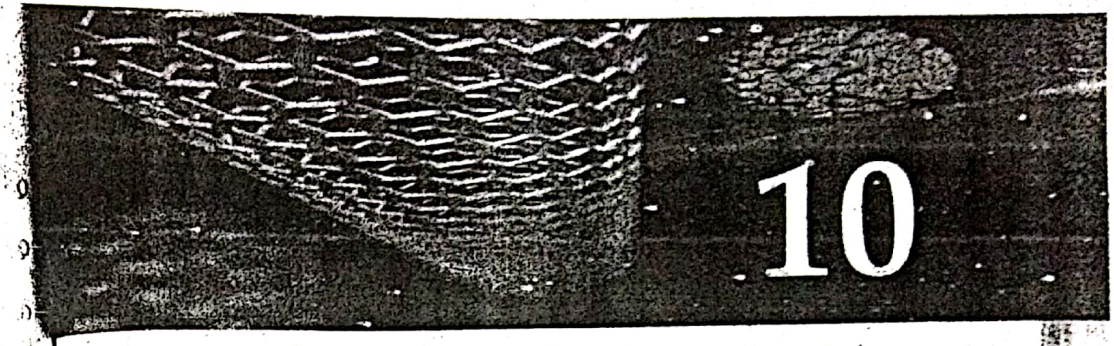


10

Electromagnetism

10.1 INTRODUCTION

Ordinary matter is made up of atoms which have positively charged nuclei and negatively charged electrons surrounding them. Charge is quantized in terms of the electronic charge $-e$. One electronic charge e is equal to 1.602×10^{-19} Coulombs. One Coulomb of charge is the charge which would flow through a 220 W light bulb (220 Vac) in one second. Do you know two charges of one Coulomb each separated by one meter would repel each other with a force of about a million tons! The separation of charges produces electric field, whereas the motion of charges generates current and hence the magnetic field. When these fields are time varying they are coupled with each other through the Maxwell's equations. With the help of the Maxwell's equations, we can derive wave equation, based on which the propagation of electromagnetic waves can be investigated in different media.

10.2 CHARGE DENSITY

If a charge is distributed continuously in a medium, it can be expressed in terms of a physical quantity known as charge density. There are three types of charge densities, namely linear charge density, surface charge density and volume charge density.

10.2.1 Linear Charge Density (λ)

If a charge is distributed continuously on a linear conductor, the charge on its unit length is called the linear charge density. It is generally represented by λ and is measured in the units of C/m. If l be the total length of a conductor and λ the linear charge density, then the total charge on the conductor will be

$$q = \int_0^l \lambda \, dl$$

For a uniform charge distribution, λ is constant, and is given as $\lambda = q/l$. Here q is the total charge, given by $q = \lambda l$.

10.2.2 Surface Charge Density (σ)

If the charge is distributed over a surface, the charge on the unit area of the conductor is called the *surface charge density* (σ) and its unit is C/m^2.

Scanned by CamScanner

If the surface charge density at a point of the conductor is σ, then the charge contained in a small element of area dS will be $\sigma\, dS$. Therefore, the total charge on the surface of the conductor,

$$q = \int_S \sigma dS$$

If the charge distribution is uniform, then the value of σ will be constant, and is given by

$$\sigma = q / S$$

where S is total area of the surface and q is the total charge, given by $q = \sigma S$.

10.2.3 Volume Charge Density (ρ)

If the charge is distributed in the volume of a conductor, the charge contained in a unit volume of the conductor is called the *volume charge density* (ρ) and its unit is C/m^3.

If the volume charge density at a point of the volume of a conductor is ρ, then the charge contained in a small element of the volume dV will be $\rho\, dV$. Therefore, the total charge contained in the conductor,

$$q = \int_V \rho\, dV$$

For the uniform distribution of the charge, ρ will be constant, and is given by

$$\rho = q / V$$

$$\text{or} \quad q = \rho V$$

where q is the total charge and V is the total volume, which is equal to $\frac{4}{3}\pi r^3$ for a spherical shape of the conductor of radius r.

10.3 DEL OPERATOR

The del operator is the differential operator, which is represented by $\vec{\nabla}$ and is given by $\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$. It is not a vector in itself, but when it operates on a scalar function it provides the resultant as a vector. For example, when $\vec{\nabla}$ is operated on a scalar function $F(x, y, z)$, we get

$$\vec{\nabla}F = \hat{i}\frac{\partial F}{\partial x} + \hat{j}\frac{\partial F}{\partial y} + \hat{k}\frac{\partial F}{\partial z}$$

$\vec{\nabla}F$ does not mean a multiplication of $\vec{\nabla}$ with F rather it is an instruction to differentiate. Here we should say that $\vec{\nabla}$ is a vector operator that acts upon F.

The del operator $\vec{\nabla}$ can act in different ways. For example, when it acts on a scalar function F, the resultant $\vec{\nabla}F$ is called the gradient of a scalar function F. When it acts on a vector function $\vec{A}$ via the dot product, the resultant is $\vec{\nabla}\cdot\vec{A}$, which is called the divergence of a vector $\vec{A}$. When it acts on a vector function $\vec{A}$ via the cross product, the resultant is $\vec{\nabla}\times\vec{A}$, which is called the curl of a vector $\vec{A}$. Finally, the Laplacian of a scalar function F is written as $\nabla^2 F$ together with $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

The del operator can be written in spherical polar coordinate system (Fig. 10.1a) as

$$\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$$

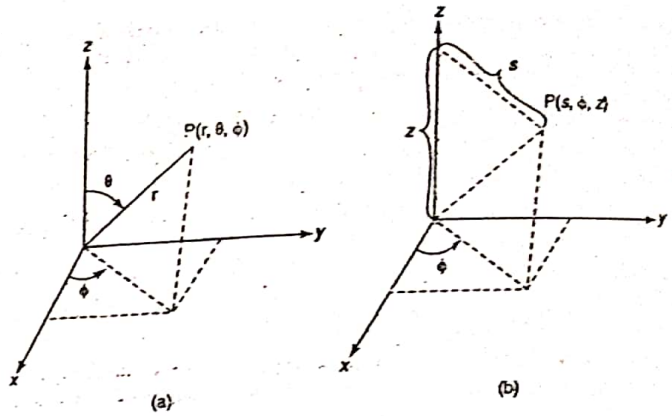


Fig. 10.1

In cylindrical coordinate system (Fig. 10.1b), it is given by

$$\vec{\nabla} = \hat{s}\frac{\partial}{\partial s} + \hat{\phi}\frac{1}{s}\frac{\partial}{\partial \phi} + \hat{z}\frac{\partial}{\partial z}$$

10.4 GRADIENT

If we think of the derivative of a function of one variable we notice that it simply tells us how fast the function varies if we move a small distance. It means the gradient is the rate of change of a quantity with distance. For example, temperature gradient in a metal bar is the rate of change of temperature along the bar. However, for a function of three variables the situation is more complicated, as it will depend on what direction we choose to move. For a function $F(x, y, z)$ of three variables, we obtain from a theorem on partial derivative

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz$$

Here dF is a measure of changes in F that occurs when we alter all three variables by small amounts dx, dy and dz. The above expression for dF can be written in terms of a dot product of vectors as

$$dF = \vec{\nabla}F \cdot d\vec{l}$$

where $\vec{\nabla}F = \hat{i}\frac{\partial F}{\partial x} + \hat{j}\frac{\partial F}{\partial y} + \hat{k}\frac{\partial F}{\partial z}$ is nothing but gradient of F. Clearly the gradient is a vector quantity, i.e. it has both magnitude and direction. The meaning of gradient becomes clearer when we write

$$dF = \vec{\nabla}F \cdot d\vec{l}$$

$$\text{or} \quad dF = \left|\vec{\nabla}F\right|\left|d\vec{l}\right|\cos\alpha$$

Scanned by CamScanner

where α is the angle between $\vec{\nabla}F$ and $d\vec{l}$. Now we fix the distance dl, i.e. magnitude $|d\vec{l}|$, and look around in various directions for the maximum change in F. Clearly the maximum change in F takes place in the direction $\alpha = 0$. It means dF is largest when we move in the direction of $\vec{\nabla}F$. This can also be said that the gradient $\vec{\nabla}F$ points in the direction of maximum increase of the function F. Hence, the gradient of a scalar field F is a vector quantity which represents both the magnitude and the direction of the maximum space rate of increase of F.

The gradient of F in Cylindrical coordinate system is written as

$$\vec{\nabla}F = \frac{\partial F}{\partial s}\hat{s} + \frac{1}{s}\frac{\partial F}{\partial \phi}\hat{\phi} + \frac{\partial F}{\partial z}\hat{z}$$

In spherical polar coordinate system, the gradient of F is written as

$$\vec{\nabla}F = \frac{\partial F}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial F}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial F}{\partial \phi}\hat{\phi}$$

10.5 DIVERGENCE

As mentioned earlier, the divergence of a vector field $\vec{A}$ is represented by $\vec{\nabla}\cdot\vec{A}$. So it is given by

$$\vec{\nabla}\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Clearly the divergence of a vector field is a scalar. Also, the divergence of a scalar cannot be obtained, as the dot product of $\vec{\nabla}$ with a scalar is not possible.

In order to make clear the meaning of the divergence, we consider the net flux $\oint \vec{A}\cdot d\vec{S}$ of a vector field $\vec{A}$ from a closed surface S. The divergence of $\vec{A}$ is defined as the net outward flux per unit volume over a closed surface. Actually the divergence of $\vec{A}$ at a given point is a measure of how much the vector $\vec{A}$ spreads out, i.e. diverges, from that point. Fig. 10.2a shows that the divergence of a vector field at point O is positive, as the vector spreads out. However, in Fig. 10.2b, the vector converges and hence the divergence at O is negative. In Fig. 10.2c, we can notice that the divergence of a vector field is zero, as the magnitude of vectors remains the same.

Now the difference between vector field and vector would have been clear to you. We call $\vec{A}$ as vector field because its values at different points are different, which are vectors with different magnitudes. For example, in Fig. 10.2d the magnitudes of the vectors get increased as we move towards right. Hence, the divergence of such a vector field is not zero but positive.

The divergence of a vector $\vec{A}$ in cylindrical coordinate system can be written as

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{s}\frac{\partial(sA_s)}{\partial s} + \frac{1}{s}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

In spherical polar coordinate system, it is written as

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

Finally, we mention that the vector field $\vec{A}$ is said to be solenoidal or divergenceless if $\vec{\nabla}\cdot\vec{A} = 0$.

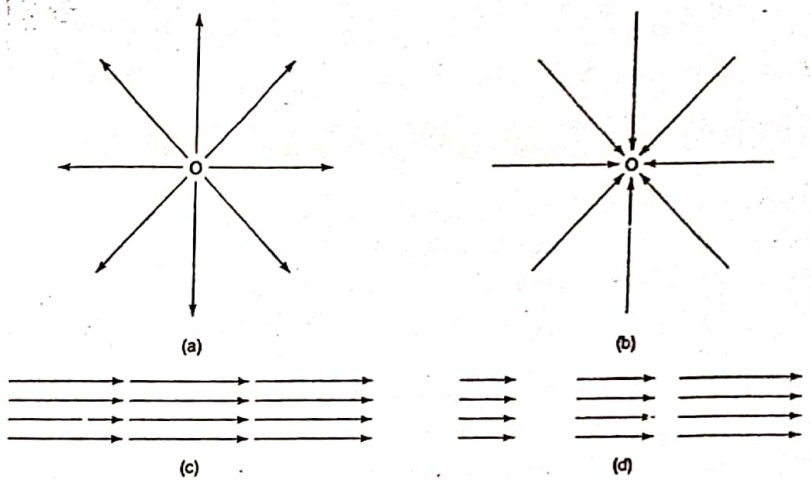


Fig. 10.2

10.6 CURL

As mentioned earlier, the curl of vector field $\vec{A}$ is represented by $\vec{\nabla}\times\vec{A}$. So it is given by

$$\vec{\nabla}\times\vec{A} = \hat{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{j}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

Clearly the curl is a vector quantity, i.e. it has both direction and magnitude. Also, it is evident that we cannot have the curl of a scalar as the cross product of $\vec{\nabla}$ with a scalar is meaningless.

In order to make clear the meaning of curl, we consider the circulation of a vector field $\vec{A}$ around a closed path, i.e. $\oint \vec{A}\cdot d\vec{l}$. It is evident that the curl of $\vec{A}$ is a rotational vector. Its magnitude would be the maximum circulation of $\vec{A}$ per unit area. Its direction is the normal direction of the area when the area is oriented so as to make the circulation maximum. Actually curl of $\vec{A}$ at some point O is a measure of how much the vector $\vec{A}$ curls around the point O. So, you may notice that the curl of vector field $\vec{A}$ in Fig. 10.2 is zero. However, in Fig. 10.3, the curl is finite and is pointing in the X-direction as per right hand rule.

The curl of a vector $\vec{A}$ in cylindrical coordinate system can be written as

$$\vec{\nabla}\times\vec{A} = \left(\frac{1}{s}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{s} + \left(\frac{\partial A_s}{\partial z} - \frac{\partial A_z}{\partial s}\right)\hat{\phi} + \frac{1}{s}\left(\frac{\partial}{\partial s}(sA_\phi) - \frac{\partial A_s}{\partial \phi}\right)\hat{z}$$

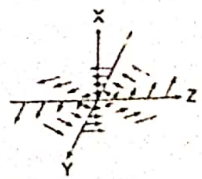


Fig. 10.3

In spherical polar coordinate system, it is written as

Scanned by CamScanner

$$\vec{\nabla}\times\vec{A}=\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\phi)-\frac{\partial A_\theta}{\partial\phi}\right]\hat{r}+\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi}-\frac{\partial}{\partial r}(rA_\phi)\right)\hat{\theta}+\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta)-\frac{\partial A_r}{\partial\theta}\right]\hat{\phi}$$

Finally, we mention that the vector field $\vec{A}$ is said to be irrotational or potential if $\vec{\nabla}\times\vec{A}=0$.

