Q1 A random variable X has the following probability function :

Values of X, x :	-2	-1	0	1	2	3
$p(x)$:	$0{\cdot}1$	k	$0{\cdot}2$	$2k$	$0{\cdot}3$	k

(i) Find the value of k, and calculate mean and variance.

(ii) Construct the c.d.f. $F(X)$ and draw its graph.

Sol

$$\sum P(x) = 1 = 0.1 + k + 0.2 + 2k + 0.3 + k$$

$$\Rightarrow 4k + 0.6 = 1$$

$$4k = 1 - 0.6 = 0.4$$

$$k = \frac{0.4}{4} = 0.1$$

$$\boxed{k = 0.1}$$

X	$p(x)$	xp	X^2	$x^2 p$
-2	0.1	-0.2	4	0.4
-1	$k = 0.1$	-0.1	1	0.1
0	0.2	0	0	0
1	$2k = 0.2$	0.2	1	0.2
2	0.3	0.6	4	1.2
3	$k = 0.1$	0.3	9	0.9
		0.8		2.8

$E(x) = \bar{x}$

$= \sum xp = 0.8$

$V(x)$

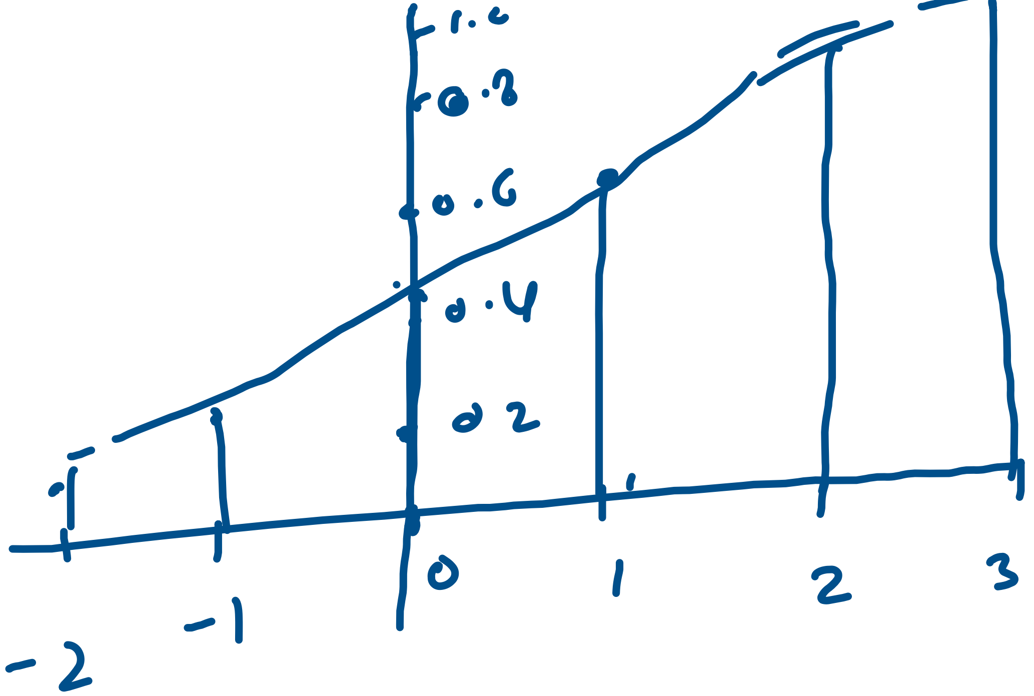
$= E(x^2) - (E(x))^2$

$= (2.8) - (0.8)^2$

$= 2.8 - 0.64$

$= \underline{2.16}$

x	$p(x)$	$F(x)$
-2	0.1	$F(-2) = 0.1$
-1	0.1	$F(-1) = 0.2$
0	0.2	$F(0) = 0.4$
1	0.2	$F(1) = 0.6$
2	0.3	$F(2) = 0.9$
3	0.1	$F(3) = 1.0$

