Q1 *Let X be a continuous random variate with p.d.f.*

$$f(x) = \begin{cases} ax, & 0 \le x \le 1 \\ a, & 1 \le x \le 2 \\ -ax + 3a, & 2 \le x \le 3 \\ 0, & \text{elsewhere} \end{cases}$$

(i) *Determine the constant a.*

(ii) *Compute $P(X \le 1{\cdot}5)$.*

$f(x) \geq 0$

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx +$$

$$\int_{1}^{2} f(x)\,dx + \int_{2}^{3} f(x)\,dx + \int_{3}^{\infty} f(x)\,dx$$

$$1 = 0 + \int_{0}^{1} ax\,dx + \int_{1}^{2} a\,dx + \int_{2}^{3} (-ax+3a)\,dx + 0$$

$$1 = \left[\frac{ax^2}{2}\right]_0^1 + \left[ax\right]_1^2 + \left[-\frac{ax^2}{2} + 3ax\right]_2^3$$

$$1 = [\frac{a}{2}(1-0) + a(2-1) + [-\frac{a}{2}(9-4) + 3a(3-2)]$$

$$1 = \left[\frac{a}{2} + a - \frac{5a}{2} + 3a\right]$$

$$1 = \left[\frac{a + 2a - 5a + 6a}{2}\right]$$

$$1 = \frac{4a}{2} \Rightarrow 1 = 2a \quad \boxed{a = \frac{1}{2} = 0.5}$$

$$\boxed{a = 0.5}$$

$$P(x \le 1.5) = P(-\infty \le x \le 1.5)$$

$$= \int_{-\infty}^{1.5} f(x)\,dx$$

$$= \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{1.5} f(x)\,dx$$

$$= 0 + \int_{0}^{1} ax\,dx + \int_{1}^{1.5} a\,dx$$

$$P(X \le 1.5) = \left[\frac{ax^2}{2}\right]_0^1 + \left[ax\right]_1^{1.5}$$

$$= \frac{a}{2}\left[1-0\right] + a\left[1.5-1\right]$$

$$= \frac{a}{2} + 0.5a$$

$$= \frac{a}{2} + \frac{a}{2} = a = 0.5$$

Q2

In a continuous distribution whose relative frequency density is given by

$$f(x) = y_0 . x(2-x),\ 0 \le x \le 2,$$

$y_0(2x - x^2)$

find mean, variance,

$$\int_0^2 f(x)\,dx = 1 = \int_0^2 y_0\, x(2-x)\,dx$$

$$= y_0 \left[\frac{\cancel{2}x^2}{\cancel{2}} - \frac{x^3}{3}\right]_0^2$$

$$1 = y_0 \left[4 - \frac{8}{3} \right]$$

$$\boxed{y_0 = \frac{3}{4}}$$

$$1 = y_0 \left(\frac{12-8}{3} \right)$$

$$1 = y_0 \left(\frac{4}{3} \right)$$

$$E(x) = \sum x\, p(x) = \sum x\, f(x)$$

$$E(x) = \int_a^b x\, p(x)\, dx = \int_a^b x\, f(x)\, dx = \bar{x}$$

$$\bar{x} = E(x) = \int_0^2 x\, f(x)\, dx = \int_0^2 x\, y_0\, x(2-x)\, dx$$

$$E(x) = \tfrac{3}{4} \int_0^2 (2x^2 - x^3)\,dx$$

$$= \frac{3}{4} \left[\frac{2x^3}{3} - \frac{x^4}{4} \right]_0^2$$

$$= \frac{3}{4} \left[\left(\frac{16}{3} - \frac{16}{4} \right) - (0-0) \right]$$

$$E(x) = \frac{3}{4}\left[\frac{16}{3} - 4\right] = \frac{3}{4}\left[\frac{16-12}{3}\right]$$

$$= \frac{3}{4} \times \frac{4}{3} = 1$$

$$\boxed{f(x) = 1}$$

$$V(x) = E(x^2) - (E(x))^2 = \sum x^2 p - \left(\sum x p\right)^2$$

$$= \sum x^2 f(x) - \left(\sum x f(x)\right)^2$$

$$V(x) = E(x^2) - (E(x))^2 = \int_a^b x^2 f(x)\,dx - \left[\int_a^b x f(x)\,dx\right]^2$$

$$V(x) = \int_0^2 x^2 y_0\, x(2-x)\,dx - \left[\int_0^2 x\, y_0\, x(2-x)\,dx\right]^2$$

$$V(x) = \frac{3}{4}\int_0^2 \left(2x^3 - x^4\right)dx - \left[\frac{3}{4}\int_0^2 \left(2x^2 - x^3\right)dx\right]^2$$

