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Linear Programming and the Simplex Method

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Overview

Review: Inequalities and Systems of Inequalities

Consider the following inequalities:

1. $4x - 5y > 3$
2. $x + y \geq 2$
3. $3x + 2y < 5$

What is the region of the plane corresponding to their solution? $(0 \leq x \leq 10,\ 0 \leq y \leq 10)$

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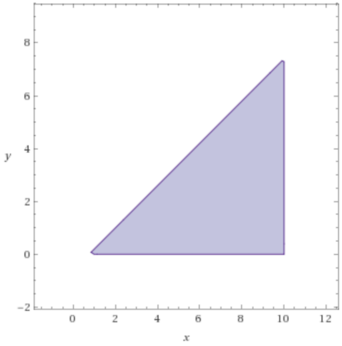
1. $4x - 5y > 3 \longrightarrow y < \frac{4}{5}x - \frac{3}{5}$



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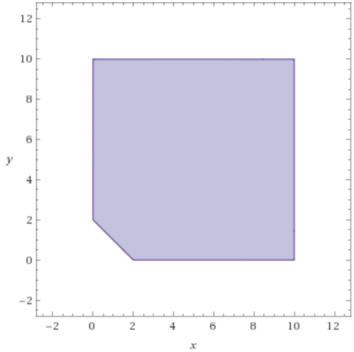
2 $x + y \geq 2 \longrightarrow y \geq -x + 2$



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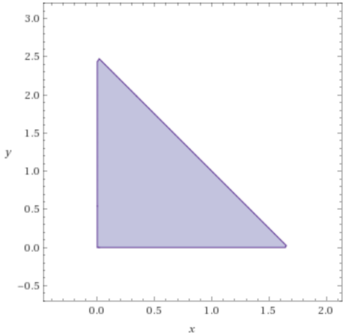
What is the region of the plane corresponding to their solution? $(0 \leq x \leq 10,\ 0 \leq y \leq 10)$

3. $3x + 2y < 5 \longrightarrow y < -\frac{3}{2}x + 5$



Review: Inequalities and Systems of Inequalities

Consider now the following systems:

1 $$\begin{cases} x + 2y \leq 8 \\ 3x - 5y \leq 13 \\ x \geq 0,\ y \geq 0 \end{cases}$$

2 $$\begin{cases} 2x + y \leq 7 \\ 4x + 2y \geq 6 \\ x \geq 0,\ y \geq 0 \end{cases}$$

What is the region of the plane corresponding to their solution?

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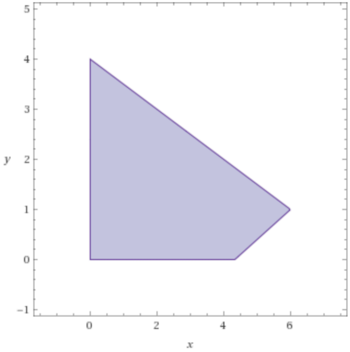
What is the region of the plane corresponding to their solution?

1 $y \leq -\frac{1}{2}x + 4,\ y \geq \frac{3}{5}x - \frac{13}{5}$



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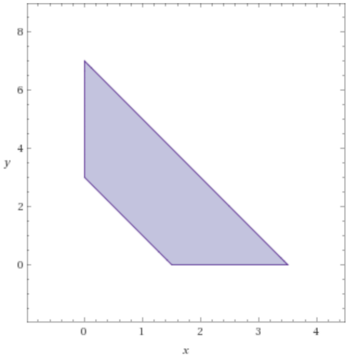
2 $$\begin{cases} 2x + y \leq 7 \\ 4x + 2y \geq 6 \\ x \geq 0,\ y \geq 0 \end{cases}$$

What is the region of the plane corresponding to their solution?

2 $y \leq -2x + 7,\ y \geq -2x + 3$



Review: Inequalities and Systems of Inequalities[1]

Two more systems with special features:

1. $$\begin{cases} x + y > 3 \\ x + y < -1 \end{cases}$$
2. $$\begin{cases} x + y < 3 \\ x + 2y > 3 \end{cases}$$

What is the region of the plane corresponding to their solution?

