CHAPTER 2

Centroid and Moment of Inertia

Under this topic first we will see how to find the areas of given figures and the volumes of given solids. Then the terms centre of gravity and centroids are explained. Though the title of this topic do not indicate the centroid of line segment, that term is also explained, since the centroid of line segment will be useful in finding the surface area and volume of solids using theorems of Pappus and Guldinus. Then the term first moment of area is explained and the method of finding centroid of plane areas and volumes is illustrated. After explaining the term second moment of area, the method of finding moment of inertia of plane figures about *x-x* or *y-y* axis is illustrated. The term product moment of inertia is defined and the mehtod of finding principal moment of inertia is presented. At the end the method of finding mass moment of inertia is presented.

2.1 DETERMINATION OF AREAS AND VOLUMES

In the school education methods of finding areas and volumes of simple cases are taught by many methods. Here we will see the general approach which is common to all cases i.e. by the method of integration. In this method the expression for an elemental area will be written then suitable integrations are carried out so as to take care of entire surface/volume. This method is illustrated with standard cases below, first for finding the areas and latter for finding the volumes:

A: Area of Standard Figures

(*i*) Area of a rectangle

Let the size of rectangle be $b \times d$ as shown in Fig. 2.1. dA is an elemental area of side $dx \times dy$.

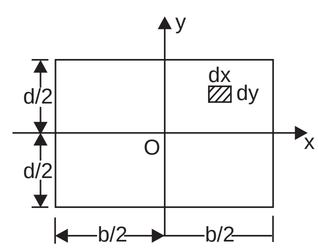


Fig. 2.1

$$\text{Area of rectangle, } A = \oint dA = \int_{-b/2}^{b/2}\int_{-d/2}^{d/2} dx\,dy$$

$$= [x]_{-b/2}^{b/2}\,[y]_{-d/2}^{d/2}$$

$$= bd.$$

If we take element as shown in Fig. 2.2,

$$A = \int_{-d/2}^{d/2} dA = \int_{-d/2}^{d/2} b \cdot dy$$

$$= b\left[y\right]_{-d/2}^{d/2}$$

$$= bd$$

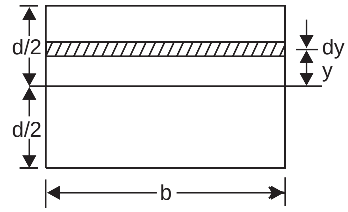


Fig. 2.2

(*ii*) Area of a triangle of base width '*b*' height '*h*'. Referring to Fig. 2.3, let the element be selected as shown by hatched lines

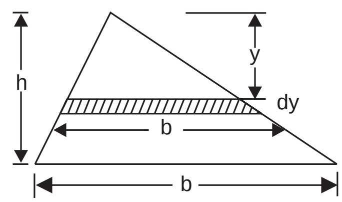


Fig. 2.3

Then $\quad dA = b'dy = b\,\dfrac{y}{h}\,dy$

$$A = \int_0^h dA = \int_0^h b\,\frac{y}{h}\,dy$$

$$= \frac{b}{h}\left[\frac{y^2}{2}\right]_0^h = \frac{bh}{2}$$

(*iii*) Area of a circle

Consider the elemental area $dA = rd\theta\ dr$ as shown in Fig. 2.4. Now,

$$dA = rd\theta\ dr$$

r varies from O to R and θ varies from O to 2π

$$\therefore \qquad A = \int_0^{2\pi}\int_0^{R} rd\theta\, dr$$

$$= \int_0^{2\pi}\left[\frac{r^2}{2}\right]_0^R d\theta$$

$$= \int_0^{2\pi}\frac{R^2}{2}\,d\theta$$

$$= \frac{R^2}{2}\left[\theta\right]_0^{2\pi}$$

$$= \frac{R^2}{2}\,.\,2\pi = \pi R^2$$

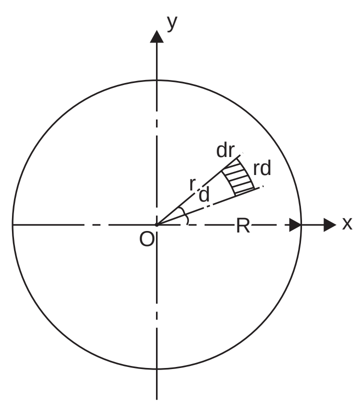


Fig. 2.4

In the above derivation, if we take variation of θ from 0 to π, we get the area of semicircle as $\dfrac{\pi R^2}{2}$ and if the limit is from 0 to $\pi/2$ the area of quarter of a circle is obtained as $\dfrac{\pi R^2}{4}$.

(*iv*) Area of a sector of a circle

Area of a sector of a circle with included angle 2α shown in Fig. 2.5 is to be determined. The elemental area is as shown in the figure

$$dA = rd\theta \cdot dr$$

θ varies from –α to α and r varies from O to R

$$\therefore \quad A = \oint dA = \int_{-\alpha}^{\alpha}\int_{0}^{R} r\,d\theta\,dr$$

$$= \int_{-\alpha}^{\alpha}\left[\frac{r^2}{2}\right]_0^R d\theta = \int_{-\alpha}^{\alpha}\frac{R^2}{2}\,d\theta$$

$$= \left[\frac{R^2}{2}\theta\right]_{-\alpha}^{\alpha} = \frac{R^2}{2}(2\alpha) = R^2\alpha$$

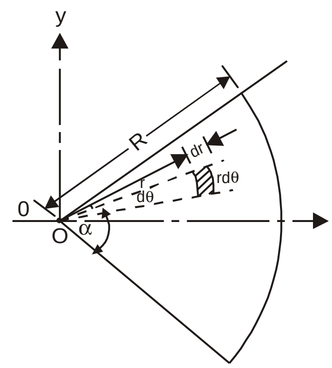


Fig. 2.5

(*v*) Area of a parabolic spandrel

Two types of parabolic curves are possible

(*a*) $y = kx^2$

(*b*) $y^2 = kx$

Case a: This curve is shown in Fig. 2.6.

