Probability — Mathematical Notion. We are now set to give the mathematical notion of the occurrence of a random phenomenon and the mathematical notion of probability. Suppose in a large number of trials the sample space S contains N sample points. The event A is defined by a description which is satisfied by N_A of the occurrences. The frequency interpretation of the probability $P(A)$ of the event A, tells us that $P(A)=N_A/N$.

A purely mathematical definition of probability cannot give us the actual value of $P(A)$ and this must be considered as a function defined on all events. With this in view, a mathematical definition of probability is enunciated as follows:

"Given a sample description space, probability is a function which assigns a non-negative real number to every event A, denoted by P(A) and is called the probability of the event A."

Probability Function. $P(A)$ is the probability function defined on a σ-field **B** of events if the following properties or axioms hold :

1. For each $A \in$ **B**, $P(A)$ is defined, is real and $P(A) \geq 0$ Non-Negativity
2. $P(S) = 1$ Certainty
3. If $\{A_n\}$ is any finite or infinite sequence of disjoint events in **B**, then

$$P\left(\bigcup_{i=1}^{n} A_n\right) = \sum_{i=1}^{n} P(A_i)$$

$S = \{1, 2, 3, 4, 5, 6\}$

$n(S) = 6$

$A = \{2, 4, 6\}$

$n(A) = 3$

$P(A) = \frac{3}{6}$

$\boxed{P(A) = \frac{1}{2}}$

$P(A_1 \cup A_2 \cup A_3 \cdots \cup A_n) = P(A_1) + P(A_2) \cdots + P(A_n)$

$S = \{H, T\}$

$B = \{\underline{\phi}, \underline{\{H\}}, \underline{\{T\}}, \underline{\{H, T\}}\}$

$A \in B$

Remarks. 1. The set function **P** defined on σ-field **B**, taking its values in the real line and satisfying the above three axioms is called the probability measure.

2. The same definition of probability applies to *uncountable sample space* except that special restrictions must be placed on S and its subsets. It is important to realise that for a complete description of a probability measure, three things must be specified, *viz.*, the sample space S, the σ-field (σ-algebra) **B** formed from certain subset of S and set function **P**. The triplet (**S**, **B**, **P**,) is often called the *probability space*. In most elementary applications, S is finite and the σ-algebra **B** is taken to be the collection of all subsets of S.

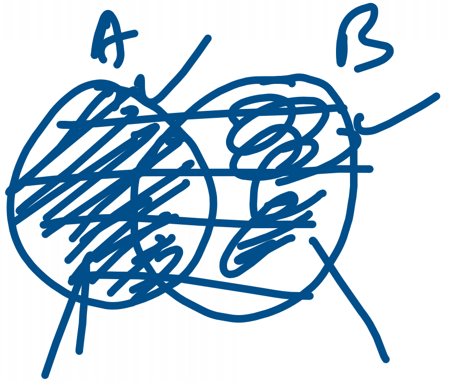
3. It is interesting to see that there are some formal statements of the properties of events derived from the frequency approach. Since $P(A)=N_A/N$, it is easy to see that $P(A) \geq 0$, as in Axiom 1. Next since $N_S = N$, $P(S)=1$, as in Axiom 2. In case of two mutually exclusive (or disjoint) events A and B defined by sample points N_A and N_B, the sample points belonging to $A \cup B$ are $N_A + N_B$. Therefore, $P(S)=N_S$

$$P(A \cup B) = \frac{N_A + N_B}{N} = \frac{N_A}{N} + \frac{N_B}{N} = P(A) + P(B), \text{ as in axiom 3.}$$

Law of Addition of Probabilities

Statement. *If A and B are any two events [subsets of sample space S] and are not disjoint, then*

