RATIONAL FUNCTIONS AND ODD ZETA VALUES

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In this note I collect some known results about the irrationality of odd zeta values $\zeta(2k+1)$. These results are elementary in nature: we construct good linear forms in odd zeta values by means of rational functions. I wish to make them more accessible for wider audiences including undergraduates. This is a draft version, I plan to complete it before 2022.

Contents

0. Introduction

We are interested in the arithmetic properties (irrationality, transcendence, ...) of some special constants. Which constants are considered to be interesting? The following is my personal taste.

Notations: We use $\overline{\mathbb{Q}}$ to denote the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$.

'First class': π, e.

- Hermite, 1873: $e \notin \overline{\mathbb{Q}}$.
- Lindemann, 1882: $\pi \notin \overline{\mathbb{Q}}$. Moreover, for any $\alpha \in \overline{\mathbb{Q}} \setminus \{0\}$, we have $e^\alpha \notin \overline{\mathbb{Q}}$.
- (Lindemann-)Weierstrass, 1885: If $\alpha_1, \ldots, \alpha_n \in \overline{\mathbb{Q}}$ are linearly independent over $\mathbb{Q}$, then $e^{\alpha_1}, \ldots, e^{\alpha_n}$ are algebraically independent over $\mathbb{Q}$.

'Second class': $\sqrt{2}^{\sqrt{3}}$, γ, $\zeta(3)$, $\Gamma(1/5)$, ...

- Gelfond/Schneider, 1934: If $\alpha, \beta \in \overline{\mathbb{Q}}$ with $\alpha \neq 0, 1$ and $\beta \notin \mathbb{Q}$, then $\alpha^\beta \notin \overline{\mathbb{Q}}$. Here $\alpha^\beta = \exp(\beta \log \alpha)$ is multi-valued, the precise statement is that any value of α^β is transcendent. As a corollary, both $\sqrt{2}^{\sqrt{2}}$ and $e^\pi = (-1)^{-i}$ are transcendent.
- Baker, 1966: If $\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n \in \overline{\mathbb{Q}}$ with $\alpha_i \neq 0, 1$ and that $1, \beta_1, \ldots, \beta_n$ are linearly independent over $\mathbb{Q}$, then $\alpha_1^{\beta_1} \cdots \alpha_n^{\beta_n} \notin \overline{\mathbb{Q}}$.
- For Euler's constant $\gamma = \lim_{n\to+\infty} \left(-\log n + \sum_{k=1}^{n} \frac{1}{k}\right)$, nothing very exciting is known. (But see [Riv12] for some explorations.)

In this note, we will focus on the Riemann zeta values

$$\zeta(k)=\frac{1}{1^k}+\frac{1}{2^k}+\frac{1}{3^k}+\cdots,\quad k\in\mathbb{Z}_{\geqslant 2}.$$

- Essentially by Euler:

$$\zeta(2k)=(-1)^{k-1}2^{2k-1}\frac{B_{2k}}{(2k)!}\pi^{2k},\quad k\in\mathbb{Z}_{\geqslant 1},$$

  where B_{2k} is Bernoulli's number defined by the generating series

$$\frac{t}{e^t-1}=\sum_{k\geqslant 0}\frac{B_k}{k!}t^k.$$

  Since $B_{2k}\in\mathbb{Q}$ and $\zeta(2k)\neq 0$, we deduce from Lindemann's theorem ($\pi\notin\overline{\mathbb{Q}}$) that $\zeta(2k)\notin\overline{\mathbb{Q}}$. (e.g. $\zeta(2)=\pi^2/6,\zeta(4)=\pi^4/90,\zeta(6)=\pi^6/945,\ldots$.) A natural question: how about odd zeta values $\zeta(2k+1)$?
- Conjecture: $\pi,\zeta(3),\zeta(5),\zeta(7),\ldots$ are algebraically independent over $\mathbb{Q}$. This conjecture is still far to reach. There are some stronger conjectures concerning about multiple zeta values, see Zagier's conjecture. Up to now, we only know some irrationality results about odd zeta values $\zeta(2k+1)$.
- Apéry, 1978: $\zeta(3)\notin\mathbb{Q}$.
- Rivoal, 2000; Ball, Rivoal, 2001: for odd $s\geqslant s_0(\varepsilon)$,

$$\dim_{\mathbb{Q}}\operatorname{Span}_{\mathbb{Q}}\left(1,\zeta(3),\zeta(5),\ldots,\zeta(s)\right)\geqslant\frac{1-\varepsilon}{1+\log 2}\log s.$$

- Zudilin, 2001: At least one of $\zeta(5),\zeta(7),\zeta(9),\zeta(11)$ is irrational.
- Fischler, Sprang, Zudilin, 2018 and Lai, Yu, 2020: for odd $s\geqslant s_0(\varepsilon)$,

$$\#\left\{k:3\leqslant k\leqslant s, k\text{ odd}, \zeta(k)\notin\mathbb{Q}\right\}\geqslant(c_0-\varepsilon)\sqrt{\frac{s}{\log s}},$$

  where the constant

$$c_0=\sqrt{\frac{4\zeta(2)\zeta(3)}{\zeta(6)}\left(1-\log\frac{\sqrt{4e^2+1}-1}{2}\right)}\approx 1.192507\ldots.$$

- There are some p-adic analogues. Calegari, 2005: for $p=2,3$, we have $\zeta_p(3)\in\mathbb{Q}_p\backslash\mathbb{Q}$. Sprang, 2020: let K be a number field, for $s\geqslant s_0(\varepsilon)$,

$$\dim_K\operatorname{Span}_K\left(1,\zeta_p(2),\zeta_p(3),\zeta_p(4),\ldots,\zeta_p(s)\right)\geqslant\frac{1-\varepsilon}{2[K:\mathbb{Q}](1+\log 2)}\log s.$$

1. Irrationality criteria

1.1. **Linear independence criteria of Siegel and Nesterenko.** We begin with an obvious criterion.

Theorem 1.1 (naive irrationality criterion). *Let $\underline{\theta} = (\theta_1, \theta_2, \dots, \theta_m) \in \mathbb{R}^m$. Suppose that there exists a sequence of linear forms (indexed by $n \in \mathbb{N}$) in $m+1$ variables with integer coefficients*

$$L_n(\underline{X}) = l_{n,0}X_0 + l_{n,1}X_1 + \cdots + l_{n,m}X_m, \quad (l_{n,j} \in \mathbb{Z}, j = 0, 1, \dots, m)$$

such that

- $L_n(1, \underline{\theta}) \neq 0$ *for each* $n \in \mathbb{N}$;
- $L_n(1, \underline{\theta}) \to 0$ *as* $n \to +\infty$.

Then, at least one of $\theta_1, \theta_2, \dots, \theta_m$ is irrational.

Proof. If all θ_i are rational, let D be a common denominator of these rational numbers. Then $|L_n(1, \underline{\theta})| \geqslant 1/D$, which contradicts that $L_n(1, \underline{\theta}) \to 0$. □

Although Theorem 1.1 is trivial, it gives us an angle of view for irrationality: if a sequence of linear forms of $\theta_1, \dots, \theta_m$ (with integer coefficients) has smaller and smaller (but nonzero) values, then we are possible to obtain some nontrivial information about the irrationality of these θ_i. In 1929, Siegel [Sie29] made a more useful and quantitative result along this perspective.

Theorem 1.2 (Siegel's linear independence criterion). *Let $\tau > 0$ be a constant and $\sigma : \mathbb{N} \to \mathbb{R}_{>0}$ a function satisfying*

$$\lim_{n \to +\infty} \sigma(n) = +\infty.$$

Let $\underline{\theta} = (\theta_1, \theta_2, \dots, \theta_m) \in \mathbb{R}^m$. Suppose that, for each $n \in \mathbb{N}$ there exists a complete system of $m+1$ linearly independent linear forms in $m+1$ variables with integer coefficients

$$L_i^{(n)}(\underline{X}) = l_{i,0}^{(n)}X_0 + l_{i,1}^{(n)}X_1 + \cdots + l_{i,m}^{(n)}X_m, \quad i = 0, 1, \dots, m, \quad l_{i,j}^{(n)} \in \mathbb{Z}$$

such that (as $n \to +\infty$)

- $\max_{0 \leqslant i,j \leqslant m} \left| l_{i,j}^{(n)} \right| \leqslant \exp((1 + o(1))\sigma(n))$;
- $\max_{0 \leqslant i \leqslant m} \left| L_i^{(n)}(1, \underline{\theta}) \right| \leqslant \exp(-(\tau + o(1))\sigma(n))$.

Then,

$$\dim_{\mathbb{Q}} \operatorname{Span}_{\mathbb{Q}} (1, \theta_1, \theta_2, \dots, \theta_m) \geqslant 1 + \tau.$$

Roughly speaking, Siegel's linear independence criterion says that, more 'faster' the values of linear forms decay than the growth of coefficients, greater is the dimension of the space spanned by θ_i. But keep in mind that Siegel's criterion requires a restrictive hypothesis, namely, we need a complete system of linearly independent linear forms.

Proof of Theorem 1.2. Denote $r = \dim_{\mathbb{Q}} \operatorname{Span}_{\mathbb{Q}}(1, \theta_1, \theta_2, \ldots, \theta_m)$. Then there exist $m+1-r$ elements $A_i = (a_{i,0}, a_{i,1}, \ldots, a_{i,m}) \in \mathbb{Q}^{m+1}$ $(i = r, r+1, \ldots, m)$, linearly independent over $\mathbb{Q}$, such that

$$a_{i,0} + a_{i,1}\theta_1 + \cdots + a_{i,m}\theta_m = 0, \quad i = r, r+1, \ldots, m.$$

We can assume that $a_{i,j} \in \mathbb{Z}$ for all i, j.

We view each linear form $L_i^{(n)}$ as an element $(l_{i,0}^{(n)}, \ldots, l_{i,m}^{(n)})$ in $\mathbb{Q}^{m+1}$. For each $n \in \mathbb{N}$, since $\left\{L_0^{(n)}, \ldots, L_m^{(n)}\right\}$ is a basis of $\mathbb{Q}^{m+1}$, after relabeling, we may assume that

$$\left\{L_0^{(n)}, L_1^{(n)}, \ldots, L_{r-1}^{(n)}, A_r, A_{r+1}, \ldots, A_m\right\}$$

forms a basis of $\mathbb{Q}^{m+1}$.

Denote

$$\Delta_n = \begin{vmatrix} l_{0,0}^{(n)} & l_{0,1}^{(n)} & \ldots & l_{0,m}^{(n)} \\ \ldots & \ldots & \ldots & \ldots \\ l_{r-1,0}^{(n)} & l_{r-1,1}^{(n)} & \ldots & l_{r-1,m}^{(n)} \\ a_{r,0} & a_{r,1} & \ldots & a_{r,m} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m,0} & a_{m,1} & \ldots & a_{m,m} \end{vmatrix}.$$

Then $\Delta_n \in \mathbb{Z} \setminus \{0\}$, it follows that $|\Delta_n| \geqslant 1$. Applying elementary column operations we obtain that

$$\begin{aligned} \Delta_n &= \begin{vmatrix} L_0^{(n)}(1, \underline{\theta}) & l_{0,1}^{(n)} & \ldots & l_{0,m}^{(n)} \\ \ldots & \ldots & \ldots & \ldots \\ L_{r-1}^{(n)}(1, \underline{\theta}) & l_{r-1,1}^{(n)} & \ldots & l_{r-1,m}^{(n)} \\ 0 & a_{r,1} & \ldots & a_{r,m} \\ \ldots & \ldots & \ldots & \ldots \\ 0 & a_{m,1} & \ldots & a_{m,m} \end{vmatrix} \\ &= \sum_{i=0}^{r-1} L_i^{(n)}(1, \underline{\theta}) \Delta_{i,0}^{(n)}, \end{aligned}$$

where $\Delta_{i,0}^{(n)}$ is the $(i,0)$-cofactor of the determinant Δ_n. Clearly, for each $k \in [0, r-1]$,

$$\begin{aligned} \left|\Delta_{k,0}^{(n)}\right| &\leqslant m! \left(\max_{0 \leqslant i,j \leqslant m} \left|l_{i,j}^{(n)}\right|\right)^{r-1} \left(\max_{\substack{r \leqslant i \leqslant m \\ 0 \leqslant j \leqslant m}} |a_{i,j}|\right)^{m+1-r} \\ &\leqslant m! A^{m+1-r} \exp\left((r - 1 + o(1))\sigma(n)\right). \qquad (A = \max_{i,j} |a_{i,j}|) \end{aligned}$$

Hence,

$$1 \leqslant |\Delta_n| \leqslant r m! A^{m+1-r} \exp\left((r - 1 - \tau + o(1))\sigma(n)\right).$$

Since $rm!A^{m+1-r}$ is independent of n and $\sigma(n) \to +\infty$ as $n \to +\infty$, we deduce that $r \geqslant 1+\tau$, as desired. □

Siegel used his criterion to prove some transcendence results (linear independence of $1, \theta, \theta^2, \theta^3, \ldots$). In 1985, Nesterenko [Nes85] introduced a similar but different criterion.

Theorem 1.3 (Nesterenko's linear independence criterion)**.** *Let $\tau_1, \tau_2 > 0$ be constants and $\sigma : \mathbb{N} \to \mathbb{R}_{>0}$ a non-decreasing function satisfying*

$$\lim_{n\to+\infty} \sigma(n) = +\infty, \quad \text{and} \quad \lim_{n\to+\infty} \frac{\sigma(n+1)}{\sigma(n)} = 1.$$

Let $\underline{\theta} = (\theta_1, \theta_2, \ldots, \theta_m) \in \mathbb{R}^m$. Suppose that, for each $n \in \mathbb{N}$ there exists a linear form in $m+1$ variables with integer coefficients

$$L_n(\underline{(X)}) = l_{n,0}X_0 + l_{n,1}X_1 + \cdots + l_{n,m}X_m, \quad l_{n,j} \in \mathbb{Z}$$

such that

- $\max_{0\leqslant j\leqslant m} |l_{n,j}| \leqslant \exp\left((1+o(1))\sigma(n)\right)$;
- $\exp\left(-(\tau_1+o(1))\sigma(n)\right) \leqslant |L_n(1,\underline{\theta})| \leqslant \exp\left(-(\tau_2+o(1))\sigma(n)\right)$.

Then,

$$\dim_{\mathbb{Q}} \operatorname{Span}_{\mathbb{Q}}(1, \theta_1, \theta_2, \ldots, \theta_m) \geqslant \frac{1+\tau_1}{1+\tau_1-\tau_2}.$$

There is a comparison between Siegel's and Nesterenko's criteria.

	Siegel's	Nesterenko's
Advantage	need only upper bounds	need only one linear form
Disadvantage	need a complete system	need both lower and upper bounds

The original proof [Nes85] of Nesterenko's criterion is quite involved. It has been simplified by Amoroso and this simplification was revisited (and translated from Italian to French) by Colmez [Col03].

In 2010, Fischler and Zudilin [FZ10] obtained a more elementary proof, which only used Minkowski's 'geometry of numbers'. In the following, I will repeat the the proof written by Chantanasiri [Cha10].

We first state a simple version of the Minkowski's convex body theorem. Recall a fact that every convex set in $\mathbb{R}^d$ is measurable with respect to the d-dimensional Lebesgue measure Vol_d.

Lemma 1.4 (Minkowski's convex body theorem)**.** *Let C be a convex set in $\mathbb{R}^d$, which is symmetric with respect to the origin (i.e. $-C = C$). If one of the following conditions holds*

- $\mathrm{Vol}_d(C) > 2^d$*; or*
- $\mathrm{Vol}_d(C) \geqslant 2^d$ *and C is compact.*

Then, C contains a nonzero integral lattice point, that is,

$$C \cap \left(\mathbb{Z}^d \setminus \{\underline{0}\}\right) \neq \emptyset.$$

Proof. Suppose first that $\mathrm{Vol}_d(C) > 2^d$, then $\mathrm{Vol}_d(C/2) = 2^{-d}\,\mathrm{Vol}_d(C) > 1$. Let

$$(C/2)_{\underline{n}} := (C/2) \cap \left([n_1, n+1) \times [n_2, n_2+1) \times \cdots \times [n_d, n_d+1)\right) - \underline{n}$$

for any $\underline{n} = (n_1, n_2, \ldots, n_d) \in \mathbb{Z}^d$. Then $(C/2)_{\underline{n}} \subset [0,1)^d$ and $\sum_{\underline{n}\in\mathbb{Z}^d} \mathrm{Vol}_d((C/2)_{\underline{n}}) = \mathrm{Vol}_d(C/2) > 1$. So there exist two distinct $\underline{n}_1 \neq \underline{n}_2 \in \mathbb{Z}^d$ such that $(C/2)_{\underline{n}_1} \cap (C/2)_{\underline{n}_2} \neq \emptyset$, which means $\underline{n}_1 - \underline{n}_2 \in (C/2) - (C/2)$. Since C is symmetric and convex, we have $(C/2) - (C/2) = (C/2) + (C/2) \subset C$, so $\underline{n}_1 - \underline{n}_2$ is a nonzero integral lattice point contained in C.

Suppose now $\mathrm{Vol}_d(C) \geqslant 2^d$ and C is compact. By the above, there exists an integral lattice point $\underline{n}_m \in \mathbb{Z}^d \setminus \{\underline{0}\}$ which is contained in $(1+m^{-1})C$, $m = 1, 2, \ldots$. The sequence $\{\underline{n}_m\}_{m=1}^{\infty}$

is bounded thus has at least one limit point $\underline{n}^*$. Then $\underline{n}^*$ is a nonzero integral lattice point contained in C. $\square$

Let $\{1, \xi_1, \dots, \xi_r\}$ be a basis of $\mathrm{Span}_{\mathbb{Q}}(1, \theta_1, \dots, \theta_m)$. To prove Theorem 1.3, the idea is that we can approximate the direction of $(\xi_1, \dots, \xi_r)$ by a direction $(x_1, \dots, x_r)$ with $x_i \in \mathbb{Z}$. The existence of such a 'good' approximation is guaranteed by Lemma 1.4.

Proof of Theorem 1.3. Let $\{\xi_0, \xi_1, \dots, \xi_r\}$ be a $\mathbb{Q}$-basis of $\mathrm{Span}_{\mathbb{Q}}(1, \theta_1, \dots, \theta_m)$ with $\xi_0 = 1$. So $\dim_{\mathbb{Q}} \mathrm{Span}_{\mathbb{Q}}(1, \theta_1, \theta_2, \dots, \theta_m) = r + 1$. (By Theorem 1.1, $r > 0$.) Then there exist $d \in \mathbb{N}$ and $c_{i,0}, c_{i,1}, \dots, c_{i,r} \in \mathbb{Z}$ $(i = 0, 1, \dots, m)$ such that

$$d\theta_i = \sum_{j=0}^{r} c_{i,j}\xi_j, \quad i = 0, 1, \dots, m.$$

Write $\underline{\xi} = (\xi_1, \xi_2, \dots, \xi_r)$. Consider the linear form

$$\begin{aligned}\Gamma_n(Y_0, Y_1, \dots, Y_r) &= L_n\left(\sum_{j=0}^{r} c_{0,j}\xi_j, \sum_{j=0}^{r} c_{1,j}\xi_j, \dots, \sum_{j=0}^{r} c_{m,j}\xi_j\right) \\ &=: \gamma_{n,0}Y_0 + \gamma_{n,1}Y_1 + \cdots + \gamma_{n,r}Y_r,\end{aligned}$$

where $\gamma_{n,j} = \sum_{k=0}^{m} l_{n,k}c_{k,j} \in \mathbb{Z},\ j = 0, 1, \dots, r$. Then $\Gamma_n(1, \underline{\xi}) = dL_n(1, \underline{\theta})$. Since $d, c_{k,j}$ are independent of n and $\sigma(n) \to +\infty$ as $n \to +\infty$, we may write (as $n \to +\infty$)

$$\max_{0 \leqslant j \leqslant r} |\gamma_{n,j}| \leqslant \exp\left((1 + o(1))\sigma(n)\right), \tag{1.1}$$

$$\exp\left(-(\tau_1 + o(1))\sigma(n)\right) \leqslant \left|\Gamma_n(1, \underline{\xi})\right| \leqslant \exp\left(-(\tau_2 + o(1))\sigma(n)\right). \tag{1.2}$$

For each sufficiently large $n \in \mathbb{N}$, we define

$$C_n := \left\{(x_0, x_1, \dots, x_r) \in \mathbb{R}^{r+1} \ :\ |x_0| \leqslant \frac{1}{2\left|\Gamma_n(1, \underline{\xi})\right|},\ |x_0\xi_j - x_j| \leqslant \left|2\Gamma_n(1, \underline{\xi})\right|^{1/r},\ j = 1, 2, \dots, r\right\}.$$

Then C_n is a symmetric and compact convex set in $\mathbb{R}^{r+1}$ with volume $\mathrm{Vol}_{r+1}(C_n) = 2^{r+1}$. By Lemma 1.4, there is a nonzero integral lattice point in C_n, we fix such a point

$$\underline{x}_n = (x_0(n), x_1(n), \dots, x_r(n)) \in C_n \cap \left(\mathbb{Z}^{r+1} \setminus \{\underline{0}\}\right).$$

We claim that

$$\lim_{n \to +\infty} |x_0(n)| = +\infty. \tag{1.3}$$

In fact, if (1.3) does not hold, then $\{x_0(n)\}_{n \geqslant 1}$ admits a bounded subsequence, so a constant subsequence, say, $x_0(n) = x^*$ along an infinite subset I of $\mathbb{N}$. We would have $\lim_{\substack{n \in I \\ n \to +\infty}} (x_0^*\xi_j - x_j(n)) = 0$, therefore, $x_j(n) = x_0^*\xi_j$ for all sufficiently large $n \in I$ (since $x_j(n) \in \mathbb{Z}$). If $x_0^* = 0$, then $x_j(n) = 0$ for all j, which is impossible. If $x_0^* \neq 0$, then $\xi_j \in \mathbb{Q}$ for all j, (so $r = 0$) which contradicts Theorem 1.1. Hence, the claim (1.3) is true.

