RATIONAL FUNCTIONS AND ODD ZETA VALUES

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In this note I collect some known results about the irrationality of odd zeta values $\zeta(2k+1)$. These results are elementary in nature: we construct good linear forms in odd zeta values by means of rational functions. I wish to make them more accessible for wider audiences including undergraduates. This is a draft version, I plan to complete it before 2022.

Contents

0. Introduction

We are interested in the arithmetic properties (irrationality, transcendence, ...) of some special constants. Which constants are considered to be interesting? The following is my personal tastes.

Notations: We use $\overline{\mathbb{Q}}$ to denote the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$.

'First class': π, e.

- Hermite, 1873: $e \notin \overline{\mathbb{Q}}$.
- Lindemann, 1882: $\pi \notin \overline{\mathbb{Q}}$. Moreover, for any $\alpha \in \overline{\mathbb{Q}} \setminus \{0\}$, we have $e^\alpha \notin \overline{\mathbb{Q}}$.
- (Lindemann-)Weierstrass, 1885: If $\alpha_1, \ldots, \alpha_n \in \overline{\mathbb{Q}}$ are linearly independent over $\mathbb{Q}$, then $e^{\alpha_1}, \ldots, e^{\alpha_n}$ are algebraically independent over $\mathbb{Q}$.

'Second class': $\sqrt{2}^{\sqrt{3}}$, γ, $\zeta(3)$, $\Gamma(1/5)$, ...

- Gelfond/Schneider, 1934: If $\alpha, \beta \in \overline{\mathbb{Q}}$ with $\alpha \neq 0, 1$ and $\beta \notin \mathbb{Q}$, then $\alpha^\beta \notin \overline{\mathbb{Q}}$. Here $\alpha^\beta = \exp(\beta \log \alpha)$ is multi-valued, the precise statement is that any value of α^β is transcendent. As a corollary, both $\sqrt{2}^{\sqrt{2}}$ and $e^\pi = (-1)^{-i}$ are transcendent.
- Baker, 1966: If $\alpha_1, \ldots, \alpha_n; \beta_1, \ldots, \beta_n \in \overline{\mathbb{Q}}$ with $\alpha_i \neq 0, 1$ and that $1, \beta_1, \ldots, \beta_n$ are linearly independent over $\mathbb{Q}$, then $\alpha_1^{\beta_1} \cdots \alpha_n^{\beta_n} \notin \overline{\mathbb{Q}}$.
- For Euler's constant $\gamma = \lim_{n \to +\infty} \left(-\log n + \sum_{k=1}^n \frac{1}{k}\right)$, nothing very exciting is known. (But see [Riv12] for some explorations.)

In this note, we will focus on the Riemann zeta values

$$\zeta(k) = \frac{1}{1^k} + \frac{1}{2^k} + \frac{1}{3^k} + \cdots, \quad k \in \mathbb{Z}_{\geqslant 2}.$$

- Essentially by Euler:

  $$\zeta(2k) = (-1)^{k-1} 2^{2k-1} \frac{B_{2k}}{(2k)!} \pi^{2k}, \quad k \in \mathbb{Z}_{\geqslant 1},$$

  where B_{2k} is Bernoulli's number defined by the generating series

  $$\frac{t}{e^t - 1} = \sum_{k \geqslant 0} \frac{B_k}{k!} t^k.$$

  Since $B_{2k} \in \mathbb{Q}$ and $\zeta(2k) \neq 0$, we deduce from Lindemann's theorem ($\pi \notin \overline{\mathbb{Q}}$) that $\zeta(2k) \notin \overline{\mathbb{Q}}$. (e.g. $\zeta(2) = \pi^2/6, \zeta(4) = \pi^4/90, \zeta(6) = \pi^6/945, \ldots$.) A natural question: how about odd zeta values $\zeta(2k+1)$?
- Conjecture: $\pi, \zeta(3), \zeta(5), \zeta(7), \ldots$ are algebraically independent over $\mathbb{Q}$. This conjecture is still far to reach. There are some stronger conjectures concerning about multiple zeta values, see Zagier's conjecture. Up to now, we only know some irrationality results about odd zeta values $\zeta(2k+1)$.
- Apéry, 1978: $\zeta(3) \notin \mathbb{Q}$.
- Rivoal, 2000; Ball, Rivoal, 2001: for odd $s \geqslant s_0(\varepsilon)$,

  $$\dim_{\mathbb{Q}} \mathrm{Span}_{\mathbb{Q}} \left(1, \zeta(3), \zeta(5), \ldots, \zeta(s)\right) \geqslant \frac{1-\varepsilon}{1+\log 2} \log s.$$

- Zudilin, 2001: At least one of $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is irrational.
- Fischler, Sprang, Zudilin, 2018 and Lai, Yu, 2020: for odd $s \geqslant s_0(\varepsilon)$,

  $$\#\left\{k : 3 \leqslant k \leqslant s, k \text{ odd}, \zeta(k) \notin \mathbb{Q}\right\} \geqslant (c_0 - \varepsilon)\sqrt{\frac{s}{\log s}},$$

  where the constant

  $$c_0 = \sqrt{\frac{4\zeta(2)\zeta(3)}{\zeta(6)} \left(1 - \log \frac{\sqrt{4e^2+1} - 1}{2}\right)} \approx 1.192507\ldots.$$

- There are some p-adic analogues. Calegari, 2005: for $p = 2, 3$, we have $\zeta_p(3) \in \mathbb{Q}_p \backslash \mathbb{Q}$. Sprang, 2020: let K be a number field, for $s \geqslant s_0(\varepsilon)$,

  $$\dim_K \mathrm{Span}_K \left(1, \zeta_p(2), \zeta_p(3), \zeta_p(4), \ldots, \zeta_p(s)\right) \geqslant \frac{1-\varepsilon}{2[K:\mathbb{Q}](1+\log 2)} \log s.$$

1. Irrationality criteria

1.1. Linear independence criteria of Siegel and Nesterenko.

We begin with an obvious criterion.

Theorem 1.1 (naive irrationality criterion)**.** *Let $\underline{\theta} = (\theta_1, \theta_2, \ldots, \theta_m) \in \mathbb{R}^m$. Suppose that there exists a sequence of linear forms (indexed by $n \in \mathbb{N}$) in $m+1$ variables with integer coefficients*

$$L_n(\underline{X}) = l_{n,0}X_0 + l_{n,1}X_1 + \cdots + l_{n,m}X_m, \quad (l_{n,j} \in \mathbb{Z}, j = 0, 1, \ldots, m)$$

such that

- $L_n(1, \underline{\theta}) \neq 0$ *for each* $n \in \mathbb{N}$*;*
- $L_n(1, \underline{\theta}) \to 0$ *as* $n \to +\infty$*.*

Then, at least one of $\theta_1, \theta_2, \ldots, \theta_m$ is irrational.

