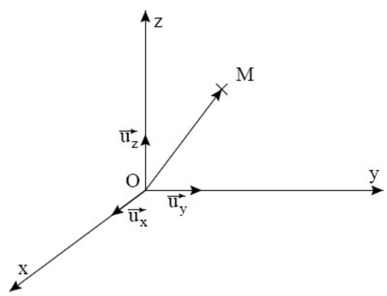
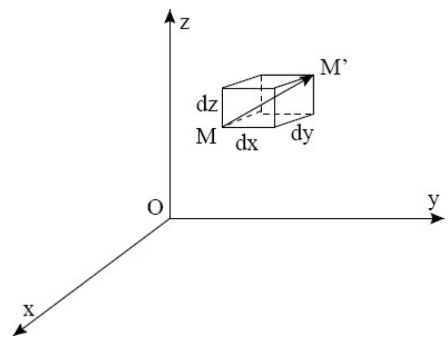
# CARTESIAN, CYLINDRICAL, SPHERICAL COORDINATES

Consider point $M$ and the reference frame $\Re = \left(O; \vec{u}_x, \vec{u}_y, \vec{u}_z\right)$. All velocities and displacements in this chapter are calculated in the $\Re$ reference frame.

## I. CARTESIAN COORDINATES

Point $M$ is identified by Cartesian coordinates $(x, y, z)$.




$-\infty < x, y, z < \infty$

$\overrightarrow{OM} = x\vec{u}_x + y\vec{u}_y + z\vec{u}_z$

$$\vec{v} = \frac{\mathrm{d}\overrightarrow{OM}}{\mathrm{d}t} = \frac{\mathrm{d}t}{\mathrm{d}t} = x\vec{u}_x + y\vec{u}_y + z\vec{u}_z = \frac{\mathrm{d}x}{\mathrm{d}t}\vec{u}_x + \frac{\mathrm{d}y}{\mathrm{d}t}\vec{u}_y + \frac{\mathrm{d}z}{\mathrm{d}t}\vec{u}_z$$

**The elementary displacement is:** $\mathrm{d}\vec{l} = \overrightarrow{MM'} = \mathrm{d}x\vec{u}_x + \mathrm{d}y\vec{u}_y + \mathrm{d}z\vec{u}_z$.
**It is used to calculate elementary surfaces and volumes.**

We deduce that: $\mathrm{d}\tau = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$.

$x = cte$ : $\mathrm{d}S_x = \mathrm{d}y\,\mathrm{d}z$

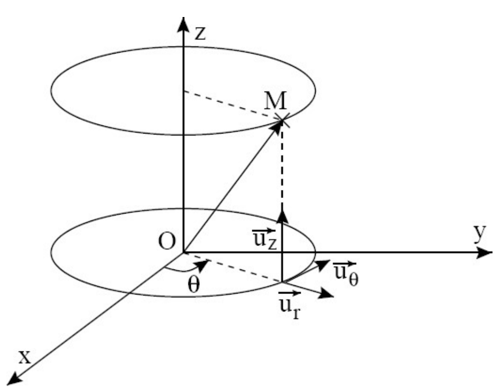
$y = cte$ : $\mathrm{d}S_y = \mathrm{d}x\,\mathrm{d}z$

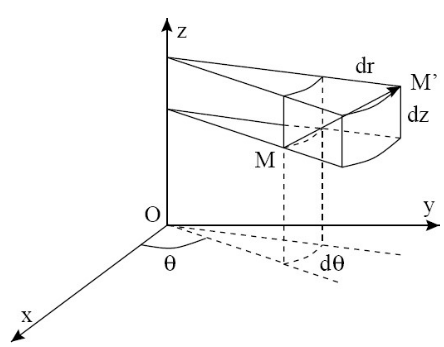
$z = cte$ : $\mathrm{d}S_z = \mathrm{d}x\,\mathrm{d}y$

## II. CYLINDRICAL COORDINATES

The point $M$ is marked by the cylindrical coordinates $(r, \theta, z)$.

**Cylindrical coordinates are used whenever distance from the $Oz$ axis plays an important role in the exercise.**



$\boxed{0 \le r < \infty, 0 \le \theta \le 2\pi, -\infty < z < +\infty}$

$\overline{OM} = r\overline{u}_r + z\overline{u}_z$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

$$\overline{v} = \frac{\mathrm{d}\overline{OM}}{\mathrm{d}t} = \frac{\mathrm{d}\overline{r}}{\mathrm{d}t} = r\overline{u}_r + r\theta\overline{u}_\theta + z\overline{u}_z = \frac{\mathrm{d}r}{\mathrm{d}t}\overline{u}_r + r\frac{\mathrm{d}\theta}{\mathrm{d}t}\overline{u}_\theta + \frac{\mathrm{d}z}{\mathrm{d}t}\overline{u}_z$$

**The elementary displacement is:** $\mathrm{d}l = \overline{MM'} = \mathrm{d}r\overline{u}_r + r\mathrm{d}\theta\overline{u}_\theta + \mathrm{d}z\overline{u}_z$**.**
**It is used to calculate elementary surfaces and volumes.**

We deduce: $\mathrm{d}\tau = (\mathrm{d}r)(r\mathrm{d}\theta)(\mathrm{d}z)$.

$r = cte$ : $\mathrm{d}S_r = r\mathrm{d}\theta\,\mathrm{d}z$

$\theta = cte$ : $\mathrm{d}S_\theta = \mathrm{d}r\,\mathrm{d}z$

$z = cte$ : $\mathrm{d}S_z = \mathrm{d}r\,r\mathrm{d}\theta$

We often need the elementary volume between cylinders of radius $r$ and radius $r + \mathrm{d}r$.