[1] Examples taken from [1]

Review: Inequalities and Systems of Inequalities

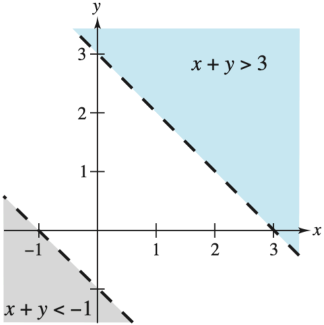
Two more systems with special features:

1. $\begin{cases} x + y > 3 \\ x + y < -1 \end{cases}$

2. $\begin{cases} x + y < 3 \\ x + 2y > 3 \end{cases}$

What is the region of the plane corresponding to their solution?

1. $y > -x + 3,\ y < -x + 1$



No Solution

Review: Inequalities and Systems of Inequalities

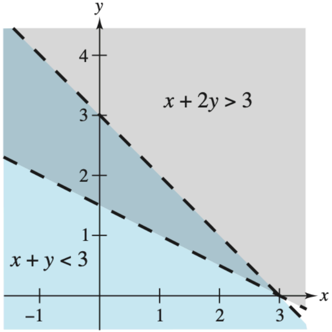
Two more systems with special features:

1. $\begin{cases} x + y > 3 \\ x + y < -1 \end{cases}$

2. $\begin{cases} x + y < 3 \\ x + 2y > 3 \end{cases}$

What is the region of the plane corresponding to their solution?

2 $y < -x + 3,\ y > -\frac{1}{2}x + \frac{3}{2}$



Unbounded Region

Review: Inequalities and Systems of Inequalities

- Solving a system of inequalities in two variables will identify a specific region of the plane.
- However there are some cases in which the system does not admit a solution (no region of the plane can be identified since the constraints are incompatible).
- In another cases we can identify an unbounded region (all the values of the plane are therefore acceptable as possible solutions).
- In these examples we sticked to 2 variables in order to obtain an easy-to-interpret graphical representation. The same concepts will hold however when extending our analysis to multiple dimensions.

Linear Programming

- Linear programming is a type of optimisation which aims at maximising or minimising a given quantity (e.g., find the minimum cost or the maximum profit).
- A linear programming problem consists of a linear objective function:

$$z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

and a set of constraints given by a system of linear inequalities:

$$\begin{aligned} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n &\geq h_1 \\ &\vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n &\leq h_m \end{aligned}$$

Linear Programming

To rewrite our problem in matrix notation, we can start by defining the following quantities:

- $c \in \mathbb{R}^n$ is a vector of coefficients (or weights)
- $A \in \mathbb{R}^{m \times n}$ is the matrix of coefficients of the constraints
- $h \in \mathbb{R}^m$ is the vector of coefficients of the constraints
- $x \in \mathbb{R}^n$ is the vector of unknowns that represent the solution of our problem

Standard Form of a Linear Program

By flipping all the inequalities so that we always have an upper bound for the maximisation problem and a lower bound for the minimisation problem, we can obtain the following **standard form of a linear program**.

Maximum problem:

$$\begin{aligned} \text{max} \quad & z = c^\mathsf{T} x \\ \text{s.t.} \quad & Ax \leq h \\ & x \geq 0 \end{aligned}$$

Minimum problem:

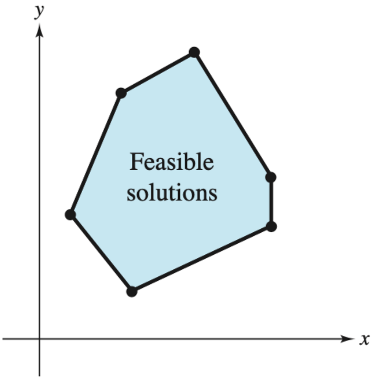
$$\begin{aligned} \text{min} \quad & z = c^\mathsf{T} x \\ \text{s.t.} \quad & Ax \geq h \\ & x \geq 0 \end{aligned}$$

Please notice that we have added an additional **non-negativity** constraint ($x \geq 0$).

Fundamental Theorem of Linear Programming

Theorem

If a linear programming problem has a solution, it must occur at a ***vertex*** *of the set of feasible solutions. If the problem has more than one solution, then at least one of them must occur at a vertex of the set of feasible solutions. In either case, the value of the objective function is* ***unique****.*



Graphical Solution of a Linear Program

Let us first consider the graphical solution of a linear program in two variables.