$$V(x) = \frac{3}{4}\left[\frac{2x^4}{4} - \frac{x^5}{5}\right]_0^2 - \left[\frac{3}{4}\left[\left(\frac{2x^3}{3} - \frac{x^4}{4}\right)\right]_0^2\right]^2$$

$$= \frac{3}{4}\left[\left(8 - \frac{32}{5}\right) - (0-0)\right] - \left[\frac{3}{4}\left(\frac{16}{3} - \frac{16}{4}\right)\right]^2$$

$$V(x) = \frac{3}{4}\left[\frac{40-32}{5}\right] - \left[\frac{3}{4}\left(\frac{4}{3}\right)\right]^2$$

$$= \frac{3}{\cancel{4}} \times \frac{\overset{2}{\cancel{8}}}{5} - 1^2$$

$$= \frac{6}{5} - 1 = \frac{6-5}{5} = \frac{1}{5}$$

$$\boxed{V(x) = 0.2}$$

Q 3

The mileage C in thousands of miles which car owners get with a certain kind of tyre is a random variable having probability density function

$$f(x) = \frac{1}{20} e^{-x/20}, \text{ for } x > 0$$

$$= 0, \text{ for } x \le 0$$

$0 < x < \infty$

Find the probabilities that one of these tyres will last

(i) at most 10,000 miles,

(ii) anywhere from 16,000 to 24,000 miles.

(iii) at least 30,000 miles.

$$P(X \le 10) = \int_0^{10} \frac{1}{20} e^{-x/20} \, dx$$

$$P(x \le 10) = \frac{1}{\cancel{20}} \left[\frac{e^{-x/20}}{-1/\cancel{20}} \right]_0^{10}$$

$$= -\left[e^{-10/20} - e^{-0} \right]$$

$$= -\left[e^{-0.5} - 1 \right] = 1 - e^{-0.5}$$

$$\boxed{P(x \le 10) = 1 - e^{-0.5}}$$

$$P(16 \le x \le 24) = \int_{16}^{24} \frac{1}{20} e^{-x/20}\, dx$$

$$= \frac{1}{\cancel{20}} \left[\frac{e^{-x/20}}{-1/\cancel{20}} \right]_{16}^{24}$$

$$= -\left[e^{-24/20} - e^{-16/20} \right]$$

$$P(16 \le X \le 24) = -\left[e^{-1.2} - e^{-0.8} \right]$$

$$= e^{-0.8} - e^{-1.2}$$

$$=$$

$$P(30 \le X) = P(30 \le X \le \infty)$$

$$= \int_{30}^{\infty} \frac{1}{20} e^{-x/20}\, dx$$

$$= \frac{1}{\cancel{20}} \left[\frac{e^{-x/20}}{-1/\cancel{20}} \right]_{30}^{\infty}$$

$$P(30 \le X) = -\left[\frac{1}{e^{x/20}}\right]_{30}^{\infty}$$

$$= -\left[\frac{1}{e^{\infty}} - \frac{1}{e^{30/20}}\right]$$

$$= -\left[0 - \frac{1}{e^{1.5}}\right] = e^{-1.5}$$

Q4 // A continuous random variable X has the probability density function :

$$f(x) = A + Bx, \quad 0 \le x \le 1.$$

$f(x) = 1 \quad 0 < x < 1$

If the mean of the distribution is $\frac{1}{2}$, find A and B.

$$E(X) = \frac{1}{2} = \int_0^1 x\, f(x)\, dx = \int_0^1 x(A + Bx)\, dx$$

$$\frac{1}{2} = \int_0^1 (Ax + Bx^2)\, dx = \left[\frac{Ax^2}{2} + \frac{Bx^3}{3} \right]_0^1$$

$$\frac{1}{2} = \left[\left(\frac{A}{2} + \frac{B}{3}\right) - (0+0)\right]$$

$$\frac{1}{2} = \left[\frac{A}{2} + \frac{B}{3}\right] \Rightarrow \boxed{\frac{A}{2} + \frac{B}{3} = \frac{1}{2}} \quad (1)$$

$$\int_0^1 f(x)\,dx = 1$$

$$\int_0^1 (A + Bx)\,dx = 1$$

$$\left[Ax + \frac{Bx^2}{2}\right]_0^1 = 1$$

$$\left(A + \frac{B}{2}\right) - (0+0) = 1$$

$$A + \frac{B}{2} = 1 \quad \text{— } 2$$

By (i) & (2)

$$\frac{1}{2} \times \left(A + \frac{B}{2} = 1\right).$$

$$A + \frac{B}{2} = 1$$

$$\boxed{A = 1}$$

$$\frac{A}{2} + \frac{B}{3} = \frac{1}{2}$$

$$\frac{A}{2} + \frac{B}{4} = \frac{1}{2}$$

$$\frac{B}{3} - \frac{B}{4} = 0$$

$$\frac{B}{12} = 0 \qquad B = 0$$