Q1 Ten coins are thrown simultaneously. Find the probability of getting at least seven ~~heads~~. heads.

Sol

$n = 10$ $\quad p = \frac{1}{2}$ $\quad q = \frac{1}{2}$

$$p(x \geq 7) = p(7) + p(8) + p(9) + p(10)$$

$$p(x) = {}^{n}C_{x}\, p^{x}\, q^{n-x}$$

$$p(x \geq 7) = {}^{10}C_{7}\left(\frac{1}{2}\right)^{7}\left(\frac{1}{2}\right)^{3} + {}^{10}C_{8}\left(\frac{1}{2}\right)^{8}\left(\frac{1}{2}\right)^{2}$$

$$+ {}^{10}C_{9}\left(\frac{1}{2}\right)^{9}\left(\frac{1}{2}\right)^{1} + {}^{10}C_{10}\left(\frac{1}{2}\right)^{10}\left(\frac{1}{2}\right)^{0}$$

$$^{10}C_7 = {}^{10}C_3 = \frac{10 \times \cancel{9}^{3} \times \cancel{8}^{4}}{\cancel{3} \times \cancel{2} \times 1} = 120$$

$$^{10}C_8 = {}^{10}C_2 = \frac{10 \times 9}{2} = 45$$

$$^{10}C_9 = {}^{10}C_1 = 10$$

$$^{10}C_{10} = 1$$

$$P(X \geq 7) = \frac{120}{2^{10}} + \frac{45}{2^{10}} + \frac{10}{2^{10}} + \frac{1}{2^{10}}$$

$$= \frac{120+45+10+1}{2^{10}} = \frac{176}{2^{10}}$$

$$= \frac{\cancel{176}\ \ \cancel{88}\ \ \cancel{44}\ \ \cancel{22}\ \ 11}{\cancel{1024}\ \ \cancel{512}\ \ \cancel{256}\ \ \cancel{128}\ \ 64} = \frac{11}{64}$$

Q2

Find the mean of Binomial dist.

Sol $f(x) = P(X=x) = {}^nC_x \, p^x \, q^{n-x}$, $x = 0, 1, 2, \ldots, n$

$$B(x; n, p) = {}^nC_x \, p^x q^{n-x}$$

$$E(X) = \bar{X} = \sum_{x=0}^{n} x \, p(x) = \sum_{x=0}^{n} x \, B(x; n, p)$$

$$E(x) = \bar{x} = \sum_{x=0}^{n} x \, {}^{n}C_{x} \, p^{x} \, q^{n-x}$$

$$\left\{ {}^{n}C_{x} = \frac{n}{x} \, {}^{n-1}C_{x-1} \right\}$$

$$E(x) = \bar{x} = \sum_{x=0}^{n} x \, \frac{n}{x} \, {}^{n-1}C_{x-1} \, p^{x-1+1} \, q^{n-1-x+1}$$

$$E(x) = \bar{x} = \sum_{x=1}^{n} n \; {}^{n-1}C_{x-1} \, p^{x-1} p^{1} q^{(n-1)-(x-1)}$$

$$= np \sum_{x=1}^{n} {}^{n-1}C_{x-1} \, p^{x-1} q^{(n-1)-(x-1)}$$

$$E(x) = np \left[{}^{n-1}C_0 \, p^0 q^{n-1} + {}^{n-1}C_1 \, p^1 q^{n-2} + \cdots + {}^{n-1}C_{n-1} \, p^{n-1} q^0 \right]$$

$$\left\{ (a+b)^n = {}^nC_0\, a^0 b^n + {}^nC_1\, a^1 b^{n-1} + {}^nC_2\, a^2 b^{n-2} + \cdots + {}^nC_n\, a^n b^{n-n} \right\}$$

$$E(x) = np\left[(p+q)^{n-1}\right] \qquad \left\{ \because p+q = 1 \right\}$$

$$E(x) = np(1)^{n-1}$$

$$\boxed{E(x) = np}$$

$E(x) = \text{mean} = np$

$\underline{x^2 = x^2 - x + x}$

$V(x) = npq$

$V(x) = E(x^2) - (E(x))^2$

$= \sum_{x=0}^{n} x^2 \, {}^{n}C_x \, p^x q^{n-1} - (np)^2$

$V(x) = npq$

Q3 A and B ~~play~~ play a game in which their chances of winning are in the ratio 3 : 2. Find A's chance of winning at least three games out of the five games played.

