Unit 2 : DIFFERENTIATION AND INTEGRATION

4.1 Introduction of Differentiation

4.1.1 Differentiation of Common Functions

Table 1.1

$y = f(x)$	$\frac{dy}{dx}$
x	1
x^n	nx^{n-1}
kx^n	nkx^{n-1}
Constant	0
e^x	e^x
e^{kx}	ke^{kx}
a^x	$a^x \ln a$
$\ln x$	$\frac{1}{x}$
$\ln kx$	$\frac{1}{x}$
$\sin x$	$\cos x$
$\sin kx$	$k \cos kx$
$\cos x$	$-\sin x$
$\cos kx$	$-k \sin kx$
$\tan x$	$sek^2 x$
$\tan kx$	$k\, sek^2\, kx$
$\cot x$	$-\cos ec^2 x$
$\sec x$	$\sec x.\tan x$
$\cos ec\, x$	$-\cos ec\, x.\cot x$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\sin^n x$	$n \sin^{n-1} x.\cos x$
$\cos^n x$	$-n \cos^{n-1} x.\sin x$
$\tan^n x$	$n \tan^{n-1} x.sek^2 x$

* k is a constant

Example 1 :

Find $\frac{dy}{dx}$ in each of the following cases :

a) $y = \frac{13}{x^3}$ b) $y = 4x^5$ c) $y = 2\sqrt{x}$

d) $y = e^{\frac{x}{2}}$ e) $y = \cos\frac{x}{2}$ f) $y = \sin^3 x$

Solutions :

a.)

d)

b)

e)

c)

f)

Example 2 :

Write down the derivatives of the following :

a) e^{3x} b) 2^x c) $2\sin 3x$

d) $3/e^{5x}$ e) $\frac{e^x + e^{-x}}{2}$ f) 4 cos $\frac{x}{2}$ + 9 - 9 x^3

Solutions :

a)

b)

c)

d)

e)

f)

Example 3 :

Find $\dfrac{dy}{dx}$ in each of the following cases :

a) $y = \dfrac{6x^2 + 4x - 3}{2x}$

b) $y = (x-5)^2$

c) $y = \sqrt{x}(x+2)$

d) $y = \dfrac{2}{7e^{2x}} + 8\ln 5x$

e) $y = x^4 + 2e^{6x} - \cos\dfrac{3}{2}x$

f) $y = \dfrac{5}{\sqrt[3]{x^2}} - 4\cos 2x$

Solutions :

a)

b)

c)

d)

e)

f)

4.2 Techniques of Differentiation

4.2.1 Techniques of Differentiation ;The Chain Rule, Product Rule and Quotient Rule)

4.2.1.1 The Chain Rule

If y is a function of x then ;

$$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$$

Example 4 :

Differentiate $y=\sqrt{\left(2x^2-3\right)}$

Solution :

Example 5 :

Differentiate $y=\left(x^2+1\right)^4$

Solution :

Example 6 :

Differentiate the following :

a) $y = (4x-5)^6$

b) $y = \left(3x^2 - 6x\right)^5$

c) $y = \sqrt{6x^2 + 4x - 2}$

d) $y = \left(x + \dfrac{1}{x}\right)^4$

e) $y = \dfrac{1}{\left(x^3 - 4\right)^2}$

f) $y = \dfrac{2}{\sqrt{x^2 + x + 1}}$

Solutions :

a)

b)

c)

d)

e)

f)

4.2.1.2 Differentiation of a Product

If $y = uv$, where u and v are functions of x, then

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

Example 7 :

Find $\dfrac{dy}{dx}$ in each of the following cases :

a) $y = (x-2)(x+1)^2$ b) $f(x) = (6x^3)(7x^4)$ c) $f(x) = 2x^3(x^2+3)^4$

Solutions:

a)

b)

c)

4.2.1.3 Differentiation of a Quotient

If $y = \dfrac{u}{v}$, where u and v are functions of x, then,

$$\frac{dy}{dx} = \frac{u'v - uv'}{v^2}$$

Example 8 :

Find $\dfrac{dy}{dx}$ in each of the following cases :

a) $y = \dfrac{x^2-1}{x^2+1}$ b) $y = \dfrac{x}{\sqrt{1+2x}}$ c) $f(x) = \dfrac{x^2+x-2}{x^3+6}$

Solutions :

a) b)

c)

4.2.2 Implicit Differentiation

A method of finding the derivative of an implicit function by taking the derivative of each term with respect to the independent variable while keeping the derivative of the dependent variable with respect to the independent variable in symbolic form and then solving for that derivative.