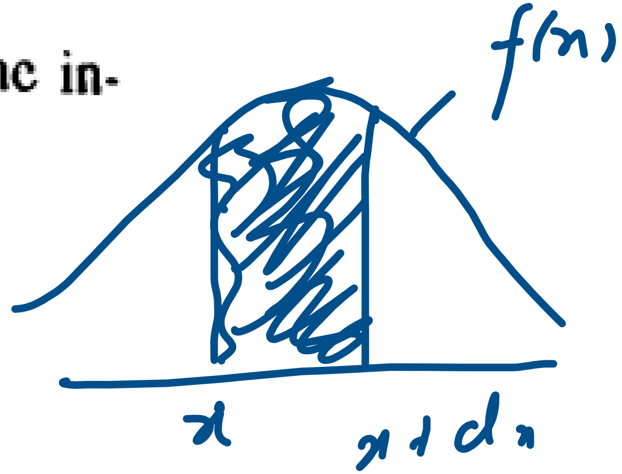


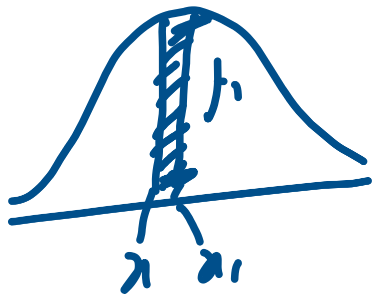
Continuous Random Variable. A random variable X is said to be continuous if it can take all possible values between certain limits. *In other words, a random variable is said to be continuous when its different values cannot be put in 1-1 correspondence with a set of positive integers.*

A continuous random variable is a random variable that (at least conceptually) can be measured to any desired degree of accuracy. Examples of continuous random variables are age, height, weight etc.

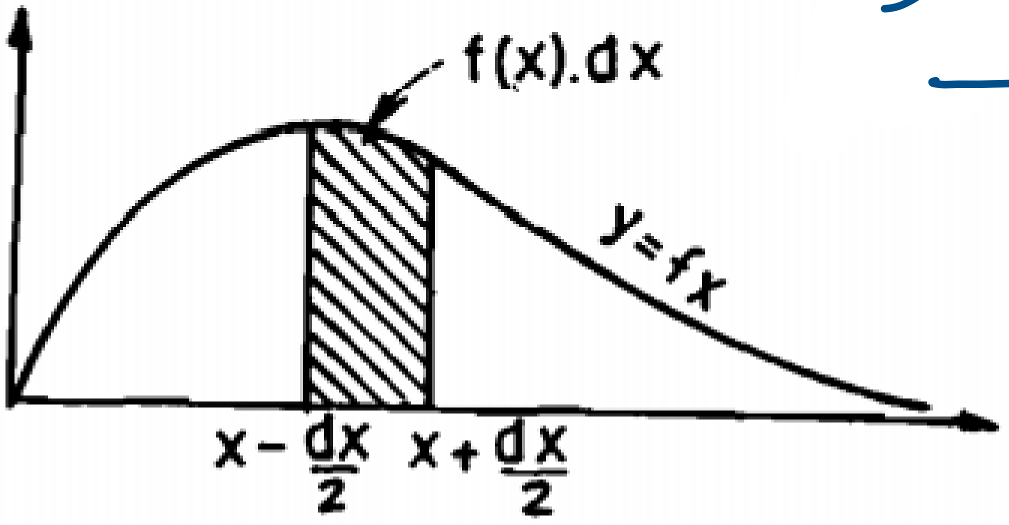
Probability Density Function *(Concept and Definition).* Consider the small interval $(x, x+dx)$ of length dx round the point x. Let $f(x)$ be any continuous function of x so that $f(x)\,dx$ represents the probability that X falls in the infinitesimal interval $(x, x+dx)$. Symbolically

$$P(x \le X \le x + dx) = f_X(x)\,dx$$





In the figure, $f(x)\,dx$ represents the area bounded by the curve $y = f(x)$, x–axis and the ordinates at the points x and $x + dx$. The function $f_X(x)$ so defined is known as *probability density function or simply density function of random variable* X *and is usually abbreviated as p.d.f.* The expression, $f(x)\,dx$, usually written as $dF(x)$, is known as the *probability differential* and the curve $y = f(x)$ is known as the *probability density curve* or simply *probability curve.*



