Empirical Methods in Finance MFE230E
Week 6: Testing Linear Factor Models

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Haas School of Business

THIS WEEK'S LECTURE

1. Quick review of the CAPM
2. Testing the CAPM: Theory
3. Testing the CAPM: Empirical evidence
4. Fama-French style multifactor models
5. Testing general factor models

Reading: Ruppert ch. 16, 17.4-17.6

The cross-section of stock returns

- Question: How many companies are publicly traded in the U.S./world-wide?
- Objectives:
  - Summarize the most important patterns in a parsimonious way
  - Formulate and test specific theories about what risk factors drive stocks
- Historically most important model: CAPM
- We will also discuss:
  - Multifactor models
  - General linear factor models
  - Models with macroeconomic risk factors
- 2013 Nobel Prize: Gene Fama for economic models of risk, Lars Hansen for econometric tests

Outline

1. **Review of CAPM**
2. Testing the CAPM: Theory
3. Testing the CAPM: Empirical evidence
4. Tests of Fama-French style multifactor models

The Capital Asset Pricing Model (CAPM) I

- Derivation of CAPM: See Investments class
- According to the CAPM, there is only one source of risk: **Market risk**
- Investors are compensated for exposure to undiversifiable market risk
- Only market risk matters for expected returns

- CAPM equation:

$$E[R_{i,t} - R_f] = \beta_i E[R_{M,t} - R_f]$$

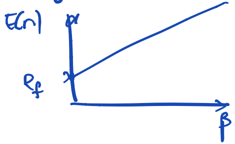
$$\beta_i = \frac{\text{Cov}(R_{i,t}, R_{M,t})}{\text{Var}(R_{M,t})}$$

- Question: Is it possible that $E[R_{i,t}] < R_f$?

Insurance, hedging
mutual fund that shorts the market
put options

- The CAPM has two dimensions:
  1. Time series given an asset i
  2. Cross-section: Do assets with different β's have different excess returns?

Stock with higher β should have higher returns



The CAPM can be written as a linear time-series regression:

$$R_{i,t} - R_f = \alpha_i + \beta_i (R_{M,t} - R_f) + \epsilon_{i,t}$$

$$\text{Cov}(R_{M,t}, \epsilon_{i,t}) = 0$$

α_i is called the **pricing error**

- If the CAPM is true:

$$\alpha_i = 0$$

- Note: The CAPM should hold for **any** asset!

Objective of today's class: Derive statistical tests

Cross-sectional implications of the CAPM

The CAPM can also be written as a **cross-sectional** relationship:

$$\text{CAPM:} \quad \mathrm{E}[R_{i,t} - R_f] = \beta_i \mathrm{E}[R_{M,t} - R_f]$$

- Suppose you know average risk premia of 100 assets and their β's.
  How could you check whether the data is consistent with the CAPM?
- *Only* market risk measured by β determines an asset's risk premium
  Question: How can we check whether this implication is true in the data?

Outline

1. Review of CAPM
2. **Testing the CAPM: Theory**
3. Testing the CAPM: Empirical evidence
4. Tests of Fama-French style multifactor models

- CAPM for risk premia $R^e_{i,t} = R_{i,t} - R_{f,t}$:

$$E[R^e_{i,t}] = \beta_i E[R^e_{M,t}]$$

- Regression

$$R^e_{i,t} = \alpha_i + \beta_i R^e_{M,t} + \varepsilon_{i,t}$$

- The CAPM hypothesis:

$$H_0 : \alpha_i = 0 \ \ \forall i$$

- N assets: Test whether α_i are **jointly** zero
- Test statistics can be derived if
  - $\epsilon_{i,t}$ are jointly normally distributed, finite sample
  - No distributional assumptions, asymptotic test

We can apply the results from OLS:

$$R^e_{i,t} = \alpha_i + \beta_i R^e_{M,t} + \varepsilon_{i,t+1}$$

$$y_t = \mathbf{x}_t' \boldsymbol{b} + e_t$$

$$y_t = R^e_{i,t}$$

$$\mathbf{x}_t = (1, R^e_{M,t})'$$

$$\boldsymbol{b} = (\alpha_i, \beta_i)'$$

If we assume that the errors are i.i.d and homoskedastic:

$$\sqrt{T}\,(\hat{\boldsymbol{b}} - \boldsymbol{b}) \xrightarrow{d} N(0, \sigma^2 \boldsymbol{\Sigma}_{xx}^{-1})$$

$$S_{xx} \equiv \frac{1}{T}\mathbf{X}'\mathbf{X} \rightarrow \boldsymbol{\Sigma}_{xx} = \mathrm{E}[\mathbf{x}_t'\mathbf{x}_t]$$

The statistical test is a straightforward application of OLS. But it will be useful to understand the intuition of this test in more detail.

The χ^2 test statistic for $H_0 : \alpha_i = 0$ is $\frac{\alpha_i^2}{\mathrm{Var}(\alpha_i)}$.

Let's derive $\mathrm{Var}(\alpha_i)$ explicitly:

$$\mathbf{\Sigma}_{xx} = \begin{pmatrix} 1 & \mathrm{E}[R^e_{M,t}] \\ \mathrm{E}[R^e_{M,t}] & \mathrm{E}[(R^e_{M,t})^2] \end{pmatrix} \qquad \mathbf{x}_t = (1, R^e_{M,t})' \quad \mathbf{\Sigma}_{xx} = \mathrm{E}[\mathbf{x}_t'\mathbf{x}_t]$$

$$\mathbf{\Sigma}_{xx}^{-1} = \frac{1}{\mathrm{Var}\left(R^e_{M,t}\right)} \begin{pmatrix} \mathrm{E}[(R^e_{M,t})^2] & -\mathrm{E}[R^e_{M,t}] \\ -\mathrm{E}[R^e_{M,t}] & 1 \end{pmatrix}$$

$$\text{Var}\begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \end{pmatrix} = \frac{\sigma_\varepsilon^2}{T \times \text{Var}\left(R^e_{M,t}\right)} \begin{pmatrix} \text{E}[(R^e_{M,t})^2] & -\text{E}[R^e_{M,t}] \\ -\text{E}[R^e_{M,t}] & 1 \end{pmatrix}$$

because

$$\cancel{\sqrt{T}}\,(\hat{\mathbf{b}} - \mathbf{b}) \xrightarrow{d} N(0, \frac{\sigma^2}{T}\boldsymbol{\Sigma}_{xx}^{-1})$$

$$\text{Var}(\hat{\alpha}_i) = \frac{\hat{\sigma}_\epsilon^2}{T \times \text{Var}\left(R^e_{M,t}\right)} \text{E}[(R^e_{M,t})^2]$$

top left term

$$= \left(\frac{\text{E}[(R^e_{M,t})]^2 + \text{Var}\left(R^e_{M,t}\right)}{\text{Var}\left(R^e_{M,t}\right)} \right) \frac{\hat{\sigma}_\epsilon^2}{T}$$

$$= \left(1 + \frac{\text{E}[(R^e_{M,t})]^2}{\text{Var}\left(R^e_{M,t}\right)} \right) \frac{\hat{\sigma}_\epsilon^2}{T} = \left(1 + \text{SR}_M^2\right) \frac{\hat{\sigma}_\epsilon^2}{T}$$

SR_M is the Sharpe-ratio of the mkt. portfolio and $\hat{\sigma}_\epsilon^2$ is an estimator of σ_ϵ^2.

THE CAPM TEST STATISTIC FOR ONE ASSET

What is the appropriate t-test statistic for the null hypothesis $\alpha_i = 0$?

$$\sqrt{T}\,(\hat{\boldsymbol{b}} - \boldsymbol{b}) \xrightarrow{d} N(0, \sigma^2 \boldsymbol{\Sigma}_{xx}^{-1})$$

$$\frac{\hat{\alpha}_i}{\sqrt{\text{Var}\left(\hat{\alpha}_i\right)}} = \sqrt{T}\,\frac{\hat{\alpha}_i}{\hat{\sigma}_\epsilon\,\sqrt{1 + \widehat{\text{SR}}_M^2}} \xrightarrow{d} N(0, 1)$$

THE CAPM TEST STATISTIC FOR ONE ASSET

We could also use a χ^2 test for the square of α_i:

$$T \frac{\hat{\alpha}_i^2}{\hat{\sigma}_\epsilon^2 \left(1 + \widehat{\mathrm{SR}}_M^2\right)} \xrightarrow{d} \chi_1^2$$

Note: This test can be applied to any linear factor model as long the factor F_t is an excess return

$$R_{i,t}^e = \alpha_i + \beta_i F_t + \varepsilon_{i,t}$$

1. We have N assets
2. Run N separate regressions for each asset i:

$$R^e_{i,t} = \alpha_i + \beta_i R^e_{M,t} + \varepsilon_{i,t}$$

3. Calculate the sample covariance matrix of the residuals $\hat{\boldsymbol{\Sigma}}$

4. Gibbons, Ross, and Shanken (GRS) (1987):

$$\frac{T-N-1}{N}\,\frac{\hat{\boldsymbol{\alpha}}'\hat{\boldsymbol{\Sigma}}^{-1}\hat{\boldsymbol{\alpha}}}{1+\widehat{SR}_M^2} \sim F(N, T-N-1)$$

$(N\times1)'$ 100×100

$$T\,\frac{\hat{\boldsymbol{\alpha}}'\hat{\boldsymbol{\Sigma}}^{-1}\hat{\boldsymbol{\alpha}}}{1+\widehat{SR}_M^2} \rightarrow \chi_N^2$$

χ^2 version

Note: The χ^2 statistic can be derived by GMM. Idea: Stack all N moment conditions in a vector.