1. Solve the system of inequalities given by the constraints.
2. Identify the part of the plane corresponding to the feasible region. All the points inside the feasible region or on its boundary are feasible solutions, but, according to the theorem above, at least one optimal solution must occur at a vertex.
3. Evaluate the objective function at the vertices and pick the one which optimises its value. If the feasible region is bounded, we can find both a minimum and a maximum value.

Graphical Solution of a Linear Program[2]: Example 1

Let us consider the maximisation problem:

$$
\begin{aligned}
\text{max} \quad & z = 4x + 6y \\
\text{s.t.} \quad & -x + y \leq 11 \\
& x + y \leq 27 \\
& 2x + 5y \leq 90 \\
& x,\, y \geq 0
\end{aligned}
$$

[2] The following examples/exercises are taken from [1]

Graphical Solution of a Linear Program: Example 1

Let us consider the maximisation problem:

$$\begin{aligned} \text{max} \quad & z = 4x + 6y \\ \text{s.t.} \quad & -x + y \leq 11 \\ & x + y \leq 27 \\ & 2x + 5y \leq 90 \\ & x,\, y \geq 0 \end{aligned}$$

1. Solve the system of inequalities given by the constraints:

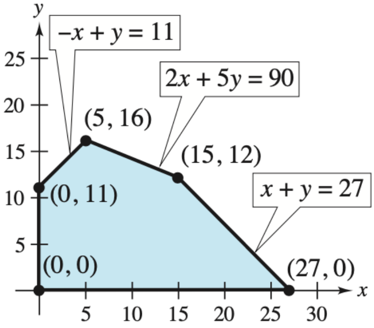
$$\begin{cases} -x + y \leq 11 & \longrightarrow y \leq x + 11 \\ x + y \leq 27 & \longrightarrow y \leq -x + 27 \\ 2x + 5y \leq 90 & \longrightarrow y \leq -\frac{2}{5}x + 18 \\ x,\, y \geq 0 \end{cases}$$

Graphical Solution of a Linear Program: Example 1

Let us consider the maximisation problem:

$$
\begin{aligned}
\text{max} \quad & z = 4x + 6y \\
\text{s.t.} \quad & -x + y \leq 11 \\
& x + y \leq 27 \\
& 2x + 5y \leq 90 \\
& x, \ y \geq 0
\end{aligned}
$$

2 The feasible region is given by:



Graphical Solution of a Linear Program: Example 1

Let us consider the maximisation problem:

$$\begin{aligned} \text{max} \quad & z = 4x + 6y \\ \text{s.t.} \quad & -x + y \leq 11 \\ & x + y \leq 27 \\ & 2x + 5y \leq 90 \\ & x,\, y \geq 0 \end{aligned}$$

3. Evaluate the objective function at the vertices and pick the maximum value:

$$\begin{aligned} &\text{At vertex } (0,0): \ z = 0 \\ &\text{At vertex } (0,11): \ z = 66 \\ &\text{At vertex } (5,16): \ z = 116 \\ &\boxed{\text{At vertex } (15,12): \ z = 132} \\ &\text{At vertex } (27,0): \ z = 108 \end{aligned}$$

The optimal $z^* = 132$ is at vertex $(15, 12)$.

Graphical Solution of a Linear Program: Example 2

Let us consider the minimisation problem:

$$
\begin{aligned}
\text{min} \quad & z = 5x + 7y \\
\text{s.t.} \quad & 2x + 3y \geq 6 \\
& 3x - y \leq 15 \\
& -x + y \leq 4 \\
& 2x + 5y \leq 27 \\
& x,\, y \geq 0
\end{aligned}
$$

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\text{min} \quad & z = 5x + 7y \\
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& -x + y \leq 4 \\
& 2x + 5y \leq 27 \\
& x,\, y \geq 0
\end{aligned}
$$

1. Solve the system of inequalities given by the constraints:

$$
\begin{cases}
2x + 3y \geq 6 & \longrightarrow y \geq -\frac{2}{3}x + 2 \\
3x - y \leq 15 & \longrightarrow y \geq 3x - 15 \\
-x + y \leq 4 & \longrightarrow y \leq x + 4 \\
2x + 5y \leq 27 & \longrightarrow y \leq -\frac{2}{5}x + \frac{27}{5} \\
x,\, y \geq 0
\end{cases}
$$

