Time-Series Analysis: Trend Extraction

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Outline

- Trend extraction (filtering) is a major task of time series analysis
- Trend of a time-series is considered to contain the global change, which contrasts with local changes due to noise
- Denoise + track the dynamics of the underlysing process

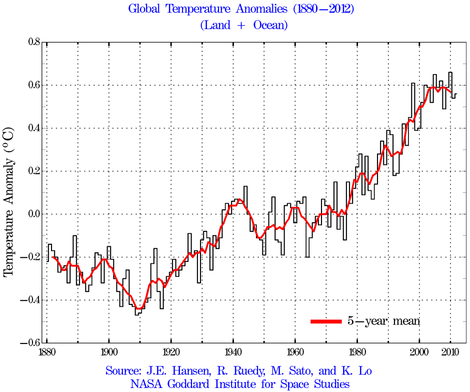

Figure – Example of a climatological time-series with trend

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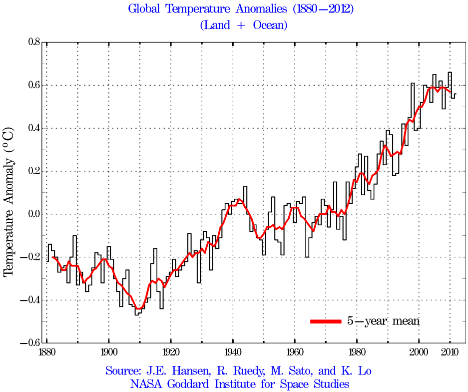

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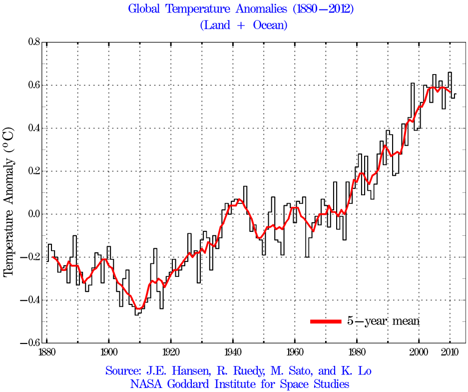

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$$y_t = x_t + \varepsilon_t,$$

where x_t is a trend, ε_t is a stochastic (or noise) process

- "*[...] the essentil idea of trend is that it shall be smooth.*" (Kendall, 1973). In statistical terms

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- Finite moving average filtering for trend estimation has been used in actuarial science since the beginning of the 20th century
- Modern theory was developed independently by Andrei Kolmogorov and Norbert Wiener in 1941
- H. Wold, P. Whittle, R. Kalman, G. Box etc. extensively developed the field extraction methods are applied in different areas: economics, climatology, etc.
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- Let $y = \{y_1, \ldots, y_m, \ldots\}$ be observations y_t in discrete moments $t_i = i\Delta$ (for simplicity we assume that $\Delta = 1$)
- A filtering procedure consists of applying a filter $\mathcal{L}$ to data y

$$\widehat{x} = (\widehat{x}_1, \ldots, \widehat{x}_m, \ldots) = \mathcal{L}(y)$$

- We consider time invariant and causal filters

$$\widehat{x}_t = \sum_{i=0}^{n-1} \mathcal{L}_i y_{t-i}$$

- The well-known Moving Average filter of length n

$$\mathcal{L}_i = \frac{1}{n} 1\{i < n\},$$

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- The drift

$$\widehat{\mu}_t \approx \frac{d}{dt}\widehat{x}_t \approx \sum_{i=0}^{n} \mathfrak{l}_i y_{t-i}$$

where

$$\mathfrak{l}_i = \begin{cases} \mathcal{L}_0 & \text{if } i = 0, \\ \mathcal{L}_i - \mathcal{L}_{i-1} & \text{if } i = 1, \dots, n-1, \\ -\mathcal{L}_{n-1} & \text{if } i = n, \end{cases}$$