Proof. If all θ_i are rational, let D be a common denominator of these rational numbers. Then $|L_n(1, \underline{\theta})| \geqslant 1/D$, which contradicts that $L_n(1, \underline{\theta}) \to 0$. □

Although Theorem 1.1 is trivial, it gives us an angle of view for irrationality: if a sequence of linear forms of $\theta_1, \ldots, \theta_m$ (with integer coefficients) has smaller and smaller (but nonzero) values, then we are possible to obtain some nontrivial information about the irrationality of these θ_i. In 1929, Siegel [Sie29] made a more useful and quantitative result along this perspective.

Theorem 1.2 (Siegel's linear independence criterion)**.** *Let $\tau > 0$ be a constant and $\sigma : \mathbb{N} \to \mathbb{R}_{>0}$ a function satisfying*

$$\lim_{n \to +\infty} \sigma(n) = +\infty.$$

Let $\underline{\theta} = (\theta_1, \theta_2, \ldots, \theta_m) \in \mathbb{R}^m$. Suppose that, for each $n \in \mathbb{N}$ there exists a complete system of $m+1$ linearly independent linear forms in $m+1$ variables with integer coefficients

$$L_i^{(n)}(\underline{X}) = l_{i,0}^{(n)}X_0 + l_{i,1}^{(n)}X_1 + \cdots + l_{i,m}^{(n)}X_m, \quad i = 0, 1, \ldots, m, \quad l_{i,j}^{(n)} \in \mathbb{Z}$$

such that (as $n \to +\infty$)

- $\max_{0 \leqslant i,j \leqslant m} \left| l_{i,j}^{(n)} \right| \leqslant \exp((1 + o(1))\sigma(n))$*;*
- $\max_{0 \leqslant i \leqslant m} \left| L_i^{(n)}(1, \underline{\theta}) \right| \leqslant \exp(-(\tau + o(1))\sigma(n))$*.*

Then,

$$\dim_{\mathbb{Q}} \operatorname{Span}_{\mathbb{Q}} (1, \theta_1, \theta_2, \ldots, \theta_m) \geqslant 1 + \tau.$$

Roughly speaking, Siegel's linear independence criterion says that, more 'faster' the values of linear forms decay than the growth of coefficients, greater is the dimension of the space spanned by θ_i. But keep in mind that Siegel's criterion requires a restrictive hypothesis, namely, we need a complete system of linearly independent linear forms.

Proof of Theorem 1.2. Denote $r = \dim_{\mathbb{Q}} \operatorname{Span}_{\mathbb{Q}}(1, \theta_1, \theta_2, \ldots, \theta_m)$. Then there exist $m+1-r$ elements $A_i = (a_{i,0}, a_{i,1}, \ldots, a_{i,m}) \in \mathbb{Q}^{m+1}$ $(i = r, r+1, \ldots, m)$, linearly independent over $\mathbb{Q}$, such that

$$a_{i,0} + a_{i,1}\theta_1 + \cdots + a_{i,m}\theta_m = 0, \quad i = r, r+1, \ldots, m.$$

We can assume that $a_{i,j} \in \mathbb{Z}$ for all i, j.

We view each linear form $L_i^{(n)}$ as an element $(l_{i,0}^{(n)}, \ldots, l_{i,m}^{(n)})$ in $\mathbb{Q}^{m+1}$. For each $n \in \mathbb{N}$, since $\left\{L_0^{(n)}, \ldots, L_m^{(n)}\right\}$ is a basis of $\mathbb{Q}^{m+1}$, after relabeling, we may assume that

$$\left\{L_0^{(n)}, L_1^{(n)}, \ldots, L_{r-1}^{(n)}, A_r, A_{r+1}, \ldots, A_m\right\}$$

forms a basis of $\mathbb{Q}^{m+1}$.

Denote

$$\Delta_n = \begin{vmatrix} l_{0,0}^{(n)} & l_{0,1}^{(n)} & \ldots & l_{0,m}^{(n)} \\ \ldots & \ldots & \ldots & \ldots \\ l_{r-1,0}^{(n)} & l_{r-1,1}^{(n)} & \ldots & l_{r-1,m}^{(n)} \\ a_{r,0} & a_{r,1} & \ldots & a_{r,m} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m,0} & a_{m,1} & \ldots & a_{m,m} \end{vmatrix}.$$

Then $\Delta_n \in \mathbb{Z} \setminus \{0\}$, it follows that $|\Delta_n| \geqslant 1$. Applying elementary column operations we obtain that

$$\begin{aligned} \Delta_n &= \begin{vmatrix} L_0^{(n)}(1, \underline{\theta}) & l_{0,1}^{(n)} & \ldots & l_{0,m}^{(n)} \\ \ldots & \ldots & \ldots & \ldots \\ L_{r-1}^{(n)}(1, \underline{\theta}) & l_{r-1,1}^{(n)} & \ldots & l_{r-1,m}^{(n)} \\ 0 & a_{r,1} & \ldots & a_{r,m} \\ \ldots & \ldots & \ldots & \ldots \\ 0 & a_{m,1} & \ldots & a_{m,m} \end{vmatrix} \\ &= \sum_{i=0}^{r-1} L_i^{(n)}(1, \underline{\theta}) \Delta_{i,0}^{(n)}, \end{aligned}$$

where $\Delta_{i,0}^{(n)}$ is the $(i,0)$-cofactor of the determinant Δ_n. Clearly, for each $k \in [0, r-1]$,

$$\begin{aligned} \left|\Delta_{k,0}^{(n)}\right| &\leqslant m! \left(\max_{0 \leqslant i,j \leqslant m} \left|l_{i,j}^{(n)}\right|\right)^{r-1} \left(\max_{\substack{r \leqslant i \leqslant m \\ 0 \leqslant j \leqslant m}} |a_{i,j}|\right)^{m+1-r} \\ &\leqslant m! A^{m+1-r} \exp\left((r-1+o(1))\sigma(n)\right). \qquad (A = \max_{i,j} |a_{i,j}|) \end{aligned}$$

Hence,

$$1 \leqslant |\Delta_n| \leqslant r m! A^{m+1-r} \exp\left((r-1-\tau+o(1))\sigma(n)\right).$$

Since $rm!A^{m+1-r}$ is independent of n and $\sigma(n) \to +\infty$ as $n \to +\infty$, we deduce that $r \geqslant 1+\tau$, as desired. $\square$

Siegel used his criterion to prove some transcendence results (linear independence of $1, \theta, \theta^2, \theta^3, \ldots$). In 1985, Nesterenko [Nes85] introduced a similar but different criterion.