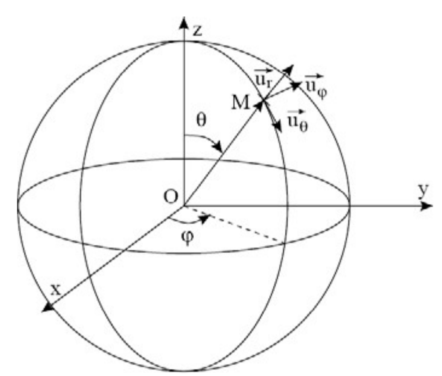
$$\pi(r+\mathrm{d}r)^2 H - \pi r^2 H = \pi r^2\left(1+\frac{\mathrm{d}r}{r}\right)^2 H - \pi r^2 H = \pi r^2\left(1+\frac{2\mathrm{d}r}{r}\right)H - \pi r^2 H = 2\pi r\mathrm{d}rH$$

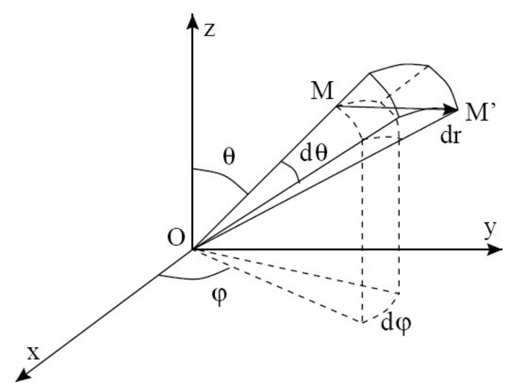
**The elementary volume between cylinders of radius *r* and radius *r* + d*r* is the area of the cylinder of radius *r* and height *H* multiplied by d*r* :** $\mathrm{d}\tau = 2\pi r\mathrm{d}rH$

## III. SPHERICAL COORDINATES

The point $M$ s marked by the cylindrical coordinates $(r,\theta,\phi)$.

**Spherical coordinates are used whenever distance from the center plays an important role in the exercise.**




Terrestrial geography:

$\overline{u}_r$ is directed vertically upwards from the location.

$\overline{u}_\theta$ faces south.

$\overline{u}_\phi$ faces east.

$\theta$ is called colatitude. $\phi$ is the longitude.

$\boxed{0 \le r < \infty, 0 \le \theta \le \pi, 0 \le \phi \le 2\pi}$

$\overline{OM} = r\overline{u}_r$

$$\begin{cases} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{cases}$$

$$\vec{v} = \frac{\mathrm{d}\vec{OM}}{\mathrm{d}t} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \dot{r}\vec{u}_r + r\dot{\theta}\vec{u}_\theta + r\sin\theta\dot{\phi}\vec{u}_\phi = \frac{\mathrm{d}r}{\mathrm{d}t}\vec{u}_r + r\frac{\mathrm{d}\theta}{\mathrm{d}t}\vec{u}_\theta + r\sin\theta\frac{\mathrm{d}\phi}{\mathrm{d}t}\vec{u}_\phi$$

**The elementary displacement is:** $\mathrm{d}\vec{l} = \vec{MM'} = \mathrm{d}r\vec{u}_r + r\mathrm{d}\theta\vec{u}_\theta + r\sin\theta\mathrm{d}\phi\vec{u}_\phi$**.**

**It is used to calculate elementary surfaces and volumes.**

We deduce : $\mathrm{d}\tau = (\mathrm{d}r)(r\mathrm{d}\theta)(r\sin\theta\,\mathrm{d}\phi)$ .

$r = cte$ : $\mathrm{d}S_r = (r\mathrm{d}\theta)(r\sin\theta\mathrm{d}\phi)$

$\theta = cte$ : $\mathrm{d}S_\theta = (\mathrm{d}r)(r\sin\theta\mathrm{d}\phi)$

$\phi = cte$ : $\mathrm{d}S_\phi = \mathrm{d}r\, r\mathrm{d}\theta$

We often need the elementary volume between spheres of radius $r$ and radius $r + \mathrm{d}r$.

$$\frac{4}{3}\pi(r+\mathrm{d}r)^3 - \frac{4}{3}\pi r^3 = \frac{4}{3}\pi r^3\left(1+\frac{\mathrm{d}r}{r}\right)^3 - \frac{4}{3}\pi r^3 = \frac{4}{3}\pi r^3\left(1+\frac{3\mathrm{d}r}{r}\right) - \frac{4}{3}\pi r^3 = 4\pi r^2\mathrm{d}r$$

**The elementary volume between spheres of radius *r* and radius *r* + d*r* is the area of the sphere of radius *r* multiplied by d*r* :** $\mathrm{d}\tau = 4\pi r^2\mathrm{d}r$

## IV. VECTOR PRODUCT WITH A DIRECT ORTHONORMAL BASIS

Exercises often involve calculating $\vec{u}_z \wedge \vec{u}_\theta$ in the exercises.

**A mnemonic is to write the 6 unit vectors in sequence:** $\vec{u}_r, \vec{u}_\theta, \vec{u}_z, \vec{u}_r, \vec{u}_\theta, \vec{u}_z$ **.**

**If three unit vectors follow, then** $\vec{u}_3 = \vec{u}_1 \wedge \vec{u}_2$ **:** $\vec{u}_r = \vec{u}_\theta \wedge \vec{u}_z$ **or** $\vec{u}_\theta = \vec{u}_z \wedge \vec{u}_r$

**If not, enter a negative sign:** $\vec{u}_z \wedge \vec{u}_\theta = -\vec{u}_r$

**It's very convenient to use without having to use all three fingers of your hand all the time!!!!**