$A : B \quad , \quad 3 : 2$

$P(A) = \frac{3}{3+2} = \frac{3}{5}$

$n = 5$

$P(B) = \frac{2}{3+2} = \frac{2}{5}$

$p = \frac{3}{5} \qquad q = 1 - \frac{3}{5} = \frac{2}{5}$

$P(x \geq 3) = P(3) + P(4) + P(5)$

$= {}^5C_3 \left(\frac{3}{5}\right)^3 \left(\frac{2}{5}\right)^2 + {}^5C_4 \left(\frac{3}{5}\right)^4 \left(\frac{2}{5}\right)^1 + {}^5C_5 \left(\frac{3}{5}\right)^5 \left(\frac{2}{5}\right)^0$

$$P(X \geq 3) = \frac{5\times 4}{2!}^{\,2} \times \frac{3}{5} \times \frac{3}{5} \times \frac{3}{5} \times \frac{2}{5} \times \frac{2}{5}$$

$$+\ 5 \times \frac{3}{5} \times \frac{3}{5} \times \frac{3}{5} \times \frac{3}{5} \times \frac{2}{5}$$

$$+\ 1 \times \frac{3\times 3\times 3\times 3\times 3}{5\times 5\times 5\times 5\times 5}$$

$$= \frac{1080 + 810 + 243}{3125} = \frac{2133}{3125}$$

Q.4 The mean and variance of binomial distribution are 4 and $\frac{4}{3}$ respectively. Find $P(X \geq 1)$.

Sol mean = 4 Variance = $\frac{4}{3}$

$np = 4$

$npq = \frac{4}{3}$

$$\frac{\cancel{np}q}{\cancel{np}} = \frac{4/3}{4}$$

$$\boxed{q = \frac{1}{3}}$$

$$p = 1 - \frac{1}{3} = \frac{2}{3}$$

$$\boxed{p = \frac{2}{3}}$$

$np = 4$

$n \times \frac{2}{3} = 4$

$n = 6$

$$P(x \geq 1) = 1 - P(0)$$

$$= 1 - \left[{}^6C_0 \left(\frac{2}{3}\right)^0 \left(\frac{1}{3}\right)^{6-0} \right]$$

$$= 1 - \left[1 \times 1 \times \frac{1}{3^6} \right]$$

$$= 1 - \left[\frac{1}{729} \right] = \frac{729-1}{729} = \frac{728}{729}$$

Q 5

Determine the binomial distribution for which the mean is 4 and variance 3 and find its mode.

Sol

mean = 4 $\qquad$ $V(x) = 3$

$np = 4$ $\qquad$ $npq = 3$

$$\frac{npq}{np} = \frac{3}{4}$$

$$\boxed{q = \frac{3}{4}}$$

$$\boxed{p = 1 - \frac{3}{4} = \frac{1}{4}}$$

$$n \times \frac{1}{4} = 4$$

$$\boxed{n = 16}$$

$$(n+1)p = (16+1)\frac{1}{4} = \frac{17}{4} = 4.25$$

$$\boxed{\text{Mode} = 4}$$

Q6 The mean and variance of a binomial variate X with parameters n and p are 16 and 8. Find

(i) $P(X = 0)$, (ii) $P(X = 1)$, (iii) $P(X \geq 2)$.