Sharpe-ratio of the market: SR_M

Highest Sharpe-ratio of all portfolios constructed by N assets: SR_N

GRS show

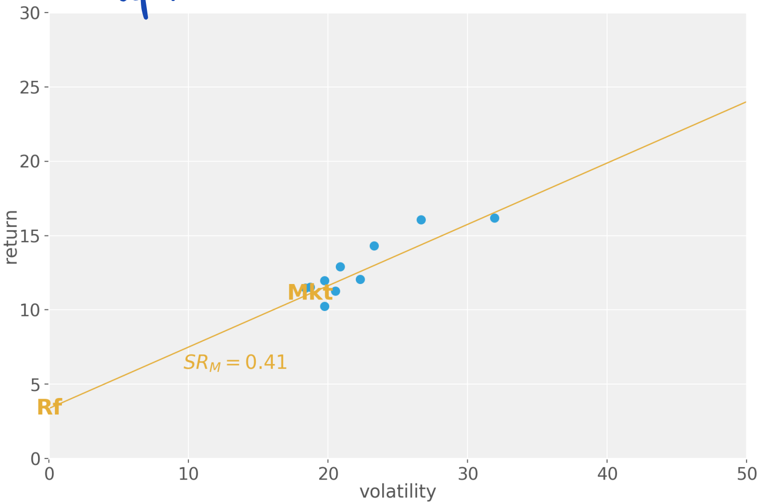
$$\frac{\hat{\boldsymbol{\alpha}}'\hat{\boldsymbol{\Sigma}}^{-1}\hat{\boldsymbol{\alpha}}}{1+\widehat{\mathrm{SR}}_M^2} = \frac{1+\widehat{\mathrm{SR}}_N^2}{1+\widehat{\mathrm{SR}}_M^2} - 1 \Leftrightarrow$$

testing whether sharp ratio of markets is smaller than sharp ratio of portfolio

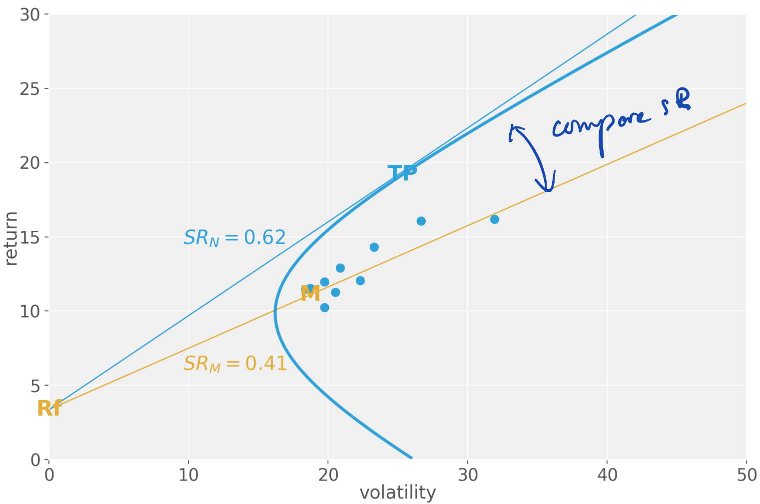
reject if SR_p is $\gg SR_M$

Thus the GRS test is equivalent to testing whether the market portfolio is mean-variance efficient!

Capital market line



Interpretation of GRS test statistic



MULTIFACTOR MODELS

- The model can be easily generalized to include multiple risk factors:

$$R^e_{i,t} = \alpha_i + \beta_{1,i} f_{1,t} + \ldots + \beta_{K,i} f_{K,t} + \varepsilon_{i,t}$$

- For now: all factors are excess returns
- Multifactor models come in different flavors:
  - Multi-period extensions of the CAPM: Intertemporal CAPM or I-CAPM
    Theory often implies additional structure of the factors
  - Arbitrage Pricing Theory: statistical descriptions of the data
    only assumption: no arbitrage

GRS test for multifactor models:

$$T \frac{\hat{\boldsymbol{\alpha}}' \hat{\boldsymbol{\Sigma}}^{-1} \hat{\boldsymbol{\alpha}}}{1 + \underbrace{\boldsymbol{\mu}_F' \widehat{\boldsymbol{\Omega}}_F^{-1} \boldsymbol{\mu}_F}_{\text{sharp ratio created by the factors}}} \rightarrow \chi_N^2$$

sum of N normals

where $\boldsymbol{\mu}_F$ and $\widehat{\boldsymbol{\Omega}}_F$ are mean and the variance covariance matrix of $\boldsymbol{F}_t = (f_{1,t}, \ldots, f_{k,t})'$.

MULTIFACTOR MODELS

- Important question: What are these factors?
  - Economic factors, business cycle variables (Chen, Roll, Ross, 1986)
  - Statistical factors: principal components, dynamic factor analysis
  - Fama-French style factors (Question: Who knows what the 3-factor Fama-French model is?)
- We will look at some of these models later

Empirical tests of the CAPM

- How do we measure the market portfolio R_M?
- How do we measure the risk free interest rate R_f?
- We know that the equity premium is changing over time. How are CAPM tests affected?
- We know that the variance of stock returns is varying over time. How about covariances and beta?
- How does the CAPM perform in practice?
- We assumed so far that the risk factors are excess returns. How can we test models in which this is not the case (e.g. models of macroeconomic risk)?

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Cross-sectional tests in practice: Estimating a firm's β

Suppose we want to estimate the β of IBM using monthly data.

Tradeoff: Longer vs. shorter sample?

What are the pros/cons of longer/shorter samples?

Example: IBM was listed on the NYSE on 11/11/1915 $\longrightarrow$ Over 100 years of available data

International Business Machines Corporation (IBM)
NYSE - NYSE Delayed Price. Currency in USD

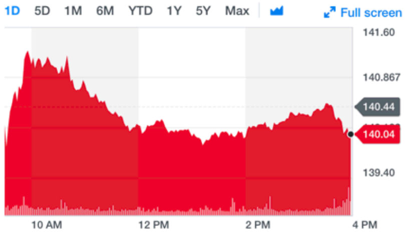
Add to watchlist

139.95 -0.49 (-0.35%) **139.63** -0.32 (-0.23%) Buy Sell

At close: 4:01PM EDT After hours: 7:39PM EDT

Summary | Chart | Conversations | Statistics | Historical Data | Profile | Financials | Analysis | Options | Holders | Sustainability

Previous Close	140.44	Market Cap	124.537B
Open	140.60	Beta (3Y Monthly)	1.68
Bid	139.64 x 800	PE Ratio (TTM)	14.73
Ask	139.80 x 800	EPS (TTM)	9.50
Day's Range	139.78 - 141.31	Earnings Date	Jul 17, 2019
52 Week Range	105.94 - 154.36	Forward Dividend & Yield	6.28 (4.45%)
Volume	2,380,494	Ex-Dividend Date	2019-02-07
Avg. Volume	3,863,104	1y Target Est	143.72



CAPM β: IBM

- Let's see whether we get the same beta as Yahoo
- Pull data from Yahoo API using pandas datareader
- Tickers:
  - IBM: IBM
  - ˆGSPC: S&P500
  - ˆIRX: Yield of 13-week T-bill

```
start = datetime.datetime(1963, 3, 31)
stocks = ['IBM', '^GSPC', '^IRX']
df = web.DataReader(stocks, 'yahoo', start)
```

ESTIMATING A FIRM'S β: 3-YEAR SAMPLE

Dep. Variable:	IBM	**R-squared:**	0.587
Model:	OLS	**Adj. R-squared:**	0.575
Method:	Least Squares	**F-statistic:**	17.83
Date:	Thu, 25 Apr 2019	**Prob (F-statistic):**	0.000171
Time:	16:36:51	**Log-Likelihood:**	62.935
No. Observations:	36	**AIC:**	-121.9
Df Residuals:	34	**BIC:**	-118.7
Df Model:	1		
Covariance Type:	HC0		

	coef	**std err**	**z**	**P>|z|**	**[0.025**	**0.975]**
const	-0.0115	0.009	-1.294	0.196	-0.029	0.006
GSPC	1.6349	0.387	4.222	0.000	0.876	2.394

Omnibus:	2.107	**Durbin-Watson:**	1.828
Prob(Omnibus):	0.349	**Jarque-Bera (JB):**	1.266
Skew:	-0.445	**Prob(JB):**	0.531
Kurtosis:	3.230	**Cond. No.**	32.6

ESTIMATING A FIRM'S β: 5-YEAR SAMPLE

Dep. Variable:	IBM	**R-squared:**	0.306
Model:	OLS	**Adj. R-squared:**	0.294
Method:	Least Squares	**F-statistic:**	19.19
Date:	Wed, 03 May 2017	**Prob (F-statistic):**	5.03e-05
Time:	14:56:09	**Log-Likelihood:**	105.24
No. Observations:	60	**AIC:**	-206.5
Df Residuals:	58	**BIC:**	-202.3
Df Model:	1		
Covariance Type:	HC0		

	coef	**std err**	**z**	**P>|z|**	**[0.025**	**0.975]**
const	-0.0120	0.005	-2.493	0.013	-0.021	-0.003
MKT	0.9559	0.218	4.380	0.000	0.528	1.384

