Solution Write $y_2(x) = u(x)y_1(x) = u(x)/x$. Here, $p(x) = a_1(x)/a_0(x) = 4/x$. Hence,

$$v(x) = \frac{1}{y_1^2} e^{-\int p(x)dx} = x^2 e^{-\int (4/x)dx} = x^2\left(\frac{1}{x^4}\right) = \frac{1}{x^2}.$$

$$u(x) = \int v(x)\,dx = \int \frac{dx}{x^2} = -\frac{1}{x}, \quad \text{and} \quad y_2(x) = u(x)y_1(x) = -\frac{1}{x^2}.$$

The general solution is $y(x) = Ay_1(x) + By_2(x) = \dfrac{A}{x} + \dfrac{B}{x^2}$.

Exercise 5.2

Show that the given set of functions $\{y_1(x), y_2(x)\}$ forms a basis of the equation and hence solve the initial value problem.

1. e^x, e^{4x}, $y'' - 5y' + 4y = 0$, $y(0) = 2$, $y'(0) = 1$.
2. e^{2x}, e^{-2x}, $y'' - 4y = 0$, $y(0) = 1$, $y'(0) = 4$.
3. e^{-3x}, xe^{-3x}, $y'' + 6y' + 9y = 0$, $y(0) = 1$, $y'(0) = 2$.
4. $x^2, 1/x^2$, $x^2y'' + xy' - 4y = 0$, $y(1) = 2$, $y'(1) = 6$.
5. $x, x\ln x$, $x^2y'' - xy' + y = 0$, $y(1) = 3$, $y'(1) = 4$.

Find a general solution of the following differential equations.

6. $y'' - 4y = 0$.
7. $y'' - y' - 2y = 0$.
8. $y'' + y' - 2y = 0$.
9. $y'' - 4y' - 12y = 0$.

10. $y'' + 4y' + y = 0$.

11. $4y'' - 9y' + 2y = 0$.

12. $4y'' + 8y' - 5y = 0$.

13. $y'' + 2y' + y = 0$.

14. $y'' + 2\pi y' + \pi^2 y = 0$.

15. $9y'' - 12y' + 4y = 0$.

16. $4y'' + 4y' + y = 0$.

17. $25y'' - 20y' + 4y = 0$.

18. $y'' + 25y = 0$.

19. $y'' + 4y' + 5y = 0$.

20. $y'' - 2y' + 2y = 0$.

21. $(4D^2 - 4D + 17)y = 0$.

22. $(D^2 - 6D + 18)y = 0$.

23. $(D^2 + 9D)y = 0$.

24. $[D^2 - 2aD + (a^2 + b^2)]y = 0$.

Find a differential equation of the form $ay'' + by' + cy = 0$, for which the following functions are solutions.

25. e^{3x}, e^{-2x}.

26. $e^{x/4}, e^{-(3x)/4}$.

27. $1, e^{-2x}$.

28. e^{2x}, xe^{2x}.

29. e^{-x}, xe^{-x}.

30. e^{-3ix}, e^{3ix}.

31. $e^{-(a+ib)x}, e^{-(a-ib)x}$.

32. $e^{(5+3i)x}, e^{(5-3i)x}$.

Solve the following initial value problems.

33. $y'' - y = 0,\ y(0) = 0,\ y'(0) = 2$.

34. $y'' - y' - 12y = 0,\ y(0) = 4,\ y'(0) = -5$.

35. $y'' + y' - 2y = 0,\ y(0) = 0,\ y'(0) = 3$.

36. $\dfrac{d^2\theta}{dt^2} + g\theta = 0$, g constant, $\theta(0) = a$, constant, $\dfrac{d\theta}{dt}(0) = 0$.

37. $y'' - 4y' + 5y = 0,\ y(0) = 2,\ y'(0) = -1$.

38. $25y'' - 10y' + 2y = 0,\ y(0) = 1,\ y'(0) = 0$.

39. $4y'' + 12y' + 9y = 0,\ y(0) = -1,\ y'(0) = 2$.

40. $9y'' + 6y' + y = 0,\ y(0) = 0,\ y'(0) = 1$.

Solve the following boundary value problems.

41. $y'' + 25y = 0,\ y(0) = 1,\ y(\pi) = -1$.

42. $y'' - 36y = 0,\ y(0) = 2,\ y(1/6) = 1/e$.

43. $y'' + 2y' + 2y = 0,\ y(0) = 1,\ y(\pi/2) = e^{-\pi/2}$.

44. $9y'' - 6y' + y = 0,\ y(1) = e^{1/3},\ y(2) = 1$.

45. $y'' - 4y' + 3y = 0,\ y(0) = 1,\ y(1) = 0$.

46. Verify that $(D - 2)(D + 3)\sin x = (D + 3)(D - 2)\sin x = (D^2 + D - 6)\sin x$.

47. Show that $x^2 Dy \neq D(x^2 y)$.

48. Find the conditions under which the following equations hold.

(i) $(D + a)[D + b(x)]f(x) = [D + b(x)][D + a]f(x)$, a constant.

(ii) $[D + a(x)][D + b(x)]f(x) = [D + b(x)][D + a(x)]f(x)$.

Factorize the operator and find the solution of the following differential equations using the method of reduction of order or by the direct method.

49. $(D^2 + 5D + 4)y = 0$.

50. $(4D^2 + 8D + 3)y = 0$.

51. $(4D^2 + 12D + 9)y = 0.$

52. $(D^2 + 6D + 9)y = 0.$

53. $(D^2 - 4)y = 0.$

54. $(9D^2 + 6D + 1)y = 0.$

55. The displacement $x(t)$ of a particle is governed by the differential equation $\ddot{x} + \dot{x} + bx = c\dot{x}$, $b > 0$. For what values of b and c is the motion of the particle oscillatory?

56. Find all non-trivial solutions of the boundary value problem

$$y'' + \omega^2 y = 0,\ y(0) = 0,\ y(\pi) = 0.$$

57. Find all the non-trivial solutions of the boundary value problem

$$y'' + \omega^2 y = 0,\ y'(0) = 0,\ y'(\pi) = 0.$$

58. Find all non-trivial solutions of the boundary value problem

$$y'' + \omega^2 y = 0,\ y(0) = 0,\ y'(\pi) = 0.$$

59. If $a^2 > 4b$, then show that the solution of the differential equation $y'' + ay' + by = 0$ can be expressed as $y(x) = e^{px}(A \cosh qx + B \sinh qx)$ where $p = -a/2$ and $q = \sqrt{a^2 - 4b}/2$.