Definition. p.d.f. $f_X(x)$ of the *r.v.* X is defined as :

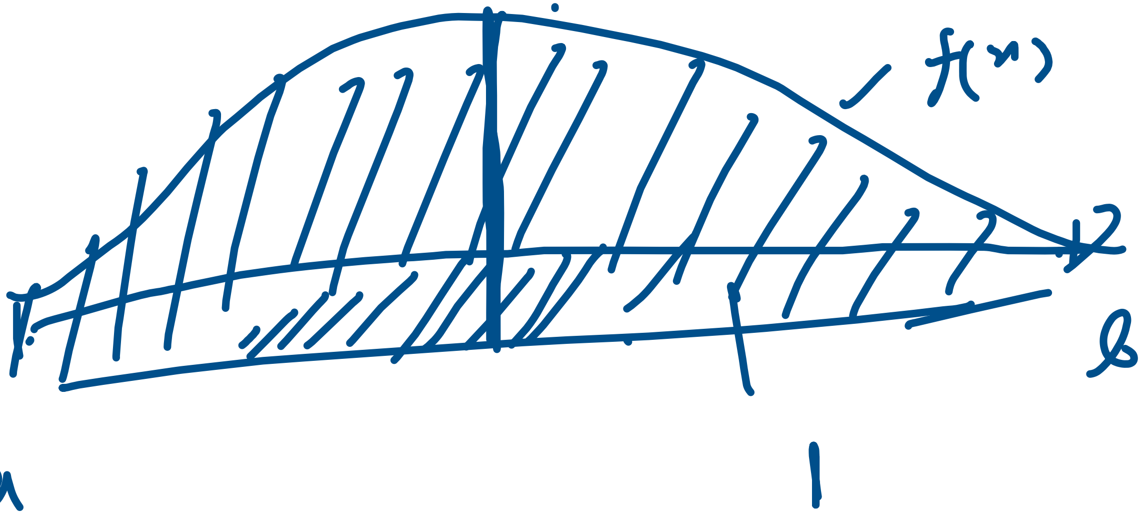
$$f_X(x) = \lim_{\delta x \to 0} \frac{P(x \leq X \leq x + \delta x)}{\delta x}$$

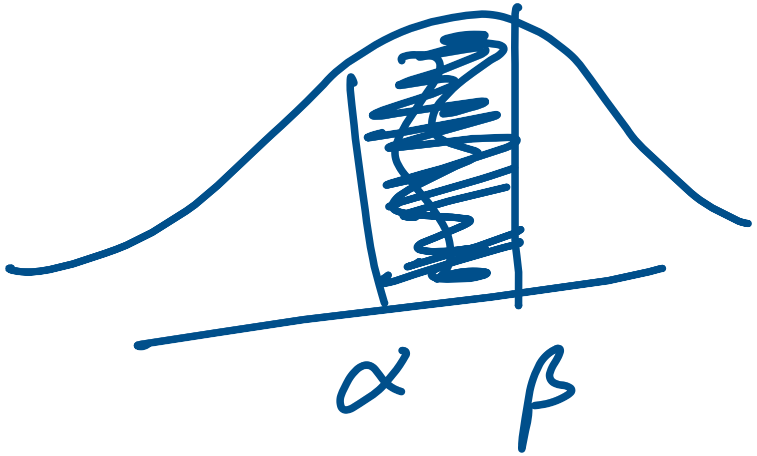
The probability for a variate value to lie in the interval dx is $f(x)\,dx$ and hence the probability for a variate value to fall in the finite interval $[\alpha, \beta]$ is :

$$P(\alpha \leq X \leq \beta) = \int_{\alpha}^{\beta} f(x)\,dx$$

which represents the area between the curve $y = f(x)$, x–axis and the ordinates at $x = \alpha$ and $x = \beta$. Further since total probability is unity, we have $\int_a^b f(x)\,dx = 1$, where $[a, b]$ is the range of the random variable X. The range of the variable may be finite or infinite.

