MOMENT GENERATING FUNCTION

(I) Moment generating function about origin.

The moment generating function (mgf) of a random variable X (about origin) having the probability function $f(x)$ is given by

$$M_X(t) = E\left(e^{tX}\right) = \begin{cases} \sum_x e^{tx} f(x) & \text{(for discrete random variable} \\ \int e^{tx} f(x)\,dx & \text{(for continuous random variable} \end{cases} \quad \text{—— (1)}$$

the integration or summation being extended to entire range of x, t being the real parameter and it is being assumed that right hand side of equation (1) is absolutely convergent for some positive h such that $-h < t < h$

$$M_X(t) = E\left(e^{tX}\right)$$

$$= E\left(1 + tX + \frac{t^2X^2}{2!} + \cdots + \frac{t^rX^r}{r!} + \cdots\right)$$

$$= E(1) + E(tX) + E\left(\frac{t^2X^2}{2!}\right) + \cdots + E\left(\frac{t^rX^r}{r!}\right) + \cdots$$

$$= E(1) + tE(X) + \frac{t^2}{2!}E(X^2) + \cdots + \frac{t^r}{r!}E(X^r)$$

If we represent

$E(x) = \mu_1'$ First moment about origin

$E(x^2) = \mu_2'$ second moment about origin.

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$E(x^r) = \mu_r'$ r^{th} moment about origin

and $E(1) = 1$

then

$$M_x(t) = 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \dots + \frac{t^r}{r!}\mu_r'$$

$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r' \quad \text{———— } (2)$$

$$\text{where } \mu_r' = E(x^r) = \begin{cases} \sum_x x^r f(x) & \text{for DRV} \\ \int_{-\infty}^{\infty} x^r f(x)\,dx, & \text{for CRV} \end{cases}$$

DRV — Discrete random variable

CRV — Continuous random variable

so μ_r' is r^{th} moment of X about origin that is coefficient of $\frac{t^r}{r!}$ in $M_x(t)$ gives μ_r' (about origin)

Since $M_x(t)$ generates moment, it is Known as moment generating function

On differentiating equation (2) with respect to 't' we have and put $t=0$

$$\frac{d}{dt}(M_x(t)) = \frac{d}{dt}\left(1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \cdots + \frac{\mu_r' t^r}{r!} + \cdots\right)$$

$$= 0 + (1)\mu_1' + \frac{2t}{2!}\mu_2' + \cdots + \frac{r t^{r-1}}{r!}\mu_r' + \cdots$$

Put $t=0$

$$\frac{d}{dt}(M_x(t)) = 0 + \mu_1' + 0 + 0 + 0 + \cdots + 0 + \cdots$$

$$\frac{d}{dt}(M_x(t)) = \mu_1'$$

So on differentiating $M_x(t)$ one time we have obtained μ_1' and substituting $t=0$

Similarly differentiating again

$$\frac{d^2}{dt^2}(M_x(t)) = \frac{d}{dt}\left[\mu_1' + \frac{2t}{2!}\mu_2' + \cdots + \frac{r t^{r-1}}{r!}\mu_r' + \cdots\right]$$

$$= \left[0 + \frac{2(1)}{2!}\mu_2' + \frac{3\times 2\times t}{3!}\mu_3' + \cdots + \frac{r(r-1)t^{r-2}}{r!}\mu_r' + \cdots\right]$$

Put $t=0$

$$\frac{d^2}{dt^2}M_x(t) = \left[0 + \mu_2' + 0 + 0 + \cdots + 0 + \cdots\right]$$

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$$\frac{d^2}{dt^2} M_x(t) = \mu_2'$$

So on differentiating r time we have obtained

$$\frac{d^r}{dt^r}(M_x(t)) = \frac{d}{dt}\left[0+0+0 \cdots + (r-1)(r-2)\cdots \frac{\mu_{r-1}'}{(r-1)!} + r(r-1)(r-2)\cdots 3.2(t)\frac{\mu_r'}{r!} + \cdots\right]$$

$$\frac{d^r}{dt^r} M_x(t) = 0+0+0+\cdots+\frac{r(r-1)(r-2)\cdots 3.2\times 1\ \mu_r'}{r!} + \cdots \infty$$

put $t=0$ we have

$$\boxed{\frac{d^r}{dt^r} M_x(0) = \mu_r'}$$

That is on differentiating $M_x(t)$ r times and substituting $t=0$ we will obtained r^{th} moment μ_r' about origin.