60. The motion of a damped mechanical system is governed by the linear differential equation $m\ddot{y} + c\dot{y} + ky = 0$ in which m (mass), k (spring modulus), c (damping factor) are positive constants and dot denotes derivative with respect to time t. Discuss the behaviour of the general solution when $t \to \infty$ in the following three cases: (i) $c^2 > 4mk$ (over damping), (ii) $c^2 < 4mk$ (under damping), (iii) $c^2 = 4mk$ (critical damping). In each case, obtain the solution subject to the initial conditions $y(0) = 0$, $\dot{y}(0) = v_0$.

Find the solution of the following differential equations, if one of its solutions is known.

61. $y'' - y' - 6y = 0,\ y_1 = e^{-2x}.$

62. $y'' + 3y' - 4y = 0,\ y_1 = e^x.$

63. $(x^2 - 1)y'' - 2xy' + 2y = 0,\ y_1 = x,\ x \neq \pm 1.$

64. $x^2y'' + xy' + (x^2 - 1/4)y = 0,\ x > 0,\ y_1 = x^{-1/2} \sin x.$

65. $(x - 2)y'' - xy' + 2y = 0,\ x \neq 2,\ y_1 = e^x.$

5.3.4 Solution of Higher Order Homogeneous Linear Equations with Constant Coefficients

In this section, we shall extend the methods discussed in section 5.3.2, for the solution of higher order linear homogeneous equations with constant coefficients.

Consider the nth order homogeneous linear equation with constant coefficients

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \ldots + a_{n-1} y' + a_n y = 0. \tag{5.38}$$

We attempt to find a solution of the form $y = e^{mx}$, as in the case of second order equations. Substituting $y = e^{mx}$, $y^{(k)} = m^k e^{mx}$, $k = 1, 2, \ldots, n$ in Eq. (5.38) and cancelling e^{mx}, we obtain the characteristic equation as

$$a_0 m^n + a_1 m^{n-1} + \ldots + a_{n-1} m + a_n = 0. \tag{5.39}$$

The degree of this algebraic equation is same as the order of the differential equation. This equation has n roots. All the roots may be real and distinct, all or some of the roots may be equal, all or some of the roots may be complex. Consider the following cases.

Real and distinct roots

Let the polynomial equation (5.39) have all real and distinct roots as $m_1, m_2, \ldots m_n$. Then the n solutions

Exercise 5.3

Find the general solution of the following differential equations.

1. $y''' - 9y' = 0.$
2. $2y''' + y'' - 13y' + 6y = 0.$
3. $3y''' - 2y'' - 3y' + 2y = 0.$
4. $y^{iv} - 13y'' + 36y = 0.$
5. $4y^{iv} - 12y''' + 7y'' + 3y' - 2y = 0.$
6. $y^{iv} + y''' - 4y'' - 4y' = 0.$
7. $8y^{iv} - 6y''' - 7y'' + 6y' - y = 0.$
8. $144y^{iv} - 25y'' + y = 0.$
9. $y''' - 2y'' + y' = 0.$
10. $y''' + 4y'' + 5y' + 2y = 0.$
11. $y''' - 2y'' - 4y' + 8y = 0.$
12. $27y''' - 27y'' + 9y' - y = 0.$
13. $y^{iv} - 11y''' + 35y'' - 25y' = 0.$
14. $y^{iv} - 3y''' + 3y'' - y' = 0.$
15. $4y^{iv} + 4y''' - 3y'' - 2y' + y = 0.$
16. $9y^{iv} - 66y''' + 157y'' - 132y' + 36y = 0.$
17. $y''' + y' = 0.$
18. $y''' - 2y'' + 4y' - 8y = 0.$
19. $y''' + 5y'' + 8y' + 6y = 0.$
20. $y''' - 7y'' + 19y' - 13y = 0.$
21. $y^{iv} + 8y'' - 9y = 0.$
22. $y^{iv} + y''' + 14y'' + 16y' - 32y = 0.$
23. $4y^{iv} + 101y'' + 25y = 0.$
24. $y^{iv} + 2y''' - 9y'' - 10y' + 50y = 0.$
25. $y^{iv} + 50y'' + 625y = 0.$
26. $y^{iv} + 2y'' + y = 0.$

Find a homogeneous linear differential equation with real constant coefficients of lowest order which has t following particular solution.

27. $5 + e^{x} + 2e^{3x}.$
28. $e^{-x} + \cos 5x + 3 \sin 5x.$
29. $xe^{-x} + e^{2x}.$
30. $1 + x + e^{x} - 3e^{3x}.$
31. $x^2e^{2x} + 2e^{-2x}.$
32. $3 \cos 2x + 5 \sinh 3x.$

Solve the following initial value problems.

33. $y''' - 2y'' - 5y' + 6y = 0,\ y(0) = 0,\ y'(0) = 0,\ y''(0) = 1.$
34. $4y''' - 4y'' - 9y' + 9y = 0,\ y(0) = 1,\ y'(0) = 0,\ y''(0) = 0.$
35. $y''' - 5y'' + 7y' - 3y = 0,\ y(0) = 1,\ y'(0) = 0,\ y''(0) = -5.$
36. $y^{iv} - 2y''' - 3y'' + 4y' + 4y = 0,\ y(0) = 3,\ y'(0) = 3,\ y''(0) = 3,\quad y'''(0) = 6.$
37. $y^{iv} + y'' = 0,\ y(0) = 1,\ y'(0) = 2,\ y''(0) = -1,\ y'''(0) = -1.$
38. $y''' - y'' + 4y' - 4y = 0,\ y(0) = 0,\ y'(0) = 3,\ y''(0) = -5.$
39. $y''' + y'' - 2y = 0,\ y(0) = 2,\ y'(0) = 2,\ y''(0) = -3.$
40. $y^{iv} - 3y''' = 0,\ y(0) = 2,\ y'(0) = 5,\ y''(0) = 15,\ y'''(0) = 27.$

Find the solution of the following differential equations satisfying the given conditions.

41. $y''' + \pi^2 y' = 0,\ y(0) = 0,\ y(1) = 0,\ y'(0) + y'(1) = 0.$
42. $y''' - 36y' = 0,\ y(0) = 2,\ y'(0) = 12,\ y'(1) = 6 \sinh(6) + 12 \cosh(6).$
43. $y^{iv} + 13y'' + 36y = 0,\ y(0) = 0,\ y''(0) = 0,\ y(\pi/2) = -1,\ y'(\pi/2) = -4.$
44. $y^{iv} - \omega^4 y = 0,\ \omega \neq 0,\ y(0) = 0,\ y''(0) = 0,\ y(\pi) = 0,\ y''(\pi) = 0.$
45. $y^{iv} + 10y'' + 9y = 0,\ y'(0) = 0,\ y'''(0) = 0,\ y'(\pi/2) = 5,\ y'''(\pi/2) = -53.$

5.4 Solution of Non-Homogeneous Linear Equations

$$y_p(x) = 108[D^2(D^2+3)]^{-1}(x^2) = 108[D^{-2}]\frac{1}{3}\left[1+\frac{\quad}{3}\right] \quad (x^2)$$

$$= 36[D^{-2}]\left[1-\frac{D^2}{3}+\frac{D^4}{9}-\dots\right](x^2) = 36D^{-2}\left[x^2-\frac{2}{3}\right]$$

$$= 36\left[\frac{x^4}{12}-\frac{x^2}{3}\right] = 3x^4 - 12x^2.$$

The general solution is $y(x) = A + Bx + (C\cos\sqrt{3}x + D\sin\sqrt{3}x) + 3x^4 - 12x^2$.

