L30 31 March 2021

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Laplace Equation :→

Two dimensional heat eqn. $u(x,y,t)$

$$\frac{\partial u}{\partial t} = c^2\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]$$

steady state temperature

$$\frac{\partial u}{\partial t} = 0$$

$$0 = c^2\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]$$

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0} \rightarrow \text{Laplace equation}$$

$0 < x < a$

$0 < y < b$

Let us Consider $u(x,y) = X(x)\,Y(y)$

$$\frac{\partial u}{\partial x} = X'Y \qquad \frac{\partial u}{\partial y} = XY'$$

$$\frac{\partial^2 u}{\partial x^2} = X''Y \qquad \frac{\partial^2 u}{\partial y^2} = XY''$$

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2}$$

$$X''Y = -XY''$$

$$\Rightarrow \quad \frac{X''}{X} = -\frac{Y''}{Y} = K \text{ (say)}$$

Case 1 $\quad K = 0$

$$\frac{X''}{X} = 0 \Rightarrow X'' = 0 \Rightarrow X' = a$$
$$\Rightarrow \boxed{X = ax + b}$$

$$-\frac{Y''}{Y} = 0 \Rightarrow Y'' = 0 \Rightarrow Y' = c$$
$$\boxed{Y = cy + d}$$

Y

$\boxed{Y = cy + d}$

Required sol

$$\boxed{u(x,y) = (ax+b)(cy+d)}$$

Case No ② $K > 0$

$K = p^2$ (positive)

$$\frac{X''}{X} = p^2 \Rightarrow X'' = p^2 X \Rightarrow X'' - p^2 X = 0$$

A.E $D^2 - p^2 = 0 \Rightarrow D = \pm p$

$$\boxed{X = c_1 e^{px} + c_2 e^{-px}}$$

$$-\frac{Y''}{Y} = p^2 \Rightarrow Y'' + p^2 Y = 0$$

A.E $D^2 + p^2 = 0 \Rightarrow D^2 = -p^2 \Rightarrow D = \pm ip$ ✓

$$\boxed{Y = c_3 \cos py + c_4 \sin py}$$

$$\boxed{u(x,y) = \left(c_1 e^{px} + c_2 e^{-px}\right)\left(c_3 \cos py + c_4 \sin py\right)}$$

Case No:- ③ $K = -ve$

$K = -p^2$

$$\frac{X''}{X} = -p^2 \Rightarrow X'' = -p^2 X$$

$$X'' + p^2 X = 0$$

A.E $D^2 + p^2 = 0 \Rightarrow D = \pm ip$

$$\boxed{X = c_1 \cos px + c_2 \sin px}$$

$$\cancel{-}\frac{Y''}{Y} = \cancel{+}p^2 \Rightarrow Y'' = p^2 Y$$

$$Y'' - p^2 Y = 0$$

$Y'' - p^2 Y = 0$

A.E $\quad D^2 - p^2 = 0 \quad D = \pm p$

$$\boxed{Y = c_3 e^{py} + c_4 e^{-py}}$$

$$\boxed{u(x,y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py})}$$

Note: Laplace Equation

Equation: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Nature: Elliptic

Solution: **1.** $u(x,y) = (ax+b)(cy+d)$

2. $u(x,y) = (ae^{px} + be^{-px})(c\cos py + d\sin py)$

X'' $\quad$ Y''

or

$u(x,y) = (a\cosh px + b\sinh px)(c\cos py + d\sin py)$

3. $u(x,y) = (a\cos px + b\sin px)(ce^{py} + de^{-py})$

or

$u(x,y) = (a\cos px + b\sin px)\,(c\cosh py + d\sinh py)$

Q38. The only suitable general solution of Lapalce's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ is

(a) $u = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$ (b) $u = (c_1 \cos px + c_2 \sin px)\, c_3 e^{-c^2p^2t}$

(c) $u = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py})$ (d) None of these

$c=1$

Q46. Solution of given heat flow equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ where $u(0,t) = u(l,t) = 0$, $u(x,0) = \sin x$, $0 < x < l$, $t > 0$ is

(a) $u(x,t) = \sin x\, e^{-(l)^2 t}$ (b) $u(x,t) = \sin x \cos 2\pi ct$ (c) $u(x,t) = (\sin x) e^{-t}$ (d) None of these

$c=1$

$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ $\qquad$ $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ $\qquad$ $u(0,t) = u(l,t) = 0$

$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$

$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$

$u(0,t) = u(l,t) = 0$

$u(x,0) = \sin x$

$\boxed{(A\cos px + B\sin px) \cdot e^{-c^2p^2t}}$

Q44. Which of the following is the solution of given partial differential equation $y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = 0$

(a) $Ae^{\lambda\left(\frac{x^2-y^2}{2}\right)}$ (b) $Ae^{\lambda\left(\frac{x^2+y^2}{2}\right)}$ (c) $Ae^{\lambda(x^2-4y^2)}$ (d) $Ae^{\lambda(4x^2+y^2)}$

$y\frac{\partial u}{\partial x} = -x\frac{\partial u}{\partial y}$

$\frac{\partial u}{\partial x} = Ae^{\lambda\left(\frac{x^2-y^2}{2}\right)} \cdot \lambda x$

$\frac{\partial u}{\partial y} = Ae^{\lambda\left(\frac{x^2-y^2}{2}\right)} \cdot (-\lambda y)$

$\cancel{\lambda xyA = + \lambda xyA}$

$u = Ae^{\lambda\left(\frac{x^2-y^2}{2}\right)}$

$\frac{\partial u}{\partial x} = A\, e^{\lambda\left(\frac{x^2-y^2}{2}\right)} \cdot \lambda \cdot \left[\frac{2x}{2}\right] = \cancel{y\lambda xA\, e^{\lambda\left(\frac{x^2-y^2}{2}\right)}}$

$\frac{\partial u}{\partial y} = A \cdot e^{\lambda\left(\frac{x^2-y^2}{2}\right)} \cdot \lambda\left(\frac{2y}{2}\right) = \cancel{x\lambda yA\, e^{\lambda\left(\frac{x^2-y^2}{2}\right)}}$

$= 0$