10.7 FUNDAMENTAL THEOREM OF CALCULUS

If $g(x)$ be a function of one variable, then dg would represent infinitesimal change in $g(x)$ when we move from x to $x+dx$. This change in $g(x)$ is given by $dg=\left(\frac{dg}{dx}\right)dx$. If we move from point a_1 to point a_2, then the total change in $g(x)$ can be obtained by using the fundamental theorem of calculus. This theorem states that

$$\int_{a_1}^{a_2}\left(\frac{dg}{dx}\right)dx=g(a_2)-g(a_1)$$

It means the total change in the function can be obtained by simply subtracting the values of the function at the points a_2 and a_1. For the movement from point a_1 to point a_2, these points can be treated as the end points. In view of this, the fundamental theorem says that the integral of a derivative $\left(\frac{dg}{dx}\right)$ over an interval $a_1\rightarrow a_2$ is given by the value of the function at the end points. The end points represent the boundaries.

Since we have three types of derivatives, namely gradient, divergence and curl in vector calculus, there are three fundamental theorems for these derivatives. The fundamental theorem for divergence is also known as Gauss's theorem or Green's theorem. Similarly, the fundamental theorem for curl is also known as Stokes' theorem.

10.8 FUNDAMENTAL THEOREM FOR GRADIENT

We know that the gradient is defined only for a scalar function. So we consider a scalar function of three variables $F(x, y, z)$. As discussed earlier, the change in this function is defined as $dF=\vec{\nabla}F\cdot d\vec{l}$. If we move from point a_1 to point a_2 then the total change in F can be calculated using fundamental theorem of calculus. Therefore, the total change in F in moving from a_1 to a_2 is given by

$$\int_{a_1}^{a_2}\vec{\nabla}F\cdot d\vec{l}=F(a_2)-F(a_1)$$

This is called the fundamental theorem for gradient, according to which the integral of a derivative over end points a_1 and a_2 is given by the value of the function at the boundaries, i.e. points a_1 and a_2. Here it can be noticed that the integral is line integral, the derivative is the gradient and the boundaries are the points a_1 and a_2. Moreover, it can be seen that the integral or the total change in function F is independent of path taken from a_1 to a_2. Also, if we take both the end points same, i.e. we evaluate the close integral, then the total change in the function F comes out to be zero (as the beginning and end points are identical).

10.9 GAUSS'S OR GREEN'S THEOREM

According to this theorem,

$$\int_V(\vec{\nabla}\cdot\vec{F})dV=\oint_S\vec{F}\cdot d\vec{S}$$

Here dV is the volume element and $d\vec{S}$ is the surface element. In the same manner as we stated fundamental theorem for gradient, the Green's theorem states that the integral of a derivative (here the divergence) over a region (here the volume) is equal to the value of the function at the boundary (here the surface). Since the boundary of a volume is always a closed surface, the R.H.S. is the integral over closed surface.

Evidently this theorem converts the volume integral into the surface integral. Therefore, this theorem is very useful in the situations where it is difficult to calculate the volume integral.

10.10 STOKES' THEOREM

Stokes' theorem states that the integral of the curl of a vector function over a patch of surface is equal to the value of the function at the perimeter of the patch. So here the derivative is the curl, region is the surface and the boundary is the perimeter of the patch of the surface. Therefore,

$$\int_S(\vec{\nabla}\times\vec{F})\cdot d\vec{S}=\oint_C\vec{F}\cdot d\vec{l}$$

Clearly, this theorem converts the surface integral into the line integral. Here the L.H.S. is the surface integral whereas the R.H.S. is the closed line integral. So a point of confusion is that which way we should go around, i.e. clockwise or anticlockwise when we integrate the line integral. Moreover, we should know about the direction of the surface element $d\vec{S}$. For example, for a closed surface $d\vec{S}$ points outwards normal but for an open surface which way is out? In order to overcome this confusion, we apply right hand rule. So if our fingers point in the direction of line integral, then the thumb gives the direction of $d\vec{S}$.

Based on the statement of Stokes' theorem, we can make some more observations. For example, $\int_S(\vec{\nabla}\times\vec{F})\cdot d\vec{S}$ does not depend on the shape of the surface rather it depends on the boundary line. Also for a closed surface, $\oint(\vec{\nabla}\times\vec{F})\cdot d\vec{S}=0$ as the boundary line shrinks down to a point.

10.11 ELECTRIC FIELD AND ELECTRIC POTENTIAL

Electric field is the region around a charge in which another charge experiences a force. The electrostatic field $\vec{E}$ is a special kind of field whose curl is always zero, i.e. $\vec{\nabla}\times\vec{E}=0$. Since any vector whose curl is zero is equal to the gradient of some scalar quantity, we make use of this property to introduce the scalar quantity as the electric potential V. In order to find this, we use the Stokes' theorem $\int(\vec{\nabla}\times\vec{E})\cdot d\vec{S}=\oint\vec{E}\cdot d\vec{l}$. This gives

$$\oint\vec{E}\cdot d\vec{l}=0 \qquad (i)$$

It means the line integral of $\vec{E}$ from point a_1 to point a_2 will be the same for all the paths between these points. Hence, the line integral of $\vec{E}$ is independent of path. Since changing the path would not alter the value of integral, we can define a function, say V, such that

$$V=-\int_a^r\vec{E}\cdot d\vec{l} \qquad (ii)$$

The differential form of the above equation is written as

$$\vec{E}=-\vec{\nabla}V \qquad (iii)$$

Scanned by CamScanner

e V is called the electric potential. Actually all the potentials are relative and there is no absolute zero ential. However, convention is that the potential is zero at infinite distance from the charge. In view of the lower limit a in Eq. (ii) is called as a standard reference point where V is zero. The upper limit is ing but the point where V is to be calculated. So V depends only on the point $\vec{r}$.

1.1 Superposition Principle

rding to the original principle of superposition of electrodynamics, the total force $\vec{F}$ on a charge q (test ge) is equal to the vector sum of the forces due to all the source charges (considering them individually), ans

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \ldots \quad \text{(iv)}$$

$\vec{F} = q\vec{E}$, from Eq.(iv) we find the following for the electric field $\vec{E}$

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \ldots \quad \text{(v)}$$

write a for the common reference point, the above equation can be written as

$$-\int_a^r \vec{E}\cdot d\vec{l} = -\int_a^r \vec{E}_1\cdot d\vec{l} - \int_a^r \vec{E}_2\cdot d\vec{l} - \int_a^r \vec{E}_3\cdot d\vec{l} - \ldots \quad \text{(vi)}$$

$$\vec{\nabla}V = \vec{\nabla}V_1 + \vec{\nabla}V_2 + \vec{\nabla}V_3 + \ldots \quad \text{(vii)}$$

it is clear from Eq. (vii) that

$$V = V_1 + V_2 + V_3 + \ldots \quad \text{(viii)}$$

above equation reveals that the potential V at a given point $\vec{r}$ is the sum of the potentials due to all the ges. It means the electric potential also satisfies the principle of superposition and the sum is simply an ary sum. However, from Eq.(v) it is clear that in case of the electric field $\vec{E}$ this sum is the vector sum.

10.12 POISSON'S AND LAPLACE'S EQUATIONS

deriving these equations, we start with the following Gauss's law for a linear medium

$$\vec{\nabla}\cdot\vec{D} = \rho$$

$$\text{or} \quad \vec{\nabla}\cdot\varepsilon\vec{E} = \rho \quad \text{(i)}$$

e $\vec{E} = -\vec{\nabla}V$, the above equation for a homogeneous medium (where ε is constant) can be written as

$$\nabla^2 V = -\rho/\varepsilon \quad \text{(ii)}$$

s equation is called as Poisson's equation. For a charge free region, i.e. where $\rho = 0$, the Poisson's ation takes the form

$$\nabla^2 V = 0. \quad \text{(iii)}$$

is equation is known as Laplace's equation. This equation is much useful in solving electrostatic problems ere a set of conductors are maintained at different potentials; for example, capacitors and vacuum tube ides.

ing the expressions for Laplacian operator ∇^2 in Cartesian, cylindrical and spherical coordinate systems, e can write Laplace's Eq. (iii) in these coordinate systems, respectively, as

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

$$\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial V}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0$$

10.13 CAPACITOR

In order to construct a capacitor, we need at least two conductors that carry equal but opposite charges. If it is so, then flux lines leaving one conductor terminate at the surface of the other conductor. These conductors are also known as the plates of the capacitor, which may be separated by a dielectric or simply by free space. The ratio of the magnitude of the charge on one of the plates (say q) and the potential difference (say V) between the plates is known as the capacitance of the capacitor (represented by C). So

$$C = \frac{q}{V}$$

Some important capacitor configurations with two conductors are parallel plate capacitor, coaxial capacitor and spherical capacitor. These are discussed below.

10.13.1 Parallel Plate Capacitor

This type of capacitor has two plane conductors, which are parallel to each other. If the area of each plate be S and they are separated by a distance d, then the capacitance of this capacitor is given as

$$C = \varepsilon\frac{S}{d}$$

Here ε is the permittivity of the dielectric, which is filled between the plates. If a potential difference V is applied between the plates then the energy stored in the capacitor would be

$$U = \frac{1}{2}CV^2 = \frac{q^2}{2C}$$

10.13.2 Coaxial Capacitor

Coaxial capacitor is simply a coaxial cable. This is also referred to as coaxial cylindrical capacitor. If L be the length of the coaxial conductors, and the radii of inner and outer conductors be r_{in} and r_{out}, then the capacitance of the capacitor is obtained as

$$C = \frac{2\pi\varepsilon L}{\ln\frac{r_{out}}{r_{in}}}$$

Here also ε is the permittivity of the dielectric filling the space between the two conductors.

Scanned by CamScanner

10.13.3 Spherical Capacitor

As name suggests, in this case the two conductors are in the form of spheres and these are concentric. Let the radius of inner sphere be r_{in} and of the outer sphere be r_{out}. Also these spheres are separated by a dielectric medium of permittivity ε. Then the capacitance of this type of capacitor is obtained as

$$C = \frac{4\pi\varepsilon}{\frac{1}{r_{in}} - \frac{1}{r_{out}}}$$

10.14 GAUSS'S THEOREM

The net outward electric flux through any closed surface drawn in an electric field is equal to $\frac{1}{\varepsilon_0}$ times the total charge enclosed within the surface. It is expressed as

$$\phi = \int_S \vec{E}\cdot d\vec{S} = \frac{1}{\varepsilon_0}\sum q = \frac{1}{\varepsilon_0}Q$$

where Q is the sum of all the charges present within the surface. The charge outside of the surface is not counted and therefore the flux ϕ due to the charge q sitting outside the surface

$$\phi = \int_S \vec{E}\cdot d\vec{S} = 0$$

10.14.1 Differential Form of Gauss's Theorem

When a charge is distributed over a volume such that ρ is the volume charge density, then the charge enclosed by the surface enclosing the volume is given by

$$q = \int_V \rho \, dV \qquad \text{(i)}$$

Substituting this expression of q in $\int_S \vec{E}\cdot d\vec{S} = \frac{q}{\varepsilon_0}$, we get

$$\varepsilon_0 \oint_S \vec{E}\cdot d\vec{S} = \int_V \rho \, dV \qquad \text{(ii)}$$

According to Gauss's divergence theorem, we can convert the surface integral into the volume integral as

$$\oint_S \vec{E}\cdot d\vec{S} = \int_V \operatorname{div}\vec{E}\, dV \qquad \text{(iii)}$$

Therefore, from Eqs. (ii) and (iii), we have

$$\varepsilon_0 \int_V \operatorname{div} E\, dV = \int_V \rho\, dV$$

Since the above equality is true for every volume, so the integrands of left and right sides should be equal, i.e.

$$\varepsilon_0 \operatorname{div}\vec{E} = \rho$$

$$\vec{\nabla}\cdot(\varepsilon_0\vec{E}) = \rho$$

$$\vec{\nabla}\cdot\vec{D} = \rho$$

where, $\vec{D}$ is the electric flux density, given by $\vec{D} = \varepsilon_0\vec{E}$. Eq. (v) is the differential form of Gauss's theorem.

10.15 MAGNETIC FLUX DENSITY ($\vec{B}$)

When a magnetic material is placed in an external magnetic field, it gets magnetised. The magnetism thus produced in the material is known as induced magnetism and this phenomenon is referred to as magnetic induction. The magnetic lines of force inside such magnetised materials are called magnetic lines of induction. The number of magnetic lines of induction crossing unit area at right angles to the flux is called the magnetic flux density $\vec{B}$. Its unit is the Tesla which is equal to 1 Wb/m².

10.16 MAGNETIC FIELD STRENGTH ($\vec{H}$)

As mentioned earlier, a magnetic material becomes magnetised when placed in a magnetic field. The actual magnetic field inside the material is the sum of external field and the field due to its magnetisation.

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M} \qquad \text{or} \qquad \vec{B} = \mu_0(\vec{H} + \vec{M})$$

Magnetic field strength at a point in a magnetic field is the magnitude of the force experienced by a unit pole situated at that point. The SI unit, corresponding to force of 1 Newton, is the A/m. The CGS unit, corresponding to a force of 1 dyne is the Oersted which is equal to 79.6 A/m.

10.17 AMPERE'S CIRCUITAL LAW

Ampere's circuital law in magnetostatics is analogous to the Gauss's law in electrostatics. This law states that the line integral of magnetic field $\vec{B}$ around any closed loop is equal to μ_0 times the net current I flowing through the area enclosed by the loop i.e.

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I$$

Here, μ_0 is the permeability of the free space.

Proof: Consider a long straight conductor carrying a current I. By Biot-Savart law, the magnitude of the magnetic field at a point O, at a distance r from the conductor, is given by

$$B = \frac{\mu_0}{4\pi}\frac{2I}{r} \qquad \text{(i)}$$

Let us draw a circle with a radius r taking C as centre around the current carrying conductor Fig. 10.4. $\vec{B}$ will be the same in magnitude at all points on this circle. Again we consider a circle element of length dl at the point O. From the figure it is clear that $d\vec{l}$ and $\vec{B}$ are in the same direction.