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- Improvement of the Uniform Moving Average Filter

$$\mathfrak{l}_i = \frac{4}{n^2}\operatorname{sgn}\left(\frac{n}{2} - i\right) \Leftrightarrow \mathcal{L}_i = \frac{4}{n^2}\left(\frac{n}{2} - \left|i - \frac{n}{2}\right|\right)$$

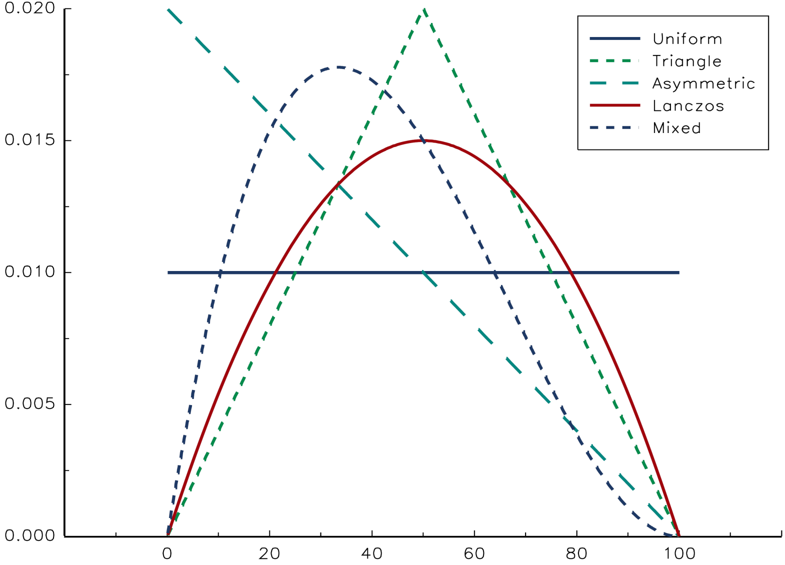
- Assymetric window function with a triangular form

$$\mathfrak{l}_i = \frac{2}{n}\left(\delta_i - 1\{i < n\}\right) \Leftrightarrow \mathcal{L}_i = \frac{2}{n^2}(n-i)1\{i < n\}$$

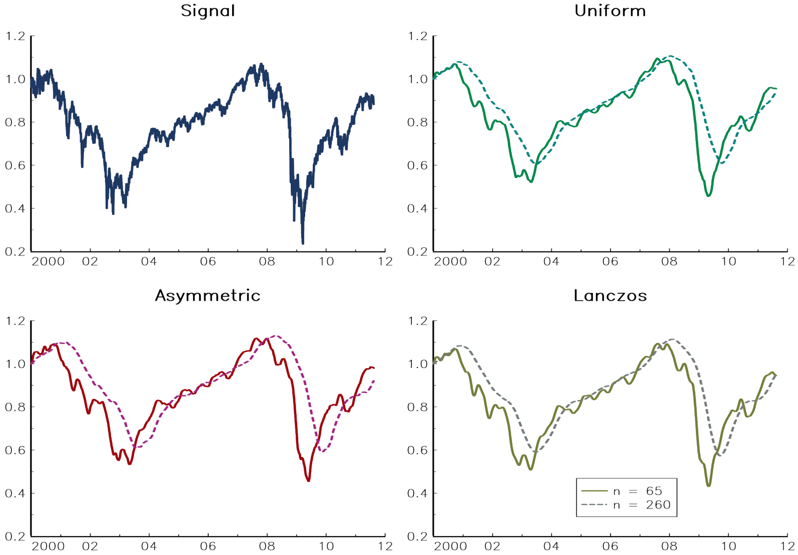
- The Lanczos derivative $\frac{d^L}{dt}f(t) = \lim_{h \to 0} \frac{\sum_{k=-n}^{n} k f(x+kh)}{2\sum_{k=1}^{n} k^2 n}$, so estimating the derivative of the trend at the point $t - n/2$ we get

$$\mathfrak{l}_i = \frac{12}{n^3}\left(\frac{n}{2} - i\right)1\{0 \le i \le n\} \Leftrightarrow \mathcal{L}_i = \frac{6}{n^3}(n-i)1\{0 \le i \le n\}$$