Exercise 5.7

Find the general solution of the following differential equations.

1. $(D^2+5D+4)y = 18e^{2x}$.
2. $(D^2-1)y = 8e^{3x}$.
3. $(D^2-3D-4)y = e^x + 6e^{5x}$.
4. $(D^2+D+2)y = e^{x/2}$.
5. $(D^2+3D+3)y = 7e^x$.
6. $(D^2-2D+1)y = 5e^{4x} + 4e^{2x}$.
7. $(9D^2-6D+1)y = 4e^{-x}$.
8. $(D^2-6D+9)y = 14e^{3x}$.
9. $(D^2+D-6)y = e^{2x}$.
10. $(2D^2-3D-2)y = xe^{-x/2}$.
11. $(D^2-1)y = 6xe^x$.
12. $(4D^2+9D+2)y = xe^{-2x}$.
13. $(9D^2+6D+1)y = e^{-x/3}$.
14. $(2D^2+7D-4)y = xe^{-4x}$.
15. $(D^3+2D^2-5D-6)y = 4e^x$.
16. $(2D^3+3D^2-3D-2)y = 10e^{2x}$.
17. $(D^3-2D^2-D+2)y = e^{3x}$.
18. $(D^3-6D^2+12D-8)y = 18e^{2x}$.

19. $(2D^3 - 3D^2 + 1)y = 16e^x$.

20. $(D^3 + 3D^2 - 4D - 12)y = 12xe^{-2x}$.

21. $(D^2 + 16)y = \cos 2x$.

22. $(2D^2 - 5D + 3)y = \sin x$.

23. $(3D^2 - 7D + 2)y = \sin x + \cos x$.

24. $(2D^2 - 7D + 3)y = \sin 2x$.

25. $(D^2 + D + 1)y = 16 \cos x$.

26. $(8D^2 - 12D + 5)y = 16 \sin x$.

27. $(D^2 + 9)y = \sin 3x$.

28. $(D^2 + 3)y = \cos \sqrt{3}x$.

29. $(D^2 + 2D + 5)y = e^{-x} \cos 2x$.

30. $(D^2 - 4D + 5)y = 24e^{2x} \sin x$.

31. $(D^2 - 6D + 13)y = 28e^{3x} \sin 2x$.

32. $(D^2 - 2D + 10)y = 16e^x \cos 3x + 24e^x \sin 3x$.

33. $(D^3 - 3D^2 + D - 3)y = 6 \cos x$.

34. $(D^3 - D^2 + 9D - 9)y = 30 \cos 3x$.

35. $(D^3 - 4D^2 + 9D - 10)y = 24e^x \sin 2x$.

36. $(4D^3 - 12D^2 + 13D - 10)y = 16e^{x/2} \cos x$.

37. $(D^4 + 5D^2 + 4)y = 16 \sin x + 64 \cos 2x$.

38. $(D^2 + 25)y = 9x^3 + 4x^2$.

39. $(D^2 + 6D + 9)y = 4x^2 - 1$.

40. $(D^2 - 2D - 3)y = 2x^2 + 6x$.

41. $(D^2 - 5D + 6)y = x \cos 2x$.

42. $(D^2 + D - 2)y = x^2 \sin x$.

43. $(D^2 - D - 6)y = xe^{-2x}$.

44. $(D^2 + 7D + 12)y = e^x \sin 2x$.

45. $(D^2 + 4D + 3)y = e^{2x} \cos x$.

46. $(D^2 + 3D + 4)y = e^x \cos (\sqrt{7}x/2)$.

47. $(D^2 + 3D + 2)y = xe^x \sin x$.

48. $(D^2 + 9)y = xe^{2x} \cos x$.

49. $(4D^2 + 8D + 3)y = xe^{-x/2} \cos x$.

50. $(D^4 + 3D^2 + 2)y = 16x^2 \cos x$.

51. If $(2D - 1)y = e^{3x}$, then prove that $(D - 3)(2D - 1)y = 0$. Find the general solution of the second equ and substituting in the first equation obtain the general solution of the first order equation.

52. If $F(D)y = (D - m)y = r(x)$, then show that the particular integral can be written as

$$y_p(x) = e^{mx} \int e^{-mx} r(x)\, dx.$$

53. Show that $y = \dfrac{1}{n} \displaystyle\int_a^x r(t) \sin n(x - t)\, dt$ is the solution of the equation $y'' + n^2 y = r(x)$.

54. If u is a function of x, then show that

$$F(D)xu = xF(D)u + F'(D)u$$

where $F(D) = a_0 D^n + a_1 D^{n-1} + \ldots + a_n$, and a_i are constants.

55. Let a given differential equation be of the form $F(D)y = r(x) = xu(x)$.
Then, using the result in problem 54 prove that the particular integral $y(x)$ can be written as

$$y(x) = [F(D)]^{-1} xu(x) = x[F(D)]^{-1} u(x) - [F'(D)\{F(D)\}^{-2}]u(x).$$

56. The particular integral of the equation $F(D)y = e^{mx}$ is

42. Normal in $(-\infty, \infty)$, $W = 12\sqrt{3}$. $\{y_1, y_2, y_3\}$ forms a basis.

43. Normal in $(0, \infty)$, $W = -2/x$. $\{y_1, y_2\}$ forms a basis.

44. $W(u, v) = (ad - bc)(y_1 y_2' - y_2 y_1')$. Since $y_1 y_2' - y_2 y_1' \neq 0$, $W(u, v) \neq 0$ if $ad - bc \neq 0$, (the deter of the coefficient matrix of the transformation). Take $a = 1$, $b = 1$, $c = 1$, $d = -1$, $ad - bc$ $u = e^{kx}$, $v = e^{-kx}$.

