IN , LASEC: Bachelor Project #1

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Abstract

This Hill cipher is a polygraphic substitution cipher based on linear algebra , invented by Lester S. in 1929. Each letter is represented by a number modulo 26, it breaks the plaintext into blocks of size d and then applies a matrix $d \times d$ to thiese blocks to yield ciphertext blocks. As it's a linear encryption, it can be simply broken with Know PlainText Attacks. The author takes the previous paper about a new Ciphertext-only Attacks on Hill , and try to improve it's complexity to get a better result that $O(d13^d)$.

The goal of this project is to actually study the algorithm to get the key matrix modulo 26 and then to improve the algorithm to get the key matrix modulo 2.

The project report is organized as follows: Section1 presents the Hill cipher and the work done in the previous repor. In section2 , the author studies the complexity and try to improve the algorithm to get the key matrix modulo 26 . Section 3 presents the possible enhancement of the FFT of algorithm 1. Experimental results and algorithm are presented at the end.

Contents

Introduction

The motivation of this project is first and foremost to improve the Linear attack on the Hill cipher , by changing the recovery of the key modulo 26 and then see the possible algorithm to improve the FFT.
Let's briefly recall how this attack works.
You get the plaintext modulo 2 , then with the aid of vectors , and bias(X)= $\varphi_X(\frac{2\pi}{p})$ in $\mathbb{Z}/26\mathbb{Z}$, we found correspondence between λ and μ (the last is the same vector but for the cipher text). We actually get $\mu = (K^T)^{-1} \times \lambda$
Then with this formula and the approximation of all the vector μ , we get the vectors column of the key matrix in $Z/2\mathbb{Z}$.
You just need to reorder them with the correlation , you find the last one and first one easily, and you do it recursively to find all the vectors in the correct order.
All this process is described by algorithm 1 at the end of the page.

Key recovery modulo 26

So now that we have the key matrix in $\mathbb{Z}/2\mathbb{Z}$, we can have the plain text in $\mathbb{Z}/2\mathbb{Z}$ using the linearity of the cipher.
To get the key matrix in $\mathbb{Z}/26\mathbb{Z}$, we can use the Chinese Reminder Theorem , but we would get a complexity of $O(13^d)$. In the previous paper , it was believed that it's possible to get the key matrix in $\mathbb{Z}/26\mathbb{Z}$ without considering $\mathbb{Z}/13\mathbb{Z}$.
First of all , we create a hash table using long text , and search mapping between segments of reference text and plain text modulo 2
#(seg in reference) = len(reference text) - n +1 , with n the segment size.
Indeed , if you take the following text : *thisisatest* , with n = 5 , you get the following segment:
$thisi, hisis, isisa, sisat, isate, sates, atest$ which is 7 segments $11 - 5 + 1 = 7$

We get the same thing for $\#(strinplain) = len(plaintext) - n + 1$, with n the string size.
Then we define the good matching : segments are equals before and after modulo 2 , and bad matching segment which are not equal but equal modulo 2.
We use Rényi entropy to get the good matching and all matching as it find the collision , with the following formula :

$$H_\alpha(X) = \frac{1}{1-\alpha} log_2(\sum_{i=1}^{n} Pr(X = i)^\alpha)$$

When alpha has the value 2 , we just get the following :

$$-log_2(\sum_{i=1}^{n} Pr(X = i)^2)$$

that gives us the probability that a segment equals another one.

For good matching , we have $E(\#goodmatching) = (\#segmentsinreference) \times (\#segmentinplaintext) \times 2^{-H_2(X)}$, as the number of good matching is actually the collision between segment in plaintext and segment in reference text multiplied by the entropy of rényi of this segment (which represents the rate of collision for a given block X).
Then you do the same for $E(\#allmatching)$, the difference is that you do it this way : $E(\#goodmatching) = (\#segmentsinreference) \times (\#segmentinplaintext) \times 2^{-H_2(Xmod2)}$. And indeed you understand that if 2 words modulo 2 are equals, these words are not always equals modulo 26.

For the $E(\#allmatching)$ the calculation is really simple , you must take $(\#segmentsinreference) \times$

$(\#segmentinplaintext) \times 2^{-H_2(Xmod2)}$ as we do all the possibles matching.