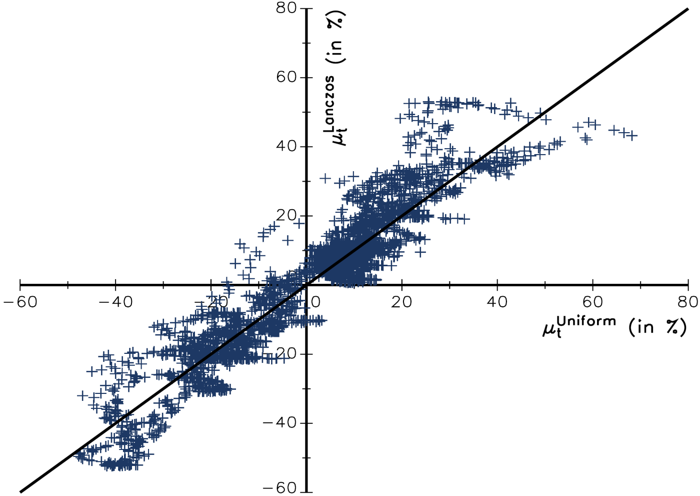
Window function $\mathcal{L}_i$ $(n = 100)$



Trend estimate for the $S\&P$ 500 index



Correlation between uniform and Lanczos derivatives



- Least squares methods are often used to define trend estimators:

$$\{\widehat{x}_1, \dots, \widehat{x}_n\} = \arg \min_{\{\widehat{x}_t\}_{t=1}^n} \frac{1}{2} \sum_{t=1}^{n} (y_t - \widehat{x}_t)^2 .$$

- The problem is not well-defined $\Rightarrow$ impose restrictions on the underlying process y_t or on the filtered trend $\widehat{x}_t$, e.g.:
  - deterministic constant model $x_t = x_{t-1} + \mu \Leftrightarrow y_t = \mu t + \varepsilon_t$
  - smooth trend condition

$$\frac{1}{2} \sum_{t=1}^{n} (y_t - \widehat{x}_t)^2 + \lambda \sum_{t=2}^{n-1} (\widehat{x}_{t+1} - 2\widehat{x}_t + \widehat{x}_{t-1})^2 \to \min_{\{\widehat{x}_t\}_{t=1}^n}$$

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- The objective function is equivalent to

$$\frac{1}{2}\|y-\widehat{x}\|_2^2+\lambda\,\|D\widehat{x}\|_2^2\,,$$

where $y=(y_1,\dots,y_n)$, $\widehat{x}=(\widehat{x}_1,\dots,\widehat{x}_n)$, $D\in\mathbb{R}^{(n-2)\times n}$,

$$D=\begin{pmatrix}1 & -2 & 1 & & & & \\ & 1 & -2 & 1 & & & \\ & & & \ddots & & & \\ & & & 1 & -2 & 1 & \\ & & & & 1 & -2 & 1\end{pmatrix}.$$

- The estimator (Hodrick-Prescott filter) is then given by the following solution

$$\widehat{x}=(I+2\lambda D^{\mathrm{T}}D)^{-1}y.$$

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- The trend μ_t is a hidden process which follows a given dynamic

$$\begin{cases} y_t = x_t + \sigma_\xi \xi_t, \\ x_t = x_{t-1} + \sigma_\eta \eta_t \end{cases}$$

- We define $\widehat{x}_{t|t-1} = \mathbb{E}_{t-1} x_t$, $P_{t|t-1} = \mathbb{E}_{t-1} \left(\widehat{x}_{t|t-1} - x_t \right)^2$.
- Optimal in L_2 sense estimate is

$$\widehat{x}_{t+1|t} = (1 - K_t) \widehat{x}_{t|t-1} + K_t y_t,$$

where the Kalman gain

$$K_t = \frac{P_{t|t-1}}{P_{t|t-1} + \sigma_\xi^2},$$

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$$P^* = \frac{\sigma_\eta}{2}\left(\sigma_\eta + \sqrt{\sigma_\eta^2 + 4\sigma_\xi^2}\right).$$