If $y = f(x, y)$, where $f(x, y)$ is implicit function , then,

$$\frac{dy}{dx} = \frac{d}{dx} f(x, y) + \frac{d}{dy} f(x, y) \bullet \frac{dy}{dx}$$

Example 9 :

Assume that y is a function of x . Find $y' = \dfrac{dy}{dx}$ for $x^3 + y^3 = 4$

Solution :

Example 10 :

Assume that y is a function of x . Find $y' = \dfrac{dy}{dx}$ for $y = \sin(3x + 4y)$

Solution :

4.2.3 Parametric Differentiation

> Parametric differentiation:
>
> if x = x(t) and y = y(t) then
>
> $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}; \qquad \text{provided} \frac{dx}{dt} \neq 0$$

Example 11 :

Find $\dfrac{dy}{dx}$ when $x = t^3 - t$ and $y = 4 - t^2$.

Solution :

Example 12 :

Find $\frac{dy}{dx}$ when x = 3 cos t, y = 3 sin t

Solution :

4.2.4 Logarithmic and Trigonometric Differentiation

Table 1.2

***y* (*u* is a function of *x*)**	$\frac{dy}{dx}$
u^n	$nu^{n-1}\frac{du}{dx}$
ku^n	$nku^{n-1}\frac{du}{dx}$
e^u	$e^u\frac{du}{dx}$
a^u	$a^u \ln(a)\frac{du}{dx}$
ln (*u*)	$\frac{1}{u}\frac{du}{dx}$
sin (u)	$\cos(u)\frac{du}{dx}$
cos (u)	$-\sin(u)\frac{du}{dx}$
tan (u)	$\sec^2(u)\frac{du}{dx}$
sec (u)	$\sec(u)\tan(u)\frac{du}{dx}$
cosec (u)	$-\cos ec(u)\cot(u)\frac{du}{dx}$
sinh (u)	$\cosh(u)\frac{du}{dx}$
cosh (u)	$\sinh(u)\frac{du}{dx}$

- ***Properties of ln*** ;

a) $\ln a^n = n \ln a$

b) $\ln ab = \ln a + \ln b$

c) $\ln \frac{a}{b} = \ln a - \ln b$

Example 13 :

Differentiate :

a) $y = \sin(x^2)$ b) $y = \sin^2 x$ c) $y = \ln(x^2)$

d) $y = e^{\sin 3x}$ d) $y = \ln(\cos 2x)$ f) $y = \cos^3 5x$

Solutions :

a)

b)

c)

d)

e)

f)

Example 14 :

Determine

a) $\dfrac{d}{dx}\left[\sin\left(x^4 + 3x\right)\right]$ b) $\dfrac{d}{dx}e^{x^2+\cos x}$ c) $\dfrac{d}{dx}\ln(3+4x)$

Solutions :

a)

b)

c)

Example 15 :

Find $\dfrac{dy}{dx}$ in each of the following cases :

a) $y = x^2 \tan x$ b) $y = \sqrt{x^3}\,\ln 3x$

c) $y = x^3 \sin 5x$ d) $y = e^{3x}\ln x$

e) $y = e^{4x}\ln 3x$ f) $y(t) = \dfrac{\sin(t)}{3 - 2\cos(t)}$

Solutions :

a)

b)

c)

d)

e)

f)

4.2.5 Tangents and Normals

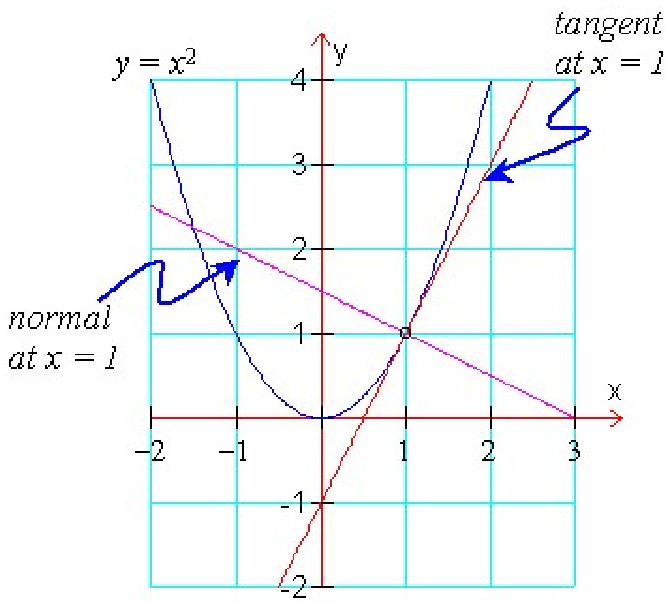
Theory:

