L12 12 Feb 2021

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Solution of 2nd order Homogeneous Linear Diff eqn with Constant Coefficient

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

or

$$ay'' + by' + cy = 0$$

Let $\frac{d}{dx} = D$

$$aD^2y + bDy + cy = 0 \quad \text{or} \quad (aD^2+bD+c)y = 0$$

↳ Symbolic form of the eqn.

Auxiliary equation (A.E)

$$aD^2+bD+c = 0$$

$$\Rightarrow D = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = m_1, m_2$$

Case 1) If roots are real and distinct $(b^2-4ac>0)$ m_1, m_2

Sol $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

Case 2) If roots are real but equal. $(b^2-4ac=0)$ m_1, m_1

Sol $y = (c_1 + c_2 x)e^{m_1 x}$

Case 3) If roots are complex conjugate $(b^2-4ac<0)$ $\alpha \pm i\beta$

Sol $y = e^{\alpha x}(c_1\cos\beta x + c_2 \sin\beta x)$

m_1, m_2

$D = \alpha \pm i\beta$

$2 \pm i3$

$3 \pm i4$

$(D-m_1)(D-m_1)$

$y = c_1 e^{m_1 x}$

$y = c_2 x e^{m_1 x}$

$aD^2+bD+c=0$

$(D-m_1)(D-m_2)y=0$

$(D-m_1)y=0$ $\quad y = c_2 e^{m_2 x}$

$\frac{dy}{dx} - m_1 y = 0$

$\frac{dy}{dx} = m_1 y$

$\frac{1}{y}\frac{dy}{dx} = m_1$

$\log y = m_1 x + c$

$y = e^{m_1 x} \cdot e^{c}$

$D^2y - 4y = 0$

Problem 1. $y''-4y = 0$

A.E $\quad D^2 - 4 = 0 \quad$ sol $\quad y = c_1 e^{-2x} + c_2 e^{2x}$

Problem 1. $y'' - 4y = 0$

A.E $D^2-4=0$

$D^2=4.$

$D=\pm 2$

$D=-2,2$

Sol $y=c_1e^{-2x}+c_2e^{2x}$

$\log y = m_1x + c$

$y=e^{m_1x}\cdot e^{c}$

$y=c_1e^{m_1x}$

Problem 2. $y'' - 4y' - 12y = 0$

A.E $D^2-4D-12=0$

$D^2-6D+2D-12=0$

$D(D-6)+2(D-6)=0$

$(D-6)(D+2)=0$

$D=6,-2$

Sol $y=c_1e^{6x}+c_2e^{-2x}$

Problem 3

$y''+2y'+y=0$

A.E $D^2+2D+1=0$

$(D+1)^2=0 \Rightarrow D=-1,-1$

Sol

$y=(c_1+c_2x)e^{-x}$

Problem 4

$y''+25y=0$

$D^2+25=0$

$D^2=-25$

$D=\pm i5 = 0\pm i5 \quad (\alpha\pm i\beta)$

Sol $y=e^{0x}(c_1\cos 5x+c_2\sin 5x)$

Problem no 5

$y''+4y'+5y=0$

$D^2+4D+5=0$

$D=\dfrac{-4\pm\sqrt{16-4\times5}}{2}=\dfrac{-4\pm\sqrt{16-20}}{2}=\dfrac{-4\pm\sqrt{-4}}{2}=-2\pm i$

Sol $y=e^{-2x}(c_1\cos 1x+c_2\sin x)$

Q16. If the roots of the auxiliary equation of a linear differential equation are 4,4 then. its complementary function is

(a) $C_1e^{4x} + C_2e^{4x}$ (b) $(C_1 + C_2x)e^{4x}$ (c) $(C_1 + C_2x^2)e^{4x}$ (d) $(C_1x + C_2x^2)e^{4x}$

(a) 2 (b) 3

Q19. Consider the differential equation $\frac{d^2y}{dx^2} - 7y = 0$, which of the following is correct?

(a) The roots of the auxiliary equation are 0 and 7.

(b) There is no auxiliary equation for a differential equation of this type.

(c) The auxiliary equation has a repeated root of $\sqrt{7}$.

(d) The roots of the auxiliary equation $\sqrt{7}$ and $-\sqrt{7}$.

$D^2 - 7 = 0$

$D^2 = 7$

$D = \pm\sqrt{7}$

Formulation of LDE of the form: $ay'' + by' + cy = 0$ when Roots are given:

Let the two given roots be: m_1 and m_2.

Then required 2nd order homogeneous LDE is:

$y'' - (\text{sum of roots})\, y' + (\text{Product of roots})y = 0$

i.e. $y'' - (m_1 + m_2)y' + (m_1m_2)y = 0$

$ax^2 + bx + C = 0$

Sum $= -b/a$

$P = c/a$

Find a LDE of the form: $ay'' + by' + cy = 0$ for which the following functions are solutions:

Problem 1. (e^{3x}, e^{-2x})

$y'' - (3-2)y' + (-6)y = 0 \Rightarrow \boxed{y'' - y' - 6y = 0}$

Problem 2. $(1, e^{-2x})$

$y'' + 2y' = 0$

e^{0x}, e^{-2x}

$m = 0, -2$

$y'' - (-2)y' + 0y = 0$

$y'' + 2y' = 0$