- In the long run the filter equation becomes

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- This Kalman filter can be approximated by an exponential moving average filter with parameter $\lambda = -\log(1-k)$

$$\widehat{x}_t = (1-e^{-\lambda})\sum_{i=0}^{\infty} e^{-\lambda i} y_{t-i}$$

and the drift coefficient

$$\widehat{\mu}_t = (1-e^{-\lambda})y_t - (1-e^{-\lambda})(e^{\lambda}-1)\sum_{i=1}^{\infty} e^{-\lambda i} y_{t-i}.$$

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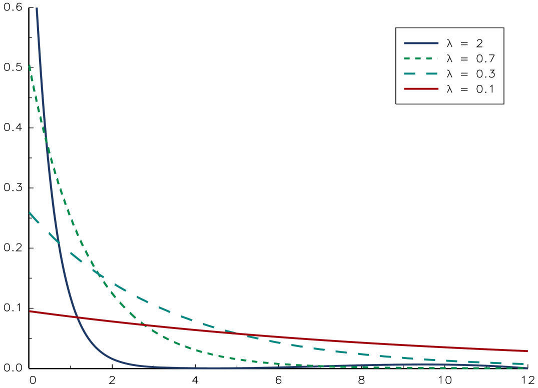
- The half-life of this filter is approximately equal to $\left\lceil \left(\lambda^{-1} - 2^{-1}\right) \log 2 \right\rceil$. For example, the half-life for $\lambda = 5\%$ is 14 days
- Window function $\mathcal{L}_i$



- More general (local linear) trend model (the slope of the trend is stochastic)

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- Remarks
  - The Kalman filter is optimal in the case of the linear Gaussian model
  - Efficient computational solution of the least squares method
  - Can be applied to more sophisticated models
  - Kalman smoother improves the estimate of $\widehat{x}_{t-i}$ by using all the information between $t-i$ and t

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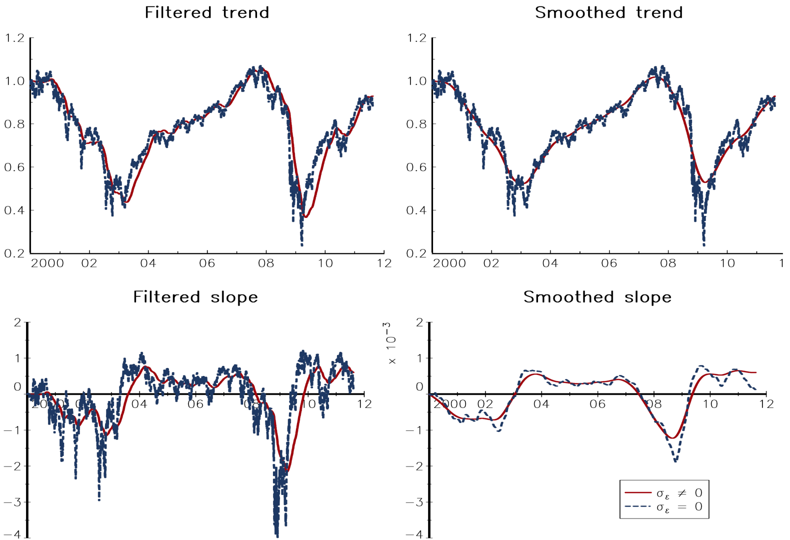
- The filtered and smoothed components x_t and μ_t
- The Kalman smoother: more information $\Rightarrow$ less noise
- $\sigma_\varepsilon = 0 \Rightarrow$ bigger variance of the trend and slope estimators