$$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$



We have

$$A \cup B = A \cup (\bar{A} \cap B)$$

Since A and $(\bar{A} \cap B)$ are disjoint,

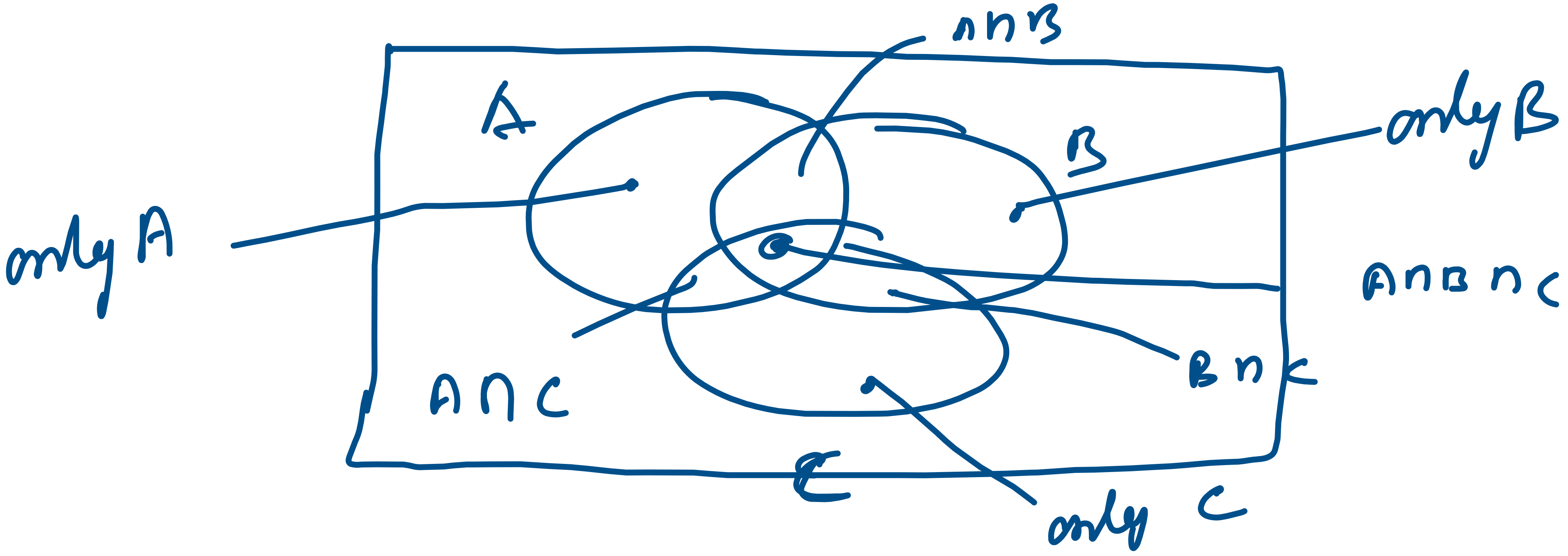
$$\begin{aligned} P(A \cup B) &= P(A) + P(\bar{A} \cap B) \\ &= P(A) + [P(\bar{A} \cap B) + P(A \cap B)] - P(A \cap B) \\ &= P(A) + P[(\bar{A} \cap B) \cup (A \cap B)] - P(A \cap B) \end{aligned}$$

[$\because (\bar{A} \cap B)$ and $(A \cap B)$ *are disjoint*]

$$\Rightarrow \quad \boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

For three events A, B and C

$$\boxed{\begin{aligned}P(A\cup B\cup C) = {} & P(A) + P(B) + P(C) - P(A\cap B) \\ & - P(A\cap C) - P(B\cap C) + P(A\cap B\cap C)\end{aligned}}$$



Remark: If A and B are mutually Exclusive events then

$$A \cap B = \phi \qquad P(A \cap B) = 0$$

$$n(A \cap B) = 0$$

$$P(A \cup B) = P(A) + P(B) - 0$$

$$\boxed{P(A \cup B) = P(A) + P(B)}$$

2. For three mutually Exclusive events A, B and C we have

$$P(A\cap B) = P(B\cap C) = P(A\cap C) = P(A\cap B\cap C) = 0$$

$$\boxed{P(A\cup B\cup C) = P(A) + P(B) + P(C)}$$

(3) If A and B are Exhaustive events then $A \cup B = S \Rightarrow \dfrac{n(A \cup B)}{n(S)} = \dfrac{n(S)}{n(S)}$

$$\boxed{P(A \cup B) = 1}$$

(4) If A, B and C are exhaustive

$$\boxed{P(A \cup B \cup C) = 1}$$

(5) For A and B equally likely

$$P(A) = P(B)$$

(6) For A, B and C equally likely

$$P(A) = P(B) = P(C)$$

Q1 If A, B and C are mutually Exclusive and Exhaustive then

Sol $P(A \cup B \cup C) = P(A) + P(B) + P(C)$

$P(A \cup B \cup C) = 1$ —— Exhaustive

$$\boxed{P(A) + P(B) + P(C) = 1}$$

Q2 If A, B and C are mutually Exclusive Exhaustive and equally likely then

$P(A)$

Sol

(i) $P(A \cup B \cup C) = P(A) + P(B) + P(C)$ M E

Exhaustive

(ii) $P(A \cup B \cup C) = 1$

(iii) $P(A) = P(B) = P(C)$ Equally likely

By (I) & (II)