The area of the element

$$dA = y\,dx$$
$$= kx^2\,dx$$

$$\therefore \quad A = \int_0^a dA = \int_0^a kx^2\,dx$$

$$= k\left[\frac{x^3}{3}\right]_0^a = \frac{ka^3}{3}$$

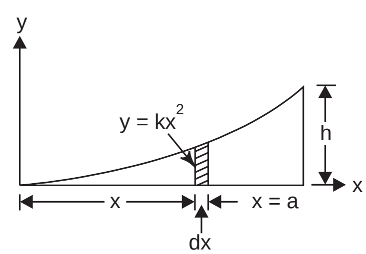


Fig. 2.6

We know, when $x = a,\ y = h$

i.e., $h = ka^2$ or $k = \dfrac{h}{a^2}$

$$\therefore \quad A = \frac{ka^3}{3} = \frac{h}{a^2}\frac{a^3}{3} = \frac{1}{3}ha = \frac{1}{3}rd$$ the area of rectangle of size $a \times h$

Case b: In this case $y^2 = kx$

Referring to Fig. 2.7

$$dA = y\,dx = \sqrt{kx}\,dx$$

$$A = \int_0^a y\,dx = \int_0^a \sqrt{kx}\,dx$$

$$= \sqrt{k}\left[x^{3/2}\,\frac{2}{3}\right]_0^a = \sqrt{k}\,\frac{2}{3}\,a^{3/2}$$

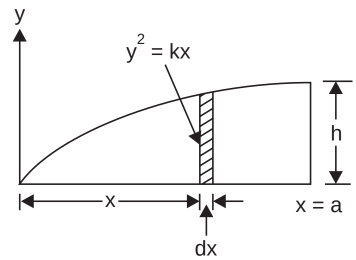


Fig. 2.7

We know that, when $x = a$, $y = h$

$$\therefore \qquad h^2 = ka \quad \text{or} \quad k = \frac{h^2}{a}$$

Hence $\quad A = \dfrac{h}{\sqrt{a}} \cdot \dfrac{2}{3} \cdot a^{3/2}$

i.e., $\quad A = \dfrac{2}{3}ha = \dfrac{2}{3}$rd the area of rectangle of size $a \times h$

(*vi*) Surface area of a cone

Consider the cone shown in Fig. 2.8. Now,

$$y = \frac{x}{h}R$$

Surface area of the element,

$$dA = 2\pi y\, dl = 2\pi \frac{x}{h} R\, dl$$

$$= 2\pi \frac{x}{h} R \frac{dx}{\sin\alpha}$$

$$\therefore \qquad A = \frac{2\pi R}{h\sin\alpha}\left[\frac{x^2}{2}\right]_0^h$$

$$= \frac{\pi R h}{\sin\alpha} = \pi R l$$

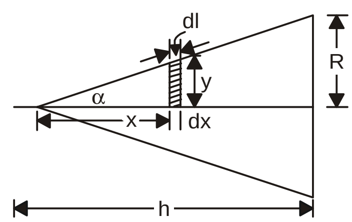


Fig. 2.8

(*vii*) Surface area of a sphere

Consider the sphere of radius R shown in Fig. 2.9. The element considered is the parallel circle at distance y from the diametral axis of sphere.

$$dS = 2\pi x\, R d\theta$$

$$= 2\pi R\cos\theta\, R d\theta, \text{ since } x = R\cos\theta$$

$$\therefore \qquad S = 2\pi R^2 \int_{-\pi/2}^{\pi/2} \cos\theta\, d\theta$$

$$= 2\pi R^2 \left[\sin\theta\right]_{-\pi/2}^{\pi/2}$$

$$= 4\pi R^2$$

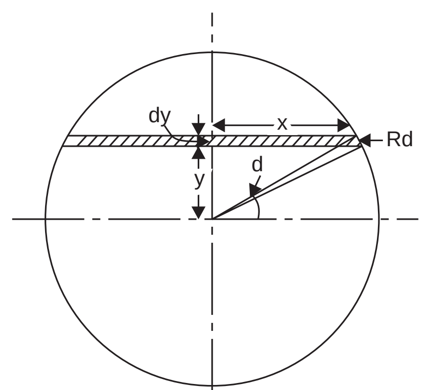


Fig. 2.9

B: Volume of Standard Solids

(*i*) Volume of a parallelpiped.

Let the size of the parallelpiped be $a \times b \times c$. The volume of the element is

$$dV = dx\, dy\, dz$$

$$V = \int_0^a \int_0^b \int_0^c dx\, dy\, dz$$

$$= [x]_0^a\, [y]_0^b\, [z]_0^c \;=\; abc$$

(*ii*) Volume of a cone:

Referring to Fig. 2.8

$$dV = \pi y^2 \cdot dx = \pi \frac{x^2}{h^2} R^2 dx, \quad \text{since } y = \frac{x}{h} R$$

$$V = \frac{\pi}{h^2} R^2 \int_0^h x^2 dx = \frac{\pi}{h^2} R^2 \left[\frac{x^3}{3}\right]_0^h$$

$$= \frac{\pi}{h^2} R^2 \frac{h^3}{3} = \frac{\pi R^2 h}{3}$$

(*iii*) Volume of a sphere

Referring to Fig. 2.9

$$dV = \pi x^2\, dy$$

But $\quad x^2 + y^2 = R^2$

i.e., $\quad x^2 = R^2 - y^2$

$\therefore \quad dV = \pi\,(R^2 - y^2)dy$

$$V = \int_{-R}^{R} \pi \left(R^2 - y^2\right) dy$$

$$= \pi \left[R^2 y - \frac{y^3}{3}\right]_{-R}^{R}$$

$$= \pi \left[R^2 \cdot R - \frac{R^3}{3} - \left\{-R^3 - \frac{(-R)^3}{3}\right\}\right]$$

$$= \pi R^3 \left[1 - \frac{1}{3} + 1 - \frac{1}{3}\right] = \frac{4}{3} \pi R^3$$

The surface areas and volumes of solids of revolutions like cone, spheres may be easily found using theorems of Pappus and Guldinus. This will be taken up latter in this chapter, since it needs the term centroid of generating lines.

2.2 CENTRE OF GRAVITY AND CENTROIDS

Consider the suspended body shown in Fig. 2.10a. The self weight of various parts of this body are acting vertically downward. The only upward force is the force T in the string. To satisfy the equilibrium condition the resultant weight of the body W must act along the line of string 1–1. Now, if the position is changed and the body is suspended again (Fig. 2.10b), it will reach equilibrium condition in a particular position. Let the line of action of the resultant weight be 2–2 intersecting 1–1 at G. It is obvious that if the body is suspended in any other position, the line of action of resultant weight W passes through G. This point is called the centre of gravity of the body. Thus *centre of gravity can be defined as the point through which the resultant of force of gravity of the body acts*.