- Parametric model $x_t = \mu t \to$ Nonparametric model $x_t = f(t)$
- Local polynomial regression

$$y_t = f(t) + \varepsilon_t = \beta_0(t_0) + \sum_{j=1}^{p} \beta_j(t_0)(t_0 - t)^j + \varepsilon_t$$

- For a given t_0, a kernel function $K(t)$ and a kernel width h parameters $\beta_j(t_0)$ are estimated as

$$\sum_{t=1}^{n} \left(y_t - \beta_0(t_0) - \sum_{j=1}^{p} \beta_j(t_0)(t_0 - t)^j \right)^2 \omega_t \to \min_{\{\beta_j(t_0)\}_{j=0}^{p}},$$

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- Improvement to the kernel regression: two-stage procedure
- First, fit a local polynomial regression to estimate the residuals $\widehat{\varepsilon}_t$
- Second, compute $\delta_t = (1 - u_t^2) 1 \{|u_t| \leq 1\}$ with

$$u_t = \frac{\widehat{\varepsilon}_t}{6 \cdot \mathrm{median}\,(|\widehat{\varepsilon}|)}$$

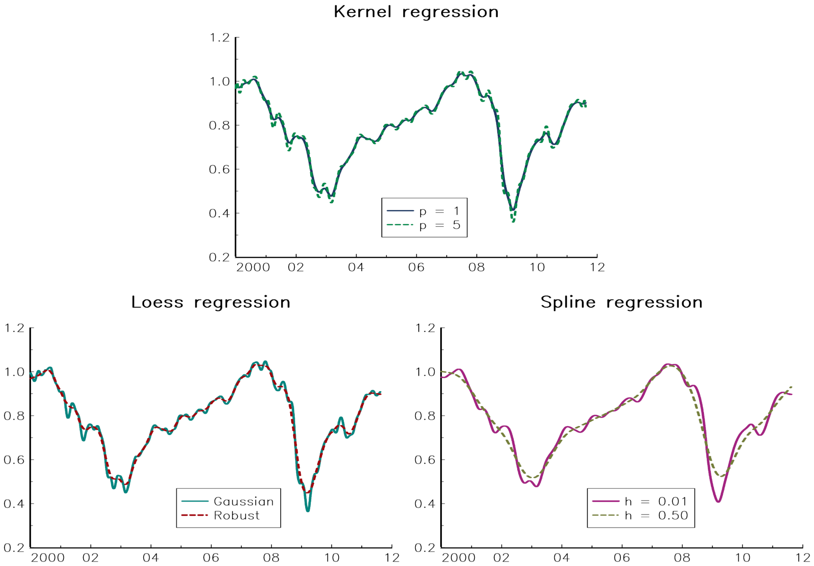
and run a local polynomial regression with weights $\delta_t \omega_t$

- Spline $= C^2$ function $S(t)$ which corresponds to a cubic polynomial function on each interval $[t, t+1)$.
- Let $\mathcal{SP}$ be the set of spline functions
- Spline regression

$$(1-h)\sum_{t=0}^{n}\omega_t(y_t - S(t))^2 + h\int_0^T \omega_\tau[S''(\tau)]^2 d\tau,$$

  where $T = n \cdot \mathit{\Delta}$
- $h = 0 \Leftrightarrow$ interpolation and $h = 1 \Leftrightarrow$ linear regression