Moment generating function of X about the point x = a

(5)

In general the moment generating function of X about the point $x = a$ is define as

$$M_x(t) = E\left[e^{t(x-a)}\right]$$

$$M_x(t) = E\left[1 + t(x-a) + \frac{t^2}{2!}(x-a)^2 + \dots + \frac{t^r}{r!}(x-a)^r + \dots\right]$$

$$= 1 + t\,E(x-a) + \frac{t^2}{2!}E(x-a)^2 + \dots + \frac{t^r}{r!}E(x-a)^r + \dots$$

If $\mu_1' = E(x-a)$

$\mu_2' = E\left[(x-a)^2\right]$

$\vdots$

$\mu_r' = E\left[(x-a)^r\right]$

Then

$$M_x(t) = 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \dots + \frac{t^r}{r!}\mu_r' + \dots \quad (3)$$

where $\mu_r' = E\left[(x-a)^r\right]$ is the r^{th} moment about the point $x = a$.

If in place of a we take mean($\bar{x}$) (6)

put $a = \bar{x}$

Then
$\mu_r' = E((x-\bar{x})^r)$ is the r^{th} moment about the mean.

Remark

(1) We know that that $E(x) = $ mean, and by moment generating function (about origin)

$$\boxed{\mu_1' = E(x)}$$

so First moment about origin is mean.

(2) We know that the $E[(x-\bar{x})^2]$ = variance so by moment generating function (about mean)

$$\boxed{\mu_2' = E(x-\bar{x})^2}$$

so second moment about mean is Variance.

Properties of Moment Generating function

① $\boxed{M_{cx}(t) = M_x(ct)}$

$\because \quad M_{cx}(t) = E\left(e^{tcx}\right)$

$M_x(ct) = E\left(e^{ctx}\right)$

② The moment generating function of sum of a number of independent variables is equal to the product of their respective moment generating function

i.e $\boxed{M_{x_1+x_2+\dots+x_n}(t) = M_{x_1}(t)\, M_{x_2}(t) \dots M_{x_n}(t)}$

③ The moment generating function of a distribution if it exist uniquelly determine the distribution

④ First moment about origin $\mu_1' = E(x)$ is mean

⑤ Second moment about mean $\mu_2' = E\left[(x-\bar{x})^2\right]$ is variance

Example ① Let random variable x assume the value 'r' with probability law

$$\boxed{p(x=r) = q^{r-1}p} \; ; \; r=1,2,3\ldots\ldots \quad (1)$$

Find moment generating function of x and hence its mean and variance. (given $p+q=1$)

Solution We know that $M_x(t) = E(e^{tx})$

$$M_x(t) = \sum_{r=1}^{\infty} e^{tx}\, p(x=r)$$

$$\{\because x=r\}$$

$$= \sum_{r=1}^{\infty} e^{tr}\, q^{r-1}\, p$$

$$= \sum_{r=1}^{\infty} e^{tr}\, q^{r}\, q^{-1}\, p$$

$$= \sum_{r=1}^{\infty} e^{tr}\, \frac{q^{r}}{q}\, p$$

$$= \frac{p}{q} \sum_{r=1}^{\infty} e^{tr} q^{r}$$

$$M_x(t) = \frac{p}{q}\left[e^{t} q + e^{2t} q^{2} + e^{3t} q^{3} + \ldots\ldots \infty \right]$$

which is infinite Geometric progression

{ By applying formula $S_\infty = \frac{a}{1-R}$ }

{ $a = e^t q$ $\quad R = \text{Common ratio} = e^{2t} q$ }

$$\Rightarrow M_x(t) = \frac{p}{q}\left[\frac{e^t q}{1-e^t q}\right]$$

$$\boxed{M_x(t) = \frac{pe^t}{1-e^t q}} \quad (2)$$

which is moment generating function

<u>For calculation of mean</u>

We know that

$$\text{mean} = \mu_1' = \frac{d}{dt}(M_x(t)) \text{ at } t=0$$

On differentiating equation (2) we have

$$\frac{d}{dt}(M_x(t)) = \frac{d}{dt}\left[\frac{pe^t}{1-e^t q}\right]$$

$$= p\left[\frac{(1-e^t q)[e^t] - e^t(-q e^t)}{(1-e^t q)^2}\right]$$

$$\frac{d}{dt}(M_x(t)) = p\left[\frac{e^t - e^{2t} q + e^{2t} q}{(1-e^t q)^2}\right]$$

$$\frac{d}{dt} M_x(t) = p\frac{\left[e^t - 0\right]}{(1-e^t q)^2}$$

$$\frac{d}{dt}\left(M_x(t)\right) = \frac{p e^t}{(1-e^t q)^2} \quad \text{——— (3)}$$