$$P(A) + P(B) + P(C) = 1 \quad \text{——} \quad (IV)$$

By (III) & (IV)

$$P(A) + P(A) + P(A) = 1$$

$$3\,P(A) = 1$$

$$\boxed{P(A) = 1/3}$$

Q3 *A card is drawn from a well-shuffled pack of playing cards. What is the probability that it is either a spade or an ace?*

S A

$$P(S \text{ or } A) = P(S \cup A)$$

$$= P(S) + P(A) - P(S \cap A)$$

$$= \frac{13}{52} + \frac{4}{52} - \frac{1}{52}$$

$$= 16/52.$$

S — 13

A — 4

S and A — 1

13 + 4 − 1

= 16

$\frac{16}{52}$

Let A and B be two events such that

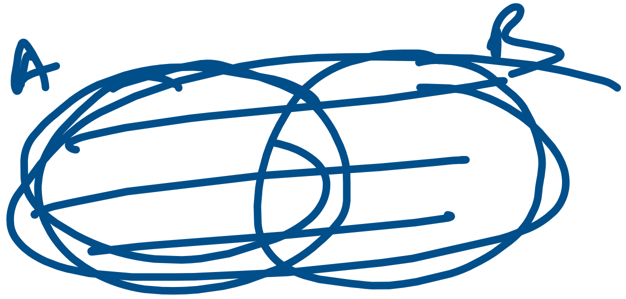
$P(A) = \frac{3}{4}$

$P(B) = 5/8$

show that

(a) $P(A \cup B) \geq \frac{3}{4}$

(b) $\frac{3}{8} \leq P(A \cap B) \leq \frac{5}{8}$



Sol

$A \subseteq A \cup B$

$n(A) \leq n(A \cup B)$

$\frac{n(A)}{n(S)} \leq \frac{n(A \cup B)}{n(S)} \Rightarrow$

$P(A) \leq P(A \cup B)$

$\frac{3}{4} \leq P(A \cup B)$

(b)

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \le 1$$

$$P(A \cup B) \le 1$$

$$P(A) + P(B) - P(A \cap B) \le 1$$

$$\frac{3}{4} + \frac{5}{8} - P(A \cap B) \le 1$$

$$\frac{3}{4} + \frac{5}{8} - 1 \le P(A \cap B)$$

$$\frac{6+5-8}{8} \leq P(A \cap B)$$

$$\boxed{\frac{3}{8} \leq P(A \cap B)} \quad \text{—} \quad 1$$

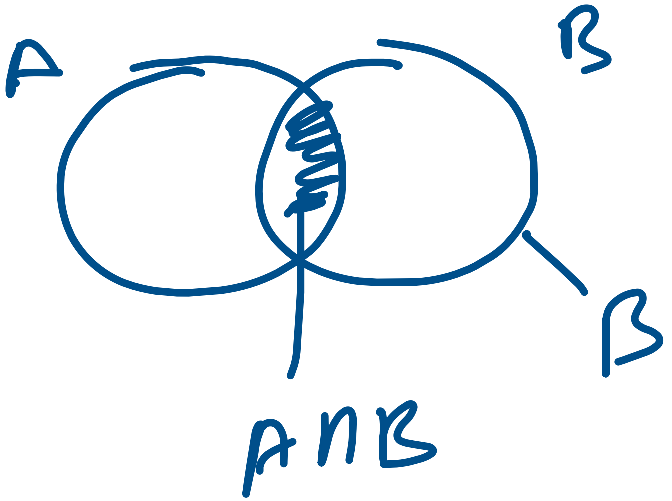
$A \cap B \subseteq B$

$\therefore$

$$\frac{n(A \cap B)}{n(S)} \leq \frac{n(B)}{n(S)}$$

$$P(A \cap B) \leq P(B)$$

$$P(A \cap B) \leq \frac{5}{8} \quad \text{———} \quad (2)$$



By ① & ②

$$\boxed{\frac{3}{8} \leq P(A \cap B) \leq \frac{5}{8}}$$