3
Linear Programming II: Simplex Method

The simplex method is a step-by-step procedure for finding the optimal solution to a linear programming problem. To apply simplex method to solving a linear programming problem, it has to be ensured that

(i) all the variables in the problem are non-negative, and

(ii) all the values on the RHS of constraints are non-negative.

For a constraint involving a negative b_i value, it is multiplied on each side by –1, and the direction of inequality is reversed. Thus a constraint $2x_1 - 7x_2 \geq -66$ would be replaced as $-2x_1 + 7x_2 \leq 66$. In case a variable is unrestricted in sign, it is replaced by the difference of two non-negative variables. Thus, if x_3 is given to be unrestricted in sign, it may be replaced in the problem by $x_4 - x_5$, where $x_4, x_5 \geq 0$.

To apply simplex method, the constraints are first expressed as equations. The *slack variables* convert "≤" type of inequalities into equations while *surplus variables* are used to convert "≥" type of inequalities into equations. Once all equations are obtained, writing variable names on the top row and the a_{ij} values below it draws up a simplex tableau. The b_i values are written on the RHS of the tableau and the c_j values in a bottom row. The three steps involved in a simplex algorithm are:

1. Obtain an initial solution.
2. Test if the solution is optimal. If it is optimal, then stop. Otherwise, proceed to step 3.
3. Obtain an improved solution. Repeat step 2.

Obtaining Initial Solution To obtain a solution in a simplex tableau, it is necessary to look for an identity matrix within the matrix of a_{ij} values. If such a matrix cannot be located by any combination of variables, then *artificial variables* (each with a large co-efficient equal to M in the objective function of minimisation type and equal to $-M$ in the objective function of maximisation type) are added. In any event, the variables constituting the identity matrix are then entered in the basis in the same order in which they yield the matrix. Along with them, their co-efficients in the objective function are entered. Finally, the solution is written in a row below the c_j values. Each of the variables in the basis is assigned the corresponding value in the b_i column while the remaining variables are all set equal to zero.

Testing the Optimality Once an initial solution is obtained with the aid of identity matrix, it is tested for optimality in terms of $\Delta_j = c_j - z_j$. The z_j value for each variable is the summation of the products of the values under the variable and the corresponding values in the basis. For a maximisation problem, if all $\Delta_j \leq 0$, the solution is optimal and for a minimisation problem, the condition is all $\Delta_j \geq 0$.

Improving a Non-optimal Solution A non-optimal solution is improved by

(i) selecting the variable with largest Δ_j (for a maximisation problem) and with most negative Δ_j (for a minimisation problem) as the incoming variable (called the *key column*);

(ii) selecting the variable with least non-negative replacement ratio as the outgoing variable (known as the *key row*); and

(iii) obtaining a new simplex tableau. For obtaining new tableau, first derive the row corresponding to the key row by dividing each element by the key element (which lies at the intersection of key column and key row). For every other element of the matrix, use the rule:

$$\text{New element} = \text{Old element} - \frac{\begin{array}{c}\text{Corresponding element} \\ \text{in key row}\end{array} \times \begin{array}{c}\text{Corresponding element} \\ \text{in key column}\end{array}}{\text{Key element}}$$

The revised solution obtained from the new tableau is also tested for optimality. Similar improvements in the solutions are done until an optimal solution is reached.

Two-phase Method Where artificial variables are involved, the two-phase method can also be used. In Phase-I, all artificial variables are eliminated while in the other phase the optimal solution is obtained.

Some points are notable:

(i) Basic variables in any solution have a Δ_j value equal to zero. In an optimal solution, if some non-basic variable(s) has $\Delta_j = 0$, the problem has multiple optimal solutions, while if none of the non-basic variables have $\Delta_j = 0$, the solution is unique optimal solution.

(ii) Infeasibility is present when a problem has no feasible solution. In solving an LPP by simplex method, if a final solution is reached in terms of the Δ_j values and an artificial variable is present in the basis, then infeasibility is indicated.

(iii) In improving a non-optimal solution, the outgoing variable is chosen as the one with the least non-negative replacement ratio. If all such ratios are negative, or a mix of negative and infinity type, then the problem has unbounded solution.

(iv) A solution is said to be degenerate if some basic variable(s) has a solution value equal to zero. A degenerate solution may be optimal or non-optimal. When a degenerate solution is non-optimal and the outgoing variable happens to be a degenerate variable, there is no change in the objective function value for the solution obtained in the next tableau.