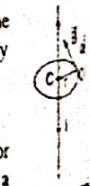


Fig. 10.4

Scanned by CamScanner

$$\therefore \oint \vec{B}\cdot d\vec{l} = \oint B\,dl\cos\theta$$
$$= B\oint dl \qquad [\because \theta = 0°]$$
$$= \frac{\mu_0}{4\pi}\frac{2I}{r}2\pi r = \mu_0 I$$
$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I$$

is the required Ampere's circuital law.

10.18 ELECTROSTATIC BOUNDARY CONDITIONS

nsider the situation where the electric field exists in a region, which has two different media with mittivities as ε_1 (in region 1) and ε_2 (in region 2). Then the conditions, which should be satisfied by the d at the interface separating the two media or at the common boundary of these media, are called boundary ditions. It is seen that when we cross a boundary surface charge σ, the electric field does not remain tinuous and it always undergoes a discontinuity.

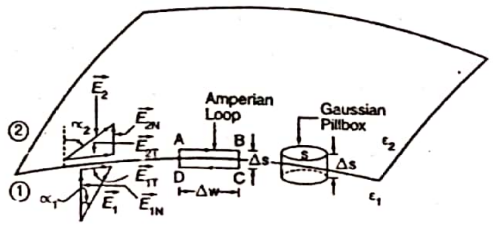


Fig. 10.5

can calculate the amount by which the electric field $\vec{E}$ changes at such a boundary, shown in Fig. 10.5. is the field in the region 1 and $\vec{E}_2$ is the field in the region 2. These fields can be decomposed into two mponents, out of which one is tangential to the boundary (say $\vec{E}_T$) and the other is perpendicular to the ndary (say $\vec{E}_N$). So we can write

$$\vec{E}_1 = \vec{E}_{1N} + \vec{E}_{1T} \quad \text{and} \quad \vec{E}_2 = \vec{E}_{2N} + \vec{E}_{2T}$$

w we can apply Ampere's and Gauss's laws for calculating the amount of discontinuity. For example, pere's law $\oint \vec{E}\cdot d\vec{l} = 0$ for the closed path ABCDA, whose length is Δw and the width is Δs, follows

$$0 = E_{2T}\Delta w - E_{2N}\frac{\Delta s}{2} - E_{1N}\frac{\Delta s}{2} - E_{1T}\Delta w + E_{1N}\frac{\Delta s}{2} + E_{2N}\frac{\Delta s}{2} \qquad \text{(i)}$$

the limit $\Delta s \to 0$, i.e. when the width of the loop is small so that we are well close to the boundary, the ve equation says

$$E_{2T}\Delta w = E_{1T}\Delta w$$

$$\text{or} \quad E_{2T} = E_{1T} \qquad \text{(ii)}$$

ce $\vec{D} = \varepsilon\vec{E}$, the above equation reads for the displacement vector $\vec{D}_T$

$$\frac{D_{2T}}{\varepsilon_2} = \frac{D_{1T}}{\varepsilon_1} \qquad \text{(iii)}$$

Eq. (ii) says that the tangential component of the electric field remains continuous across the boundary, as its values just below and just above the boundary are equal. It means the electric field component $\vec{E}_T$ undergoes no change on the boundary. However, you can see from Eq. (iii) that the field component $\vec{D}_T$ undergoes some change across the boundary. It means $\vec{D}_T$ is discontinuous across the boundary.

In order to check the continuity of the normal component $\vec{E}_N$ of the field $\vec{E}$, we select Gaussian pillbox of area $\vec{S}$ (upper and lower surfaces) and the thickness Δs. Now we apply Gauss's law $\oint \vec{D}\cdot d\vec{S} = q$ and obtain the following under the limit of $\Delta s \to 0$

$$D_{2N}S - D_{1N}S = \sigma S$$

$$\text{or} \quad D_{2N} - D_{1N} = \sigma \qquad \text{(iv)}$$

Here σ is the free charge density placed at the boundary. It is clear from Eq. (iv) that the normal component of $\vec{D}$ is discontinuous and this discontinuity amounts to the free charge density σ. If there is no free charge, the normal component of the field $\vec{D}$ will be continuous at the boundary, as in the absence of σ Eq. (iv) follows

$$D_{2N} - D_{1N} = 0 \qquad \text{(v)}$$

From the above equation, we can find the condition for the electric field component $\vec{E}_N$ as

$$\varepsilon_2 E_{2N} - \varepsilon_1 E_{1N} = 0 \qquad \text{(vi)}$$

The above equation shows that the normal component $\vec{E}_N$ of the field $\vec{E}$ will be discontinuous at the boundary. If we write Eqs. (ii), (iii), (iv) and (vi) together, these equations represent the boundary conditions, named dielectric – dielectric boundary conditions.

The boundary conditions are useful in finding the electric field on one side of the boundary if the field on the other side is given. In addition to this, we can determine the refraction of the electric field across the boundary. If the field $\vec{E}_1$ (or $\vec{D}_1$) in the region 1 makes an angle α_1 and the field $\vec{E}_2$ (or $\vec{D}_2$) in the region 2 makes an angle α_2 with the normal to the boundary, then from Eq. (ii) we get

$$E_1 \sin\alpha_1 = E_2 \sin\alpha_2 \qquad \text{(vii)}$$

Similarly, Eq. (vi) yields

$$\varepsilon_1 E_1 \cos\alpha_1 = \varepsilon_2 E_2 \cos\alpha_2 \qquad \text{(viii)}$$

From Eqs. (vii) and (viii), we finally get

$$\frac{\tan\alpha_1}{\varepsilon_1} = \frac{\tan\alpha_2}{\varepsilon_2}$$

$$\text{or} \quad \frac{\tan\alpha_1}{\varepsilon_{r1}} = \frac{\tan\alpha_2}{\varepsilon_{r2}} \qquad [\because \varepsilon_1 = \varepsilon_0\varepsilon_{r1}, \varepsilon_2 = \varepsilon_0\varepsilon_{r2}] \qquad \text{(ix)}$$

Here ε_{r1} is the dielectric constant of medium 1 and ε_{r2} is the dielectric constant of medium 2. The above equation is known as law of refraction of the electric field at the interface which is free of charge. This is also clear from Eq. (ix) that in general a boundary between two dielectrics produces bending of the flux lines. This is attributed to unequal polarisation charges accumulated on the sides of the boundary.

Now we can discuss some special cases of the above boundary conditions. For example, if the dielectric in the medium 2 is replaced with a conductor whose conductivity is infinite. In this case, it is obtained that the field

Scanned by CamScanner

does not exist within the conductor, i.e. $\vec{E} = 0$. Since $\vec{E} = -\vec{\nabla}V$, a potential difference between any two points in the conductor cannot exist. In other words, we say that a conductor is an equipotential body. Under this situation, only the normal component of the electric field survives, i.e. $D_T = \epsilon_0\epsilon_r E_T = 0, D_N = \epsilon_0\epsilon_r E_N = \sigma$. Similarly, when the dielectric in medium 1 is replaced with a conductor and the dielectric in medium 2 is replaced with free space ($\epsilon_r = 1$), then the boundary conditions take the form $D_T = \epsilon_0 E_T = 0, \quad D_N = \epsilon_0 E_N = \sigma$.

10.19 SCALAR AND VECTOR POTENTIALS

As mentioned earlier, the zero curl of electrostatic field $\vec{E}$, i.e. $\vec{\nabla}\times\vec{E} = 0$, introduces a scalar potential V such that $\vec{E} = -\vec{\nabla}V$. When we analyze $\vec{\nabla}\cdot\vec{B} = 0$, we find that the field $\vec{B}$ can be written as a curl of another vector (say $\vec{A}$), i.e.

$$\vec{B} = \vec{\nabla}\times\vec{A} \tag{i}$$

Since a field can be completely determined if we know its divergence as well as its curl, the divergence of $\vec{A}$ remains to be explored. In this context, with the use of Eq. (i), Ampere's law reads

$$\vec{\nabla}\times\vec{B} = \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A} = \mu_0\vec{J} \tag{ii}$$

It is clear that Eq.(ii) will resemble Poisson's equation, if $\vec{\nabla}\cdot\vec{A} = 0$. This condition is known as Coulomb gauge. With the application of this condition, the Ampere's law simply yields

$$\nabla^2\vec{A} = -\mu_0\vec{J} \tag{iii}$$

The solution of the above equation can be obtained if the current density $\vec{J}$ vanishes at infinity. Then the solution comes out to be

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int\frac{\vec{J}}{r}dX \tag{iv}$$

Here dX is the volume element and the vector $\vec{A}$ is called magnetic vector potential. Like the electric scalar potential V, the magnetic vector potential $\vec{A}$ cannot be uniquely defined as we can add to it another vector whose curl is zero. This addition does not change the field $\vec{B}$. On the other side, it is a point of observation that we cannot introduce a magnetic scalar potential U such that $\vec{B} = -\vec{\nabla}U$. The reason is that it is incompatible with Ampere's law, since the curl of a gradient is always zero.

10.20 CONTINUITY EQUATION

The continuity equation says that the total current flowing out of some volume must be equal to the rate of decrease of the charge within that volume, if the charge is neither being created nor lost. Since the charge is flowing, we consider that the charge density ρ is a function of time. The transportation of the charge constitutes the current, i.e.

$$I = \frac{dq}{dt} = \frac{d}{dt}\int_V \rho dV \tag{i}$$

Here, it is assumed that the current is extended in space of volume V closed by a surface S. The net amount of charge which crosses a unit area normal to the directed surface in unit time is defined as the *current density* $\vec{J}$. This current density $\vec{J}$ is related to the total current I flowing through the surface S as

$$I = \oint_S \vec{J}\cdot d\vec{S} \tag{ii}$$

Here the integral is over closed surface, as the surface bounding the volume is closed surface. From Eqs. (i) and (ii), we have

$$\oint_S \vec{J}\cdot d\vec{S} = -\frac{dq}{dt} = \frac{-d}{dt}\oint_V \rho dV \tag{iii}$$

The minus sign above is needed in view of decreasing charge ρ in the volume V. So

$$\oint_S \vec{J}\cdot d\vec{S} = -\oint_V \frac{\partial\rho}{\partial t}dV \tag{iv}$$

From Gauss's divergence theorem, we have

$$\oint_S \vec{J}\cdot d\vec{S} = \int_V (\text{div}\,\vec{J})dV$$

$$\text{or} \quad \int_V (\text{div}\,\vec{J})dV = -\int_V \frac{\partial\rho}{\partial t}dV \tag{v}$$

Since the Eq. (v) holds good for any arbitrary volume, we can put the integrands to be equal. Then

$$\text{div}\,\vec{J} + \frac{\partial\rho}{\partial t} = 0 \tag{vi}$$

This is the continuity equation.

In case of stationary currents, i.e. when the charge density at any point within the region remains constant

$$\frac{\partial\rho}{\partial t} = 0, \tag{vii}$$

$$\text{so that div}\,\vec{J} = 0 \quad \text{or} \quad \vec{\nabla}\cdot\vec{J} = 0$$

which expresses the fact that there is no net outward flux of current density $\vec{J}$.

10.21 MAXWELL'S EQUATIONS: DIFFERENTIAL FORM

When the charges are in motion, the electric and magnetic fields are associated with this motion which will have variations in both the space and time. These electric and magnetic fields are inter related. This phenomenon is called *electromagnetism* which is summarised by the set of equations, known as Maxwell's equations. The Maxwell's equations are nothing but are the representation of the basic laws of electromagnetism.

In differential form, four Maxwell's equations are given below (S.I. units)

$$\vec{\nabla}\cdot\vec{D} = \rho \quad \text{or} \quad \vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0 \tag{i}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \tag{ii}$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial B}{\partial t} \tag{iii}$$

$$\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t} \tag{iv}$$

Scanned by CamScanner

10.21.1 Derivation of Maxwell's First Equation

Let us consider a surface S bounding a volume V in a dielectric medium, which is kept in the $\vec{E}$ field. The application of external field $\vec{E}$ polarises the dielectric medium and charges are induced, called bound charges or charges due to polarisation. The total charge density at a point in a small volume element dV would then be $\rho - \rho_p$, where ρ_p is the polarisation charge density, given by $\rho_p = -\operatorname{div}\vec{P}$ and ρ is the free charge density at that point in the small volume element dV.

Thus, the total charge density at that point will be $\rho - (\operatorname{div}\vec{P})$. Then Gauss's theorem can be expressed as

$$\oint_S \vec{E}\cdot d\vec{S} = \int_V (\operatorname{div}\vec{E})\,dV = \frac{1}{\varepsilon_0}\int_V (\rho - \operatorname{div}\vec{P})\,dV$$

$$\text{or} \quad \varepsilon_0 \int_V (\operatorname{div}\vec{E})\,dV = \int_V (\rho - \operatorname{div}\vec{P})\,dV$$

$$\text{or} \quad \int_V \operatorname{div}(\varepsilon_0\vec{E} + \vec{P})\,dV = \int_V \rho\,dV$$

The quantity $(\varepsilon_0\vec{E} + \vec{P})$ is denoted by a quantity $\vec{D}$, called the electric displacement. Therefore,

$$\int_V (\operatorname{div}\vec{D})\,dV = \int_V \rho\,dV$$

Since this equation is true for all the arbitrary volumes, the integrands in this equation must be equal, i.e.

$$\operatorname{div}\vec{D} = \rho$$

$$\text{or} \quad \vec{\nabla}\cdot\vec{D} = \rho$$

This is the Maxwell's first equation.

When the medium is isotropic, the three vectors $\vec{D}, \vec{E}$ and $\vec{P}$ are in the same direction and for small field $\vec{E}$, $\vec{D}$ is proportional to $\vec{E}$, i.e.

$$\vec{D} = \varepsilon\vec{E}$$

where ε is called the permittivity of the dielectric medium. The ratio $\varepsilon/\varepsilon_0$ is called the dielectric constant of the medium.