$H_2(Xmod2) = -log_2(\sum_{i=1}^{n} Pr(X=i)^2)$, where $Pr(X=i)$ declined in $Pr(X=0)$ and $Pr(X=1)$

From diverse calculation we always get 0.5^n so E(# all matching) is always equals to $(\#segmentsinreference)x(\#segmentinp$

0.5^n Then to have an idea of the complexity , you do the ratio $\frac{E(\#goodmatchings)}{E(\#allmatchings)}$, you generaly found $\frac{1}{8^n}$

In the following parts , the calculation of $E(\#allmatchings)$ are done again thanks to a Java programm.

But to have a better complexity , we need to increase the ratio of good matching as $E(\#allmatchings)$ can't be changed so we can only try on $E(\#goodmatchings)$, with different assumptions and calculations.

Study of Faster Fourrier Transform for Algorithm 1

With a fast fourrier Transform (FFT) the complexity is $O(NlogN)$ for N the input size

Deteministic Sparse Fourier Approximation via Fooling Arithmetic Progressions

If we only want to have the few significant Fourrier Coefficient , we can use this.

Here if we gave a threshold $\tau \in (0,1]$ and an oracle access to a function f , it outputs the τ-significant Fourier Coefficient. This is called SFT and runs in $log(N), \frac{1}{\tau}$

An oracle access to a function take as input x and return the f(x) of the function f.

This algorithm is robust to random noise and local (mean polynomial time)

It's based on partition of set by binary search , you have at the beginning 4 intervals , you test for the two first if the norm of f's Fourier Transform squared is equals to the set_i oracle ouput squared

Meaning more explicitly : $f(J_i)^2 = \sum_{\alpha \in J_i} |f(\alpha)|^2$ If this pass , it will output yes , and we'll be able to continue the algorithm by replacing the J and insert the J_i

The heart of the code is actually to decide which intervals potentially contain a significant Fourier coefficient.

Yes if weight on J , exceeds significant threshold τ , NO if J larger.

The threshold τ can be chosen , with the fact that a α is a $\tau - significant$ Fourier coefficient iff $|\hat{f}|^2 \geq \tau||f||_2^2$ where $\hat{f} =< f, X_\alpha >$ and $X_{alpha} = e^{2\pi i \alpha x/N}$

Nearly optimal Sparse Fourier Transform

We want here to compute the k-sparse approximation to the discrete Fourier transform of an n-dimensional signal.

There is to time in function of the number of input has at most k non-zero Fourier Coefficient.

In this case , we got $O(k.log(n))$ time , else we have $O(k.log(n).log(\frac{n}{k}))$

The basis is still the same, if a signal has a small number k of non-zero Fourier , the output of this DFT can be represented succinctly using only k coefficient.

What is required , is that the input size n is a power of 2.

This algorithm seems to restrictive and also perform the same in the worst case.

Combinatorial sub linear-Time Fourier Algorithm

You have a vector A of length $n >> k$ you identify the k largest frequencies of the transform of A , getting polynomial time $(k, log(n))$ for the algorithm.

Simple and practical algorithm for sparse Fourier transform

Here you consider a complex vector x of length n This algorithm compute the k-sparse Fourier transform in $O(\sqrt{knlog^{3/2}n})$, if x is sparse then you find it in exactly $O(klog^2 n)$, but in general estimate x is approximately $O(\sqrt{nk})$
So this algorithm is better if the ratio $\frac{n}{k} \in [2 \times 10^3, 10^6]$, but it's clearly not the best one as those before are supposed to find it in a lower complexity $(klog(n))$.