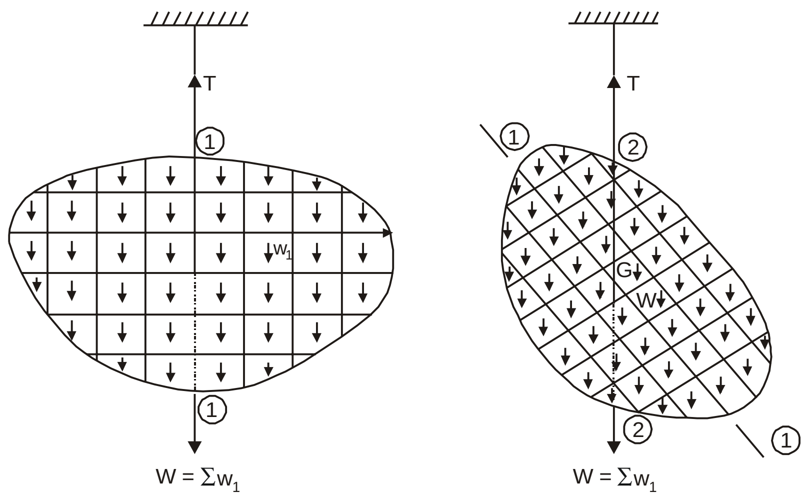


Fig. 2.10

The above method of locating centre of gravity is the practical method. If one desires to locating centre of gravity of a body analytically, it is to be noted that the resultant of weight of various portions of the body is to be determined. For this Varignon's theorem, which states the moment of resultant force is equal to the sum of moments of component forces, can be used.

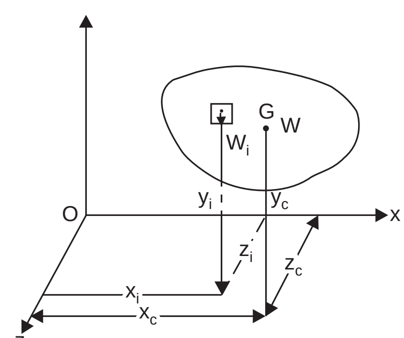


Fig. 2.11

Referring to Fig. 2.11, let W_i be the weight of an element in the given body. W be the total weight of the body. Let the coordinates of the element be x_i, y_i, z_i and that of centroid G be x_c, y_c, z_c. Since W is the resultant of W_i forces,

$$W = W_1 + W_2 + W_3 \ldots$$
$$= \Sigma W_i$$

and $$Wx_c = W_1x_1 + W_2x_2 + W_3x_3 + \ldots$$

$$\left.\begin{aligned} \therefore \quad & Wx_c = \Sigma W_i x_i = \oint x\,dw \\ \text{Similarly,} \quad & Wy_c = \Sigma W_i y_i = \oint y\,dw \\ \text{and} \quad & Wz_c = \Sigma W_i z_c = \oint z\,dw \end{aligned}\right\} \quad \text{Eqn. (2.1)}$$

If M is the mass of the body and m_i that of the element, then

$M = \frac{W}{g}$ and $m_i = \frac{W_i}{g}$, hence we get

$$\left.\begin{aligned} Mx_c &= \Sigma m_i x_i = \oint x_i dm \\ My_c &= \Sigma m_i y_i = \oint y_i dm \\ \text{and} \quad Mz_c &= \Sigma m_i z_i = \oint z_i dm \end{aligned}\right\} \quad \text{Eqn. (2.2)}$$

If the body is made up of uniform material of unit weight γ, then we know $W_i = Ui\gamma$, where U represents volume, then equation 2.1 reduces to

$$\left.\begin{aligned} Vx_c &= \Sigma V_i x_i = \oint xdV \\ Vy_c &= \Sigma V_i y_i = \oint ydV \\ Vz_c &= \Sigma V_i z_i = \oint zdV \end{aligned}\right\} \quad \text{Eqn. (2.3)}$$

If the body is a flat plate of uniform thickness, in x-y plane, $W_i = \gamma A_i t$ (Ref Fig. 2.12). Hence equation 2.1 reduces to

$$\left.\begin{aligned} Ax_c &= \Sigma A_i x_i = \oint x\, dA \\ Ay_c &= \Sigma A_i y_i = \oint y\, dA \end{aligned}\right\} \quad \text{Eqn. (2.4)}$$

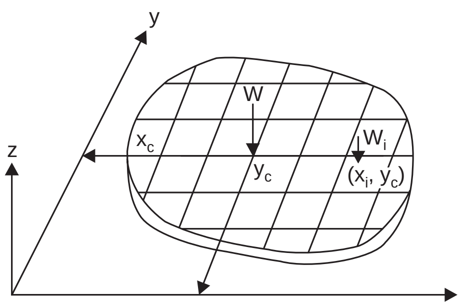


Fig. 2.12

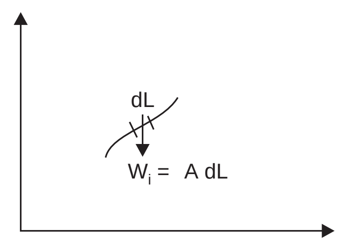


Fig. 2.13

If the body is a wire of uniform cross section in plane x, y (Ref. Fig. 2.13) the equation 2.1 reduces to

$$\left.\begin{aligned} Lx_c &= \Sigma L_i x_i = \oint x\, dL \\ Ly_c &= \Sigma L_i y_i = \oint y\, dL \end{aligned}\right\} \quad \text{Eqn. (2.5)}$$

The term centre of gravity is used only when the gravitational forces (weights) are considered. This term is applicable to solids. Equations 2.2 in which only masses are used the point obtained is termed as *centre of mass*. The central points obtained for volumes, surfaces and line segments (obtained by eqns. 2.3, 2.4 and 2.5) are termed as *centroids.*

2.3 CENTROID OF A LINE

Centroid of a line can be determined using equation 2.5. Method of finding the centroid of a line for some standard cases is illustrated below:

(*i*) Centroid of a straight line:
Selecting the x-coordinate along the line (Fig. 2.14)

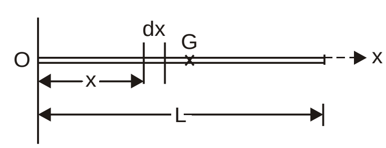


Fig. 2.14

$$Lx_c = \int_0^L x\,dx = \left[\frac{x^2}{2}\right]_0^L = \frac{L^2}{2}$$

$$\therefore \qquad x_c = \frac{L}{2}$$

Thus the centroid lies at midpoint of a straight line, whatever be the orientation of line (Ref. Fig. 2.15).