Consider the function: $y = f(x)$, with point (x_1,y_1) lying on the function graph. The tangent line to the function at (x_1,y_1) is the straight line that touches

$y = f(x)$ at that point. Both the graph of $y = f(x)$ and the tangent line pass through the point, and the tangent line has the same gradient, 'm', as the function at that point.

The normal line to function $y = f(x)$ at the point (x_1,y_1) is the straight line that passes through the point making a 90º angle with the graph. The gradient of the normal line is $^{-1}/_{m}$, where 'm' is the gradient of the tangent line at the same point.

For example, consider the function $y = x^2$. The tangent and normal lines at the point (1,1) are shown on the diagram below:



The equation of the **tangent line** to y = f(x) at the point (x_1,y_1):
('m' is the gradient at (x_1,y_1))

$$y = m(x - x_1) + y_1$$

The equation of the **normal line** to y = f(x) at (x_1,y_1) is:

$$y = -\frac{1}{m}(x - x_1) + y_1$$

The derivative of y = f(x) at (x_1,y_1) gives us the gradient 'm'.

Example 16 :

Calculate the tangent and normal lines to the function: $y = x^2$,
when x = 1.

Solution :

At x = 1, $y = 1^2 = 1$. So $(x_1,y_1) = (1,1)$
The gradient 'm' at x = 1 is found by calculating the derivative of the function at that point. For $y = x^2$, $\frac{dy}{dx} = 2x$. At x = 1, $\frac{dy}{dx} = 2(1) = 2$.

So 'm' = 2.

The tangent line is given by:

$$y = m(x - x_1) + y_1$$

Substituting, we have: $y = 2(x - 1) + 1$

Expanding and simplifying: $y = 2x - 2 + 1$

so: $y = 2x - 1$

Normal line:

$$y = -\frac{1}{m}(x - x_1) + y_1$$

$$y = -\frac{1}{2}(x - 1) + 1 = -\frac{x}{2} + \frac{3}{2}$$

Example 17 :

Find the equation of the tangent and normal line to $y = 3x^2 - 4x + 7$ at x = 1.

Solution :

4.3 Introduction of Integration

Definitions

Given a function *f(x)* an **anti-derivative** of *f(x)* is any function *F(x)* such that

$$F'(x) = f(x)$$

If *F(x)* is any anti-derivative of *f(x)* then the most general anti-derivative of *f(x)* is called an **indefinite integral** and denoted,

$$\int f(x)dx = F(x) + c\,, \quad c \text{ is any constant}$$

In this definition $\int$ the is called the **integral symbol**, *f(x)* is called the **integrand**, *x* is called the **integration variable** and the "*c*" is called the **constant of integration**.

4.3.1 The Indefinite Integrals

Table 1.3

$\int x^n \, dx$	$\frac{x^n}{n+1} + C$
$\int \frac{1}{x} \, dx$	$\ln x + C$
$\int \frac{1}{ax+b} dx$	$\frac{1}{a} \ln\lvert ax+b \rvert + C$
$\int e^x \, dx$	$e^x + C$
$\int e^{kx} \, dx$	$\frac{e^{kx}}{k} + C$
$\int a^x \, dx$	$\frac{a^x}{\ln a} + C$
$(\int ax+b)\, dx$	$\frac{1}{(n+1)(a)}(ax+b)^{n+1} + C$
$\int \sin x \, dx$	$-\cos x + C$
$\int \cos x \, dx$	$\sin x + C$
$\int \tan x \, dx$	$\ln\lvert \sec x \rvert + C$
$\int \cot x \, dx$	$\ln\lvert \sin x \rvert + C$
$\int \sec^2 x \, dx$	$\tan x + C$
$\int \cos ec^2 x \, dx$	$-\cot x + C$
$\int \sec x \tan x \, dx$	$\sec x + C$

