Q1

A random variable X has the p.d.f. :

$$f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & otherwise \end{cases}$$

Find (i) $P\left(X < \frac{1}{2}\right)$, (ii) $P\left(\frac{1}{4} < X < \frac{1}{2}\right)$, (iii) $P\left(X > \frac{3}{4} \,\middle|\, X > \frac{1}{2}\right)$, and (iv) $P\left(X < \frac{3}{4} \,\middle|\, X > \frac{1}{2}\right)$.

$$P\left(x < \tfrac{1}{2}\right) = \int_0^{1/2} f(x)\,dx = \int_0^{1/2} 2x\,dx = \left[\frac{\cancel{2}x^2}{\cancel{2}}\right]_0^{1/2}$$

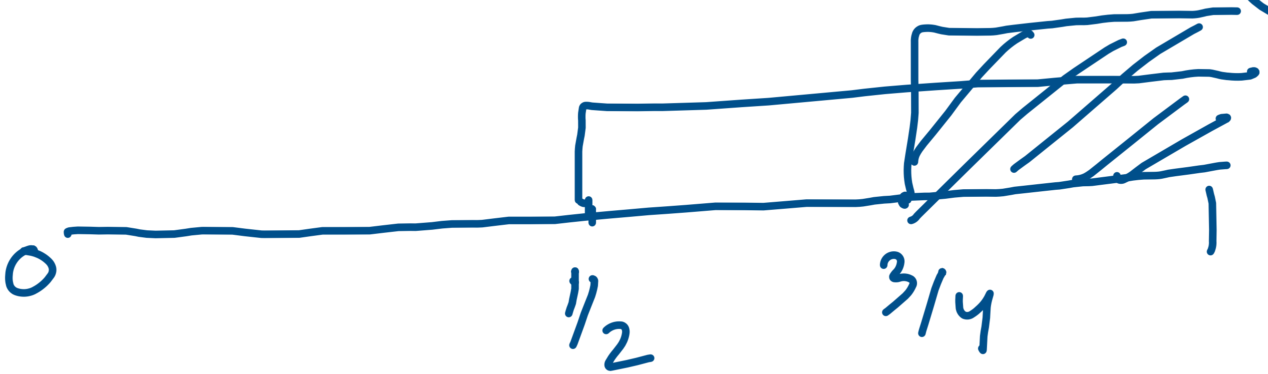
$$= \left(\tfrac{1}{2}\right)^2 - 0 = \frac{1}{4}$$

$$P\left(\frac{1}{4} < X < \frac{1}{2}\right) = \int_{1/4}^{1/2} 2x \, dx$$

$$= \cancel{2}\left[\frac{x^2}{\cancel{2}}\right]_{1/4}^{1/2} = \left(\frac{1}{2}\right)^2 - \left(\frac{1}{4}\right)^2$$

$$= \frac{1}{4} - \frac{1}{16} = \frac{4-1}{16} = \frac{3}{16}$$

$$P\left(x > 3/4 \mid x > \tfrac{1}{2}\right) = \frac{P\left(\left(x > 3/4\right) \cap \left(x > \tfrac{1}{2}\right)\right)}{P\left(x > \tfrac{1}{2}\right)}$$



$$= \frac{P\left(\tfrac{3}{4} < x < 1\right)}{P\left(x > \tfrac{1}{2}\right)}$$

$$\therefore P(A|B) = \frac{P(A \cap B)}{P(B)}$$

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$$P\left(\frac{3}{4} < X < 1\right) = \int_{3/4}^{1} f(x)\,dx = \int_{3/4}^{1} 2x\,dx$$

$$= \left[\frac{\cancel{2}\,x^2}{\cancel{2}}\right]_{3/4}^{1} = 1^2 - \left(\frac{3}{4}\right)^2$$

$$= 1 - \frac{9}{16} = \frac{16-9}{16} = \frac{7}{16}$$

$$= \frac{7}{16}$$

$$p\left(x > \frac{1}{2}\right) = p\left(\frac{1}{2} < x < 1\right)$$

$$= \int_{1/2}^{1} 2x \, dx = \left[\frac{\cancel{2}x^2}{\cancel{2}}\right]_{1/2}^{1}$$

$$= 1^2 - \left(\frac{1}{2}\right)^2 = 1 - \frac{1}{4}$$

$$p\left(x > \frac{1}{2}\right) = \frac{3}{4} = 1 - p\left(x < \frac{1}{2}\right) = 1 - \frac{1}{4}$$

