Q1 Let X be a continuous random variable with probability density function given by

$$f(x) = \begin{cases} ax, & 0 \le x \le 1 \\ a, & 1 \le x \le 2 \\ -ax + 3a, & 2 \le x \le 3 \\ 0, & \text{elsewhere} \end{cases}$$

(i) Determine the constant a.

(ii) Determine $F(x)$ — Distribution function

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \qquad \int_0^1 ax\,dx + \int_1^2 a\,dx + \int_2^3 (-ax + 3a)\,dx = 1$$

$$\left[\frac{ax^2}{2}\right]_0^1 + \left[ax\right]_1^2 + \left[-\frac{ax^2}{2} + 3ax\right]_2^3 = 1$$

$$\frac{a}{2}\left[1-0\right] + a\left[2-1\right] + \left[-\frac{a}{2}(9-4) + 3a(3-2)\right] = 1$$

$$\frac{a}{2} + a + \left(-\frac{5a}{2}\right) + 3a = 1$$

$$\frac{a + 2a - 5a + 6a}{2} = 1$$

$$\frac{4a}{2} = 1$$

$$a = \frac{1}{2}$$

$$f(x) = \begin{cases} 0 & , \quad -\infty < x < 0 \\ ax = \dfrac{x}{2} & , \quad 0 \le x < 1 \\ a = \dfrac{1}{2} & , \quad 1 \le x < 2 \\ -ax + 3a \\ = -\dfrac{x}{2} + \dfrac{3}{2} & , \quad 2 \le x \le 3 \\ 0 & , \quad 3 < x < \infty \end{cases}$$

$$F(x) = \int_{-\infty}^{x} f(t)\, dt$$

$f(x)$ — pdf

$\underline{\underline{x = t}}$

$-\infty < x < 0$

$$F(x) = \int_{-\infty}^{x} 0 \, dt = 0$$

$\underline{0 \le x \le 1}$

$$F(x) = \int_{-\infty}^{x} f(t)\,dx = \int_{-\infty}^{0} f(t)\,dt + \int_{0}^{x} f(t)\,dt$$

$$= 0 + \int_{0}^{x} \frac{t}{2}\,dt = \left[\frac{t^2}{4}\right]_0^x = \left[\frac{x^2 - 0}{4}\right]$$

$$= \frac{x^2}{4}$$

$\underline{1 \leq x \leq 2}$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{0} f(t)\,dt + \int_{0}^{1} f(t)\,dt$$

$$+ \int_{1}^{x} f(t)\,dt$$

$$= 0 + \int_{0}^{1} \frac{t}{2}\,dt + \int_{0}^{x} \frac{1}{2}\,dt$$

$$F(x) = 0 + \left[\frac{t^2}{4}\right]_0^1 + \left[\frac{t}{2}\right]_1^x$$

$$= \frac{1-0}{4} + \frac{x-1}{2} = \frac{1}{4} + \frac{x-1}{2}$$

$$= \frac{1+2x-2}{4} = \frac{2x-1}{4}$$

$\underline{2 \leq x \leq 3}$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{0} f(t)\,dt + \int_{0}^{1} f(t)\,dt$$

$$+ \int_{1}^{2} f(t)\,dt + \int_{2}^{x} f(t)\,dt$$

$$F(x) = 0 + \int_0^1 \frac{t}{2}\,dt + \int_1^2 \frac{1}{2}\,dt + \int_2 \left(-\frac{t}{2} + \frac{3}{2}\right)dt$$

$$= \left[\frac{t^2}{4}\right]_0^1 + \left[\frac{t}{2}\right]_1^2 + \left[-\frac{t^2}{4} + \frac{3t}{2}\right]_2^{x}$$

$$= \frac{1-0}{4} + \frac{2-1}{2} + \left[\left(-\frac{x^2}{4} + \frac{3x}{2}\right) - \left(-1 + 3\right)\right]$$

$$f(x) = \frac{1}{4} + \frac{1}{2} - \frac{x^2}{4} + \frac{3x}{2} - 2$$

$$= \frac{1 + 2 - x^2 + 6x - 8}{4}$$

$$= \frac{6x - x^2 - 5}{4}$$

$\underline{3 \le x < \infty}$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{0} f(t)\,dt + \int_{0}^{1} f(t)\,dt$$

$$+ \int_{1}^{2} f(t)\,dt + \int_{2}^{3} f(t)\,dt + \int_{3}^{x} f(t)\,dt$$

$$= 0 + \int_{0}^{1} \frac{t}{2}\,dt + \int_{1}^{2} \frac{1}{2}\,dt + \int_{2}^{3} \left(-\frac{t}{2} + \frac{3}{2}\right) dt + 0$$

$$F(x) = \left[\frac{t^2}{4}\right]_0^1 + \left[\frac{t}{2}\right]_1^2 + \left(\frac{-t^2}{4} + \frac{3t}{2}\right)_2^3$$

$$= \frac{1-0}{4} + \frac{2-1}{2} + \left(\frac{-(9-4)}{4} + \frac{3}{2}(3-1)\right)$$

$$= \frac{1}{4} + \frac{1}{2} - \frac{5}{4} + \frac{3}{2}$$

$$= \frac{1+2-5+6}{4} = \frac{4}{4} = 1$$

$$F(x) = \begin{cases} 0 & , \quad -\infty < x < 0 \\ \dfrac{x^2}{4} & , \quad 0 \le x \le 1 \\ \dfrac{2x-1}{4} & , \quad 1 \le x \le 2 \\ \dfrac{6x-x^2-5}{4} & , \quad 2 \le x \le 3 \\ 1 & , \quad 3 \le x < \infty \end{cases}$$