45. $W(y_1, y_2) \neq 0$. If for $x_0 \in I$, either $y_1(x_0)$, $y_2(x_0)$ vanish or $y_1'(x_0)$, $y_2'(x_0)$ vanish, then $W(y_1, y_2$

46. Simplify $W(y, y_1, y_2)$ and substitute $y_i'' = -(ay_i' + by_i)$, $i = 1, 2$. We obtain

$$W(y, y_1, y_2) = (y'' + ay' + by)(y_1 y_2' - y_2 y_1') = 0.$$

47. At the given point $y_1(x_1) = y'(x_1) = 0$. Therefore, $y_1 \equiv 0$.

48. The differential equation is $W(y, y_1, y_2) = 0$, where $y_1 = e^{3x}$, $y_2 = e^{-2x}$, $y'' - y' - 6y = 0$.

49. $y'' + 2\alpha y' + (\alpha^2 + \omega^2)y = 0$.

50. $y'' - 10y' + 25y = 0$.

Exercise 5.2

1. $(7e^x - e^{4x})/3$.
2. $(3e^{2x} - e^{-2x})/2$.
3. $(1 + 5x)e^{-3x}$.
4. $\frac{1}{2}(5x^2 - (1/x^2))$.
5. $(3 + \ln x)x$.
6. $Ae^{2x} + Be^{-2x}$.
7. $Ae^{2x} + Be^{-x}$.
8. $Ae^x + Be^{-2x}$.
9. $Ae^{6x} + Be^{-2x}$.
10. $Ae^{m_1 x} + Be^{m_2 x}$, $m_1 = -2 + \sqrt{3}, m_2 = -2 - \sqrt{3}$
11. $Ae^{2x} + Be^{x/4}$.
12. $Ae^{x/2} + Be^{-(5x)/2}$.
13. $(A + Bx)e^{-x}$.
14. $(A + Bx)e^{-\pi x}$.
15. $(A + Bx)e^{(2x)/3}$.
16. $(A + Bx)e^{-x/2}$.
17. $(A + Bx)e^{(2x)/5}$.
18. $A \cos 5x + B \sin 5x$.
19. $(A \cos x + B \sin x)e^{-2x}$.
20. $e^x(A \cos x + B \sin x)$.
21. $e^{x/2}(A \cos 2x + B \sin 2x)$.
22. $e^{3x}(A \cos 3x + B \sin 3x)$.
23. $A + Be^{-9x}$.
24. $e^{ax}(A \cos bx + B \sin bx)$.
25. $m = 3, -2$, ch. equation is $m^2 - m - 6 = 0$, diff. equation is $y'' - y' - 6y = 0$.
26. $m = 1/4, -3/4$, ch. equation is $16m^2 + 8m - 3 = 0$, diff. equation is $16y'' + 8y' - 3y = 0$.
27. $m = 0, -2$, ch. equation is $m(m + 2) = 0$, diff. equation is $y'' + 2y' = 0$.
28. $m = 2, 2$, ch. equation is $(m - 2)^2 = 0$, diff. equation is $y'' - 4y' + 4y = 0$.
29. $m = -1, -1$, ch. equation is $(m + 1)^2 = 0$, diff. equation is $y'' + 2y' + y = 0$.
30. $y'' + 9y = 0$.
31. $y'' + 2ay' + (a^2 + b^2)y = 0$.
32. $y'' - 10y' + 34y = 0$.
33. $e^x - e^{-x}$.
34. $e^{4x} + 3e^{-3x}$.
35. $e^x - e^{-2x}$.

36. $a \cos \sqrt{g}\, t$.

37. $e^{2x}(2 \cos x - 5 \sin x)$.

38. $e^{x/5}[\cos (x/5) - \sin (x/5)]$.

39. $((x/2) - 1)\, e^{-(3x)/2}$.

40. $xe^{-x/3}$.

41. $\cos 5x + B \sin 5x$, B arbitrary.

42. $[(2e^2 - 1)e^{-6x} - e^{6x}]/(e^2 - 1)$.

43. $e^{-x}(\cos x + \sin x)$.

44. $(Ax + B)e^{x/3}$, $A = e^{-2/3} - 1$, $B = 2 - e^{-2/3}$.

45. $(e^{x+2} - e^{3x})/(e^2 - 1)$.

48. (i) b = constant, (ii) $a(x) = b(x)$.

49. $(D + 4)(D + 1)y = 0$, set $(D + 1)y = v$ and $(D + 4)v = 0$; $v = A_1e^{-4x}$, $y = Ae^{-4x} + Be^{-x}$.

50. $(2D + 1)(2D + 3)y = 0$, set $(2D + 3)y = v$ and $(2D + 1)v = 0$, $v = A_1e^{-x/2}$, $y = Ae^{-x/2} + Be^{(-3x)/2}$.

51. $(2D + 3)(2D + 3)y = 0$, set $(2D + 3)y = v$, $(2D + 3)v = 0$, $v = A_1e^{-(3x)/2}$, $y = (Ax + B)e^{-(3x)/2}$.

52. $(D + 3)(D + 3)y = 0$, set $(D + 3)y = v$, $(D + 3)v = 0$, $v = A_1e^{-3x}$, $y = (Ax + B)e^{-3x}$.

53. $(D + 2)(D - 2)y = 0$, set $(D - 2)y = v$, $(D + 2)v = 0$, $v = A_1e^{-2x}$, $y = Ae^{-2x} + Be^{2x}$.

54. $(3D + 1)(3D + 1)y = 0$, set $(3D + 1)y = v$, $(3D + 1)v = 0$, $v = A_1e^{-x/3}$, $y = (Ax + B)e^{-x/3}$.

55. For oscillatory solutions, the discriminant of the characteristic equation should be less than zero. $|1 - c| < 2\sqrt{b}$, $1 - 2\sqrt{b} < c < 1 + 2\sqrt{b}$.

56. $\omega = n$, $y(x) = B_n \sin nx$, B_n arbitrary.

57. $y_n(x) = A_n \cos nx$, A_n arbitrary $y(x) = \sum_{n=1}^{\infty} y_n(x)$.

58. $y_n(x) = B_n \sin [(2n + 1)x/2]$, B_n arbitrary $y(x) = \sum_{n=1}^{\infty} y_n(x)$.

59. $y(x) = e^{px}(A'e^{qx} + B'e^{-qx}) = e^{px}[A \cosh qx + B \sinh qx]$.

60. (i) For $c^2 > 4mk$, both the characteristic roots $-p \pm q$ where $p = c/(2m)$ and $q = \sqrt{c^2 - 4mk}/(2m)$, are negative and $q < p$. Therefore, the solution $y(t) = e^{-pt}(Ae^{qt} + Be^{-qt}) \to 0$ as $t \to \infty$, that is, there exists a t_0 such that for $t > t_0$ the system is in equilibrium. $y = [av_0e^{-pt} \sinh qt]/q$.