Omnibus:	8.892	**Durbin-Watson:**	2.033
Prob(Omnibus):	0.012	**Jarque-Bera (JB):**	10.981
Skew:	-0.559	**Prob(JB):**	0.00413
Kurtosis:	4.772	**Cond. No.**	34.4

ESTIMATING A FIRM'S β: 10-YEAR SAMPLE

Dep. Variable:	IBM	**R-squared:**	0.322
Model:	OLS	**Adj. R-squared:**	0.317
Method:	Least Squares	**F-statistic:**	31.02
Date:	Wed, 03 May 2017	**Prob (F-statistic):**	1.63e-07
Time:	14:56:17	**Log-Likelihood:**	204.08
No. Observations:	120	**AIC:**	-404.2
Df Residuals:	118	**BIC:**	-398.6
Df Model:	1		
Covariance Type:	HC0		

	coef	**std err**	**z**	**P>|z|**	**[0.025**	**0.975]**
const	0.0018	0.004	0.421	0.674	-0.007	0.010
MKT	0.6948	0.125	5.569	0.000	0.450	0.939

Omnibus:	5.259	**Durbin-Watson:**	1.928
Prob(Omnibus):	0.072	**Jarque-Bera (JB):**	7.509
Skew:	0.085	**Prob(JB):**	0.0234
Kurtosis:	4.214	**Cond. No.**	22.8

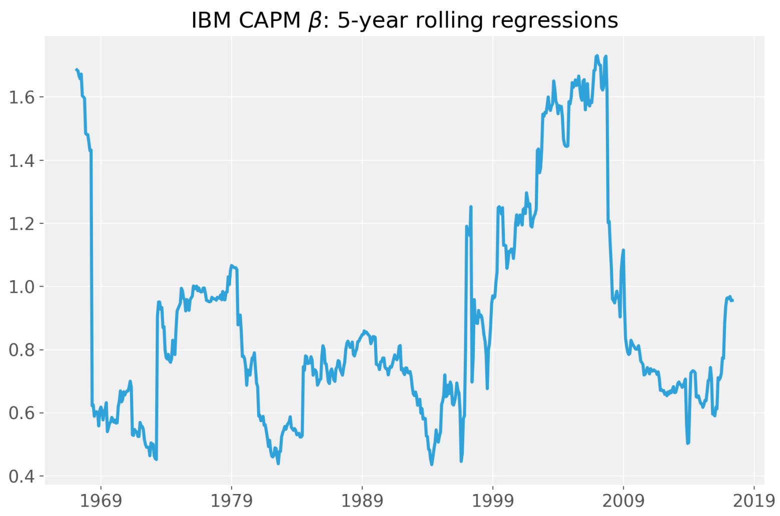
ESTIMATING A FIRM'S β: 35-YEAR SAMPLE

Dep. Variable:	IBM	**R-squared:**	0.225
Model:	OLS	**Adj. R-squared:**	0.224
Method:	Least Squares	**F-statistic:**	160.5
Date:	Wed, 03 May 2017	**Prob (F-statistic):**	4.36e-33
Time:	14:56:20	**Log-Likelihood:**	793.96
No. Observations:	662	**AIC:**	-1584.
Df Residuals:	660	**BIC:**	-1575.
Df Model:	1		
Covariance Type:	HC0		

	coef	**std err**	**z**	**P>|z|**	**[0.025**	**0.975]**
const	-0.0034	0.003	-1.249	0.212	-0.009	0.002
MKT	0.9222	0.073	12.671	0.000	0.780	1.065

Omnibus:	540.042	**Durbin-Watson:**	2.095
Prob(Omnibus):	0.000	**Jarque-Bera (JB):**	25573.242
Skew:	-3.227	**Prob(JB):**	0.00
Kurtosis:	32.757	**Cond. No.**	23.5

Estimating a firm's β: Rolling 5-year samples



- Large idiosyncratic firm volatility makes estimating beta for individual firms noisy
- Large number of firms $\Rightarrow$ inverting a large covariance matrix numerically difficult in GRS tests
- Sort firms into **portfolios** using firm characteristics.
- Typically, we do this by sorting on characteristics prior to time t.
- Then, we investigate the resulting portfolio returns until time $t + j$, when we redo the sort, etc.

- References:
  - Fama, Eugene F., and Kenneth R. French. 2004. “The Capital Asset Pricing Model: Theory and Evidence,” Journal of Economic Perspectives, (https://www.aeaweb.org/articles.php?doi=10.1257/0895330042162430)
  - Goyal, Amit. 2012. “Empirical Cross-Sectional Asset Pricing: A Survey,” Financial Markets and Portfolio Management, (http://www.hec.unil.ch/agoyal/docs/Survey_FMPM.pdf).

Portfolios based on

- Market cap (size)
- Industry (SIC codes)
- Past β: Each June, estimate β of each firm using data from the previous 60 months

If the CAPM describes the cross-section of stock returns, we expect that

1. Portfolio returns are linked to the portfolio's β.
2. Small pricing errors α_i in time series regressions

$$R^e_{i,t} = \alpha_i + \beta_i R^e_{M,t} + \varepsilon_{i,t}$$

CAPM: Size portfolios

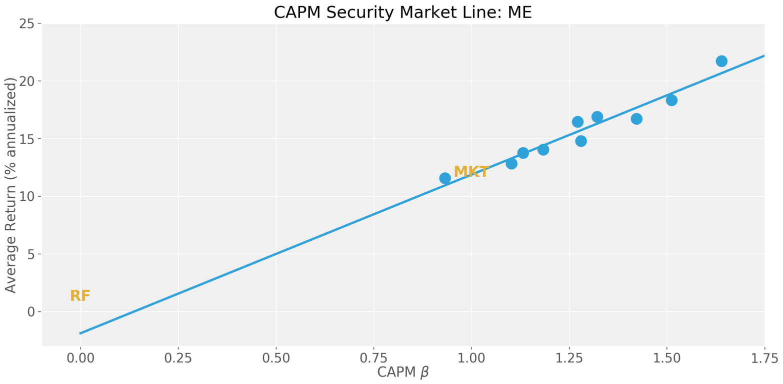
Each June: Sort on market cap and construct 10 VW-weighted portfolios

Sample: CRSP, 1926-1962

	1	2	3	4	5	6	7	8	9	10
Mean	1.81	1.53	1.40	1.41	1.23	1.37	1.17	1.15	1.07	0.97
β	1.64	1.51	1.42	1.32	1.28	1.27	1.18	1.13	1.10	0.93

The CAPM equation $E[R_{i,t} - R_{f,t}] = \beta_i E[R_{M,t} - R_{f,t}]$ implies that average excess returns and β's have a linear relationship.

CAPM: Size portfolios



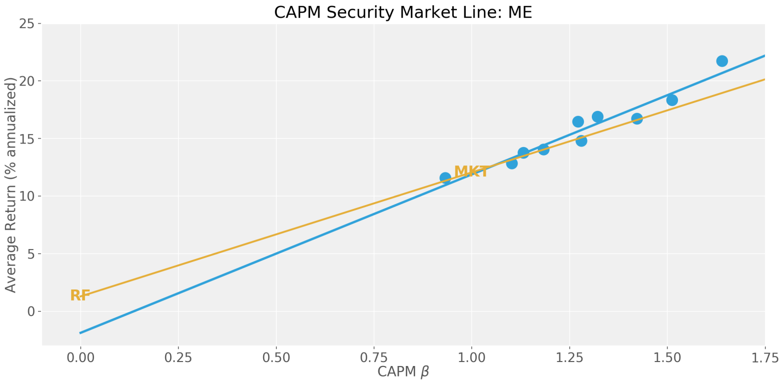
CAPM: Size portfolios - not perfect!

So, β and excess returns look indeed linear but

$$
\begin{aligned}
\mathrm{E}[R_{i,t} - R_{f,t}] &= +\beta_i \mathrm{E}[R_{M,t} - R_{f,t}] \\
\mathrm{E}[R_{i,t}] &= R_{f,t} + \beta_i \mathrm{E}[R_{M,t} - R_{f,t}]
\end{aligned}
$$

makes prediction about the intercept and slope!