Fix an index k_0 such that $|\Gamma_k(1, \underline{\xi})| > 0$ for all $k \geqslant k_0$. For sufficiently large n, we define

$$k_n := \min\left\{k \in \mathbb{N}_{\geqslant k_0} \ :\ |x_0(n)| \leqslant \frac{1}{2\left|\Gamma_k(1, \underline{\xi})\right|}\right\}.$$

By (1.3), we have $k_n \to +\infty$ as $n \to +\infty$. By the definition of $\underline{x}_n$, we have $k_n \leqslant n$. Moreover, by the minimality of k_n, it holds that

$$|x_0(n)| \leqslant \frac{1}{2\left|\Gamma_{k_n}(1,\underline{\xi})\right|}, \quad \text{and} \quad |x_0(n)| > \frac{1}{2\left|\Gamma_{k_n-1}(1,\underline{\xi})\right|}. \tag{1.4}$$

Now, we can write

$$\underbrace{\sum_{j=0}^{r} \gamma_{k_n,j} x_j(n)}_{\in\mathbb{Z}} = \underbrace{x_0(n)\sum_{j=0}^{r} \gamma_{k_n,j}\xi_j}_{\text{absolute value}\leqslant\frac{1}{2}} + \sum_{j=0}^{r} \gamma_{k_n,j}\left(x_j(n) - x_0(n)\xi_j\right). \tag{1.5}$$

By the fact that $y = y_1 + y_2$, $y \in \mathbb{Z}$, $|y_1| \leqslant 1/2$ implies $|y_2| \geqslant |y_1|$, we deduce from (1.5) that

$$\left|\sum_{j=0}^{r} \gamma_{k_n,j}\left(x_j(n) - x_0(n)\xi_j\right)\right| \geqslant \left|x_0(n)\sum_{j=0}^{r} \gamma_{k_n,j}\xi_j\right|. \tag{1.6}$$

By (1.1), the definition of C_n, and (1.4), we obtain from (1.6) that

$$(r+1)\cdot\exp\left((1+o(1))\sigma(k_n)\right)\cdot\left|2\Gamma_n(1,\underline{\xi})\right|^{1/r} \geqslant \frac{1}{2\left|\Gamma_{k_n-1}(1,\underline{\xi})\right|}\left|\Gamma_{k_n}(1,\underline{\xi})\right|. \tag{1.7}$$

Since $k_n \to +\infty$, we can write $r+1 = \exp(o(1)\sigma(k_n))$. Then, by (1.2),

$$\exp\left((1+o(1))\sigma(k_n) - \frac{1}{r}(\tau_2+o(1))\sigma(n)\right) \geqslant \exp\left((\tau_2+o(1))\sigma(k_n-1) - (\tau_1+o(1))\sigma(k_n)\right), \tag{1.8}$$

which is

$$(1+o(1)) - \frac{1}{r}(\tau_2+o(1))\frac{\sigma(n)}{\sigma(k_n)} \geqslant (\tau_2+o(1))\frac{\sigma(k_n-1)}{\sigma(k_n)} - (\tau_1+o(1)). \tag{1.9}$$

Since $k_n \leqslant n$ and $\sigma(\cdot)$ is non-decreasing, and $\sigma(k_n-1)/\sigma(k_n) = 1+o(1)$, we obtain

$$1 - \frac{1}{r}\tau_2 \geqslant \tau_2 - \tau_1,$$

that is, $r \geqslant \tau_2/(1+\tau_1-\tau_2)$. So

$$\dim_{\mathbb{Q}} \operatorname{Span}_{\mathbb{Q}}(1,\theta_1,\theta_2,\ldots,\theta_m) = r+1 \geqslant \frac{1+\tau_1}{1+\tau_1-\tau_2},$$

as desired. □

We will see the applications of Nesterenko's linear independence criterion in § 3.

1.2. **Irrationality measure.** In this subsection we introduce the concept of irrationality measure, which is one kind of coarsely quantitative measure of irrationality.

Definition 1.5 (irrationality measure/exponent)**.** *Let $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ be an irrational number. We define the irrationality measure $\mu(\alpha)$ of α to be the infimum of real numbers μ satisfying the following condition: there exists a constant $C(\mu) > 0$ such that $|\alpha - p/q| \geqslant C(\mu)/q^{\mu}$ for all $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. (If such μ does not exist, then $\mu(\alpha) = +\infty$.)*

There are some facts about irrationality measure:

- Dirichlet: $\mu(\alpha) \geqslant 2$ for all $\alpha \in \mathbb{R}\setminus\mathbb{Q}$.

- Borel-Cantelli's lemma implies that $\mu(\alpha) = 2$ for almost all $\alpha \in \mathbb{R}$ with respect to the Lebesgue measure.
- If the simple continued fraction of α is $\alpha = [a_0; a_1, a_2, \ldots]$ and p_n/q_n is the n-th convergent of α, then
$$\mu(\alpha) = 1 + \limsup_{n\to+\infty} \frac{\log q_{n+1}}{\log q_n} = 2 + \limsup_{n\to+\infty} \frac{\log a_{n+1}}{\log q_n}.$$
- A difficult theorem of Roth, 1955: $\mu(\alpha) = 2$ for every $\alpha \in \overline{\mathbb{Q}} \setminus \mathbb{Q}$.
- Euler: $e = [2; \overline{(1, 2k, 1)_{k=1}^{\infty}}]$, so $\mu(e) = 2$.
- Zeilberger, Zudilin, 2020 [ZZ20]: $\mu(\pi) \leqslant 7.10320533\ldots$.

Although we know that $\mu(\alpha) = 2$ for almost all irrational numbers α, it is difficult to decide $\mu(\alpha)$ for a given α like $\log 2$ or π. Nevertheless, if we have a sequence of 'good' linear forms in 1 and α, then we are possible to obtain an upper bound of α. Such theorems are similar in spirit to Theorem 1.2 and Theorem 1.3.

Theorem 1.6. *Let $\tau > 0$ be a constant and $\sigma : \mathbb{N} \to \mathbb{R}_{>0}$ a function satisfying*
$$\lim_{n\to+\infty} \sigma(n) = +\infty, \quad \text{and} \quad \lim_{n\to+\infty} \frac{\sigma(n+1)}{\sigma(n)} = 1.$$
Let $\alpha \in \mathbb{R}$. Assume that there exist two sequences of integers $\{a_n\}_{n\geqslant 1}$, $\{b_n\}_{n\geqslant 1}$ such that (as $n \to +\infty$)
- $|a_n| \leqslant \exp((1 + o(1))\sigma(n))$;
- $0 < |a_n\alpha - b_n| \leqslant \exp(-(\tau + o(1))\sigma(n))$;
- $a_n b_{n+1} \neq a_{n+1} b_n$ *for all* $n \in \mathbb{N}$.

Then $\alpha \notin \mathbb{Q}$ and
$$\mu(\alpha) \leqslant 1 + \frac{1}{\tau}.$$

Theorem 1.7 (Hata, 1993 [Hat93])**.** *Let $\tau > 0$ be a constant and $\sigma : \mathbb{N} \to \mathbb{R}_{>0}$ a function satisfying*
$$\lim_{n\to+\infty} \sigma(n) = +\infty, \quad \text{and} \quad \lim_{n\to+\infty} \frac{\sigma(n+1)}{\sigma(n)} = 1.$$
Let $\alpha \in \mathbb{R}$. Assume that there exist two sequences of integers $\{a_n\}_{n\geqslant 1}$, $\{b_n\}_{n\geqslant 1}$ such that (as $n \to +\infty$)
- $|a_n| = \exp((1 + o(1))\sigma(n))$;
- $0 < |a_n\alpha - b_n| \leqslant \exp(-(\tau + o(1))\sigma(n))$.

Then $\alpha \notin \mathbb{Q}$ and
$$\mu(\alpha) \leqslant 1 + \frac{1}{\tau}.$$

2. Apéry's theorem

In 1978, Apéry [Apé79] made a breakthrough.

Theorem 2.1 (Apéry's theorem)**.**

$$\zeta(3) \notin \mathbb{Q}.$$

See van der Poorten's report [Poo79] for some history and see Fischler's Bourbaki seminar notes [Fis04] for a survey. Nowadays, there are a lot of proofs.

- Apéry, 1978: original, astonishing, miraculous, ...
  - "This is marvellous! It is something Euler could have done ..." – N. Katz
  - "A proof that Euler missed..." – A. van der Poorten
- Beukers, 1979: elegant. It used integrals over $[0,1]^3$.
- Beukers, 1987: used modular forms.
  - "... it seemed that my fate ... was closely linked with work of Roger Apéry." – F. Beukers, said in 2003.
- Nesterenko, 1996: used summation of derivatives of rational functions.

We will repeat the details of Beukers' proof and take a look at other proofs including Apéry's.

2.1. Beukers' proof.

To illustrate Beukers' proof, we first go through an example.

Example 2.2. *A proof that* $\log 2 \notin \mathbb{Q}$.

Proof. Denote by $d_n = [1, 2, \ldots, n]$ the least common multiple of the first n positive integers. For any $P(X) \in \mathbb{Z}[X]$ with $\deg P \leqslant n$, we can write $P(X) = (X+1)Q(X) + P(-1)$ for some $Q(X) \in \mathbb{Z}[X]$ with $\deg Q \leqslant n-1$. Hence,

$$\begin{aligned}\int_0^1 \frac{P(x)}{1+x}\mathrm{d}x &= \int_0^1 Q(x)\mathrm{d}x + P(-1)\int_0^1 \frac{1}{1+x}\mathrm{d}x \\ &\in \frac{\mathbb{Z}}{1} + \frac{\mathbb{Z}}{2} + \cdots + \frac{\mathbb{Z}}{n} + \mathbb{Z}\log 2 \\ &\subset \frac{\mathbb{Z}}{d_n} + \mathbb{Z}\log 2 \qquad (2.1)\end{aligned}$$

We take (Legendre-type polynomials)

$$P_n(X) = \frac{1}{n!}\left(\frac{\mathrm{d}}{\mathrm{d}X}\right)^n (X^n(1-X)^n), \quad n = 1, 2, \ldots,$$

and denote $I_n = \int_0^1 (P(x)\mathrm{d}x)/(1+x)$. Note that $P_n(X) \in \mathbb{Z}[X]$ and $\deg P_n = n$, so by (2.1) we have $I_n \in \mathbb{Z}/d_n + \mathbb{Z}\log 2$. By an n-fold partial integration, we deduce that

$$I_n = \int_0^1 \frac{x^n(1-x)^n}{(1+x)^{n+1}}\mathrm{d}x.$$

A straightforward computation shows that $x(1-x)/(1+x) \leqslant 3 - 2\sqrt{2}$ for all $x \in [0,1]$, which implies $0 < I_n < (3-2\sqrt{2})^n$, $n = 1, 2, \ldots$. By the prime number theorem, $d_n = \exp((1+o(1))n)$, so $d_n \leqslant 4^n$ for all sufficiently large n. Thus,

$$\begin{cases} 0 < d_n I_n < (4(3-2\sqrt{2}))^n < 1, & \text{for all } n \geqslant n_0 \\ \quad d_n I_n \in \mathbb{Z} + \mathbb{Z} d_n \log 2 \end{cases},$$

the conclusion $\log 2 \notin \mathbb{Q}$ follows. $\square$

We see in the above example that $1/(1+x)$ is the right 'kernel' to produce $\log 2$. Beukers found a right 'kernel' for $\zeta(3)$.

Lemma 2.3. *Denote $d_n = [1, 2, \dots, n]$. Let $r, s \in \mathbb{Z}_{\geqslant 0}$.*

(1) *If $r > s$, then*
$$\int_{[0,1]^2} \frac{-\log(xy)}{1-xy} x^r y^s \mathrm{d}x\mathrm{d}y \in \frac{\mathbb{Z}}{d_r^3};$$
(2) *If $r = s$, then*
$$\int_{[0,1]^2} \frac{-\log(xy)}{1-xy} x^r y^r \mathrm{d}x\mathrm{d}y \in \frac{\mathbb{Z}}{d_r^3} + \mathbb{Z}\zeta(3).$$

Proof. Let $\sigma \geqslant 0$. We have

$$\int_{[0,1]^2} \frac{x^{r+\sigma} y^{s+\sigma}}{1-xy} \mathrm{d}x\mathrm{d}y = \sum_{k=0}^{\infty} \frac{1}{(k+r+\sigma+1)(k+s+\sigma+1)} \tag{2.2}$$

by $(1-xy)^{-1} = \sum_{k\geqslant 0} x^k y^k$ and Levi's lemma.

If $r > s$, the sum in the right hand side of (2.2) is telescoping:

$$\begin{aligned}\text{RHS of (2.2)} &= \sum_{k=0}^{\infty} \frac{1}{r-s} \left(\frac{1}{k+s+\sigma+1} - \frac{1}{k+r+\sigma+1} \right) \\ &= \frac{1}{r-s} \left(\frac{1}{s+1+\sigma} + \frac{1}{s+2+\sigma} + \cdots + \frac{1}{r+\sigma} \right).\end{aligned}$$

Differentiate (2.2) with respect to σ and put $\sigma = 0$, we obtain that

$$\int_{[0,1]^2} \frac{\log(xy)}{1-xy} x^r y^s \mathrm{d}x\mathrm{d}y = \frac{-1}{r-s} \left(\frac{1}{(s+1)^2} + \frac{1}{(s+2)^2} + \cdots + \frac{1}{r^2} \right),$$

(we can interchange the order of integration and differentiation on LHS because $-\log(xy)/(1-xy)$ is a dominating function on $[0,1]^2$ for difference quotients) which proves (1). If $r = s$, then (2.2) becomes

$$\int_{[0,1]^2} \frac{x^{r+\sigma} y^{r+\sigma}}{1-xy} \mathrm{d}x\mathrm{d}y = \sum_{k=0}^{\infty} \frac{1}{(k+r+\sigma+1)^2}. \tag{2.3}$$

Differentiate (2.3) with respect to σ and put $\sigma = 0$,

$$\begin{aligned}\int_{[0,1]^2} \frac{\log(xy)}{1-xy} x^r y^r \mathrm{d}x\mathrm{d}y &= -2 \sum_{k=0}^{\infty} \frac{1}{(k+r+1)^3} \\ &= -2 \left(\zeta(3) - \frac{1}{1^3} - \frac{1}{2^3} - \cdots - \frac{1}{r^3} \right),\end{aligned}$$

then (2) follows. $\square$

Now, we can give Beukers' proof of Apéry's theorem.

Proof of Theorem 2.1. We take the (Legendre-type) polynomials

$$P_n(X) = \frac{1}{n!}\left(\frac{\mathrm{d}}{\mathrm{d}X}\right)^n (X^n(1-X)^n) \in \mathbb{Z}[X], \quad \deg P_n = n, \quad n = 1, 2, \ldots,$$

and consider the integrals

$$I_n = \int_{[0,1]^2} \frac{-\log(xy)}{1-xy} P_n(x)P_n(y)\mathrm{d}x\mathrm{d}y.$$

By Lemma 2.3, we have $I_n \in \mathbb{Z}/d_n^3 + \mathbb{Z}\zeta(3)$.

Noticing that

$$\frac{-\log(xy)}{1-xy} = \int_0^1 \frac{1}{1-(1-xy)z}\mathrm{d}z,$$

we can write (thanks to Fubini-Tonelli's theorem)

$$I_n = \int_0^1 \int_0^1 \int_0^1 \frac{P_n(x)P_n(y)}{1-(1-xy)z}\mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

After an n-fold partial integration with respect to x, we have

$$I_n = \int_0^1 \int_0^1 \int_0^1 \frac{(xyz)^n(1-x)^n P_n(y)}{(1-(1-xy)z)^{n+1}}\mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

Now, making the following change of variable

$$w = \frac{1-z}{1-(1-xy)z}$$

we obtain that

$$I_n = \int_0^1 \int_0^1 \int_0^1 (1-x)^n(1-w)^n \frac{P_n(y)}{1-(1-xy)w}\mathrm{d}x\mathrm{d}y\mathrm{d}w.$$

Then, an n-fold partial integration with respect to y implies that

$$I_n = \int_{[0,1]^3} \frac{x^n(1-x)^n y^n(1-y)^n w^n(1-w)^n}{(1-(1-xy)w)^{n+1}}\mathrm{d}x\mathrm{d}y\mathrm{d}w. \tag{2.4}$$

It is routine to check that

$$\frac{x(1-x)y(1-y)w(1-w)}{1-(1-xy)w} \leqslant (\sqrt{2}-1)^4, \quad \text{for all } (x,y,w) \in (0,1)^3,$$

so we deduce from (2.4) that

$$0 < I_n \leqslant (\sqrt{2}-1)^{4n} \int_{[0,1]^3} \frac{\mathrm{d}x\mathrm{d}y\mathrm{d}w}{1-(1-xy)w} = 2\zeta(3)(\sqrt{2}-1)^{4n}.$$

Recall that $d_n = \exp((1+o(1))n)$ as $n \to +\infty$ by the prime number theorem, so

$$\begin{cases} 0 < d_n^3 I_n \leqslant ((\sqrt{2}-1)^4 e^3)^{(1+o(1))n} \\ \qquad d_n^3 I_n \in \mathbb{Z} + \mathbb{Z} d_n^3 \zeta(3) \end{cases}.$$

Note that $(\sqrt{2}-1)^4 e^3 < 1$, from above $\zeta(3) \notin \mathbb{Q}$ follows easily. □

2.2. **Glimpses of other proofs.** A sketch of Apéry's ideas (1978):

- We define two sequences of integers
$$a_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2,$$
$$b_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \left(\sum_{m=1}^{n} \frac{1}{m^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m}\binom{n+m}{m}} \right).$$
- $a_n \in \mathbb{Z}$, $2d_n^3 b_n \in \mathbb{Z}$. $\quad (d_n = [1, 2, \dots, n].)$
- $\{a_n\}$, $\{b_n\}$ both satisfy the recurrence
$$(n+1)^3 X_{n+1} - (34n^3 + 51n^2 + 27n + 5)X_n + n^3 X_{n-1} = 0$$
- $$a_n = (\sqrt{2}+1)^{(4+o(1))n} \text{ as } n \to +\infty,$$
$$a_n\zeta(3) - b_n = \sum_{k=n+1}^{\infty} \frac{6a_n}{k^3 a_k a_{k+1}},$$
$$0 < a_n\zeta(3) - b_n = (\sqrt{2}-1)^{(4+o(1))n} \text{ as } n \to +\infty.$$

In 1981, Beukers [Beu81] explained Apéry's numbers a_n by Hermite-Padé approximation. Let $L_k(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^k}$ be the polylogarithm ($k \in \mathbb{N}$, $|z| < 1$), then $L_k(1) = \zeta(k)$ for $k \geqslant 2$. We look for polynomials $A_n(z), B_n(z), C_n(z), D_n(z) \in \mathbb{Q}[z]$ of degree n such that
$$\begin{cases} A_n(z)L_2(z) + B_n(z)L_1(z) + C_n(z) = O(z^{2n+1}) \\ 2A_n(z)L_3(z) + B_n(z)L_2(z) + D_n(z) = O(z^{2n+1}) \\ B_n(1) = 0 \end{cases}.$$
Suppose that $A_n(z) = \sum_{k=0}^{n} \alpha_k z^k$, $B_n(z) = \sum_{k=0}^{n} \beta_k z^k$. It turns out that up to proportion, $\alpha_k = \binom{n}{k}^2 \binom{2n-k}{n}^2$, so
$$A_n(1) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2n-k}{n}^2 = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = a_n.$$
Moreover, $2A_n(1)\zeta(3) + D_n(1) =$ remainder, and
$$\sum_{k=0}^{n} \left(\frac{\alpha_k}{(t-k)^2} + \frac{\beta_k}{t-k} \right) = \frac{(t-n-1)^2(t-n-2)^2 \cdots (t-2n^2)}{t^2(t-1)^2 \cdots (t-n)^2}.$$
In 1996, Nesterenko [Nes96] gave a self-contained proof of Apéry's theorem by starting directly from
$$\sum_{k=1}^{\infty} R_n'(k), \quad \text{where } R_n(t) = \frac{(t-1)^2(t-2)^2 \cdots (t-n)^2}{t^2(t+1)^2 \cdots (t+n)^2}.$$
There is a rather different proof of Apéry's theorem by Beukers in 1987, he used modular forms [Beu87]. In 2005, Calegari [Cal05] used p-adic modular forms to show that p-adic zeta value $\zeta_p(3) \notin \mathbb{Q}$ for $p = 2, 3$.