Kernel, Loess and Spline Regression



- Lasso penalty

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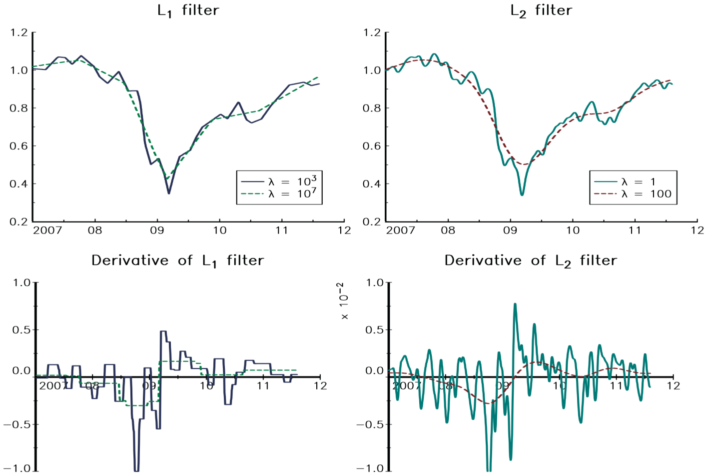
- Filtered signal comprises a set of straight trends and breaks
- Smoothing parameter λ influence on the number of breaks
- It is easy to estimate the slope $\widehat{\mu}$



- The trend is "located" in low frequencies
- Fourier transform

$$y(\omega) = \mathcal{F}(y) = \sum_{t=1}^{n} y_t e^{-i\omega t}$$

- Remove high frequencies (thresholding) and estimate the trend $\widehat{x} = \mathcal{F}^{-1}(y(\omega))$
- **Problem**: bad time location for low frequency signals and bad frequency location for the high frequency signals $\Rightarrow$ difficult to localize when the trend reverses (nonstationary/transient process)

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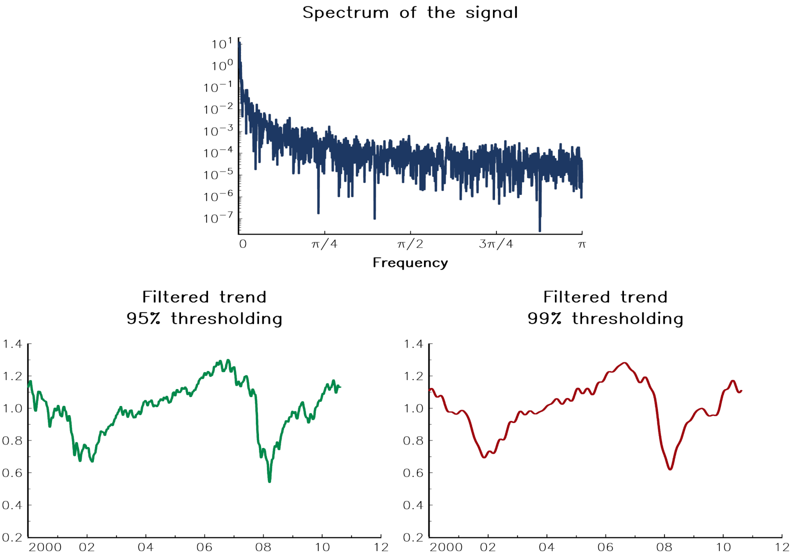
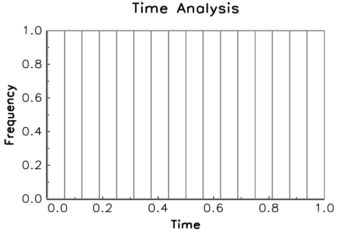
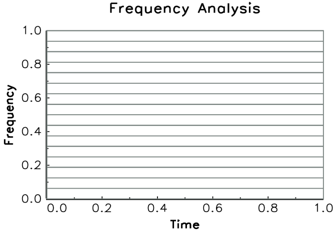
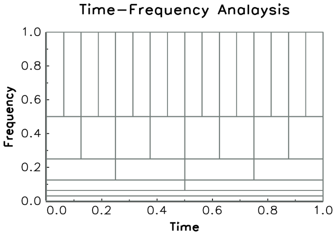
- Denoising based on Fourier transform



