1.1 Discrete Probability

1. There are M green apples and N red apples in a basket. We take apples out randomly one by one until all the apples left in the basket are red. What is the probability that at the moment we stop the basket is empty?

2. A fair coin is tossed n times. What is the expected product of the number of heads and the number of tails?

3. If $x_1, x_2, \ldots, x_9$ is a random arrangement of numbers 1, 2, ..., 9 around a circle, what is the probability that $\sum_{i=1}^{9} |x_{i+1} - x_i|$ is minimized? (Here, $x_{10} = x_1$.)

4. There are 1000 green balls and 3000 red balls in container A, and 3000 green balls and 1000 red balls in container B. You take half of the balls from A at random and transfer them to B. Then you take one ball from B at random. What is the probability that this ball is green?

5. A robot performs coin tossing. It is poorly designed, it produces a lot of sounds, lights, and vapors, and it takes one hour to toss a coin. Yet in the end, when the coin finally lands, it somehow has equal probability of showing heads and tails.

   Two scientists, A and B, enjoy observing this robot and, by analyzing its unusual and faulty behavior, they became fairly decent at guessing whether the coin will land heads or tails half an hour before the coin is released from the robot's hand. The scientist A has 80% chance of successfully predicting the outcome, while the scientist B is successful 60% of the time.

The robot started its routine, and the scientist A predicts the coin will land tails. The scientist B predicts the coin will land heads. Can you calculate the probability that the coin will land heads?

6. A player chooses a number $k \leq 52$ and the top k cards are drawn one by one from a properly shuffled standard deck of 52 cards. The player wins if the last drawn card is an Ace and if there is exactly one more Ace among the cards drawn. Which k should the player choose to maximize the chance of winning in this game?

7. Let N be a random variable whose values are positive integers. Prove that

$$\mathbb{E}[N] = \sum_{i=0}^{\infty} \mathbb{P}(N > i).$$

8. Each box of cereal contains a coupon. If there are p kinds of coupons, how many boxes of cereal have to be bought on average to obtain at least one coupon of each kind?

9. You roll a fair n-sided die repeatedly and sum the outcomes. What is the expected number of rolls until the sum is a multiple of n for the first time?

10. Is it possible to have two non-fair 6-sided dice, with sides numbered 1 through 6, with a uniform sum probability?

11. Consider 2^n players of equal skill[1] playing a game where the players are paired off against each other at random. The 2^{n-1} winners are again paired off randomly, and so on until a single winner remains.

[1]The probability of winning a game between any two players is $\frac{1}{2}$ for each player.

Find the probability that two contestants never play each other.

12. You have six identical pieces of rope. The top ends of the ropes are randomly paired up and the pairs are tied together. The same procedure is done with the bottom ends of the ropes. What is the probability that, as a result of this process, the six pieces of rope will be connected in a single closed loop of rope?

13. Consider a standard deck with 52 cards; 26 red and 26 black. A run is a maximum contiguous block of cards of the same color. For example, (R, B, R, B, ..., R, B) has 52 runs; while (R, R, ..., R, B, B, ..., B) has 2 runs. What is the expected number of runs in a shuffled deck of cards?

14. Eight boys and seven girls went to the movies and sat in the same row of 15 seats. Assuming that all the 15! permutations of their seating arrangements are equally likely, compute the expected number of pairs of neighbors of different genders.

15. A total of N balls is placed into N boxes in such a way that each ball is equally likely to be placed in each of the boxes and the placements are independent of each other. Find the expectation and the variance of the number of empty boxes.

16. In successive rolls of a pair of fair dice, what is the probability of rolling two sevens before rolling six even numbers?

17. What is the expected number of rolls of a fair 6-sided die until it shows a repeat, that is, a number you have already rolled?

18. If n is a random positive integer, what is the probability that 2^n starts with the digit 1?

19. At a bus stop you can take bus #1 or bus #2. Bus #1 passes 10 minutes after bus #2 has passed, whereas bus #2 passes 20 minutes after bus #1 has passed. What is the average waiting time to get on a bus at that bus stop?

