** Solving Linear Recurrences :- (Homogenous eq^n)

Eg. $F(N) = F(N-1) + F(N-2)$

form:-

$$f(x) = a_1 f(x-1) + a_2 f(x-2) + a_3 f(x-3) + \dots + a_n f(x-n)$$

$$\therefore f(x) = \sum_{i=1}^{n} a_i f(x-i)$$ for a_i, n is fixed.

n is the order of recurrence.

Solution :- for fifth fibonacci no.

$$f(n) = f(n-1) + f(n-2) \quad -①$$

steps:-

1) Put $f(n) = \alpha^n$ for some constant α.

$$\therefore \alpha^n = \alpha^{n-1} + \alpha^{n-2}$$

$$\alpha^n - \alpha^{n-1} - \alpha^{n-2} = 0$$

$\div$ by α^{n-2}

$$\alpha^2 - \alpha - 1 = 0 \quad -(a)$$

$\frac{1}{\alpha^{-1}} = \alpha$

$\therefore \frac{\alpha^n}{\alpha^{n-2}}$

$\frac{\alpha^n \times \alpha^2}{\alpha^n}$

This eq^n is also known as characteristics of recurrence

2) take roots of (a) by using quadratic eqn.

(1)

$$\alpha^2 - \alpha - 1 = 0$$

formula :- $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$\therefore \alpha = \frac{1 \pm \sqrt{5}}{2}$$

$$\alpha_1 = \frac{1+\sqrt{5}}{2} \qquad \alpha_2 = \frac{1-\sqrt{5}}{2}$$

(2) If α_1 & α_2 has 2 roots, you can write the eqn in such a way

$f(n) = c_1 \alpha_1^n + c_2 \alpha_2^n$ is a soln for fibonacci.

it will equal to

$= f(n-1) + f(n-2)$

Note:-

Not just for this any equation you have the number of roots, take that many no. of constants & × that many no. of roots with the power of n, add all that will equal to the Φ.

So, for any constant C_1 & C_2.

$$f(n) = C_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + C_2\left(\frac{1-\sqrt{5}}{2}\right)^n \quad \text{--- (2)}$$

Note:-

So, if you had three roots, you will write something like $C_1\alpha_1^n + C_2\alpha_2^n + C_3\alpha_3^n$.

(3) Fact

No of roots that you have = no of ans you have already

So, here we have 2 roots α_1 & α_2.

Hence, we should have 2 ans already

We know that

$$f(0) = 0 \quad \& \quad f(1) = 1$$

Note:-

When you have n no of roots or any number of roots you will have that many amount of answer.

$$f(0) = 0 = C_1\alpha_1^0 + C_2\alpha_2^0$$
$$= C_1 + C_2$$

$$\therefore \quad C_1 = -C_2 \quad \text{--- (3)}$$

for $f(1) = 1 = C_1\left(\frac{1+\sqrt{5}}{2}\right) + C_2\left(\frac{1-\sqrt{5}}{2}\right)$

from ③

$$1 = C_1\left(\frac{1+\sqrt{5}}{2}\right) - C_1\left(\frac{1-\sqrt{5}}{2}\right)$$

$$C_1 = \frac{1}{\sqrt{5}} \quad \& \quad C_2 = -\frac{1}{\sqrt{5}}$$

put in ②

$$f(n) = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

– formula for n^{th} fibonacci no.

Q. How do we get the time complexity from this?

Ans:-

① $f(n) = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$

from the formula of fibonnaci number.

② As 'n' increases or as $n \to \infty$

"$\left(\frac{1-\sqrt{5}}{2}\right)^n$" will be close to 0.

and as we know about time complexity that we should ignore the less dominating term

Hence, ~~igore~~ ignore, the "$\left(\frac{1-\sqrt{5}}{2}\right)^n$".

③ Time complexity = $O\left(\frac{1+\sqrt{5}}{2}\right)^n$. for n^{th} fibonacci no.