CAPM: Size portfolios



Consensus around 1980

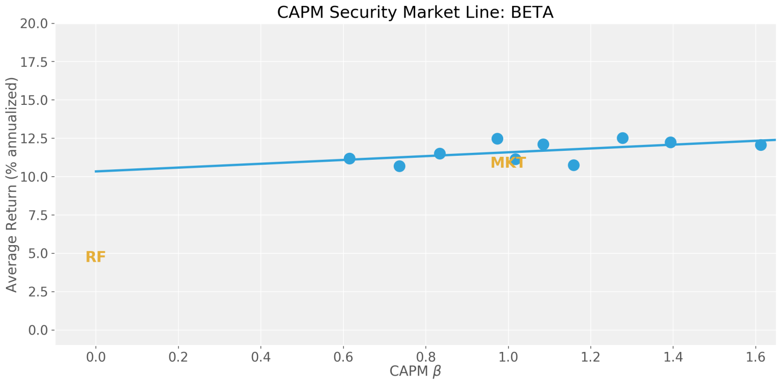
- Beta is related to average returns of stocks
- The relationship is close to linear as predicted by the CAPM
- But intercept and slope of the SML are different from CAPM predictions

CAPM: β-SORTED PORTFOLIOS

Each June: Sort on CAPM-β estimated over previous 60 months

	1	2	3	4	5	6	7	8	9	10
Mean	0.93	0.89	0.96	1.04	0.93	1.01	0.90	1.04	1.02	1.01
β	0.61	0.74	0.83	0.97	1.02	1.08	1.16	1.28	1.39	1.61

CAPM: β-SORTED PORTFOLIOS



In the 1980s Everything Changed

- Measure of value/growth: A firm's book-to-market ratio (B/M) or P/E
- Low B/M $\longrightarrow$ High market value relative to book value $\longrightarrow$ **Growth stock**
- High B/M $\Rightarrow$ low market value relative to book value $\longrightarrow$ **Value stock**
- Basu (1977): Firms with high (low) P/E ratio have low (high) future returns $\longrightarrow$ **Value Premium**
- The CAPM betas of value and growth stocks are very similar!
- The failure of the CAPM to explain the returns of value/growth stocks is known as the **Value Puzzle**
- Enormous academic literature: Risk or behavioral?
- Many hedge funds follow some sort of value/growth strategy

Next: Let's look at some plots and numbers

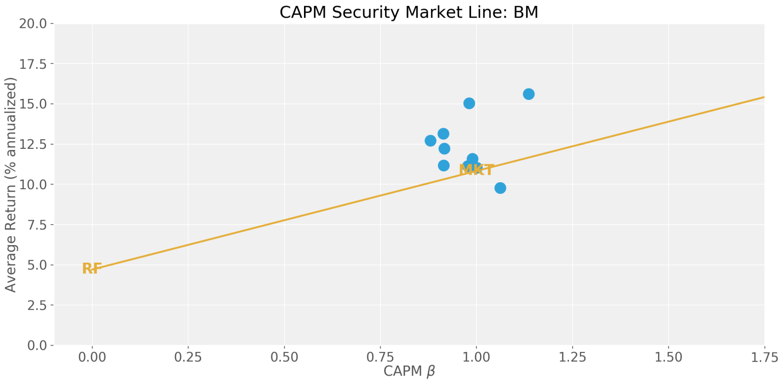
THE DEMISE OF THE CAPM: VALUE-GROWTH PORTFOLIOS

- Every June, sort firms according to their B/M and form portfolios; compute monthly portfolio returns From July to following June; re-sort according to current B/M and form new portfolios
- Data source: `http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html`
- Let's start with 10 B/M portfolios

Each June: Sort on B/M ratio

	1	2	3	4	5	6	7	8	9	10
Mean	0.82	0.92	0.97	0.93	0.93	1.06	1.02	1.10	1.25	1.30
β	1.06	1.00	0.99	0.98	0.91	0.88	0.92	0.91	0.98	1.14

The demise of the CAPM: Value-growth portfolios



- Jegadesh and Titman (1993): Individual stock returns are positively auto-correlated: Winners will continue to be winners. Losers will continue to be losers.
- Each month, sort stocks according to their past return

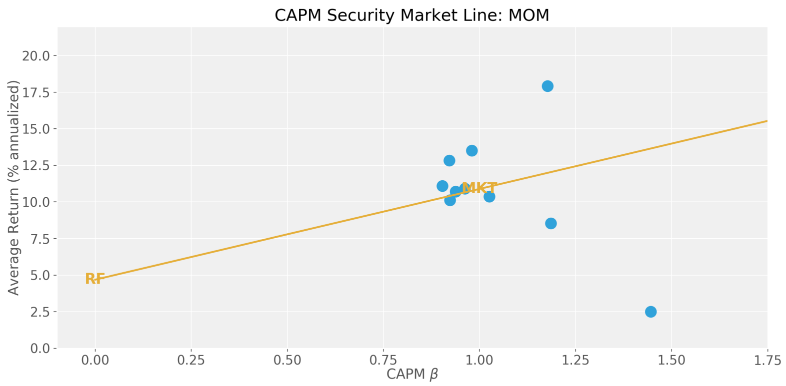
Sort stocks according to returns over past 12 to 2 months

	1	2	3	4	5	6	7	8	9	10
Mean	0.21	0.71	0.86	0.91	0.84	0.89	0.92	1.07	1.13	1.49
β	1.45	1.19	1.03	0.96	0.92	0.94	0.90	0.92	0.98	1.18

Sort stocks according to returns over past 60 to 13 months

	1	2	3	4	5	6	7	8	9	10
Mean	1.22	1.11	1.11	1.00	1.02	1.02	1.00	0.97	0.85	0.86
β	1.21	1.05	0.95	0.92	0.90	0.87	0.89	0.90	0.99	1.22

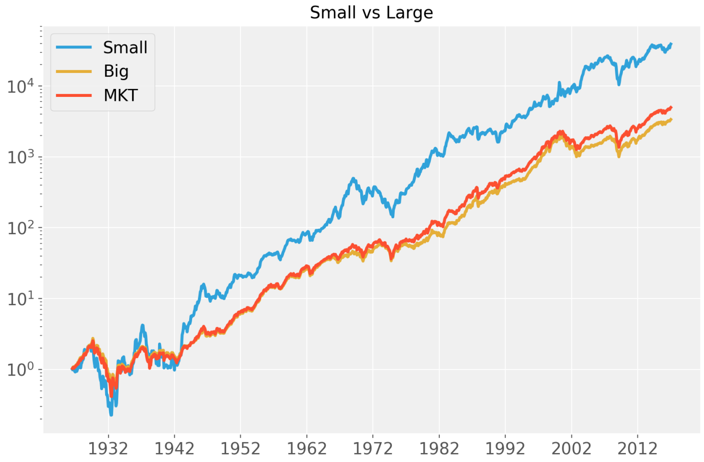
THE DEMISE OF THE CAPM: MOMENTUM PORTFOLIOS



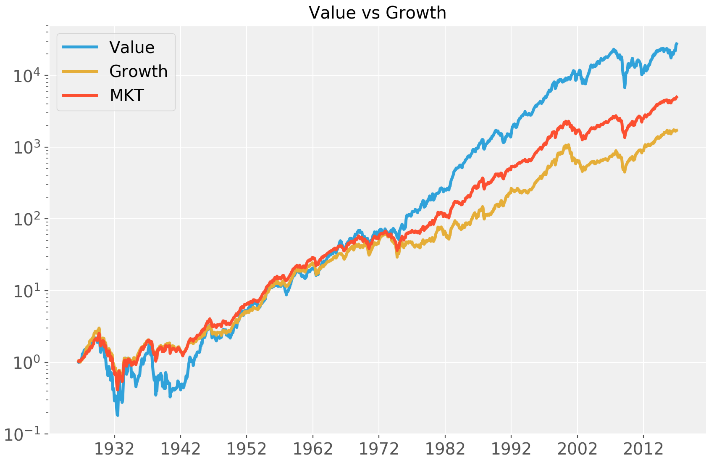
Implications for Investment Strategies

- Let's compare returns of different investment strategies over time
- Value vs. growth
- Small vs. big
- Winners vs losers

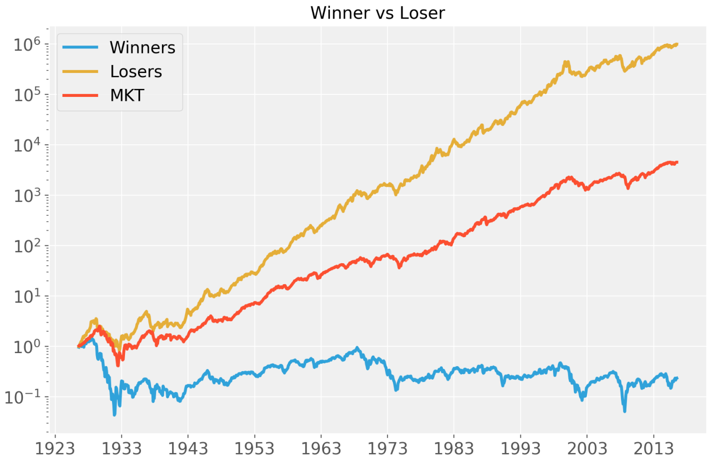
Small vs. big


Small vs Large
Small
Big
MKT
10^4
10^3
10^2
10^1
10^0
1932
1942
1952
1962
1972
1982
1992
2002
2012