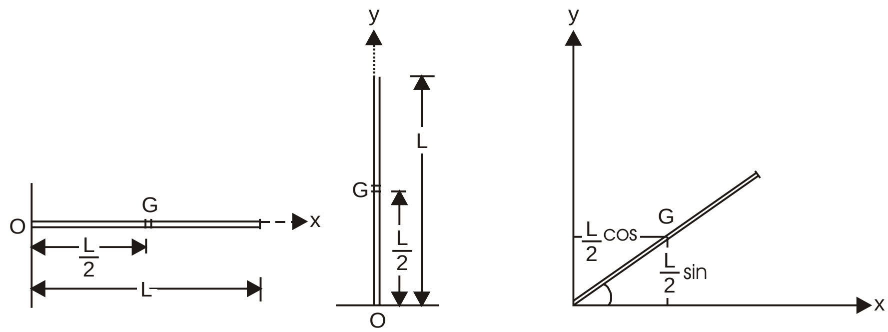


Fig. 2.15

(*ii*) Centroid of an Arc of a Circle

Referring to Fig. 2.16,

$$L = \text{Length of arc} = R\,2\alpha$$

$$dL = Rd\theta$$

Hence from eqn. 2.5

$$x_c L = \int_{-\alpha}^{\alpha} x dL$$

i.e.,

$$x_c\,R\,2\alpha = \int_{-\alpha}^{\alpha} R\ \cos\theta \cdot Rd\theta$$

$$= R^2\left[\sin\theta\right]_{-\alpha}^{\alpha} \qquad \text{(i)}$$

$$\therefore \qquad x_c = \frac{R^2 \times 2\sin\alpha}{2R\alpha} = \frac{R\sin\alpha}{\alpha}$$

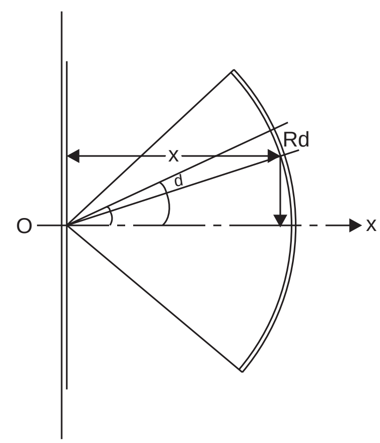


Fig. 2.16

and

$$y_c\,L\int_{-\alpha}^{\alpha} y\,dL = \int_{-\alpha}^{\alpha} R\ \sin\theta \cdot Rd\theta$$

$$= R^2\left[-\cos\theta\right]_{-\alpha}^{\alpha} \qquad \text{(ii)}$$

$$= 0$$

$$\therefore \qquad y_c = 0$$

From equation (i) and (ii) we can get the centroid of semicircle shown in Fig. 2.17 by putting $\alpha = \pi/2$ and for quarter of a circle shown in Fig. 2.18 by putting α varying from zero to $\pi/2$.

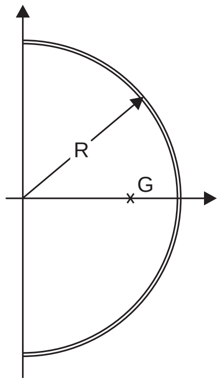


Fig. 2.17

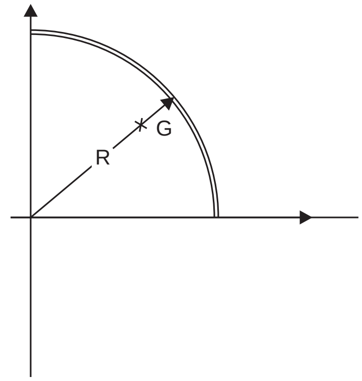


Fig. 2.18

For semicircle $x_c = \dfrac{2R}{\pi}$

$y_c = 0$

For quarter of a circle,

$$x_c = \frac{2R}{\pi}$$

$$y_c = \frac{2R}{\pi}$$

(*iii*) Centroid of composite line segments:

The results obtained for standard cases may be used for various segments and then the equations 2.5 in the form

$$x_c L = \Sigma L_i x_i$$

$$y_c L = \Sigma L_i y_i$$

may be used to get centroid x_c and y_c. If the line segments is in space the expression $z_c L = \Sigma L_i z_i$ may also be used. The method is illustrated with few examples below:

Example 2.1 Determine the centroid of the wire shown in Fig. 2.19.

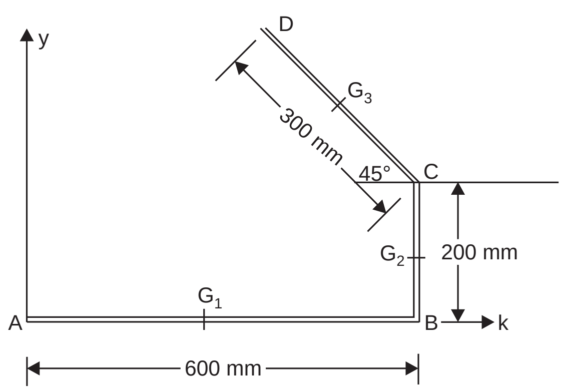


Fig. 2.19

Solution. The wire is divided into three segments *AB, BC* and *CD*. Taking *A* as origin the coordinates of the centroids of *AB, BC* and *CD* are

$$G_1(300, 0);\ G_2(600, 100) \text{ and } G_3(600 - 150 \cos 45°;\ 200 + 150 \sin 45°)$$

i.e. $G_3(493.93,\ 306.07)$

$$L_1 = 600 \text{ mm}, L_2 = 200 \text{ mm}, L_3 = 300 \text{ mm}$$

$\therefore$ Total length $L = 600 + 200 + 300 = 1100$ mm

$\therefore$ From the eqn. $Lx_c = \Sigma L_i x_i$, we get

$$1100\ x_c = L_1x_1 + L_2x_2 + L_3x_3$$
$$= 600 \times 300 + 200 \times 600 + 300 \times 493.93$$

$\therefore$ $x_c = 407.44$ mm **Ans.**

Now, $Ly_c = \Sigma L_i y_i$

$$1100\ y_c = 600 \times 0 + 200 \times 100 + 300 \times 306.07$$

$y_c = 101.66$ mm **Ans.**

Example 2.2 Locate the centroid of the uniform wire bent as shown in Fig. 2.20.

