Random Variable. Intuitively by a *random variable (r.v)* we mean a real number X connected with the outcome of a random experiment E. For example, if E consists of two tosses of a coin, we may consider the random variable which is the number of heads (0, 1 or 2).

Thus to each outcome ω, there corresponds a real number $X(\omega)$. Since the points of the sample space S correspond to outcomes, this means that a real number, which we denote by $X(\omega)$, is defined for each $\omega \in S$. From this standpoint, we define random variable to be a real function on S as follows:

Coin-1	X
T	0
H	1

Coins-2	$X(\omega)=x$	X
TT	0	0
HT, TH	1	1
HH	2	4

" Let S be the sample space associated with a given random experiment. A real-valued function defined on S and taking values in R (− ∞ , ∞) is called a one-dimensional random variable. If the function values are ordered pairs of real numbers (i.e., vectors in two-space) the function is said to be a two- dimensional random variable. More generally, an n-dimensional random variable is simply a function whose domain is S and whose range is a collection of n-tuples of real numbers (vectors in n- space)."

Def. A random variable (r.v.) is a function $X(\omega)$ with domain S and range $(-\infty, \infty)$ such that for every real number a, the event $[\omega : X(\omega) \leq a] \in \mathbf{B}$.

$X(\omega)$ — $R(-\infty, \infty)$

S

Discrete Random Variable. If a random variable takes at most a countable number of values, it is called a discrete random variable. *In other words, a real valued function defined on a discrete sample space is called a discrete random variable.*

Coins, dice, no. of persons,

Probability Mass Function *(and probability distribution of a discrete random variable).*

Suppose X is a one-dimensional discrete random variable taking at most a countably infinite number of values $x_1, x_2, \ldots$ With each possible outcome x_i, we associate a number $p_i = P(X = x_i) = p(x_i)$, called the probability of x_i. The numbers $p(x_i)$; $i = 1, 2, \ldots$ must satisfy the following conditions:

$$(i) \quad p(x_i) \geq 0 \;\; \forall\, i, \quad (ii) \quad \sum_{i=1}^{\infty} p(x_i) = 1$$

This function p is called the probability mass function of the random variable X and the set $\{x_i, p(x_i)\}$ is called the probability distribution (p.d.) of the r.v. X.

X	$p(x_i)$
x_1	$p(x_1)$
x_2	$p(x_2)$
x_3	$p(x_3)$

$p(x_i) \geq 0$

$i = 1, 2, 3$

$p(x_1) + p(x_2) + p(x_3) = 1$

Remarks: 1. The set of values which X takes is called the *spectrum* of the random variable.

2. For discrete random variable, a knowledge of the probability mass function enables us to compute probabilities of arbitrary events. In fact, if E is a set of real numbers, we have

$$P(X \in E) = \sum_{x \in E \cap S} p(x), \text{ where } S \text{ is the sample space.}$$

Coin - 1

$\{0, 1\}$

Coin - 2

$\{0, 1, 2\}$

Ex (1) Probability distribution of discrete random variable X of number of head on tossing 3 - Coins.

X	P
0	1/8
1	3/8
2	3/8
3	1/8

$2^3 = 8$

H H H
H H T
H T H
H T T
T H H
T H T
T T H
T T T

Expected value (mean)

X	$p(x_i)$
x_1	$p(x_1)$
x_2	$p(x_2)$
$\vdots$	
x_n	$p(x_3)$

$$\sum_{i=1}^{n} p(x_i) = 1$$

$$E(x) = \overline{X} = \frac{\sum_{i=1}^{n} x_i\, p(x_i)}{\sum_{i=1}^{n} p(x_i)} = \sum_{i=1}^{n} x_i\, p(x_i)$$

$$\boxed{E(x) = \sum x p}$$

X	f
x_1	f_1
x_2	f_2
$\vdots$	
x_n	f_n

$$\overline{X} = \frac{\sum_{i=1}^{n} x_i f_i}{\sum_{i=1}^{n} f_i}$$

x	p	xp
0	$1/8$	0
1	$3/8$	$3/8$
2	$3/8$	$6/8$
3	$1/8$	$3/8$

$$\Sigma xp = \frac{3}{8} + \frac{6}{8} + \frac{3}{8}$$

$$E(x) = \Sigma xp$$

$$= \frac{3}{8} + \frac{6}{8} + \frac{3}{8}$$

$$= \frac{12}{8} = 1.5$$

Remark

1. $E(x) = \sum x p$

2. $E(x^2) = \sum x^2 p$

3. $E(x^3) = \sum x^3 p$

4. $E(\underline{x-\bar{x}}) = \sum (\underline{x-\bar{x}}) \underline{p}$

Variance of random variable (x)

X	$p(x)$
x_1	$p(x_1)$
x_2	$p(x_2)$
$\vdots$	$\vdots$
x_n	$p(x_n)$

$\sum p(x_i) = 1$

$$\sigma^2 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2 \, p(x_i)}{\sum_{i=1}^{n} p(x_i)}$$

X	f
x_1	f_1
x_2	f_2
$\vdots$	$\vdots$
x_n	f_n

$$\sigma^2 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2 f_i}{\sum_{i=1}^{n} f_i}$$

$$\text{Variance} = \sigma^2 = \sum_{i=1}^{n} (x_i - \bar{x})^2 \, p(x_i) \qquad \left(E(x) = \bar{x} \right)$$

$$= E\left((x_i - \bar{x})^2 \right)$$

$$= E\left(x_i - E(x)\right)^2$$

$$= \sum_{i=1}^{n} (x_i - E(x))^2 \, p(x_i)$$

$$\sigma^2 = \frac{\sum x^2 p}{\sum p} - \left(\frac{\sum x p}{\sum p}\right)^2$$

$$\sigma^2 = \frac{\sum x^2 f}{\sum f} - \left(\frac{\sum x f}{\sum f}\right)^2$$

$$\sum p = 1$$

$$\sigma^2 = \sum x^2 p - \left(\sum x p\right)^2$$

$$\boxed{\sigma^2 = E(x^2) - \left(E(x)\right)^2}$$

$$\text{Variance}(x) = \sigma^2 = E(x-\bar{x})^2 = \sum (x-\bar{x})^2 p(x)$$

$$= E(x - E(x))^2 = \sum (x - E(x))^2 p(x)$$

$$\text{Variance} = E(x^2) - (E(x))^2$$

$$= \sum x^2 p - \left(\sum x p\right)^2$$

x	p	x^2	xp	x^2p
0	1/8	0	1/8	0
1	3/8	1	3/8	3/8
2	3/8	4	6/8	12/8
3	1/8	9	3/8	9/8
			$\Sigma xp = \frac{12}{8}$	24/8

$$V(x) = \frac{24}{8} - \left(\frac{12}{8}\right)^2$$

$$= 3 - (1.5)^2$$

$$= 3 - 2.25$$

$$= \underline{\underline{0.75}}$$

$$V(x) = \sigma^2$$

$$\sigma = \sqrt{V(x)}$$

$$= \sqrt{0.75}$$