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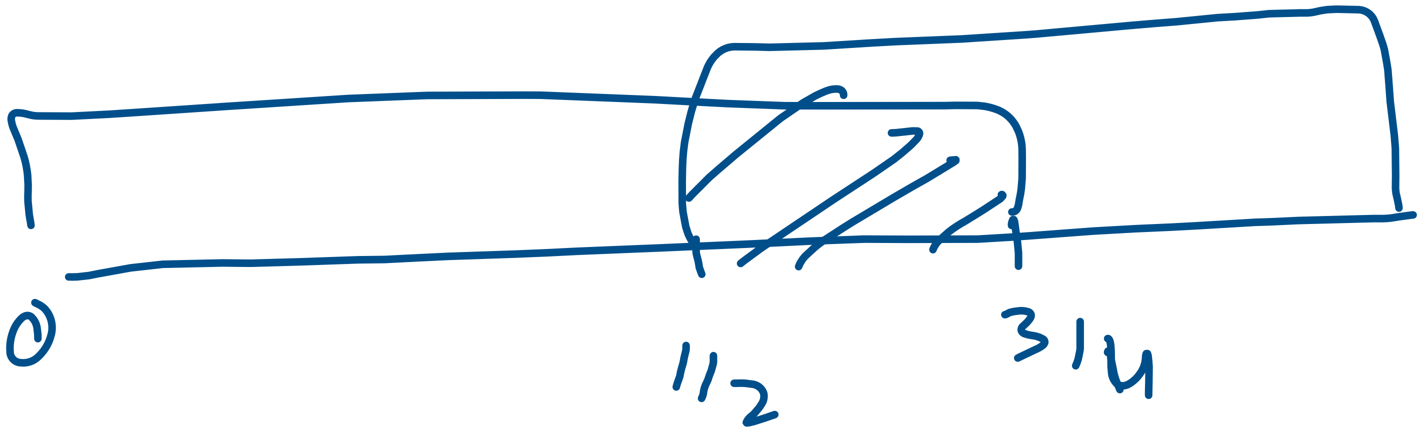
$$P(X > 3/4 \mid X > \tfrac{1}{2}) = \frac{P(\tfrac{3}{4} < X < 1)}{P(X > 1/2)}$$

$$= \frac{7/16}{3/4} = \frac{7}{16} \times \frac{4}{3}$$

$$= \frac{7}{12}$$

$$P(x < 3/4 \mid x > \tfrac{1}{2}) = \frac{P((x < 3/4) \cap (x > \tfrac{1}{2}))}{P(x > \tfrac{1}{2})}$$



$$= \frac{P(\tfrac{1}{2} < x < 3/4)}{P(x > 1/2)}$$

$$P\left(\frac{1}{2} < X < \frac{3}{4}\right) = \int_{1/2}^{3/4} 2x \, dx$$

$$= \left[\frac{\cancel{2}x^2}{\cancel{2}}\right]_{1/2}^{3/4} = \left(\frac{3}{4}\right)^2 - \left(\frac{1}{2}\right)^2$$

$$= \frac{9}{16} - \frac{1}{4} = \frac{9-4}{16} = \frac{5}{16}$$

$$P\left(x < 3/4 \mid x > \tfrac{1}{2}\right) = \frac{5/16}{3/4}$$

$$= \frac{5}{16} \times \frac{4}{3} = \frac{5}{12}$$

$$= \frac{5}{12}$$

Continuous Distribution Function. If X is a continuous random variable with the p.d.f. $f(x)$, then the function

$f(x)$ $x = t$

$$F_X(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt\,, \quad -\infty < x < \infty$$

