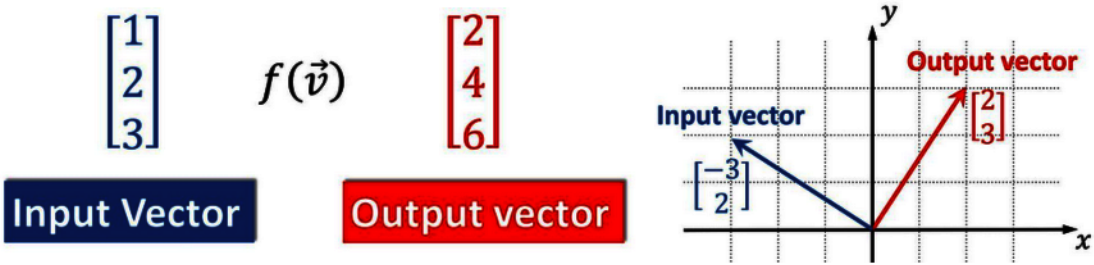
Linearly independent and dependent vectors



If from a span we can simply remove one vector because it does not add any information or we can say that the set of vectors are **linearly dependent**, if all vectors add some information, they are **linearly independent**.

Linear Transformations

Linear Transformation



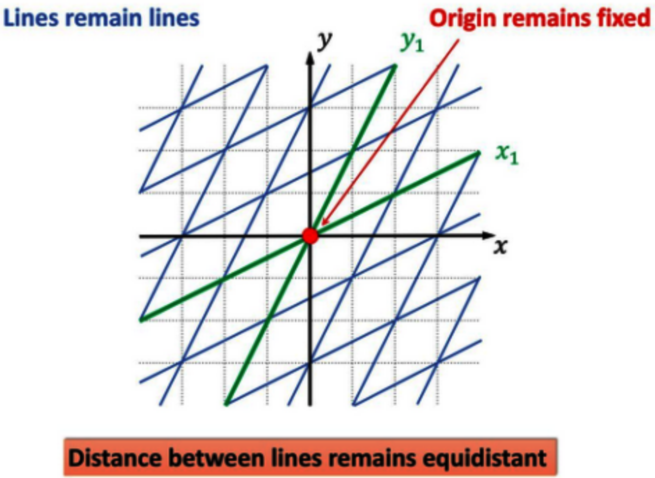
A linear transformation is actually a function that maps an input vector into an output vector.

Linear Transformation

Essential characteristics of a linear transformation

- a line should remain a line after transforming coordinate system
- an origin should remain at the fixed place
- distance between grid lines should remain equidistant and parallel

Characteristics



Hands-on Exercise

Hands-on

Colab Exercise - Warm-up