Note:-

Time complexity of fibo = $T(N) = O(1.6180)^n$

④ This is also known as golden ratio in mathematics.

⑤ So, this was the reason why our program was hanging for even small no. because exponential time complexity is very bad.

Code :-

```
public class FiboFormula{
        public static void main (String[] args){

        System.out.println(formula(50));

    }

    static int formula (intn){

        return (int)((Math.pow(((1+Math.sqrt(5))/2
                    ,n)-Math.pow(((1-Math.sqrt(5))/2
                    n))/Math.sqrt(5));
    }

    static int fibo(int n){

        if (n<2){
            return n;
        }
        return fibo(n-1) + fibo(n-2)

}
```

Q When you'll get equal no. of roots

Q $f(n) = 2f(n-1) + f(n-2)$

I put $f(n) = \alpha^n$

$\therefore \alpha^n = 2\alpha^{n-1} + \alpha^{n-2}$

$$\frac{\alpha^n - 2\alpha^{n-1} - \alpha^{n-2}}{\alpha^{n-2}} = \cancel{0} = 0$$ both LHS & RHS

$= \alpha^2 - 2\alpha + 1 = 0$

$\therefore \alpha = 1$ ~~and~~ on both. (double root)

So, In such case what will happen is that if there root is repeated twice. so, let's say in the general case:

* general case:-
If α is repeated 'r' times then, $\alpha^n, n\alpha^n, n^2\alpha^n, \ldots n^{r-1}\alpha^n$ all are solutions to the recurrence.

Note:-

So, we know that if the number of roots are 2, so we know that there should be two roots but, we're getting only ~~a~~ 1 root
So, we ~~k~~ can take extra roots from here because these are also a solⁿ. we just multiply n, we know than n divides n.

Hence, we can take two roots one itself and the other can be 'n' α^n.

as we know that $\alpha = 1$

$\therefore 1, n$

putting in formula :-

$$f(n) = C_1 (\alpha)^n + C_2 \, n \alpha^n$$

$$= C_1 + C_2 n$$

let's say the ans given us is

$f(0) = 0$ & $f(1) = 1$

$\therefore f(0) = 0 = C_1$.

$\therefore C_1 = 0 \quad - (1)$

$\therefore f(1) = 1 = C_1 + C_2$

$\therefore C_2 = 1 \quad - (2)$

So, if we put eq (1) & (2) together,

$f(n) = n$.

$\therefore$ Time complexity of the above relation is $O(n)$.

* Homogenous linear recurrence:-

Q What does how homogenous means?

Ans:- The form of a recurrance relationship where we do not have a perticular other function like g(x).
So, there is no seperat function that is why it is known as homogenous.

Eg: Solving Linear Recurrance.

** Non - Homogenous Linear recurrences:-

$$f(n) = a_1 f(n-1) + a_2 f(n-2) + a_3 f(n-3) + \dots + a_d f(n-d) + g(x)_n$$

So, when this extra function is present then it is known as non homogenous linear recurrences.

Q How to solve?

Steps:-

1) Replace g(n) by 0 & solve usually.

Exg:- $f(n) = 4f(n-1) + 3^n$, $f(0) = 1$

$\therefore f(n) = 4f(n-1) + 0.$

$\alpha^n = 4\alpha^{n-1}$

$\therefore \alpha^4 - 4\alpha^{n-1} = 0$

$$\alpha - 4 = 0$$

$$\therefore \alpha = 4$$

$\therefore$ Homogenous sol^n :-

$$f(n) = C_1 \alpha^n$$

$$f(n) = C_1 4^n \quad — (A)$$

$\therefore$ ~~$f(n) = C_1 4^n + 3^n$~~

2) Take g(n) on one side and find particular sol^n.

$$\therefore f(n) - 4f(n-1) = \underbrace{3^n}_{g(x)} \quad — (1)$$

- {as per the set step!}

Q What is particula solution?

Ans: We ~~kneed~~ need to do guess something that is similar to g(n), this is known as particular sol^n.