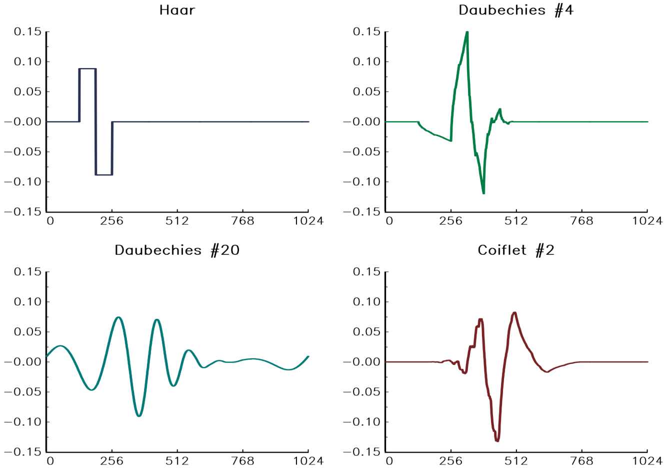
- Spectral analysis both in time and frequency







- Localized basis functions



- Compute the wavelet transform $\omega = (s_0, d_{j,k}, j = \overline{0, J-1}, k = \overline{0, 2^j - 1}) = \mathcal{W}(y)$. This corresponds to the representation

$$x(t) = s_0\phi(t) + \sum_{j=0}^{J-1} \sum_{k=0}^{2^j - 1} d_{j,k}\psi_{j,k}(t),$$

$$\psi_{j,k}(t) = 2^{\frac{j}{2}}\psi\left(2^j t - k\right)$$

- Multiscale representation

$$\begin{tikzcd}
 & & & & x \arrow[dl] \arrow[dr] & \\
 & & & s \arrow[dl] \arrow[dr] & & d \\
 & & ss \arrow[dl] \arrow[dr] & & sd & \\
 & sss \arrow[dl] \arrow[dr] & & ssd & & \\
ssss & & sssd & & &
\end{tikzcd}$$

- Modify the wavelet coefficients according to the thresholding rule

$$\omega^* = D(\omega),$$

where e.g.

  - Hard thresholding

$$\omega_i^* = \omega_i \cdot 1\left\{|\omega_i| > \omega^+\right\},$$

  - Soft thresholding

$$\omega_i^* = \mathrm{sign}(\omega_i) \cdot \max\left\{|\omega_i| - \omega^+, 0\right\}$$

- Apply the inverse wavelet transform

$$\widehat{x} = \mathcal{W}^{-1}(\omega^*)$$

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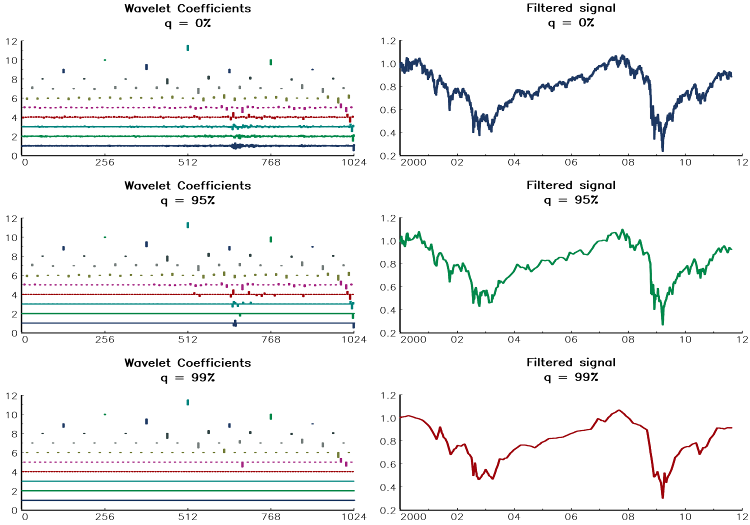
- Low-pass and high-pass filters Daubechies 6
- We remove 95% and 99% wavelet coefficients



- Singular Spectrum Analysis
- Empirical Mode Decomposition
- etc.