Theorem 1.3 (Nesterenko's linear independence criterion). *Let $\tau_1, \tau_2 > 0$ be constants and $\sigma : \mathbb{N} \to \mathbb{R}_{>0}$ a non-decreasing function satisfying*

$$\lim_{n\to+\infty} \sigma(n) = +\infty, \quad \text{and} \quad \lim_{n\to+\infty} \frac{\sigma(n+1)}{\sigma(n)} = 1.$$

Let $\underline{\theta} = (\theta_1, \theta_2, \dots, \theta_m) \in \mathbb{R}^m$. Suppose that, for each $n \in \mathbb{N}$ there exists a linear form in $m+1$ variables with integer coefficients

$$L_n((\underline{X})) = l_{n,0}X_0 + l_{n,1}X_1 + \cdots + l_{n,m}X_m, \quad l_{n,j} \in \mathbb{Z}$$

such that

- $\max_{0\leqslant j\leqslant m} |l_{n,j}| \leqslant \exp\left((1+o(1))\sigma(n)\right)$;
- $\exp\left(-(\tau_1+o(1))\sigma(n)\right) \leqslant |L_n(1,\underline{\theta})| \leqslant \exp\left(-(\tau_2+o(1))\sigma(n)\right)$.

Then,

$$\dim_{\mathbb{Q}} \operatorname{Span}_{\mathbb{Q}}(1, \theta_1, \theta_2, \dots, \theta_m) \geqslant \frac{1+\tau_1}{1+\tau_1-\tau_2}.$$

There is a comparison between Siegel's and Nesterenko's criteria.

	Siegel's	Nesterenko's
Advantage	need only upper bounds	need only one linear form
Disadvantage	need a complete system	need both lower and upper bounds

The original proof [Nes85] of Nesterenko's criterion is quite involved. It has been simplified by Amoroso and this simplification was revisited (and translated from Italian to French) by Colmez [Col03].

In 2010, Fischler and Zudilin [FZ10] obtained a more elementary proof, which only used Minkowski's 'geometry of numbers'. In the following, I will repeat the the proof written by Chantanasiri [Cha10].

We first state a simple version of the Minkowski's convex body theorem. Recall a fact that every convex set in $\mathbb{R}^d$ is measurable with respect to the d-dimensional Lebesgue measure Vol_d.

Lemma 1.4 (Minkowski's convex body theorem). *Let C be a convex set in $\mathbb{R}^d$, which is symmetric with respect to the origin (i.e. $-C = C$). If one of the following conditions holds*

- $\mathrm{Vol}_d(C) > 2^d$; *or*
- $\mathrm{Vol}_d(C) \geqslant 2^d$ *and C is compact.*

Then, C contains a nonzero integral lattice point, that is,

$$C \cap \left(\mathbb{Z}^d \setminus \{\underline{0}\}\right) \neq \emptyset.$$

Proof. Suppose first that $\mathrm{Vol}_d(C) > 2^d$, then $\mathrm{Vol}_d(C/2) = 2^{-d}\,\mathrm{Vol}_d(C) > 1$. Let

$$(C/2)_{\underline{n}} := (C/2) \cap ([n_1, n+1) \times [n_2, n_2+1) \times \cdots \times [n_d, n_d+1)) - \underline{n}$$

for any $\underline{n} = (n_1, n_2, \dots, n_d) \in \mathbb{Z}^d$. Then $(C/2)_{\underline{n}} \subset [0,1)^d$ and $\sum_{\underline{n}\in\mathbb{Z}^d} \mathrm{Vol}_d((C/2)_{\underline{n}}) = \mathrm{Vol}_d(C/2) > 1$. So there exist two distinct $\underline{n}_1 \neq \underline{n}_2 \in \mathbb{Z}^d$ such that $(C/2)_{\underline{n}_1} \cap (C/2)_{\underline{n}_2} \neq \emptyset$, which means $\underline{n}_1 - \underline{n}_2 \in (C/2) - (C/2)$. Since C is symmetric and convex, we have $(C/2) - (C/2) = (C/2) + (C/2) \subset C$, so $\underline{n}_1 - \underline{n}_2$ is a nonzero integral lattice point contained in C.

Suppose now $\mathrm{Vol}_d(C) \geqslant 2^d$ and C is compact. By the above, there exists an integral lattice point $\underline{n}_m \in \mathbb{Z}^d \setminus \{\underline{0}\}$ which is contained in $(1+m^{-1})C$, $m = 1, 2, \dots$. The sequence $\{\underline{n}_m\}_{m=1}^{\infty}$

is bounded thus has at least one limit point $\underline{n}^*$. Then $\underline{n}^*$ is a nonzero integral lattice point contained in C. $\square$

Let $\{1,\xi_1,\dots,\xi_r\}$ be a basis of $\operatorname{Span}_{\mathbb{Q}}(1,\theta_1,\dots,\theta_m)$. To prove Theorem 1.3, the idea is that we can approximate the direction of $(\xi_1,\dots,\xi_r)$ by a direction $(x_1,\dots,x_r)$ with $x_i\in\mathbb{Z}$. The existence of such a 'good' approximation is guaranteed by Lemma 1.4.

Proof of Theorem 1.3. Let $\{\xi_0,\xi_1,\dots,\xi_r\}$ be a $\mathbb{Q}$-basis of $\operatorname{Span}_{\mathbb{Q}}(1,\theta_1,\dots,\theta_m)$ with $\xi_0=1$. So $\dim_{\mathbb{Q}}\operatorname{Span}_{\mathbb{Q}}(1,\theta_1,\theta_2,\dots,\theta_m)=r+1$. (By Theorem 1.1, $r>0$.) Then there exist $d\in\mathbb{N}$ and $c_{i,0},c_{i,1},\dots,c_{i,r}\in\mathbb{Z}$ $(i=0,1,\dots,m)$ such that

$$d\theta_i=\sum_{j=0}^{r}c_{i,j}\xi_j,\quad i=0,1,\dots,m.$$

Write $\underline{\xi}=(\xi_1,\xi_2,\dots,\xi_r)$. Consider the linear form

$$\begin{aligned}\Gamma_n(Y_0,Y_1,\dots,Y_r)&=L_n\left(\sum_{j=0}^{r}c_{0,j}\xi_j,\sum_{j=0}^{r}c_{1,j}\xi_j,\dots,\sum_{j=0}^{r}c_{m,j}\xi_j\right)\\&=:\gamma_{n,0}Y_0+\gamma_{n,1}Y_1+\cdots+\gamma_{n,r}Y_r,\end{aligned}$$