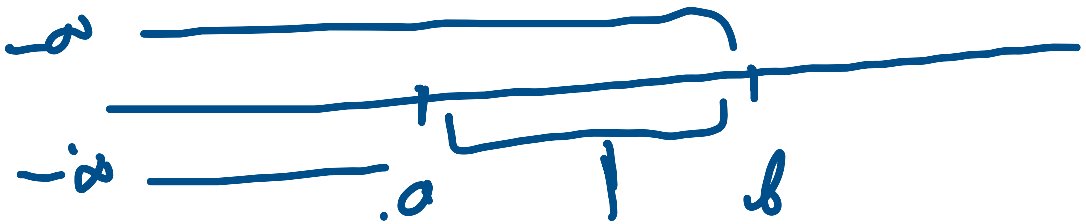
$= P(-\infty < X \le x)$

is called the *distribution function* (d.f.) or sometimes the *cumulative distribution function* (c.d.f.) of the random variable X.

It may be noted that

$$P(a \le X \le b) = \int_a^b f(x)\,dx = \int_{-\infty}^{b} f(x)\,dx - \int_{-\infty}^{a} f(x)\,dx$$

$$= P(X \le b) - P(X \le a) = F(b) - F(a)$$



Remarks 1. $0 \le F(x) \le 1, \quad -\infty < x < \infty.$

2. From analysis (Riemann integral), we know that

$$\boxed{F'(x) = \frac{d}{dx} F(x) = f(x) \ge 0} \qquad [\because f(x) \text{ is p.d.f.}]$$

$\Rightarrow$ $F(x)$ is non-decreasing function of x.

3. $F(-\infty) = \lim_{x \to -\infty} F(x) = \lim_{x \to -\infty} \int_{-\infty}^{x} f(x)\,dx = \int_{-\infty}^{-\infty} f(x)\,dx = 0$

and $F(+\infty) = \lim_{x \to \infty} F(x) = \lim_{x \to \infty} \int_{-\infty}^{x} f(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx = 1$

$F(-\infty) = \int_{-\infty}^{-\infty} f(t)\,dt = 0 \qquad F(\infty) = \int_{-\infty}^{\infty} f(t)\,dt = 1$

$$P(a < X < b) = P(a < X \le b) = P(a \le X < b) = \int_a^b f(t)\,dt$$

$$P(a < x < b) = P(a \le x < b)$$
$$= P(a < x \le b)$$
$$= P(a \le x \le b) = \int_a^b f(x)\,dt$$

x - is a continuous RV

Q2 *Verify that the following is a distribution function:*

$$F(x) = \begin{cases} 0, & x < -a \\ \frac{1}{2}\left(\frac{x}{a} + 1\right), & -a \leq x \leq a \\ 1, & x > a \end{cases}$$

$F(-\infty) = 0$

$F(\infty) = 1$

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$$f(x) = \frac{d}{dx} F(x) = \begin{cases} \frac{d}{dx}(0) & x < -a \\ \frac{d}{dx} \frac{1}{2}\left(\frac{x}{a} + 1\right) & -a \leq x \leq a \\ \frac{d}{dx}(1) & x > a \end{cases}$$

$$f(x) = \begin{cases} 0 & , \quad x < -a \\ \frac{1}{2}\left(\frac{1}{a}+0\right) = \frac{1}{2a} & , \quad -a \leq x \leq a \\ 0 & , \quad x > a \end{cases}$$

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{-a} f(x)\, dx + \int_{-a}^{a} f(x)\, dx + \int_{a}^{\infty} f(x)\, dx$$

$$\int_{-\infty}^{\infty} f(x)\,dx = 0 + \int_{-a}^{a} \frac{1}{2a}\,dx + 0$$

$$= 0 + \left[\frac{x}{2a}\right]_{-a}^{a} + 0$$

$$= \frac{a-(-a)}{2a} = \frac{2a}{2a} = 1$$

$$\boxed{\int_{-\infty}^{\infty} f(x)\,dx = 1}$$

Solution. Obviously the properties *(i)*, *(ii)*, *(iii)* and *(iv)* are satisfied. Also we observe that $F(x)$ is continuous at $x = a$ and $x = -a$ as well.