10.21.2 Derivation of Maxwell's Second Equation

Since the magnetic lines of force are either closed or go off to infinity, the number of magnetic lines of force entering any arbitrary surface is exactly the same as leaving it. It means the flux of magnetic induction $\vec{B}$ across any closed surface is always zero, i.e.

$$\oint_S \vec{B}\cdot d\vec{S} = 0$$

Transforming the surface integral to volume integral using Gauss's divergence theorem, we have

$$\oint_S \vec{B}\cdot d\vec{S} = \int_V (\operatorname{div}\vec{B})\,dV = 0$$

The integrand in the above equation should vanish for the surface boundary as the volume is arbitrary. Therefore

$$\operatorname{div}\vec{B} = 0 \quad \text{or} \quad \vec{\nabla}\cdot\vec{B} = 0$$

This is the Maxwell's second equation.

10.21.3 Derivation of Maxwell's Third Equation

According to Faraday's law, the emf induced in a closed loop is given by

$$E_{emf} = -\frac{\partial\phi}{\partial t} = -\int_S \frac{\partial\vec{B}}{\partial t}\cdot d\vec{S} = -\frac{\partial}{\partial t}\oint_S \vec{B}\cdot d\vec{S}$$

Here the flux $\phi = \oint_C \vec{B}\cdot d\vec{S}$ where S is any closed surface having the loop as boundary. The emf (E_{emf}) can also be found by calculating the work done in carrying a unit charge completely around the loop. Thus,

$$E_{emf} = \oint_C \vec{E}\cdot d\vec{l}$$

Here $\vec{E}$ is the intensity of the field associated with the induced emf on equating the above two equations, we get

$$\oint_C \vec{E}\cdot d\vec{l} = -\oint_S \frac{\partial\vec{B}}{\partial t}\cdot d\vec{S}$$

According to the Stokes' theorem, the line integral can be transformed into surface integral with the help of $\oint_C \vec{E}\cdot d\vec{l} = \int_S (\vec{\nabla}\times\vec{E})\cdot d\vec{S}$. Therefore

$$\int_S (\vec{\nabla}\times\vec{E})\cdot d\vec{S} = -\oint_S \frac{\partial\vec{B}}{\partial t}\cdot d\vec{S}$$

This equation must be true for any surface whether small or large in the field. So the two vectors in the integrands must be equal at every point, i.e.

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

This is the Maxwell's third equation.

10.21.4 Derivation of Maxwell's Fourth Equation

According to the Ampere's law, the work done in carrying a unit magnetic pole once around a closed arbitrary path linked with the current is expressed by

$$\oint_C \vec{H}\cdot d\vec{l} = I$$

$$\text{or} \quad \oint_C \vec{H}\cdot d\vec{l} = \oint_C \vec{J}\cdot d\vec{S}$$

As per Stokes' theorem,

$$\oint_C \vec{H}\cdot d\vec{l} = \int_S (\vec{\nabla}\times\vec{H})\cdot d\vec{S}$$

Therefore,

$$\int_S (\vec{\nabla}\times\vec{H})\cdot d\vec{S} = \int_S \vec{J}\cdot d\vec{S}$$

This gives

$$\vec{\nabla}\times\vec{H} = \vec{J}$$

Scanned by CamScanner

The above relation is derived on the basis of Ampere's law, which holds good only for the steady current however, for the changing electric fields, the current density should be modified. The difficulty with the above equation is that, if we take divergence of this equation, then

$$\text{div}(\vec{\nabla}\times\vec{H}) = \text{div}\,\vec{J}$$

$$\Rightarrow \quad 0 = \text{div}\,\vec{J} \qquad \text{[Since divergence of a curl} = 0]$$

$$\Rightarrow \quad \text{div}\,\vec{J} = 0$$

which conflicts with the continuity equation, as

$$\text{div}\,\vec{J} = -\frac{\partial\rho}{\partial t}$$

Therefore, Maxwell realised that the definition of the total current density is incomplete and suggested to add another density $\vec{J}'$. Therefore

$$\text{curl}\,\vec{H} = \vec{J} + \vec{J}'$$

Now, taking divergence of the above equation, we get

$$\text{div}\,(\text{curl}\,\vec{H}) = \text{div}\,\vec{J} + \text{div}\,\vec{J}'$$

$$\text{or} \quad 0 = \text{div}\,\vec{J} + \text{div}\,\vec{J}'$$

$$\text{div}\,\vec{J}' = -\text{div}\,\vec{J} = \frac{\partial\rho}{\partial t}$$

Since,

$$\rho = \vec{\nabla}\cdot\vec{D}$$

$$\text{div}\,\vec{J}' = \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{D})$$

$$\vec{\nabla}\cdot\vec{J}' = \vec{\nabla}\cdot\frac{\partial\vec{D}}{\partial t}$$

Hence $\vec{J}' = \dfrac{\partial\vec{D}}{\partial t}$

Therefore, the Maxwell's fourth equation can be written as

$$\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}$$

The last term of R.H.S. of this equation is called Maxwell's correction and is known as *displacement current density*. The above equation is called modified Ampere's law for unsteady or changing current which is responsible for electromagnetic fields.

10.22 MAXWELL'S EQUATIONS: INTEGRAL FORM

There are situations where the integral form of Maxwell's equations is useful. Therefore, now we derive these equations in integral form.

10.22.1 Maxwell's First Equation

Differential form of the Maxwell's first equation is

$$\vec{\nabla}\cdot\vec{D} = \rho \qquad \text{(i)}$$

On integrating Eq. (i) over a volume V, we have

$$\int_V (\vec{\nabla}\cdot\vec{D})dV = \int_V \rho dV$$

Using Gauss's divergence theorem, the above equation reads

$$\oint_S \vec{D}\cdot d\vec{S} = \int_V \rho dV = q$$

$$\text{or} \quad \oint_S \vec{D}\cdot d\vec{S} = q$$

Here q is the net charge contained in the volume V and S is the surface bounding the volume V. This integral form of the Maxwell's first equation says that the total electric displacement through the surface S enclosing a volume V is equal to the total charge contained within this volume.

This statement can also be put in the following form: The total outward flux corresponding to the displacement vector $\vec{D}$ through a closed surface $\vec{S}$ is equal to the total charge q within the volume V enclosed by the surface $\vec{S}$.

10.22.2 Maxwell's Second Equation

Differential form of the Maxwell's second equation is

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \text{(ii)}$$

Exactly in a manner adopted above, we can show that

$$\oint_S \vec{B}\cdot d\vec{S} = 0$$

which signifies that the total outward flux of magnetic induction $\vec{B}$ through any closed surface $\vec{S}$ is equal to zero.

10.22.3 Maxwell's Third Equation

Differential form of the Maxwell's third equation is

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \qquad \text{(iii)}$$

On integrating Eq. (iii) over a surface $\vec{S}$ bounded by a closed path, we have

$$\int_S (\vec{\nabla}\times\vec{E})\cdot d\vec{S} = -\int_S \frac{\partial\vec{B}}{\partial t}\cdot d\vec{S}$$

Converting surface integral into line integral by Stokes' theorem, we get

$$\oint_C \vec{E}\cdot d\vec{l} = -\frac{\partial}{\partial t}\int_S \vec{B}\cdot d\vec{S}$$

which signifies that the electromotive force around a closed path is equal to the time derivative of the magnetic displacement through any closed surface bounded by that path.

Scanned by CamScanner

10.22.4 Maxwell's Fourth Equation

Differential form of the Maxwell's fourth equation is

$$\vec{\nabla}\times\vec{H}=\vec{J}+\frac{\partial\vec{D}}{\partial t} \qquad \text{(iv)}$$

Exactly, in a manner adopted above, we can have this equation in the following form

$$\oint_C \vec{H}\cdot d\vec{l}=\int_S\left(\vec{J}+\frac{\partial\vec{D}}{\partial t}\right)\cdot d\vec{S}$$

The above equation signifies that the magnetomotive force around a closed path is equal to the conduction current plus the time derivative of the electric displacement through any surface bounded by that path.

10.23 SIGNIFICANCE OF MAXWELL'S EQUATIONS

Maxwell's equations represent concisely the fundamentals of electricity and magnetism. From them one can develop most of the working relationships in the field.

Maxwell's first equation represents the Gauss's law for electricity which says that the electric flux out of any closed surface is proportional to the total charge enclosed within the surface. The integral form of this equation finds applications in calculating electric fields around charge objects. It is consistent with Coulomb's law when applied to the electric field of a point charge. The area integral of the electric field gives a measure of the net charge enclosed. However, the divergence of the electric field gives a measure of the density of sources.

As mentioned, the area integral of a vector field determines the net source of the field (function). The integral form $\oint_S \vec{B}\cdot d\vec{S}=0$ of the Maxwell's second equation says that the net magnetic flux out of any closed surface is zero. This is because the magnetic flux directed inward toward the south pole, of a magnetic dipole kept in any closed surface, will be equal to the flux outward the north pole. Therefore, the net flux is zero for dipole sources. If we imagine a magnetic monopole source, the area integral $\oint_S \vec{B}\cdot d\vec{S}$ would have some finite value. Because of this and since the divergence of a vector field is proportional to the density of point source, the form of the Gauss's law for magnetic field simply says that there are no magnetic monopoles.

The Maxwell's third equation when written in the integral form states that the line integral of the electric field around a closed loop is equal to the negative of the rate of change of the magnetic flux through the area enclosed by the loop. The line integral basically is the generated voltage or emf in the loop. Therefore, the physical interpretation of Maxwell's third equation is that the changing magnetic field induces electric field. For static electric field $\vec{E}$, the second term of the R.H.S. of the Maxwell's fourth equation vanishes and then the integral form of this equation says that the line integral of the magnetic field around a closed loop is proportional to the electric current flowing through the loop. This form of the Maxwell's equation is useful for calculating the magnetic field for simple geometries. However, this equation more specifically reveals that the changing electric field induces magnetic field. This seems complimentary to the meaning of the Maxwell's third equation. Therefore, they together yield the formulation of electromagnetic fields or electromagnetic waves, where both electric and magnetic field propagate together and the change in one field induces the other field.

10.24 MAXWELL'S DISPLACEMENT CURRENT AND CORRECTION IN AMPERE'S LAW

When a current flows in a conductor, magnetic field is produced around it. A relation between the conduction current (I) and the magnetic field vector ($\vec{B}$) was given by Ampere, according to which the line integral of the field $\vec{B}$ along a closed curve in the magnetic field is equal to μ_0 times the current I flowing in the conductor.

$$\oint_C \vec{B}\cdot d\vec{l}=\mu_0 I=\mu_0\int_S \vec{J}\cdot d\vec{S} \qquad \text{(i)}$$

$$\text{or}\quad \int_S (\text{curl}\,\vec{B})\cdot d\vec{S}=\mu_0\int_S \vec{J}\cdot d\vec{S} \qquad \text{(Using Stokes' theorem)}$$

$$\text{or}\quad \int_S (\text{curl}\vec{B}-\mu_0\vec{J})\cdot d\vec{S}=0$$

For arbitrary surfaces, the above integral is true, so we get

$$\text{curl}\ \vec{B}-\mu_0\vec{J}=0$$

$$\text{or}\quad \text{curl}\ \vec{B}=\mu_0\vec{J} \qquad \text{(ii)}$$

$$\text{or}\quad \text{div}(\text{curl}\ \vec{B})=\text{div}(\mu_0\vec{J})$$

$$\text{or}\quad 0=\mu_0\text{div}(\vec{J}) \qquad [\because \text{divergence of curl of a vector is always zero}]$$

$$\text{or}\quad \text{div}\ \vec{J}=0 \qquad \text{(iii)}$$

Using continuity equation $\text{div}\ \vec{J}+\frac{\partial\rho}{\partial t}=0$ or $\text{div}\ \vec{J}=-\frac{\partial\rho}{\partial t}$, we get $\text{div}\ \vec{J}=0$ only if $\frac{\partial\rho}{\partial t}=0$, i.e. for static charge density.

Thus, the Ampere's law given by Eq. (i) is valid only for the steady state conditions, i.e. for the fields that do not vary with time. This law is not valid for the time varying fields, such as charging and discharging of a capacitor, where $\frac{\partial\rho}{\partial t}\neq 0$.

Ampere's law was modified by Maxwell for the time varying fields. His concept was based on the Faraday's law, according to which a changing magnetic field produces an electric field. On the basis of the fact that the magnetic field around a conductor is produced by the current flowing in it, Maxwell hypothesized that changing electric field should also induce a magnetic field. A changing electric field is equivalent to a current called displacement current (I_d) which flows as long as the electric field is changing. The displacement current produces the magnetic field the same way as the conduction current (I).

Thus, the total magnetic field ($\vec{B}$) will be the sum of the two terms (a) due to current I ($\vec{B_1}$) and (b) due to current I_d ($\vec{B_2}$), i.e.

$$\vec{B}=\vec{B_1}+\vec{B_2}$$

$$\therefore\quad \text{curl}\,\vec{B}=\text{curl}\,\vec{B_1}+\text{curl}\,\vec{B_2}$$

$$\text{or}\quad \text{curl}\ \vec{B}=\mu_0\vec{J}+\mu_0\vec{J_d}$$

$$\text{or}\quad \text{curl}\ \vec{B}=\mu_0(\vec{J}+\vec{J_d}) \qquad \text{(iv)}$$

where $\vec{J_d}$ is the displacement current density. Also, in the integral form

Scanned by CamScanner

$$\oint_C \vec{B}\cdot d\vec{l} = \mu_0 I + \mu_0 I_d$$

$$\text{or} \quad \oint_C \vec{B}\cdot d\vec{l} = \mu_0 (I + I_d) \qquad \text{(v)}$$

In analogy to the Faraday's law of induction $\oint_C \vec{E}\cdot d\vec{l} = -\frac{d\phi_B}{dt}$, I_d should correspond to $\varepsilon_0 \frac{d\phi_E}{dt}$. With this Eq. (v) can be written as

$$\oint_C \vec{B}\cdot d\vec{l} = \mu_0 \left(I + \varepsilon_0 \frac{d\phi_E}{dt} \right) \qquad \text{(vi)}$$

$$\text{Thus,} \quad I_d = \varepsilon_0 \frac{d\phi_E}{dt} = \varepsilon_0 S \frac{d\vec{E}}{dt} = S\frac{d\vec{D}}{dt} = S\vec{J}_d \qquad \text{(vii)}$$

where $\vec{D}$ is the electric displacement vector.