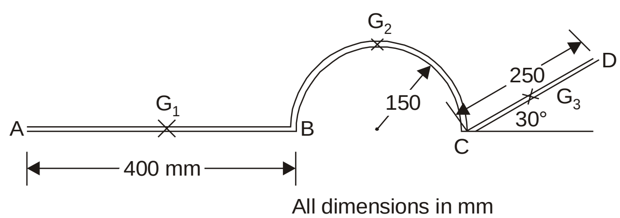


Fig. 2.20

Solution. The composite figure is divided into 3 simple figures and taking *A* as origin coordinates of their centroids noted as shown below:

AB—a straight line

$L_1 = 400$ mm, $G_1\ (200, 0)$

BC—a semicircle

$$L_2 = 150\ \pi = 471.24, \qquad G_2\left(475, \frac{2 \times 150}{\pi}\right)$$

i.e., $G_2\ (475,\ 92.49)$

CD—a straight line

$$L_3 = 250;\ x_3 = 400 + 300 + \frac{250}{2} \cos 30° = 808.25 \text{ mm}$$

$$y_3 = 125 \sin 30 = 62.5 \text{ mm}$$

$\therefore$ Total length $L = L_1 + L_2 + L_3 = 1121.24$ mm

$\therefore$ $Lx_c = \Sigma L_i x_i$ gives

$$1121.24\ x_c = 400 \times 200 + 471.24 \times 475 + 250 \times 808.25$$

$x_c = 451.20$ mm **Ans.**

$Ly_c = \Sigma L_i y_i$ gives

$$1121.24\, y_c = 400 \times 0 + 471.24 \times 95.49 + 250 \times 62.5$$

$$y_c = 54.07 \text{ mm}$$ **Ans.**

Example 2.3 Locate the centroid of uniform wire shown in Fig. 2.21. Note: portion AB is in x-z plane, BC in y-z plane and CD in x-y plane. AB and BC are semi circular in shape.

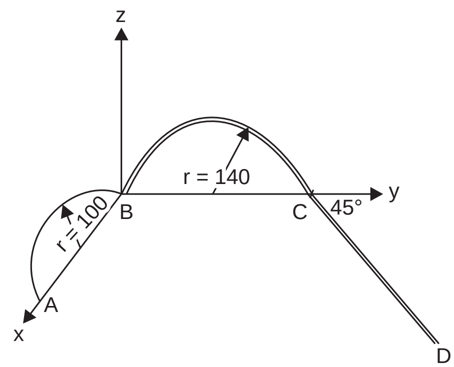


Fig. 2.21

Solution. The length and the centroid of portions *AB, BC* and *CD* are as shown in table below:

Table 2.1

Portion	L_i	x_i	y_i	z_i
AB	100π	100	0	$\frac{2\times100}{\pi}$
BC	140π	0	140	$\frac{2\times140}{\pi}$
CD	300	300 sin 45°	280 + 300 cos 45° = 492.13	0

$\therefore$ $$L = 100\pi + 140\pi + 300 = 1053.98 \text{ mm}$$

From eqn. $Lx_c = \Sigma L_i x_i$, we get

$$1053.98\, x_c = 100\pi \times 100 + 140\pi \times 0 + 300 \times 300 \sin 45°$$

$$x_c = 90.19 \text{ mm}$$ **Ans.**

Similarly, $$1053.98\, y_c = 100\pi \times 0 + 140\pi \times 140 + 300 \times 492.13$$

$$y_c = 198.50 \text{ mm}$$ **Ans.**

and $$1053.98\, z_c = 100\pi \times \frac{200}{\pi} + 140\pi \times \frac{2\times140}{\pi} + 300\times0$$

$$z_c = 56.17 \text{ mm}$$ **Ans.**

2.4 FIRST MOMENT OF AREA AND CENTROID

From equation 2.1, we have

$$x_c = \frac{\sum W_i x_i}{W}, \quad y_c = \frac{\sum W_i y_i}{W} \quad \text{and} \quad z_c = \frac{\sum W_i z_i}{W}$$

From the above equation we can make the statement that distance of centre of gravity of a body from an axis is obtained by dividing moment of the gravitational forces acting on the body, about the axis, by the total weight of the body. Similarly from equation 2.4, we have,

$$x_c = \frac{\sum A_i x_i}{A}, \quad y_c = \frac{\sum A_i y_i}{A}$$

By terming $\Sigma A_i x$: as the moment of area about the axis, we can say centroid of plane area from any axis is equal to moment of area about the axis divided by the total area. The moment of area $\Sigma A_i x$: is termed as first moment of area also just to differentiate this from the term $\Sigma A_i x_{\square}^2$, which will be dealt latter. It may be noted that since the moment of area about an axis divided by total area gives the distance of the centroid from that axis, the moment of area is zero about any centroidal axis.

Difference between Centre of Gravity and Centroid

From the above discussion we can draw the following differences between centre of gravity and centroid:

(1) The term centre of gravity applies to bodies with weight, and centroid applies to lines, plane areas and volumes.

(2) Centre of gravity of a body is a point through which the resultant gravitational force (weight) acts for any orientation of the body whereas centroid is a point in a line plane area volume such that the moment of area about any axis through that point is zero.

Use of Axis of Symmetry

Centroid of an area lies on the axis of symmetry if it exits. This is useful theorem to locate the centroid of an area.