Put $t=0$

$$\text{mean} = \frac{d}{dt}\left(M_x(0)\right) = \frac{p e^0}{(1-e^0 q)^2} = \frac{p}{(1-q)^2}$$

$\{ \because \; p+q=1 \quad p = 1-q \}$

$$\text{mean} = \frac{p}{p^2} = \frac{1}{p}$$

$$\boxed{\text{mean} = \frac{1}{p}} \qquad \text{mean} = E(x)$$

For calulation of variance

We know that

$$\text{Variance} = E(x^2) - \left[E(x)\right]^2$$

$$E(x^2) = \mu_2' = \frac{d^2}{dt^2}\left(M_x(t)\right) \{\text{at } t=0\}$$

$$E(x) = \mu_1' = \frac{d}{dt}\left(M_x(t)\right) \{\text{at } t=0\}$$

$$E(x^2) = \frac{p(1+q)}{(1-q)^3} \qquad \{\because p = 1-q\}$$

$$E(x^2) = \frac{p(1+q)}{p^3} = \frac{1+q}{p^2}$$

So variance $= E(x^2) - [E(x)]^2$

$$= \frac{1+q}{p^2} - \left[\frac{1}{p}\right]^2$$

$$= \frac{1+q}{p^2} - \frac{1}{p^2}$$

$$= \frac{1+q-1}{p^2}$$

$$\boxed{V(x) = \frac{q}{p^2}}$$

Remark

① Moment generating function of Binomial distribution, we know that binomial dist

$$b(x; n, p) = f(x) = {}^nC_x\, p^x q^{n-x}, \quad x = 0, 1, \dots n$$

$$p + q = 1$$

for $E(x^2)$ differentiating ③ with respect to t

$$\frac{d^2}{dt^2} M_x(t) = \frac{d}{dt}\left(\frac{d}{dt} M_x(t)\right)$$

$$= \frac{d}{dt}\left[\frac{d}{dt} M_x(t)\right]$$

$$= \frac{d}{dt} \frac{p e^t}{(1-e^t q)^2}$$

$$= p\left[\frac{(1-e^t q)^2 e^t - e^t\, 2(1-e^t q)(-e^t q)}{(1-e^t q)^4}\right]$$

$$= \frac{p(1-e^t q)\left[e^t - e^{2t} q + 2 e^{2t} q\right]}{(1-e^t q)^4}$$

$$= \frac{p(e^t + e^{2t} q)}{(1-e^t q)^3}$$

$$\frac{d^2}{dt^2}(M_x(t)) = \frac{p e^t (1+e^t q)}{(1-e^t q)^3}$$

at $t=0$

$$E(x^2) = \frac{d^2}{dt^2}\left[M_x(0)\right] = \frac{p e^0 (1+e^0 q)}{(1-e^0 q)^3}$$

So $M_x(t) = E(e^{tx})$

$$= \sum_{x=0}^{n} e^{tx}\, {}^nC_x\, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} {}^nC_x (pe^t)^x q^{n-x}$$

$$\boxed{M_x(t) = (q + pe^t)^n}$$

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(2) Moment generating function of the Poisson distribution

$$M_x(t) = E(e^tx) = \sum_{x=0}^{\infty} e^{tx}\, p(x=x).$$

we know that for Poisson distribution

$$p(x=x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

$$\Rightarrow M_x(t) = \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!}$$

$$= e^{-\lambda}\left[1 + \lambda e^t + \frac{(\lambda e^t)^2}{2!} + \frac{(\lambda e^t)^3}{3!} + \cdots\right]$$

$$= e^{-\lambda} e^{\lambda e^t} = e^{\lambda e^t - 1}$$

So Moment generating function for Poisson distribution

$$\boxed{M_x(t) = e^{\lambda(e^t - 1)}}$$

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(3) Moment generating function for Normal Distribution

we know that

$$N(\mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty$$

where μ = population mean
σ = standard deviation

So

$$M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} N(\mu, \sigma)\, dx$$

$$M_x(t) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx$$

on solving

$$\Rightarrow \boxed{M_x(t) = e^{\mu t + t^2\sigma^2/2}}$$