Q1

The odds that person X speaks the truth are $\overset{\text{f}}{3}:2$ $\overset{\text{of}}{}$ and the odds that person Y speaks the truth are 5:3. In what percentage of cases are they likely to contradict each other on an identical point.

Sol

$$P(X) = \frac{3}{3+2} = \frac{3}{5} \qquad P(\bar{X}) = 1 - \frac{3}{5} = \frac{2}{5}$$

$$P(Y) = \frac{5}{5+3} = \frac{5}{8} \qquad P(\bar{Y}) = 1 - \frac{5}{8} = \frac{3}{8}$$

$$P(X \cap \bar{Y}) + P(\bar{X} \cap Y)$$

$$= P(X)\,P(\bar{Y}) + P(\bar{X})\,P(Y)$$

$$= \frac{3}{5} \times \frac{3}{8} + \frac{2}{5} \times \frac{5}{8} = \frac{9}{40} + \frac{10}{40} = \frac{19}{40}$$

The probability that X and Y not contradict each other

$$P(X \cap Y) + P(\bar{X} \cap \bar{Y})$$

$$= P(X)\,P(Y) + P(\bar{X})\,P(\bar{Y})$$

$$= \frac{3}{5} \times \frac{5}{8} + \frac{2}{5} \times \frac{3}{8}$$

$$= \frac{15}{40} + \frac{6}{40} = \frac{21}{40}$$

$$1 - \frac{19}{40}$$

$$= \frac{21}{40}$$

Q2 For two events A and B we have the following probabilities:

$$P(A) = P(A \mid B) = \frac{1}{4} \text{ and } P(B \mid A) = \frac{1}{2}.$$

Check whether the following statements are true or false :

(i) A and B are mutually exclusive, (ii) A and B are independent, (iii) A is a subevent of B, and (iv) $P(\bar{A} \mid B) = \frac{3}{4}$

V $P(\bar{A} \mid \bar{B}) = 3/4$,

VI $P(A \mid B) + P(A \mid \bar{B}) = 1$

T

F

1/2

$P(A) = \frac{1}{4}$

$P(A|B) = \frac{1}{4}$

$P(B|A) = \frac{1}{2}$

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

$$\frac{1}{2} = \frac{P(A \cap B)}{1/4}$$

$$P(A \cap B) = \frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$$

If A and B are Mutually Ex.

$P(A \cap B) = 0$

(1) False.

(II) If A and B are independent events then

$$P(A \cap B) = P(A)\,P(B)$$

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$$\frac{1}{4} = \frac{1/8}{P(B)}$$

$$P(B) = \frac{1/8}{1/4} = \frac{1}{2} \checkmark$$

$$P(A \cap B) = \frac{1}{8}$$

$$P(A)\,P(B) = \frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$$

(II) — True

(iii) A is subevent of B $\longrightarrow$ False

$A \subseteq B$

$P(A) = \frac{1}{4}$ $\quad$ $P(B) = \frac{1}{2}$

$P(A) \leq P(B)$

B — E

$\{2, 4, 6\}$

$\frac{3}{6} = \frac{1}{2}$

A

52

$S = 13$

$13/52 = \frac{1}{4}$

(iv) $$P(\bar{A}/B) = \frac{P(\bar{A} \cap B)}{P(B)} = \frac{P(\text{only } B)}{P(B)}$$

$$= \frac{P(B) - P(A \cap B)}{P(B)} = \frac{\frac{1}{2} - \frac{1}{8}}{\frac{1}{2}}$$

$$= \frac{\frac{4-1}{8}}{\frac{1}{2}} = \frac{3/8}{1/2} = \frac{3}{4}$$

(v) $$P(\bar{A}/\bar{B}) = \frac{P(\bar{A} \cap \bar{B})}{P(\bar{B})} = \frac{P(\overline{A \cup B})}{P(\bar{B})}$$

$$= \frac{1 - P(A \cup B)}{1 - P(B)}$$

$$= \frac{1 - [P(A) + P(B) - P(A \cap B)]}{1 - P(B)}$$

$$P(\bar{A}/\bar{B}) = \frac{1 - \left[\frac{1}{4} + \frac{1}{2} - \frac{1}{8}\right]}{1 - \frac{1}{2}}$$

$$= \frac{1 - \left[\frac{2+4-1}{8}\right]}{\frac{1}{2}} = \frac{1 - \frac{5}{8}}{\frac{1}{2}}$$

