L2 25 Jan 2021

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Exact first order differential equations :-

Let $f(x,y)$ be a differentiable function defined on some domain $\in \mathbb{R}^2$ (plane) (belongs to)

then $\boxed{df(x,y) = 0}$ defines an exact differential equation

$x\,dy + y\,dx = 0 \Rightarrow \int d(xy) = \int 0$

$\int d(xy)$

$\Rightarrow \boxed{xy = c}$ solution

$$df(x,y) = \frac{\partial f}{\partial x}\cdot dx + \frac{\partial f}{\partial y}\cdot dy = 0 \quad \text{—— } (*)$$

Consider a first order diff. equation

$$\boxed{M(x,y)\,dx + N(x,y)\,dy = 0} \quad \text{—— } (*)$$

$$M(x,y) = \frac{\partial f}{\partial x} \qquad N(x,y) = \frac{\partial f}{\partial y}$$

$$\frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} \qquad \frac{\partial N}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

$$\boxed{\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}}$$

$$\Rightarrow \boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

$$(x^2 - ay)dx = (ax - y^2)dy$$

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

$$(x^2 - ay)\,dx - (ax - y^2)\,dy = 0$$

$$M = (x^2 - ay) \qquad N = -(ax - y^2)$$

$$\frac{\partial M}{\partial y} = 0 - a = -a \qquad \frac{\partial N}{\partial x} = -[a - 0] = -a$$

Exact

Exact

$M(x,y)\,dx + N(x,y)\,dy = 0$ is exact differential equation

if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

Solution

$$\boxed{\int_{y\ \text{Constant}} M\,dx + \int (\text{those terms of } N \text{ free from } x)\,dy = C}$$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}\,dy = 0 = M(x,y)\,dx + N(x,y)\,dy$$

$$\int d(f(x,y)) = 0$$

$$\boxed{f(x,y) = c}$$

$$\boxed{M(x,y) = \frac{\partial f}{\partial x}} \qquad N(x,y) = \frac{\partial f}{\partial y}$$

c

$$\int_{y\ \text{Cost}} \frac{\partial f}{\partial x} = \int_{y\ \text{Cost.}} M(x,y)\,dx$$

$$f(x,y) = \int M(x,y)\,dx + g(y)$$

$$K(x,y) = \int M\,dx$$

$$f(x,y) = K(x,y) + g(y)$$

$$N(x,y) = \frac{\partial f}{\partial y} = \frac{\partial K}{\partial y} + g'(y)$$

$$\Rightarrow \int g'(y) = \int \left(N(x,y) - \frac{\partial K}{\partial y}\right) dy$$

$$g(y) = \int (\text{terms of } N \text{ free from } x)\,dy$$

$$(\boldsymbol{x^2 - 4xy - 2y^2})\boldsymbol{dx} + (\boldsymbol{y^2 - 4xy - 2x^2})\boldsymbol{dy} = \boldsymbol{0}$$

$M\,dx + N\,dy = 0$

$M = x^2 - 4xy - 2y^2$

$N = y^2 - 4xy - 2x^2$

$$\frac{\partial M}{\partial y} = -4x - 4y$$

$$\frac{\partial N}{\partial x} = -4y - 4x$$

$$\Rightarrow \boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$$

$N = y^2 - 4xy - 2x^2$ $\qquad \frac{\partial N}{\partial x} = -4y - 4x$ $\qquad \Rightarrow \boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$

Exact

Solution

$$\boxed{\int_{y\ \text{constant}} M\,dx + \int (\text{terms of } N \text{ free from } x)\,dy = C}$$

$$\int_{y\ \text{constant}} (x^2 - 4xy - 2y^2)\,dx + \int y^2\,dy = C$$

$$\Rightarrow \frac{x^3}{3} - 4y\frac{x^2}{2} - 2y^2 x + \frac{y^3}{3} = C$$

The differential equation: $M(x,y)dx + N(x,y)dy = 0$ will be an exact differential equation if:

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

(A) $\frac{\partial N}{\partial x} + \frac{\partial M}{\partial y} = 0$

(B) $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$

(C) $\frac{\partial M}{\partial x} - \frac{\partial N}{\partial y} = 0$

(D) None of these

What is the relationship between a and b, so that the given differential equ.: $(x^2 + ay)dx + (y^2 + bx)dy = 0$ is exact:

(A) $a = 2b$

(B) $a = b$

(C) $a \neq b$

(D) $a \neq 2b$

$M = x^2 + ay$ $\qquad N = y^2 + bx$

$\frac{\partial M}{\partial y} = a$ $\qquad \frac{\partial N}{\partial x} = b$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow a = b$

For the following differential equations, check whether the equation is exact and obtain its general solution

1. $(1 + e^x)\,dx + y\,dy = 0$.
2. $y\,dx + x(1 + y)\,dy = 0$.
3. $2\cosh x\,dx + \sinh x\,dy = 0$.
4. $\sinh x \cos y\,dx - \cosh x \sin y\,dy = 0$.
5. $(3x^2 y + (y/x))\,dx + (x^3 + \ln x)\,dy = 0$.
6. $(xe^{xy} + 2y)\,dy + ye^{xy}\,dx = 0$.
7. $x\,dy + 2y\,dx = xy\,dy$.
8. $x\,dy - y\,dx = e^y(x^2 + y^2)\,dy$.

For the following differential equations, check whether the equation is exact and obtain its general solution

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4. $\sinh x \cos y\, dx - \cosh x \sin y\, dy = 0$.
5. $(3x^2y + (y/x))\, dx + (x^3 + \ln x)\, dy = 0$.
6. $(xe^{xy} + 2y)\, dy + ye^{xy}\, dx = 0$.
7. $x\, dy + 2y\, dx = xy\, dy$.
8. $x\, dy - y\, dx = e^y(x^2 + y^2)\, dy$.
9. $x\, dx + y\, dy = 2y(x^2 + y^2)\, dy$.
10. $x\, dy - y\, dx + y^2 dx = 0$.
11. $y(1 + 6xy)\, dx + (4y - x)\, dy = 0$.
12. $(2x + e^y)\, dx + xe^y dy = 0$.
13. $(1 + x^2)\, dy + 2xy\, dx = 0$.
14. $2xy\, dx + (x^2 + 1)\, dy = 0$.
15. $(e^{2y} + 1) \cos x\, dx + 2e^{2y} \sin x\, dy = 0$.

... the following differential equations are ...