3. Ball-Rivoal's theorem

In 2000, Rivoal [Riv00], Ball and Rivoal [BR01] made a significant progress.

Theorem 3.1 (Ball-Rivoal's theorem)**.** *For any $\varepsilon > 0$, there exists $s_0(\varepsilon)$ such that for all odd integers $s \geqslant s_0(\varepsilon)$, the dimension of the space spanned by $1, \zeta(3), \zeta(5), \ldots, \zeta(s)$ over $\mathbb{Q}$ is at least that*

$$\dim_{\mathbb{Q}} \operatorname{Span}_{\mathbb{Q}} \left(1, \zeta(3), \zeta(5), \ldots, \zeta(s)\right) \geqslant \frac{1-\varepsilon}{1+\log 2} \log s.$$

As a (much weaker) corollary, there are infinitely many positive integers k such that $\zeta(2k+1) \notin \mathbb{Q}$. In this section, we will repeat Ball and Rivoal's proof of Theorem 3.1.

3.1. **Possible motivations.** In this subsection, we (try to) explain some possible motivations of Ball-Rivoal's theorem. Recall Beukers' integrals (see (2.4))

$$\int_{[0,1]^3} \frac{x^n(1-x)^n y^n(1-y)^n z^n(1-z)^n}{(1-(1-(1-x)y)z)^{n+1}} \mathrm{d}x\mathrm{d}y\mathrm{d}z \in \mathbb{Q} + \mathbb{Q}\zeta(3). \tag{3.1}$$

One natural generalization of (3.1) is that (for odd $s \geqslant 3$)

$$\int_{[0,1]^s} \frac{x_1^n(1-x_1)^n x_2^n(1-x_2)^n \cdots x_s^n(1-x_s)^n}{(1-(\cdots(1-(1-x_1)x_2)\cdots)x_s)^{n+1}} \mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_s \in \operatorname{Span}_{\mathbb{Q}} \left(1, \zeta(3), \zeta(5), \ldots, \zeta(s)\right). \tag{3.2}$$

However, this conclusion (3.2) is not easy. It was proved by Vasilyev [Vas96] for $s = 5$ and by Zudilin [Zud03, §8] for general cases by an identity between a certain multiple integral and a non-terminating very-well-poised hypergeometric series.

But there is a trivial 'generalization' of (3.1). For any $P(X_1, X_2, \ldots, X_s) \in \mathbb{Q}[X_1, X_2, \ldots, X_s]$ we have

$$\int_{[0,1]^s} \frac{P(x_1, x_2, \ldots, x_s)}{(1-x_1x_2\ldots x_s)^N} \mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_s \in \operatorname{Span}_{\mathbb{Q}} \left(1, \zeta(2), \zeta(3), \ldots, \zeta(s)\right), \tag{3.3}$$

provided that the left hand side is integrable. To see this, after expanding $(1-x_1x_2\cdots x_s)^{-N} = \sum_{k=0}^{\infty} \binom{N+k-1}{N-1} x_1^k x_2^k \cdots x_s^k$ and integrating term by term, the left hand side of (3.3) can be written as $\sum_{k=0}^{\infty} R(k)$ for some rational function $R(t)$, whose poles lie in $\mathbb{Z}_{\leqslant 0}$ and the order of any pole is not greater than s. Then it is easy to show that (by partial fraction decomposition) $\sum_{k=0}^{\infty} R(k)$ is a linear form in 1 and zeta values (see later Lemma).

What if we start directly from $\sum_{k=0}^{\infty} R(k)$ for such rational function $R(t)$ above? Recall Nesterenko had already considered

$$R_n(t) = \frac{(t-1)^2(t-2)^2\cdots(t-n)^2}{t^2(t+1)^2\cdots(t+n)^2},$$

he showed that $\sum_{k=1}^{\infty} R_n'(k) = a_n'\zeta(3) - b_n'$ is small, and the denominators of the rational coefficients a_n', b_n' are not too large. Then Nesterenko managed to give a new proof that $\zeta(3) \notin \mathbb{Q}$. In an email from Ball to Rivoal (see Rivoal's thesis), Ball considered

$$R_n^{\mathrm{B}}(t) = n!^2 \left(t + \frac{n}{2}\right) \frac{(t-n)_n(t+n+1)_n}{(t)_{n+1}^4},$$

where $(\alpha)_m = (\alpha)(\alpha+1)\cdots(\alpha+m-1)$ is the rising factorial (Pochhammer symbol). By last paragraph, we know that $\sum_{k=1}^{\infty} R_n^{\mathrm{B}}(k) \in \mathbb{Q} + \mathbb{Q}\zeta(2) + \mathbb{Q}\zeta(3) + \mathbb{Q}\zeta(4)$. Ball and Rivoal found that the symmetry of $R_n^{\mathrm{B}}(t)$ will eliminate the even zeta values $\zeta(2), \zeta(4)$, so $\sum_{k=1}^{\infty} R_n^{\mathrm{B}}(k)$ is in fact belonging to $\mathbb{Q} + \mathbb{Q}\zeta(3)$. To generalize $R_n^{\mathrm{B}}(t)$ (importantly, keep the symmetry), Rivoal considered

$$R_n^{\mathrm{BR}}(t) = n!^{s-2r} \frac{(t-rn)_{rn}(t+n+1)_{rn}}{(t)_{n+1}^s},$$

where $r < s/2$ is an integer parameter. An overview of the proof of Theorem 3.1:

- Symmetry ($R_n^{\mathrm{BR}}(t) = -R_n^{\mathrm{BR}}(-t-n)$) kills all even zeta values $\zeta(2k)$. We have (for any even n)
$$S_n := \sum_{k=1}^{\infty} R_n^{\mathrm{BR}}(k) = \rho_0 + \sum_{\substack{3 \leqslant i \leqslant s \\ i \text{ odd}}} \rho_i \zeta(i),$$
where $\rho_i = \rho_i^{(n)}$ are rational coefficients depend on n.
- The common denominator of ρ_i is not large: $d_n^s \rho_i \in \mathbb{Z}$ for all $i = 0, 3, 5, \dots, s$. ($d_n = [1, 2, \dots, n]$.)
- We have good estimates for S_n and ρ_i:
$$\limsup_{n \to +\infty} |\rho_i|^{1/n} \leqslant 2^{s-2r}(2r+1)^{2r+1}, \text{for all } i;$$
$$\lim_{n \to +\infty} |S_n|^{1/n} \overset{\text{exists}}{=\!=\!=} c = c(s,r) \text{ and } 0 < c \leqslant \frac{2^{r+1}}{r^{s-2r}}.$$
- By applying Nesterenko's linear independence criterion (Theorem 1.3) to $\{d_n^s S_n\}_{n \text{ even}}$ and optimizing the parameter r, we will complete the proof of Theorem 3.1.

How to make S_n small and nonzero? Our method is that letting $R_n(k) = 0$ for $k = 1, 2, \dots, rn$ and $R_n(k) > 0$ for $k > rn$. Thus $R_n(t)$ needs to have the factor $(t-rn)_{rn}$. In order to keep symmetry, $R_n(t)$ also has to have the factor $(t+n+1)_{rn}$. In this sense, the construction of $R_n^{\mathrm{BR}}(t)$ is perhaps the most natural choice.

3.2. **From rational functions to odd zeta values.**

Lemma 3.2. *Let s, n be positive integers. Let*

$$R(t) = \frac{P(t)}{t^s(t+1)^s \cdots (t+n)^s}$$

be a rational function, where $P(t)$ is a polynomial with rational coefficients. Suppose that $\deg R \leqslant -2$ *and*

$$R(t) = \sum_{k=0}^{n} \sum_{i=1}^{s} \frac{a_{i,k}}{(t+k)^i} \tag{3.4}$$

is the unique partial fraction decomposition of $R_n(t)$. Then

$$\sum_{m=1}^{\infty} R(m) = \rho_0 + \sum_{i=2}^{s} \rho_i \zeta(i)$$

is a linear form in 1 *and zeta values, where*

$$\rho_i = \sum_{k=0}^{n} a_{i,k} \in \mathbb{Q},\ \ i = 2, 3, \ldots, s;$$

$$\rho_0 = -\sum_{k=0}^{n}\sum_{i=1}^{s}\sum_{\ell=1}^{k} \frac{a_{i,k}}{\ell^i} \in \mathbb{Q}.$$

Proof. Since $\deg R \leqslant -2$, we have $R(m) = O(m^{-2})$ as $m \to +\infty$. Hence the summation $\sum_{m=1}^{\infty} R(m)$ is absolutely convergent.

By substituting (3.4) into $\sum_{m=1}^{\infty} R(m)$ and separating the terms with $i = 1$ and $i \neq 1$, we obtain that

$$\begin{aligned}\sum_{m=0}^{\infty} R(m) &= \sum_{m=1}^{\infty}\sum_{k=0}^{n} \frac{a_{1,k}}{m+k} + \sum_{m=1}^{\infty}\sum_{k=0}^{n}\sum_{i=2}^{s} \frac{a_{i,k}}{(m+k)^i}\\ &= I_1 + I_2.\end{aligned}$$

The summation for I_2 is absolutely convergent, we can interchange the order of summation. Thus,

$$\begin{aligned}I_2 &= \sum_{k=0}^{n}\sum_{i=2}^{s}\sum_{m=1}^{\infty} \frac{a_{i,k}}{(m+k)^i}\\ &= \sum_{k=0}^{n}\sum_{i=2}^{s} a_{i,k}\left(\zeta(i) - \sum_{\ell=1}^{k}\frac{1}{\ell^i}\right).\end{aligned} \tag{3.5}$$

For I_1, note firstly that (by (3.4) and $\deg R \leqslant -2$)

$$\sum_{k=0}^{n} a_{1,k} = \lim_{m \to +\infty} mR(m) = 0.$$

We have

$$\sum_{m=1}^{M}\sum_{k=0}^{n}\frac{a_{1,k}}{m+k} = -\sum_{k=0}^{n} a_{1,k}\sum_{\ell=1}^{k}\frac{1}{\ell} + \underbrace{\left(\sum_{k=0}^{n} a_{1,k}\right)}_{=0}\sum_{\ell=1}^{M+n}\frac{1}{\ell} - \sum_{k=0}^{n} a_{1,k}\underbrace{\sum_{\ell=M+k+1}^{M+n}\frac{1}{\ell}}_{\to 0 \text{ as } M \to +\infty},$$

so letting $M \to +\infty$ we deduce that

$$I_1 = -\sum_{k=0}^{n} a_{1,k}\sum_{\ell=1}^{k}\frac{1}{\ell}. \tag{3.6}$$

Combining (3.5) and (3.6), we complete the proof of Lemma 3.2. □

The following lemma is one key observation of Ball and Rivoal, it says that symmetry can eliminate all even zeta values.

Lemma 3.3 (Symmetry kills even zetas)**.** *Under the hypothesis of Lemma 3.2, if we assume in addition that* $R(t) = -R(-t-n)$*, then* $\rho_i = 0$ *for all* $i \in \{2, 4, 6, \ldots, 2\lfloor s/2 \rfloor\}$*.*

Proof. We have

$$
\begin{aligned}
R(t) &= \sum_{k=0}^{n} \sum_{i=1}^{s} \frac{a_{i,k}}{(t+k)^i}, \\
-R(-t-n) &= -\sum_{k=0}^{n} \sum_{i=1}^{s} \frac{a_{i,k}}{(-t-n+k)^i} \\
&= \sum_{k=0}^{n} \sum_{i=1}^{s} (-1)^{i+1} \frac{a_{i,n-k}}{(t+k)^i}.
\end{aligned}
$$

The symmetry $R(t) = -R(-t-n)$ implies (since the partial fraction decomposition is unique) that $a_{i,k} = (-1)^{i+1} a_{i,n-k}$ holds for all i, k. Thus,

$$
2\rho_i = \sum_{k=0}^{n} (a_{i,k} + a_{i,n-k}) = 0 \text{ for all even } i > 0.
$$

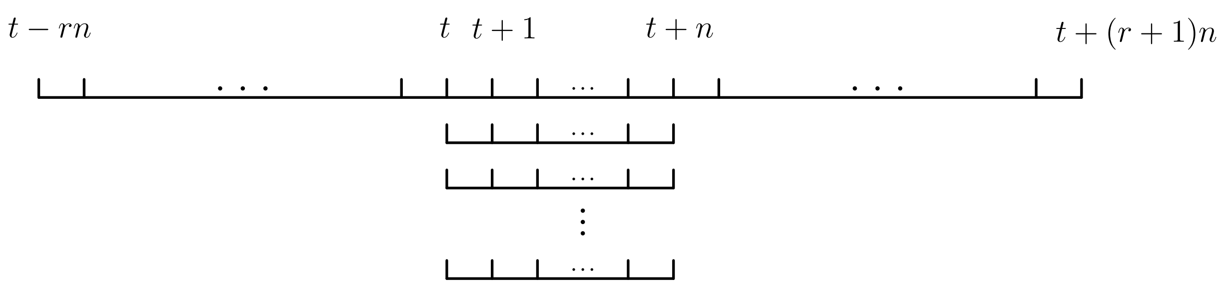
□

In the sequel, Ball and Rivoal took a sequence of particular rational functions $\{R_n(t)\}$.

Definition 3.4 ($R_n^{(\mathrm{BR})}(t)$)**.** *Let $s \geqslant 3$ be an odd integer and n be an even positive integer. Let $r \in \mathbb{N}$, $r < s/2$ be an integer parameter. We define the rational function*

$$
\begin{aligned}
R_n(t) &= n!^{s+1-2r} \frac{\prod_{j=0}^{(2r+1)n} (t - rn + j)}{\prod_{j=0}^{n} (t+j)^{s+1}} \\
&= n!^{s+1-2r} \frac{(t-rn)(t-rn+1)\cdots(t-1)\cdot(t+n+1)(t+n+2)\cdots(t+(r+1)n)}{t^s (t+1)^s \cdots (t+n)^s}.
\end{aligned}
$$

The factor $n!^{s+1-2r}$ is a normalizing factor. The following diagram illustrates the shape of $R_n(t)$.



One easily check that $\deg R_n \leqslant -2$ and $R_n(t)$ possesses the symmetry $R_n(t) = -R_n(-t-n)$. Thus we can apply Lemma 3.3 to $R_n(t)$.

Definition 3.5 ($a_{i,k}$)**.** *We denote by*

$$
R_n(t) = \sum_{k=0}^{n} \sum_{i=1}^{s} \frac{a_{i,k}}{(t+k)^i}
$$

the unique partial fraction decomposition of $R_n(t)$. The coefficients $a_{i,k} = a_{n,i,k}$ also depend on n but we omit the subscript n for convenience.

Lemma 3.6 (linear forms)**.** *Let*

$$S_n = \sum_{m=1}^{\infty} R_n(m). \tag{3.7}$$

Then

$$S_n = \rho_0 + \sum_{\substack{3\leqslant i\leqslant s\\ i\ \text{odd}}} \rho_i \zeta(i) \tag{3.8}$$

is a linear form in 1 and odd zeta values, where the coefficients

$$\rho_i = \sum_{k=0}^{n} a_{i,k} \qquad (3 \leqslant i \leqslant s,\ i\ \text{odd}),\ \text{and}$$

$$\rho_0 = -\sum_{k=0}^{n}\sum_{i=1}^{s}\sum_{\ell=1}^{k} \frac{a_{i,k}}{\ell^i}$$

depend on n but we omit the subscript n.

Proof. A direct application of Lemma 3.2 and Lemma 3.3. □

3.3. Arithmetic of coefficients.

Lemma 3.7 (arithmetic of $a_{i,k}$)**.** *For the coefficients $a_{i,k}$ in the partial fraction decomposition of $R_n(t)$ in Definition 3.5, we have*

$$d_n^{s-i} a_{i,k} \in \mathbb{Z},$$

where $d_n = [1, 2, \ldots, n]$ is the least common multiple of the first n positive integers.

Proof. We first split $R_n(t)$ into a product of building blocks. We have

$$R_n(t) = H(t)^{s-2r} \prod_{i=1}^{r} G_i^-(t) \prod_{i=1}^{r} G_i^+(t), \tag{3.9}$$

where

$$H(t) = \frac{n!}{t(t+1)\cdots(t+n)},$$

$$G_i^-(t) = \frac{(t-in)(t-in+1)\cdots(t-(i-1)n-1)}{t(t+1)\cdots(t+n)},$$

$$G_i^+(t) = \frac{(t+in+1)(t+in+2)\cdots(t+(i+1)n)}{t(t+1)\cdots(t+n)}.$$

Suppose that the partial fraction decomposition of the rational function $H(t)$ is

$$H(t) = \frac{b_0}{t} + \frac{b_1}{t+1} + \cdots + \frac{b_n}{t+n},$$

then

$$b_i = (t+i)H(t)\big|_{t=-i} = (-1)^i \binom{n}{i} \in \mathbb{Z}.$$

So

$$H(t) \in \frac{\mathbb{Z}}{t} + \frac{\mathbb{Z}}{t+1} + \cdots + \frac{\mathbb{Z}}{t+n}.$$

Suppose that

$$G_i^-(t) = \frac{c_{i,0}}{t} + \frac{c_{i,1}}{t+1} + \cdots + \frac{c_{i,n}}{t+n},$$

then

$$c_{i,j} = (t+j)G_i^-(t)\big|_{t=-j} = (-1)^j \frac{(-j-in)_n}{j!(n-j)!}.$$

Here $(-j-in)_n = (-j-in)(-j-in+1)\cdots(-j-(i-1)n-1)$ is a produce of n consecutive integers, so it is a multiple of $n!$. Thus, $c_{i,j} \in \mathbb{Z}$ for all i,j and

$$G_i^-(t) \in \frac{\mathbb{Z}}{t} + \frac{\mathbb{Z}}{t+1} + \cdots + \frac{\mathbb{Z}}{t+n}.$$

Similarly, we also have

$$G_i^+(t) \in \frac{\mathbb{Z}}{t} + \frac{\mathbb{Z}}{t+1} + \cdots + \frac{\mathbb{Z}}{t+n}.$$

In conclusion, $R_n(t)$ is a product of s rational functions of the form

$$\frac{\mathbb{Z}}{t} + \frac{\mathbb{Z}}{t+1} + \cdots + \frac{\mathbb{Z}}{t+n}.$$

Notice that the rule

$$\frac{1}{(t+k)(t+k')} = \frac{1}{k-k'}\left(\frac{1}{t+k'} - \frac{1}{t+k}\right) \text{ for } k \neq k'.$$

A denominator appears each time the above rule is applied, and the denominator is always a divisor of d_n. This happens $s-i$ times for each term that finally contributes to $a_{i,k}$. Therefore,

$$d_n^{s-i} a_{i,k} \in \mathbb{Z}$$

for all i, k, as desired. □

The arithmetic property of $a_{i,k}$ induces the arithmetic property of ρ_i.

