L28 26 March 2021

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Solve the eqn. $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ with boundary Conditions $u(x,0) = 3\sin n\pi x$

$u(0,t) = 0$ and $u(1,t) = 0$

where $0 < x < 1$, $t > 0$.

$$\boxed{u(x,t) = (A\cos px + B\sin px) \cdot e^{-p^2 t}}$$

$u(0,t) = 0$

$$0 = (A\cos 0 + B\sin 0) \cdot e^{-p^2 t}$$

$$0 = A\, e^{-p^2 t} \Rightarrow \boxed{A = 0}$$

$$u(x,t) = (B\sin px)\, e^{-p^2 t}$$

$u(1,t) = 0$

$\downarrow \; \downarrow$

$x \quad t$

$$0 = B\sin p \cdot e^{-p^2 t} = 0$$

$B \neq 0$

$$\Rightarrow \sin p = 0$$

$$\Rightarrow \boxed{p = n\pi}$$

$$\boxed{u(x,t) = B\sin n\pi x \cdot e^{-n^2\pi^2 t}}$$

The General sol

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-n^2\pi^2 t} \sin n\pi x$$

$u(x,0) = 3\sin n\pi x$

$$u(x,0) = \sum_{n=1}^{\infty} B_n e^{-n^2\pi^2(0)} \sin n\pi x$$

$$3\sin n\pi x = \sum_{n=1}^{\infty} B_n \sin n\pi x$$

$$\boxed{B_n = 3}$$

Sol

$$\boxed{u(x,t) = 3\sum_{n=1}^{\infty} \sin n\pi x\, e^{-n^2\pi^2 t}}$$

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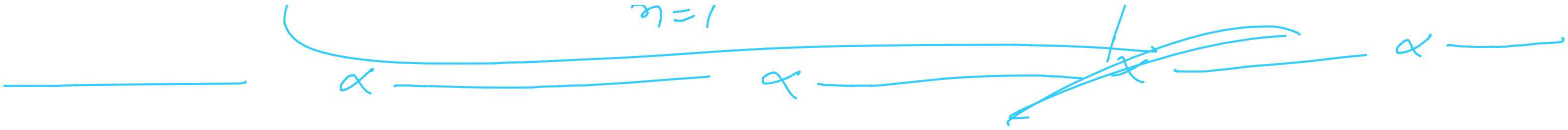
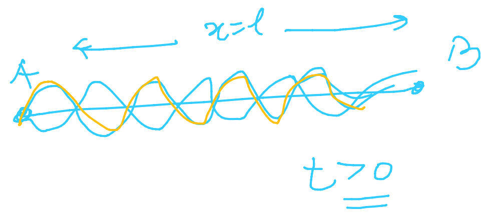


Solve $\dfrac{\partial^2 u}{\partial t^2} = C^2 \dfrac{\partial^2 u}{\partial x^2}$ (Wave Equation)



$u(x,t) = X(x)\,T(t)$

$\dfrac{\partial u}{\partial x} = X'T \qquad \dfrac{\partial u}{\partial t} = XT'$

$\dfrac{\partial^2 u}{\partial x^2} = X''T \qquad \dfrac{\partial^2 u}{\partial t^2} = XT''$

Given eqn becomes

$XT'' = c^2 X'' T$

$$\Rightarrow \boxed{\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = k} \text{ (say)}$$

Depending upon the value of k, we have different case

① If $k = 0$ then

$\dfrac{X''}{X} = 0 \Rightarrow X'' = 0 \Rightarrow X' = a \Rightarrow X = ax+b$

$\dfrac{1}{c^2}\dfrac{T''}{T} = 0 \Rightarrow T'' = 0 \Rightarrow T' = c \Rightarrow T = cx+d$

$$\boxed{u(x,t) = XT = (ax+b)(cx+d)}$$

② If $k = +ve$

$k = p^2$

$\dfrac{X''}{X} = p^2 \Rightarrow X'' - p^2 X = 0$

A.E $D^2 - p^2 = 0 \Rightarrow D^2 = p^2 \Rightarrow D = \pm p$

$$\boxed{X = c_1 e^{px} + c_2 e^{-px}}$$

$\dfrac{1}{c^2}\dfrac{T''}{T} = p^2 \Rightarrow T'' - p^2 c^2 T = 0$

A.E $D^2 - p^2 c^2 = 0 \qquad D^2 = p^2 c^2 \Rightarrow D = \pm pc$

A.E $\quad D^2 - p^2c^2 = 0 \qquad D = pc \Rightarrow D = \pm pc$

$$\boxed{T = c_3 e^{pct} + c_4 e^{-pct}}$$

$$u(x,t) = \left(c_1 e^{px} + c_2 e^{-px}\right)\left(c_3 e^{pct} + c_4 e^{-pct}\right)$$

(3) If $k = -ve$

$k = -p^2$

$$\frac{X''}{X} = -p^2 \Rightarrow X'' + p^2 X = 0$$

A.E $\quad D^2 + p^2 = 0 \Rightarrow D^2 = -p^2 \Rightarrow D = \pm ip$

$$\boxed{X = c_1 \cos px + c_2 \sin px}$$

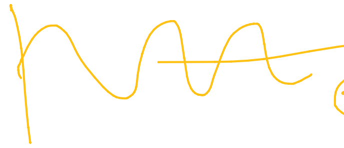
$$\frac{1}{c^2}\frac{T''}{T} = -p^2 \Rightarrow T'' + p^2c^2 T = 0$$

A.E $\quad D^2 + p^2c^2 = 0 \Rightarrow D^2 = -p^2c^2 \Rightarrow D = \pm ipc$

$$\boxed{T = c_3 \cos pct + c_4 \sin pct}$$

$$\boxed{u(x,t) = \left(c_1 \cos px + c_2 \sin px\right)\left(c_3 \cos pct + c_4 \sin pct\right)}$$

<u>**Note:**</u> <u>**Wave Equation**</u>



<u>Equation:</u> $\frac{\partial^2 u}{\partial t^2} = C^2 \frac{\partial^2 u}{\partial x^2}$

<u>Nature:</u> Hyperbolic

<u>Solution:</u> **1.** $u(x,t) = (ax+b)(ct+d)$ K=0

2. $u(x,t) = (ae^{px} + be^{-px})(ce^{Cpt} + de^{-Cpt})$ K= +ve

or

$u(x,t) = (a\cosh px + b\sinh px)(c\cosh Cpt + d\sinh Cpt)$

K=-ve

3. $u(x,t) = XT = (a\cos px + b\sin px)\,(c\cos Cpt + d\sin Cpt)$ **(Most suitable one)**