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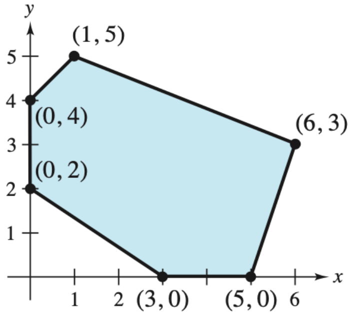

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& x,\ y \geq 0
\end{aligned}
$$

3. Evaluate the objective function at the vertices and pick the minimum value:

$$\boxed{\text{At vertex } (0,2) : \ z = 14}$$

At vertex $(0,4)$: $z = 28$

At vertex $(1,5)$: $z = 40$

At vertex $(6,3)$: $z = 51$

At vertex $(5,0)$: $z = 25$

At vertex $(3,0)$: $z = 15$

The optimal $z^* = 14$ is at vertex $(0,2)$.

Formulating a Problem as a Linear Program

The liquid portion of a diet is to provide at least 300 calories, 36 units of vitamin A, and 90 units of vitamin C daily. A cup of dietary drink X provides 60 calories, 12 units of vitamin A, and 10 units of vitamin C. A cup of dietary drink Y provides 60 calories, 6 units of vitamin A, and 30 units of vitamin C. Now, suppose that dietary drink X costs \$0.12 per cup and drink Y costs \$0.15 per cup. How many cups of each drink should be consumed each day to minimise the cost and still meet the stated daily requirements?

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Formulating a Problem as a Linear Program

We obtain the minimisation problem:

$$
\begin{aligned}
\text{min} \quad & z = 0.12x + 0.15y \\
\text{s.t.} \quad & 60x + 60y \geq 300 \\
& 12x + 6y \geq 36 \\
& 10x + 30y \geq 90 \\
& x,\ y \geq 0
\end{aligned}
$$

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\end{aligned}
$$

1. Solve the system of inequalities given by the constraints:

$$
\begin{cases}
60x + 60y \geq 300 & \longrightarrow y \geq -x + 5 \\
12x + 6y \geq 36 & \longrightarrow y \geq -2x + 6 \\
10x + 30y \leq 90 & \longrightarrow y \geq -\frac{1}{3}x + 3 \\
x,\ y \geq 0
\end{cases}
$$

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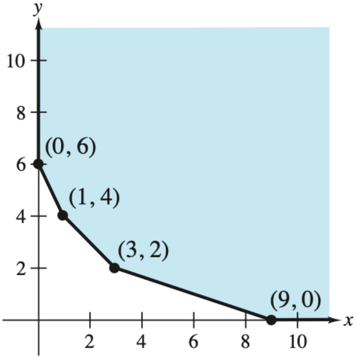

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& 10x + 30y \geq 90 \\
& x,\ y \geq 0
\end{aligned}
$$

3. Evaluate the objective function at the vertices and pick the minimum value:

$$
\begin{aligned}
&\text{At vertex } (0,6): \quad z = 0.90 \\
&\text{At vertex } (1,4): \quad z = 0.72 \\
&\boxed{\text{At vertex } (3,2): \quad z = 0.66} \\
&\text{At vertex } (9,0): \quad z = 1.08
\end{aligned}
$$

The optimal $z^* = 0.66$ is at vertex $(3, 2)$.

The Simplex Method

- The graphical solution of a linear program can be useful when we have only two variables and we can easily identify both the feasible region (given by the constraints) and the vertices to consider as possible optimal solutions.
- When we have, however, a large number of variables to optimise with several constraints, it is better to adopt a more efficient procedure.
- Here we will consider the simplex method, introduced by Dantzig in 1947.

Maximisation: Introducing the Slack Variables

We consider the maximisation of a linear program in standard form and we add m **slack variables** to turn the inequality constraints in equalities, going from:

$$
\begin{aligned}
a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n &\leq h_1 \\
&\vdots \\
a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n &\leq h_m
\end{aligned}
$$

to the following new system of constraints:

$$
\begin{aligned}
a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n + s_1 &= h_1 \\
&\vdots \\
a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n + s_m &= h_m
\end{aligned}
$$