This theorem can be proved as follows:

Consider the area shown in Fig. 2.22. In this figure y-y is the axis of symmetry. From eqn. 2.4, the distance of centroid from this axis is given by:

$$\frac{\sum A_i x_i}{A}$$

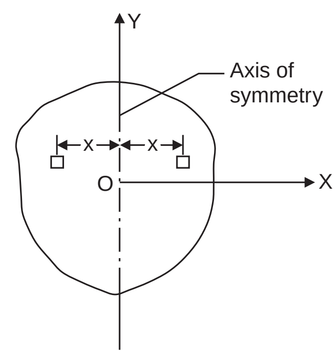


Fig. 2.22

Consider the two elemental areas shown in Fig. 2.22, which are equal in size and are equidistant from the axis, but on either side. Now the sum of moments of these areas cancel each other since the areas and distances are the same, but signs of distances are opposite. Similarly, we can go on considering an area on one side of symmetric axis and corresponding image area on the other side, and prove that total moments of area ($\Sigma A_i x_i$) about the symmetric axis is zero. Hence the distance of centroid from the symmetric axis is zero, i.e. centroid always lies on symmetric axis.

Making use of the symmetry we can conclude that:

(1) Centroid of a circle is its centre (Fig. 2.23);

(2) Centroid of a rectangle of sides b and d is at distance $\frac{b}{2}$ and $\frac{d}{2}$ from the corner as shown in Fig. 2.24.

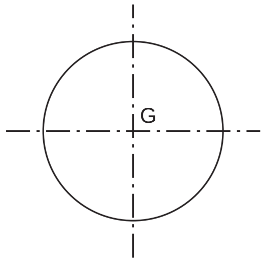


Fig. 2.23

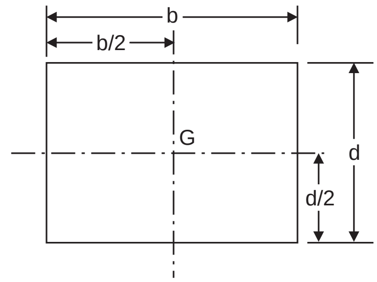


Fig. 2.24

Determination of Centroid of Simple Figures From First Principle

For simple figures like triangle and semicircle, we can write general expression for the elemental area and its distance from an axis. Then equations 2.4

$$\bar{y} = \frac{\int y dA}{A}$$

$$\bar{x} = \frac{\int x dA}{A}$$

The location of the centroid using the above equations may be considered as finding centroid from first principles. Now, let us find centroid of some standard figures from first principles.

Centroid of a Triangle

Consider the triangle *ABC* of base width *b* and height *h* as shown in Fig. 2.25. Let us locate the distance of centroid from the base. Let b_1 be the width of elemental strip of thickness *dy* at a distance *y* from the base. Since ΔAEF and ΔABC are similar triangles, we can write:

$$\frac{b_1}{b} = \frac{h-y}{h}$$

$$b_1 = \left(\frac{h-y}{h}\right)b = \left(1-\frac{y}{h}\right)b$$

$\therefore$ Area of the element

$$= dA = b_1 dy$$

$$= \left(1-\frac{y}{h}\right)b\,dy$$

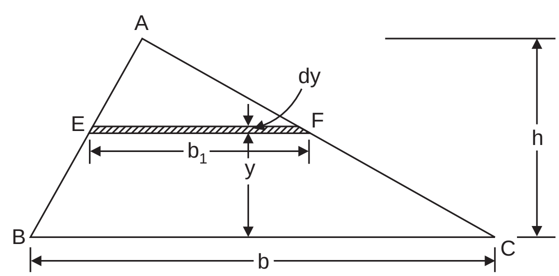


Fig. 2.25

Area of the triangle $A = \frac{1}{2}bh$

$\therefore$ From eqn. 2.4

$$\bar{y} = \frac{\text{Movement of area}}{\text{Total area}} = \frac{\int y dA}{A}$$

Now, $$\int y dA = \int_0^h y\left(1-\frac{y}{h}\right)b\,dy$$

$$= \int_0^h \left(y - \frac{y^2}{h} \right) b\, dy$$

$$= b\left[\frac{y^2}{2} - \frac{y^3}{3h} \right]_0^h$$

$$= \frac{bh^2}{6}$$

$$\therefore \qquad \bar{y} = \frac{\int y dA}{A} = \frac{bh^2}{6} \times \frac{1}{\frac{1}{2}bh}$$

$$\therefore \qquad \bar{y} = \frac{h}{3}$$

Thus the centroid of a triangle is at a distance $\frac{h}{3}$ from the base (or $\frac{2h}{3}$ from the apex) of the triangle where h is the height of the triangle.

Centroid of a Semicircle

Consider the semicircle of radius R as shown in Fig. 2.26. Due to symmetry centroid must lie on y axis. Let its distance from diametral axis be $\bar{y}$. To find $\bar{y}$, consider an element at a distance r from the centre O of the semicircle, radial width being dr and bound by radii at θ and θ + dθ.

Area of element = $r\, d\theta\, dr$.

Its moment about diametral axis x is given by:

$$r d\theta \times dr \times r \sin\theta = r^2 \sin\theta\, dr\, d\theta$$

∴ Total moment of area about diametral axis,

$$\int_0^{\pi}\int_0^{R} r^2 \sin\theta\, dr\, d\theta = \int_0^{\pi} \left[\frac{r^3}{3} \right]_0^R \sin\theta\, d\theta$$

$$= \frac{R^3}{3}\left[-\cos\theta \right]_0^{\pi}$$

$$= \frac{R^3}{3}\left[1 + 1 \right] = \frac{2R^3}{3}$$

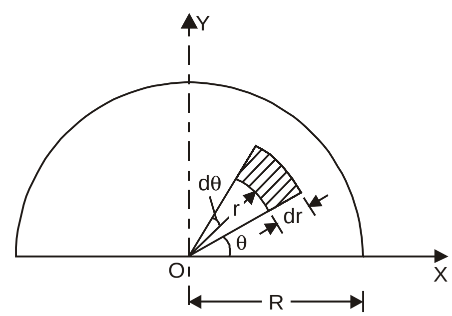


Fig. 2.26

Area of semicircle $\qquad A = \frac{1}{2}\pi R^2$

$$\therefore \qquad \bar{y} = \frac{\text{Moment of area}}{\text{Total area}} = \frac{\frac{2R^3}{3}}{\frac{1}{2}\pi R^2}$$

$$= \frac{4R}{3\pi}$$

Thus, the centroid of the circle is at a distance $\frac{4R}{3\pi}$ from the diametral axis.

Centroid of Sector of a Circle

Consider the sector of a circle of angle 2α as shown in Fig. 2.27. Due to symmetry, centroid lies on x axis. To find its distance from the centre O, consider the elemental area shown.