Lemma 3.8 (arithmetic of ρ_i). *We have $d_n^{s-i}\rho_i \in \mathbb{Z}$ for all $i \in \{3,5,\ldots,s\}$ and $d_n^s \rho_0 \in \mathbb{Z}$.*

Proof. Since $\rho_i = \sum_{k=0}^n a_{i,k}$ for $i \in \{3,5,\ldots,s\}$, Lemma 3.7 immediately implies that $d_n^{s-i}\rho_i \in \mathbb{Z}$. For ρ_0, we have

$$d_n^s \rho_0 = -\sum_{i=1}^{s}\sum_{k=0}^{n} \underbrace{d_n^{s-i} a_{i,k}}_{\in\mathbb{Z}} \sum_{\ell=1}^{k} \underbrace{\frac{d_n^i}{\ell^i}}_{\in\mathbb{Z}}.$$

Thus, $d_n^s\rho_0 \in \mathbb{Z}$ as desired. □

3.4. Asymptotics.

Lemma 3.9 (asymptotics of S_n)**.** *We have (as $n \to +\infty$ along even integers)*

$$\lim_{n\to+\infty} S_n^{1/n} =: c(s,r) \text{ exsits,}$$

and

$$0 < c(s,r) \leqslant \frac{2^{r+1}}{r^{s-2r}}.$$

Proof. We first claim that

$$S_n = \frac{((2r+1)n+1)!}{n!^{s-2r}} \int_{[0,1]^{s+1}} \left(\frac{\prod_{i=1}^{s+1}(x_i^r(1-x_i))}{(1-x_1x_2\cdots x_{s+1})^{2r+1}} \right)^n \frac{\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_{s+1}}{(1-x_1x_2\cdots x_{s+1})^2}. \tag{3.10}$$

In fact, by expanding

$$\frac{1}{(1-x_1x_2\cdots x_{s+1})^{(2r+1)n+2}} = \sum_{k\geqslant 1} \binom{(2r+1)n+k}{k-1} x_1^{k-1}x_2^{k-1}\cdots x_{s+1}^{k-1},$$

integrating term by term, and using

$$\int_0^1 x^{rn+k-1}(1-x)^n \mathrm{d}x = B(rn+k, n+1) = \frac{(rn+k-1)!n!}{((r+1)n+k)!},$$

A calculation shows that the right hand side of (3.10) equals to

$$\sum_{k\geqslant 1} R_n(rn+k).$$

Note that $R_n(k) = 0$ for $k = 1, 2, \ldots, rn$, so (3.10) is proved.

We can view $f(x_1, x_2, \ldots, x_{s+1}) := \left(\prod_{i=1}^{s+1}(x_i^r(1-x_i))\right)/(1-x_1x_2\ldots x_{s+1})^{2r+1}$ as a continuous function on the hypercube $[0,1]^{s+1}$. (Note that $1 - x_1x_2\ldots x_{s+1} \geqslant 1 - x_i$ for each i, so $(1-x_1x_2\ldots x_{s+1})^{2r+1} \geqslant (\prod_{i=1}^{s}(1-x_i))^{(2r+1)/(s+1)}$, hence $f(x_1, x_2, \ldots, x_{s+1}) \to 0$ when $(x_1, x_2, \ldots, x_{s+1}) \to (1, 1, \ldots, 1)$.)

As $\mathrm{d}\mu := \mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_{s+1}/(1-x_1x_2\cdots x_{s+1})^2$ is a finite measure on $[0,1]^{s+1}$ ($\mu([0,1]^{s+1}) = \zeta(s) < +\infty$) and $f(x_1, x_2, \ldots, x_{s+1})$ is continuous on $[0,1]^{s+1}$, we have

$$\lim_{n\to+\infty} \left(\int_{[0,1]^{s+1}} f(\underline{x})^n \mathrm{d}\mu \right)^{\frac{1}{n}} = \max_{\underline{x}\in[0,1]^{s+1}} f(\underline{x}).$$

Therefore, Stirling's formula and (3.10) imply that $\lim_{n\to+\infty} S_n^{1/n}$ exists and

$$c(s,r) := \lim_{n\to+\infty} S_n^{1/n} = (2r+1)^{2r+1} \max_{\underline{x}\in[0,1]^{s+1}} f(\underline{x}).$$

Clearly $c(s,r) > 0$. To see why $c(s,r) \leqslant 2^{r+1}/r^{s-2r}$, we take another approach. Denote $\widetilde{R_n}(t) = t^s R_n(t)$. Then for $k > rn$,

$$\begin{aligned}
\widetilde{R_n}(k) &< n^{(s-2r)n} \frac{k^{rn}(k+(r+1)n)^{rn}}{k^{sn}} \\
&< n^{(s-2r)n} \frac{k^{rn}(2^{1+1/r}k)^{rn}}{k^{sn}} \quad \text{(because } k+(r+1)n < 2^{1+1/r}k \text{ for } k > rn) \\
&= \left(2^{r+1} \left(\frac{n}{k}\right)^{s-2r} \right)^n < \left(\frac{2^{r+1}}{r^{s-2r}} \right)^n \quad \text{(because } k > rn).
\end{aligned}$$

Note also $R_n(k) = 0$ for $k = 1, 2, \ldots, rn$, so

$$S_n = \sum_{k=rn+1}^{\infty} \frac{1}{k^s} \widetilde{R_n}(k) < \zeta(s) \left(\frac{2^{r+1}}{r^{s-2r}} \right)^n .$$

Therefore, $c(s,r) \leqslant 2^{r+1}/r^{s-2r}$, as desired. □

We remark that the estimates for S_n are asymptotically best in the sense that any better estimates for S_n will not improve the final result in Theorem 3.1. Same remark for the next lemma.

Now, we estimate the coefficients ρ_i.

Lemma 3.10 (estimates for ρ_i)**.** *We have (as $n \to +\infty$ along even integers)*

$$\limsup_{n \to +\infty} |\rho_i|^{1/n} \leqslant 2^{s-2r}(2r+1)^{2r+1}.$$

Proof. Recall the expression for ρ_i in Lemma 3.6, it is sufficient to show that

$$\limsup_{n \to +\infty} \left(\max_{i,k} |a_{i,k}| \right)^{1/n} \leqslant 2^{s-2r}(2r+1)^{2r+1}. \tag{3.11}$$

By the partial fraction decomposition in Definition 3.5, we have

$$a_{i,k} = \frac{1}{2\pi i} \int_{|z+k|=\frac{1}{2}} R_n(z)(z+k)^{i-1} \mathrm{d}z.$$

Note that

$$R_n(z)(z+k)^{i-1} = n!^{s-2r} \frac{(z-rn)_{rn}(z+n+1)_{rn}}{(z)_{n+1}^s} \cdot (z+k)^{i-1},$$

where $(\alpha)_m = (\alpha)(\alpha+1)\cdots(\alpha+m-1)$ is the Pochhammer symbol. On the circle $|z+k| = 1/2$, we have

$$\begin{aligned} |(z-rn)_{rn}| &\leqslant (k+2)_{rn}, \\ |(z+n+1)_{rn}| &\leqslant (n-k+2)_{rn}, \\ |(z)_{n+1}| &\geqslant 2^{-3}(k-1)!(n-k-1)!. \end{aligned}$$

Therefore, for all $i \in \{1, 2, \ldots, s\}$ and $k \in \{0, 1, \ldots, n\}$,

$$\begin{aligned} |a_{i,k}| &\leqslant n!^{s-2r} \frac{(k+2)_{rn}(n-k+2)_{rn}}{(2^{-3}(k-1)!(n-k-1)!)^s} \\ &= \left(\frac{n!}{k!(n-k)!} \right)^{s-2r} \cdot \frac{(k+rn+1)!}{(k+1)!(k!(n-k)!)^r} \cdot \frac{((r+1)n-k+1)!}{(n-k+1)!(k!(n-k!)^r)} \cdot (8k(n-k))^s \\ &\leqslant 2^{(s-2r)n} \cdot (2r+1)^{k+rn+1} \cdot (2r+1)^{(r+1)n-k+1} \cdot (2n^2)^s \\ &= 2^{(s-2r)n}(2r+1)^{(2r+1)n+2}(2n^2)^s. \end{aligned}$$

Thus, (3.11) is true and the proof of Lemma 3.10 is complete. □

3.5. **Proof of Theorem 3.1.**

Proof of Theorem 3.1. By Lemma 3.8, Lemma 3.9, and Lemma 3.10, we have (we assume n is even in the following three items, for the symmetry of $R_n(t)$)

- $d_n^s S_n = d_n^s \rho_0 + \sum_{\substack{3\leqslant i\leqslant s \\ i \text{ odd}}} d_n^s \rho_i \zeta(3)$ is a linear form with integer coefficients.
- $|d_n^s \rho_i| \leqslant \exp\left((s + (s-2r)\log 2 + (2r+1)\log(2r+1) + o(1))n\right)$ as $n \to +\infty$.
- $d_n^s S_n = \exp\left((s + \log c(s,r) + o(1))n\right)$ as $n \to +\infty$. (Here we used the prime number theorem: $d_n = \exp((1+o(1))n)$.)

Applying Nesterenko's linear independence criterion (Theorem 1.3, with $\tau_1 = \tau_2$ and $\sigma(n) = 2n$) to $\{d_{2n}^s S_{2n}\}_{n\geqslant 1}$, we deduce that

(3.12)

$$D(s) := \dim_{\mathbb{Q}} \operatorname{Span}_{\mathbb{Q}}\left(1, \zeta(3), \zeta(5), \ldots, \zeta(s)\right) \geqslant 1 + \frac{-\log c(s,r) - s}{s + (s-2r)\log 2 + (2r+1)\log(2r+1)}.$$

Since $0 < c(s,r) \leqslant 2^{r+1}/r^{s-2r}$, we have

$$D(s) \geqslant 1 + \frac{(s-2r)\log r - (r+1)\log 2 - s}{s + (s-2r)\log 2 + (2r+1)\log(2r+1)}.$$

Choose $r = \lfloor s/\log^2 s \rfloor$, we obtain that, as $s \to +\infty$,

$$\begin{aligned} D(s) &\geqslant 1 + \frac{\log s - 1 + o(1)}{1 + \log 2 + o(1)} \\ &= \frac{1+o(1)}{1+\log 2}\log s. \end{aligned}$$

This completes the proof of Theorem 3.1. □

We give some final remarks. Ball and Rivoal [BR01] took $s = 169$ and $r = 10$, then $\log c(s,r) \approx -505.734$ and (3.12) implies $D(169) > 2.001$, so $D(169) \geqslant 3$. In other words, there exists $j \in \{5, 7, \ldots, 169\}$ such that $1, \zeta(3), \zeta(j)$ are linearly independent over $\mathbb{Q}$. In 2010, Fischler and Zudilin [FZ10] lowed 169 to 139 by saving some common factor of the coefficients $d_n^s \rho_i$ (they used some different but similar rational functions) and refining Nesterenko's criteria correspondingly.

In 2003, Fischler and Rivoal [FR03] showed that

$$S(z) = \sum_{t=1}^{\infty} \frac{(t-rn)_{rn}(t+n+1)_{rn}}{(t)_{n+1}^s} z^{-t}$$

is the unique solution (up to proportionality) of the following Padé approximation problem: find polynomials $P_0, P_1, \ldots, P_{s+2}$ of degree $\leqslant n$ such that

$$\begin{cases} S(z) := P_{s+2}(z) + \sum_{i=1}^{s} P_i(z) \operatorname{Li}_i(1/z) = O(z^{-rn-1}), \ z \to \infty \\ \widetilde{S}(z) := P_{s+1}(z) + \sum_{i=1}^{s} P_i(z)(-1)^i \operatorname{Li}_i(z) = O(z^{(r+1)n+1}), \ z \to 0 \\ T(z) := \sum_{i=1}^{s} P_i(z)(-1)^{i-1} \frac{(\log z)^{i-1}}{(i-1)!} = O((z-1)^{(s-2r)n+s-1}), \ z \to 1 \end{cases}.$$

(Where $\mathrm{Li}_s(z) = \sum_{k \geqslant 1} z^k/k^s$ is the s-th polylogarithm of z.)

The methods of Ball-Rivoal can be applied to some related problems. In 2006, Marcovecchio [Mar06] showed that for any $\alpha \in \overline{\mathbb{Q}}$ with $0 < |\alpha| < 1$, it holds that

$$\dim_{\mathbb{Q}} \mathrm{Span}_{\mathbb{Q}} (1, \mathrm{Li}_1(\alpha), \mathrm{Li}_2(\alpha), \ldots, \mathrm{Li}_s(\alpha)) \geqslant \frac{1 + o(1)}{(1 + \log 2)[\mathbb{Q}(\alpha) : \mathbb{Q}]} \log s, \quad s \to +\infty.$$

In 2020, Fischler and Rivoal [FR20] generalized Ball-Rivoal' theorem (in some sense) to G-functions. (See also Lepetit [Lep21] for a further generalization.)

4. FSZ-LY: 'MANY' ODD ZETA VALUES ARE IRRATIONAL

During 2018-2020, there are some new progresses about the irrationality of odd zeta values.

Theorem 4.1 ([FSZ19][LY20]). *For any $\varepsilon > 0$, there exists $s_0(\varepsilon)$ such that for all odd integers $s \geqslant s_0(\varepsilon)$, it holds that*

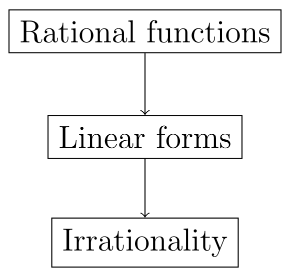
$$\#\left\{k : 3 \leqslant k \leqslant s, k \text{ odd}, \zeta(k) \notin \mathbb{Q}\right\} \geqslant (c_0 - \varepsilon)\sqrt{\frac{s}{\log s}},$$

where the constant

$$c_0 = \sqrt{\frac{4\zeta(2)\zeta(3)}{\zeta(6)}\left(1 - \log\frac{\sqrt{4e^2+1}-1}{2}\right)} \approx 1.192507\ldots.$$

Note that the magnitude of $\sqrt{s/\log s}$ is larger than that of $\log s$ in Ball-Rivoal's theorem (Theorem 3.1), but there is no information in Theorem 4.1 for linear independence.

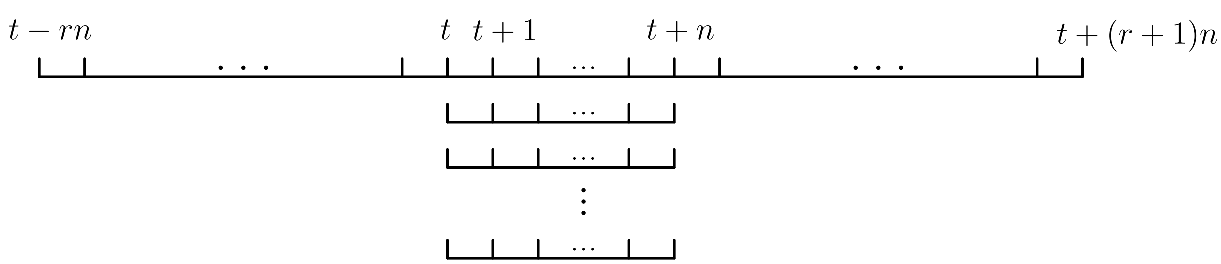
Below we will explain the ideas (and some history) of the proof of Theorem 4.1. Recall that the strategy of the proof of Ball-Rivoal's theorem (Theorem 3.1) can be sketched as



This approach is rather elementary. We have fun in constructing rational functions. Ball and Rivoal used the rational functions

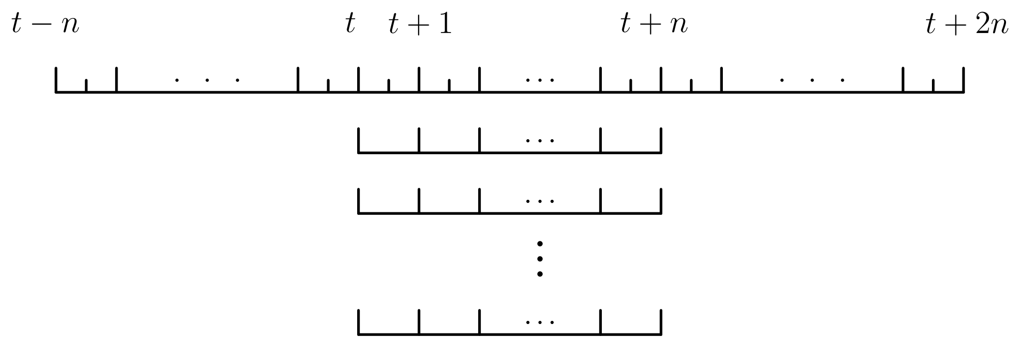
$$R_n^{(\mathrm{BR})}(t) = n!^{s-2r}\frac{(t-rn)_{(2r+1)n+1}}{(t)_{n+1}^{s+1}},$$

which has the shape



In 2018, Zudilin [Zud18] considered the rational functions

$$R_n^{(\mathrm{Z})}(t) = 2^{6n}n!^{s-5}\frac{(t-n)_{3n+1}(t-n+1/2)_{3n}}{(t)_{n+1}^{s+1}}.$$



Zudilin showed that

$$S_n = \sum_{m=1}^{\infty} R_n^{(\mathrm{Z})}(m) = \rho_0 + \sum_{\substack{3\leqslant i\leqslant s \\ i \text{ odd}}} \rho_i \zeta(i), \text{ and}$$

$$S_n' = \sum_{m=1}^{\infty} R_n^{(\mathrm{Z})}(m+1/2) = \rho_0' + \sum_{\substack{3\leqslant i\leqslant s \\ i \text{ odd}}} (2^i-1)\rho_i \zeta(i)$$

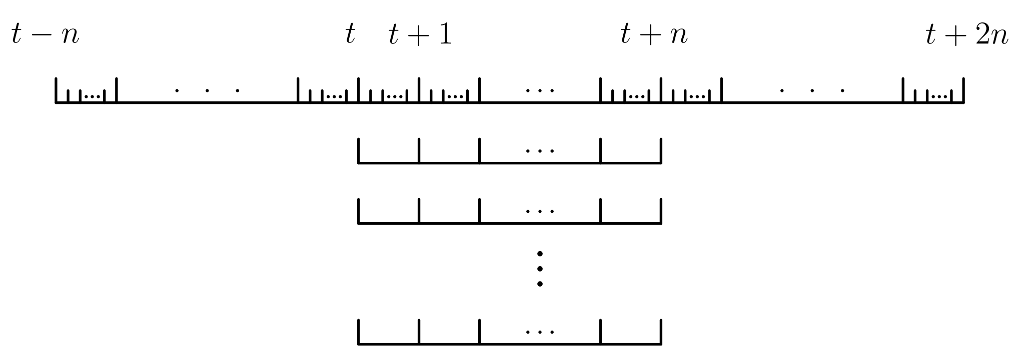
are linear forms in 1 and odd zeta values with *related coefficients*. We can *eliminate* one odd zeta value by some combination of S_n and S_n', for example, $7S_n - S_n'$ eliminates $\zeta(3)$. It is somewhat routine to show that $d_n^{s-i}\rho_i \in \mathbb{Z}$ for $i > 0$. Amazingly, it also holds that $d_n^s\rho_0, d_n^s\rho_0' \in \mathbb{Z}$. Zudilin took $s = 25$ to show that there is at least one irrational numbers among $\zeta(5), \zeta(7), \ldots, \zeta(25)$.