Maximisation: Introducing the Slack Variables

The quantities will then be redefined as follows (notice the new dimensions except for h):

$$c = \underbrace{\begin{bmatrix} c_1 \\ \vdots \\ c_n \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{(n+m)\times 1}, \quad x = \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_n \\ s_1 \\ \vdots \\ s_m \end{bmatrix}}_{(n+m)\times 1}, \quad A = \underbrace{\begin{bmatrix} a_{1,1} & \dots & a_{1,n} & 1 & 0 & \dots & 0 \\ a_{2,1} & \dots & a_{2,n} & 0 & 1 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & \dots & a_{m,n} & 0 & 0 & \dots & 1 \end{bmatrix}}_{m\times(n+m)}, \quad h = \underbrace{\begin{bmatrix} h_1 \\ \vdots \\ h_m \end{bmatrix}}_{m\times 1}$$

and we will add the non-negativity constraint also for the slack variables ($s_1, \dots, s_m \geq 0$).

Minimisation: Introducing the Surplus Variables

We consider the minimisation of a linear program in standard form and we subtract m **surplus variables** to turn the inequality constraints in equalities, going from:

$$\begin{aligned} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n &\geq h_1 \\ &\vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n &\geq h_m \end{aligned}$$

to the following new system of constraints:

$$\begin{aligned} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n - s_1 &= h_1 \\ &\vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n - s_m &= h_m \end{aligned}$$

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Minimisation: Introducing the Surplus Variables

The quantities will then be given by (notice the sign of A compared to the maximisation):

$$c = \underbrace{\begin{bmatrix} c_1 \\ \vdots \\ c_n \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{(n+m)\times 1}, \quad x = \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_n \\ s_1 \\ \vdots \\ s_m \end{bmatrix}}_{(n+m)\times 1}, \quad A = \underbrace{\begin{bmatrix} a_{1,1} & \dots & a_{1,n} & -1 & 0 & \dots & 0 \\ a_{2,1} & \dots & a_{2,n} & 0 & -1 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & \dots & a_{m,n} & 0 & 0 & \dots & -1 \end{bmatrix}}_{m\times(n+m)}, \quad h = \underbrace{\begin{bmatrix} h_1 \\ \vdots \\ h_m \end{bmatrix}}_{m\times 1}$$

and we will add the non-negativity constraint also for the surplus variables ($s_1, \dots, s_m \geq 0$).

Basic and Nonbasic Variables

- We can freely swap the rows of A, as long as we swap also the elements of vector h
- We can freely swap the columns of A, as long as we swap also the elements of vector x
- Matrix A can be split into two sub-matrices as follows:

$$A = \begin{bmatrix} B & D \end{bmatrix}$$

  $B \in \mathbb{R}^{m \times m}$ is the matrix formed by m linearly independent columns of A (it has, therefore, full rank and it is invertible). $D \in \mathbb{R}^{m \times n}$ contains the remaining columns.
- In the same way, we can also split x and c:

$$x = \begin{bmatrix} x_B \\ x_D \end{bmatrix}, \quad c = \begin{bmatrix} c_B \\ c_D \end{bmatrix}, \quad \text{with } x_B, c_B \in \mathbb{R}^{m \times 1} \text{ and } x_D, c_D \in \mathbb{R}^{n \times 1}$$

Basic and Nonbasic Variables

- We can rewrite our constraints system $Ax = h$ by using the new decomposition:

$$Bx_B + Dx_D = h \longrightarrow Bx_B = h - Dx_D$$

  The variables in x_B are called *basic variables*, while those in x_D are *nonbasic variables*.
- By inverting matrix B (always nonsingular since its rank is full) we obtain:

$$x_B = B^{-1}h - B^{-1}Dx_D$$

  This represents a general solution of our problem and the value of the basic variables x_B depends on the values assigned to the nonbasic variables x_D.
- Please notice that x_B can contain both the original unknowns $(x_1, \ldots, x_n)$ and the slack variables $(s_1, \ldots, s_m)$.

Basic Solutions

- We call *basic solution* the one obtained by setting $x_D = 0$

$$x_B = B^{-1}h$$

  If the non-negativity condition $x_B \geq 0$ is satisfied, then the basic solution is *feasible*.
- In case one or more components of x_B have value 0, the solution is *degenerate*.
- The basic solutions correspond to the vertices of the polytope (the points defined by the intersection of the constraints).