20. Each of the seven dwarfs has his own bed in a common dormitory. Every night, they retire to bed one at a time, always in the same sequential order according to their age. On a particular evening, the youngest dwarf, who always retires first, has had too much to drink. He randomly chooses one of the seven beds to fall asleep in. As each of the other dwarfs retires, he chooses his own bed if it is not occupied, and otherwise randomly chooses another unoccupied bed. For each of the other dwarfs, what is the probability that he will sleep in his own bed?

21. Initially, there are N cars on a one-lane highway, spaced far apart. They all start driving at the same time in the same direction. Each car drives at a constant speed, with their speeds being random and independent from each other. In the beginning, all the cars have different speeds; however, when a faster car approaches a slower one, it gets stuck behind it and starts driving at the speed of the slower car. After a certain time, all the cars get separated into several clusters, with all the cars within the same cluster driving at the same speed. What is the expected number of clusters?

22. A total of 25 ants are scattered on a horizontal meter stick. Simultaneously, each ant picks a random direction (left or right) independently of the others, and starts marching at 1 cm/second in the chosen direction. Whenever two ants meet, they switch the directions of their movement. When an ant reaches

the end of the stick, it falls off. The ant in the middle is sick with cold. A sick ant will spread the cold to any ant it meets. By the time all ants have fallen off the stick, what is the expected number of sick ants?

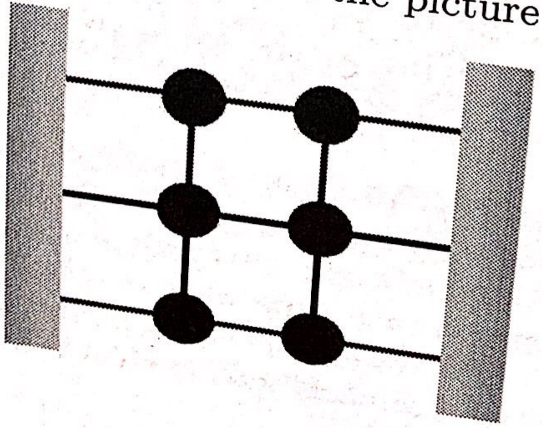
23. A standard six-sided die is rolled until two consecutive outcomes are equal. All the numbers obtained up to that point are summed. What is the probability that this sum is an even number?

24. Find the variance of the number of tosses of a fair coin until one obtains n heads in a row.

25. Three frogs are jumping on the vertices of an equilateral triangle. A vertex can be occupied by more than one frog. Every minute, each frog jumps from the vertex where it is located to one of the other two vertices, each being equally likely. The frogs choose where to jump independently of each other. If initially each vertex contains exactly one frog, how long does it take on average for all of the frogs to meet at the same vertex?

26. Is it possible to select a letter from $\{a, b, c\}$ uniformly at random using a finite number of tosses of a fair coin?

27. Two riverbanks and six islands are connected by thirteen bridges as shown in the picture below.



When a flood occurs, each bridge is destroyed with probability $\frac{1}{2}$, independent of the others. What is the probability that it will still be possible to cross the river after the flood using the bridges that remain?

28. Write a program in C++ that creates a random sequence from the numbers provided by the user, according to the following requirements:

    - The user first provides the desired length n for the sequence to be created.

    - The user then supplies positive numbers through the standard input. The total number N of the numbers provided by the user is not known in advance. It is only known that $N \geq n$. Once the user provides a negative number, the input is over.

    - The final sequence that the program outputs must consist of the numbers provided by the user, with each of the numbers appearing in the sequence with equal probability.

    For example, if the user provides $n = 2$ for the length of the sequence and the standard input $[1, 2, 3, -1]$ (corresponding to $N = 3$), the program output is $\{[1, 2], [1, 3], [2, 3], [2, 1], [3, 1], [3, 2]\}$, with each output having probability $\frac{1}{6}$.

    The desired random sequence has to be of length n. The memory usage of the program must be $O(n)$, that is, you are not allowed to store all of the N input numbers in the memory.

29. What is the expected last digit of the number of heads obtained when a fair coin is tossed 30 times?

30. What is the variance of $\det(A)$, where A is an $n \times n$ matrix whose entries are chosen from $\{-1, 1\}$ independently at random?

31. Two boxes, A and B, contain n balls each. In every step, you randomly choose a box and then draw one ball from it. Repeat this until the box you choose actually turns out to be empty. What is the expected number of remaining balls in the other box at the end of this process?