Later, Sprang [Spr18] generalized Zudilin's ideas. He constructed the rational functions

$$R_n^{(\mathrm{S})}(t) = D^{6(D-1)n} n!^{s+1-3D} \frac{(t-n)_{3n+1} \prod_{j=1}^{D-1}(t-n+j/D)_{3n}}{(t)_{n+1}^{s+1}}.$$

Sprang used a clever argument to show that $d_n^s \rho_{0,j/D} \in \mathbb{Z}$, which used the zeros of the rational functions in the 'middle part'. Fischler, Sprang and Zudilin [FSZ19] used

$$R_n^{(\mathrm{FSZ})}(t) = D^{3Dn} n!^{s+1-3D} \frac{(t-n)_{3n+1} \prod_{j=1}^{D-1}(t-n+j/D)_{3n}}{(t)_{n+1}^{s+1}}.$$

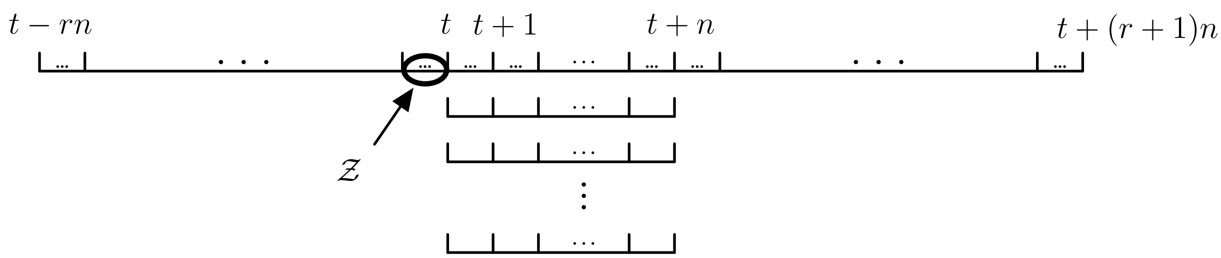


They showed that there are at least $2^{(1-o(1))\log s/\log\log s}$ irrational numbers amongst $\zeta(3)$, $\zeta(5)$, …, $\zeta(s)$, as $s \to +\infty$.

As you might see, the 'zero set' of $R_n(t)$ matters. In 2020, Lai and Yu [LY20] considered rational functions of the form

$$R_n(t) = (\text{some normalizing factors}) \cdot \frac{(t-rn)\prod_{\theta\in\mathcal{Z}}(t-rn+\theta)_{(2r+1)n}}{(t)_{n+1}^{s+1}},$$

where $\mathcal{Z} \subset (0,1] \cap \mathbb{Q}$ is the 'zero set' of $R_n(t)$. For a class of 'zero set' $\mathcal{Z}$, FSZ's proof still works without new essential difficulties.



An outline of the proof of Theorem 4.1:

- We will consider rational functions of the form:
$$R_n(t) = (\text{some normalizing factors}) \cdot \frac{(t-rn)\prod_{\theta\in\mathcal{Z}}(t-rn+\theta)_{(2r+1)n}}{(t)_{n+1}^{s+1}},$$
where $\mathcal{Z} \subset (0,1] \cap \mathbb{Q}$ is the 'zero set' of $R_n(t)$ and $r \in \mathbb{Q}$ is the 'length parameter'.
- One more b such that $\{1/b, 2/b, \dots, b/b\} \subset \mathcal{Z}$, one more $\zeta(2k+1)$ we can eliminate. ($\sum_{k=1}^{b}\zeta(i,k/b) = b^i\zeta(i)$.) Thus, we take
$$\mathcal{Z} = \bigcup_{b\in\mathcal{B}}\left\{\frac{1}{b}, \frac{2}{b}, \dots, \frac{b}{b}\right\}$$
for some set $\mathcal{B}$.
- The major 'arithmetic wasting' caused by denominators of rational zeros of $R_n(t)$ is
$$A_1(\mathcal{Z}) = \left(\prod_{\theta\in\mathcal{Z}}\operatorname{den}(\theta)\right)^{(2r+1)}.$$
We want to make $\#\mathcal{B}$ large while $|A_1(\mathcal{Z})|$ small. It turns out that the asymptotically best choice (as $s \to +\infty$) is
$$\mathcal{B} = \{b \in \mathbb{N} \ : \ \varphi(b) \leqslant B\} \ \text{for some } B;$$
$$\mathcal{Z} = \left\{\frac{a}{b} \in \mathbb{Q} \ : \ b \in \mathcal{B}, 1 \leqslant a \leqslant b, \gcd(a,b) = 1\right\}.$$
- Define $S_{n,\theta} := \sum_{m=1}^{\infty} R_n(m+\theta)$ for every $\theta \in \mathcal{Z}$. Then
$$S_{n,\theta} = \rho_{0,\theta}^{(n)} + \sum_{\substack{3\leqslant i\leqslant s\\ i \text{ odd}}} \underbrace{\rho_i^{(n)}}_{\text{indep. on }\theta} \zeta(i,\theta)$$

is a linear form in 1 and odd Hurwitz zeta values. For the arithmetic of coefficients, we have
  - $d_n^{s+1-i}\rho_i \in \mathbb{Z}$ (routine);
  - $d_n^{s+1}\rho_{0,\theta} \in \mathbb{Z}$ (Sprang's lemma, new and tricky).
- For the asymptotic of $S_{n,\theta}$, we have (as $n \to +\infty$ along a subsequence of $\mathbb{N}$)
  - $\lim_{n\to\infty} (S_{n,\theta})^{1/n}$ exits and has a precise expression;
  -
$$\lim_{n\to\infty} \frac{S_{n,1}}{S_{n,\theta}} = 1, \text{ for all } \theta \in \mathcal{Z}.$$
- Some combinations of $S_{n,\theta}$ produce good linear forms in 1 and odd zeta values, eliminate any prescribed $\#\mathcal{B} - 1$ odd zeta values. Thus, $\#(\text{irrationals}) \geqslant \#\mathcal{B}$.

4.1. **Rational functions and linear forms.** Let $r = \mathrm{num}(r)/\mathrm{den}(r)$ be a positive rational number, where $\mathrm{num}(r)$ and $\mathrm{den}(r)$ are the numerator and denominator of r in reduced form, respectively. We refer to r the 'length ratio parameter'. Eventually we will take the rational number r arbitrarily close to $r_0 = (\sqrt{4e^2+1} - 1)/2 \approx 2.26388$ in order to maximize certain quantity.

Let s be a positive odd integer and B be a positive real number. We will always assume that

(1) Both s and B are larger than some absolute constant.
(2) $s \geqslant 10(2r+1)B^2$.

Eventually we will take $B = c\sqrt{s/\log s}$ for some constant c.

Definition 4.2. *We define the following two sets which depend only on B:*

(1) *the* Denominator Set
$$\mathcal{B}_B = \{b \in \mathbb{N} \mid \varphi(b) \leqslant B\},$$
where $\varphi(\cdot)$ is the Euler totient function.
(2) *the* Zero Set
$$\mathcal{Z}_B = \left\{ \frac{a}{b} \in \mathbb{Q} \;\middle|\; b \in \mathcal{B}_B, 1 \leqslant a \leqslant b, \text{ and } \gcd(a,b) = 1 \right\}.$$

The zero set $\mathcal{Z}_B$ consists of the zeros in the interval $(0,1]$ of our auxiliary rational functions (defined later), and the denominator set $\mathcal{B}_B$ consists of different denominators of the zeros. We collect some properties of these two sets. The first property is known in the topic about inverse totient problem.

Lemma 4.3. (1) *The size of the set $\mathcal{B}_B$ is*
$$|\mathcal{B}_B| = \left(\frac{\zeta(2)\zeta(3)}{\zeta(6)} + o_{B\to+\infty}(1) \right) B.$$
(2) *For any $b \in \mathcal{B}_B$, we have*
$$\left\{ \frac{1}{b}, \frac{2}{b}, \cdots, \frac{b}{b} \right\} \subset \mathcal{Z}_B.$$
(3) *If B is larger than some absolute constant, then*
$$|\mathcal{Z}_B| \leqslant B^2.$$

Proof. For the first proposition, we refer the readers to [Dre70] or [Bat72]. Since

$$|\mathcal{Z}_B| = \sum_{b\in\mathcal{B}_B}\varphi(b) = \sum_{m=1}^{\lfloor B\rfloor} m\left(|\mathcal{B}_m| - |\mathcal{B}_{m-1}|\right) = \lfloor B\rfloor \left|\mathcal{B}_{\lfloor B\rfloor}\right| - \sum_{m=1}^{\lfloor B\rfloor -1}|\mathcal{B}_m|,$$

the first proposition implies that $|\mathcal{Z}_B| = \left(\zeta(2)\zeta(3)/2\zeta(6) + o(1)\right)B^2$. Now, $\zeta(2)\zeta(3)/\zeta(6) = 1.94\ldots < 2$, the third proposition follows. For the second proposition, note that if $b\in\mathcal{B}_B$ and b' is any divisor of b, then $\varphi(b')\leqslant\varphi(b)\leqslant B$, so $b'\in\mathcal{B}_B$. Therefore, for any $k\in\{1,2,\cdots,b\}$, we have $k/b = (k/\gcd(k,b))/(b/\gcd(k,b))\in\mathcal{Z}_B$. □

We define the integer

$$P_{B,r} = 2\mathrm{den}(r)\cdot \mathrm{LCM}_{\substack{b\in\mathcal{B}_B\\ p\mid b}}\{p-1\}, \tag{4.1}$$

where LCM means taking the least common multiple. As a convention, the letter p always denotes prime numbers.

For given r, s and B, we define the following auxiliary rational functions.

Definition 4.4 (rational functions). *For any positive integer n which is a multiple of $P_{B,r}$, we define the rational function*

$$R_n(t) = A_1(B)^n A_2(B)^n \frac{n!^{s+1}}{\left(\frac{n}{\mathrm{den}(r)}\right)!^{\,\mathrm{den}(r)(2r+1)|\mathcal{Z}_B|}} \frac{(t-rn)\prod_{\theta\in\mathcal{Z}_B}\prod_{j=0}^{(2r+1)n-1}(t-rn+j+\theta)}{\prod_{j=0}^{n}(t+j)^{s+1}},$$

where

$$A_1(B) = \prod_{b\in\mathcal{B}_B} b^{(2r+1)\varphi(b)},$$

and

$$A_2(B) = \prod_{b\in\mathcal{B}_B}\prod_{p\mid b} p^{\frac{(2r+1)\varphi(b)}{p-1}}.$$

We refer to $A_1(B)^n$ (respectively, $A_2(B)^n$) the *major arithmetic (wasting) factor* (respectively, *minor arithmetic (wasting) factor*).

Notice that by (4.1), both $A_1(B)^n$ and $A_2(B)^n$ are integers, also, $n/\mathrm{den}(r)$, rn, and $(2r+1)n$ are integers. The factors $A_1(B)^n$ and $A_2(B)^n$ are technical: approximately speaking, they are designed to remedy the arithmetic loss from the denominators of rational zeros (it will be clear later in Section 4.2). In the following lemma we estimate $A_1(B)$ and $A_2(B)$:

Lemma 4.5. *We have*

$$A_1(B) = \exp\left(\left(\frac{1}{2}\frac{\zeta(2)\zeta(3)}{\zeta(6)} + o_{B\to+\infty}(1)\right)(2r+1)\,B^2\log B\right),$$

and for any B larger than some absolute constant,

$$A_2(B) \leqslant \exp\left(10(2r+1)B^2(\log\log B)^2\right).$$

Proof. We start by $\log A_1(B) = (2r+1)\sum_{b\in\mathcal{B}_B}\varphi(b)\log b$. Firstly,

$$\begin{aligned}\log A_1(B) &\geqslant (2r+1)\sum_{b\in\mathcal{B}_B}\varphi(b)\log\varphi(b)\\ &= (2r+1)\int_{1^-}^{B} x\log x\ \mathrm{d}|\mathcal{B}_x|,\end{aligned}$$

an integration by parts argument with the fact $|\mathcal{B}_x| = (\zeta(2)\zeta(3)/\zeta(6) + o_{x\to+\infty}(1))\,x$ (see Lemma 4.3 (1)) gives $\log A_1(B) \geqslant (2r+1)\,(\zeta(2)\zeta(3)/2\zeta(6) + o_{B\to+\infty}(1))\,B^2\log B$. On the other hand, it is well known (see, for instance, [MV06, Thm 2.9]) that

$$\varphi(m) \geqslant \left(e^{-\gamma} + o_{m\to+\infty}(1)\right)\frac{m}{\log\log m},$$

where $\gamma = 0.577\ldots$ is Euler's constant. For any $b\in\mathcal{B}_B$, since $\varphi(b)\leqslant B$, we derive that

$$b \leqslant (e^{\gamma} + o_{B\to+\infty}(1))B\log\log B, \tag{4.2}$$

thus $\log A_1(B) \leqslant (2r+1)(1+o_{B\to+\infty}(1))\log B\sum_{b\in\mathcal{B}_B}\varphi(b)$, a summation by parts argument as above gives $\log A_1(B) \leqslant (2r+1)\,(\zeta(2)\zeta(3)/2\zeta(6) + o_{B\to+\infty}(1))\,B^2\log B$. Combining the two parts we obtain the estimate for $A_1(B)$.

Now, for $A_2(B)$, by (4.2) and $e^{\gamma} = 1.78\ldots < 2$, when B is larger than some absolute constant, we have

$$\begin{aligned}\log A_2(B) &\leqslant (2r+1)\sum_{b\leqslant 2B\log\log B}\varphi(b)\sum_{p\mid b}\frac{\log p}{p-1}\\ &= (2r+1)\sum_{p\leqslant 2B\log\log B}\frac{\log p}{p-1}\sum_{\substack{b\leqslant 2B\log\log B\\ p\mid b}}\varphi(b)\end{aligned}$$

Since $\sum_{\substack{b\leqslant 2B\log\log B\\ p\mid b}}\varphi(b) \leqslant \sum_{\substack{b\leqslant 2B\log\log B\\ p\mid b}} b \leqslant \frac{4B^2(\log\log B)^2}{p}$, it implies that

$$\log A_2(B) \leqslant 4(2r+1)B^2(\log\log B)^2\sum_{p}\frac{\log p}{p(p-1)}.$$

Because $\sum_p(\log p)/p(p-1) < 2$, the estimate for $A_2(B)$ follows. $\square$

We proceed to construct linear forms in Hurwitz zeta values. Since the numerator and denominator of $R_n(t)$ have a common factor $\prod_{j=0}^{n}(t+j)$, it can be rewritten as $R_n(t) = Q_n(t)/\prod_{j=0}^{n}(t+j)^s$, where $Q_n(t)$ is a polynomial in t with rational coefficients. As a rational function, the degree of $R_n(t)$ is

$$\deg R_n = 1 + (2r+1)|\mathcal{Z}_B|n - (s+1)(n+1) \leqslant -2$$

(due to $|\mathcal{Z}_B| \leqslant B^2$ and $s \geqslant 10(2r+1)B^2$). So we know that $R_n(t)$ has a unique partial fraction expansion.

Definition 4.6 ($a_{i,k}$)**.** *We denote by*

$$R_n(t) = \sum_{i=1}^{s}\sum_{k=0}^{n}\frac{a_{i,k}}{(t+k)^i} \tag{4.3}$$

the unique partial fraction expansion of the rational function $R_n(t)$.

Clearly the coefficients $a_{i,k} \in \mathbb{Q}$. Note that these coefficients $a_{i,k}$ also depend on n, r, s, and B, but we omit these subscripts.

Definition 4.7 ($S_{n,\theta}$)**.** *For all $\theta \in \mathcal{Z}_B$, we define the following quantities:*

$$S_{n,\theta} = \sum_{m=1}^{\infty} R_n(m+\theta). \tag{4.4}$$

We recall the definition of the Hurwitz zeta values:

$$\zeta(i,\alpha) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^i},$$

where $i \geqslant 2$ is an integer and α is a positive real number.

The following lemma is similar to . It tells us that $S_{n,\theta}$ is a linear form in 1 and odd Hurwitz zeta values.

Lemma 4.8 (linear forms)**.** *For all $\theta \in \mathcal{Z}_B$, we have*

$$S_{n,\theta} = \rho_{0,\theta} + \sum_{\substack{3 \leqslant i \leqslant s \\ i \text{ odd}}} \rho_i \zeta(i,\theta),$$

where the rational coefficient

$$\rho_i = \sum_{k=0}^{n} a_{i,k} \quad \text{for } 3 \leqslant i \leqslant s,\ i \text{ odd},$$

does not depend on $\theta \in \mathcal{Z}_B$, and

$$\rho_{0,\theta} = -\sum_{k=0}^{n} \sum_{\ell=0}^{k} \sum_{i=1}^{s} \frac{a_{i,k}}{(\ell+\theta)^i}.$$

Proof. It is similar to the proof of Lemma 3.2 and Lemma 3.3. ☐

4.2. **Arithmetic lemmas.** The following lemma is elementary, we omit the proof:

Lemma 4.9. *Let $L \in \mathbb{N} \cup \{0\}$. Suppose $x_1, x_2, \cdots, x_L$ be any L consecutive terms in an integer arithmetic progression with common difference $b \in \mathbb{N}$, then for any prime $q \nmid b$, we have*

$$v_q(x_1 x_2 \cdots x_L) \geqslant \sum_{i=1}^{\infty} \left\lfloor \frac{L}{q^i} \right\rfloor.$$

In the degenerate case of $L = 0$, we view $x_1 x_2 \cdots x_L = 1$.

For any $a/b \in \mathcal{Z}_B$ with $\gcd(a,b) = 1$, we define the following polynomials:

$$\begin{aligned} F_{b,a}(t) &= \frac{\prod_{p|b} p^{\frac{(2r+1)n}{p-1}}}{\left(\frac{n}{\mathrm{den}(r)}\right)!^{\,\mathrm{den}(r)(2r+1)}} \cdot b^{(2r+1)n} \prod_{j=0}^{(2r+1)n-1} \left(t - rn + j + \frac{a}{b}\right) \\ &= \frac{\prod_{p|b} p^{\frac{(2r+1)n}{p-1}}}{\left(\frac{n}{\mathrm{den}(r)}\right)!^{\,\mathrm{den}(r)(2r+1)}} \prod_{j=0}^{(2r+1)n-1} (bt - brn + a + bj). \end{aligned} \tag{4.5}$$

Then we define

$$\widetilde{F}_{b,a}(t) = \begin{cases} F_{b,a}(t) & \text{if } \frac{a}{b} \neq 1, \\ (t-rn)F_{1,1}(t) & \text{if } \frac{a}{b} = 1. \end{cases}$$

Notice that since $n \in P_{B,r}\mathbb{N}$, by (4.1), all of $(2r+1)n/(p-1)$, $n/\mathrm{den}(r)$, rn, and $(2r+1)n$ are integers. By Definition 4.4, we have

$$R_n(t) = n!^{s+1} \frac{\prod_{\frac{a}{b}\in \mathcal{Z}_B} \widetilde{F}_{b,a}(t)}{\prod_{j=0}^{n}(t+j)^{s+1}}. \tag{4.6}$$

(In fact, this is the reason why the factors $A_1(B)^n$, $A_2(B)^n$ appear in Definition 4.4.)

For a formal series $U(t) = \sum_{\ell=0}^{\infty} u_\ell t^\ell \in \mathbb{Q}[[t]]$, we denote by $[t^\ell](U(t))$ the ℓ-th coefficient of $U(t)$, i.e., $[t^\ell](U(t)) = u_\ell$.

As usual, we denote by $d_n = \mathrm{LCM}\{1, 2, \cdots, n\}$ the least common multiple of the first n positive integers. By the prime number theorem, we have $\lim_{n\to+\infty} d_n^{1/n} = e$. We first establish the following arithmetic property of $\widetilde{F}_{b,a}(t)$:

Lemma 4.10. *For any nonnegative integers ℓ and k, we have*

$$d_n^\ell \cdot [t^\ell](\widetilde{F}_{b,a}(t-k)) \in \mathbb{Z}$$

Proof. Note that we only need to prove the proposition with $\widetilde{F}_{b,a}$ replaced by $F_{b,a}$. If $\ell > \deg F_{b,a} = (2r+1)n$, the proposition trivially holds. In the rest of the proof, we assume $\ell \leqslant \deg F_{b,a}$.

For a prime $q \mid b$, the q-adic order of the factor $\prod_{p|b} p^{\frac{(2r+1)n}{p-1}} / \left(\frac{n}{\mathrm{den}(r)}\right)!^{\,\mathrm{den}(r)(2r+1)}$ is nonnegative (recall that $v_q(m!) \leqslant \frac{m}{q-1}$). So by (4.5), the q-adic order of every coefficient of $F_{b,a}(t-k)$ is nonnegative. Therefore, for any prime $q \mid b$, we have $v_q(d_n^\ell \cdot [t^\ell](F_{b,a}(t-k))) \geqslant 0$.