mean = 16

$np = 16$

$V(x) = 8$

$npq = 8$

$np = 16$

$\frac{npq}{np} = \frac{8}{16}$

$p = 1 - \frac{1}{2} = \frac{1}{2}$

$n \times \frac{1}{2} = 16$

$q = \frac{1}{2}$

$n = 32$

$$P(x) = {}^{n}C_{x} \; p^{x} \; q^{n-x}$$

$$P(0) = {}^{32}C_{0} \left(\frac{1}{2}\right)^{0} \left(\frac{1}{2}\right)^{32-0} = \frac{1}{2^{32}}$$

$$P(x=1) = {}^{32}C_{1} \left(\frac{1}{2}\right)^{1} \left(\frac{1}{2}\right)^{32-1} = \frac{32}{2^{1} \; 2^{31}}$$

$$= \frac{32}{2^{32}}$$

$$P(X \geqslant 2) = 1 - P(X < 2)$$

$$= 1 - \left[P(0) + P(1) \right]$$

$$= 1 - \left[\frac{1}{2^{32}} + \frac{32}{2^{32}} \right]$$

$$= 1 - \frac{33}{2^{32}} = \frac{2^{32} - 33}{2^{32}}$$

Q7 Determine the probability of getting sum 9 exactly twice in 3 throw with a pair of fair dice.

Sol

$n = 3$

$6 \times 6 = 36$

$p = \frac{4}{36} = \frac{1}{9}$

$(6,3)\ (3,6)\ (4,5)\ (5,4)$

$q = 1 - \frac{1}{9} = \frac{8}{9}$

$$P(x=2) = {}^{n}C_{2}\, p^{2}\, q^{n-2}$$

$$= {}^{3}C_{2} \left(\frac{1}{9}\right)^{2} \left(\frac{8}{9}\right)^{3-2}$$

$$= \cancel{3} \times \frac{1}{\underset{3}{\cancel{9}}} \times \frac{1}{9} \times \frac{8}{9}$$

$$P(x=2) = \frac{8}{243}$$

Q 8 Out of 800 families with 5 children each, how many families would you expect to have (a) 3 Boys (b) 5 girls (c) either 2 or 3 Boys.

Assume equal probabilities for Boys and Girls.

<u>Sol</u> $n = 5 \quad p = \frac{1}{2} \quad q = \frac{1}{2} \quad N = 800$

$$E(3) = N\,P(3) = 800\ {}^5C_3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2$$

$$= \overset{\cancel{100}\ 50}{8\cancel{0}0} \times \frac{5 \times \cancel{4} \times \cancel{3}}{\cancel{3} \times \cancel{2} \times 1} \times \frac{1}{\cancel{2} \times \cancel{2} \times \cancel{2}} \times \frac{1}{\cancel{2} \times \cancel{2}}$$

$$= 250$$

$$E(0\ \text{Boys}) = E(5\ \text{girls})$$

$$= N\,P(0)$$

$$= 800\ {}^{5}C_{0}\left(\tfrac{1}{2}\right)^{0}\left(\tfrac{1}{2}\right)^{5-0}$$

$$= \underset{25}{\cancel{800}} \times \frac{1}{\cancel{2}} \times \frac{1}{\cancel{2}} \times \frac{1}{\cancel{2}} \times \frac{1}{\cancel{2}} \times \frac{1}{\cancel{2}}$$

$$= 25$$

$$E(2 \text{ or } 3) = N \; P(2 \text{ or } 3)$$

$$= 800 \left(P(2) + P(3) \right)$$

$$= 800 \; P(2) + 800 \; P(3)$$

$$= 800 \; {}^{5}C_{2} \left(\frac{1}{2}\right)^{2} \left(\frac{1}{2}\right)^{5-2}$$

$$+ 800 \; {}^{5}C_{3} \left(\frac{1}{2}\right)^{3} \left(\frac{1}{2}\right)^{5-3}$$

$$E(2023) = \overset{50\ \cancel{400}}{8\cancel{0}0} \times \frac{5 \times 4}{\cancel{2}} \times \frac{1}{\cancel{2}} \times \frac{1}{\cancel{2}} \times \frac{1}{\cancel{2}} \times \frac{1}{\cancel{2}} \times \frac{1}{\cancel{2}}$$

$$+ \overset{\cancel{100}\ 50}{8\cancel{0}0} \times \frac{5 \times 4 \times \cancel{3}}{\cancel{3} \times \cancel{2} \times 1} \times \frac{1}{\cancel{2}} \times \frac{1}{\cancel{2}} \times \frac{1}{\cancel{2}} \times \frac{1}{\cancel{2}} \times \frac{1}{\cancel{2}}$$

$$= 250 + 250$$

$$= 500$$