    (ii) For $c^2 < 4mk$, the characteristic roots are $-p \pm iq$, where $p = c/(2m)$ and $q = \sqrt{4mk - c^2}/(2m)$ are complex. The solutions are oscillatory in this case. The solution is $y(t) = e^{-pt}(A \cos qt + B \sin qt)$. The oscillations are damped and they decay as $t \to \infty$. $y = (e^{-pt}v_0 \sin qt)/q$.

    (iii) For $c^2 = 4mk$, the characteristic roots are repeated roots $-p$. The solution is $y(t) = (A + Bt)e^{-pt}$. $y = v_0te^{-pt}$.

61. $Ae^{3x} + Be^{-2x}$.

62. $Ae^{x} + Be^{-4x}$.

63. $u = x + 1/x$, $y_2 = 1 + x^2$, $Ax + B(1 + x^2)$.

64. $u = -\cot x$, $y_2 = -x^{-1/2} \cos x$, $x^{-1/2}(A \cos x + B \sin x)$.

65. $u = -e^{-x}(x^2 - 2x + 2)$, $y_2 = -(x^2 - 2x + 2)$, $Ae^{x} + B(x^2 - 2x + 2)$.

Exercise 5.3

1. $A + Be^{3x} + Ce^{-3x}$.

2. $Ae^{x/2} + Be^{2x} + Ce^{-3x}$.

3. $Ae^{x} + Be^{-x} + Ce^{2x/3}$.

4. $Ae^{2x} + Be^{-2x} + Ce^{3x} + De^{-3x}$.

5. $Ae^{x} + Be^{2x} + Ce^{-x/2} + De^{x/2}$.

6. $A + Be^{2x} + Ce^{-2x} + De^{-x}$.

7. $Ae^{x/4} + Be^{x/2} + Ce^{x} + De^{-x}$.

8. $Ae^{x/3} + Be^{-x/3} + Ce^{x/4} + De^{-x/4}$.

9. $A + (Bx + C)e^x$.
10. $Ae^{-2x} + (Bx + C)e^{-x}$.
11. $Ae^{-2x} + (Bx + C)e^{2x}$.
12. $(A + Bx + Cx^2)e^{x/3}$.
13. $A + Be^x + (Cx + D)e^{5x}$.
14. $A + (Bx^2 + Cx + D)e^x$.
15. $(Ax + B)e^{-x} + (Cx + D)e^{x/2}$.
16. $(Ax + B)e^{3x} + (Cx + D)e^{2x/3}$.
17. $A + B\cos x + C\sin x$.
18. $Ae^{2x} + B\cos 2x + C\sin 2x$.
19. $Ae^{-3x} + e^{-x}(B\cos x + C\sin x)$.
20. $Ae^x + e^{3x}(B\cos 2x + C\sin 2x)$.
21. $Ae^x + Be^{-x} + C\cos 3x + D\sin 3x$.
22. $Ae^x + Be^{-2x} + C\cos 4x + D\sin 4x$.
23. $A\cos 5x + B\sin 5x + C\cos (x/2) + D\sin (x/2)$.
24. $e^{2x}(A\cos x + B\sin x) + e^{-3x}(C\cos x + D\sin x)$.
25. $(A + Bx)\cos 5x + (C + Dx)\sin 5x$.
26. $(A + Bx)\cos x + (C + Dx)\sin x$.
27. $m = 0, 1, 3$, $y''' - 4y'' + 3y' = 0$.
28. $m = -1, \pm 5i$, $y''' + y'' + 25y' + 25y = 0$.
29. $m = -1, -1, 2$, $y''' - 3y' - 2y = 0$.
30. $m = 0, 0, 1, 3$, $y^{iv} - 4y''' + 3y'' = 0$.
31. $m = 2, 2, 2, -2$, $y^{iv} - 4y''' + 16y' - 16y = 0$.
32. $m = \pm 3, \pm 2i$, $y^{iv} - 5y'' - 36y = 0$.
33. $(3e^{3x} + 2e^{-2x} - 5e^x)/30$.
34. $(9e^x - 5e^{3x/2} + e^{-3x/2})/5$.
35. $(2 + x)e^x - e^{3x}$.
36. $(1 + x)e^{-x} + (2 - x)e^{2x}$.
37. $x + \cos x + \sin x$.
38. $\cos 2x + 2\sin 2x - e^x$.
39. $e^x + e^{-x}(\cos x + 2\sin x)$.
40. $1 + 2x + 3x^2 + e^{3x}$.
41. $A\sin \pi x$, A arbitrary.
42. $1 + 2\sinh 6x + \cosh 6x$.
43. $2\sin 2x + \sin 3x$.
44. $D_n \sin nx$, $\Sigma D_n \sin nx$.
45. $2\cos 3x + \cos x$.

Exercise 5.4

1. $A(x) = -e^{2x}/8$, $B(x) = -e^{-2x}/8$, $y = c_1e^{-x} + c_2e^{3x} - (e^x/4)$.
2. $A(x) = -e^{-4x}/4$, $B(x) = (4x + 1)e^{-4x}/16$, $y = (c_1x + c_2)e^{2x} + e^{-2x}/16$.
3. $A(x) = \cos^3 x/3$, $B(x) = (\sin 3x + 3\sin x)/12$, $y_p = (\cos x)/3$, $y = c_1\cos 2x + c_2\sin 2x + y_p$.
4. $A(x) = \ln|\cos x|$, $B(x) = x$, $y_p = \cos x\ln|\cos x| + x\sin x$, $y = c_1\cos x + c_2\sin x + y_p$.
5. $A(x) = -x$, $B(x) = \ln|\sin x|$, $y_p = \sin x\ln|\sin x| - x\cos x$, $y = c_1\cos x + c_2\sin x + y_p$.
6. $A(x) = \sin x - \ln|\sec x + \tan x|$, $B(x) = -\cos x$, $y_p = -\cos x\ln|\sec x + \tan x|$.

   $y = c_1\cos x + c_2\sin x + y_p$.
7. $A(x) = -x/2$, $B(x) = -e^{-2x}/4$, $y(x) = c_1e^x + c_2e^{3x} - (xe^x)/2$.
8. $A(x) = \frac{1}{4}\ln|\cos 2x|$, $B(x) = x/2$, $y_p = \frac{1}{4}\cos 2x\ln|\cos 2x| + \frac{1}{2}x\sin 2x$.

   $y(x) = c_1\cos 2x + c_2\sin 2x + y_p$.
9. $A(x) = (\cos 4x)/16$, $B(x) = (4x + \sin 4x)/16$, $y_p = (\cos 2x + 4x\sin 2x)/16$.