$\int \operatorname{cosec} x \cot x \, dx$	$-\operatorname{cosec} x + C$
$\int \sinh x \, dx$	$\cosh x + C$
$\int \cosh x \, dx$	$\sinh x + C$
$\int \sin(ax+b)dx$	$-\frac{1}{a}\cos(ax+b)+C$
$\int \cos(ax+b)dx$	$\frac{1}{a}\sin(ax+b)+C$
$\int \sec^2(ax+b)dx$	$\frac{1}{a}\tan(ax+b)+C$
$\int \frac{1}{\sqrt{1-x^2}}dx$	$\sin^{-1} x + C$
$\int \frac{-1}{\sqrt{1-x^2}}dx$	$\cos^{-1} x + C$
$\int \frac{1}{1+x^2}dx$	$\tan^{-1} x + C$
$\int \frac{1}{\sqrt{x^2+1}}dx$	$\sinh^{-1} x + C$
$\int \frac{1}{\sqrt{x^2-1}}dx$	$\cosh^{-1} x + C$
$\int \frac{1}{1-x^2}dx$	$\tanh^{-1} x + C$

Let p and q be the functions of x then ;

$$\int (p+q)dx = \int p\,dx + \int q\,dx$$

$$\int (p-q)dx = \int p\,dx - \int q\,dx$$

$$\int kp\,dx = k\int p\,dx \text{ where } k \text{ is a constant}$$

Example 18 :

Evaluate each of the following indefinite integrals.

a) $\int (5t^3 + 6t^2)dt$

b) $\int (x^4 - x^{-4})dx$

c) $\int dy$

d) $\int [(w + \sqrt[3]{w})(4 - w^2)]dw$

Solution :

a) b)

c) d)

4.3.2 The Definite Integrals

A definite integral is an integral

$$\int_a^b f(x)dx$$

with upper and lower limits it mean *f(x)* continuous on closed interval [a, b] and *F(x)* is the antiderivative (indefinite integral) of *f(x)* on [a,b], then

$$\int_a^b f(x)dx = [F(x)]_a^b F(b) - F(a) \qquad \text{(Fundamental Theorems of Calculus)}$$

Example 19 :

Evaluate the integrals

a) $\int_1^2 (5t^3 + 6t^2)dt$

c) $\int_0^5 dy$

b) $\int_1^4 (x^4 - x^{-4})dx$

d) $\int_0^1 [(w + \sqrt[3]{w})(4 - w^2)]dw$

Solutions :

a) b)

c) d)

4.3.3 The u-substitution; change of variables

If $u = g(x)$ is a differentiable function whose range is an interval I and $f(x)$ continuous on I, then

$$\int f(g(x))g'(x)dx = \int f(u)du \text{ where } I \in (-\infty, \infty)$$

Example 20 :

Evaluate each of the following integrals.

a) $\int \sqrt{6x+1}\,dx$

b) $\int \frac{x}{\sqrt{1-4x^2}}dx$

c) $\int_1^2 \frac{dy}{(3-5y)^2}$

d) $\int_{-2}^{-1}\left[\frac{2w^3+1}{(w^4+2w)^3}\right]dw$

Solutions :

a)

b)

c)

d)

4.4 Techniques of Integration

4.4.1 Integration by-parts

Integration by parts formula is given by ;

$$\int uv'\,dx = uv - \int u'v\,dx$$

Or

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$

Example 21 :

Determine $\int (5x+1)\cos 2x\,dx$

Solution:

Example 22 :

Evaluate the integrals

a) $\int x^2 \sin(x)dx$

c) $\int_0^{\pi} t(\sin(3t))dt$

b) $\int x^2(\sqrt{1+x})dx$

Solutions :

a)

c)

4.4.2 Using Partial Fraction

Consider the rational function

$$f(x) = \frac{P(x)}{Q(x)} \text{ where } P \text{ and } Q \text{ are polynomials}$$

The rules of partial fractions are as follows ;

a) The degree of $P(x)$ must be less than the degree of $Q(x)$. We factorize the denominator, Q(x) into its prime factor. It is important to determine the shape of the partial fractions.