$$P(\bar{A}/\bar{B}) = \frac{3/8}{1/2} = 3/4$$

$$P(A|\bar{B}) = \frac{P(A \cap \bar{B})}{P(\bar{B})} = \frac{P(A) - P(A \cap B)}{1 - P(B)}$$

$$= \frac{\frac{1}{4} - \frac{1}{8}}{1 - \frac{1}{2}} = \frac{\frac{2-1}{8}}{\frac{1}{2}}$$

$$= \frac{1/8}{1/2} = \frac{1}{4}$$

$$P(A|B) + P(A|\bar{B}) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

Q3

A, B and C are independent witnesses of an event which is known to have occurred. A speaks the truth three times out of four, B four times out of five and C five times out of six. What is the probability that the occurrence will be reported truthfully by majority of three witnesses?

Sol

$P(A) = \frac{3}{4}$

$P(B) = \frac{4}{5}$

$P(C) = \frac{5}{6}$

$P(\bar{A}) = 1 - \frac{3}{4} = \frac{1}{4}$

$P(\bar{B}) = 1/5$ $\quad P(\bar{C}) = 1/6$

$P(A\cap B\cap C) + P(A\cap B\cap \bar{C}) + P(A\cap \bar{B}\cap C) + P(\bar{A}\cap B\cap C)$

$= P(A)\,P(B)\,P(C) + P(A)\,P(B)\,P(\bar{C}) + P(A)\,P(\bar{B})\,P(C) + P(\bar{A})\,P(B)\,P(C)$

$= \frac{3}{4}\times\frac{4}{5}\times\frac{5}{6} + \frac{3}{4}\times\frac{4}{5}\times\frac{1}{6} + \frac{3}{4}\times\frac{1}{5}\times\frac{5}{6} + \frac{1}{4}\times\frac{4}{5}\times\frac{5}{6} =$

Q 4 The odds that a book will be favourably reviewed by 3 independent critics are 5 to 2, 4 to 3 and 3 to 4 respectively. What is the probability that, of the three reviews, a majority will be favourable?

$P(A) = \frac{5}{5+2} = \frac{5}{7}$, $P(B) = \frac{4}{7}$, $P(C) = \frac{3}{3+4} = \frac{3}{7}$

$P(\bar{A}) = 1-\frac{5}{7} = \frac{2}{7}$ $P(\bar{B}) = \frac{3}{7}$ $P(\bar{C}) = 1-\frac{3}{7} = \frac{4}{7}$

$P(A \cap B \cap C) + P(A \cap B \cap \bar{C}) + P(A \cap \bar{B} \cap C) + P(\bar{A} \cap B \cap C)$

$= \frac{5}{7} \times \frac{4}{7} \times \frac{3}{7} + \frac{5}{7} \times \frac{4}{7} \times \frac{4}{7} + \frac{5}{7} \times \frac{3}{7} \times \frac{4}{7} +$

$\frac{2}{7} \times \frac{4}{7} \times \frac{3}{7} =$

Bayes Theorem

If $E_1, E_2, \ldots, E_n$ are mutually disjoint events with

$P(E_i) \neq 0$, $(i = 1, 2, \ldots, n)$ *then for any orbitrary event A which is a subset of*

$\bigcup_{i=1}^{n} E_i$ *such that* $P(A) > 0$, *we have*

$A \subseteq \bigcup_{i=1}^{n} E_i$

$$P(E_i \mid A) = \frac{P(E_i)\, P(A \mid E_i)}{\sum_{i=1}^{n} P(E_i)\, P(A \mid E_i)}, \quad i = 1, 2, \ldots, n.$$

$$P(E_1/A) = \frac{P(E_1)\, P(A/E_1)}{P(E_1)\, P(A/E_1) + P(E_2)\, P(A/E_2) + \cdots\cdots + P(E_n)\, P(A/E_n)}$$

In 1989 there were three candidates for the position of principal – Mr. Chatterji, Mr. Ayangar and Dr. Singh – whose chances of getting the appointment are in the proportion 4:2:3 respectively. The probability that Mr. Chatterji if selected would introduce co-education in the college is 0·3. The probabilities of Mr. Ayangar and Dr. Singh doing the same are respectively 0·5 and 0·8. What is the probability that there was co-education in the college in 1990?