Value vs. growth



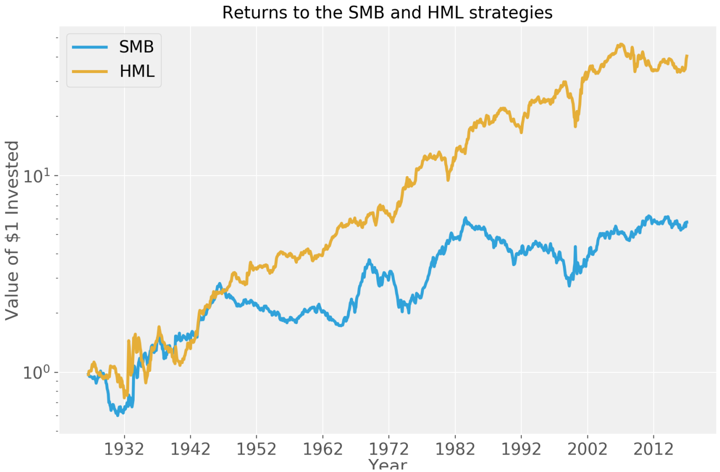
WINNERS VS. LOSERS



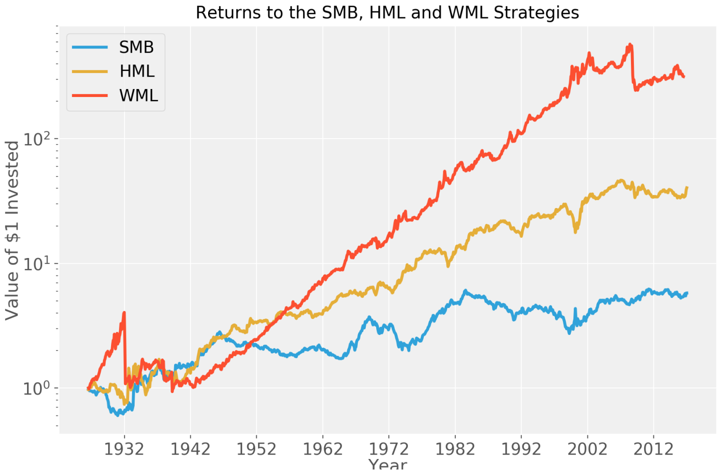
We can form long-short strategies to form zero-cost portfolios:

- HML: long \$1 in high B/M stocks, short \$1 in low B/M stocks
- SMB: long \$1 in small stocks, short \$1 in large stocks
- MOM: long \$1 in winners, short \$1 in losers

Long-short strategies



Long-short strategies

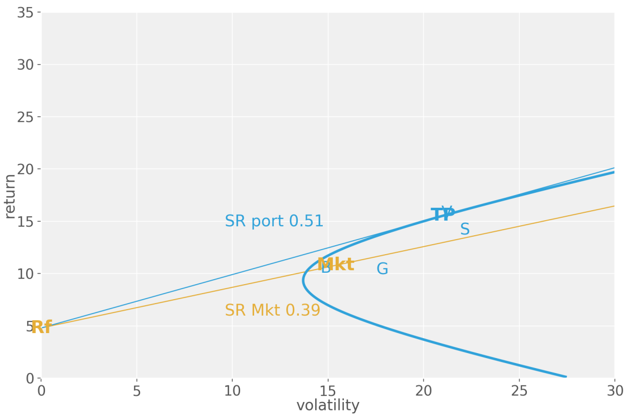


Formal tests of the CAPM: Intuition

CAPM implies that the market portfolio is mean-variance efficient

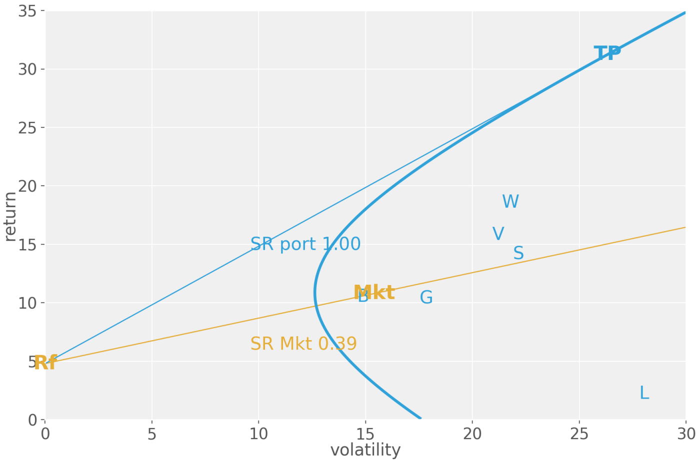
Data: Annualized monthly returns from 1967-2016

	MKT	SMB	HML	MOM
Mean	7.79	2.62	4.83	7.87
Std. Dev.	19.74	11.15	12.15	16.42
Sharpe-ratio	0.39	0.23	0.40	0.48



Recall interpretation of GRS statistic $\dfrac{\hat{\boldsymbol{\alpha}}'\hat{\boldsymbol{\Sigma}}^{-1}\hat{\boldsymbol{\alpha}}}{1+\widehat{SR}_M^2} = \dfrac{1+\widehat{SR}_N^2}{1+\widehat{SR}_M^2}$

MEAN-VARIANCE FRONTIER FOR MKT, SMB, HML, MOM



Outline

1. Review of CAPM
2. Testing the CAPM: Theory
3. Testing the CAPM: Empirical evidence
4. **Tests of Fama-French style multifactor models**

Fama-French (1993) "Common risk factors in the returns on stocks and bonds," *Journal of Financial Economics* suggests a multifactor extension of the CAPM known as the "FF 3-factor model":

$$\mathrm{E}[R_i - R_f] = \beta_{i,mkt}\mathrm{E}[R_M - R_f] + \beta_{i,smb}\mathrm{E}[\mathrm{SMB}] + \beta_{i,hml}\mathrm{E}[\mathrm{HML}]$$

- FF argue that SMB and HML represent undiversifiable risk factors
- $\beta_{i,smb}$ and $\beta_{i,hml}$ measure the exposure of asset i to these risk factors
- The interpretation of these factors is (still) hotly debated
- Issues:
  - No theoretical foundation
  - FF do not explain why SMB and HML should be risk factors
  - What is the underlying economic reason that give rise to SMB and HML?

Formal Tests of CAPM and FF model I

Data: 25 B/M-size sorted portfolio from Ken French's website, sample 1963-2017

Average returns; columns: growth to value; rows: small to large

	low	2	3	4	high
small	0.63	1.17	1.18	1.40	1.49
2	0.87	1.14	1.27	1.33	1.40
3	0.89	1.18	1.15	1.28	1.43
4	0.99	0.99	1.10	1.25	1.22
big	0.87	0.91	0.93	0.89	1.07

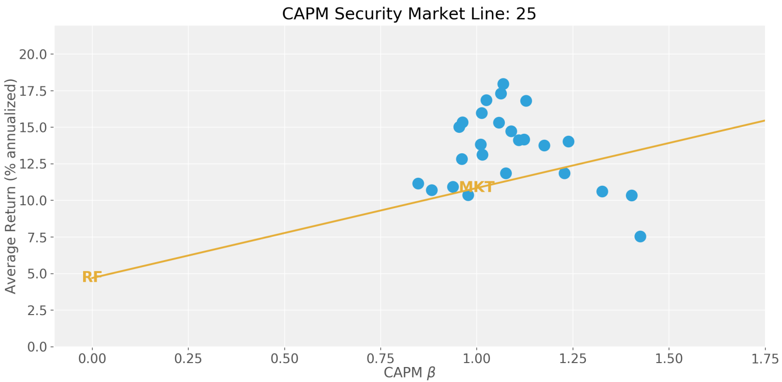
α_i

	low	2	3	4	high
small	-0.49	0.14	0.22	0.48	0.55
2	-0.24	0.15	0.34	0.42	0.42
3	-0.18	0.21	0.24	0.39	0.49
4	-0.03	0.04	0.18	0.37	0.27
big	-0.03	0.04	0.10	0.05	0.18

β_i

	low	2	3	4	high
small	1.42	1.24	1.11	1.02	1.07
2	1.40	1.17	1.06	1.01	1.13
3	1.33	1.12	1.01	0.96	1.06
4	1.23	1.08	1.01	0.95	1.09
big	0.98	0.94	0.85	0.88	0.96

SML



CAPM Security Market Line: 25
Average Return (% annualized)
20.0
17.5
15.0
12.5
10.0
7.5
5.0
2.5
0.0
0.00
0.25
0.50
0.75
1.00
1.25
1.50
1.75
CAPM β
RF
MKT

$t(\alpha_i)$ (White)

	low	2	3	4	high
small	-2.66	0.89	1.63	3.57	3.77
2	-1.69	1.26	3.10	3.89	3.14
3	-1.59	2.32	2.71	4.01	3.97
4	-0.37	0.57	2.10	4.09	2.24
big	-0.40	0.64	1.25	0.44	1.40

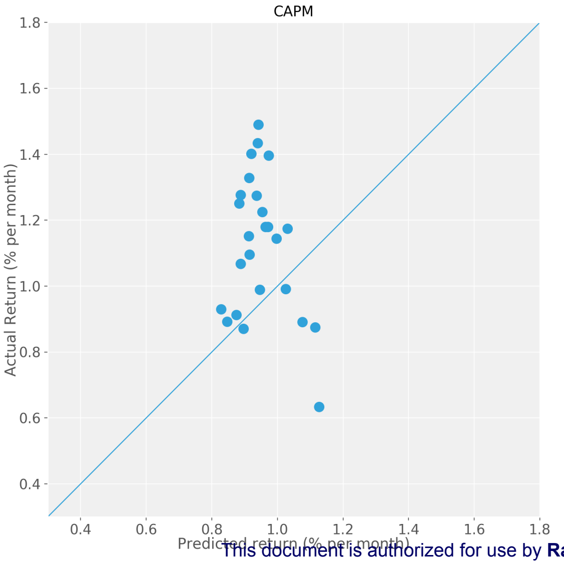
$t(\alpha_i)$ (Newey-West)

	low	2	3	4	high
small	-2.51	0.87	1.58	3.44	3.56
2	-1.66	1.25	3.13	3.83	3.02
3	-1.59	2.30	2.64	3.92	3.73
4	-0.36	0.53	1.96	3.94	2.07
big	-0.38	0.63	1.20	0.42	1.37

Recall that the α_i have the interpretations of **pricing errors**.