$$\text{Example : } \int \frac{x^2 + 3x - 1}{2x^3 + 3x^2 - 2x}\, dx$$

b) The degree of $P(x)$ greater than the degree of $Q(x)$. Divide out by long division.

$$\text{Example : } \int \frac{x^3 + x}{x - 1}\, dx$$

c) Factorize the denominator into its prime factors.

d) A linear factor $(ax+b)$ gives a partial fraction of the form : $\dfrac{A}{ax+b}$

e) Factors $(ax+b)^2$ give partial fractions : $\dfrac{A}{ax+b}+\dfrac{B}{(ax+b)^2}$

f) Factors $(ax+b)^3$ give partial fraction **:** $\dfrac{A}{ax+b}+\dfrac{B}{(ax+b)^2}+\dfrac{C}{(ax+b)^3}$

g) A quadratic factor (ax^2+bx+c) gives a partial fraction $\dfrac{Ax+B}{ax^2+bx+c}$

Example 23 :

Find $\int \dfrac{3x^2+x+1}{x+1}dx$

Solution :

Use long division

$$\begin{array}{r|l} & 3x-2 \\ \hline x+1 & 3x^2+x+1 \\ & \underline{3x^2+3x} \\ & \quad -2x+1 \\ & \quad \underline{-2x-2} \\ & \qquad\quad 3 \end{array}$$

$$\frac{3x^2+x+1}{x+1}=3x-2+\frac{3}{x+1}$$

$$\int \frac{3x^2+x+1}{x+1}=\int\left(3x-2+\frac{3}{x+1}\right)dx=\frac{3x^2}{2}-2x+3\ln|x+1|+c$$

Example 24 :

Find $\int \dfrac{x^3+x}{x-1}dx$

Solution :

Example 25 :

Evaluate the following integrals

a) $\displaystyle\int \frac{x^2+1}{x^3+6x^2+11x+6}dx$

b) $\displaystyle\int \frac{-2x^2-14x-49}{x^3-7x^2}dx$

c) $\displaystyle\int \frac{x^3+x^2+x+3}{x^4+3x^2+3}dx$

d) $\displaystyle\int \frac{1-x+2x^2-x^3}{x^5+2x^3+x}$

Solution :

a)

b)

c)

d)

4.4.3 Integration of Trigonometric Functions

Example 26 :

Determine

a) $\int \cos 3x \, dx$ b) $\int \sin 4x \cos 2x \, dx$

Solutions :

a)

b.)

Example 27 :

Find

a) $\int \frac{1}{\sqrt{4-36x^2}} dx$ b) $\int \frac{x}{x^4+9} dx$ c) $\int_0^{\frac{1}{4}} \frac{-1}{\sqrt{1-4x^2}} dx$

Solutions :

a)

b)

c)

Given an equation for the displacement of a moving object, find an equation for its velocity and an equation for its acceleration, and use the equations to analyze the motion.

PROPERTIES: Velocity, Speed, and Acceleration
If x is the displacement of a moving object from a fixed plane (such as the ground), and t is time, then

Velocity: $v = x' = dx/dt$

Acceleration: $a = v' = dv/dt = x'' = d^2x/dt^2$

Speed: $|v|$

Example 1
Suppose a football is punted into the air. As it rises and falls, its **displacement** (directed distance) from the ground is a function of the number of seconds since it was punted.

$$y = -16t^2 + 37t + 37$$

where y is the football's displacement in feet and t is the number of seconds since it was punted.

The **velocity** of the ball gives its speed and the direction in which it's going. Because velocity is the instantaneous rate of change, it is a derivative.

$$\text{velocity} = dy/dt = y' = -32t + 37$$

Find velocity at $t = 1$

at $t = 2$

The dy/dt symbol reminds you of the units for velocity (ft/sec).

Speed is the absolute value of velocity. Speed tells how fast an object is going without regard to its direction.
Describe the speed and velocity at $t = 1$ and $t = 2$ sec.

Note that the velocity changes from $t = 1$ to $t = 2$ sec. The instantaneous rate of change in velocity is called **acceleration.** Using v for velocity, $v = -32t + 37$.
Find acceleration.

The dv/dt symbol for the derivative gives the units of acceleration. dv/dt is in (feet/second)/sec and written "ft/sec^2"
Interpret the idea of negative acceleration.