(I) **Mathematical Expectation.** Let X be a random variable (r.v.) with p.d.f. (p.m.f.) $f(x)$. Then its mathematical expectation, denoted by $E(X)$ is given by :

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx, \quad (\text{ for continuous } r.v.) \quad = \text{mean} = \bar{x}$$

$$= \sum_{x} x\, f(x), \quad (\text{ for discrete } r.v.)$$

Remarks.

1. $E(X)$ exists iff $E|X|$ exists.

2. The expectation of a random variable is thought of as a long-term average.

(II)

Expectation of a Function of a Random Variable. Consider a r.v. X with p.d.f. (p.m.f.) $f(x)$ and distribution function $F(x)$. If $g(\cdot)$ is a function such that $g(X)$ is a r.v. and $E[g(X)]$ exists (*i.e.*, is defined), then

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)\, dF(x) = \int_{-\infty}^{\infty} g(x) f(x)\, dx$$

(*For continuous r.v.*)

$$= \sum_{x} g(x) f(x)$$

(*For discrete r.v.*)

$E(X) = \sum x\, p(x) = \sum x\, f(x)$

$E(g(x)) = \sum g(x) f(x)$ $\quad$ x is DRV

$= \int g(x)\, f(x)$ $\quad$ x is CRV

$g(x) = (2x+5)$

$$E(2x+5) = \sum (2x+5)\,p(x) = \sum (2x+5)\,f(x).$$

$$= \int (2x+5)\,p(x)$$

III

Addition Theorem of Expectation

If X and Y are random variables then

$$E(X+Y) = E(X) + E(Y),$$

provided all the expectations exist.

$E(X) = 2$

$E(Y) = 3$

$E(X+Y) = E(X) + E(Y)$

$= 2 + 3 = 5$

(IV)

The mathematical expectation of the sum of n random variables is equal to the sum of their expectations, provided all the expectations exist.

Symbolically, if $X_1, X_2, \ldots, X_n$ are random variables then

$$E(X_1 + X_2 + \ldots + X_n) = E(X_1) + E(X_2) + \ldots + E(X_n)$$

or
$$E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i),$$

if all the expectations exist.

$$E(x_1 + x_2 + x_3 + \cdots x_n) = E(x_1) + E(x_2) + \cdots + E(x_n)$$

$$E\left(\sum_{i=1}^{n} x_i\right) = \sum_{i=1}^{n} E(x_i)$$

(V) Multiplication Theorem of Expectation

If X and Y are independent random variables, then

$$E(XY) = E(X) \cdot E(Y)$$

$E(X) = 2$

$E(Y) = 3$

$E(XY) = E(X)\,E(Y)$

$= 2 \times 3$

$= 6$

VI Generalisation to n-variables.

The mathematical expectation of the product of a number of independent random variables is equal to the product of their expectations. Symbolically, if $X_1, X_2, ..., X_n$ are n independent random variables, then

$$E(X_1 X_2 \ldots X_n) = E(X_1) E(X_2) \ldots E(X_n)$$

i.e.,

$$E\left(\prod_{i=1}^{n} X_i\right) = \prod_{i=1}^{n} E(X_i)$$

provided all the expectations exist.

$$E(x_1 x_2 \ldots x_n) = E(x_1)\, E(x_2) \ldots E(x_n)$$

$$E\left(\prod_{i=1}^{n} x_i\right) = \prod_{i=1}^{n} E(x_i)$$

VII

If X is a random variable and 'a' is constant, then

(i) $E[a\Psi(X)] = a\,E[\Psi(X)]$

(ii) $E[\Psi(X) + a] = E[\Psi(X)] + a,$

where $\Psi(X)$, a function of X, is a r.v. and all the expectations exist.

$E(a) = a$

$E(a\psi(x)) = a\,E(\psi x)$

$E(\psi(x) + a) =$

$= E(\psi(x)) + E(a)$

$= E(\psi x) + a$

$E(a) = \sum a\,p(x) = a\sum p(x)$

$= a(1) = a$

$E(a) = 1$

$E(a) = \int_{-\infty}^{\infty} a\,p(x)\,dx = a\int_{-\infty}^{\infty} p(x)\,dx$

$= a(1)$

$= a$

VIII If X is a random variable and a and b are constants, then

$$E(aX + b) = a\,E(X) + b$$

provided all the expectations exist.

$$E(2x+5) = E(2x) + E(5)$$

$$= 2E(x) + 5$$

(IX)

Expectation of a Linear Combination of Random Variables

Let $X_1, X_2, \ldots, X_n$ be any n random variables and if $a_1, a_2, \ldots, a_n$ are any n constants, then

$$E\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i E(X_i)$$

provided all the expectations exist.

$$E(a_1 x_1 + a_2 x_2 + a_3 x_3 + \cdots + a_n x_n)$$

$$= a E(x_1) + a_2 E(x_2) + a_3 E(x_3) + \cdots + a_n E(x_n)$$

$\bar{X}$

$\sum x\,p(x) \geq 0$

$\int x\,p(x) \geqslant 0$

1. *If* $X \geq 0$ *then* $E(X) \geq 0$.

