L36 10 April 2021

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Angle between two Scalar Surface

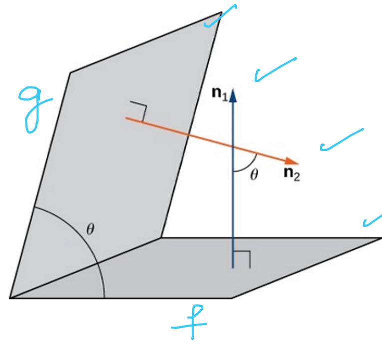
Angle between two surfaces is equal to the angle between their normal.

Let f and g be two given scalar surfaces. Let $\overrightarrow{n_1}$ and $\overrightarrow{n_2}$ be the vectors normal to the surfaces f and g respectively.

Then angle between surfaces f and g is given by:

$$\cos\theta = [\widehat{n_1}.\widehat{n_2}] = \left[\frac{\overrightarrow{n_1}}{|\overrightarrow{n_1}|}.\frac{\overrightarrow{n_2}}{|\overrightarrow{n_2}|}\right]$$

$$\Rightarrow \theta = cos^{-1}\left[\frac{\vec{\nabla} f}{|\vec{\nabla} f|}.\frac{\vec{\nabla} g}{|\vec{\nabla} g|}\right]$$



$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$

$\cos\theta = \dfrac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}||\vec{n_2}|}$

Find the angle b/w the surface $x\log z = y^2-1$ and $x^2y = 2-z$ at the point $(1,1,1)$

$f = x\log z - y^2 + 1 = 0$

$\nabla f = \hat{i}(\log z) + \hat{j}(-2y) + \hat{k}\left(\frac{x}{z}\right)$

$\vec{n_1} = \nabla f\Big|_{(1,1,1)} = 0\cdot\hat{i} - 2\hat{j} + \hat{k}$

$g = x^2y - 2 + z = 0$

$\nabla g = \hat{i}(2xy) + \hat{j}(x^2) + \hat{k}(1)$

$\vec{n_2} = \nabla g\Big|_{(1,1,1)} = 2\hat{i} + \hat{j} + \hat{k}$

$$\cos\theta = \left|\frac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}||\vec{n_2}|}\right| = \left|\frac{0-2+1}{\sqrt{0+4+1}\;\sqrt{4+1+1}}\right| = \left|\frac{-1}{\sqrt{30}}\right|$$

$$\cos\theta = \frac{1}{\sqrt{30}} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{\sqrt{30}}\right)$$

$x^2+y^2+z^2 = 9$, $z+3 = x^2+y^2$ at $(-2,1,2)$

$f = x^2+y^2+z^2-9 = 0$

$\nabla f = \hat{i}(2x) + \hat{j}(2y) + \hat{k}(2z)$

$g = z+3-x^2-y^2 = 0$

$\nabla g = \hat{i}(-2x) + \hat{j}(-2y) + \hat{k}(1)$

$\nabla f = \hat{i}(2x) + \hat{j}(2y) + \hat{k}(2z)$ $\nabla g = \hat{i}(-2x) + \hat{j}(-2y) + \hat{k}(1)$

$\vec{n}_1 = \nabla f_{(-2,1,2)} = -4\hat{i} + 2\hat{j} + 4\hat{k}$ $\vec{n}_2 = 4\hat{i} - 2\hat{j} + \hat{k}$

$$\cos\theta = \left| \frac{-16 - \cancel{4} + \cancel{4}}{\sqrt{16+4+16}\ \sqrt{16+4+1}} \right| = \frac{\cancel{16}\ 8}{3\ \cancel{6} \cdot \sqrt{21}}$$

$$\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$$

Directional Derivatives

$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$

$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\hat{b} = \frac{\vec{b}}{|\vec{b}|}$

$f(x,y,z) = K$

$P_0(x_0, y_0, z_0)$

$\frac{\partial f}{\partial x}\Big|_{P_0} =$

$(\nabla f) \cdot \hat{b}$ ∇f

Let $f(x, y, z)$ be a given scalar surface.

Let the given direction be $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$

Then directional derivative of f in the direction of vector $\vec{b}$ is given by:

$\boxed{\nabla f \cdot \hat{b}} = D_{\hat{b}}(f)$

$D_{\vec{b}}(f)$

$$D_{\vec{b}}(f) = \vec{\nabla} f.\,\hat{b} = \vec{\nabla} f.\left(\frac{\vec{b}}{|\vec{b}|}\right)$$

$$= (f_x\hat{\imath} + f_y\hat{\jmath} + f_z\hat{k}).\left(\frac{b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}}{\sqrt{(b_1)^2 + (b_2)^2 + (b_3)^2}}\right)$$

$$= \frac{f_x b_1 + f_y b_2 + f_z b_3}{\sqrt{(b_1)^2 + (b_2)^2 + (b_3)^2}}$$

Problem 1. Find the directional derivative of the scalar function $(x^2y - y^2z - xyz)$at the point $(1, -1, 0)$ in the direction $(\hat{\imath} - \hat{\jmath} + 2\hat{k})$.

$$\nabla f = \hat{i}(2xy - yz) + \hat{j}(x^2 - 2yz - xz) + \hat{k}(-y^2 - xy)$$

$$\nabla f_{(1,-1,0)} = \hat{i}(-2-0) + \hat{j}(1-0-0) + \hat{k}(\cancel{-1+1})$$

$$= -2\hat{i} + \hat{j}$$

$$D_{\vec{b}}(f) = \nabla f \cdot \hat{b} = (-2\hat{i} + \hat{j}) \cdot \left(\frac{\hat{i} - \hat{j} + 2\hat{k}}{\sqrt{1+1+4}}\right) = \frac{-2-1}{\sqrt{6}}$$

$$= \frac{-3}{\sqrt{6}}$$

Discuss

$$D_{\vec{b}}(f) = \nabla f \cdot \hat{b}$$

$$D_{\vec{b}}(f) = |\nabla f| \, |\hat{b}| \cos\theta = \text{Maximum ?}$$

$-1 \le \cos\theta \le 1$

$\boxed{\theta = 0^\circ}$

$D_{\vec{b}}(f) = |\nabla f| \rightarrow$ Maximum directional derivative

The maximum value of directional derivative is $|\nabla f|$ and it occurs when $\theta = 0$ or when $\vec{b}$ has the direction of ∇f

Minimum value of directional derivative is $-|\nabla f|$ and it occurs when $\theta = \pi$ i.e $\hat{b}$ and $|\nabla f|$ have opposite directions.

Problem 3. Find a direction that gives the direction of maximum rate of increase of scalar function f: $(3x^2 + y^2 + 2z^2)$ at (0,1,2). Find the maximum rate too.

$f = 3x^2 + y^2 + 2z^2$ at $(0,1,2)$

$\nabla f = \hat{i}(6x) + \hat{j}(2y) + \hat{k}(4z)$

$\nabla f_{(0,1,2)} = 0\hat{i} + 2\hat{j} + 8\hat{k}$ $\longrightarrow$ Maximum in this direction

Maximum value $= |\nabla f| = \sqrt{0+4+64} = \sqrt{68}$

1. grad(f) ∇f
2. Normal to the surface ∇f
3. eqn of tangent plane $(x-x_0)\frac{\partial f}{\partial x}$
4. Directional Derivative $\rightarrow \nabla f \cdot \vec{b}$
5. Max. value of Directional Derivative $|\nabla f|$
6. Min value $-|\nabla f|$