Graphical illustration of fit: Plot model-predicted returns on x-axis and realized returns on y-axis.

TESTING THE CAPM: $R_{i,t} - R_{f,t} = \alpha_i + \beta_i (R_{m,t} - R_{f,t}) + \varepsilon_t$



Testing the FF 3-model I

The CAPM fits the returns of the 25 portfolios poorly and produces large pricing errors.

Let's try to test the 3-factor FF model:

Estimation

$$R_{i,t} - R_{f,t} = \alpha_i + \beta_i \left(R_{m,t} - R_{f,t}\right) + s_i SMB_t + h_i HML_t + \varepsilon_t.$$

TESTING THE FF 3-MODEL: FACTOR BETAS

β_i

	low	2	3	4	high
small	1.10	0.98	0.93	0.88	0.96
2	1.14	1.01	0.96	0.96	1.08
3	1.10	1.04	0.99	0.98	1.08
4	1.08	1.06	1.05	1.00	1.17
big	0.97	1.00	0.95	1.03	1.12

s_i

	low	2	3	4	high
small	1.36	1.31	1.08	1.06	1.08
2	0.99	0.88	0.76	0.70	0.88
3	0.74	0.54	0.43	0.41	0.55
4	0.39	0.21	0.17	0.20	0.26
big	-0.24	-0.21	-0.25	-0.23	-0.10

Testing the FF 3-model: Factor betas

h_i

	low	2	3	4	high
small	-0.29	0.05	0.29	0.44	0.68
2	-0.38	0.12	0.39	0.57	0.79
3	-0.43	0.17	0.42	0.59	0.80
4	-0.41	0.18	0.43	0.55	0.79
big	-0.37	0.09	0.31	0.64	0.81

α_i

	low	2	3	4	high
small	-0.53	-0.03	-0.04	0.16	0.12
2	-0.19	-0.01	0.08	0.08	-0.03
3	-0.08	0.07	0.00	0.08	0.08
4	0.10	-0.06	-0.03	0.10	-0.11
big	0.17	0.02	-0.00	-0.21	-0.17

TESTING THE FF 3-MODEL; t-STATS

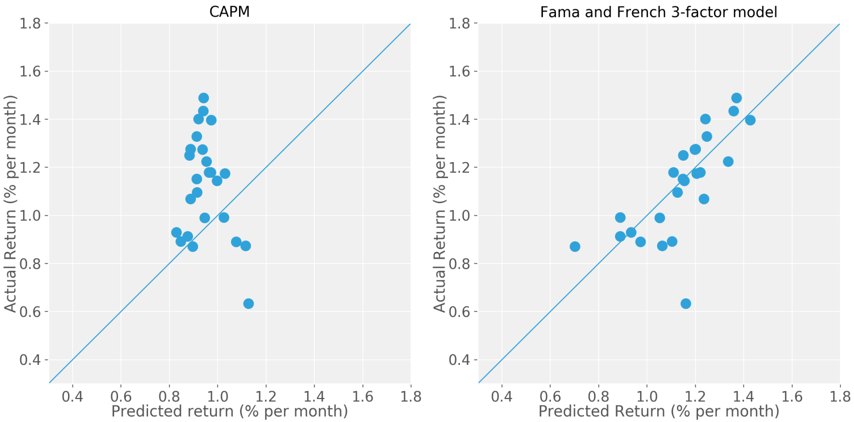
t_i-White

	low	2	3	4	high
small	-5.80	-0.44	-0.76	3.01	2.27
2	-2.82	-0.17	1.33	1.65	-0.55
3	-1.34	1.11	0.05	1.20	1.01
4	1.65	-0.91	-0.41	1.52	-1.31
big	3.82	0.45	-0.06	-3.18	-1.75

t_i-Newey-West

	low	2	3	4	high
small	-5.65	-0.46	-0.78	2.96	2.31
2	-2.72	-0.17	1.30	1.63	-0.54
3	-1.33	1.08	0.05	1.19	0.99
4	1.61	-0.85	-0.40	1.58	-1.30
big	3.76	0.44	-0.06	-3.03	-1.69

Testing the FF 3-model



Testing the CAPM and the FF 3-model: GRS statistics

Finally, testing the joint null $H_0 : \alpha_i = 0 \ \forall i$

$$\text{CAPM: GRS} = T \frac{\hat{\boldsymbol{\alpha}}' \hat{\boldsymbol{\Sigma}}^{-1} \hat{\boldsymbol{\alpha}}}{1 + \widehat{\text{SR}}_M^2} \to \chi_N^2$$

$$GRS = 120.4, \quad p\text{-value} = 0.00$$

$$\text{FF3: GRS} = T \frac{\hat{\boldsymbol{\alpha}}' \hat{\boldsymbol{\Sigma}}^{-1} \hat{\boldsymbol{\alpha}}}{1 + \boldsymbol{\mu}_F' \widehat{\boldsymbol{\Omega}}_F^{-1} \boldsymbol{\mu}_F} \to \chi_N^2$$

$$GRS = 98.22, \quad p\text{-value} = 0.00$$

where $\boldsymbol{\mu}_F$ and $\widehat{\boldsymbol{\Omega}}_F$ are mean and the variance covariance matrix of $F_t = (\text{MKT}_t, \text{SMB}_t, \text{HML}_t)'$.

Comments on Fama-French model

- The precise meaning of the FF model is (still) hotly debated
- It is agreed that the CAPM is dead and the FF model produces smaller pricing errors (even though the FF model is statistically rejected)
- But are HML and SMB true risk factors?
- Fama-French: Yes, they are
- Others are more skeptical
- My (subjective) view:
  - HML and SMB are summaries of the value and size puzzles but they are not explanations of the puzzles.
  - Indeed, they should be left-hand-side variables, i.e. portfolios to be explained. However, the FF model is useful in practice as a reduced-form model.

Richardson, Tuna and Wysocki (2010): Survey of 201 investment managers and 63 academics

Q1: Which risk model is most appropriate for risk calibration of an equity trading strategy?

	Practitioner Opinions	*Academic Opinions*
CAPM with size & industry adjustments	35%	*7%* **
Fama-French 3-factor model (Market, Size, Book Value/Market Value)	24%	*22%*
Multifactor model	11%	*4%* **
Other model	11%	*15%*
CAPM	10%	*4%* *
Fama-French 3-factor model plus other factors	5%	*33%* **
CAPM with size adjustments	4%	*15%* **

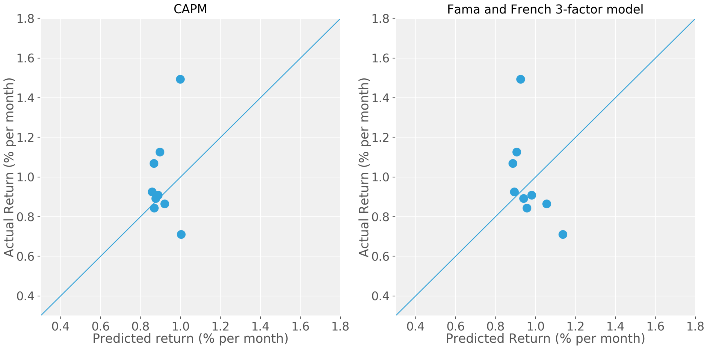
* and ** indicate difference in means across practitioner and academic sample answers are significant at 5 and 1% levels, respectively.

Asness, Moskowitz and Pedersen (2013), Koijen, Moskowitz, Pedersen and Vrugt (2013) and Lettau, Maggiori and Weber (2014) study other asset classes

Idea: The value/growth strategy implies that stocks that are “cheap” relative to their fundamentals outperform “”expensive” stocks (M/B, P/E).

Cheap assets outperform expensive assets in many other asset classes:

- Global equities
- Currencies
- Government Bonds
- Commodities
- Options



"Solution": Fama-French-Carhart 4-factor model

$$R_{i,t} - R_{f,t} = \alpha_i + b_i \left(R_{m,t} - R_{f,t} \right) + s_i SMB_t + h_i HML_t + w_i MOM_t + \varepsilon_t.$$

OTHER PUZZLES

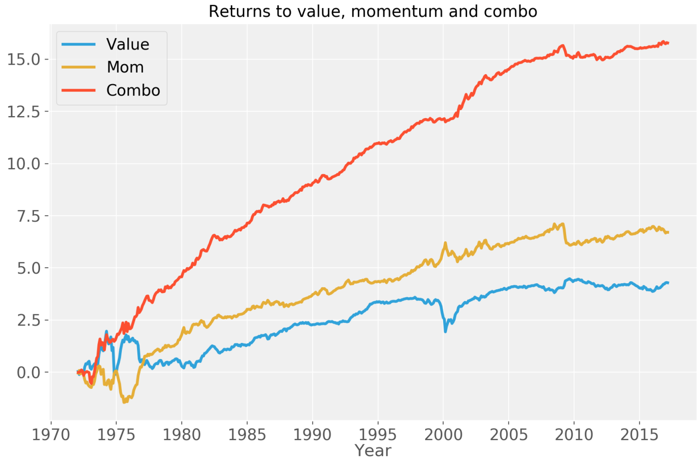
Survey: A. Goyal "Empirical cross-sectional asset pricing: A survey" *Financial Markets and Portfolio Management*, 2012

- **Accruals**: firms with high accruals have abnormally low future returns
- **New stock issues**: companies that issue new stock (IPO or seasoned) have abnormally low future returns
- **Asset growth**: firms with high asset growth and investment have abnormally low future returns
- **Idiosyncratic risk**: firms with high idiosyncratic risk and investment have abnormally low future returns
- **Liquidity**: firms with high turnover have abnormally low returns
- **Analyst's forecast**: stocks with higher dispersion of analyst's forecast have low future returns

Frazzini, Israel and Moskowitz (2014), "Trading Costs of Asset Pricing Anomalies"

Panel A: All Stocks 1926 - 2011 - U.S.

