Fundamental Algorithm Techniques
Problem Set 5

Umargaliyeva Leila 25-1-19

Problem 1

Introduction

The goal of this problem is to show that seven different definitions of a tree in graph theory are equivalent. To avoid proving all 7×6 implications, I choose definition (1) as the central reference point:

$$(1) \quad \text{A tree is a connected and acyclic graph.}$$

I will show that each of the remaining six definitions is equivalent to (1). Before that, I recall several key lemmas about the number of edges in finite graphs.

Lemmas

Lemma 1. If a finite graph is acyclic, then $E \leq V - 1$.
 Lemma 2. If a finite graph is connected, then $E \geq V - 1$.
 Lemma 3. If a graph is connected and acyclic, then $E = V - 1$.

Implications

(1) $\Rightarrow$ (3) and (5). From Lemma 3, a connected acyclic graph satisfies $E = V - 1$. Thus it is a connected graph with at most $V - 1$ edges (definition (3)), and an acyclic graph with at least $V - 1$ edges (definition (5)).

 (3) $\Rightarrow$ (4). If a graph is connected and has $E \leq V - 1$ edges, Lemma 2 implies $E = V - 1$. Removing any edge leaves a graph with $V - 2$ edges, which

cannot be connected by Lemma 2. Thus, removing any edge disconnects the graph, which is definition (4).

(4) $\Rightarrow$ (1). If removing any edge disconnects the graph, then the graph is connected. Assume a cycle exists. Removing an edge from this cycle leaves the graph connected, contradicting minimal connectivity. Thus the graph is acyclic and connected $\rightarrow$ (1).

(5) $\Leftrightarrow$ (6).
(5) $\Rightarrow$ (6): Acyclic and $E \geq V - 1$, together with Lemma 1, gives $E = V - 1$. Adding any new edge joins two vertices that already have a unique path, producing a cycle. Hence the graph is maximally acyclic.
(6) $\Rightarrow$ (5): If adding any edge creates a cycle, the graph is acyclic. If $E < V - 1$, the graph is disconnected, and adding an edge between components cannot create a cycle—contradiction. Thus $E \geq V - 1$.

(1) $\Leftrightarrow$ (7).
(1) $\Rightarrow$ (7): A connected acyclic graph has a unique path between any two vertices, because two distinct paths would form a cycle.
(7) $\Rightarrow$ (1): A unique path between every two vertices implies connectivity. If a cycle existed, then two vertices on the cycle would have two distinct paths—contradiction. Thus the graph is acyclic.

(1) $\Leftrightarrow$ (2). A connected acyclic graph is a connected component of an acyclic graph (i.e., a forest). Conversely, a connected component of a forest is connected and acyclic.

Implication Map

$$(3) \Rightarrow (4) \Rightarrow (1) \Rightarrow (7)$$
$$\downarrow$$
$$(5) \Leftrightarrow (6)$$
$$(1) \Leftrightarrow (2)$$

Conclusion

All seven definitions are equivalent and describe the same notion of a tree: a connected acyclic graph with $E = V - 1$ edges.
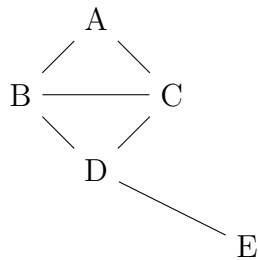
Problem 2

We are given two graphs on vertices A, B, C, D, E, indexed as $0, 1, 2, 3, 4$. Using the CSC representation, we reconstruct the adjacency matrices and draw the graphs using TikZ.

Graph 1 (Undirected)

$$A_1 = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Diagram of Graph 1

Graph 2 (Directed)

$$A_2 = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Directed edges:

$$A \rightarrow B, \ A \rightarrow C, \ B \rightarrow C, \ B \rightarrow E, \ C \rightarrow D, \ D \rightarrow B, \ D \rightarrow E.$$

The unique directed cycle is:

$$B \rightarrow C \rightarrow D \rightarrow B.$$

Diagram of Graph 2