Solution. Let the events and probabilities be defined as follows:

A : Introduction of co-education

E_1 : Mr. Chatterji is selected as principal

E_2 : Mr. Ayangar is selected as principal

E_3 : Dr. Singh is selected as principal.

$4:2:3$

$4+2+3 = 9$

$P(E_1) = \frac{4}{9}$

$P(E_3) = \frac{3}{9}$

$P(A/E_1) = 0\cdot3$

$P(A/E_2) = 0\cdot5$

$P(E_2) = \frac{2}{9}$

$P(A/E_3) = 0\cdot8$

$P(A \cap B) = P(A)\ P(B/A)$

Then

$$P(E_1) = \frac{4}{9}, \quad P(E_2) = \frac{2}{9} \quad \text{and} \quad P(E_3) = \frac{3}{9}$$

$$P(A \mid E_1) = \frac{3}{10}, \quad P(A \mid E_2) = \frac{5}{10} \quad \text{and} \quad P(A \mid E_3) = \frac{8}{10}$$

$$\therefore \quad P(A) = P[(A \cap E_1) \cup (A \cap E_2) \cup (A \cap E_3)]$$

$$= P(A \cap E_1) + P(A \cap E_2) + P(A \cap E_3)$$

$$= P(E_1)P(A \mid E_1) + P(E_2)P(A \mid E_2) + P(E_3)P(A \mid E_3)$$

$$= \frac{4}{9} \cdot \frac{3}{10} + \frac{2}{9} \cdot \frac{5}{10} + \frac{3}{9} \cdot \frac{8}{10} = \frac{23}{45}$$

$P(A) = P(E_1 \cap A) + P(E_2 \cap A) + P(E_3 \cap A)$

$= P(E)\ P(A/E_1) + P(E_2)\ P(A/E_2) + P(E_3)\ P(A/E_3)$

$$P(E_1/A) = \frac{P(E_1)\,P(A/E_1)}{P(A)}$$

$$= \frac{P(E_1)\,P(A/E_1)}{P(E_1)\,P(A/E_1) + P(E_2)\,P(A/E_2) + P(E_3)\,P(A/E_3)}$$

. In a bolt factory machines A, B and C manufacture respectively 25%, 35% and 40% of the total. Of their output 5, 4, 2 per cent are defective bolts. A bolt is drawn at random from the product and is found to be defective. What are the probabilities that it was manufactured by machines A, B and C?

$A \subseteq E_1 \cup E_2 \cup E_3$

Solution. Let E_1, E_2 and E_3 denote the events that a bolt selected at random is manufactured by the machines A, B and C respectively and let E denote the event of its being defective. Then we have

$$P(E_1) = 0{\cdot}25,\ P(E_2) = 0{\cdot}35,\ P(E_3) = 0{\cdot}40$$

The probability of drawing a defective bolt manufactured by machine A is $P(E \mid E_1) = 0{\cdot}05$.

Similarly, we have

$$P(E \mid E_2) = 0{\cdot}04, \text{ and } P(E \mid E_3) = 0{\cdot}02$$

Hence the probability that a defective bolt selected at random is manufactured by machine A is given by

$$P(E_1 \mid E) = \frac{P(E_1)\,P(E \mid E_1)}{\sum_{i=1}^{3} P(E_i)\,P(E \mid E_i)}$$

$$= \frac{0.25 \times 0.05}{0.25 \times 0.05 + 0.35 \times 0.04 + 0.40 \times 0.02} = \frac{125}{345} = \frac{25}{69}$$

Similarly

$$P(E_2 \mid E) = \frac{0.35 \times 0.04}{0.25 \times 0.05 + 0.35 \times 0.04 + 0.40 \times 0.02} = \frac{140}{345} = \frac{28}{69}$$

and

$$P(E_3 \mid E) = 1 - [P(E_1 \mid E) + P(E_2 \mid E)] = 1 - \frac{25}{69} - \frac{28}{69} = \frac{16}{69}$$

$$P(E_1/E) = \frac{P(E_1)\,P(E/E_1)}{P(E_1)\,P(E/E_1) + P(E_2)\,P(E/E_2) + P(E_3)\,P(E/E_3)}$$

(A)

(B)

$$P(E_2/E) = \frac{P(E_2)\,P(E/E_2)}{P(E_1)\,P(E/E_1) + P(E_2)\,P(E/E_2) + P(E_3)\,P(E/E_3)}$$