	SMB	HML	UMD	STR	ValMom	Combo
Realized cost	0.55	0.97	2.49	6.25	1.36	1.89
Break-even cost	2.61	3.88	7.85	8.27	6.23	5.39
Realized minus breakeven	**-2.07**	**-2.92**	**-5.37**	**-2.02**	**-4.87**	**-3.51**
t-statistics	-(104.98)	-(90.97)	-(74.27)	-(16.61)	-(142.46)	-(81.19)
Break-even fund size (billions)	$354.27	$189.56	$65.92	$9.45	$129.47	$54.36
Return (Gross)	**2.61**	**3.88**	**7.85**	**8.27**	**6.23**	**5.39**
	(2.13)	(2.65)	(4.41)	(6.37)	(8.79)	(9.99)
Return (Net)	2.07	**2.92**	**5.37**	2.02	**4.87**	**3.51**
	(1.68)	(2.00)	(2.99)	(1.56)	(6.84)	(6.47)
Turnover (monthly)	0.27	0.45	1.13	3.06	0.69	0.99
MI (bps)	16.92	17.71	18.32	16.99	16.46	15.93
Sharpe ratio (gross)	0.27	0.34	0.48	0.69	1.13	1.28
Sharpe ratio (net)	0.22	0.26	0.32	0.17	0.88	0.83
Obs	729	729	1,023	1,034	729	729


Returns to value, momentum and combo
Value
Mom
Combo
15.0
12.5
10.0
7.5
5.0
2.5
0.0
1970
1975
1980
1985
1990
1995
2000
2005
2010
2015
Year

Today's lecture

1. Quick review of the CAPM
2. Diversifiable vs. undiversifiable risk
3. Testing the CAPM: Theory
4. Testing the CAPM: Empirical evidence
5. Fama-French style multifactor models
6. **Testing general factor models**

General risk factors

- So far, factors were excess returns $\Rightarrow$ Times series estimation
- How can we test models when risk factors are not excess returns?
- Examples of possible risk factors?
- The time-series based tests developed by Gibbons-Ross-Shanken only apply to factors that are excess returns

Linear factor model when F_t is **not** an excess return:

$$E\left[R^e_{i,t}\right] = \beta_i \lambda, \qquad \beta_i = \frac{\mathrm{Cov}\left(F_{t+1}, R^e_{i,t+1}\right)}{\mathrm{Var}\left(F_{t+1}\right)} \qquad (*)$$

- λ: risk premium associated with F_t
- β_i: exposure to risk factor F_t, OLS coefficient of **time series** regression

$$R^e_{i,t+1} = a_i + \beta_i F_{t+1} + \varepsilon_{i,t+1}$$

General linear factor models II

Take expectations and combine with $(*)$:

$$\begin{aligned} E\left[R^e_{i,t}\right] &= \beta_i\lambda = a_i + \beta_i E\left[F_t\right] \\ a_i &= \beta_i(\lambda - \mathrm{E}[F_t]) \end{aligned}$$

- Nothing in the model implies that a_i is necessarily zero!
- Note: if F_t is excess return, $a_i = 0$. Question: Why?

Step 1:

For each asset $i = 1, \ldots, N$, estimate the **time-series (TS)** regression

$$R^e_{i,t+1} = a_i + \beta_i F_{t+1} + e_{i,t+1}$$

Save the $\hat{\beta}_i$'s, the average excess returns $\overline{R}^e_i$ and the residuals $\{e_{i,t}\}_{t=1}^T$

Step 2:

Estimate the price of risk (λ) in the single **cross-sectional (XS)** regression:

$$\overline{R}_i^e = \lambda \hat{\beta}_i + \alpha_i$$

using the N sample average asset returns and the estimated β_i's.

- The residual from the XS regression (α_i) is the pricing errors of asset i.
- Under H_0: No intercept in XS regression
- We can then test the null hypothesis that these are jointly zero.
- We can also test whether the factor is ”priced”, i.e. $\hat{\lambda} \neq 0$.

Appendix: Derivation of correct test that corrects for the fact that $\widehat{\beta}_i$ are estimated in step 1.

Step 1: Estimate β's in time series regression

$$R_{i,t} - R_{f,t} = \alpha_i + \beta_i \left(R_{m,t} - R_{f,t}\right) + \varepsilon_t.$$

Step 2: Run cross-sectional regression

$$\overline{R}_i^e = \lambda_b \hat{\beta}_i + \alpha_i$$

Step 1: Estimate β's in time series regression

$$R_{i,t} - R_{f,t} = \alpha_i + b_i \left(R_{m,t} - R_{f,t}\right) + s_i SMB_t + h_i HML_t + \varepsilon_t.$$

Step 2: Run cross-sectional regression

$$\overline{R}_i^e = \lambda_b \hat{b}_i + \lambda_s \hat{s}_i + \lambda_h \hat{h}_i + \alpha_i$$

Question: If the CAPM/FF3 models were correct, what estimates do you expect for the λ's?'

$\overline{R}_i^e$

	low	2	3	4	high
small	0.63	1.17	1.18	1.40	1.49
2	0.87	1.14	1.27	1.33	1.40
3	0.89	1.18	1.15	1.28	1.43
4	0.99	0.99	1.10	1.25	1.22
big	0.87	0.91	0.93	0.89	1.07

β_i

	low	2	3	4	high
small	1.42	1.24	1.11	1.02	1.07
2	1.40	1.17	1.06	1.01	1.13
3	1.33	1.12	1.01	0.96	1.06
4	1.23	1.08	1.01	0.95	1.09
big	0.98	0.94	0.85	0.88	0.96

Step 2: Run cross-sectional regression

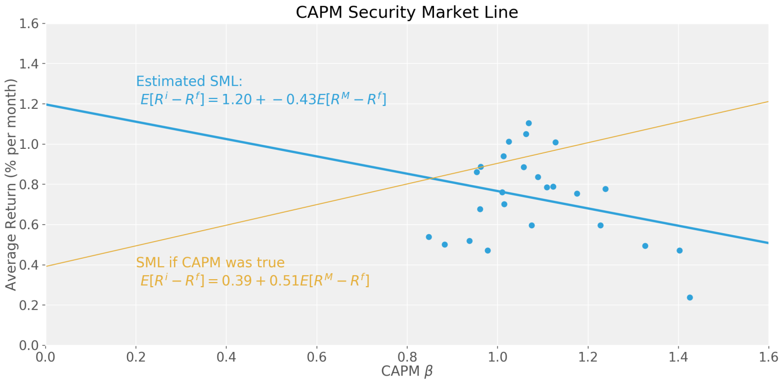
$$\overline{R}_i^e = const. + \lambda_b \hat{\beta}_i + \alpha_i$$

Dep. Variable:	ER	**R-squared:**	0.086
Model:	OLS	**Adj. R-squared:**	0.047
Method:	Least Squares	**F-statistic:**	2.173
Date:	Mon, 01 May 2017	**Prob (F-statistic):**	0.154
Time:	16:27:40	**Log-Likelihood:**	4.2728
No. Observations:	25	**AIC:**	-4.546
Df Residuals:	23	**BIC:**	-2.108
Df Model:	1		
Covariance Type:	nonrobust		

	coef	**std err**	**t**	**P>|t|**	**[0.025**	**0.975]**
const	1.1971	0.319	3.748	0.001	0.536	1.858
Mkt-RF	-0.4307	0.292	-1.474	0.154	-1.035	0.174

Omnibus:	1.473	**Durbin-Watson:**	1.192
Prob(Omnibus):	0.479	**Jarque-Bera (JB):**	0.960
Skew:	-0.088	**Prob(JB):**	0.619
Kurtosis:	2.056	**Cond. No.**	15.0

Testing the CAPM: Factor betas



Testing the FF 3-model: Factor betas

MKT

	low	2	3	4	high
small	1.29	1.08	1.05	0.95	0.99
2	1.09	1.02	0.98	0.97	1.07
3	1.13	1.02	0.99	0.98	1.13
4	1.08	1.02	1.02	1.03	1.22
big	1.02	0.99	0.96	1.04	1.18

SMB

	low	2	3	4	high
small	1.44	1.53	1.24	1.22	1.30
2	1.12	0.98	0.83	0.81	0.91
3	0.82	0.51	0.44	0.46	0.59
4	0.33	0.23	0.20	0.20	0.29
big	-0.15	-0.20	-0.25	-0.19	-0.15