Value of J_d can also be determined by taking divergence of Eq. (iv), i.e.

$$\text{div curl}\,\vec{B} = \text{div}(\mu_0 \vec{J} + \mu_0 \vec{J}_d) = 0 \qquad [\because \text{div curl}\,\vec{B} = 0]$$

$$\text{or} \quad \text{div}\,\vec{J}_d = -\text{div}\,\vec{J} = \frac{\partial \rho}{\partial t} \qquad \left[\because \text{div}\,\vec{J} + \frac{\partial \rho}{\partial t} = 0\right]$$

$$\text{or} \quad \text{div}\,\vec{J}_d = \frac{\partial}{\partial t}(\text{div}\,\vec{D}) = \text{div}\left(\frac{\partial \vec{D}}{\partial t}\right) \qquad [\because \text{div}\,\vec{D} = \rho]$$

$$\therefore \vec{J}_d = \frac{\partial \vec{D}}{\partial t} \qquad \text{(viii)}$$

Therefore, the modified form of Ampere's law is

$$\text{curl}\,\vec{B} = \mu_0 \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \qquad \text{(ix)}$$

$$\text{or} \quad \text{curl}\,\vec{B} = \mu_0 \left(\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$$

$$\text{Also,} \quad \text{curl}\,\vec{H} = \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \qquad (\because \vec{B} = \mu_0 \vec{H})$$

$$\text{or} \quad \text{curl}\,\vec{H} = \vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \qquad \text{(x)}$$

The Eqs. (v), (ix) or (x) represent the Maxwell's fourth equation in different form which is nothing but the modified form of Ampere's law.

10.25 ELETROMAGNETIC WAVE PROPAGATION

Maxwell's equations for free space are given as follows

$$\text{div}\,\vec{E} = 0 \qquad \text{(i)}$$

$$\text{div}\,\vec{H} = 0 \qquad \text{(ii)}$$

$$\text{curl}\,\vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t} \qquad \text{(iii)}$$

$$\text{curl}\,\vec{H} = \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \qquad \text{(iv)}$$

Taking curl of Eq. (iii), we get $\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\mu_0 \frac{\partial}{\partial t}(\vec{\nabla}\times\vec{H})$ or $\vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - \nabla^2\vec{E} = -\mu_0 \frac{\partial}{\partial t}\left[\varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right]$

Here we have used Eq. (iv) for the value of $\vec{\nabla}\times\vec{H}$. Now from Eq. (i) $\vec{\nabla}\cdot\vec{E} = 0$. Hence

$$\nabla^2\vec{E} - \mu_0\varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

This is the wave equation governing the field $\vec{E}$. In view of the dimensions of $\mu_0\varepsilon_0$ as of velocity (say, v), we can write this as

$$\nabla^2\vec{E} - \frac{1}{v^2}\frac{\partial^2 \vec{E}}{\partial t^2} = 0 \qquad \text{(v)}$$

Similarly the curl of Eq. (iv) gives the wave equation for the field $\vec{H}$ as

$$\nabla^2\vec{H} - \mu_0\varepsilon_0 \frac{\partial^2 \vec{H}}{\partial t^2} = 0$$

$$\text{or} \quad \nabla^2\vec{H} - \frac{1}{v^2}\frac{\partial^2 \vec{H}}{\partial t^2} = 0 \qquad \text{(vi)}$$

The plane wave solutions of Eqs. (v) and (vi) may be written as

$$\vec{E}(\vec{r},t) = \vec{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)} \quad \text{and} \quad \vec{H}(\vec{r},t) = \vec{H}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

where ω is the angular frequency of the variation of the fields $\vec{E}$ and $\vec{H}$ and $\vec{k}$ is the wave vector which tells the direction of propagation of the wave. The ratio ω/k gives the phase velocity of the wave.

$$\text{Now,} \quad \vec{\nabla} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right), \quad \vec{E}_0 = (E_{0x}\hat{i} + E_{0y}\hat{j} + E_{0z}\hat{k})$$

$$\vec{k} = (k_x\hat{i} + k_y\hat{j} + k_z\hat{k}), \quad \vec{r} = (x\hat{i} + y\hat{j} + z\hat{k})$$

$$\Rightarrow \vec{k}\cdot\vec{r} = (k_x x + k_y y + k_z z)$$

$$\therefore \text{curl}\,\vec{E} = \vec{\nabla}\times\left[\vec{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right]$$

$$= \vec{\nabla}\times\left\{(E_0 e^{i\vec{k}\cdot\vec{r}})e^{-i\omega t}\right\}$$

$$= \vec{\nabla}\times\left\{\left[(E_{0x}\hat{i} + E_{0y}\hat{j} + E_{0z}\hat{k})e^{i(k_x x + k_y y + k_z z)}\right]e^{-i\omega t}\right\}$$

Here note that i appeared in exponential term is such that $i = \sqrt{-1}$, whereas $\hat{i}$ is the unit vector along the X-direction. When we solve the above equation with the help of expansion of curl, we obtain

Scanned by CamScanner

$$\text{curl}\,\vec{E} = i\left\{\hat{i}[E_{0z}k_y - E_{0y}k_z] + \hat{j}[E_{0x}k_z - E_{0z}k_x] + \hat{k}[E_{0y}k_x - E_{0x}k_y]\right\}e^{i(k_x x + k_y y + k_z z)}e^{-i\omega t}$$

$$= i\left[\vec{k}\times\vec{E}_0\right]e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$\text{or curl}\,\vec{E} = i\left[\vec{k}\times\vec{E}\right] \qquad \text{(vii)}$$

we have used

$$\vec{E}(\vec{r},t) = \vec{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

g Eqs. (iii) and (vii),

$$-\mu_0\frac{\partial\vec{H}}{\partial t} = i\left[\vec{k}\times\vec{E}\right]$$

S. of the above equation is written as

$$-\mu_0\frac{\partial}{\partial t}\left[\vec{H}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}\right] = i\omega\mu_0\vec{H}$$

Hence $\vec{k}\times\vec{E} = \omega\mu_0\vec{H}$ (viii)

ilarly, it can be shown using Eq. (iv) that

$$\vec{k}\times\vec{H} = -\omega\varepsilon_0\vec{E} \qquad \text{(ix)}$$

m Eq (viii) it is obvious that the magnetic field vector $\vec{H}$ is perpendicular to both the propagation vector nd the electric field vector $\vec{E}$ and according to Eq. (ix) $\vec{E}$ is perpendicular to both $\vec{k}$ and $\vec{H}$. Therefore, it y be concluded that the electric and magnetic vectors are normal to each other as well as to the direction propagation of the wave or $\vec{E},\vec{H}$ and direction of wave propagation $\vec{k}$ form a set of orthogonal vectors.

ther, we can prove that the electromagnetic wave travels at the speed of light c in free space. For this, the ss product of $\vec{k}$ with Eq. (viii) gives

$$\vec{k}\times(\vec{k}\times\vec{E}) = \omega\mu_0(\vec{k}\times\vec{H})$$

$$\vec{k}(\vec{k}\cdot\vec{E}) - k^2\vec{E} = \omega\mu_0\left[-\omega\varepsilon_0\vec{E}\right]$$ [Putting the value of $\vec{k}\times\vec{H}$ from Eq. (ix)]

nce $\vec{k}$ and $\vec{E}$ are perpendicular to each other, the above equation reads

$$k^2\vec{E} - \omega^2\mu_0\varepsilon_0\vec{E} = 0$$

$$(k^2 - \omega^2\mu_0\varepsilon_0)\vec{E} = 0$$

ince $\vec{E}$ cannot be zero for the wave, $k^2 - \omega^2\mu_0\varepsilon_0 = 0$

$$\Rightarrow \frac{\omega}{k} = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = 3\times10^8\ \text{m/sec} = c, \text{ the speed of light.}$$

Therefore, the phase velocity $\frac{\omega}{k}$ of the electromagnetic wave is equal to the speed of light c in free space or vacuum.

10.26 TRANSVERSE NATURE OF ELECTROMEGNETIC WAVES

Transverse nature of electromagnetic waves can be proved with the help of Maxwell's equations div $\vec{E} = 0$ and div $\vec{H} = 0$ for free space. Using different relations as discussed in the previous section div $\vec{E}$ and div $\vec{H}$ can be calculated as follows.

$$\vec{\nabla}\cdot\vec{E} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\vec{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left[(E_{0x}\hat{i} + E_{0y}\hat{j} + E_{0z}\hat{k})e^{i(k_x x + k_y y + k_z z - \omega t)}\right]$$

$$= \frac{\partial}{\partial x}\left[E_{0x}e^{i(k_x x + k_y y + k_z z - \omega t)}\right] + \frac{\partial}{\partial y}\left[E_{0y}e^{i(k_x x + k_y y + k_z z - \omega t)}\right] + \frac{\partial}{\partial z}\left[E_{0z}e^{i(k_x x + k_y y + k_z z - \omega t)}\right]$$

$$= (E_{0x}ik_x + E_{0y}ik_y + E_{0z}ik_z)e^{i(k_x x + k_y y + k_z z - \omega t)}$$

$$= i(k_x E_{0x} + k_y E_{0y} + k_z E_{0z})e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$= i(\vec{k}\cdot\vec{E}_0)e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$

$$= i\vec{k}\cdot\vec{E}$$

$\therefore$ For free space, $\vec{\nabla}\cdot\vec{E} = 0$

$$i\vec{k}\cdot\vec{E} = 0 \text{ or } \vec{k}\cdot\vec{E} = 0$$

It means the wave vector $\vec{k}$ is perpendicular to $\vec{E}$.

Similarly, from $\vec{\nabla}\cdot\vec{H} = 0$ it can be shown that $\vec{k}\cdot\vec{H} = 0$.

Hence, the wave vector $\vec{k}$ is perpendicular to $\vec{H}$. Therefore, the relations $\vec{k}\cdot\vec{E} = 0$ and $\vec{k}\cdot\vec{H} = 0$ indicate that the electromagnetic field vectors $\vec{E}$ and $\vec{H}$ (or $\vec{B}$, as $\vec{B} = \mu_0\vec{H}$) both are perpendicular to the direction of propagation vector $\vec{k}$. It means that the electromagnetic waves are transverse in nature.

10.27 MAXWELL'S EQUATIONS IN ISOTROPIC DIELECTRIC MEDIUM

In an isotropic dielectric medium, the current density $\vec{J}$ and volume charge density ρ are zero. Further, the displacement vector $\vec{D}$ and the magnetic field $\vec{B}$ are defined as $\vec{D} = \varepsilon\vec{E}$ and $\vec{B} = \mu\vec{H}$. In fact $\vec{D} = \varepsilon_0\vec{E} + \vec{P} \equiv \varepsilon\vec{E}$ and $\vec{B} = \mu_0\vec{H} + \mu_0\vec{M}$ for the isotropic linear dielectric (polarizable and magnetic) medium. Here, the vectors $\vec{P}$ and $\vec{M}$ give respectively the measure of polarization and magnetization of the medium. However, for the

Scanned by CamScanner

dielectric medium, it would be sufficient to remember that ε_0 and μ_0 of free space have been simply replaced with ε and μ. Hence, for dielectric medium

$$\vec{J} = 0 \text{ (or } \sigma = 0), \rho = 0, \vec{D} = \varepsilon\vec{E} \text{ and } \vec{B} = \mu\vec{H}$$

where ε and μ are respectively the absolute permittivity and permeability of the medium. Under this situation, we can express the Maxwell's equation as

$$\vec{\nabla}\cdot\vec{E} = 0$$

$$\vec{\nabla}\cdot\vec{H} = 0 \qquad \text{(i)}$$

$$\vec{\nabla}\times\vec{E} = -\mu\frac{\partial\vec{H}}{\partial t} \qquad \text{(ii)}$$

$$\vec{\nabla}\times\vec{H} = \varepsilon\frac{\partial\vec{E}}{\partial t} \qquad \text{(iii)}$$

Taking curl of Eq. (iii), we get

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}\times\left[-\mu\frac{\partial\vec{H}}{\partial t}\right] \qquad \text{(iv)}$$

or $$\vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - \nabla^2\vec{E} = -\mu\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{H})$$

or $$0 - \nabla^2 E = -\mu\varepsilon\frac{\partial^2\vec{E}}{\partial t}$$ [Using Eqs. (i) and (iv)]

$$\nabla^2\vec{E} = \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} \qquad \text{(v)}$$

Similarly, taking curl of Eq. (iv) and using Eqs. (ii) and (iii), we get

$$\nabla^2\vec{H} = \mu\varepsilon\frac{\partial^2\vec{H}}{\partial t^2} \qquad \text{(vi)}$$

As discussed earlier, $\frac{1}{\sqrt{\mu\varepsilon}}$ gives the phase velocity of the wave in the medium. If we represent this as v, we obtain from Eqs. (v) and (vi)

$$\nabla^2\vec{E} - \frac{1}{v^2}\frac{\partial^2\vec{E}}{\partial t^2} = 0$$

and $$\nabla^2\vec{H} - \frac{1}{v^2}\frac{\partial^2\vec{H}}{\partial t^2} = 0$$

Eqs. (v) and (vi) are the wave equations in an isotropic linear dielectric medium.

$$\text{Now, } v = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu_0\mu_r\varepsilon_0\varepsilon_r}}$$

$$\text{or} \quad v = \frac{c}{\sqrt{\mu_r\varepsilon_r}} \quad \left[\because c = \frac{1}{\sqrt{\mu_0\varepsilon_0}}\right]$$