Eg: If $g(n) = n^2$, then guess a polynomial of degree 2.

Kunal's guess :-

$$f(n) = C\,3^n \quad — (2)$$

put in eq^n (1)

$$\therefore C3^n - 4C3^{n-1} = 3^n$$

$\therefore \; c = -3 \qquad - (a)$

$\therefore$ the particular sol^n : put (a) in (2)

$$\therefore \; f(n) = -3 \times 3^n$$

$$\therefore \; f(n) = -3^{n+1} \qquad - (B)$$

3) Add both the sol^n together.

$$f(n) = C_1 4^n + (-3^{n+1})$$

$$f(1) = 1 \qquad - \{\text{we know}\}$$

$$C_1 4 - 3^2 = 1$$

$$\therefore \; C_1 = \frac{5}{2} \; //.$$

put the value of C_1 & in Orignal eq^n.

$$f(n) = \frac{5}{2} 4^n - 3^{n+1} \; //.$$

Short:

1) Replace $g(n)$ by 0 & solve usually.

2) Take $g(n)$ on one side & find particular sol^n

3) Add both the sol^n together

Abbrevation :-

1) First, put $g(n) = 0$ & take the normal sol^n then guess the particular sol^n and get the answer for that.

2) After that put $g(n)$ on one side, guess the ~~positiv~~ particular solution, put that sol^n in the eq^n where you $g(n)$ one one side you'll get the value of 'c' and then you can put it in the orignal eq^n so, got your particular sol^n.

3) So, doing this we can get 'c' & then find answer of a particular sol^n.
Then just add the ① eq^n with the particular eq^n and put it the ans ("which is already provided") use it then you'll get your orignal answer.

Q How do we guess a particular solution?

Ans:-

1) If $g(x)$ is exponential, guess of the same type.

Eg:- $g(n) = 2^n + 3^n$

guess:- $f(n) = a2^n + b3^n$

So, this is your particular solution.

2) But, if it is polynomial, $g(n)$, in that case guess of same degree.

① Eg:- $g(n) = n^2 - 1$

then guess should be of same degree. Here it is 2 so, 2.

guess:- $an^2 + bn + c = f(n)$

② Eg:- $g(n) = 2^n + n$

guess: $f(n) = a2^n + (bn + c)$

Note:-

1) If it's exponential, just multiply with the constant

2) OR, if it's a polynomial take eq^n of that degree

So, that's how you guess the particular sol^n

3) let say you guessed, eg:- $f(n) = a2^n$ & it fails, then try $\underline{(an + b)2^n}$, if this also fails increase the degree.

$\therefore (a^2n + bn + c)2^n$

keep trying.

Eg: $f(n) = 2f(n-1) + 2^n$, $f(1) = 1$

① put $2^n = 0$

$\therefore f(n) = 2f(n-1)$

put $f(n) = \alpha^n$.

$\therefore \alpha^n - 2\alpha^{n-1} = 0$

$\alpha = 2$ //

② guess p.s

$g(n) = 2^n$

guess: $f(n) = a2^n$

put it in main eq^n

$a\cancel{2^n} = 2a2^{n-1} + \cancel{2^n}$

$a = a + 1$ × wrong

Hence, guess another one from our rules because this is not working

So, $f(n) = (an+b)2^n$

$(an+b)\cancel{2^n} = \cancel{2}(a(n-1)+b)\cancel{2^{n-1}} + \cancel{2^n}$

$a\cancel{n} + \cancel{b} = a\cancel{n} - a + \cancel{b} + 1$

$\therefore \quad a = 1$

discard b.

$$f(n) = n\,2^n$$

our particular soln

3) Genral answer :-

$f(n) = c_1 2^n + n 2^n$ — { some of both the eqn }

$f(0) = 1 = c_1 + 0$

$c_1 = 1$

$\therefore \quad f(n) = 2^n + n2^n$ //

Complexity $= O(n2^n)$