The Simplex Method: Algorithm

Algorithm 1 Simplex Method

1: Write the problem in standard form and add slack/surplus variables
2: Consider a feasible starting basic solution (through solution of an *auxiliary problem*)
3: **while** the *optimality criterion* is not satisfied **do**
4: Apply the *iterative rule* by exchanging one basic and nonbasic variable.
5: **end while**
6: **return** the basic solution satisfying the optimality criterion

Existence of an Optimal Basic Solution

- The existence of a feasible solution according to the Fundamental Theorem of Linear Programming implies also the existence of a feasible basic solution. Moreover, the existence of an optimal solution, implies also the existence of an optimal basic solution.
- The simplex method will then iterate through the corners of the polytope delimited by the constraints of the problem, until the optimality criterion is satisfied.
- **Downside:** The number of possible basic solutions grows exponentially with the number of unknowns and constraints. The possible *maximum number of iterations* is given by:

$$N = \frac{(m+n)!}{m!n!}$$

The Optimality Condition

- The optimality condition that needs to be checked at every iteration is based on the **reduced cost coefficients**:

$$r_D = c_D^\intercal - c_B^\intercal B^{-1} D$$

  where c_D represents the nonbasic coefficients of the objective function, c_B the basic coefficients of the objective function, B^{-1} the inverse of the basic matrix and D the nonbasic matrix.
- The optimality conditions are given by:

Maximisation:	Minimisation:
$r_D \leq 0$	$r_D \geq 0$

The Iterative rule

- If the optimality condition is not met, the variable with the highest value of the reduced cost coefficient r_D for a maximisation (or the one with the lowest value in case of a minimisation) needs to be brought inside the basis.
- The departing variable is determined by taking the ratio:

$$\frac{B^{-1}h}{B^{-1}D} \tag{1}$$

  for the column referring to the entering variable and by selecting the variable with the smallest **positive** value.

Feasible Starting Basic Solution: Auxiliary Problem

In order to find a feasible initial basic solution, the two phases method requires the solution of an **auxiliary problem**. The first thing to do is to ensure the positivity of the right hand side: in other words, we want all $h_1, \dots, h_m$ to be positive. We will then add slack variables or subtract surplus variables to ensure that this condition is met:

$$\begin{aligned} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n \pm s_1 &= h_1 \\ &\vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n \pm s_m &= h_m \end{aligned}$$

with all $h_1, \dots, h_m \geq 0$ by construction.

Feasible Starting Basic Solution: Auxiliary Problem

The auxiliary problem is obtained by adding m artificial variables $u_1, \ldots, u_m$ to the constraints:

$$\begin{aligned} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n \pm s_1 + u_1 &= h_1 \\ \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n \pm s_m + u_m &= h_m \end{aligned}$$

and by defining the new objective function:

$$z_{\mathsf{aux}} = u_1 + u_2 + \cdots + u_m.$$

The auxiliary problem is always a **minimisation** problem. The starting feasible basic solution is given by the set of variables for which $z_{\mathsf{aux}} = 0$. If z_{aux} does not reach the value 0, then both the auxiliary problem and the original LP problem do not admit a feasible solution.

Simplex Method: Minimisation Example

Let us consider the following linear programming problem:

$$
\begin{aligned}
\text{min} \quad & z = 9x_1 + 11x_2 + 8x_3 \\
\text{s.t.} \quad & 2x_1 + x_2 + 3x_3 \geq 210 \\
& 3x_1 + 4x_2 + 2x_3 \geq 360 \\
& x_1,\, x_2,\, x_3 \geq 0
\end{aligned}
$$

The problem is already in standard form for a minimisation, so we can directly subtract the surplus variables to turn the inequalities into equality constraints.

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Simplex Method: Minimisation Example

After subtracting the surplus variables, the problem becomes:

$$
\begin{aligned}
\text{min} \qquad & z = 9x_1 + 11x_2 + 8x_3 \\
\text{s.t.} \qquad & 2x_1 + x_2 + 3x_3 - s_1 = 210 \\
& 3x_1 + 4x_2 + 2x_3 - s_2 = 360 \\
& x_1,\, x_2,\, x_3 \geq 0; \;\; s_1,\, s_2 \geq 0
\end{aligned}
$$

with the following quantities:

$$
c = \underbrace{\begin{bmatrix} 9 \\ 11 \\ 8 \\ 0 \\ 0 \end{bmatrix}}_{5\times 1}, \quad x = \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ s_1 \\ s_2 \end{bmatrix}}_{5\times 1}, \quad A = \underbrace{\begin{bmatrix} 2 & 1 & 3 & -1 & 0 \\ 3 & 4 & 2 & 0 & -1 \end{bmatrix}}_{2\times 5}, \quad h = \underbrace{\begin{bmatrix} 210 \\ 360 \end{bmatrix}}_{2\times 1}.
$$