Experiment

Probability of the independent English letters

From the frequency letter given by Wikipédia , in english we got the following result :
Proba sum = 0.9999999999999999
Sum of probability squared = 0.06549717159999999 , which corresponds to $(\sum_{i=0}^{26} Pr(i = y)^2)^n, y \in \{alphabet\}$
Sum of probability that gives 0 modulo 2 squared = 0.32298762240000006 which corresponds to which $(\sum_{i=0}^{26} Pr(i = 0)^2)^n, i \in \{alphabetmodulo2\}$
Sum of probability that gives 1 modulo 2 squared = 0.18634762239999997 which corresponds to which $(\sum_{i=0}^{26} Pr(i = 1)^2)^n, i \in \{alphabetmodulo2\}$
Ration of good matching and all matching=0.1285934407027314^n
So $\frac{1}{7,77644^n}$

Another site , with some novel and book from Edgar Poe , and articles from encyclopedia :

proba sum = 0.9999000000000001
sum of probability squared = 0.06609151 which corresponds to $(\sum_{i=0}^{26} Pr(i = y)^2)^n, y \in \{alphabet\}$
sum of probability that gives 0 modulo 2 squared = 0.32001649 which corresponds to which $(\sum_{i=0}^{26} Pr(i = 0)^2)^n, i \in \{alphabetmodulo2\}$
sum of probability that gives 1 modulo 2 squared = 0.18852964 which corresponds to which $(\sum_{i=0}^{26} Pr(i = 1)^2)^n, i \in \{alphabetmodulo2\}$
Ration of good matching and all matching=0.12996168115565054^n
So $\frac{1}{7,69457^n}$

Probability if you consider blocks of size d

So calculation are done on a text of approximately 860000 characters to see the evolution of the ratio good matchingbad matching.
A program is ran to see the evolution for a block size between 1 and 25 , and give the ratio , and also the probability that a block appears.It is completely heuristic as I'm just counting the number of block that appears and do some manipulation with it. So the basic is to choose a block size , then it'll count every different blocks that appears modulo 26 and modulo 2. Then it'll compute the probability that a good matching happen with the following : $\sum_{X \in block} \frac{\#X-1}{\#block-1}^2$ With this , the evolution of the ratio in function of the block size looks like this:

Algorithm

You hash a reference text.
You take the key matrix that you get from algorithm 1 , find plain text in $\mathbb{Z}/2\mathbb{Z}$, and create an array.
find the list of all matchings : find all pairs(seg,str) such that seg is a segment of plaintext modulo 2 and str $\in hash(seg)$ and save it in a list.

```
repeat
    select d matching form list (you'll get a dxd key matrix)
    for each of these matchings ($seg_i$, $str_i$)
        extract $block_i$ from $seg_i$ and $str'_i$ from $str_i$,
        then find $ciphertext_i$ such that $K^{-1}$ x $ciphertext_i$ mod 2 = $block_i$
    solve $ciphertext_i = K * str'_i$ for i=1 to d
    compute $K^{-1}$*ciphertext
until it makes sense
```

number of iteration is $\frac{1}{ratio^{nd}} = 8^{nd}$

The following algorithm is to recover the key matrix in $\mathbb{Z}/2\mathbb{Z}$
Part1:

```
You require Ciphertext $Y_1, Y_2, ..., Y_n$
for all $\mu$ do compute $S_n(\mu) = \sum_{k=1}^{n} (-1)^{\mu.y} \times n_y$ where $n_y = \#\{k; Y_k = y\}$
endfor
set all $\mu$ to the d values of $\mu$ with largest $S_n(\mu) = bias(\mu.Y)$
```

Part2:

```
for all (i,i') do
    compute $n_{00}(i,i') = \#\{k < n : (\mu_i.Y_k, \mu'_i.Y_{k+1}) = (0,0)\}$ endfor
set $(i_d, i_1)$ to the first pair with lowest $n_{00}$
```

Part3:

```
for all $t = 2$ to $d - 1$ do
    for all i $\notin \{i_1, i_2, .., i_{t-1}, i_d\}$ do
        compute $n_{00}(i,i') = \#\{k : (\mu^T_{i_{t-1}} Y_k, \mu_i^T Y_k) = (0,0)\}$
    endfor
    take i such that $n_{00}$ is minimum and set $i_t = i$
endfor
set $\mu = (\mu_{i1}, \mu_{i1}, ..., \mu_{id})$ and $K = (\mu^{-}1)^T$ output K
```

Here to be faster we store n_y in a table and we do a FFT on this table to get S_n. With this operation the total complexity drop from $O(d^2 \times 2^d)$ to $O(d \times 2^d)$ But it seems with some other techniques we could do better.