L27 24 March 2021

24 March 2021 12:02

Seperation of Variables Method :-

$u(x,y) = XY$

$\hookrightarrow X = f(x)$

$Y = g(y)$

Problem 1. $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y}$

Let us Consider the sol. as

$u(x,y) = X.Y$

$\frac{\partial u}{\partial x} = X'Y \qquad \frac{\partial u}{\partial y} = XY'$

⇒ n becomes $X'Y = XY'$

$\frac{X'}{X} = \frac{Y'}{Y} = K$ (say)

$\int \frac{f'(x)}{f(x)} dx = \log(f(x))$

$\frac{X'}{X} = K \Rightarrow \int \frac{X'}{X} = \int K dx$

$\Rightarrow \log X = Kx + C_1$

$\Rightarrow \boxed{X = e^{Kx+C_1}}$

IIly $\frac{Y'}{Y} = K \Rightarrow \int \frac{Y'}{Y} = \int K dy$

$\Rightarrow \log Y = Ky + C_2 \Rightarrow \boxed{Y = e^{Ky+C_2}}$

$u(x,y) = XY = e^{Kx+C_1} . e^{Ky+C_2}$

$= e^{C_1+C_2} . e^{Kx+Ky}$

$\boxed{u(x,y) = C . e^{Kx+Ky}}$ Soln.

(2)

$y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = 0$

Let us suppose the solution $u(x,y) = XY$

$$y\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = 0$$

Let us suppose the solution

$$u(x,y) = XY$$

$$\frac{\partial u}{\partial x} = X'Y \qquad \frac{\partial u}{\partial y} = XY'$$

$$y\frac{\partial u}{\partial x} = -x\frac{\partial u}{\partial y}$$

$$y X'Y = -x \cancel{X} Y'$$

$$\frac{X'}{xX} = -\frac{Y'}{yY} = k \text{ (say)}$$

$$\frac{X'}{xX} = k \Rightarrow \frac{X'}{X} = kx \Rightarrow \log X = k\cdot\frac{x^2}{2} + c_1$$

$$\Rightarrow \boxed{X = e^{\frac{kx^2}{2} + c_1}}$$

$$-\frac{Y'}{yY} = k \Rightarrow \frac{Y'}{Y} = -ky \Rightarrow \log Y = -\frac{ky^2}{2} + c_2$$

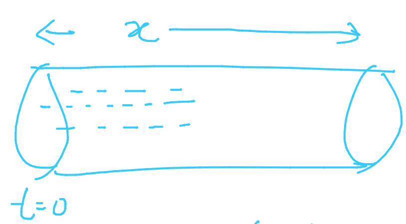
$$\Rightarrow \boxed{Y = e^{-\frac{ky^2}{2} + c_2}}$$

$$u(x,y) = e^{\frac{kx^2}{2} - \frac{ky^2}{2}} \cdot e^{c_1 + c_2}$$

$$\boxed{u(x,y) = c\, e^{\frac{k}{2}(x^2 - y^2)}}$$

One-Dimensional Heat Equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 \le x \le l, \quad t > 0$$



$u(x,t) \to$ heat flow

Let the solution be

$$u(x,t) = XT$$

$$\frac{\partial u}{\partial x} = X'T \qquad \frac{\partial u}{\partial t} = XT'$$

$$\frac{\partial^2 u}{\partial x^2} = X''T$$

The given eqn

$$XT' = c^2 X''T$$

$\frac{\partial^2 u}{\partial x^2} = X''T$ The given eqn $XT' = c^2 X''T$

$\Rightarrow \frac{T'}{T} = \frac{c^2 X''}{X}$

$\Rightarrow \frac{X''}{X} = \frac{1}{c^2}\frac{T'}{T} = k$ (say)

As k can take these 3 values, we have certain cases

Case ① If $k = \underline{0}$

$\frac{X''}{X} = 0 \Rightarrow X'' = 0 \Rightarrow \boxed{X = ax + b}$

$X' = a$

$X = ax + b$

$\frac{1}{c^2}\frac{T'}{T} = 0 \Rightarrow T' = 0 \Rightarrow T = \text{Constant} = c$