Now consider a prime $q \nmid b$. Notice that $[t^\ell]\left(\prod_{j=0}^{(2r+1)n-1}(b(t-k) - brn + a + bj)\right)$ is a sum of finitely many terms all of the form

$$b^\ell \prod_{i=1}^{\ell+1} \prod_{j\in J_i} (-bk - brn + a + bj), \tag{4.7}$$

where J_i is a set consisting of $L_i \in \mathbb{N}\cup\{0\}$ consecutive integers such that $L_1+L_2+\cdots+L_{\ell+1} = (2r+1)n - \ell$. By Lemma 4.9, we derive that the q-adic order of the expression (4.7) is

$$v_q((4.7)) \geqslant \sum_{i=1}^{\infty} \sum_{j=1}^{\ell+1} \left\lfloor \frac{L_j}{q^i} \right\rfloor.$$

For a fixed $i \geqslant 1$, we have $\sum_{j=1}^{\ell+1} \left\lfloor \frac{L_j}{q^i} \right\rfloor \geqslant \sum_{j=1}^{\ell+1} \frac{L_j-(q^i-1)}{q^i} = \frac{(2r+1)n+1}{q^i} - \ell - 1 > \left\lfloor \frac{(2r+1)n}{q^i} \right\rfloor - \ell - 1$, but the left hand side is a nonnegative integer, so we obtain that $\sum_{j=1}^{\ell+1} \left\lfloor \frac{L_j}{q^i} \right\rfloor \geqslant$

$\max(0, \left\lfloor \frac{(2r+1)n}{q^i} \right\rfloor - \ell)$. Therefore,

$$\begin{aligned} v_q((4.7)) &\geqslant \sum_{i=1}^{\lfloor \log_q n \rfloor} \left(\left\lfloor \frac{(2r+1)n}{q^i} \right\rfloor - \ell \right) \\ &\geqslant \sum_{i=1}^{\lfloor \log_q n \rfloor} \left(\mathrm{den}(r)(2r+1) \left\lfloor \frac{n/\mathrm{den}(r)}{q^i} \right\rfloor - \ell \right) \\ &= v_q \left(\left(\frac{n}{\mathrm{den}(r)} \right)!^{\,\mathrm{den}(r)(2r+1)} \right) - \ell v_q(d_n). \end{aligned} \tag{4.8}$$

(The non-trivial part is for cases $q \leqslant n$, for $q > n$, the above derivation is also valid but degenerates to trivial results.) In conclusion, for any prime $q \nmid b$, by equation (4.5) and inequality (4.8), we find that $d_n^{\ell} \cdot [t^{\ell}](F_{b,a}(t-k))$ is a sum of finitely many terms, each of these terms has nonnegative q-adic order, this completes the proof of Lemma 4.10. ☐

Now, we prove the following lemma about the arithmetic of ρ_i and $\rho_{0,\theta}$. The conclusion for $\rho_{0,\theta}$ is somewhat tricky, and is referred to Sprang's lemma, see the origin in [Spr18, Lemma 1.4]. It will use the zeros of $R_n(t)$ in the 'middle part'.

Lemma 4.11 (arithmetic lemma)**.** *We have*

$$d_n^{s+1-i}\rho_i \in \mathbb{Z}$$

for all odd integers i with $3 \leqslant i \leqslant s$, and we have

$$d_{n+1}^{s+1}\rho_{0,\theta} \in \mathbb{Z}$$

for all $\theta \in \mathcal{Z}_B$.

Proof. For any $k \in \{0, 1, \cdots, n\}$ and any $i \in \{1, 2, \cdots, s\}$, by comparing (4.6) with the partial fraction expansion (4.3) of $R_n(t)$, and by viewing $t^{s+1}R_n(t-k) \in \mathbb{Q}[[t]]$ as a formal series, we have

$$\begin{aligned} a_{i,k} &= [t^{s+1-i}]\left(t^{s+1}R_n(t-k)\right) \\ &= (-1)^{(s+1)k}\frac{n!^{s+1}}{k!^{s+1}(n-k)!^{s+1}}[t^{s+1-i}]\left(\prod_{\frac{a}{b}\in\mathcal{Z}_B}\widetilde{F}_{b,a}(t-k)\prod_{\substack{0\leqslant j\leqslant n\\ j\neq k}}\left(1+\frac{t}{j-k}\right)^{-s-1}\right) \\ &= \binom{n}{k}^{s+1}\sum_{\substack{\underline{\ell}\\ \mathrm{sum}(\underline{\ell})=s+1-i}}\prod_{\frac{a}{b}\in\mathcal{Z}_B}[t^{\ell_{b,a}}]\left(\widetilde{F}_{b,a}(t-k)\right)\prod_{\substack{0\leqslant j\leqslant n\\ j\neq k}}\frac{(-1)^{\ell_j}\binom{s+\ell_j}{\ell_j}}{(j-k)^{\ell_j}}, \end{aligned}$$

where the sum is taken for all tuples $\underline{\ell}$ consisting of nonnegative integers $\ell_{b,a}$ and ℓ_j such that

$$\mathrm{sum}(\underline{\ell}) = \sum_{\frac{a}{b}\in\mathcal{Z}_B}\ell_{b,a} + \sum_{\substack{0\leqslant j\leqslant n\\ j\neq k}}\ell_j = s+1-i.$$

By Lemma 4.10 and the fact $d_n^{\ell_j}\frac{1}{(j-k)^{\ell_j}} \in \mathbb{Z}$, we derive that

$$d_n^{s+1-i}a_{i,k} \in \mathbb{Z}.$$

Once $d_n^{s+1-i}a_{i,k} \in \mathbb{Z}$ is established, $d_n^{s+1-i}\rho_i \in \mathbb{Z}$ follows immediately from the expression of ρ_i. The proof for $d_{n+1}^{s+1}\rho_{0,\theta} \in \mathbb{Z}$ is more tricky: we prove by contradiction that

$$\sum_{i=1}^{s} \frac{d_{n+1}^{s+1} a_{i,k}}{(\ell+\theta)^i} \in \mathbb{Z} \tag{4.9}$$

holds for any $0 \leqslant \ell \leqslant k \leqslant n$ and $\theta \in \mathcal{Z}_B$. If $\theta = 1$, then (4.9) follows from $d_{n+1}^{s+1-i}a_{i,k} \in \mathbb{Z}$. Now we assume $\theta \in \mathcal{Z} \setminus \{1\}$. If (4.9) is not true, then there exist $k_0 \in [0,n]$ and $\ell_0 \in [0,k_0]$ such that

$$\sum_{i=1}^{s} \frac{d_{n+1}^{s+1} a_{i,k_0}}{(\ell_0+\theta)^i} \notin \mathbb{Z}.$$

Note that $-k_0+\ell_0+\theta$ is a zero of $R_n(t)$, by (4.3), we have

$$\sum_{i=1}^{s} \frac{d_{n+1}^{s+1} a_{i,k_0}}{(\ell_0+\theta)^i} = -\sum_{\substack{k=0\\k\neq k_0}}^{n}\sum_{i=1}^{s} \frac{d_{n+1}^{s+1} a_{i,k}}{(\ell_0+k-k_0+\theta)^i} \notin \mathbb{Z}.$$

Therefore, there exist a prime number p, an index $k_1 \in [0,n]$ with $k_1 \neq k_0$, and two indices $i_0, i_1 \in [1,s]$ such that

$$v_p\left(\frac{d_{n+1}^{s+1} a_{i_0,k_0}}{(\ell_0+\theta)^{i_0}}\right) < 0, \quad v_p\left(\frac{d_{n+1}^{s+1} a_{i_1,k_1}}{(\ell_0+k_1-k_0+\theta)^{i_1}}\right) < 0.$$

Since $d_{n+1}^{s+1-i}a_{i,k} \in \mathbb{Z}$ for all i,k, we deduce that

$$v_p\left(\frac{d_{n+1}^{i_0}}{(\ell_0+\theta)^{i_0}}\right) < 0, \quad v_p\left(\frac{d_{n+1}^{i_1}}{(\ell_0+k_1-k_0+\theta)^{i_1}}\right) < 0.$$

So $v_p(\ell_0+\theta) > v_p(d_{n+1})$ and $v_p(\ell_0+k_1-k_0+\theta) > v_p(d_{n+1})$. It follows that $v_p(k_1-k_0) > v_p(d_{n+1})$, which is a contradiction because $0 < |k_1-k_0| \leqslant n$. Hence (4.9) is true, and $d_{n+1}^{s+1}\rho_{0,\theta} \in \mathbb{Z}$ follows. □

4.3. **Analysis lemmas.** Under our assumptions $s \geqslant 10(2r+1)B^2$ and B is larger than some absolute constant, we have the following:

Lemma 4.12 (analysis lemma)**.** *AS $n \to +\infty$ along $P_{B,r}\mathbb{N}$, we have*

$$\lim_{n\to+\infty} (S_{n,1})^{\frac{1}{n}} = g(x_0),$$

where

$$g(X) := A_1(B)A_2(B)\mathrm{den}(r)^{(2r+1)|\mathcal{Z}_B|}(X+2r+1)^{(2r+1)|\mathcal{Z}_B|}\left(\frac{(X+r)^r}{(X+r+1)^{r+1}}\right)^{s+1},$$

and x_0 is the unique positive real solution of the equation $f(x)=1$ with

$$f(X) := \left(\frac{X+2r+1}{X}\right)^{|\mathcal{Z}_B|}\left(\frac{X+r}{X+r+1}\right)^{s+1}.$$

Moreover, for any $\theta \in \mathcal{Z}_B$, we have

$$\lim_{n\to+\infty} \frac{S_{n,1}}{S_{n,\theta}} = 1.$$

Despite the complicate expression for $\lim_{n\to+\infty}(S_{n,1})^{1/n}$, there is a simple heuristic reason to see why Lemma 4.12 is true, we explain it below. Suppose for the moment that $k=\kappa n$ for a fixed constant κ, then by Stirling's formula, we have

$$\lim_{n\to+\infty} R_n(rn+\kappa n)^{1/n} = f(\kappa)^{\kappa} g(\kappa).$$

We will see that the function $f(\kappa)^{\kappa}g(\kappa)$ attains its maximal value on $(0,+\infty)$ only at $x=x_0$. So we may convince ourselves that the sum $S_{n,\theta}=\sum_{k=0}^{\infty} R_n(rn+k+\theta)$ is dominated by the terms in the index range $(x_0-\varepsilon)n \leqslant k \leqslant (x_0+\varepsilon)n$, and then Lemma 4.12 will follow easily. In the sequel, we will make the above heuristic reason rigorously.

We first collect some properties of the functions f and g. Note that these two functions does not depend on n (they depend only on r, s and B).

Lemma 4.13. *Let $f(x)$ and $g(x)$ be the functions in Lemma 4.12 (defined on $x\in(0,+\infty)$). Then*

(1) *There exists a unique $x_0\in(0,+\infty)$ such that $f(x_0)=1$, $f(x)>1$ on $(0,x_0)$ and $f(x)<1$ on $(x_0,+\infty)$. Moreover,*

$$x_0 < \frac{r(r+1)|\mathcal{Z}_B|}{s+1-(2r+1)|\mathcal{Z}_B|}.$$

(2) *If we fix $r\in\mathbb{Q}_+$ and assume in addition that $B=c\sqrt{s/\log s}$ for some positive constant c, when $s\to+\infty$, we have*

$$\lim_{\substack{s\to+\infty\\ B=c\sqrt{s/\log s}}} g(x_0)^{\frac{1}{s+1}} = \exp\left(\frac{\zeta(2)\zeta(3)}{4\zeta(6)}(2r+1)c^2\right)\frac{r^r}{(r+1)^{r+1}}.$$

Proof. For the first proposition, by calculating $f'(x)/f(x)$, we find that $f'(x)=0$ has a unique positive solution x_1 which satisfies

$$(s+1-(2r+1)|\mathcal{Z}_B|)x_1^2+(2r+1)(s+1-(2r+1)|\mathcal{Z}_B|)x_1-r(r+1)(2r+1)|\mathcal{Z}_B|=0, \tag{4.10}$$

and f is decreasing on $(0,x_1)$, increasing on $(x_1,+\infty)$. Since $f(0^+)=+\infty$ and $f(+\infty)=1$, there exists a unique x_0 satisfying all the requirements. The last (very weak) bound for x_0 comes from $x_0<x_1$ and (4.10).

The second proposition follows from the estimates for $A_1(B)$, $A_2(B)$, $|\mathcal{Z}_B|$ (see Lemma 4.3 and Lemma 4.5), and $x_0\to 0$. $\square$

Now we prove Lemma 4.12.

Proof of Lemma 4.12. For any $\theta\in\mathcal{Z}_B$, since $R_n(m+\theta)=0$ for $m=1,2,\cdots,rn-1$, we define the shift version of the auxiliary rational functions:

$$\widehat{R}_n(t)=R_n(t+rn),$$

then by (4.4) we have

$$S_{n,\theta}=\sum_{k=0}^{\infty}\widehat{R}_n(k+\theta). \tag{4.11}$$

We have the following two expressions for $\widehat{R}_n(t)$:

$$\widehat{R}_n(t) = A_1(B)^n A_2(B)^n \frac{n!^{s+1}}{\left(\frac{n}{\mathrm{den}(r)}\right)!^{\,\mathrm{den}(r)(2r+1)|\mathcal{Z}_B|}} \cdot \frac{t\ \prod_{\theta'\in\mathcal{Z}_B}\prod_{j=0}^{(2r+1)n-1}(t+j+\theta')}{\prod_{j=0}^{n}(t+rn+j)^{s+1}} \tag{4.12}$$

$$\begin{aligned} &= A_1(B)^n A_2(B)^n \frac{n!^{s+1}}{\left(\frac{n}{\mathrm{den}(r)}\right)!^{\,\mathrm{den}(r)(2r+1)|\mathcal{Z}_B|}} \\ &\times t\cdot\left(\prod_{\theta'\in\mathcal{Z}_B}\frac{\Gamma(t+(2r+1)n+\theta')}{\Gamma(t+\theta')}\right)\cdot\left(\frac{\Gamma(t+rn)}{\Gamma(t+(r+1)n+1)}\right)^{s+1}. \end{aligned} \tag{4.13}$$

We define $c_1 = \min(e^{-10s/r}/2, x_0/2)$, which is independent of n. To estimate the series (4.11) for $S_{n,\theta}$, we divide it into three parts:

$$S_{n,\theta} = \left(\sum_{0\leqslant k<c_1 n} + \sum_{c_1 n\leqslant k\leqslant n^{10}} + \sum_{k>n^{10}}\right)\left(\widehat{R}_n(k+\theta)\right).$$

For the first part, by (4.12), for all $t\in(0,2c_1 n]$ we have

$$\begin{aligned} \frac{\widehat{R}_n'(t)}{\widehat{R}_n(t)} &> \sum_{j=0}^{(2r+1)n}\frac{1}{t+j} - (s+1)\sum_{j=0}^{n}\frac{1}{t+rn+j} \\ &> \log\left(\frac{t+(2r+1)n}{t}\right) - (s+1)\frac{n+1}{rn} \\ &> \log\left(\frac{1}{2c_1}\right) - \frac{4s}{r} \\ &> 0. \end{aligned}$$

So $\widehat{R}_n(t)$ is increasing on $t\in(0,2c_1 n]$, we derive that (when $n>c_1^{-1}$)

$$\sum_{0\leqslant k<c_1 n}\widehat{R}_n(k+\theta) < (c_1 n+1)\,\widehat{R}_n(\lfloor c_1 n\rfloor+\theta). \tag{4.14}$$

To deal with the middle part, for all $c_1 n\leqslant k\leqslant n^{10}$, we denote by $\kappa=\kappa(k,n)=k/n\in[c_1,+\infty)$. By applying Stirling's formula in the weak form

$$\Gamma(x) = x^{O_{x\to+\infty}(1)}\left(\frac{x}{e}\right)^x$$

for the equation (4.13), a calculation shows that as $n \to +\infty$:

$$\begin{aligned}
\widehat{R}_n(k+\theta) &= n^{O(1)} \cdot A_1(B)^n A_2(B)^n \frac{(n/e)^{(s+1)n}}{\left(\frac{n}{\operatorname{den}(r)}/e\right)^{(2r+1)|\mathcal{Z}_B|n}} \\
&\quad \times \left(\frac{((k+(2r+1)n)/e)^{k+(2r+1)n}}{(k/e)^k}\right)^{|\mathcal{Z}_B|} \frac{((k+rn)/e)^{(s+1)(k+rn)}}{((k+(r+1)n)/e)^{(s+1)(k+(r+1)n)}} \\
&= n^{O(1)} \cdot A_1(B)^n A_2(B)^n \operatorname{den}(r)^{(2r+1)|\mathcal{Z}_B|n} \\
&\quad \times \left(\frac{(\kappa+2r+1)^{\kappa+2r+1}}{\kappa^\kappa}\right)^{|\mathcal{Z}_B|n} \left(\frac{(\kappa+r)^{\kappa+r}}{(\kappa+r+1)^{\kappa+r+1}}\right)^{(s+1)n} \\
&= n^{O(1)} \cdot \left(f(\kappa)^\kappa g(\kappa)\right)^n \\
&= n^{O(1)} \cdot h(\kappa)^n \qquad (4.15)
\end{aligned}$$

uniformly for any $k \in [c_1 n, n^{10}]$ and any $\theta \in \mathcal{Z}_B$ (the absolute bound for $O(1)$ depends only on s, B, r and $\operatorname{den}(r)$); here the function $h(x)$ is defined for $x > 0$ as $h(x) = f(x)^x g(x)$, a direct computation shows that $h'(x)/h(x) = \log f(x)$. Hence $h(x)$ achieves its maximum only at $x = x_0$ with maximal value $h(x_0) = g(x_0)$.

In particular, we have the following bound for each $k \in [c_1 n, n^{10}]$:

$$\widehat{R}_n(k+\theta) \leqslant n^{O(1)} \cdot g(x_0)^n. \qquad (4.16)$$

Finally, we treat the tail part. For any $k > n^{10}$, when n is sufficiently large in terms of r, s and B (more precisely, when $n \geqslant \max(10(2r+1), 10A_1(B)A_2(B), 10/g(x_0)))$, by (4.12) and our assumption $s \geqslant 10(2r+1)B^2$, we have

$$\begin{aligned}
\widehat{R}_n(k+\theta) &< A_1(B)^n A_2(B)^n n^{(s+1)n} \cdot \frac{(2k)\prod_{\theta' \in \mathcal{Z}_B} \prod_{j=0}^{(2r+1)n-1}(2k)}{\prod_{j=0}^{n}(k)^{s+1}} \\
&< \frac{(2A_1(B)A_2(B)n)^{(s+1)n}}{k^{\frac{9}{10}(s+1)n+2}} \\
&< \left(\frac{2A_1(B)A_2(B)}{n^8}\right)^{(s+1)n} \frac{1}{k^2} \\
&< \left(\frac{g(x_0)}{2}\right)^n \frac{1}{k^2}.
\end{aligned}$$

As a conclusion, we obtain the following bound for the tail part for all sufficiently large n:

$$\sum_{k > n^{10}} \widehat{R}_n(k+\theta) \leqslant \left(\frac{g(x_0)}{2}\right)^n. \qquad (4.17)$$

Now, in view of the estimates (4.14), (4.16) and (4.17), we have $S_{n,1} \leqslant n^{O(1)} g(x_0)^n$. On the other hand (4.15) implies that $S_{n,1} \geqslant \widehat{R}_n(\lfloor x_0 n \rfloor) = n^{O(1)} h\left(x_0 + o(1)\right)^n$. Therefore,

$$\lim_{n \to +\infty} (S_{n,1})^{\frac{1}{n}} = g(x_0).$$

To prove the last statement in the lemma, we first fix an arbitrary (sufficiently) small $\varepsilon_0 > 0$. For all $\theta \in \mathcal{Z}_B$, we have

$$S_{n,\theta} \geqslant \sum_{(x_0-\varepsilon_0)n \leqslant k \leqslant (x_0+\varepsilon_0)n} \widehat{R}_n(k+\theta). \tag{4.18}$$

In view of the estimates (4.14), (4.15) and (4.17), we also have

$$\begin{aligned} S_{n,\theta} &\leqslant n^{O(1)} \max\left(h(x_0-\varepsilon_0), h(x_0+\varepsilon_0)\right)^n \quad + \sum_{(x_0-\varepsilon_0)n \leqslant k \leqslant (x_0+\varepsilon_0)n} \widehat{R}_n(k+\theta) \\ &< (1+\varepsilon_0) \sum_{(x_0-\varepsilon_0)n \leqslant k \leqslant (x_0+\varepsilon_0)n} \widehat{R}_n(k+\theta), \end{aligned} \tag{4.19}$$

provided n is sufficiently large with respect to ε_0, i.e., $n \geqslant n_0(\varepsilon_0)$. For all k with $(x_0-\varepsilon_0)n \leqslant k \leqslant (x_0+\varepsilon_0)n$, let $\kappa = \kappa(n,k) = k/n$ as before. We now use the fact that, for any fixed real number τ,

$$\frac{\Gamma(x+\tau)}{\Gamma(x)} = \left(1 + o_{x\to+\infty}(1)\right) x^{\tau}. \tag{4.20}$$

Applying (4.20) to (4.13), we derive that

$$\begin{aligned} \frac{\widehat{R}_n(k+1)}{\widehat{R}_n(k+\theta)} &= (1+o(1)) \cdot \left(\frac{\kappa+2r+1}{\kappa}\right)^{|\mathcal{Z}_B|(1-\theta)} \left(\frac{\kappa+r}{\kappa+r+1}\right)^{(s+1)(1-\theta)} \\ &= (1+o(1)) \cdot f(\kappa)^{1-\theta} \end{aligned} \tag{4.21}$$

uniformly for $k \in [(x_0-\varepsilon_0)n, (x_0+\varepsilon_0)n]$ as $n \to +\infty$. By (4.18), (4.19) and (4.21) we find that

$$(1+o(1)) \frac{1}{1+\varepsilon_0} f(x_0+\varepsilon_0)^{1-\theta} \leqslant \frac{S_{n,1}}{S_{n,\theta}} \leqslant (1+o(1))\,(1+\varepsilon_0) f(x_0-\varepsilon_0)^{1-\theta},$$

thus

$$\frac{1}{1+\varepsilon_0} f(x_0+\varepsilon_0)^{1-\theta} \leqslant \liminf_{n\to+\infty} \frac{S_{n,1}}{S_{n,\theta}} \leqslant \limsup_{n\to+\infty} \frac{S_{n,1}}{S_{n,\theta}} \leqslant (1+\varepsilon_0) f(x_0-\varepsilon_0)^{1-\theta}.$$

It is true for all sufficiently small $\varepsilon_0 > 0$. Letting $\varepsilon_0 \to 0^+$, we deduce that

$$\lim_{n\to+\infty} \frac{S_{n,1}}{S_{n,\theta}} = 1.$$

This completes the proof of Lemma 4.12. □

4.4. **Elimination procedure and proof of the theorem.** We prove Theorem 4.1 in this subsection by an elimination procedure.