   $y(x) = c_1\cos 2x + c_2\sin 2x + (x\sin 2x)/4$.
10. $A(x) = \sin x + x\cos x$, $B(x) = -\cos x$, $y_p = -e^{-2x}\sin x$, $y(x) = (c_1x + c_2)e^{-2x} + y_p$.
11. $A(x) = -x$, $B(x) = \ln|x|$, $y_p = x[\ln|x| - 1]e^{-3x}$, $y(x) = (c_1x + c_2)e^{-3x} + y_p$.

12. $A(x) = (\cos 2x)/4$, $B(x) = (2x + \sin 2x)/4$, $y(x) = c_1e^{-x}\cos x + c_2e^{-x}\sin x + (xe^{-x}\sin x)/2$.
13. $g(x) = x$, $A(x) = x^2/4$, $B(x) = -x^4/8$, $y_p = x^3/8$, $y(x) = c_1x + (c_2/x) + y_p$.
14. $g(x) = \ln|x|$, $A(x) = [\ln|x|]^2/8$, $B(x) = -x^4[4\ln|x| - 1]/64$.
$y_p = x^2[8(\ln|x|)^2 - 4\ln|x| + 1]/64$, $y(x) = c_1x^2 + c_2/x^2 + y_p$.
15. $g(x) = 1/x^6$, $A(x) = [1 + 5\ln|x|]/(25x^5)$, $B(x) = -1/(5x^5)$,
$y_p = 1/(25x^4)$, $y(x) = c_1x + c_2x\ln|x| + y_p$.
16. $g(x) = x + (1/x)$, $A(x) = -[(x^2/2) + \ln|x|]$, $B(x) = x - (1/x)$,
$y_p = (x^3/2) - x(1 + \ln|x|)$, $y(x) = c_1x + c_2x^2 + y_p$.
17. $g(x) = 16e^{-2x}\operatorname{cosec}^2 2x$, $A(x) = 4\ln|\operatorname{cosec} 2x + \cot 2x|$, $B(x) = -4/\sin 2x$.
$y_p = 4e^{-2x}\cos 2x\ln|\operatorname{cosec} 2x + \cot 2x| - 4e^{-2x}$, $y(x) = e^{-2x}(c_1\cos 2x + c_2\sin 2x) + y_p$.
18. $A(x) = (\ln|\sec 2x + \tan 2x|)/8$, $B(x) = -x/4$, $C(x) = (\ln|\cos 2x|)/8$,
$y(x) = c_1 + c_2\cos 2x + c_3\sin 2x - (x\cos 2x)/4 + (\sin 2x\ln|\cos 2x|)/8 + (\ln|\sec 2x + \tan 2x|)/8$.
19. $A(x) = x^2/4$, $B(x) = -x$, $C(x) = (\ln|x|)/2$,
$y(x) = (c_1 + c_2x + c_3x^2)e^{2x} + (x^2\ln|x|e^{2x})/2$.
20. $y_p = \dfrac{1}{k}\displaystyle\int_0^x g(t)[\sin kx\cos kt - \cos kx\sin kt]\,dt = \dfrac{1}{k}\int_0^x g(t)\sin[k(x-t)]\,dt$.

Exercise 5.5

1. $y_p = -(50x^2 - 30x + 69)/500$, $y_c = Ae^{-2x} + Be^{5x}$.
2. $y_p = (20 - 51x + 9x^2 - 9x^3)/27$, $y_c = Ae^{-x} + Be^{3x/2}$.
3. $y_p = (35e^x + 3e^{3x})/105$, $y_c = Ae^{x/2} + Be^{-x/2}$.
4. $y_p = (e^{-2x} - 7x - 14)/7$, $y_c = Ae^{-x} + Be^{x/3}$.
5. $y_p = -e^{-3x} + e^x/15$, $y_c = Ae^{-2x} + Be^{-4x}$.
6. $y_p = 3xe^{-x}$, $y_c = Ae^{-x} + Be^{-3x}$.
7. $y_p = -xe^{-2x} + e^x/3$, $y_c = Ae^{-2x} + Be^{x/2}$.
8. $y_p = 2xe^{3x} - xe^{-2x}$, $y_c = Ae^{-2x} + Be^{3x}$.
9. $y_p = 2xe^{x/3}$, $y_c = Ae^{-2x} + Be^{x/3}$.
10. $y_p = (2\sin x - \cos x)/5$, $y_c = Ae^{-x} + Be^{-2x}$.
11. $y_p = (\sin 3x - 5\cos 3x)/2$, $y_c = Ae^{2x} + Be^{-3x}$.
12. $y_p = 2(\sin 2x - \cos 2x)$, $y_c = Ae^x + Be^{-5x}$.
13. $y_p = x(-3\cos 5x + 5\sin 5x)$, $y_c = A\cos 5x + B\sin 5x$.
14. $y_p = -2x\cos 4x$, $y_c = A\cos 4x + B\sin 4x$.
15. $y_p = 4x^2e^{2x} + e^{3x}$, $y_c = (Ax + B)e^{2x}$.
16. $y_p = 3x^2e^{(x/2)}/4$, $y_c = (Ax + B)e^{x/2}$.
17. $y_p = 13x^2e^{-3x} + e^{2x}/5$, $y_c = (Ax + B)e^{-3x}$.
18. $y_p = e^x(\sin x - 2\cos x)/5$, $y_c = A\cos x + B\sin x$.
19. $y_p = -(xe^{-x}\cos 3x)/6$, $y_c = e^{-x}(A\cos 3x + B\sin 3x)$.
20. $y_p = 8xe^{2x}\sin x$, $y_c = e^{2x}(A\cos x + B\sin x)$.