where $\gamma_{n,j}=\sum_{k=0}^{m}l_{n,k}c_{k,j}\in\mathbb{Z},\ j=0,1,\dots,r$. Then $\Gamma_n(1,\underline{\xi})=dL_n(1,\underline{\theta})$. Since $d,c_{k,j}$ are independent of n and $\sigma(n)\to+\infty$ as $n\to+\infty$, we may write (as $n\to+\infty$)

$$\max_{0\leqslant j\leqslant r}|\gamma_{n,j}|\leqslant\exp\left((1+o(1))\sigma(n)\right),\tag{1.1}$$

$$\exp\left(-(\tau_1+o(1))\sigma(n)\right)\leqslant\left|\Gamma_n(1,\underline{\xi})\right|\leqslant\exp\left(-(\tau_2+o(1))\sigma(n)\right).\tag{1.2}$$

For each sufficiently large $n\in\mathbb{N}$, we define

$$C_n:=\left\{(x_0,x_1,\dots,x_r)\in\mathbb{R}^{r+1}\ :\ |x_0|\leqslant\frac{1}{2\left|\Gamma_n(1,\underline{\xi})\right|},\ |x_0\xi_j-x_j|\leqslant\left|2\Gamma_n(1,\underline{\xi})\right|^{1/r},\ j=1,2,\dots,r\right\}.$$

Then C_n is a symmetric and compact convex set in $\mathbb{R}^{r+1}$ with volume $\operatorname{Vol}_{r+1}(C_n)=2^{r+1}$. By Lemma 1.4, there is a nonzero integral lattice point in C_n, we fix such a point

$$\underline{x}_n=(x_0(n),x_1(n),\dots,x_r(n))\in C_n\cap\left(\mathbb{Z}^{r+1}\setminus\{\underline{0}\}\right).$$

We claim that

$$\lim_{n\to+\infty}|x_0(n)|=+\infty.\tag{1.3}$$

In fact, if (1.3) does not hold, then $\{x_0(n)\}_{n\geqslant1}$ admits a bounded subsequence, so a constant subsequence, say, $x_0(n)=x^*$ along an infinite subset I of $\mathbb{N}$. We would have $\lim_{\substack{n\in I\\ n\to+\infty}}(x_0^*\xi_j-x_j(n))=0$, therefore, $x_j(n)=x_0^*\xi_j$ for all sufficiently large $n\in I$ (since $x_j(n)\in\mathbb{Z}$). If $x_0^*=0$, then $x_j(n)=0$ for all j, which is impossible. If $x_0^*\neq0$, then $\xi_j\in\mathbb{Q}$ for all j, (so $r=0$) which contradicts Theorem 1.1. Hence, the claim (1.3) is true.

Fix an index k_0 such that $|\Gamma_k(1,\underline{\xi})|>0$ for all $k\geqslant k_0$. For sufficiently large n, we define

$$k_n:=\min\left\{k\in\mathbb{N}_{\geqslant k_0}\ :\ |x_0(n)|\leqslant\frac{1}{2\left|\Gamma_k(1,\underline{\xi})\right|}\right\}.$$

By (1.3), we have $k_n \to +\infty$ as $n \to +\infty$. By the definition of $\underline{x}_n$, we have $k_n \leqslant n$. Moreover, by the minimality of k_n, it holds that

$$|x_0(n)| \leqslant \frac{1}{2\left|\Gamma_{k_n}(1,\underline{\xi})\right|}, \quad \text{and} \quad |x_0(n)| > \frac{1}{2\left|\Gamma_{k_n-1}(1,\underline{\xi})\right|}. \tag{1.4}$$

Now, we can write

$$\underbrace{\sum_{j=0}^{r} \gamma_{k_n,j} x_j(n)}_{\in\mathbb{Z}} = \underbrace{x_0(n)\sum_{j=0}^{r}\gamma_{k_n,j}\xi_j}_{\text{absolute value}\leqslant\frac{1}{2}} + \sum_{j=0}^{r}\gamma_{k_n,j}\left(x_j(n) - x_0(n)\xi_j\right). \tag{1.5}$$

By the fact that $y = y_1 + y_2$, $y \in \mathbb{Z}$, $|y_1| \leqslant 1/2$ implies $|y_2| \geqslant |y_1|$, we deduce from (1.5) that

$$\left|\sum_{j=0}^{r}\gamma_{k_n,j}\left(x_j(n) - x_0(n)\xi_j\right)\right| \geqslant \left|x_0(n)\sum_{j=0}^{r}\gamma_{k_n,j}\xi_j\right|. \tag{1.6}$$

By (1.1), the definition of C_n, and (1.4), we obtain from (1.6) that

$$(r+1)\cdot\exp\left((1+o(1))\sigma(k_n)\right)\cdot\left|2\Gamma_n(1,\underline{\xi})\right|^{1/r} \geqslant \frac{1}{2\left|\Gamma_{k_n-1}(1,\underline{\xi})\right|}\left|\Gamma_{k_n}(1,\underline{\xi})\right|. \tag{1.7}$$

Since $k_n \to +\infty$, we can write $r+1 = \exp(o(1)\sigma(k_n))$. Then, by (1.2),

$$\exp\left((1+o(1))\sigma(k_n) - \frac{1}{r}(\tau_2+o(1))\sigma(n)\right) \geqslant \exp\left((\tau_2+o(1))\sigma(k_n-1) - (\tau_1+o(1))\sigma(k_n)\right), \tag{1.8}$$

which is

$$(1+o(1)) - \frac{1}{r}(\tau_2+o(1))\frac{\sigma(n)}{\sigma(k_n)} \geqslant (\tau_2+o(1))\frac{\sigma(k_n-1)}{\sigma(k_n)} - (\tau_1+o(1)). \tag{1.9}$$

Since $k_n \leqslant n$ and $\sigma(\cdot)$ is non-decreasing, and $\sigma(k_n-1)/\sigma(k_n) = 1+o(1)$, we obtain

$$1 - \frac{1}{r}\tau_2 \geqslant \tau_2 - \tau_1,$$

that is, $r \geqslant \tau_2/(1+\tau_1-\tau_2)$. So

$$\dim_{\mathbb{Q}}\operatorname{Span}_{\mathbb{Q}}(1,\theta_1,\theta_2,\ldots,\theta_m) = r+1 \geqslant \frac{1+\tau_1}{1+\tau_1-\tau_2},$$

as desired. □

We will see the applications of Nesterenko's linear independence criterion in § 3.