We denote by $I_s = \{3, 5, 7, \cdots, s\}$. For any subset $J \subset I_s$ with $|J| = |\mathcal{B}_B| - 1$, since the following general Vandermonde matrix (see, for instance, [GK02, pp. 76-77])

$$\left[b^j\right]_{b\in\mathcal{B}_B,\ j\in\{1\}\cup J}$$

is invertible, there exist integers $w_b \in \mathbb{Z}$ for all $b \in \mathcal{B}_B$ such that $\sum_{b\in\mathcal{B}_B} w_b b^j = 0$ for any $j \in J$ and $\sum_{b\in\mathcal{B}_B} w_b b \neq 0$. (Note that these w_b depend only on J and $\mathcal{B}_B$.) Since

$$\sum_{k=1}^{b} \zeta\left(i, \frac{k}{b}\right) = \sum_{k=1}^{b} \sum_{m=0}^{\infty} \frac{b^i}{(mb+k)^i} = b^i \zeta(i), \tag{4.22}$$

we derive that (recall Lemma 4.3 (2), $k/b \in \mathcal{Z}_B$)

$$\widehat{S}_{n,b} := \sum_{k=1}^{b} S_{n,\frac{k}{b}} = \sum_{k=1}^{b} \rho_{0,\frac{k}{b}} + \sum_{i \in I_s} \rho_i b^i \zeta(i)$$

is a linear combination of 1 and odd zeta values. By Lemma 4.12, we have $\widehat{S}_{n,b} = (b+o(1))S_{n,1}$ as $n \to +\infty$ (along $n \in P_{B,r}\mathbb{N}$). Let

$$\widetilde{S}_n := \sum_{b \in \mathcal{B}_B} w_b \widehat{S}_{n,b},$$

then

$$\widetilde{S}_n = \sum_{b \in \mathcal{B}_B} w_b \sum_{k=1}^{b} \rho_{0,\frac{k}{b}} + \sum_{i \in I_s \setminus J} \left(\sum_{b \in \mathcal{B}_B} w_b b^i \right) \rho_i \zeta(i), \tag{4.23}$$

and as $n \to +\infty$,

$$\widetilde{S}_n = \left(\sum_{b \in \mathcal{B}_B} w_b b + o(1) \right) S_{n,1} \text{ with } \sum_{b \in \mathcal{B}_B} w_b b \neq 0. \tag{4.24}$$

Equation (4.23) shows that we can eliminate any $|\mathcal{B}_B| - 1$ odd zeta values.

Lemma 4.14. *If $g(x_0) < e^{-(s+1)}$, then the number of irrationals in the odd zeta values set $\{\zeta(i)\}_{i \in I_s}$ is at least $|\mathcal{B}_B|$.*

Proof. We argue by contradiction. Suppose the number of irrationals in $\{\zeta(i)\}_{i \in I_s}$ is less than $|\mathcal{B}_B|$, then we can take a subset $J \subset I_s$ with $|J| = |\mathcal{B}_B| - 1$ such that $\zeta(i) \in \mathbb{Q}$ for all $I_s \setminus J$; let A be the common denominator of these rational zeta values. Define $\widetilde{S}_n$ as above for this J, then by (4.23) and Lemma 4.11, for all $n \in P_{B,r}\mathbb{N}$, we derive that

$$A d_{n+1}^{s+1} \widetilde{S}_n \in \mathbb{Z}.$$

But by (4.24), Lemma 4.12 and the hypothesis $g(x_0) < e^{-(s+1)}$, we have

$$0 < \lim_{n \to +\infty} \left| A d_{n+1}^{s+1} \widetilde{S}_n \right|^{\frac{1}{n}} = e^{s+1} g(x_0) < 1,$$

this is a contradiction. ☐

So we seek for parameters r, s and B to meet the requirement $g(x_0) < e^{-(s+1)}$, and at the same time to make $|\mathcal{B}_B| \sim (\zeta(2)\zeta(3)/\zeta(6))B$ as large as possible. By Lemma 4.13 (2), for a fixed r (such that $r^r/(r+1)^{r+1} < e^{-1}$), if we take $B = c\sqrt{s/\log s}$ for some constant c, then $\lim_{s \to +\infty} g(x_0)^{\frac{1}{s+1}} < e^{-1}$ if and only if

$$c < \sqrt{\frac{4\zeta(6)}{\zeta(2)\zeta(3)} \frac{(r+1)\log(r+1) - r\log(r) - 1}{2r+1}}.$$

The maximum point of the function $r \mapsto \frac{(r+1)\log(r+1) - r\log(r) - 1}{2r+1}$ is

$$r_0 = \frac{\sqrt{4e^2+1} - 1}{2} \approx 2.26388,$$

with maximal value $1 - \log r_0$. The constant c_0 in Theorem 4.1 is designed by

$$c_0 = \sqrt{\frac{4\zeta(2)\zeta(3)}{\zeta(6)}(1 - \log r_0)}.$$

This leads to the following proof:

Proof of Theorem 4.1. Given any small $\varepsilon > 0$, we first fix a rational number $r = r(\varepsilon)$ sufficiently close to r_0 such that

$$\frac{c_0 - \varepsilon/10}{\zeta(2)\zeta(3)/\zeta(6)} < \sqrt{\frac{4\zeta(6)}{\zeta(2)\zeta(3)} \frac{(r+1)\log(r+1) - r\log(r) - 1}{2r+1}}.$$

Take $B = c\sqrt{s/\log s}$ with constant $c = (c_0 - \varepsilon/10)/(\zeta(2)\zeta(3)/\zeta(6))$, by Lemma 4.13 (2) and Lemma 4.3 (1), there exists $s_0(r, \varepsilon)$ such that for any odd integer $s \geqslant s_0(r, \varepsilon)$, we have $g(x_0) < e^{-(s+1)}$ and $|\mathcal{B}_B| > (\zeta(2)\zeta(3)/\zeta(6) - \varepsilon/10)B$. Therefore, by Lemma 4.14, the number of irrationals among $\zeta(3), \zeta(5), \ldots, \zeta(s)$ is at least

$$|\mathcal{B}_B| > (c_0 - \varepsilon)\sqrt{\frac{s}{\log s}}.$$

$\square$

5. Zudilin's theorem

Theorem 5.1 ([Zud01])**.** *At least one of the four numbers*

$$\zeta(5), \zeta(7), \zeta(9), \zeta(11)$$

is irrational.

6. Sprang's theorem: p-adic analogue of Ball-Rivoal's theorem

In 2020, Sprang obtained the p-adic analogue of Ball-Rivoal's theorem (Theorem 3.1).

Theorem 6.1 ([Spr20])**.** *Let p be a prime number. For any $\varepsilon > 0$, there exists $s_0(\varepsilon, p)$ such that for all odd positive integer $s \geqslant s_0(\varepsilon, p)$, we have*

$$\dim_{\mathbb{Q}} \operatorname{Span}_{\mathbb{Q}} (1, \zeta_p(3), \zeta_p(5), \ldots, \zeta_p(s)) \geqslant \frac{1-\varepsilon}{2(1+\log 2)} \log s,$$

where $\zeta_p(i) := L_p(i, \omega^{1-i}) \in \mathbb{Q}_p$ $(i = 3, 5, \ldots, s)$ are Kubota-Leopoldt p-adic zeta values.

In fact, Sprang showed a more general result about Kubota-Leopoldt p-adic L-values. We only repeat his proof for the above special case, which is most interesting and already contains the main ideas for all general cases.

6.1. A quick definition of Kubota-Leopoldt L-functions.

Kubota-Leopoldt p-adic L-functions

- interpolate Dirichlet L-values at negative integers.
- give good explanation about Kummer's congruences.

We will take the definitions in Cohen GTM 240 [Coh07] and Washington GTM 83 [Wah97].

Definition 6.2 (Volkenborn integration)**.** *Let $f : \mathbb{Z}_p \to \mathbb{Q}_p$ be a function. We say that f is Volkenborn integrable if the following limit exists:*

$$\lim_{n\to+\infty} \frac{1}{p^n} \sum_{k=0}^{p^n-1} f(k).$$

In the case that the above limit exists, we denote by

$$\int_{\mathbb{Z}_p} f(t)\mathrm{d}t := \lim_{n\to+\infty} \frac{1}{p^n} \sum_{k=0}^{p^n-1} f(k)$$

the Volkenborn integration of f.

We use the notation $C(\mathbb{Z}_p, \mathbb{Q}_p)$ to denote the set of continuous functions from $\mathbb{Z}_p$ to $\mathbb{Q}_p$. There exists $f \in C(\mathbb{Z}_p, \mathbb{Q}_p)$ which is not Volkenborn integrable, and also there exists $f \notin C(\mathbb{Z}_p, \mathbb{Q}_p)$ which is Volkenborn integrable. However, if functions are smooth enough in the sense below, they are always Volkenborn integrable.

Definition 6.3 (strictly differentiable functions)**.** *Let $f : \mathbb{Z}_p \to \mathbb{Q}_p$ be a function. We say that f is strictly differentiable at a point $a \in \mathbb{Z}_p$ if the difference quotients*

$$\Phi f(x, y) := \frac{f(x) - f(y)}{x - y}$$

have a limit $\ell =: f'(a)$ as $(x, y) \to (a, a)$ in $\mathbb{Z}_p \times \mathbb{Z}_p$ along $x \neq y$. We say that f is strictly differentiable on $\mathbb{Z}_p$ – notation: $f \in S^1(\mathbb{Z}_p, \mathbb{Q}_p)$ – if f is strictly differentiable at every point in $\mathbb{Z}_p$.

Lemma 6.4. *For any $f \in S^1(\mathbb{Z}_p, \mathbb{Q}_p)$, it is Volkenborn integrable. Moreover, it holds that*

$$\int_{\mathbb{Z}_p} f(t+1)\mathrm{d}t = \int_{\mathbb{Z}_p} f(t)\mathrm{d}t + f'(0).$$

Proof. See $\square$

We use Volkenborn integration as a tool to define p-adic Hurwitz zeta functions and then p-adic L-functions.

Definition 6.5 (Teichmüller character). *Let*

$$q_p = \begin{cases} p, & \text{if } p \neq 2, \\ 4, & \text{if } p = 2. \end{cases}$$

Then $\mathbb{Z}_p^\times$ decomposes canonically as

$$\mathbb{Z}_p^\times \cong \mu_{\varphi(q_p)}(\mathbb{Z}_p) \times (1 + q_p\mathbb{Z}_p)\,,$$

where $\mu_{\varphi(q_p)}(\mathbb{Z}_p)$ is the group of $\varphi(q_p)$th roots of unity in $\mathbb{Z}_p$. The canonical projection

$$\omega : \mathbb{Z}_p^\times \to \mu_{\varphi(q_p)}(\mathbb{Z}_p)$$

is called the Teichmüller character. We extend ω to be a map $\mathbb{Q}_p^\times \to \mathbb{Q}_p^\times$ by setting

$$\omega(x) := p^{v_p(x)}\omega\left(\frac{x}{p^{v_p(x)}}\right)$$

for every $x \in \mathbb{Q}_p^\times$. Also, we define

$$\langle x \rangle := \frac{x}{\omega(x)}, \quad x \in \mathbb{Q}_p^\times.$$

Definition 6.6 (p-adic Hurwitz zeta function). *Let $x \in \mathbb{Q}_p$ with $|x|_p \geqslant q_p$. We define the p-adic Hurwitz zeta function $\zeta_p(\cdot, x)$ on*

$$\left\{ s \in \mathbb{C}_p \;:\; |s|_p < \frac{q_p}{p^{1/(p-1)}} \right\} \setminus \{1\} \tag{6.1}$$

to be

$$\zeta_p(s, x) := \frac{1}{s-1}\int_{\mathbb{Z}_p} \langle t + x \rangle^{1-s} \mathrm{d}t.$$

Equivalently,

$$\omega(x)^{1-s}\zeta_p(s, x) = \frac{1}{s-1}\int_{\mathbb{Z}_p} (t + x)^{1-s} \mathrm{d}t.$$

Some facts:

- $\zeta_p(\cdot, x)$ is analytic in the range (6.1).
- It is indeed the p-adic analogue of Hurwitz zeta function in the sense that

$$\zeta(1-n, x) = -\frac{B_n(x)}{n}, \quad n \in \mathbb{Z}_{\geqslant 1}, x \in (0, 1],$$
$$\omega(x)^n\zeta_p(1-n, x) = -\frac{B_n(x)}{n}, \quad n \in \mathbb{Z}_{\geqslant 1}, x \in \mathbb{Q}_p, |x|_p \geqslant q_p.$$

(Here $B_n(x)$ is the Bernoulli polynomial.)

Definition 6.7 (p-adic L-functions)**.** *Let χ be a primitive Dirichlet character of conductor f. Let D be any common multiple of f and q_p. We define*

$$L_p(s,\chi) := \frac{\langle D\rangle^{1-s}}{D}\sum_{\substack{j=1\\ p\nmid j}}^{D}\chi(j)\zeta_p\left(i,\frac{j}{D}\right)$$

for s in the range (6.1).

It can be shown that the RHS does not depend on the choice of D, see [Coh07, p. 303, Remarks (1)]. Some facts:

- $L_p(\cdot,\chi)$ is analytic in the range (6.1).
- Dirichlet L-function $L(s,\chi) = D^{-s}\sum_{j=1}^{D}\chi(j)\zeta(s,j/D)$.
- $L_p(1-n,\chi\omega^n) = -(1-\chi(p)p^{n-1})B_{n,\chi}/n$ while $L(1-n,\chi) = -B_{n,\chi}/n$ for $n\in\mathbb{Z}_{\geqslant 1}$. In particular, $\zeta_p(1-n) := L_p(1-n,\omega^n) = (1-p^{n-1})\zeta(1-n)$ for $n\in\mathbb{Z}_{\geqslant 1}$.
- By continuity, we have $\zeta_p(2k) := L_p(2k,\omega^{1-2k}) = 0$ for $k\in\mathbb{Z}_{\geqslant 1}$. We know little about $\zeta_p(2k+1)$. In fact, the conjecture "$\zeta_p(2k+1)\neq 0$ for all $k\in\mathbb{Z}_{\geqslant 1}$" is still open.

An outline of proof of Theorem 6.1:

$$\text{Rational functions} \xrightarrow{\text{Volkenborn integration}} \text{Linear forms in 1 and } \zeta_p(i)\text{'s}$$
$$\xrightarrow{\text{Nesterenko's criterion}} \text{Theorem 6.1}$$

6.2. p-adic version of Nesterenko's linear independence criterion.

Theorem 6.8 (p-adic version of Nesterenko's linear independence criterion)**.** *Let $\tau_1,\tau_2>0$ be constants and $\sigma:\mathbb{N}\to\mathbb{R}_{>0}$ be a non-decreasing function satisfying*

$$\lim_{n\to+\infty}\sigma(n) = +\infty \quad \text{and} \quad \lim_{n\to+\infty}\frac{\sigma(n+1)}{\sigma(n)} = 1.$$

Let $\underline{(\theta)} = (\theta_1,\theta_2,\ldots,\theta_m)\in\mathbb{Q}_p^m$. Suppose that for each $n\in\mathbb{N}$ there exists a linear form in $m+1$ variables with integer coefficients

$$\Lambda_n(\underline{X}) = \lambda_{n,0}X_0 + \lambda_{n,1}X_1 + \cdots + \lambda_{n,m}X_m, \quad \lambda_{n,j}\in\mathbb{Z}$$

such that

- $\max_{0\leqslant j\leqslant m}|\lambda_{n,j}|_\infty \leqslant \exp((1+o(1))\sigma(n))$*;*
- $\exp(-(\tau_1+o(1))\sigma(n)) \leqslant |\Lambda_n(1,\underline{\theta})|_p \leqslant \exp(-(\tau_2+o(1))\sigma(n))$.

Then,

$$\dim_{\mathbb{Q}}\operatorname{Span}_{\mathbb{Q}}(1,\theta_1,\ldots,\theta_m) \geqslant \frac{\tau_1}{1+\tau_1-\tau_2}.$$

Proof. See [Nes12, Theorem 1]. ∎

6.3. **rational functions and linear forms.** Let s be an odd positive integer. Let $\ell \in \mathbb{N}$ be a parameter to be determined, but we always assume that $p^{2\ell+1} < s$. For each $n \in \mathbb{N}$, let

$$N(n) := p^{\ell}\left(p^{\lfloor \frac{\log n}{\log p} \rfloor + 1} - 1\right).$$

We define the rational function

$$R_n(t) = n!^s \binom{N(n)}{\underline{n}}^{p^{\ell}} \binom{p^{\ell} t + N(n)}{N(n)}^{p^{\ell}} \frac{1}{(t)_{n+1}^s},$$

where

$$\binom{N(n)}{\underline{n}} := \binom{N(n)}{\underbrace{n, n, \ldots, n}_{\lfloor N(n)/n \rfloor}, N(n) - n\lfloor N(n)/n \rfloor} = \frac{N(n)!}{n!^{\lfloor N(n)/n \rfloor} (N(n) - n\lfloor N(n)/n \rfloor)!}.$$

Some remarks:

- We do not need the symmetry, because $\zeta_p(2k) = 0$ in the p-adic case.
- The length $N(n)$ and the multiplicity p^{ℓ} are designed to make $\int_{\mathbb{Z}_p} R_n(t + j/p^{\ell})\mathrm{d}t$ easy to control.
- $R_n(t)$ has the double roots in the 'middle part', which will be used to prove $d_n^{s+1}\rho_{0,j/p^{\ell}} \in \mathbb{Z}$ (Sprang's lemma).