Now

$$\frac{d}{dx} F(x) = \begin{cases} \frac{1}{2a}, & -a \leq x \leq a \\ 0, & \text{otherwise} \end{cases}$$

$$= f(x), \text{ say}$$

In order that $F(x)$ is a distribution function, $f(x)$ must be a p.d.f. Thus we have to show that

$$\int_{-\infty}^{\infty} f(x)\ dx = 1$$

Now $\displaystyle\int_{-\infty}^{\infty} f(x)\ dx = \int_{-a}^{a} f(x)\ dx = \frac{1}{2a}\int_{-a}^{a} 1 \,.\, dx = 1$

Hence $F(x)$ is a d.f.

Q3

Suppose that the time in minutes that a person has to wait at a certain station for a train is found to be a random phenomenon, a probability function specified by the distribution function,

$$
\begin{aligned}
F(x) &= 0, && \text{for } x \le 0 \\
&= \frac{x}{2}, && \text{for } 0 \le x < 1 \\
&= \frac{1}{2}, && \text{for } 1 \le x < 2 \\
&= \frac{x}{4}, && \text{for } 2 \le x < 4 \\
&= 1, && \text{for } x \ge 4
\end{aligned}
$$

(a) Is the Distribution Function continuous ? If so, give the formula for its probability density function ?

(b) What is the probability that a person will have to wait (i) more than 3 minutes, (ii) less than 3 minutes, and (iii) between 1 and 3 minutes ?

$$F(x) = \begin{cases} 0 & x \le 0 \\ \frac{x}{2} & 0 \le x < 1 \\ \frac{1}{2} & 1 \le x < 2 \\ \frac{x}{4} & 2 \le x < 4 \\ 1 & x \geq 4 \end{cases}$$

	I	II
0	0	$\frac{x}{2} = \frac{0}{2} = 0$
1	$\frac{x}{2} = \frac{1}{2}$	$\frac{1}{2}$
2	$\frac{1}{2}$	$\frac{x}{4} = \frac{2}{4} = \frac{1}{2}$
4	$\frac{x}{4} = \frac{4}{4} = 1$	1

So $F(x)$ is continuous

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$$f(x) = \frac{d}{dx} F(x) = \begin{cases} \frac{d}{dx}(0), & x \le 0 \\ \frac{d}{dx}\left(\frac{x}{2}\right), & 0 \le x < 1 \\ \frac{d}{dx}\left(\frac{1}{2}\right), & 1 \le x \le 2 \\ \frac{d}{dx}\left(\frac{x}{4}\right), & 2 \le x < 4 \\ \frac{d}{dx}(1), & x \geq 4 \end{cases}$$

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$$f(x) = \begin{cases} 0 , & x \leq 0 \\ \frac{1}{2} , & 0 \leq x < 1 \\ 0 , & 1 \leq x < 2 \\ \frac{1}{4} , & 2 \leq x < 4 \\ 0 , & x \geq 4 \end{cases}$$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx$$

$$\int_{2}^{4} f(x)\,dx + \int_{4}^{\infty} f(x)\,dx$$

$$= 0 + \int_0^1 \frac{1}{2}\,dx + 0 + \int_2^4 \frac{1}{4}\,dx$$

$$+ \; 0$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \left[\frac{x}{2}\right]_0^1 + \left[\frac{x}{\cancel{2}4}\right]_2^4$$

$$= \frac{1-0}{2} + \left(\frac{4-2}{\cancel{2}4}\right) = \frac{1}{2} + \frac{1}{2} = 1$$

$$P(X \geq 3) = \int_3^\infty f(x)\,dx$$

$$= \int_3^4 f(x)\,dx + 0$$

$$= \int_3^4 \left(\frac{1}{4}\right) dx = \left[\frac{x}{4}\right]_3^4$$

$$= \frac{1}{4}$$

$$p(x<3) = 1 - p(x \geq 3)$$

$$= 1 - \frac{1}{4}$$

$$= 3/4$$

$$p(1<x<3) = \int_1^3 f(x)\,dx$$

$$= \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx$$

$$= 0 + \int_2^3 \frac{1}{4}\, dx$$

$$= \left[\frac{x}{4}\right]_2^3 = \frac{3-2}{4}$$

$$= \frac{1}{4}$$