Eq. (vii) shows that the propagation velocity of an electromagnetic wave in a dielectric medium is less than that in free space.

$$\text{Also, refractive index} = \frac{c}{v} = \sqrt{\mu_r\varepsilon_r}$$

For non-magnetic dielectric medium $\mu_r \approx 1$. Hence, refractive index $= \sqrt{\varepsilon_r}$ or Refractive index $= \sqrt{\text{Relative permittivity}}$

10.28 MAXWELL'S EQUATIONS IN CONDUCTING MEDIUM

We consider a conducting linear and isotropic medium whose permeability is μ, permittivity is ε and the conductivity is σ. Further, we assume that there is no free charge available, i.e., $\rho = 0$. Under this situation, we can write the Maxwell's equation as

$$\vec{\nabla}\cdot\vec{E} = 0$$

$$\vec{\nabla}\cdot\vec{H} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} = -\mu\frac{\partial\vec{H}}{\partial t}$$

$$\vec{\nabla}\times\vec{H} = \vec{J} + \varepsilon\frac{\partial\vec{E}}{\partial t}$$

or $$\vec{\nabla}\times\vec{H} = \sigma\vec{E} + \varepsilon\frac{\partial\vec{E}}{\partial t} \quad [\because \vec{J} = \sigma\vec{E}]$$

Taking curl of Eq. (iii), we have

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}\times\left[-\mu\frac{\partial\vec{H}}{\partial t}\right]$$

$$= -\mu\left[\vec{\nabla}\times\frac{\partial\vec{H}}{\partial t}\right]$$

$$= -\mu\frac{\partial}{\partial t}[\vec{\nabla}\times\vec{H}]$$

$$= -\mu\frac{\partial}{\partial t}\left[\sigma\vec{E} + \varepsilon\frac{\partial\vec{E}}{\partial t}\right]$$ [using Eq. (iv)]

or $$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\mu\sigma\frac{\partial\vec{E}}{\partial t} - \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}$$

Scanned by CamScanner

Also, $\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - (\vec{\nabla}\cdot\vec{\nabla})\vec{E} = -\nabla^2\vec{E}$ $\quad$ [$\because \vec{\nabla}\cdot\vec{E} = 0$ from Eq. (i)]

$$\therefore -\nabla^2\vec{E} = -\mu\sigma\frac{\partial\vec{E}}{\partial t} - \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}$$

$$\text{or} \quad \nabla^2\vec{E} = +\mu\sigma\frac{\partial\vec{E}}{\partial t} + \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} \qquad \text{(v)}$$

(v) is the electromagnetic wave equation for the electric field $\vec{E}$ in a conducting medium.

case of non-conducting medium $\sigma = 0$. Hence, from Eq. (v)

$$\nabla^2\vec{E} = \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} \qquad \text{(vi)}$$

(v) and (vi) show that the term $\mu\sigma\frac{\partial\vec{E}}{\partial t}$ is the dissipative term which allows the current to flow through medium.

, by taking curl of Eq. (iv), we obtain

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{H}) = \vec{\nabla}\times\left[\sigma\vec{E} + \varepsilon\frac{\partial\vec{E}}{\partial t}\right]$$

$$= \sigma(\vec{\nabla}\times\vec{E}) + \varepsilon\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{E})$$

$$= \sigma\left(-\mu\frac{\partial\vec{H}}{\partial t}\right) + \varepsilon\frac{\partial}{\partial t}\left(-\mu\frac{\partial\vec{H}}{\partial t}\right) \qquad \text{[Using Eq. (iii)]}$$

$$= -\mu\sigma\frac{\partial\vec{H}}{\partial t} - \varepsilon\mu\frac{\partial^2\vec{H}}{\partial t^2}$$

$$\because \vec{\nabla}\times(\vec{\nabla}\times\vec{H}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{H}) - (\vec{\nabla}\cdot\vec{\nabla})\vec{H}$$

$$= 0 - \nabla^2\vec{H} \qquad [\because \vec{\nabla}\cdot\vec{H} = 0 \text{ from Eq. (ii)}]$$

$$\therefore -\nabla^2\vec{H} = -\mu\sigma\frac{\partial\vec{H}}{\partial t} - \mu\varepsilon\frac{\partial^2\vec{H}}{\partial t^2}$$

$$\text{or} \quad \nabla^2\vec{H} = \mu\sigma\frac{\partial\vec{H}}{\partial t} + \mu\varepsilon\frac{\partial^2\vec{H}}{\partial t^2} \qquad \text{(vii)}$$

(vii) represents the electromagnetic wave equation for magnetic field ($\vec{H}$) in conducting medium.

case of non-conducting medium $\sigma = 0$, the wave equation takes the form

$$\nabla^2\vec{H} = \mu\varepsilon\frac{\partial^2\vec{H}}{\partial t^2} \qquad \text{(viii)}$$

(vii) and (viii) show that the term $\mu\sigma\frac{\partial\vec{H}}{\partial t}$ is the dissipative term which allows the current to flow through conducting medium.

10.29 ELECTROMAGNETIC ENERGY DENSITY

It can be proved that the work done in assembling a static charge distribution (number n) against the Coulomb repulsion of like charges is

$$W_E = \frac{1}{2}\sum_{j=1}^{n} q_j V(\vec{r}_j) \qquad \text{(i)}$$

For a volume charge density ρ, this equation takes the form

$$W_E = \frac{1}{2}\int \rho V dX \qquad \text{(ii)}$$

Here dX is the volume element. Now the above equation can be written in terms of the resulting electric field $\vec{E}$ if we apply Gauss's law $\varepsilon_0\vec{\nabla}\cdot\vec{E} = \rho$ and mention the potential V in terms of $\vec{E}$. This yields the following relation where the integration is over all the space containing the whole charge distribution.

$$W_E = \frac{\varepsilon_0}{2}\int E^2 dX \qquad \text{(iii)}$$

The same way we can derive an expression for the work done on a unit charge against the back emf in one trip around the circuit, as follows

$$W_B = \frac{1}{2\mu_0}\int B^2 dX \qquad \text{(iv)}$$

Here B is the resulting magnetic field. Eqs. (iii) and (iv) suggest that the total energy stored in electromagnetic field would be

$$W_{EM} = \frac{1}{2}\int\left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right)dX \qquad \text{(v)}$$

Therefore, the electromagnetic energy density can be obtained as

$$U_{EM} = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right) \qquad \text{(vi)}$$

For a monochromatic plane electromagnetic wave, $B = \frac{E}{c}$, where $c\left(= \frac{1}{\sqrt{\mu_0\varepsilon_0}}\right)$ is the speed of light. Hence, it can be seen that the contribution of magnetic field $\vec{B}$ to the energy density is the same as that of the electric field $\vec{E}$.

10.30 POYNTING VECTOR AND POYNTING THEOREM

The electromagnetic waves carry energy when they propagate and there is an energy density associated with both the electric and magnetic fields.

The amount of energy flowing through unit area, perpendicular to the direction of energy propagation per unit time, i.e. the rate of energy transport per unit area, is called the *poynting vector*. It is also termed as instantaneous energy flux density and is represented by $\vec{S}$ (or $\vec{P}$, sometimes). Mathematically it is defined

$$\vec{S} = \vec{E}\times\vec{H}$$

Scanned by CamScanner

where $\vec{E}$ and $\vec{H}$ represent the instantaneous values of electric and magnetic field vectors. This is clear that the rate of energy transport $\vec{S}$ is perpendicular to both $\vec{E}$ and $\vec{H}$ and is in the direction of propagation of the wave, as $\vec{E}\times\vec{H}$ is in the direction of $\vec{k}$. Since the poynting vector represents the rate of energy transport per unit area. Its units are W/m².

Derivation: We can calculate the energy density carried by electromagnetic waves with the help of Maxwell's equations given below.

$$\text{div}\,\vec{D}=0 \qquad \text{(i)}$$

$$\text{div}\,\vec{B}=0 \qquad \text{(ii)}$$

$$\text{curl}\,\vec{E}=-\frac{\partial\vec{B}}{\partial t} \qquad \text{(iii)}$$

$$\text{curl}\,\vec{H}=\vec{J}+\frac{\partial\vec{D}}{\partial t} \qquad \text{(iv)}$$

Take scalar (dot) product of Eq. (iii) and Eq. (iv) with $\vec{H}$ and $\vec{E}$ respectively, i.e.

$$\vec{H}\cdot\text{curl}\,\vec{E}=-\vec{H}\cdot\frac{\partial\vec{B}}{\partial t} \qquad \text{(v)}$$

$$\text{and}\quad \vec{E}\cdot\text{curl}\,\vec{H}=\vec{E}\cdot\vec{J}+\vec{E}\cdot\frac{\partial\vec{D}}{\partial t} \qquad \text{(vi)}$$

Subtract Eq. (vi) from Eq. (v), i.e.

$$\vec{H}\cdot\text{curl}\,\vec{E}-\vec{E}\cdot\text{curl}\,\vec{H}=-\vec{H}\cdot\frac{\partial\vec{B}}{\partial t}-\vec{E}\cdot\vec{J}-\vec{E}\cdot\frac{\partial\vec{D}}{\partial t}$$

$$=-\left(\vec{H}\cdot\frac{\partial\vec{B}}{\partial t}+\vec{E}\cdot\frac{\partial\vec{D}}{\partial t}\right)-\vec{E}\cdot\vec{J}$$

$$\text{or}\quad \text{div}\left(\vec{E}\times\vec{H}\right)=-\left(\vec{H}\cdot\frac{\partial\vec{B}}{\partial t}+\vec{E}\cdot\frac{\partial\vec{D}}{\partial t}\right)-\vec{E}\cdot\vec{J} \qquad \text{(vii)}$$

$$\left[\because \text{div}\left(\vec{A}\times\vec{B}\right)=\vec{B}\cdot\text{curl}\,\vec{A}-\vec{A}\cdot\text{curl}\,\vec{B}\right]$$

Using the relations $\vec{B}=\mu\vec{H}$ and $\vec{D}=\varepsilon\vec{E}$, we can get

$$\vec{E}\cdot\frac{\partial\vec{D}}{\partial t}=\vec{E}\cdot\frac{\partial}{\partial t}\left(\varepsilon\vec{E}\right)=\frac{1}{2}\varepsilon\frac{\partial}{\partial t}(E)^2=\frac{\partial}{\partial t}\left(\frac{1}{2}\vec{E}\cdot\vec{D}\right) \qquad \left[\because E^2=\vec{E}\cdot\vec{E}\right]$$

$$\vec{H}\cdot\frac{\partial\vec{B}}{\partial t}=\vec{H}\cdot\frac{\partial}{\partial t}\left(\mu\vec{H}\right)=\frac{1}{2}\mu\frac{\partial}{\partial t}(H)^2=\frac{\partial}{\partial t}\left(\frac{1}{2}\vec{H}\cdot\vec{B}\right) \qquad \left[\because H^2=\vec{H}\cdot\vec{H}\right]$$

Now Eq. (vii) can be written as

$$\text{div}\left(\vec{E}\times\vec{H}\right)=-\frac{\partial}{\partial t}\left[\frac{1}{2}\left(\vec{H}\cdot\vec{B}+\vec{E}\cdot\vec{D}\right)\right]-\vec{E}\cdot\vec{J}$$

$$\text{or}\quad \vec{E}\cdot\vec{J}=-\frac{\partial}{\partial t}\left[\frac{1}{2}(\vec{H}\cdot\vec{B}+\vec{E}\cdot\vec{D})\right]-\text{div}(\vec{E}\times\vec{H})$$

Integrating Eq. (viii) over a volume V enclosed by a surface S, we get

$$\int_V\vec{E}\cdot\vec{J}\,dV=-\int_V\left\{\frac{\partial}{\partial t}\left[\frac{1}{2}(\vec{H}\cdot\vec{B}+\vec{E}\cdot\vec{D})\right]\right\}dV-\int_V\text{div}(\vec{E}\times\vec{H})\,dV$$

$$\text{or}\quad \int_V\vec{E}\cdot\vec{J}\,dV=-\int_V\left[\left(\frac{1}{2}\mu H^2+\frac{1}{2}\varepsilon E^2\right)\right]dV-\int_S(\vec{E}\times\vec{H})\cdot d\vec{S}$$

$$\left[\because\ \vec{B}=\mu\vec{H},\vec{D}=\varepsilon\vec{E}\ \text{ and }\ \int_V\text{div}(\vec{E}\times\vec{H})\,dV=\int_S(\vec{E}\times\vec{H})\cdot d\vec{S}\right]$$

Eq. (ix) can also be written as

$$\int_V\vec{E}\cdot\vec{J}\,dV=-\frac{\partial}{\partial t}\int_V\left[\frac{1}{2}\mu H^2+\frac{1}{2}\varepsilon E^2\right]dV-\int_S(\vec{E}\times\vec{H})\cdot d\vec{S}$$

Interpretation

(a) $\int_V(\vec{E}\cdot\vec{J})dV$: This term represents the rate of energy transferred into the electromagnetic field through the motion of charges in the volume V, i.e. the total power dissipated in a volume V.

(b) $\frac{\partial}{\partial t}\int_V\left[\frac{1}{2}\mu H^2+\frac{1}{2}\varepsilon E^2\right]dV$: The terms $\frac{1}{2}\mu H^2$ and $\frac{1}{2}\varepsilon E^2$ represent the energy stored in electric and magnetic fields respectively and their sum will be equal to the total energy stored in electromagnetic field. Therefore, this total expression represents the rate of decrease of energy stored in volume V due to electric and magnetic fields.

(c) $\int_S(\vec{E}\times\vec{H})\cdot d\vec{S}$: This term represents the amount of electromagnetic energy crossing the closed surface per second or the rate of flow of outward energy through the surface S enclosing volume V. The vector $(\vec{E}\times\vec{H})$ is known as the *poynting vector* $\vec{S}$ or $\vec{S}=(\vec{E}\times\vec{H})$.