Simplex Method: Minimisation Example

We can split the matrices in the following basic variables:

$$c_B = \underbrace{\begin{bmatrix} 8 \\ 0 \end{bmatrix}}_{2\times 1}, \quad x_B = \underbrace{\begin{bmatrix} x_3 \\ s_1 \end{bmatrix}}_{2\times 1}, \quad B = \underbrace{\begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}}_{2\times 2}, \quad h = \underbrace{\begin{bmatrix} 210 \\ 360 \end{bmatrix}}_{2\times 1},$$

and nonbasic variables:

$$c_D = \underbrace{\begin{bmatrix} 9 \\ 11 \\ 0 \end{bmatrix}}_{3\times 1}, \quad x_D = \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ s_2 \end{bmatrix}}_{3\times 1}, \quad D = \underbrace{\begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & -1 \end{bmatrix}}_{2\times 3}.$$

Simplex Method: Minimisation Example

We obtain the following basic solution (remember to set $x_D = 0$):

$$x_B = B^{-1}h - B^{-1}Dx_D = \begin{bmatrix} 0 & 0.5 \\ -1 & 1.5 \end{bmatrix} \begin{bmatrix} 210 \\ 360 \end{bmatrix} = \begin{bmatrix} 180 \\ 330 \end{bmatrix},$$

which is feasible, since also the non-negativity constraint is satisfied. The corresponding value of the objective function is given by:

$$z = c_B^\intercal x_B = \begin{bmatrix} 8 & 0 \end{bmatrix} \begin{bmatrix} 180 \\ 330 \end{bmatrix} = 1440.$$

Is the solution optimal? Let us apply the simplex method to find it out.

Simplex Method: Minimisation Example

We start by computing the reduced cost coefficients:

$$r_D = c_D^\intercal - c_B^\intercal B^{-1} D = \begin{bmatrix} 9 & 11 & 0 \end{bmatrix} - \begin{bmatrix} 8 & 0 \end{bmatrix} \begin{bmatrix} 1.5 & 2 & -0.5 \\ 2.5 & 5 & -1/5 \end{bmatrix} = \begin{bmatrix} -3 & -5 & 4 \end{bmatrix}.$$

According to the iterative rule, we pick the coefficient with the smallest value, since we are dealing with a minimisation (-5 in this case). In other words, the entering variable will be x_2, since - according to the algorithm - it should improve the value of the objective function.

Which should be the departing variable?

Simplex Method: Minimisation Example

We keep applying the iterative rule by computing the ratios for the second column:

$$\frac{B^{-1}h}{B^{-1}D} = \begin{bmatrix} 90 \\ 66 \end{bmatrix}$$

and by selecting the smallest positive value (in this case 66). The corresponding variable is s_1, which will be now leaving the basis.

How is our new system (basic and nonbasic variables) defined?

Simplex Method: Minimisation Example

The basic variables are:

$$c_B = \underbrace{\begin{bmatrix} 8 \\ 11 \end{bmatrix}}_{2\times 1}, \quad x_B = \underbrace{\begin{bmatrix} x_3 \\ x_2 \end{bmatrix}}_{2\times 1}, \quad B = \underbrace{\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}}_{2\times 2}, \quad h = \underbrace{\begin{bmatrix} 210 \\ 360 \end{bmatrix}}_{2\times 1}$$

while the nonbasic variables are:

$$c_D = \underbrace{\begin{bmatrix} 9 \\ 0 \\ 0 \end{bmatrix}}_{3\times 1}, \quad x_D = \underbrace{\begin{bmatrix} x_1 \\ s_1 \\ s_2 \end{bmatrix}}_{3\times 1}, \quad D = \underbrace{\begin{bmatrix} 2 & -1 & 0 \\ 3 & 0 & -1 \end{bmatrix}}_{2\times 3}$$

