L25 20 March 2021

20 March 2021 11:04

Unit-4 Partial Differential Equations

Ordinary diff. eqn

$y = f(x)$

dependent | independent

$\frac{dy}{dx} = ?$

$\boxed{\frac{dy}{dx} + Py = 0}$

$Z = f(x, y)$

Dependent Variable

independent variable

$\frac{\partial z}{\partial x}$ $\frac{\partial z}{\partial y}$

$$f\left(z, x, y, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) = 0$$

$z = f(x, y)$

$$\left.Z_x = \frac{\partial z}{\partial x} = p, \quad Z_y = \frac{\partial z}{\partial y} = q \right\} \rightarrow \text{first order derivates}$$

$$\frac{\partial^2 z}{\partial x^2} = Z_{xx} = r$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = Z_{xy} = Z_{yx} = s$$

$$\frac{\partial^2 z}{\partial y^2} = Z_{yy} = t$$

$\rightarrow$ 2nd order derivates

$z = x^2 + y^2$

$\frac{\partial z}{\partial x}$ $\frac{\partial z}{\partial y}$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}$$

Linear?

\# An equation of the form $f(x, y, z, p, q) = 0$ is called first order partial diff. equation

$\cancel{z \cdot \frac{\partial z}{\partial x}} \times \cancel{\left(\frac{\partial z}{\partial x}\right)^2}$

\# An equation of the form $f(x, y, z, p, q, r, s, t) = 0$ is called 2nd order partial diff. eqns.

$y\frac{dy}{dx} + 2 = 0$

Non Linear

Formulation of PDEs

① By elimination of arbitrary constants

Problem 1. **Form the PDE for:** $\boldsymbol{u = ax + by,}$ $\boldsymbol{a}$ **and** $\boldsymbol{b}$ **are constants**

$\frac{\partial u}{\partial x} = a(1) + b(0) \Rightarrow \frac{\partial u}{\partial x} = \boxed{a = p}$

$\frac{\partial u}{\partial y} = a(0) + b(1) \Rightarrow \frac{\partial u}{\partial y} = \boxed{b = q}$

$\boxed{u = px + qy}$

$y = ax^2 + b$

$\frac{dy}{dx} = 2ax$

$\boxed{\frac{d^2y}{dx^2} = 2a}$

$\boxed{y' = y''x}$

Problem 2. **Form the PDE for:** $\boldsymbol{u = ax + by + a^4 + b^4,}$ $\boldsymbol{a}$ **and** $\boldsymbol{b}$ **are constants**

$\frac{\partial u}{\partial x} = a = p$, $\frac{\partial u}{\partial y} = b = q$

$u = px + qy + p^4 + q^4$

Problem 3. **Form the PDE for:** $\boldsymbol{u = (x-\alpha)^2 + (y-\beta)^2, \alpha}$ **and** $\boldsymbol{\beta}$ **are constants**

$\frac{\partial u}{\partial x} = 2(x-\alpha) \Rightarrow p = 2(x-\alpha)$

$\Rightarrow \boxed{(x-\alpha) = \frac{p}{2}}$

$\frac{\partial u}{\partial y} = 2(y-\beta) \Rightarrow q = 2(y-\beta)$

$\Rightarrow \boxed{(y-\beta) = \frac{q}{2}}$

$u = \left(\frac{p}{2}\right)^2 + \left(\frac{q}{2}\right)^2$

$$u = \left(\frac{p}{2}\right)^2 + \left(\frac{q}{2}\right)^2 \Rightarrow \boxed{4u = p^2 + q^2}$$

The PDE for function: $\boldsymbol{u = ax + by + a^2b^2}$, ***a*** **and** ***b*** **are constants, is:**

(A) $u = aq + bp + q^2p^2$

(B) $u = ap + bq + a^2b^2$

(C) $u = px + qy + p^2q^2$

$$\frac{\partial u}{\partial x} = a \Rightarrow p = a$$

$$\frac{\partial u}{\partial y} = b \Rightarrow q = b$$

$$u = px + qy + p^2q^2$$

By elimination of arbitrary functions

<u>**Problem 1.**</u> **Form the PDE for:** $\boldsymbol{u = f(x^2 + y^2)}$

$$f(x^2+y^2) = e^{x^2+y^2}$$

$$\downarrow$$

$$\sin(x^2+y^2)$$

$$(x^2+y^2)^2$$

$$\frac{\partial u}{\partial x} = f'(x^2+y^2)(2x)$$

$$\Rightarrow p = f'(x^2+y^2)(2x)$$

$$\Rightarrow \boxed{\frac{p}{2x} = f'(x^2+y^2)} \quad (1)$$

$$\frac{\partial u}{\partial y} = f'(x^2+y^2)\cdot(2y) \Rightarrow q = f'(x^2+y^2)(2y)$$

$$\Rightarrow \boxed{\frac{q}{2y} = f'(x^2+y^2)} \quad (2)$$

$$\frac{p}{\cancel{2}x} = \frac{q}{\cancel{2}y} \Rightarrow \boxed{yp - xq = 0}$$

<u>**Problem 2.**</u> **Form the PDE for:** $\boldsymbol{u = f\left(\frac{x}{y}\right)}$

$$\frac{x}{y}$$

$$\frac{\partial u}{\partial x} = f'\left(\frac{x}{y}\right)\left[\frac{1}{y}\right] \Rightarrow p = f'\left(\frac{x}{y}\right)\cdot\left(\frac{1}{y}\right) \Rightarrow \boxed{py = f'(x/y)} \quad (1)$$

$$\frac{\partial u}{\partial y} = f'\left(\frac{x}{y}\right)\cdot\left(-\frac{x}{y^2}\right) \Rightarrow q = f'(x/y)\left(\frac{-x}{y^2}\right) \Rightarrow \boxed{-\frac{y^2q}{x} = f'(x/y)} \quad (2)$$

$$\frac{\partial u}{\partial y} = f'\left(\frac{x}{y}\right)\cdot\left(-\frac{x}{y^2}\right) \Rightarrow \quad q = f'\left(\frac{x}{y}\right)\left(\frac{-x}{y^2}\right) \Rightarrow \boxed{-\frac{y^2 q}{x} = f'\left(\frac{x}{y}\right)} \quad (2)$$

$$\Rightarrow \quad py = -\frac{y^2 q}{x}$$

$$\Rightarrow \quad x p \cancel{y} = -y^{\cancel{2}} q \quad \Rightarrow \quad \boxed{xp + yq = 0}$$

The PDE for function: $\boldsymbol{u = f\left(\frac{ax}{by}\right)}$**,** $\boldsymbol{a}$ **and** $\boldsymbol{b}$ **are constants, is:**

(A) $px = qy$

(B) $px + qy = 0$

(C) $py = qx$

$$\frac{\partial u}{\partial x} = f'\left(\frac{ax}{by}\right)\cdot\left(\frac{a}{by}\right)$$

$$\boxed{py = \frac{a}{b} f'\left(\frac{ax}{by}\right)}$$

$$\Rightarrow \quad \frac{\partial u}{\partial y} = f'\left(\frac{ax}{by}\right)\cdot\left(\frac{-ax}{by^2}\right)$$

$$\Rightarrow \quad \boxed{-\frac{qy^2}{x} = \frac{a}{b} f'\left(\frac{ax}{by}\right)}$$

$$py = -\frac{qy^2}{x}$$

$$px\cancel{y} = -qy^{\cancel{2}}$$

$$\boxed{px + qy = 0}$$

$u(x, y, z)$