1.2. **Irrationality measure.** In this subsection we introduce the concept of irrationality measure, which is one kind of coarsely quantitative measure of irrationality.

Definition 1.5 (irrationality measure/exponent)**.** *Let $\alpha \in \mathbb{R}\setminus\mathbb{Q}$ be an irrational number. We define the irrationality measure $\mu(\alpha)$ of α to be the infimum of real numbers μ satisfying the following condition: there exists a constant $C(\mu) > 0$ such that $|\alpha - p/q| \geqslant C(\mu)/q^{\mu}$ for all $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. (If such μ does not exist, then $\mu(\alpha) = +\infty$.)*

There are some facts about irrationality measure:

- Dirichlet: $\mu(\alpha) \geqslant 2$ for all $\alpha \in \mathbb{R}\setminus\mathbb{Q}$.

- Borel-Cantelli's lemma implies that $\mu(\alpha) = 2$ for almost all $\alpha \in \mathbb{R}$ with respect to the Lebesgue measure.
- If the simple continued fraction of α is $\alpha = [a_0; a_1, a_2, \ldots]$ and p_n/q_n is the n-th convergent of α, then
$$\mu(\alpha) = 1 + \limsup_{n\to+\infty} \frac{\log q_{n+1}}{\log q_n} = 2 + \limsup_{n\to+\infty} \frac{\log a_{n+1}}{\log q_n}.$$
- A difficult theorem of Roth, 1955: $\mu(\alpha) = 2$ for every $\alpha \in \overline{\mathbb{Q}} \setminus \mathbb{Q}$.
- Euler: $e = [2; \overline{(1, 2k, 1)_{k=1}^{\infty}}]$, so $\mu(e) = 2$.
- Zeilberger, Zudilin, 2020 [ZZ20]: $\mu(\pi) \leqslant 7.10320533\ldots$.

Although we know that $\mu(\alpha) = 2$ for almost all irrational numbers α, it is difficult to decide $\mu(\alpha)$ for a given α like $\log 2$ or π. Nevertheless, if we have a sequence of 'good' linear forms in 1 and α, then we are possible to obtain an upper bound of α. Such theorems are similar in spirit to Theorem 1.2 and Theorem 1.3.

Theorem 1.6. *Let $\tau > 0$ be a constant and $\sigma : \mathbb{N} \to \mathbb{R}_{>0}$ a function satisfying*
$$\lim_{n\to+\infty} \sigma(n) = +\infty, \quad \text{and} \quad \lim_{n\to+\infty} \frac{\sigma(n+1)}{\sigma(n)} = 1.$$
Let $\alpha \in \mathbb{R}$. Assume that there exist two sequences of integers $\{a_n\}_{n\geqslant1}$, $\{b_n\}_{n\geqslant1}$ such that (as $n \to +\infty$)

- $|a_n| \leqslant \exp((1 + o(1))\sigma(n))$;
- $0 < |a_n\alpha - b_n| \leqslant \exp(-(\tau + o(1))\sigma(n))$;
- $a_n b_{n+1} \neq a_{n+1} b_n$ *for all* $n \in \mathbb{N}$.

Then $\alpha \notin \mathbb{Q}$ and
$$\mu(\alpha) \leqslant 1 + \frac{1}{\tau}.$$

Theorem 1.7 (Hata, 1993 [Hat93])**.** *Let $\tau > 0$ be a constant and $\sigma : \mathbb{N} \to \mathbb{R}_{>0}$ a function satisfying*
$$\lim_{n\to+\infty} \sigma(n) = +\infty, \quad \text{and} \quad \lim_{n\to+\infty} \frac{\sigma(n+1)}{\sigma(n)} = 1.$$
Let $\alpha \in \mathbb{R}$. Assume that there exist two sequences of integers $\{a_n\}_{n\geqslant1}$, $\{b_n\}_{n\geqslant1}$ such that (as $n \to +\infty$)

- $|a_n| = \exp((1 + o(1))\sigma(n))$;
- $0 < |a_n\alpha - b_n| \leqslant \exp(-(\tau + o(1))\sigma(n))$.

Then $\alpha \notin \mathbb{Q}$ and
$$\mu(\alpha) \leqslant 1 + \frac{1}{\tau}.$$

2. Apéry's theorem: $\zeta(3) \notin \mathbb{Q}$

In 1978, Apéry [Apé79] made a breakthrough.

Theorem 2.1 (Apéry's theorem)**.**

$$\zeta(3) \notin \mathbb{Q}.$$

See van der Poorten's report [Poo79] for some history and see Fischler's Bourbaki seminar notes [Fis04] for a survey. Nowadays, there are a lot of proofs.

- Apéry, 1978: original, astonishing, miraculous, ...
  - "This is marvellous! It is something Euler could have done ..." – N. Katz
  - "A proof that Euler missed..." – A. van der Poorten
- Beukers, 1979: elegant. It used integrals over $[0,1]^3$.
- Beukers, 1987: used modular forms.
  - "... it seemed that my fate ... was closely linked with work of Roger Apéry." – F. Beukers, said in 2003.
- Nesterenko, 1996: used summation of derivatives of rational functions.

We will repeat the details of Beukers' proof and take a look at other proofs including Apéry's.

2.1. Beukers' proof.

To illustrate Beukers' proof, we first go through an example.

Example 2.2. *A proof that* $\log 2 \notin \mathbb{Q}$.

Proof. Denote by $d_n = [1, 2, \dots, n]$ the least common multiple of the first n positive integers. For any $P(X) \in \mathbb{Z}[X]$ with $\deg P \leqslant n$, we can write $P(X) = (X+1)Q(X) + P(-1)$ for some $Q(X) \in \mathbb{Z}[X]$ with $\deg Q \leqslant n-1$. Hence,

$$\begin{aligned} \int_0^1 \frac{P(x)}{1+x}\mathrm{d}x &= \int_0^1 Q(x)\mathrm{d}x + P(-1)\int_0^1 \frac{1}{1+x}\mathrm{d}x \\ &\in \frac{\mathbb{Z}}{1} + \frac{\mathbb{Z}}{2} + \cdots + \frac{\mathbb{Z}}{n} + \mathbb{Z}\log 2 \\ &\subset \frac{\mathbb{Z}}{d_n} + \mathbb{Z}\log 2 \end{aligned} \tag{2.1}$$