Equation (x) is also known as *poynting theorem* or *work-energy theorem*, according to which the power transferred into the electromagnetic field is equal to the sum of the time rate of change of electromagnetic energy within a certain volume and the time rate of the energy flowing out through the boundary surface. This is also called as the *energy conservation* law in electromagnetism.

10.31 WAVE PROPAGATION IN BOUNDED SYSTEM: WAVEGUIDE

The first waveguide was proposed by *Thomson* in 1893 and was experimentally verified by *Lodge* in 1894. A structure that can guide waves like electromagnetic waves, light waves or sound waves is called waveguide. For each type of the wave there are different types of waveguides. For example, depending on the frequency of electromagnetic wave the waveguide can be constructed from either conductive or dielectric material.

Scanned by CamScanner

Such electromagnetic waveguides are especially useful in the microwave and optical frequency ranges. The waveguides used at optical frequencies (optical waveguides) are typically dielectric waveguides. In such waveguides, a dielectric material with high permittivity (i.e. high index of refraction) is surrounded by a material with lower permittivity. This type of structure guides optical waves by the process of total internal reflection. The most common optical waveguide is optical fibre. On the other hand, a structure that guides sound waves is called an *acoustic waveguide*. A duct for sound propagation also behaves as a transmission line. The duct contains some medium like air which supports the propagation of sound waves.

10.31.1 Electromagnetic Waveguides

The original and the most common meaning of waveguide is a hollow metal pipe used for guiding the waves. The electromagnetic waves in such waveguides may be imagined as waves travelling down the guide in a zig-zag path as these waves are repeatedly reflected between opposite walls of the guide (for example, rectangular waveguide). The first mathematical analysis of the propagating modes (waves) within a hollow metal cylinder was performed by *Rayleigh* in 1897.

To function properly, a waveguide must have a certain minimum diameter relative to the wavelength of the signal. If the waveguide is too narrow or the frequency is too low (the wavelength is too long), the electromagnetic field cannot propagate. There is a minimum frequency, known as *cutoff frequency* for the propagation of the wave, i.e. a wave can propagate only if its frequency is larger than the cutoff frequency. The cutoff frequency is decided by the dimensions of the waveguide.

10.31.2 Modes in Waveguides

In order to analyze the mode (wave) propagation in the waveguide, we solve the Maxwell's equations along with appropriate bounding conditions determined by the properties of the materials and their interfaces. These equations admit multiple solutions, or modes, which are origin functions of the equation system. The propagation of the waveguide modes depends on the operating wavelength and polarization, and shape and size of the waveguide. The longitudinal mode can be realised in a cavity (closed end waveguide), the longitudinal mode is particularly standing wave pattern formed by the waves confined in the cavity. However, a number of transverse modes can be excited in the waveguide, which are classified below.

(1) ***Transverse Electric (TE) Modes:*** These modes do not have electric field in the direction of propagation. So electric field vector is in transverse direction.

(2) ***Transverse Magnetic (TM) Modes:*** These modes have no magnetic field in the direction of propagation. So magnetic field vector is in transverse direction.

(3) ***Transverse Electromagnetic (TEM) Modes:*** These modes have no electric and magnetic fields in the direction of mode propagation. In hollow waveguides, TEM modes are not possible because as per Maxwell's equation the electric field then must have zero divergence, zero curl and be zero at the boundaries. This will result in a zero field or $\nabla^2 \vec{E} = 0$. However, TEM modes can propagate in a coaxial cable.

(4) ***Hybrid Modes:*** These modes have both electric and magnetic field components in the direction of propagation. The mode for which the cutoff frequency is the minimum is called the *fundamental mode*. For example, Transverse Electric TE_{10} mode is the fundamental mode for rectangular waveguide whereas TE_{11} mode is the fundamental mode for circular waveguide.

10.32 COAXIAL CABLE

A transmission line is the material medium that forms a path from one place to another for directing the transmission of energy like electromagnetic waves, acoustic waves or electric power. Open wire transmission lines have the property that the electromagnetic wave propagating down the line extends into the space surrounding the parallel wires. The transmission lines cannot be bent, twisted or otherwise shaped without changing their characteristic impedance. They also cannot be attached to anything conductive, as the extended fields will induce currents in the nearby conductors. This will cause unwanted radiation and detuning of the line. However, this problem is solved by coaxial lines that confine the electromagnetic wave to the area inside the cable, as the coaxial cable consists of a round conducting wire surrounded by an insulating spaces (called the dielectric), surrounded by a cylindrical conducting sheath that is usually surrounded by a final insulating layer (called jacket). Here the transmission of energy occurs totally through the dielectric inside the cable between the conductors. Coaxial lines (cables) can therefore be bent and moderately twisted without negative effects. Also, they can be strapped to conductive supports without inducing unwanted currents in them. Coaxial cable is used as a high-frequency transmission line to carry a high frequency or broadband signals. Coaxial cables may be flexible or rigid depending on the type of sheath. Rigid types have a solid sheath, whereas flexible types have an interwoven sheath usually of thin copper wire. The inner insulator has a significant effect on the cable's properties such as its characteristic impedance and its attenuation. The characteristic impedance in ohms is calculated from the ratio of the inner diameter d and outer diameter D of the dielectric and the dielectric constant ε_r as below

$$Z_0 = \frac{138}{\sqrt{\varepsilon_r}} \log\left(\frac{D}{d}\right)$$

The most common impedances that are widely used are 50 or 52 Ω for industrial and commercial radio frequency applications, and 75 Ω for domestic television and radio, although other impedances are available for specified applications

10.32.1 Signal Propagation in Coaxial Cables

In radio-frequency applications up to a few GHz (10^9Hz) the wave propagates only in the transverse electromagnetic (TEM) mode which means the electric and magnetic fields are both perpendicular to the direction of propagation. However, above a certain cutoff frequency, transverse electric (TE) and/or transverse magnetic (TM) modes can also propagate, as they do in a waveguide. It is usually undesirable to transmit signals above the cutoff frequency, since it may cause multiple modes with different phase velocities to propagate, interfering with each other. This cutoff frequency is roughly inversely proportional to the outer diameter.

10.32.2 Uses of Coaxial Cables

Short length coaxial cables are commonly used to connect home video equipment and in measurement electronics. They are used to be common for implementing computer networks (particularly ethernet) but twisted pair cables have replaced them in most applications except in the growing consumer cable modem market for broadband internet access.

Long distance coaxial cable is used to connect radio networks and television networks, though this has largely been superseded by other more high-tech methods which use fibre optics, satellite etc. Micro coaxial cables are used in a range of consumer devices, military, equipment and also in ultrasound scanning equipment.

Scanned by CamScanner

10.33 SUMMARY

The topics covered in this chapter are summarised below.

(1) The separation of charges produces electric field, whereas the motion of charges generates current and hence the magnetic field. When these fields are time varying they are coupled with each other through the Maxwell's equations. With the help of the Maxwell's equations, we can derive wave equation, based on which the propagation of electromagnetic waves can be investigated in different media.

(2) If a charge is distributed continuously in a medium, it can be expressed in terms of a physical quantity known as charge density. There are three types of charge densities, namely linear charge density λ, surface charge density σ and volume charge density ρ.

(3) Gauss's law of electricity states that the net outward electric flux through any closed surface drawn in an electric field is equal to $1/\epsilon_0$ times the total charge enclosed within the surface. Here it is clear that the charge located outside the surface does not contribute to the electric flux and hence the electric field.

(4) The del operator is the differential operator, which is represented by $\vec{\nabla}$ and is given by $\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$. It is not a vector in itself, but when it operates on a scalar function it provides the resultant as a vector. $\vec{\nabla}F$ does not mean a multiplication of $\vec{\nabla}$ with F rather it is an instruction to differentiate. Here we should say that $\vec{\nabla}$ is a vector operator that acts upon F.

(5) The gradient of a scalar field F is a vector quantity which represents both the magnitude and the direction of the maximum space rate of increase of F. The gradient of a vector field cannot be realised.

(6) The divergence of a vector field $\vec{A}$ is defined as the net outward flux per unit volume over a closed surface. The gradient at a given point is a measure of how much the vector $\vec{A}$ spreads out, i.e. diverges, from that point.

(7) The curl of a vector field $\vec{A}$ is a rotational vector. Its magnitude would be the maximum circulation of $\vec{A}$ per unit area. Its direction is the normal direction of the area when the area is oriented so as to make the circulation maximum. The curl of $\vec{A}$ at some point O is a measure of how much the vector $\vec{A}$ curls around the point O.

(8) For a function $g(x)$, the fundamental theorem of calculus states that $\int_{a_1}^{a_2}\left(\frac{dg}{dx}\right)dx = g(a_2) - g(a_1)$. It means the total change in the function $g(x)$ can be obtained by simply subtracting the values of the function at the points a_1 and a_2.

(9) For a scalar function F, $\int_{a_1}^{a_2}\vec{\nabla}F\cdot d\vec{l} = F(a_2) - F(a_1)$. This is called the fundamental theorem for gradient, according to which the integral of a derivative over end points a_1 and a_2 is given by the value of the function at the boundaries, i.e. the points a_1 and a_2.

(10) According to Gauss's or Green's theorem, $\int_V(\vec{\nabla}\cdot\vec{F})dV = \oint_S \vec{F}\cdot d\vec{S}$. Here dV is the volume element and $d\vec{S}$ is the surface element. This theorem states that the integral of a derivative (here the divergence) over a region (here the volume) is equal to the value of the function at the boundary (here the surface). Since the boundary of a volume is always a closed surface, the R.H.S. is the integral over closed surface. Evidently this theorem converts the volume integral into the surface integral.

(11) Stokes' theorem states that the integral of the curl of a vector function over a patch of surface is equal to the value of the function at the perimeter of the patch. So here the derivative is the curl, region is the surface and the boundary is the perimeter of the patch of the surface. Therefore,

$$\int_S(\vec{\nabla}\times\vec{F})\cdot d\vec{S} = \oint_C \vec{F}\cdot d\vec{l}$$

Clearly, this theorem converts the surface integral into the line integral.

(12) If we represent the charge density by ρ and the current density by $\vec{J}$, then the continuity equation is written as $\vec{\nabla}\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0$. In this equation, the second term is the time rate of change of charge density, i.e. the current. The first term represents the divergence of $\vec{J}$, i.e. the measure of spreading of the current through a surface. This spreading is balanced by the time rate of change of charge density. Therefore, this equation just tells us the conservation of charges.

(13) When the charges are in motion, the electric and magnetic fields are associated with this motion which will have variations in both the space and time. These electric and magnetic fields are interrelated. This phenomenon is called electromagnetism, which is summarised by the set of equations, known as Maxwell's equations. The Maxwell's equations are nothing but are the representation of the basic laws of electromagnetism. If the electric field is represented by $\vec{E}$, magnetic field by $\vec{B}$, current density by $\vec{J}$, then the Maxwell's equations are, $\vec{\nabla}\cdot\vec{D} = \rho$, $\vec{\nabla}\cdot\vec{B} = 0$, $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$ and $\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}$. Here for the linear media $\vec{B} = \mu\vec{H}$ and $\vec{D} = \epsilon\vec{E}$.

(14) The Maxwell's first equation $\vec{\nabla}\cdot\vec{D} = \rho$ represents the Gauss's Law for electricity.

(15) The area integral of a vector field determines the net source of the field (function). Since the Maxwell's second equation $\vec{\nabla}\cdot\vec{B} = 0$ can be written as $\oint\vec{B}\cdot d\vec{S} = 0$, it says that the net magnetic flux out of any closed surface is zero. This is because the magnetic flux directed inward towards the south pole, of a magnetic dipole kept in any closed surface, will be equal to the flux outward the north pole. Therefore, the net flux is zero for dipole sources. For a magnetic monopole source the value of the area integral $\oint\vec{B}\cdot d\vec{S}$ would be finite. Since the divergence of a vector field is proportional to the density of point source, the Maxwell's second equation simply says that there are no magnetic monopoles.

(16) The Maxwell's third equation $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$ when written in the integral form states that the line integral of the electric field around a closed loop is equal to the negative rate of change of the magnetic flux through the area enclosed by the loop. The line integral basically is the generated voltage or emf in the loop. Therefore, the physical interpretation of Maxwell's third equation is that the changing magnetic field induces electric field.

(17) For static electric field $\vec{E}$, i.e. when $\vec{E}$ does not change with time, the second term of the R.H.S. of the Maxwell's fourth equation $\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}$ vanishes and then the integral form of this equation says that the line integral of the magnetic field around a closed loop is proportional to

Scanned by CamScanner

the electric current flowing through the loop. This form of the Maxwell's equation is useful for calculating the magnetic field for simple geometries. However, this equation more specifically reveals that the changing electric field induces magnetic field as $\frac{\partial \vec{D}}{\partial t}$ is nothing but the rate of change of electric field. This is complimentary to the meaning of the Maxwell's third equation. Therefore, they together form the electromagnetic fields or electromagnetic waves, where both electric and magnetic fields propagate together and the change in one field induces the other field.

(18) The vector $\vec{D}(=\varepsilon\vec{E})$ is called the displacement vector and the term $\frac{\partial \vec{D}}{\partial t}$ is called the displacement current. The displacement current is postulated in a dielectric when electric stress or potential gradient is varied. It is different from a normal or conduction current, as it is not accompanied by the motion of current carriers in the dielectrics. The concept of introducing this term in the Ampere's law was given by Maxwell for the completion of his electromagnetic equations, as the Ampere's law is valid only for the steady state or static fields. In order to make it consistent with the time varying fields, i.e. also for satisfying the continuity equation, this additional term is required.

(19) Electromagnetic fields, i.e. the electric field and magnetic field are transverse in nature that always remain perpendicular to each other and also in the transverse direction to the direction of wave propagation.

(20) The electromagnetic waves carry energy when they propagate and there is an energy density associated with both the electric and magnetic fields. The amount of energy flowing through unit area, perpendicular to the direction of energy propagation per unit time, i.e. the rate of energy transport per unit area, is called the poynting vector. It is also known as instantaneous energy flux density and is represented by $\vec{S} = \vec{E}\times\vec{H}$.