2. *Let X and Y be two random variables such that* $Y \leq X$ *then*

$$E(Y) \leq E(X)$$

provided the expectations exist. ✓

3. $$|E(X)| \leq E|X|$$ *provided the expectations exist.*

Q1 *Let X be a random variable with the following probability distribution :*

x	:	-3	6	9
$P_r(X = x)$	:	$1/6$	$1/2$	$1/3$

Find $E(X)$ and $E(X^2)$ and using the laws of expectation, evaluate $E(2X+1)^2$.

x	$p(x)$	$x\,p(x)$	x^2	$x^2 p(x)$
-3	$1/6$	$-\frac{1}{2}$	9	$\frac{9}{6} = \frac{3}{2}$
6	$1/2$	3	36	$\frac{36}{2} = 18$
9	$1/3$	3	81	$\frac{81}{3} = 27$

$$E(x) = \sum x\,p(x)$$

$$= 3+3-\frac{1}{2}$$

$$= 5.5 = \frac{11}{2}$$

$$E(x^2) = \sum x^2 p(x^2$$

$$= \frac{3}{2} + 18 + 27$$

$$= \frac{3+36+54}{2}$$

$$= \frac{93}{2}$$

x	$p(x)$	$(2x+1)^2$	$(2x+1)^2 p(x)$
-3	$1/6$	25	$\frac{25}{6}$
6	$1/2$	169	$\frac{169}{2}$
9	$1/3$	$19^2 = 361$	$\frac{361}{3}$

$$E(2x+1)^2$$

$$= \sum (2x+1)^2 p(x)$$

$$= \frac{25}{6} + \frac{169}{2} + \frac{361}{3}$$

$$= \frac{25 + 507 + 722}{6}$$

$$= \frac{1254}{6} = 209$$

$$E(2x+1)^2 = E\left[(2x)^2 + 1^2 + 2(2x)\right]$$

$$= E\left[4x^2 + 1 + 4x\right]$$

$$= 4E(x^2) + E(1) + 4E(x)$$

$$= \cancel{4}^{2}\left(\frac{93}{\cancel{2}}\right) + 1 + \cancel{4}^{2}\left(\frac{11}{\cancel{2}}\right)$$

$$= 186 + 1 + 22$$

$$= 209$$

$V(x) = E(x^2) - (E(x))^2$

$V(b) = E(b^2) - (E(b))^2$

$= b^2 - b^2 = 0$

$V(b) = 0$

If X is a random variable, then

$$V(aX + b) = a^2 V(X)$$

where a and b are constants.

(i) If $b = 0$, then $V(aX) = a^2 V(X)$

$\Rightarrow$ Variance is not independent of change of scale.

(ii) If $a = 0$, then $V(b) = 0$

$\Rightarrow$ Variance of a constant in zero.

(iii) If $a = 1$, then $V(X + b) = V(X)$

$\Rightarrow$ Variance is independent of change of origin.

$a x + b$

scale

origin

$x = 2u x + 5$

$$V(4x+5) = V(4x) + V(5)$$

$$= 4^2 V(x) + 0$$

$$= 16\, V(x)$$

Given the following table :

x	-3	-2	-1	0	1	2	3
$p(x)$	0·05	0·10	0·30	0	0·30	0·15	0·10

Compute (i) $E(X)$, (ii) $E(2X \pm 3)$, (iii) $E(4X+5)$, (iv) $E(X^2)$

(v) $V(X)$, and (vi) $V(2X \pm 3)$.

x	$p(x)$	$x\,p(x)$	x^2	$x^2 p(x)$
-3	0.05	-0.15	9	0.45
-2	0.10	-0.20	4	0.40
-1	0.30	-0.30	1	0.30
0	0	0	0	0
1	0.30	0.30	1	0.30
2	0.15	0.30	4	0.60
3	0.10	0.30	9	0.90

$$E(x) = \sum x\,p(x)$$
$$= 0.9 - 0.65$$
$$= 0.25$$

$$E(x^2) = \sum x^2 p(x)$$
$$= 2.95$$

$$E(2X+3) = E(2x) + E(3)$$

$$= 2E(X) + 3$$

$$= 2(0.25) + 3 = 0.5 + 3 = 3.5$$

$$E(2X-3) = E(2x) - E(3)$$

$$= 2E(x) - 3$$

$$= 2(0.25) - 3 = 0.5 - 3 = -2.5$$

$E(4X+5) = 4E(X) + 5$

$E(X) = 0.25$

$= 4(0.25) + 5$

$= 1 + 5 = 6$

$$V(X) = E(x^2) - E(X)^2$$

$$= 2.95 - (0.25)^2$$

$$= 2.95 - 0.0625$$

$$= 2.8875$$

$$V(2x+3) = 2^2\,V(X) = 4(2.8875) = 11.55$$

$$V(2x-3) = 2^2\,V(X) = 4(2.8875) = 11.55$$