We take (Legendre-type polynomials)

$$P_n(X) = \frac{1}{n!}\left(\frac{\mathrm{d}}{\mathrm{d}X}\right)^n (X^n(1-X)^n), \quad n = 1, 2, \dots,$$

and denote $I_n = \int_0^1 (P(x)\mathrm{d}x)/(1+x)$. Note that $P_n(X) \in \mathbb{Z}[X]$ and $\deg P_n = n$, so by (2.1) we have $I_n \in \mathbb{Z}/d_n + \mathbb{Z}\log 2$. By an n-fold partial integration, we deduce that

$$I_n = \int_0^1 \frac{x^n(1-x)^n}{(1+x)^{n+1}}\mathrm{d}x.$$

A straightforward computation shows that $x(1-x)/(1+x) \leqslant 3 - 2\sqrt{2}$ for all $x \in [0,1]$, which implies $0 < I_n < (3-2\sqrt{2})^n$, $n = 1, 2, \dots$. By the prime number theorem, $d_n = \exp((1+o(1))n)$, so $d_n \leqslant 4^n$ for all sufficiently large n. Thus,

$$\begin{cases} 0 < d_n I_n < (4(3-2\sqrt{2}))^n < 1, & \text{for all } n \geqslant n_0 \\ \quad d_n I_n \in \mathbb{Z} + \mathbb{Z} d_n \log 2 \end{cases},$$

the conclusion $\log 2 \notin \mathbb{Q}$ follows. □

We see in the above example that $1/(1+x)$ is the right 'kernel' to produce $\log 2$. Beukers found a right 'kernel' for $\zeta(3)$.

Lemma 2.3. *Denote $d_n = [1, 2, \ldots, n]$. Let $r, s \in \mathbb{Z}_{\geqslant 0}$.*

(1) *If $r > s$, then*
$$\int_{[0,1]^2} \frac{-\log(xy)}{1-xy} x^r y^s \mathrm{d}x\mathrm{d}y \in \frac{\mathbb{Z}}{d_r^3};$$

(2) *If $r = s$, then*
$$\int_{[0,1]^2} \frac{-\log(xy)}{1-xy} x^r y^r \mathrm{d}x\mathrm{d}y \in \frac{\mathbb{Z}}{d_r^3} + \mathbb{Z}\zeta(3).$$

Proof. Let $\sigma \geqslant 0$. We have

$$\int_{[0,1]^2} \frac{x^{r+\sigma} y^{s+\sigma}}{1-xy} \mathrm{d}x\mathrm{d}y = \sum_{k=0}^{\infty} \frac{1}{(k+r+\sigma+1)(k+s+\sigma+1)} \tag{2.2}$$

by $(1-xy)^{-1} = \sum_{k\geqslant 0} x^k y^k$ and Levi's lemma.

If $r > s$, the sum in the right hand side of (2.2) is telescoping:

$$\begin{aligned} \text{RHS of (2.2)} &= \sum_{k=0}^{\infty} \frac{1}{r-s} \left(\frac{1}{k+s+\sigma+1} - \frac{1}{k+r+\sigma+1} \right) \\ &= \frac{1}{r-s} \left(\frac{1}{s+1+\sigma} + \frac{1}{s+2+\sigma} + \cdots + \frac{1}{r+\sigma} \right). \end{aligned}$$

Differentiate (2.2) with respect to σ and put $\sigma = 0$, we obtain that

$$\int_{[0,1]^2} \frac{\log(xy)}{1-xy} x^r y^s \mathrm{d}x\mathrm{d}y = \frac{-1}{r-s} \left(\frac{1}{(s+1)^2} + \frac{1}{(s+2)^2} + \cdots + \frac{1}{r^2} \right),$$

(we can interchange the order of integration and differentiation on LHS because $-\log(xy)/(1-xy)$ is a dominating function on $[0,1]^2$ for difference quotients) which proves (1). If $r = s$, then (2.2) becomes

$$\int_{[0,1]^2} \frac{x^{r+\sigma} y^{r+\sigma}}{1-xy} \mathrm{d}x\mathrm{d}y = \sum_{k=0}^{\infty} \frac{1}{(k+r+\sigma+1)^2}. \tag{2.3}$$

Differentiate (2.3) with respect to σ and put $\sigma = 0$,

$$\begin{aligned} \int_{[0,1]^2} \frac{\log(xy)}{1-xy} x^r y^r \mathrm{d}x\mathrm{d}y &= -2 \sum_{k=0}^{\infty} \frac{1}{(k+r+1)^3} \\ &= -2 \left(\zeta(3) - \frac{1}{1^3} - \frac{1}{2^3} - \cdots - \frac{1}{r^3} \right), \end{aligned}$$

then (2) follows. □

Now, we can give Beukers' proof of Apéry's theorem.

Proof of Theorem 2.1. We take the (Legendre-type) polynomials

$$P_n(X) = \frac{1}{n!}\left(\frac{\mathrm{d}}{\mathrm{d}X}\right)^n (X^n(1-X)^n) \in \mathbb{Z}[X], \quad \deg P_n = n, \quad n = 1, 2, \ldots,$$

and consider the integrals

$$I_n = \int_{[0,1]^2} \frac{-\log(xy)}{1-xy} P_n(x)P_n(y)\mathrm{d}x\mathrm{d}y.$$

By Lemma 2.3, we have $I_n \in \mathbb{Z}/d_n^3 + \mathbb{Z}\zeta(3)$.

Noticing that

$$\frac{-\log(xy)}{1-xy} = \int_0^1 \frac{1}{1-(1-xy)z}\mathrm{d}z,$$

we can write (thanks to Fubini-Tonelli's theorem)

$$I_n = \int_0^1\int_0^1\int_0^1 \frac{P_n(x)P_n(y)}{1-(1-xy)z}\mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

After an n-fold partial integration with respect to x, we have

$$I_n = \int_0^1\int_0^1\int_0^1 \frac{(xyz)^n(1-x)^nP_n(y)}{(1-(1-xy)z)^{n+1}}\mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

Now, making the following change of variable

$$w = \frac{1-z}{1-(1-xy)z}$$

we obtain that

$$I_n = \int_0^1\int_0^1\int_0^1 (1-x)^n(1-w)^n\frac{P_n(y)}{1-(1-xy)w}\mathrm{d}x\mathrm{d}y\mathrm{d}w.$$

Then, an n-fold partial integration with respect to y implies that

$$I_n = \int_{[0,1]^3} \frac{x^n(1-x)^ny^n(1-y)^nw^n(1-w)^n}{(1-(1-xy)w)^{n+1}}\mathrm{d}x\mathrm{d}y\mathrm{d}w. \tag{2.4}$$