$f(x)$ $\qquad a \le x \le b$





$f(x)$ $\qquad \alpha \le x \le \beta.$

$$P(\alpha \le x \le \beta) = \int_{\alpha}^{\beta} f(x)\,dx$$

The probability density function (*p.d.f.*) of a random variable (*r.v.*) X usually denoted by $f_X(x)$ or simply by $f(x)$ has the following obvious properties

(i) $f(x) \geq 0, \; -\infty < x < \infty$

(ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

(iii) The probability $P(E)$ given by

$$P(E) = \int_E f(x)\,dx$$

is well defined for any event E.

Q 2

Example *The diameter of an electric cable, say X, is assumed to be a continuous random variable with p.d.f.* $f(x) = 6x(1-x),\ 0 \le x \le 1$.

(i) *Check that above is p.d.f.,*

(ii) *Determine a number b such that* $P(X < b) = P(X > b)$

$6x - 6x^2$

Sol

$f(x) = 6x(1-x)$, $0 \le x \le 1$

$\boxed{f(x) \geq 0}$

$f(0) = 0$

$f(1) = 0$

$f(0.5) = 6(0.5)(1-0.5)$

$= 6 \times (0.5)(0.5)$

$= 1.5$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \quad , \quad 0 \leq x \leq 1$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{\infty} f(x)\,dx$$

$$= 0 + \int_{0}^{1} f(x)\,dx + 0$$

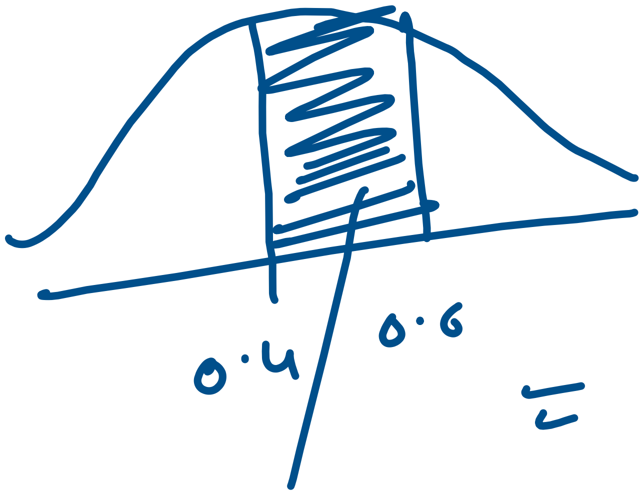
$$\int_0^1 f(x)\,dx = \int_0^1 6x(1-x)\,dx = 1$$

$$= \int_0^1 (6x - 6x^2)\,dx = \left[\frac{\cancel{6}^{3} x^2}{\cancel{2}} - \frac{\cancel{6}^{2} x^3}{\cancel{3}}\right]_0^1$$

$$= \left(3(1)^2 - 2(1)^3\right) - (0-0)$$

$$= 3 - 2 - 0 = 1$$

$$P(0.4 \le x \le 0.6) = \int_{0.4}^{0.6} f(x)\,dx$$



$$= \int_{0.4}^{0.6} (6x - 6x^2)\,dx$$

$$= \left[\frac{\overset{3}{\cancel{6}}\,x^2}{\cancel{2}} - \frac{\overset{2}{\cancel{6}}\,x^3}{\cancel{3}}\right]_{0.4}^{0.6}$$

$\underline{\underline{0.296}}$

$$P(0.4 \le x \le 0.6) = \Big[3x^2 - 2x^3\Big]_{0.4}^{0.6}$$

$$= \left(3(0.6)^2 - 2(0.6)^3\right) - \left(3(0.4)^2 - 2(0.4)^3\right)$$

$$= 3(0.36) - 2(0.216) - 3(0.16)$$

$$+ 2(0.064)$$

$$= 1.08 - 0.432 - 0.48 + 0.128$$

$$= \underline{0.296}$$

$$P(X \le 0.3) = P(0 \le X \le 0.3)$$

$$0 \le X \le 1 \qquad = \int_0^{0.3} f(x)\,dx = \int_0^{0.3} (6x - 6x^2)\,dx$$

$$= \left[\frac{\overset{3}{\cancel{6}}x^2}{\cancel{2}} - \frac{\overset{2}{\cancel{6}}x^3}{\cancel{3}} \right]_0^{0.3}$$