Simplex Method: Minimisation Example

We obtain the following basic solution (remember to set $x_D = 0$):

$$x_B = B^{-1}h - B^{-1}Dx_D = \begin{bmatrix} 0.4 & -0.1 \\ -0.2 & 0.33 \end{bmatrix} \begin{bmatrix} 210 \\ 360 \end{bmatrix} = \begin{bmatrix} 48 \\ 66 \end{bmatrix}$$

which is again feasible, as guaranteed by the algorithm. The corresponding value of the objective function is given by:

$$z = c_B^\intercal x_B = \begin{bmatrix} 8 & 11 \end{bmatrix} \begin{bmatrix} 48 \\ 66 \end{bmatrix} = 1100.$$

The value is lower than the one obtained before (1440), so the objective function value is improving. Let us check if it is the optimal solution by computing the reduced cost coefficients.

Simplex Method: Minimisation Example

The reduced cost coefficients are given by:

$$r_D = c_D^\intercal - c_B^\intercal B^{-1} D = \begin{bmatrix} 9 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 8 & 11 \end{bmatrix} \begin{bmatrix} 0.5 & -0.4 & 0.1 \\ 0.5 & 0.2 & -0.33 \end{bmatrix} = \begin{bmatrix} -0.5 & 1 & 2.5 \end{bmatrix}$$

The solution is still not optimal, since we have a negative reduced cost coefficient. The entering variable will be x_1, which corresponds to the only negative value (-0.5).

Which should be the departing variable?

Simplex Method: Minimisation Example

We keep applying the iterative rule by computing the ratios for the first column:

$$\frac{B^{-1}h}{B^{-1}D} = \begin{bmatrix} 96 \\ 132 \end{bmatrix}$$

and by selecting the smallest positive value (in this case 96). The corresponding variable is x_3, which will be now leaving the basis.

How is our new system (basic and nonbasic variables) defined?

Simplex Method: Minimisation Example

The basic variables are:

$$c_B = \underbrace{\begin{bmatrix} 9 \\ 11 \end{bmatrix}}_{2\times 1}, \quad x_B = \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{2\times 1}, \quad B = \underbrace{\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}}_{2\times 2}, \quad h = \underbrace{\begin{bmatrix} 210 \\ 360 \end{bmatrix}}_{2\times 1}$$

while the nonbasic variables are:

$$c_D = \underbrace{\begin{bmatrix} 8 \\ 0 \\ 0 \end{bmatrix}}_{3\times 1}, \quad x_D = \underbrace{\begin{bmatrix} x_3 \\ s_1 \\ s_2 \end{bmatrix}}_{3\times 1}, \quad D = \underbrace{\begin{bmatrix} 3 & -1 & 0 \\ 2 & 0 & -1 \end{bmatrix}}_{2\times 3}$$

Simplex Method: Minimisation Example

We obtain the following basic solution (remember to set $x_D = 0$):

$$x_B = B^{-1}h - B^{-1}Dx_D = \begin{bmatrix} 0.8 & -0.2 \\ -0.66 & 0.4 \end{bmatrix} \begin{bmatrix} 210 \\ 360 \end{bmatrix} = \begin{bmatrix} 96 \\ 18 \end{bmatrix}$$

which is again feasible, as guaranteed by the algorithm. The corresponding value of the objective function is given by:

$$z = c_B^\intercal x_B = \begin{bmatrix} 9 & 11 \end{bmatrix} \begin{bmatrix} 96 \\ 18 \end{bmatrix} = 1062.$$

We obtained an even lower value of the objective function $(1062 < 1100)$. Can we still do better or we reached the optimal solution?

Simplex Method: Minimisation Example

The reduced cost coefficients are given by:

$$r_D = c_D^\intercal - c_B^\intercal B^{-1} D = \begin{bmatrix} 9 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 9 & 11 \end{bmatrix} \begin{bmatrix} 2 & -0.8 & 0.2 \\ -1 & 0.66 & -0.4 \end{bmatrix} = \begin{bmatrix} 1 & 0.6 & 2.6 \end{bmatrix}$$

All the reduced cost coefficients are positive and the algorithm stops: the simplex method converged to the optimal solution, given by:

$$x^* = \begin{bmatrix} 96 \\ 18 \\ 0 \end{bmatrix}, \text{ with } z^* = 1062.$$

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