- $y = (y_1, \ldots, y_t)$ is transformed into Hankel matrix H of the m concatenated lag vectors of y, where the window length $n = t - m + 1$, $m < t/2$

$$H = \begin{pmatrix} y_1 & y_2 & y_3 & \cdots & y_m \\ y_2 & y_3 & y_4 & \cdots & y_{m+1} \\ y_3 & y_4 & y_5 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & y_{t-1} \\ y_n & y_{n+1} & y_{n+2} & \cdots & y_t \end{pmatrix}$$

- The time series can be recovered from the matrix H as

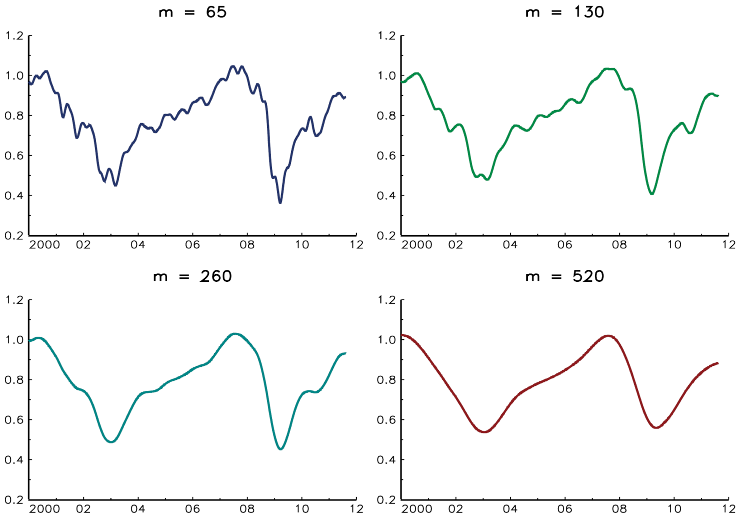
$$y_p = \frac{1}{\alpha_p} \sum_{j=1}^{m} H^{(i,j)},$$

where $i = p - j + 1$, $0 < i < n + 1$ and

$$\alpha_p = \begin{cases} p & \text{if } p < m, \\ t - p + 1 & \text{if } p > t - m + 1, \\ m & \text{otherwise }. \end{cases}$$

- Let $C = H^{\mathrm{T}} H$ be the covariance matrix of H
- We recover $\widehat{H}$ using $k < m$ singular vectors and values
- We remove noise and obtain $\widehat{x}$ by recovering it not from H but from $\widehat{H}$

- SSA: Only the first eigenvector is used



- Very often trend and residual models are defined implicitly by the computational procedure used for a trend extraction
- Selection of the particular model/method is determined by the subsequent usage of the extracted trend:
  - Filtering for ex-post analysis and separation of positive/negative trends
  - Prediction of the future signal values

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- **Local Polynomial Regression**. Trend is globally smooth and locally approximated by a polynomial
- **Hodrick-Prescott**. No model
- **SSA**. Large window length m: deterministic (finite rank time series); short m: no model
- **Wavelets**. Semi-parametric, specified by the wavelet

- **Local Polynomial Regression**. A stationary and invertible ARMA process, usually $NID(0, \sigma^2)$
- **Hodrick-Prescott**. No model
- **SSA**. Large window length m: typically a combination of cycle and seasonal components with varying amplitudes plus irregular component; short m: irregular component
- **Wavelets**. Very general, could include a combination of cycles and irregular components

- **Local Polynomial Regression**. Polynomial degree, kernel type, kernel width
- **Hodrick-Prescott**. Regularization coefficient
- **SSA**. Window length m, the number of SVD components
- **Wavelets**. Wavelet basis, levels used for trend reconstruction

- **Local Polynomial Regression**. Pros: fast, simple, a few prespecifications is required. Cons: a residual of a complex structure is not allowed, only seasonally adjusted data
- **Hodrick-Prescott**. Pros: the same as above. Cons: the same as above
- **SSA**. Pros: a few prespecifications, can separate a trend from a complex residual, good for time-seris with a large noise. Cons: few theoretical studies of trend estimators, computational complexity of SVD, small $m \to$ seasonally adjusted data
- **Wavelets**. Pros: efficients algorithms, many available wavelet bases, good smoothing properties. Cons: subjective choice of levels used for trend reconstruction, boundary effects