(21) There are structures that can guide waves like electromagnetic waves, light waves or sound waves. These structures are called waveguides. For each type of the wave there are different types of waveguides. For example, depending on the frequency of electromagnetic wave the waveguide can be constructed from either conductive or dielectric material. Such electromagnetic waveguides are especially useful in the microwave and optical frequency ranges. The waveguides used at optical frequencies (optical waveguides) are typically dielectric waveguides.

(22) The waveguides support different types of wave propagation, which are called modes. In order to analyze the mode propagation, we solve the Maxwell's equations along with appropriate boundary conditions determined by the properties of the materials and their interfaces. These equations admit multiple solutions, or modes, which are origin functions of the equation system. The mode propagation in the waveguides depends on the operating wavelength and polarization, and shape and size of the waveguide.

(23) A number of transverse modes can be excited in the waveguide. For example, transverse electric (TE) modes which do not have electric field in the direction of propagation, transverse magnetic (TM) modes which have no magnetic field in the direction of propagation, transverse electromagnetic (TEM) modes which have no electric and magnetic fields in the direction of mode propagation. In hollow waveguides, TEM modes are not possible. However, TEM modes can propagate in a coaxial cable. Also hybrid modes can be excited which have both electric and magnetic field components in the direction of propagation.

(24) The mode for which the cutoff frequency is the minimum is called the fundamental mode. For example, TE_{10} mode is the fundamental mode for rectangular waveguide whereas TE_{11} mode is the fundamental mode for circular waveguide.

(25) A transmission line forms a path from one place to another for directing the transmission of energy like electromagnetic waves, acoustic waves or electric power. The transmission lines cannot be bent, twisted or otherwise shaped without changing their characteristic impedance. They cannot be attached to anything conductive, as the extended fields will induce currents in the nearby conductors. This will cause unwanted radiation and detuning of the line. However, coaxial lines that confine the electromagnetic wave to the area inside the cable solve this problem. Here the transmission of energy occurs totally through the dielectric inside the cable between the conductors. Coaxial lines (cables) can therefore be bent and moderately twisted without negative effects. Also, they can be strapped to conductive supports without inducing unwanted currents in them.

(26) In radio frequency applications up to a few GHz (10^9Hz) the wave propagates only in the form of transverse electromagnetic (TEM) mode. However, above a certain cutoff frequency, transverse electric (TE) and/or transverse magnetic (TM) modes can also propagate, as they do in a waveguide. It is usually undesirable to transmit signals above the cutoff frequency, since it may cause multiple modes with different phase velocities to propagate, interfering with each other.

SOLVED EXAMPLES

Example 1 Find the value of $\vec{\nabla} r^n$ where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Solution

$$\vec{\nabla} r^n = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)^{n/2}$$

where $r = (x^2+y^2+z^2)^{1/2}$

$$\vec{\nabla} r^n = \left\{\frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}2x\right\}\hat{i} + \left\{\frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}2y\right\}\hat{j} + \left\{\frac{n}{2}(x^2+y^2+z^2)2z\right\}\hat{k}$$

$$= n(x^2+y^2+z^2)^{\left(\frac{n}{2}-1\right)}(\hat{i}x + \hat{j}y + \hat{k}z)$$

$$= nr^{n-2}\vec{r}$$

Example 2 Prove that $\vec{\nabla}\left(\frac{1}{r^n}\right) = -\frac{n}{r^{n+2}}\vec{r}$, where $r = \sqrt{x^2+y^2+z^2}$.

Solution

$$\vec{\nabla}\left(\frac{1}{r^n}\right) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)r^{-n}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)^{\frac{-n}{2}}$$

$$= \left\{\frac{-n}{2}(x^2+y^2+z^2)^{\frac{-n}{2}-1}2x\right\}\hat{i} + \left\{\frac{-n}{2}(x^2+y^2+z^2)^{\frac{-n}{2}-1}2y\right\}\hat{j} + \left\{\frac{-n}{2}(x^2+y^2+z^2)^{\frac{-n}{2}-1}2z\right\}\hat{k}$$

$$= -n(x^2+y^2+z^2)^{-\left(\frac{n+2}{2}\right)}(\hat{i}x + \hat{j}y + \hat{k}z)$$

$$= \frac{-n}{r^{n+2}}\vec{r}$$

Hence proved.

Scanned by CamScanner

Example 29 If the magnitude of $\vec{H}$ in a plane wave is 1 A/m, find the magnitude of $\vec{E}$ for plane wave in free space.

Solution We know that, $\frac{H_0}{E_0} = \sqrt{\frac{\varepsilon_0}{\mu_0}}$ or $E_0 = \left(\sqrt{\frac{\mu_0}{\varepsilon_0}}\right) H_0$

Here $H_0 = 1$ A/m, $\mu_0 = 4\pi \times 10^{-7}$ Wb/A-m

and $\varepsilon_0 = 8.85 \times 10^{-12}$ C/Nm²

$$\therefore E_0 = 1 \times \sqrt{\frac{4\pi \times 10^{-7}}{8.85 \times 10^{-12}}} = 376.72 \text{ V/m}$$

Example 30 Show that equation of continuity div $\vec{J} + \frac{\partial \rho}{\partial t} = 0$ is contained in Maxwell's equations.

Solution According to Maxwell's fourth equation,

$$\text{curl } \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

Taking divergence of both sides, we get

$$\text{div (curl } \vec{H}) = \text{div} \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right)$$

But, div (curl $\vec{H}$) = 0

$$\therefore \text{div} \left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right) = 0 \text{ or div } \vec{J} + \text{div} \left(\frac{\partial \vec{D}}{\partial t}\right) = 0$$

$$\text{or} \quad \text{div } \vec{J} + \frac{\partial}{\partial t}(\text{div} \vec{D}) = 0$$

[∴ space and time operations are interchangeable]

From Maxwell's first equation,

$$\text{div } \vec{D} = \rho,$$

where ρ is the surface charge density.

$$\therefore \text{div } \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

Example 31 Considering that all the energy from a 1000 W lamp is radiated uniformly, calculate the average value of the intensity of electric field of radiation at a distance of 2 m from the lamp.

Solution As the total power (P) is radiated uniformly, energy flux per unit area per second at a distance r from the lamp is given by

$$S = \frac{P_0}{4\pi r^2} = \frac{1000}{4\pi (2)^2} \text{ W/m}^2$$

From the poynting theorem, $|\vec{S}| = |\vec{E} \times \vec{H}| = EH \sin 90°$

∵ E and H are perpendiculars to each other,

$$\therefore EH = \frac{1000}{16\pi} \quad \text{(i)}$$

$$\text{But} \quad \frac{E}{H} = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 376.72 \text{ ohm} \quad \text{(ii)}$$

Multiplying Eqs. (i) and (ii), we get

$$EH \times \frac{E}{H} = \frac{376.72 \times 1000}{16\pi} \Rightarrow E = \left(\frac{376.72 \times 1000}{16 \times 3.14}\right)^{1/2} = 86.59 \text{ V/m}$$

Example 32 If the relative permittivity of distilled water be 81. Calculate refractive index and velocity of light in it.

Solution We know that, $\mu' = \sqrt{\frac{\mu\varepsilon}{\mu_0 \varepsilon_0}}$ and $v = \frac{c}{\mu}$

Here $\varepsilon = 81\varepsilon_0$ and for distilled water $\mu \approx \mu_0$

$$\therefore \mu' = \sqrt{81} = 9 \quad \text{and} \quad v = \frac{3 \times 10^8}{9} = 3.33 \times 10^7 \text{ m/sec}$$

OBJECTIVE TYPE QUESTIONS

Q.1 The direction of $\vec{\nabla}\phi$ is always
(a) parallel to the surface
(b) perpendicular to the surface
(c) depends upon the shape of surface
(d) none of these

Q.2 The vector field $\vec{A}$ is solenoidal if
(a) $\vec{\nabla} \times \vec{A} = 0$
(b) $\vec{A} = 0$
(c) $\vec{\nabla} \cdot \vec{A} = 0$
(d) none of these

Q.3 A vector field $\vec{A}$ is conservative if
(a) $\vec{\nabla} \times \vec{A} \neq 0$
(b) $\vec{\nabla} \cdot \vec{A} = 0$
(c) $\vec{A} = \vec{\nabla}\phi$
(d) none of these

Q.4 The vector function $\vec{\nabla} \cdot \vec{A}$ represents
(a) the total flux over any arbitrary closed surface
(b) the inward flux density at the point (x,y,z)
(c) the outward flux density at the point (x,y,z)
(d) none of these

Q.5 The vector field is irrotational if
(a) $\vec{\nabla} \times \vec{A} = 0$
(b) $\vec{\nabla} \cdot \vec{A} = 1$
(c) $\vec{\nabla} \cdot \vec{A} = 0$
(d) $\vec{\nabla} \times \vec{A} = 1$

Q.6 If $\vec{a}$ is a constant vector then $\vec{\nabla} \times (\vec{a} \times \vec{r})$ is equal to
(a) zero
(b) $2\vec{a}$
(c) $\vec{a}/2$
(d) $\vec{a}$

Q.7 For a conservative field $\vec{E}$
(a) $\oint \vec{E} \cdot d\vec{l} = 0$
(b) $\oint \vec{E} \cdot d\vec{s} = 0$
(c) $\vec{\nabla} \cdot \vec{E} \neq 0$
(d) $\vec{\nabla} \times \vec{E} \neq 0$

Scanned by CamScanner

Q.8 The divergence of curl of a vector is always
(a) 1
(b) $\pi/2$
(c) 1/2
(d) zero

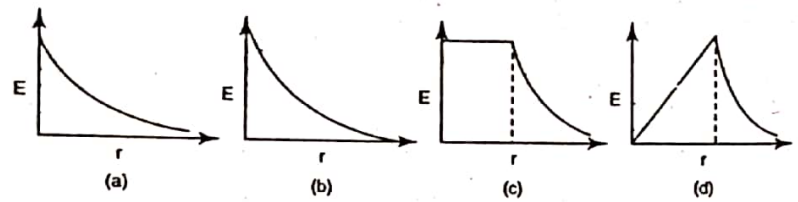
Q.9 The unit of ε_0 in SI units is
(a) Nm^2/C^2
(b) C/S
(c) C^2/Nm^2
(d) N/C

Q.10 The unit of electric flux in SI system of units is
(a) Weber
(b) Gauss
(c) Nm^2/C
(d) N/C

Q.11 A charge of 15 μC is placed at one corner of a cube. The electric flux will be
(a) zero through each face
(b) same through each face
(c) zero through three faces meeting at the location
(d) none of these.

Q.12 The variation of electric field intensity E with distance r from centre of hollow spherical shell of radius R is

Q.13 The electric field intensity E inside a uniformly charged sphere varies with distance r of the observation



point as
(a) $E \propto r$
(b) $E \propto \frac{1}{r}$
(c) $E \propto r^2$
(d) $E \propto \frac{1}{r^2}$

Q.14 The electric field between two oppositely charged plates having equal charge density σ is given by
(a) σ/ε_0
(b) $\sigma/2\varepsilon_0$
(c) zero
(d) $2\sigma/\varepsilon_0$

Q.15 Which of the following is zero
(a) grad div
(b) div grad
(c) curl grad
(d) curl curl

Q.16 The relation between electric field and potential is
(a) $\vec{\nabla}\cdot E = V$
(b) $E = -\nabla^2 V$
(c) $\vec{E} = -\vec{\nabla} V$
(d) $E = \nabla^2 V$

Q.17 The work done in displacing a charge 2C through 0.5m on an equipotential surface is
(a) zero
(b) 4 J
(c) 1J
(d) none of these

Q.18 A point charge q is located at the origin. The amount of work done in bringing a unit positive charge from infinity to the origin is
(a) zero
(b) finite
(c) infinite
(d) none of these

Q.19 Which of the following equations tells us about the existence of the magnetic monopole?
(a) $\text{curl}\,\vec{E} = -\frac{\partial \vec{B}}{\partial t}$
(b) $\text{div}\,\vec{B} = 0$
(c) $\text{div}\,\vec{D} = \rho$
(d) $\text{curl}\,\vec{H} = \vec{J} + \frac{\partial \vec{B}}{\partial t}$

Q.20 Displacement current appears because of
(a) time varying electric field
(b) time varying magnetic field
(c) negative charge only
(d) positive charge only

SHORT-ANSWER QUESTIONS

Q.1 Discuss conservative versus nonconservative fields by giving examples.
Q.2 How do you find the magnetic flux crossing an infinitesimal surface?
Q.3 Provide physical interpretations for the closed surface integrals of any two vectors.
Q.4 What are different ways in which an emf is induced around a loop?
Q.5 State Ampere's circuital law.
Q.6 What is displacement current?
Q.7 State Gauss's law for electric field.
Q.8 State Gauss's law for magnetic field.
Q.9 What is the physical interpretation of Gauss' law for magnetic field?
Q.10 State the law of conservation of charge.
Q.11 How is Gauss's law dependent on Ampere's circuital law?
Q.12 What is Gaussian surface?
Q.13 What is the electric flux through a closed surface surrounding a dipole?
Q.14 Can we apply Gauss's law to calculate the electric field due to electric dipole? Explain.
Q.15 A Gaussian surface encloses no net charge. Does it mean E = 0 on its surface.
Q.16 What do you understand by pointing vector?
Q.17 Write Maxwell's equations.
Q.18 State Faraday's laws of electromagnetic induction.
Q.19 What do you mean by a waveguide?
Q.20 Write an expression for characteristic impedance of a co-axial cable.

PRACTICE PROBLEMS

General Questions

Q.1 Describe gradient of a scalar field in Cartesian co-ordinates. Explain its physical significance.
Q.2 The gradient of a scalar field is a vector. Hence explain how can you produce a vector from a scalar field.

Scanned by CamScanner