HML

	low	2	3	4	high
small	0.43	0.23	0.52	0.57	0.91
2	-0.22	0.14	0.35	0.57	0.88
3	-0.23	0.04	0.31	0.55	0.88
4	-0.36	0.08	0.35	0.57	0.94
big	-0.26	0.02	0.34	0.65	1.02

	Mkt-RF	SMB	HML
lambda	-1.07	0.12	0.42
se	0.31	0.05	0.06
factor premia	0.65	0.21	0.40

Recall:

$$E\left[R^e_{i,t}\right] = \beta_i \lambda, \qquad \beta_i = \frac{\text{Cov}\left(F_{t+1}, R^e_{i,t+1}\right)}{\text{Var}\left(F_{t+1}\right)}$$

Since, $R_{M,t} - R_{f,t}$, HML and SMB are excess returns, their λ's should be close to their factor premia

CAPM and FF2 XS regressions

	const	Mkt-RF	SMB	HML
Factor premium		0.52	0.22	0.36
$\widehat{\lambda}$		0.59		
s.e.		0.07		
$\widehat{\lambda}$	1.47	-0.70		
s.e.	0.49	0.44		
$\widehat{\lambda}$		0.57	0.45	-0.72
s.e.		0.12	0.18	0.40
$\widehat{\lambda}$	-0.84	-1.70	0.36	2.51
s.e.	0.20	0.28	0.09	0.30

Alternative method: Fama-MacBeth I

The cross-section OLS regression assumes that β's are constant

Fama and MacBeth (1973) suggest an alternative that allows for time-varying β's

The idea is to recursively estimate β_t and then run a cross-sectional regression every period using the estimated β_t:

- Step 1: Start at the beginning of the sample and use the first 60 months to estimate the time series regression to obtain $\hat{\beta}_{60,i}$:

$$R^e_{it} = \hat{\alpha}_i + \hat{\beta}_{60,i} F_t + e_{it} \;\; \text{for } t = 1, \ldots, 60$$

$\hat{\beta}_{60,i}$: 25 portfolios

25 beta × 4 feature
portfolio

- Step 2: estimate a cross-sectional regressions for period 61:

$$R^e_{i,61} = \lambda_{0,61} + \hat{\beta}_{60,i}\lambda_{1,61} + \alpha_{i,61}, \qquad \hat{\lambda}_{61} = \left(\hat{\lambda}_{0,61}\; \hat{\lambda}_{1,61}\right)'$$

4 features: Mkt-Rf, HML...

portfolios

$R^e_{i,61}$: 25 portfolios, excess return 1

portfolio

excess return

calculated over 60 month

calculated by regression

excess return = α + βX ← ×4

← (1,4)

- Repeat Steps 1 and 2 by shifting the sample forward one month to estimate $\beta_{61,i}$, α_{62} and λ_{62}
- Repeat until you reach the end of the sample
- This yields time series $\alpha_{61}, \alpha_{62}, \ldots, \alpha_T$ and $\lambda_{61}, \lambda_{62}, \ldots, \lambda_T$
- Fama-MacBeth: use time series mean of $\hat{\lambda}_t$ to estimate risk premium:

$$\hat{\lambda} = \frac{1}{T-59} \sum_{t=61}^{T} \hat{\lambda}_t$$

- The variance-covariance matrix of the sample mean is then $\frac{1}{T-61}$ times the sample variance of $\hat{\lambda}_t$ (since the variance of the sample mean is the variance of the sample divided by T):

$$\mathrm{Var}\left(\hat{\lambda}\right) = \frac{1}{T-59} \hat{\mathrm{Var}}(\lambda_t) = \frac{1}{(T-59)^2} \sum_{t=59}^{T} \left(\hat{\lambda}_t - \hat{\lambda}\right)\left(\hat{\lambda}_t - \hat{\lambda}\right)'$$

The test of the residuals α_t (a $N \times 1$ vector) are jointly zero:

$$\hat{\boldsymbol{\alpha}} = \frac{1}{T-59} \sum_{t=61}^{T} \hat{\boldsymbol{\alpha}}_t$$

$$\mathrm{Var}\left(\hat{\boldsymbol{\alpha}}\right) = \frac{1}{(T-59)^2} \sum_{t=61}^{T} \left(\hat{\boldsymbol{\alpha}}_t - \hat{\boldsymbol{\alpha}}\right)\left(\hat{\boldsymbol{\alpha}}_t - \hat{\boldsymbol{\alpha}}\right)'$$

Test statistic:

$$\hat{\boldsymbol{\alpha}}' \, \mathrm{Var}\left(\hat{\boldsymbol{\alpha}}\right)^{-1} \hat{\alpha} \sim \chi^2 \left(N-K-1\right)$$

Statistical test of linear factor models: Summary

- If factors are excess returns: Run time series regression and test whether intercepts are jointly zero
- If factors are not excess returns: Run 2-stage procedure (Either cross-sectional OLS/GLS of Fama-MacBeth)
- Further results:
  - If factors are iid and errors are uncorrelated over time (but are allowed to be cross-correlated): Fama-MacBeth and OLS are identical
  - Note: If β's are varying over time, Fama-MacBeth is preferable to OLS
  - If errors are normal and factors are iid and excess returns: MLE identical to time series OLS
  - If errors are normal and factors are iid and not excess returns: MLE identical to cross sectional GLS (not OLS!)

Empirical evidence: Summary

- The CAPM is an appealing theoretical model
- But it does not capture risk and returns in (most) asset classes
- Stock markets: Value-growth, momentum, ...
- Fama-French multifactor models capture expected returns better than the CAPM but have no theoretical foundation
- 3-factor FF model is statistically rejected
- Open question: Risk or behavioral biases
- Other asset classes: “Cheap” assets outperform “expensive” ones

- Note that regressors in 2nd stage (β_i) are estimated in 1st stage $\Rightarrow$ need to take regression uncertainty into account when testing XS regression
- GMM is ideally suited for this problem
- Derivation: Cochrane *Asset Pricing*, ch. 12, simplest case: i.i.d. factors and errors

Moment condition for first pass time series regression:

$$E\left(R^e_{i,t} - a_i - \beta_i F_t\right) = 0$$
$$E\left[\left(R^e_{i,t} - a_i - \beta_i f_t\right) F_t\right] = 0$$

The moment condition for the cross sectional regression is

$$E\left(R^e_{i,t} - \beta_i \lambda\right) = 0,$$

Stack these moment conditions as

$$g_T = \begin{bmatrix} E_T\left(\underset{N\times 1}{R_t^e} - \underset{N\times 1}{a} - \underset{(N\times 1)(1\times 1)}{\beta \quad F_t}\right) \\ E_T\left[\left(R_t^e - a - \beta F_t\right) F_t\right] \\ E_T\left(\underset{N\times 1}{R_t^e} - \underset{(N\times 1)(1\times 1)}{\beta \quad \lambda}\right) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

- The first two moment conditions are $2N$ equations, and there are N values of a_i and β_i to estimate, so $2N$ parameters; thus the first two moment conditions are exactly identified.
- The last moment condition has only one parameter to estimate, λ, and N moment conditions. Thus we have $3N$ total moments and only $2N + 1$ total parameters.

Test statistics for $H_0 : \hat{\alpha} = 0$ I

If we assume that

1. The errors, $\varepsilon_{i,t}$ are i.i.d. and independent of the factors F_t
2. The factors, F_t, are uncorrelated over time,

Let

$$\mathbf{\Sigma} = \frac{1}{T}\sum_{t=1}^{T} e_t e_t'$$

$$\mathbf{\Sigma}_F = \frac{1}{T}\sum_{t=1}^{T} F_t F_t'$$

Test statistic for K factors:

$$\hat{\alpha}' \mathrm{Cov}\left(\hat{\alpha}\right)^{-1} \hat{\alpha} \sim \chi^2_{N-K}$$

$$\mathrm{Cov}\left(\hat{\alpha}\right) = \frac{1}{T}\left(I_N - \beta\left(\beta'\beta\right)^{-1}\beta'\right)\mathbf{\Sigma}\left(I_N - \beta\left(\beta'\beta\right)^{-1}\beta'\right)' \times \left(1 + \lambda'\mathbf{\Sigma}_f^{-1}\lambda\right)$$

Variance-covariance matrix of $\hat{\lambda}$:

$$\mathrm{Cov}\left(\hat{\lambda}\right) = \frac{1}{T}\left[\left(\beta'\beta\right)^{-1}\beta'\mathbf{\Sigma}\beta\left(\beta'\beta\right)^{-1} \times \left(1 + \lambda'\mathbf{\Sigma}_f^{-1}\lambda\right) + \mathbf{\Sigma}_f\right]$$

These test statistics are derived in Cochrane's *Asset Pricing* book (pp. 235-245).

No, you don't have to memorize these formulas for the exam. Note that the tests are very easy to code in Matlab, R, ...

Comments

- The term $\left(1 + \lambda' \boldsymbol{\Sigma}_f^{-1} \lambda\right)$ is due to the correction for the fact that the β's are estimated in the first stage (Shanken (1992) correction)
- If assumptions
  1. The errors, $\varepsilon_{i,t}$, are i.i.d. and independent of the factors F_t
  2. The factors, F_t, are uncorrelated over time,

  are not satisfied, the GMM procedure can easily be adjusted.

  How?
- Also straightforward to run GLS instead of OLS (see Cochrane ch. 12)
- We can also estimate linear factor models using maximum likelihood (see again Cochrane)