Q2 A random variable X has the density function :

$$f(x) = K \cdot \frac{1}{1+x^2}, \quad \text{if } -\infty < x < \infty$$
$$= 0, \quad \textit{otherwise}$$

Determine K and the distribution function.

Evaluate the probability $P(X \geq 0)$

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

$$K \int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx = 1$$

$$K \left[\tan^{-1}(x) \right]_{-\infty}^{\infty}$$

$$K \left[\tan^{-1}(\infty) - \tan^{-1}(-\infty) \right] = 1$$

$$K\left[\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right]=1$$

$$K\left[\frac{\pi}{2}+\frac{\pi}{2}\right]=1$$

$$K\left[2\frac{\pi}{2}\right]=1 \qquad \boxed{K=\frac{1}{\pi}}$$

$$f(x) = \frac{1}{\pi}\left(\frac{1}{1+x^2}\right), \quad -\infty < x < \infty$$

$$F(x) = \int_{-\infty}^{x} f(t)\, dx = \int_{-\infty}^{x} \frac{1}{\pi}\left(\frac{1}{1+t^2}\right) dt$$

$$= \frac{1}{\pi}\left[\tan^{-1}(t)\right]_{-\infty}^{x}$$

$$F(x) = \frac{1}{\pi}\left[\tan^{-1} x - \tan^{-1}(-\infty)\right]$$

$$= \frac{1}{\pi}\left[\tan^{-1} x - \left(-\frac{\pi}{2}\right)\right]$$

$$\boxed{F(x) = \frac{1}{\pi}\left[\tan^{-1} x + \frac{\pi}{2}\right], \quad -\infty < x < \infty}$$

$$P(X \geq 0) = P(0 \leq X < \infty) = \frac{1}{2}$$

$$= \frac{1}{\pi} \int_0^{\infty} \left(\frac{1}{1+x^2} \right) dx$$

$$= \frac{1}{\pi} \left[\tan^{-1} x \right]_0^{\infty} = \frac{1}{\pi} \left[\tan^{-1}\infty - \tan^{-1} 0 \right]$$

$$= \frac{1}{\pi} \left[\frac{\pi}{2} - 0 \right] = \frac{1}{2}$$

Q3 A continuous random variable X has the distribution function

$$F(x) = \begin{cases} 0, & \text{if } x \le 1 \\ k(x-1)^4, & \text{if } 1 < x \le 3 \\ 1, & \text{if } x > 3 \end{cases}$$

Find *(i)* k, *(ii)* the probability density function $f(x)$, and *(iii)* the mean

Let pdf $f(x)$

$$f(x) = \frac{d}{dx} F(x) = \begin{cases} \frac{d}{dx}(0) = 0 & x \le 1 \\ \frac{d}{dx} k(x-1)^4 = 4k(x-1)^3, & 1 < x \le 3 \\ \frac{d}{dx}(1) = 0, & x > 3 \end{cases}$$

$$f(x) = \begin{cases} 0 & -\infty < x \leq 1 \\ 4K(x-1)^3 & 1 < x \leq 3 \\ 0 & 3 < x < \infty \end{cases}$$

$$\int_1^3 4K(x-1)^3\,dx = 4K\left[\frac{(x-1)^4}{4}\right]_1^3 = 1$$

$$= 4K\left[\frac{2^4}{4} - 0\right] = 4K(4) = 1$$

$$16K = 1 \qquad K = \frac{1}{16}$$

$$E(X) = \bar{X} = \int_1^3 x\, f(x)\, dx$$

$$= \int_1^3 x \quad 4 \times \frac{1}{16} (x-1)^3\, dx$$

$$= \frac{1}{4} \int_1^3 x \left(x^3 - 1^3 - 3x^2 + 3x\right) dx$$

$$f(x) = \frac{1}{4}\int_1^3 (x^4 - x - 3x^3 + 3x^2)\,dx$$

$$= \frac{1}{4}\left[\frac{x^5}{5} - \frac{x^2}{2} - \frac{3x^4}{4} + \frac{\cancel{3}x^3}{\cancel{3}}\right]_1^3$$

$$= \frac{1}{4}\left(\left(\frac{243}{5} - \frac{9}{2} - \frac{3\times 81}{4} + 27\right) - \left(\frac{1}{5} - \frac{1}{2} - \frac{3}{4} + 1\right)\right)$$