$u(x,t) = XT = (ax+b)c = acx + bc$

$\boxed{u(x,t) = Ax + B}$

$D^2 = 4 = 2^2$

$D =$

Case ② If k is positive. Let $k = p^2$ (positive)

$\frac{X''}{X} = p^2 \Rightarrow X'' - p^2 X = 0$

S.E $(D^2 - p^2)X = 0$

$D^2 - p^2 = 0$

$D^2 = p^2$

$\boxed{D = \pm p}$

$\boxed{X = c_1 e^{px} + c_2 e^{-px}}$

$\frac{1}{c^2}\frac{T'}{T} = p^2 \Rightarrow T' - p^2c^2 T = 0$

$D - p^2c^2 = 0 \Rightarrow D = p^2c^2$

$\boxed{T = c_3 e^{p^2c^2t}}$

$u(x,t) = XT = (c_1 e^{px} + c_2 e^{-px})(c_3 e^{p^2c^2t})$

$u(x,t) = XT = \left(c_1 e^{px} + c_2 e^{-px}\right)\left(c_3 e^{p^2c^2t}\right)$

$$\boxed{u(x,t) = \left(Ae^{px} + Be^{-px}\right) e^{p^2c^2t}}$$

Case ③ If $K = -ve$

$K = -p^2$

$\frac{X''}{X} = -p^2 \Rightarrow X'' + p^2X = 0$

$D^2 + p^2 = 0 \Rightarrow D = \pm ip$

$X = e^{0x}\left(c_1 \sin px + c_2 \cos px\right)$

$$\boxed{X = c_1 \sin px + c_2 \cos px}$$

lly $\frac{1}{c^2}\frac{T'}{T} = -p^2$

$\Rightarrow T' + p^2c^2T = 0$

$\Rightarrow D + p^2c^2 = 0 \Rightarrow D = -p^2c^2$

$$\boxed{T = c_3 e^{-p^2c^2t}}$$

$u(x,t) = \left(c_1 \sin px + c_2 \cos px\right)\left(c_3 e^{-p^2c^2t}\right)$

$$\boxed{u(x,t) = \left(A \sin px + B \cos px\right) e^{-p^2c^2t}}$$

Note: Heat Equation

Equation: $\frac{\partial u}{\partial t} = C^2 \frac{\partial^2 u}{\partial x^2}$

Nature: Parabolic

Solution: **1.** $u(x,t) = Ax + B$

2. $u(x,t) = (Ae^{px} + Be^{-px})\, e^{C^2p^2t}$

or

$u(x,t) = (A\cosh px + B\sinh px)e^{C^2p^2t}$

3. $u(x,t) = (A\cos px + B\sin px)e^{-C^2p^2t}$ → **(Most suitable one)**

Q69. Let u(x, t) satisfy $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$, $u(0,t) = 0$, $u(1,t) = 0$,, $u(x,0) = \sin \pi x$ than

a) $u(x,t) = \sum \sin(\pi x)\, e^{-n^2\pi^2 t}$

b) $u(x,t) = \sum \sin(\pi x)\, e^{-n\pi^2 t/l^2}$

c) $u(x,t) = \sin(\pi x)\, e^{-\pi^2 t}$

d) None of these

$u(0,t) = \sum \sin(\pi 0)\, e^{-\pi^2 n^2 t}$ (c)

$u(x,t) = \sin(\pi x)\, e^{-\pi^2 t}$

$u(0,t) = \sin(\pi 0)\, e^{-\pi^2 t}$

$= 0$

$u(1,t) = \sin(\pi)\, e^{-\pi^2 t}$

$= 0$

$\cancel{\{\sin \pi x + \cos \pi x\}}$

$u(x,0) = \sin \pi x\, e^{-\pi^2(0)}$

$= \sin \pi x$

$6 + 6 = 12 \times 40$

5