It is routine to check that

$$\frac{x(1-x)y(1-y)w(1-w)}{1-(1-xy)w} \leqslant (\sqrt{2}-1)^4, \quad \text{for all } (x,y,w) \in (0,1)^3,$$

so we deduce from (2.4) that

$$0 < I_n \leqslant (\sqrt{2}-1)^{4n}\int_{[0,1]^3}\frac{\mathrm{d}x\mathrm{d}y\mathrm{d}w}{1-(1-xy)w} = 2\zeta(3)(\sqrt{2}-1)^{4n}.$$

Recall that $d_n = \exp((1+o(1))n)$ as $n \to +\infty$ by the prime number theorem, so

$$\begin{cases} 0 < d_n^3 I_n \leqslant ((\sqrt{2}-1)^4e^3)^{(1+o(1))n} \\ \qquad d_n^3 I_n \in \mathbb{Z} + \mathbb{Z}d_n^3\zeta(3) \end{cases}.$$

Note that $(\sqrt{2}-1)^4e^3 < 1$, from above $\zeta(3) \notin \mathbb{Q}$ follows easily. $\square$

2.2. **Glimpses of other proofs.** A sketch of Apéry's ideas (1978):

- We define two sequences of integers

$$a_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2,$$
$$b_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \left(\sum_{m=1}^{n} \frac{1}{m^3} + \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3 \binom{n}{m}\binom{n+m}{m}} \right).$$

- $a_n \in \mathbb{Z}$, $2d_n^3 b_n \in \mathbb{Z}$. $\quad (d_n = [1, 2, \ldots, n].)$
- $\{a_n\}$, $\{b_n\}$ both satisfy the recurrence

$$(n+1)^3 X_{n+1} - (34n^3 + 51n^2 + 27n + 5)X_n + n^3 X_{n-1} = 0$$

-

$$a_n = (\sqrt{2}+1)^{(4+o(1))n} \text{ as } n \to +\infty,$$
$$a_n\zeta(3) - b_n = \sum_{k=n+1}^{\infty} \frac{6a_n}{k^3 a_k a_{k+1}},$$
$$0 < a_n\zeta(3) - b_n = (\sqrt{2}-1)^{(4+o(1))n} \text{ as } n \to +\infty.$$

In 1981, Beukers [Beu81] explained Apéry's numbers a_n by Hermite-Padé approximation. Let $L_k(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^k}$ be the polylogarithm ($k \in \mathbb{N}$, $|z| < 1$), then $L_k(1) = \zeta(k)$ for $k \geqslant 2$. We look for polynomials $A_n(z), B_n(z), C_n(z), D_n(z) \in \mathbb{Q}[z]$ of degree n such that

$$\begin{cases} A_n(z)L_2(z) + B_n(z)L_1(z) + C_n(z) = O(z^{2n+1}) \\ 2A_n(z)L_3(z) + B_n(z)L_2(z) + D_n(z) = O(z^{2n+1}) \\ B_n(1) = 0 \end{cases}.$$

Suppose that $A_n(z) = \sum_{k=0}^{n} \alpha_k z^k$, $B_n(z) = \sum_{k=0}^{n} \beta_k z^k$. It turns out that up to proportion, $\alpha_k = \binom{n}{k}^2\binom{2n-k}{n}^2$, so

$$A_n(1) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2n-k}{n}^2 = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = a_n.$$

Moreover, $2A_n(1)\zeta(3) + D_n(1) =$ remainder, and

$$\sum_{k=0}^{n} \left(\frac{\alpha_k}{(t-k)^2} + \frac{\beta_k}{t-k} \right) = \frac{(t-n-1)^2(t-n-2)^2 \cdots (t-2n^2)}{t^2(t-1)^2 \cdots (t-n)^2}.$$

In 1996, Nesterenko [Nes96] gave a self-contained proof of Apéry's theorem by starting directly from

$$\sum_{k=1}^{\infty} R_n'(k), \quad \text{where } R_n(t) = \frac{(t-1)^2(t-2)^2 \cdots (t-n)^2}{t^2(t+1)^2 \cdots (t+n)^2}.$$

There is a rather different proof of Apéry's theorem by Beukers in 1987, he used modular forms [Beu87]. In 2005, Calegari [Cal05] used p-adic modular forms to show that p-adic zeta value $\zeta_p(3) \notin \mathbb{Q}$ for $p = 2, 3$.

3. Ball-Rivoal's theorem: $\dim_{\mathbb{Q}} \mathrm{Span}_{\mathbb{Q}}\left(1, \zeta(3), \zeta(5), \ldots, \zeta(s)\right) \geqslant \frac{1-o(1)}{1+\log 2} \log s$

In 2000, Rivoal [Riv00], Ball and Rivoal [BR01] made a significant progress.

Theorem 3.1 (Ball-Rivoal's theorem)**.** *For any $\varepsilon > 0$, there exists $s_0(\varepsilon)$ such that for all odd integers $s \geqslant s_0(\varepsilon)$, the dimension of the space spanned by $1, \zeta(3), \zeta(5), \ldots, \zeta(s)$ over $\mathbb{Q}$ is at least that*

$$\dim_{\mathbb{Q}} \mathrm{Span}_{\mathbb{Q}}\left(1, \zeta(3), \zeta(5), \ldots, \zeta(s)\right) \geqslant \frac{1-\varepsilon}{1+\log 2} \log s.$$

As a (much weaker) corollary, there are infinitely many positive integers k such that $\zeta(2k+1) \notin \mathbb{Q}$.

3.1. **Possible motivations.** In this subsection, we (try to) explain some possible motivations of Ball-Rivoal's theorem. Recall Beukers' integrals (see (2.4))

$$\int_{[0,1]^3} \frac{x^n(1-x)^n y^n(1-y)^n z^n(1-z)^n}{(1-(1-(1-x)y)z)^{n+1}} \mathrm{d}x\mathrm{d}y\mathrm{d}z \in \mathbb{Q} + \mathbb{Q}\zeta(3). \tag{3.1}$$

One natural generalization of (3.1) is that (for odd $s \geqslant 3$)

$$\int_{[0,1]^s} \frac{x_1^n(1-x_1)^n x_2^n(1-x_2)^n \cdots x_s^n(1-x_s)^n}{(1-(\cdots(1-(1-x_1)x_2)\cdots)x_s)^{n+1}} \mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_s \in \mathrm{Span}_{\mathbb{Q}}\left(1, \zeta(3), \zeta(5), \ldots, \zeta(s)\right). \tag{3.2}$$

However, this conclusion (3.2) is not easy. It was proved by Vasilyev [Vas96] for $s = 5$ and by Zudilin [Zud03, §8] for general cases by an identity between a certain multiple integral and a non-terminating very-well-poised hypergeometric series.