Area of the element $= r d\theta \, dr$

Its moment about y axis

$$= r d\theta \times dr \times r \cos\theta$$

$$= r^2 \cos\theta \, dr d\theta$$

$\therefore$ Total moment of area about y axis

$$= \int_{-\alpha}^{\alpha}\int_{0}^{R} r^2 \cos\theta \, dr d\theta$$

$$= \left[\frac{r^3}{3}\right]_0^R \left[\sin\theta\right]_{-\alpha}^{\alpha}$$

$$= \frac{R^3}{3}\, 2\sin\alpha$$

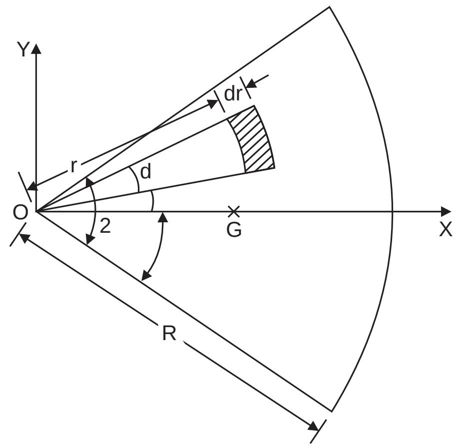


Fig. 2.27

Total area of the sector

$$= \int_{-\alpha}^{\alpha}\int_{0}^{R} r dr d\theta$$

$$= \int_{-\alpha}^{\alpha}\left[\frac{r^2}{2}\right]_0^R d\theta$$

$$= \frac{R^2}{2}\left[\theta\right]_{-\alpha}^{\alpha}$$

$$= R^2\alpha$$

$\therefore$ The distance of centroid from centre O

$$= \frac{\text{Moment of area about } y \text{ axis}}{\text{Area of the figure}}$$

$$= \frac{\dfrac{2R^3}{3}\sin\alpha}{R^2\alpha} = \frac{2R}{3\alpha}\sin\alpha$$

Centroid of Parabolic Spandrel

Consider the parabolic spandrel shown in Fig. 2.28. Height of the element at a distance x from O is $y = kx^2$

Width of element $= dx$

$\therefore$ Area of the element $= kx^2dx$

$\therefore$ Total area of spandrel $= \int_0^a kx^2dx$

$$= \left[\frac{kx^3}{3}\right]_0^a = \frac{ka^3}{3}$$

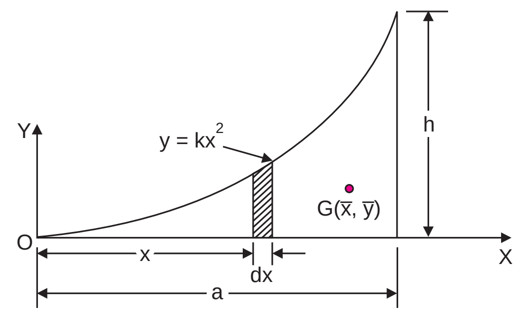


Fig. 2.28

Moment of area about y axis

$$= \int_0^a kx^2dx \times x$$

$$= \int_0^a kx^3\, dx$$

$$= \left[\frac{kx^4}{4}\right]_0^a$$

$$= \frac{ka^4}{4}$$

Moment of area about x axis $= \int_0^\alpha dAy/2$

$$= \int_0^a kx^2dx\,\frac{kx^2}{2} = \int_0^a \frac{k^2x^4}{2}dx = \left[\frac{k^2x^5}{2\times 5}\right]_0^a$$

$$= \frac{k^2a^5}{10}$$

$\therefore$ $\bar{x} = \frac{ka^4}{4} \div \frac{ka^3}{3} = \frac{3a}{4}$

$\bar{y} = \frac{k^2a^5}{10} \div \frac{ka^3}{3} = \frac{3}{10}\ ka^2$

From the Fig. 2.28, at $x = a$, $y = h$

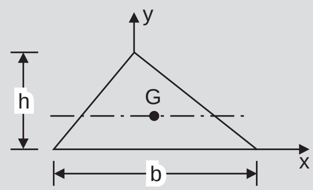
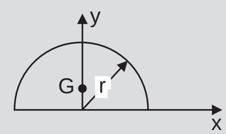
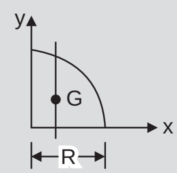
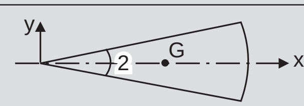
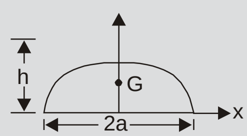
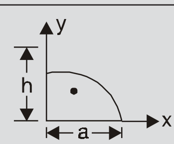
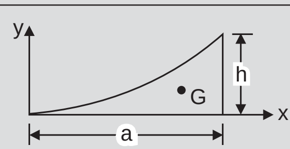
$\therefore$ $h = ka^2$ or $k = \frac{h}{a^2}$

$\therefore$ $\bar{y} = \frac{3}{10} \times \frac{h}{a^2} a^2 = \frac{3h}{10}$

Thus, centroid of spandrel is $\left(\frac{3a}{4}, \frac{3h}{10}\right)$

Centroids of some common figures are shown in Table 2.2.

Table 2.2. Centroid of Some Common Figures

Shape	***Figure***	$\bar{x}$	$\bar{y}$	***Area***
Triangle		—	$\frac{h}{3}$	$\frac{bh}{2}$
Semicircle		0	$\frac{4R}{3\pi}$	$\frac{\pi R^2}{2}$
Quarter circle		$\frac{4R}{3\pi}$	$\frac{4R}{3\pi}$	$\frac{\pi R^2}{4}$
Sector of a circle		$\frac{2R}{3\alpha}$ sin a	0	αR^2
Parabola		0	$\frac{3h}{5}$	$\frac{4ah}{3}$
Semi parabola		$\frac{3a}{8}$	$\frac{3h}{5}$	$\frac{2ah}{3}$
Parabolic spandrel		$\frac{3a}{4}$	$\frac{3h}{10}$	$\frac{ah}{3}$

Centroid of Composite Sections

So far, the discussion was confined to locating the centroid of simple figures like rectangle, triangle, circle, semicircle, etc. In engineering practice, use of sections which are built up of many simple sections is very common. Such sections may be called as built-up sections or composite sections. To locate the centroid of composite sections, one need not go for the first principle (method of integration). The given composite section can be split into suitable simple figures and then the centroid of each simple figure can be found by inspection or using the standard formulae listed in Table 2.2. Assuming the area of the simple figure as concentrated at its centroid, its moment about an axis can be found by multiplying the area with distance of its centroid from the reference axis. After determining moment of each area about reference axis, the distance of centroid from the axis is obtained by dividing total moment of area by total area of the composite section.