21. $y_p = -3xe^{3x}\cos 2x/4$, $y_c = e^{3x}(A\cos 2x + B\sin 2x)$.
22. $r(x) = 3e^{-2x}(1+\cos 2x)$, $y_p = e^{-2x}(c_1x^2 + c_2\cos 2x + c_3\sin 2x) = [3e^{-2x}(2x^2 - \cos 2x)]/4$, $y_c = (Ax+B)e^{-2x}$.
23. $r(x) = 3e^{-x}(3\sin x - \sin 3x)$, $y_p = e^{-x}[-45(\cos x + \sin x) + (\cos 3x + 3\sin 3x)]/10$, $y_c = Ae^{-x} + Be^{-2x}$.
24. $r(x) = 2(e^{3x} + e^{-3x})$, $y_p = (e^{-3x} + 12xe^{3x})/12$, $y_c = Ae^{x} + Be^{3x}$.
25. $y_p = -3xe^{-x}$, $y_c = Ae^{x} + Be^{-x} + Ce^{-4x}$.
26. $y_p = xe^{x} - 2x^2e^{-2x}$, $y_c = (Ax+B)e^{-2x} + Ce^{x}$.
27. $y_p = 6x^3e^{3x}$, $y_c = (Ax^2 + Bx + C)e^{3x}$.
28. $y_p = 2(\cos 2x - 2\sin 2x)/5$, $y_c = Ae^{x} + B\cos x + C\sin x$.
29. $y_p = -[2(x^2+x) + x(\cos 2x + \sin 2x)]/2$, $y_c = Ae^{2x} + B\cos 2x + C\sin 2x$.
30. $y_p = -x\sin 4x/2$, $y_c = Ae^{4x} + Be^{-4x} + C\cos 4x + D\sin 4x$.
31. $y_p = -(x^4 + 25)$, $y_c = Ae^{x} + Be^{-x} + C\cos x + D\sin x$.
32. $y_p = x^2 - 2x$, $y_c = A + (Bx^2 + Cx + D)e^{-x}$.
33. $y_p = 3xe^{2x}$, $y_c = Ae^{2x} + Be^{-2x} + C\cos x + D\sin x$.
34. $y_p = -5x^3e^{-2x}$, $y_c = A + (Bx^2 + Cx + D)e^{-2x}$.
35. $y_p = -(x^3 + 6x^2)/12$, $y_c = Ax + B + Ce^{4x} + De^{-4x}$.

Exercise 5.6

1. $y = Ax^2 + B/x^2$.
2. $y = (A/x) + (B/x^2)$.
3. $y = Ax + B/x$.
4. $y = (A + B\ln x)x^{-1/3}$.
5. $y = (A + B\ln x)x^{-3/2}$.
6. $y = A\cos(\ln x/\sqrt{2}) + B\sin(\ln x/\sqrt{2})$.
7. $y = (A + B\ln x)/x$.
8. $y = x[A\cos(2\ln x) + B\sin(2\ln x)]$.
9. $y = x^{-1}[A\cos(3\ln x) + B\sin(3\ln x)]$.
10. $y = x^{1/3}[A\cos(\ln x) + B\sin(\ln x)]$.
11. $y = A + Bx + C\ln x$.
12. $y = [A + B\ln x + C\ln^2 x]x$.
13. $y = Ax + x^{-1}[B\cos(\ln x) + C\sin(\ln x)]$.
14. $y = (A/x) + (B/x^2) + (C/x^3)$.
15. $y = (A/x) + (B + C\ln x)x^2$.
16. $y = (A/x^2) + x[B\cos(4\ln x) + C\sin(4\ln x)]$.
17. $y = A + Bx + Cx^2 + D\ln x$.
18. $y = Ax^2 + (B/x^2) + C\cos(\ln x) + D\sin(\ln x)$.
19. $y = A\sqrt{x} + (B/\sqrt{x}) + C\cos(2\ln x) + D\sin(2\ln x)$.
20. $y = (A + B\ln x)x + (C + D\ln x)/x$.
21. $y = Ax^2 + (B/x) - x - 3$.
22. $y = Ax + Bx^3 + \ln x + 2$.
23. $y = Ax + (B/x^2) + 2x\ln x + 7$.
24. $y = Ax^2 + (B/x^3) + 3x^2\ln x$.
25. $y = A + (B/x) + [\sin(\ln x) - \cos(\ln x)]/2$.
26. $y = Ax + (B/x^5) + 2x(3\ln^2 x - \ln x)/3$.
27. $y = (A + B\ln x)x^{1/2} + 4\cos(\ln x) - 3\sin(\ln x)$.
28. $y = (A + B\ln x)x^2 + x^3$.
29. $y = (A + B\ln x)x^{-3/2} + 2\sin(\ln x) - \cos(\ln x)$.
30. $y = Ax + (B/x^2) - x[3\cos(\ln x) + \sin(\ln x)]/10$.
31. $y = (A/x) + Bx^4 - x^2 - \ln x + 3/4$.
32. $y = Ax + (B/x) + (C/x^5) + 2x^2$.

33. $y = Ax^2 + (B/x^2) + (C/x^3) - (3 \ln x)/x^2$.

34. $y = (A + B \ln x + C \ln^2 x)x^2 + 3x^3 - 8x$.

35. $y = (A + B \ln x)x^{1/2} + (C/x) + \sin(\ln x) + 7\cos(\ln x)$.

36. Set $3x + 1 = z$, $y = [A + B \ln(3x + 1)](3x + 1)^{1/3} + \frac{3}{2}(x - 1)$.

37. Set $x + 2 = z$, $y = A(x + 2) + (x + 2)^{1/2}[B\cos t + C\sin t] + 8(x + 2)^2 - 96(x + 2)\ln(x + 2) - 96$,

where $t = \sqrt{3}\ln(x + 2)/2$.

38. $y = Ax + (B/x) + Cx^2 + (D/x^2) + 1/(4x^3)$.

39. $y = Ax^{3/2} + Bx^{-3/2} + (C + D\ln x)x + 2x^2 - 1/9$.

40. $y = A\cos(\ln x) + B\sin(\ln x) + C\cos(2\ln x) + D\sin(2\ln x) + 1/(20x^2)$.

41. $y = \frac{1}{4}\left(\sqrt{x} + \frac{1}{x}\right) + \frac{x}{2}$.

42. $y = 4(\ln x - 1)\sqrt{x} + \ln x + 4$.

43. $y = [7x - 10x^2 + 5x^3 + x\ln x]/2$.

44. $y = x[4\sin(\ln x) - 2\cos(\ln x)] + 3$.

45. $y = \frac{1}{x}[2\cos(3\ln x) + 3\sin(3\ln x) + \frac{x^2}{2}$.

Exercise 5.7

1. $Ae^{-x} + Be^{-4x} + e^{2x}$.

2. $Ae^{x} + Be^{-x} + e^{3x}$.

3. $Ae^{-x} + Be^{4x} + e^{5x} - (e^{x})/6$.

4. $e^{-x/2}[A\cos(\sqrt{7}x/2) + B\sin(\sqrt{7}x/2)] + \frac{4}{11}e^{x/2}$.

5. $e^{-3x/2}[A\cos(\sqrt{3}x/2) + B\sin(\sqrt{3}x/2)] + e^{x}$.

6. $(A + Bx)e^{x} + 4e^{2x} + (5e^{4x})/9$.

7. $(A + Bx)e^{x/3} + (e^{-x})/4$.

8. $(A + Bx)e^{3x} + 7x^2e^{3x}$.

9. $Ae^{2x} + Be^{-3x} + (xe^{2x})/5$.

10. $Ae^{2x} + Be^{-x/2} - e^{-x/2}(4x + 5x^2)/50$.