Since $p^{2\ell+1} < s$, we have $\deg R_n < 0$. So $R_n(t)$ has the unique partial fraction decomposition

$$R_n(t) = \sum_{i=1}^{s} \sum_{k=0}^{n} \frac{a_{i,k}}{(t+k)^i}.$$

Lemma 6.9. *For $x \in \mathbb{Q}_p$ with $|x|_p \geqslant q_p$, we have*

$$\int_{\mathbb{Z}_p} R_n(t+x)\mathrm{d}t = \rho_{0,x} + \sum_{i=1}^{s} \rho_i \omega(x)^{-i} \zeta_p(i+1, x),$$

where

$$\rho_i = \sum_{k=0}^{n} i a_{i,k}, \quad 1 \leqslant i \leqslant s,$$

$$\rho_{0,x} = -\sum_{i=1}^{s} \sum_{k=0}^{n} \sum_{\nu=0}^{k-1} \frac{i a_{i,k}}{(\nu + x)^{i+1}}.$$

Proof. We have

$$\begin{aligned}
\int_{\mathbb{Z}_p} R_n(t+x)\mathrm{d}t &= \int_{\mathbb{Z}_p} \sum_{i=1}^{s} \sum_{k=0}^{n} \frac{a_{i,k}}{(t+k+x)^i}\mathrm{d}t \\
&= \sum_{i=1}^{s} \sum_{k=0}^{n} a_{i,k} \int_{\mathbb{Z}_p} \frac{\mathrm{d}t}{(t+k+x)^i} \\
&= \sum_{i=1}^{s} \sum_{k=0}^{n} a_{i,k} \left(\int_{\mathbb{Z}_p} \frac{\mathrm{d}t}{(t+x)^i} - i \sum_{\nu=0}^{k-1} \frac{1}{(\nu+x)^{i+1}} \right),
\end{aligned}$$

where the last equality comes from Lemma 6.4. Since $\int_{\mathbb{Z}_p} \frac{\mathrm{d}t}{(t+x)^i} = i\omega(x)^{-i}\zeta_p(i+1,x)$, the result follows. ◻

Lemma 6.10 (linear forms)**.** *Let*

$$\Lambda_n = d_n^{s+1} \sum_{\substack{j=1 \\ p\nmid j}}^{p^\ell} \int_{\mathbb{Z}_p} R_n\left(t + \frac{j}{p^\ell}\right) \mathrm{d}t.$$

Then Λ_n is a linear form in 1 and $\zeta_p(2), \zeta_p(3), \ldots, \zeta_p(s+1)$:

$$\Lambda_n = \lambda_0 + \sum_{i=1}^{s} \lambda_i \zeta_p(i+1),$$

where the coefficients are

$$\lambda_i = d_n^{s+1}(p^\ell)^{i+1}\rho_i, \quad 1 \leqslant i \leqslant s,$$

$$\lambda_0 = d_n^{s+1} \sum_{\substack{j=1 \\ p\nmid j}}^{p^\ell} \rho_{0,j/p^\ell}.$$

Proof. This directly follows from Lemma 6.9 and

$$\sum_{\substack{j=1 \\ p\nmid j}}^{p^\ell} \omega\left(\frac{j}{p^\ell}\right)^{-i} \zeta_p\left(i+1, \frac{j}{p^\ell}\right) = (p^\ell)^{i+1}\zeta_p(i+1).$$

◻

6.4. **arithmetic of coefficients.** As usual, we denote $d_n := [1, 2, \ldots, n]$.

Lemma 6.11. *We have*

$$d_n^{s-i} a_{i,k} \in \mathbb{Z}, \quad \textit{for all } i, k.$$

Consequently, $\lambda_i \in \mathbb{Z}$ for all $1 \leqslant i \leqslant s$.

Lemma 6.12 (Sprang's lemma)**.** *We have*

$$d_n^{s+1}\rho_{0,j/p^\ell} \in \mathbb{Z}, \quad 1 \leqslant j \leqslant p^\ell, p \nmid j.$$

Consequently, $\lambda_0 \in \mathbb{Z}$.

6.5. $|\cdot|_\infty$ **estimates for coefficients.**

Lemma 6.13. *For all i, k, we have*

$$\limsup_{n\to+\infty} |a_{i,k}|_\infty^{\frac{1}{n}} \leqslant \left(p^{\ell+1}\right)^{p^{2\ell+1}} 2^s.$$

Consequently,

$$\max_{0\leqslant i\leqslant s} |\lambda_i|_\infty \leqslant \exp\left((\tau_\infty(\ell, s) + o(1))n\right),$$

where

$$\tau_\infty(\ell, s) = s + 1 + s\log 2 + p^{2\ell+1}\log\left(p^{\ell+1}\right).$$

Proof. we start by

$$a_{i,k} = \frac{1}{2\pi i} \int_{|t+k|=1/p^\ell} n!^s \binom{N(n)}{\underline{n}}^{p^\ell} \binom{p^\ell t + N(n)}{N(n)}^{p^\ell} \frac{(t+k)^{i-1}}{(t)_{n+1}^s} \mathrm{d}t.$$

□

6.6. $|\cdot|_p$ **estimates for** Λ_n**.**

Definition 6.14 (van der Put base)**.** *Define* $\chi_0 := \mathbf{1}_{\mathbb{Z}_p}$*. For each* $k \in \mathbb{Z}_{\geqslant 1}$*, we define* $l(k) := \lfloor \log k / \log p \rfloor$ *and*

$$\chi_k = \mathbf{1}_{k + p^{l(k)+1}\mathbb{Z}_p}.$$

Then every $f \in C(\mathbb{Z}_p, \mathbb{Q}_p)$ *can be written as a convergent series*

$$f = \sum_{k \geqslant 0} b_k \chi_k,$$

where $b_0 = f(0)$ *and* $b_k = f(k) - f(k_-)$*. Here,* $k_- = \sum_{i=0}^{l(k)-1} k_i p^i$ *if the* p*-adic expansion of* k *is* $k = \sum_{i=0}^{l(k)} k_i p^i$*.*

Lemma 6.15. *If* $f \in S^1(\mathbb{Z}_p, \mathbb{Q}_p)$ *and* $f = \sum_{k \geqslant 0} b_k \chi_k$ *as in Definition 6.14, then*

$$\int_{\mathbb{Z}_p} f(t)\mathrm{d}t = b_0 + \lim_{n \to +\infty} \sum_{k=1}^{p^n - 1} b_k p^{-l(k)-1}.$$

Proof. It is direct to check that

$$b_0 + \sum_{k=1}^{p^n-1} b_k p^{-l(k)-1} = \frac{1}{p^n} \sum_{k=0}^{p^n-1} f(k).$$

Since $f \in S^1(\mathbb{Z}_p, \mathbb{Q}_p)$, the limit exists by Lemma 6.4. □

Definition 6.16. *If* $f \in S^1(\mathbb{Z}_p, \mathbb{Q}_p)$ *and* $f = \sum_{k \geqslant 0} b_k \chi_k$ *as in Definition 6.14, we define*

$$\triangle(f) := \min \left\{ v_p(b_0), \min_{k \geqslant 1} \{ v_p(b_k) - l(k) - 1 \} \right\}.$$

Lemma 6.17. *For* $f, g \in S^1(\mathbb{Z}_p, \mathbb{Q}_p)$*, if* $\triangle(f-g) \geqslant c \in \mathbb{Z}$*, then*

$$\int_{\mathbb{Z}_p} f(t)\mathrm{d}t \equiv \int_{\mathbb{Z}_p} g(t)\mathrm{d}t \pmod{p^c \mathbb{Z}_p}.$$

Proof. This follows directly from Definition 6.16 and Lemma 6.15. □

Lemma 6.18. *Let* $f(t), g(t) \in S^1(\mathbb{Z}_p, \mathbb{Z}_p)$*. Then*

(1) *For constants* $b, c \in \mathbb{Z}_{\geqslant 1}$*,*

$$\triangle\left(\binom{t+c}{b}\right) \geqslant -\left\lfloor \frac{\log b}{\log p} \right\rfloor - 1.$$

(2) *For* $\ell \in \mathbb{Z}_{>0}$*, if* $v_p(f(0)) \geqslant \triangle(f) + \ell$*, then* $\triangle(f(p^\ell t)) = \triangle(f(t)) + \ell$*.*

(3) *We have* $\triangle(f+g) \geqslant \min\{\triangle(f), \triangle(g)\}$ *and* $\triangle(f \cdot g) \geqslant \min\{\triangle(f), \triangle(g)\}$.
(4) *If* $f \equiv g \pmod{p\mathbb{Z}_p}$, *then for* $\ell \in \mathbb{Z}_{>0}$,

$$\triangle\left(f^{p^\ell} - g^{p^\ell}\right) \geqslant \min\{\triangle(f), \triangle(g)\} + \ell.$$

(5) *If* $f(t) = \sum_{\nu \geqslant 0} c_\nu t^\nu \in \mathbb{Z}_p[[t]]$ *is a power series with* $\lim_{\nu \to +\infty} |c_\nu|_p = 0$, *then* $\triangle(f) \geqslant \min_{\nu \geqslant 0} v_p(c_\nu) - 1$.

Proof. (1) For any two positive integers A_1, A_2, by comparing the coefficients of t^b on both sides of $(1+t)^{A_1+A_2} = (1+t)^{A_1}(1+t)^{A_2}$, we obtain the combinatorial identity $\binom{A_1+A_2}{b} = \sum_{i=0}^{b} \binom{A_1}{i}\binom{A_2}{b-i}$. Thus, for $k > 0$, we have

$$\binom{k+c}{b} - \binom{k_- + c}{b} = \sum_{i=1}^{b} \binom{k_- + c}{b-i}\binom{k-k_-}{i} = \sum_{i=1}^{b} \binom{k_- + c}{b-i}\binom{k-k_- -1}{i-1}\frac{k-k_-}{i}.$$

Since $v_p(k-k_-) = l(k)$ and $v_p(i) \leqslant \lfloor \log i/\log p \rfloor \leqslant \lfloor \log b/\log p \rfloor$ for $1 \leqslant i \leqslant b$, we deduce that

$$v_p\left(\binom{k+c}{b} - \binom{k_- + c}{b}\right) - l(k) - 1 \geqslant -\lfloor \log b/\log p \rfloor - 1.$$

Also, $v_p\left(\binom{c}{b}\right) \geqslant 0 > -\lfloor \log b/\log p \rfloor - 1$, the proof for (1) is complete.

(2) By definition.

(3) $\triangle(f+g) \geqslant \min\{\triangle(f), \triangle(g)\}$ follows directly by definition. Now consider $\triangle(fg)$. For $k \in \mathbb{Z}_{>0}$,

$$\begin{aligned} v_p\left(f(k)g(k) - f(k_-)g(k_-)\right) &= v_p\left(f(k)(g(k) - g(k_-)) - g(k_-)(f(k) - f(k_-))\right) \\ &\geqslant \min\{v_p(f(k) - f(k_-)), v_p(g(k) - g(k_-)\} \geqslant \min\{\triangle(f), \triangle(g)\} + l(k) + 1. \end{aligned}$$

Also, $v_p(f(0)g(0)) \geqslant v_p(f(0)) \geqslant \triangle(f) \geqslant \min\{\triangle(f), \triangle(g)\}$. So $\triangle(fg) \geqslant \min\{\triangle(f), \triangle(g)\}$.

(4) Let $h = (f-g)/p \in S^1(\mathbb{Z}_p, \mathbb{Z}_p)$. Then $\triangle(h) = \triangle(f-g) - 1 \geqslant \min\{\triangle(f), \triangle(g)\} - 1$ and

$$f^{p^\ell} - g^{p^\ell} = \sum_{i=1}^{p^\ell} \binom{p^\ell}{i} p^i h^i g^{p-i}.$$

Since $v_p\left(\binom{p^\ell}{i} p^i\right) \geqslant \ell + 1$ for every $i \geqslant 1$, we deduce that

$$\begin{aligned} \triangle\left(f^{p^\ell} - g^{p^\ell}\right) &\geqslant \ell + 1 + \min_{1 \leqslant i \leqslant p^\ell} \triangle(h^i g^{p-i}) \\ &\geqslant \ell + 1 + \min\{\triangle(h), \triangle(g)\} \\ &\geqslant \min\{\triangle(f), \triangle(g)\} + \ell. \end{aligned}$$

(5) For $k \in \mathbb{Z}_{>0}$, for each $\nu \geqslant 0$, $v_p(k^\nu - k_-^\nu) \geqslant l(k)$, so $v_p(f(k) - f(k_-)) - l(k) - 1 \geqslant \min_{\nu \geqslant 0} v_p(c_\nu) - 1$. Also, $v_p(f(0)) = v_p(c_0)$. Therefore, (5) is proved. $\square$

Now, we start to estimate $|\Lambda_n|_p$.

Lemma 6.19. *We have* $|\Lambda_n|_p = \exp\left(-(\tau_p(\ell, s) + o(1))n\right)$ *as* $n \equiv -1 \pmod{(p-1)p^{\ell-1}}$ *and* $n \to +\infty$. *Here*

$$\tau_p(\ell, s) = s \log p \left(\ell + \frac{1}{p-1}\right).$$

Proof. Note that

$$\begin{aligned}\Lambda_n &= d_n^{s+1} \sum_{\substack{j=1 \\ p \nmid j}}^{p^\ell} \int_{\mathbb{Z}_p} R_n\left(t + \frac{j}{p^\ell}\right) \mathrm{d}t \\ &= d_n^{s+1} n!^s \binom{N(n)}{\underline{n}}^{p^\ell} p^{\ell s(n+1)} \sum_{\substack{j=1 \\ p \nmid j}}^{p^\ell} \int_{\mathbb{Z}_p} f_j(t) \mathrm{d}t, \end{aligned} \tag{6.2}$$

where

$$f_j(t) := \binom{p^\ell t + p^\ell(p^m - 1) + j}{p^\ell(p^m - 1) + j}^{p^\ell} \frac{1}{\prod_{i=0}^{n} \left(p^\ell(t+i) + j\right)^s},$$

with $m = \lfloor \log n / \log p \rfloor + 1$.

By Lemma 6.18 (1), we have

$$\triangle\left(\binom{t + p^\ell(p^m - 1) + j}{p^\ell(p^m - 1) + j}\right) \geqslant -m - \ell,$$

then, by Lemma 6.18 (2),

$$\triangle\left(\binom{p^\ell t + p^\ell(p^m - 1) + j}{p^\ell(p^m - 1) + j}\right) \geqslant -m.$$

It is elementary (by Kummer's and Lucas' theorem) to show that

$$\binom{p^\ell t + p^\ell(p^m - 1) + j}{p^\ell(p^m - 1) + j} \equiv \mathbf{1}_{p^m\mathbb{Z}_p}(t) \pmod{p\mathbb{Z}_p}.$$

(This is the reason for defining $N(n)$.) Thus, by Lemma 6.18 (4),

$$\begin{aligned} &\triangle\left(\binom{p^\ell t + p^\ell(p^m - 1) + j}{p^\ell(p^m - 1) + j}^{p^\ell} - \mathbf{1}_{p^m\mathbb{Z}_p}(t)\right) \\ &\geqslant \min\left\{\triangle\left(\binom{p^\ell t + p^\ell(p^m - 1) + j}{p^\ell(p^m - 1) + j}\right), \triangle\left(\mathbf{1}_{p^m\mathbb{Z}_p}(t)\right)\right\} + \ell \\ &\geqslant -m + \ell. \end{aligned}$$

(It is easy to check that $\triangle\left(\mathbf{1}_{p^m\mathbb{Z}_p}\right) = -m$ by definition.)

By Lemma 6.18 (5),

$$\triangle\left(\frac{1}{\prod_{i=0}^{n} \left(p^\ell(t+i) + j\right)^s}\right) \geqslant -1.$$

We may assume n is sufficiently large, so $m = \lfloor \log n / \log p \rfloor + 1$ is sufficiently large. Thus, by Lemma 6.18 (3),

$$\triangle\left(f_j(t) - \mathbf{1}_{p^m\mathbb{Z}_p}(t) \frac{1}{\prod_{i=0}^{n} \left(p^\ell(t+i) + j\right)^s}\right) \geqslant -m + \ell.$$

Therefore, Lemma 6.17 implies that

$$\int_{\mathbb{Z}_p} f_j(t) \mathrm{d}t \equiv \int_{\mathbb{Z}_p} \mathbf{1}_{p^m\mathbb{Z}_p}(t) \frac{1}{\prod_{i=0}^{n} \left(p^\ell(t+i) + j\right)^s} \mathrm{d}t \pmod{p^{-m+\ell}}. \tag{6.3}$$

Now, by definition of Volkenborn integration we have

$$\int_{\mathbb{Z}_p} \mathbf{1}_{p^m\mathbb{Z}_p}(t) \frac{1}{\prod_{i=0}^{n} \left(p^\ell(t+i)+j\right)^s} \mathrm{d}t = p^{-m} \int_{\mathbb{Z}_p} \frac{1}{\prod_{i=0}^{n} \left(p^\ell(p^m t+i)+j\right)^s} \mathrm{d}t. \tag{6.4}$$

By Lemma 6.18 (5) we have

$$\triangle\left(\frac{1}{\prod_{i=0}^{n} \left(p^\ell(p^m t+i)+j\right)^s} - \frac{1}{\prod_{i=0}^{n} \left(p^\ell i+j\right)^s} \right) \geqslant \ell+m-1,$$

so Lemma 6.17 implies

$$\int_{\mathbb{Z}_p} \frac{1}{\prod_{i=0}^{n} \left(p^\ell(p^m t+i)+j\right)^s} \mathrm{d}t \equiv \frac{1}{\prod_{i=0}^{n} \left(p^\ell i+j\right)^s} \pmod{p^{\ell+m-1}\mathbb{Z}_p}. \tag{6.5}$$

Combining (6.3),(6.4), and (6.5) we deduce that

$$\begin{aligned} \int_{\mathbb{Z}_p} f_j(t)\mathrm{d}t &\equiv p^{-m} \frac{1}{\prod_{i=0}^{n} \left(p^\ell i+j\right)^s} \pmod{p^{-m+\ell}\mathbb{Z}_p} \\ &\equiv p^{-m} \frac{1}{j^{s(n+1)}} \pmod{p^{-m+\ell}\mathbb{Z}_p}. \end{aligned}$$

We assume $n \equiv -1 \pmod{(p-1)p^{\ell-1}}$, so $\int_{\mathbb{Z}_p} f_j(t)\mathrm{d}t \equiv p^{-m} \pmod{p^{-m+\ell}\mathbb{Z}_p}$. Thus,

$$\sum_{\substack{j=1 \\ p\nmid j}}^{p^n} \int_{\mathbb{Z}_p} f_j(t)\mathrm{d}t \equiv p^{-m} \cdot p^{\ell-1}(p-1) \pmod{p^{-m+\ell}\mathbb{Z}_p},$$

so

$$v_p\left(\sum_{\substack{j=1 \\ p\nmid j}}^{p^n} \int_{\mathbb{Z}_p} f_j(t)\mathrm{d}t \right) = -m+\ell-1 = -\lfloor \log n/\log p \rfloor + \ell - 2.$$

Now, by (6.2),

$$v_p(\Lambda_n) = (s+1)v_p(d_n) + s v_p(n!) + p^\ell v_p\left(\binom{N(n)}{\underline{n}} \right) + \ell s(n+1) - \lfloor \log n/\log p \rfloor + \ell - 2.$$

It is elementary to show that (as $n \to +\infty$)

$$v_p(d_n) = O(\log n) = o(n), \quad v_p(n!) = \frac{n}{p-1} + o(n), \quad v_p\left(\binom{N(n)}{\underline{n}} \right) = O(\log n) = o(n),$$

then $v_p(\Lambda_n) = s(\ell + 1/(p-1))n + o(n)$ and this completes the proof of Lemma 6.19. ☐

6.7. **proof of Theorem 6.1.**

Proof of Theorem 6.1. Applying p-adic version of Nesterenko's linear independence criterion (Theorem 6.8) to $\{\Lambda_{(p-1)p^{\ell-1}n-1}\}_{n\in\mathbb{N}}$, by Lemma 6.13 and Lemma 6.19, we obtain that

$$\begin{aligned} \dim_{\mathbb{Q}} \mathrm{Span}_{\mathbb{Q}}\left(1, \zeta_p(2), \zeta_p(3), \ldots, \zeta_p(s+1)\right) &\geqslant \frac{\tau_p(\ell,s)}{\tau_\infty(\ell,s)} \\ &= \frac{s\log p\left(\ell + \frac{1}{p-1}\right)}{s+1+s\log 2 + p^{2\ell+1}\log\left(p^{\ell+1}\right)}. \end{aligned}$$

Choose the parameter

$$\ell = \left\lfloor \frac{\frac{1}{2}\log s - \log\log s}{\log p} \right\rfloor,$$

then we have (as $s \to +\infty$)

$$\begin{aligned} \frac{s\log p\left(\ell + \frac{1}{p-1}\right)}{s+1+s\log 2 + p^{2\ell+1}\log\left(p^{\ell+1}\right)} &= \frac{(\frac{1}{2}+o(1))s\log s}{(1+\log 2+o(1))s} \\ &= \frac{1+o(1)}{2(1+\log 2)}\log s. \end{aligned}$$

Since $\zeta_p(2k) = 0$ for $k \in \mathbb{Z}_{\geqslant 1}$, we complete the proof of Theorem 6.1.

$\square$

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