Example 2.4 Locate the centroid of the T-section shown in the Fig. 2.29.

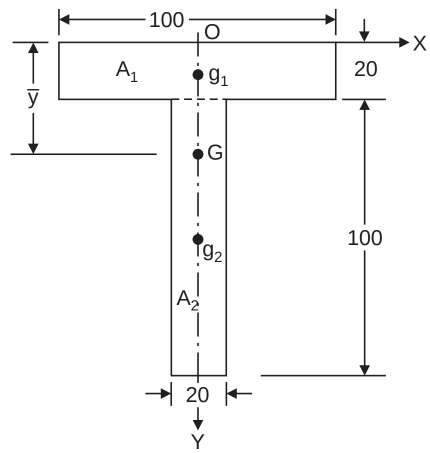
Solution. Selecting the axis as shown in Fig. 2.29, we can say due to symmetry centroid lies on y axis, *i.e.* $\bar{x} = 0$. Now the given T-section may be divided into two rectangles A_1 and A_2 each of size 100 × 20 and 20 × 100. The centroid of A_1 and A_2 are $g_1(0, 10)$ and $g_2(0, 70)$ respectively.

∴ The distance of centroid from top is given by:

$$\bar{y} = \frac{100 \times 20 \times 10 + 20 \times 100 \times 70}{100 \times 20 + 20 \times 100}$$

$$= 40 \text{ mm}$$

Hence, *centroid of T-section is on the symmetric axis at a distance 40 mm from the top.* **Ans.**



All dimensions in mm

Fig. 2.29

Example 2.5 Find the centroid of the unequal angle 200 × 150 × 12 mm, shown in Fig. 2.30.

Solution. The given composite figure can be divided into two rectangles:

$$A_1 = 150 \times 12 = 1800 \text{ mm}^2$$

$$A_2 = (200 - 12) \times 12 = 2256 \text{ mm}^2$$

Total area $A = A_1 + A_2 = 4056 \text{ mm}^2$

Selecting the reference axis x and y as shown in Fig. 2.30. The centroid of A_1 is g_1 (75, 6) and that of A_2 is:

$$g_2\left[6, 12 + \frac{1}{2}(200 - 12)\right]$$

i.e., g_2 (6, 106)

$$\therefore \quad \bar{x} = \frac{\text{Movement about } y \text{ axis}}{\text{Total area}}$$

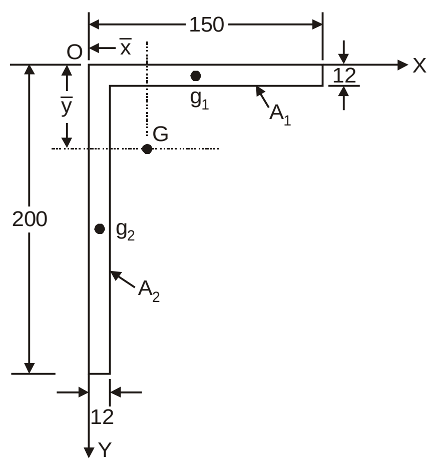
$$= \frac{A_1x_1 + A_2x_2}{A}$$

$$= \frac{1800 \times 75 + 2256 \times 6}{4056} = 36.62 \text{ mm}$$

$$\bar{y} = \frac{\text{Movement about } x \text{ axis}}{\text{Total area}}$$

$$= \frac{A_1y_1 + A_2y_2}{A}$$

$$= \frac{1800 \times 6 + 2256 \times 106}{4056} = 61.62 \text{ mm}$$



All dimensions in mm

Fig. 2.30

Thus, the centroid is at $\bar{x}$ = 36.62 mm and $\bar{y}$ = 61.62 mm as shown in the figure **Ans.**

Example 2.6 Locate the centroid of the I-section shown in Fig. 2.31.

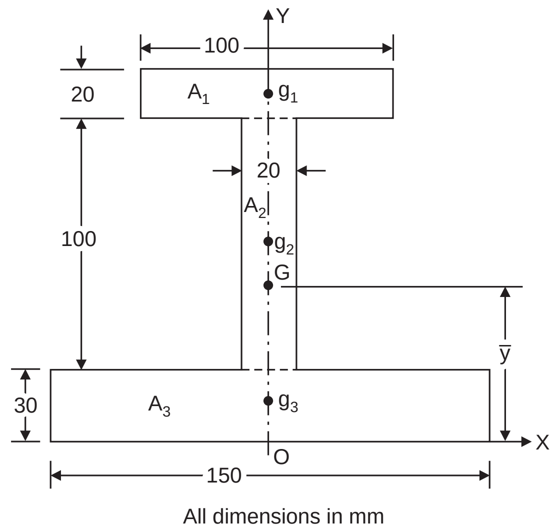


Fig. 2.31

Solution. Selecting the co-ordinate system as shown in Fig. 2.31, due to symmetry centroid must lie on y axis,

i.e., $\bar{x} = 0$

Now, the composite section may be split into three rectangles

$$A_1 = 100 \times 20 = 2000 \text{ mm}^2$$

Centroid of A_1 from the origin is:

$$y_1 = 30 + 100 + \frac{20}{2} = 140 \text{ mm}$$

Similarly $A_2 = 100 \times 20 = 2000 \text{ mm}^2$

$$y_2 = 30 + \frac{100}{2} = 80 \text{ mm}$$

$$A_3 = 150 \times 30 = 4500 \text{ mm}^2\text{, and}$$

$$y_3 = \frac{30}{2} = 15 \text{ mm}$$

$$\therefore \quad \bar{y} = \frac{A_1 y_1 + A_2 y_2 + A_3 y_3}{A}$$

$$= \frac{2000 + 140 + 2000 \times 80 + 4500 \times 15}{2000 + 2000 + 4500}$$

$$= 59.71 \text{ mm}$$

Thus, *the centroid is on the symmetric axis at a distance 59.71 mm from the bottom* as shown in Fig. 2.31. **Ans.**