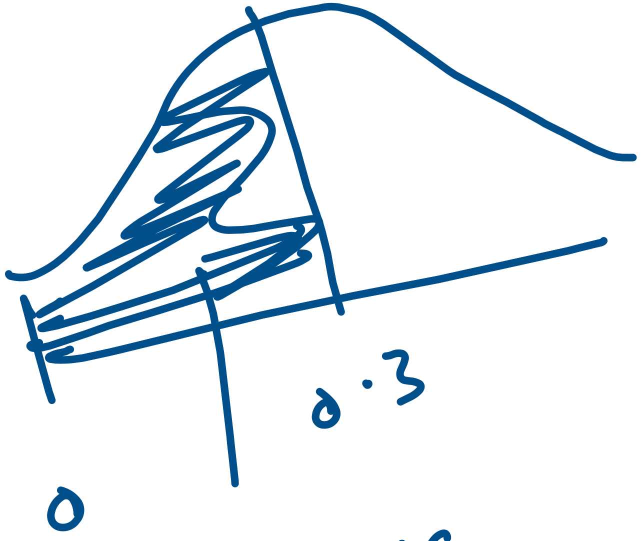
$$= \left[3x^2 - 2x^3 \right]_0^{0.3}$$

$$= \left(3(0.3)^2 - 2(0.3)^3\right) - (0 - 0)$$

$$= 3(0.09) - 2(0.027)$$

$$= 0.27 - 0.054$$

$$= \underline{0.216}$$



$\underline{\underline{0.216}}$

$$P(X < b) = P(X > b) \qquad 0 \le x \le 1$$

$$\Rightarrow P(0 \le x \le b) = P(b < x < 1)$$

$$\int_0^b (6x - 6x^2)\,dx = \int^1 (6x - 6x^2)\,dx$$

$$6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^b = 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_b^1$$

$$\left(\frac{\beta^2}{2} - \frac{\beta^3}{3}\right) - (0 - 0) = \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{\beta^2}{2} - \frac{\beta^3}{3}\right)$$

$$\frac{\beta^2}{2} - \frac{\beta^2}{3} = \frac{1}{6} - \frac{\beta^2}{2} + \frac{\beta^3}{3}$$

$$0 = \frac{1}{6} - \frac{2\beta^2}{2} + \frac{2\beta^3}{3}$$

$$\beta = ?$$

Example *A continuous random variable X has a p.d.f.* $f(x) = 3x^2,\ 0 \le x \le 1$. *Find a and b such that*

(i) $P\{X \le a\} = P\{X > a\}$, and

(ii) $P\{X > b\} = 0.05$.

Sol

$$P(x \le a) = P(x > a)$$

$$P(0 \le x \le a) = P(a < x < 1)$$

$$\int_0^a 3x^2\,dx = \int_a^1 3x^2\,dx$$

$$\left[\frac{x^3}{\cancel{3}}\right]_0^a = \left[\frac{x^3}{\cancel{3}}\right]_a^1$$

$$a^3 - 0 = 1 - a^3$$

$$2a^3 = 1$$

$$a^3 = \frac{1}{2} \qquad a = \left(\frac{1}{2}\right)^{1/3}$$

$$P(X > b) = 0.05$$

$$P(b < X < 1) = 0.05$$

$$\int_b^1 3x^2\,dx = 0.05$$

$$\left[\frac{3x^3}{3}\right]_b^1 = 0.05$$

$$1 - b^3 = 0.05$$

$$1 - 0.05 = b^3$$

$$b^3 = 0.95$$

$$\boxed{b = (0.95)^{1/3}}$$