11. $Ae^{x} + Be^{-x} + [3e^{x}(x^2 - x)]/2$.

12. $Ae^{-2x} + Be^{-x/4} - \frac{1}{98}(7x^2 + 8x)e^{-2x}$.

13. $(A + Bx)e^{-x/3} + (x^2e^{-x/3})/18$.

14. $Ae^{x/2} + Be^{-4x} - e^{-4x}(9x^2 + 4x)/162$.

15. $Ae^{-x} + Be^{2x} + Ce^{-3x} - (e^{x})/2$.

16. $Ae^{x} + Be^{-2x} + Ce^{-x/2} + (e^{2x})/2$.

17. $Ae^{x} + Be^{-x} + Ce^{2x} + (e^{3x})/8$.

18. $(A + Bx + Cx^2)e^{2x} + 3x^3e^{2x}$.

19. $(A + Bx)e^{x} + Ce^{-x/2} + (8x^2e^{x})/3$.

20. $Ae^{2x} + Be^{-2x} + Ce^{-3x} - 3e^{-2x}(2x^2 - 3x)/4$.

21. $A\cos 4x + B\sin 4x + (\cos 2x)/12$.

22. $Ae^{x} + Be^{3x/2} + (\sin x + 5\cos x)/26$.

23. $Ae^{2x} + Be^{x/3} + (3\cos x - 4\sin x)/25$.

24. $Ae^{3x} + Be^{x/2} + (14\cos 2x - 5\sin 2x)/221$.

25. $e^{-x/2}[A\cos(\sqrt{3}x/2) + B\sin(\sqrt{3}x/2)] + 16\sin x$.

26. $e^{3x/4}[A\cos(x/4) + B\sin(x/4)] + 16(4\cos x - \sin x)/51$.

27. $A\cos 3x + B\sin 3x - (x\cos 3x)/6$.

28. $A\cos(\sqrt{3}x) + B\sin(\sqrt{3}x) + (x\sin\sqrt{3}x)/(2\sqrt{3})$.

29. $e^{-x}(A\cos 2x + B\sin 2x) + (xe^{-x}\sin 2x)/4$.

30. $e^{2x}(A\cos x + B\sin x) - 12\, x\cos x\, e^{2x}$.

31. $e^{3x}(A\cos 2x + B\sin 2x) - 7x\cos 2x\, e^{3x}$.

32. $e^{x}[A\cos 3x + B\sin 3x + x\,(8\sin 3x - 12\cos 3x)/3]$.

33. $Ae^{3x} + B\cos x + C\sin x - 3x(\cos x + 3\sin x)/10$.

34. $Ae^{x} + B\cos 3x + C\sin 3x - x(3\cos 3x + \sin 3x)/2$.

35. $Ae^{2x} + e^{x}(B\cos 2x + C\sin 2x) - 6xe^{x}(2\sin 2x - \cos 2x)/5$.

36. $Ae^{2x} + e^{x/2}(B\cos x + C\sin x) - 4xe^{x/2}(2\cos x + 3\sin x)/13$.

37. $A\cos x + B\sin x + C^*\cos 2x + D^*\sin 2x - 8x(\cos x + 2\sin 2x)/3$.

38. $A\cos 5x + B\sin 5x + (225x^3 + 100x^2 - 54x - 8)/625$.

39. $(A + Bx)e^{-3x} + (12x^2 - 16x + 5)/27$.

40. $Ae^{-x} + Be^{3x} - (18x^2 + 30x - 8)/27$.

41. $Ae^{2x} + Be^{3x} + [(52x + 25)(\cos 2x - 5\sin 2x) - 21(5\cos 2x + \sin 2x)]/2704$.

42. $Ae^{x} + Be^{-2x} - [(25x^2 + 5x - 9)(3\sin x + \cos x) + (35x + 12)(3\cos x - \sin x)]/250$.

43. $Ae^{3x} + Be^{-2x} - e^{-2x}(5x^2 + 2x)/50$.

44. $Ae^{-3x} + Be^{-4x} + e^{x}(8\sin 2x - 9\cos 2x)/290$.

45. $Ae^{-x} + Be^{-3x} + e^{2x}(7\cos x + 4\sin x)/130$.

46. $e^{-3x/2}[A\cos p + B\sin p] + 4e^{x}(25\cos p + 10\sqrt{7}\sin p)/1325,\; p = \sqrt{7}x/2$.

47. Write $xe^{x}\sin x = \operatorname{Im}[xe^{(1+i)x}]$, $Ae^{-x} + Be^{-2x} + e^{x}[5(1 - x)\cos x + (5x - 2)\sin x]/50$.

48. Write $xe^{2x}\cos x = \operatorname{Re}[xe^{(2+i)x}]$, $A\cos 3x + B\sin 3x + e^{2x}[(30x - 11)\cos x + (10x - 2)\sin x]/400$.

49. $Ae^{-x/2} + Be^{-3x/2} - e^{-x/2}[(x - 2)\cos x - (x + 1)\sin x]/8$.

50. $A\cos x + B\sin x + C^*\cos\sqrt{2}x + D^*\sin\sqrt{2}x - 4[9x^2\cos x - (2x^3 - 51x)\sin x]/3$.

51. $y = Ae^{x/2} + Be^{3x},\quad B = 1/5$.

52. $\int e^{-mx}r(x)\,dx = \int e^{-mx}(D - m)y\,dx = e^{-mx}y$, or $y = e^{mx}\int e^{-mx}r(x)\,dx$.

53. Use the result

$$\frac{d}{dx}\int_a^b f(x,t)\,dt = f(x,b)\frac{db}{dx} - f(x,a)\frac{da}{dx} + \int_a^b \frac{\partial f}{\partial x}\,dt$$

$$\frac{dy}{dx} = \int_a^x r(t)\cos n(x-t)\,dt,\quad \frac{d^2y}{dx^2} = r(x) - n\int_a^b r(t)\sin n(x-t)\,dt = r(x) - n^2 y.$$

54. $D^m(x\,u) = xD^m u + mD^{m-1}u = xD^m u + \left[\frac{d}{dD}D^m\right]u \quad m = 1, 2, \ldots$

$$F(D)(x\,u) = x[a_0D^n + a_1D^{n-1} + \ldots + a_n]u + \frac{d}{dD}[a_0D^n + a_1D^{n-1} + \ldots + a_n]u$$

$$= xF(D)u + F'(D)u.$$

55. $F(D)(xv) = x\,F(D)v + F'(D)v$. Let $F(D)v = u$.

$F(D)[x\{F(D)\}^{-1}u] = xF(D)[F(D)]^{-1}u + F'(D)[F(D)]^{-1}u = xu + F'(D)[F(D)]^{-1}u$