But there is a trivial 'generalization' of (3.1). For any $P(X_1, X_2, \ldots, X_s) \in \mathbb{Q}[X_1, X_2, \ldots, X_s]$ we have

$$\int_{[0,1]^s} \frac{P(x_1, x_2, \ldots, x_s)}{(1-x_1x_2\ldots x_s)^N} \mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_s \in \mathrm{Span}_{\mathbb{Q}}\left(1, \zeta(2), \zeta(3), \ldots, \zeta(s)\right), \tag{3.3}$$

provided that the left hand side is integrable. To see this, after expanding $(1-x_1x_2\cdots x_s)^{-N} = \sum_{k=0}^{\infty} \binom{N+k-1}{N-1} x_1^k x_2^k \cdots x_s^k$ and integrating term by term, the left hand side of (3.3) can be written as $\sum_{k=0}^{\infty} R(k)$ for some rational function $R(t)$, whose poles lie in $\mathbb{Z}_{\leqslant 0}$ and the order of any pole is not greater than s. Then it is easy to show that (by partial fraction decomposition) $\sum_{k=0}^{\infty} R(k)$ is a linear form in 1 and zeta values (see later Lemma).

What if we start directly from $\sum_{k=0}^{\infty} R(k)$ for such rational function $R(t)$ above? Recall Nesterenko had already considered

$$R_n(t) = \frac{(t-1)^2(t-2)^2\cdots(t-n)^2}{t^2(t+1)^2\cdots(t+n)^2},$$

he showed that $\sum_{k=1}^{\infty} R_n'(k) = a_n'\zeta(3) - b_n'$ is small, and the denominators of the rational coefficients a_n', b_n' are not too large. Then Nesterenko managed to give a new proof that $\zeta(3) \notin \mathbb{Q}$. In an email from Ball to Rivoal (see Rivoal's thesis), Ball considered

$$R_n^{\mathrm{B}}(t) = n!^2 \left(t + \frac{n}{2}\right) \frac{(t-n)_n(t+n+1)_n}{(t)_{n+1}^4},$$

where $(\alpha)_m = (\alpha)(\alpha+1)\cdots(\alpha+m-1)$ is the rising factorial (Pochhammer symbol). By last paragraph, we know that $\sum_{k=1}^{\infty} R_n^{\mathrm{B}}(k) \in \mathbb{Q} + \mathbb{Q}\zeta(2) + \mathbb{Q}\zeta(3) + \mathbb{Q}\zeta(4)$. Ball and Rivoal found that the symmetry of $R_n^{\mathrm{B}}(t)$ will eliminate the even zeta values $\zeta(2), \zeta(4)$, so $\sum_{k=1}^{\infty} R_n^{\mathrm{B}}(k)$ is in fact belonging to $\mathbb{Q} + \mathbb{Q}\zeta(3)$. To generalize $R_n^{\mathrm{B}}(t)$ (importantly, keep the symmetry), Rivoal considered

$$R_n^{\mathrm{BR}}(t) = n!^{s-2r} \frac{(t-rn)_{rn}(t+n+1)_{rn}}{(t)_{n+1}^s},$$

where $r < s/2$ is an integer parameter. An overview of the proof of Theorem 3.1:

- Symmetry ($R_n^{\mathrm{BR}}(t) = -R_n^{\mathrm{BR}}(-t-n)$) kills all even zeta values $\zeta(2k)$. We have (for any even n)
$$S_n := \sum_{k=1}^{\infty} R_n^{\mathrm{BR}}(k) = \rho_0 + \sum_{\substack{3 \leqslant i \leqslant s \\ i \text{ odd}}} \rho_i \zeta(i),$$
where $\rho_i = \rho_i^{(n)}$ are rational coefficients depend on n.
- The common denominator of ρ_i is not large: $d_n^s \rho_i \in \mathbb{Z}$ for all $i = 0, 3, 5, \ldots, s$. ($d_n = [1, 2, \ldots, n]$.)
- We have good estimates for S_n and ρ_i:
$$\limsup_{n \to +\infty} |\rho_i|^{1/n} \leqslant 2^{s-2r}(2r+1)^{2r+1}, \text{for all } i;$$
$$\lim_{n \to +\infty} |S_n|^{1/n} \overset{\text{exists}}{=\!=\!=} c = c(s,r) \text{ and } 0 < c \leqslant \frac{2^{r+1}}{r^{s-2r}}.$$
- By applying Nesterenko's linear independence criterion (Theorem 1.3) to $\{d_n^s S_n\}_{n \text{ even}}$ and optimizing the parameter r, we complete the proof of Theorem 3.1.

How to make S_n small and nonzero? Our method is that letting $R_n(k) = 0$ for $k = 1, 2, \ldots, rn$ and $R_n(k) > 0$ for $k > rn$. Thus $R_n(t)$ needs to have the factor $(t-rn)_{rn}$. In order to keep symmetry, $R_n(t)$ also has to have the factor $(t+n+1)_{rn}$. In this sense, the construction of $R_n^{\mathrm{BR}}(t)$ is perhaps the most natural choice.

3.2. **From rational functions to odd zeta values.**

3.3. **Arithmetic of coefficients.**

3.4. **Asymptotics.**

3.5. **Proof of Theorem 3.1.**

4. FSZ-LY: many odd zeta values are irrational

Theorem 4.1 ([FSZ19][LY20]). *For any $\varepsilon > 0$, there exists $s_0(\varepsilon)$ such that for all odd integers $s \geqslant s_0(\varepsilon)$, it holds that*

$$\#\left\{k : 3 \leqslant k \leqslant s, k \text{ odd}, \zeta(k) \notin \mathbb{Q}\right\} \geqslant (c_0 - \varepsilon)\sqrt{\frac{s}{\log s}},$$

where the constant

$$c_0 = \sqrt{\frac{4\zeta(2)\zeta(3)}{\zeta(6)}\left(1 - \log\frac{\sqrt{4e^2+1}-1}{2}\right)} \approx 1.192507\ldots.$$

5. A result of Zudilin: at least one of $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is irrational

Theorem 5.1 ([Zud01])**.** *At least one of the four numbers*

$$\zeta(5), \zeta(7), \zeta(9), \zeta(11)$$

is irrational.

6. A result of Sprang: p-adic analogue of Ball-Rivoal's theorem

Theorem 6.1 ([Spr20]).

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