32. A box contains 100 green, 100 white, and one red ball. A player draws balls from the box without replacement and earns \$1 for each green ball and no money for white balls. The game is over once the player chooses to stop or once the red ball is drawn. If the game ends with the red ball, the player loses all the money. However, if the game ends because the player chooses to stop, the player can keep the money. What is the best strategy that the player should use to maximize the gain?

33. A wheel of fortune has 100 sectors labeled with 1, 2, ..., 100. When rotated, it can stop at each of the sectors with equal probability. The wheel is rotated repeatedly until the total sum of all obtained numbers is odd. Which odd number is the most likely to appear as the total sum?

34. What is the probability that two randomly chosen positive integers are relatively prime?

35. The rooms A and B initially contain 1 person each. Every second, a new person arrives to one of the rooms. If there are a people in room A and b people in room B, then the person arriving chooses to go to room A with probability $\frac{a}{a+b}$, or to room B with probability $\frac{b}{a+b}$.

    Determine the distribution of the random variable $\min\{A_n, B_n\}$, where A_n is the number of people in room A and B_n is the number of people in room B, after n seconds.

36. Evil Commander took the cell-phones from all of his one hundred soldiers. He then correctly wrote the names of soldiers on the phones, but intentionally placed phones randomly in boxes labeled by 1, 2, ..., 100. One by one the soldiers are taken to the room with the boxes. Once in the room, a soldier is allowed to perform the following 3-step procedure at most 50 times:

- Step 1. Choose one of the boxes;
- Step 2. Open the box;
- Step 3. If the box contains the soldier's own cell-phone, the soldier uses the fingerprint technology to unlock it. Then he can send a message to the President voicing the discontent with Evil Commander.

After repeating the procedure at most 50 times, the soldier must close all the boxes and leave the room without taking any phones regardless whether the soldier succeeded in finding his/her own device.

If the President receives 100 messages (one from each soldier), then the Evil Commander will be required to return the phones to the soldiers. However, if at least one of the soldiers fails to find the phone in 50 attempts or fewer, then the President will believe Evil Commander who will deny any mischief and none of the soldiers will get their phone back.

The soldiers are allowed to discuss and decide on a strategy. Find a strategy with success rate greater than 25%.

37. You throw a die until you get a 6. What is the expected number of throws conditioned on the event that all throws gave even numbers?

38. Starting with an empty $1 \times n$ board (a row of n squares), we successively place 1×2 dominoes to

cover two adjacent squares. At each stage, the placement of the new domino is chosen at random, with all available pairs of adjacent empty squares being equally likely. The process continues until no further dominoes can be placed. Find the limit, as n goes to infinity, of the expected fraction of the board that is covered when the process ends.

39. Seven dwarfs are captured by the evil queen who decides to play the following game: The queen puts a red hat or a green hat on the head of each of the dwarfs. The hats are chosen randomly and every configuration is equally likely. The dwarfs can see all the hats except for their own.

    At a signal, each dwarf can stay silent, or guess the color of his hat. The queen will free all seven dwarfs if at least one dwarf guesses his hat correctly and no one guesses the hat incorrectly. If all the dwarfs are silent, or some dwarfs say an incorrect color, the dwarfs remain captured. Find a strategy for the dwarfs to go free with probability greater than 85%.

40. A standard deck of 52 playing cards is shuffled and cards are flipped over, in sequence, one at a time. Immediately before each flip, you have the opportunity to bet any amount of money that you have, from \$0 to everything you have, on the color of the card that the dealer is about to flip.[2] A correct bet of $\$x$ wins you $\$x$; an incorrect bet costs you $\$x$. You begin the game with \$100. At any point in the game, you can recall perfectly the sequence of cards that has been flipped. Assume that dollars are continuously divisible and you can bet any real

[2]For instance, if you have \$5 and the dealer is about to flip a card, you may either do nothing, bet any amount of money up to \$5 that the card will be red, or bet any amount of money up to \$5 that the card will be black.

number of dollars. What is the maximum amount of money you can be guaranteed to have once the deck is through, and what betting strategy should you use to achieve this outcome?

1.2 Random Walks and Martingales

1. A fair coin is tossed 3 times. For every head, the gambler earns a dollar, and for every tail, the gambler loses a dollar. Denote by M_i the amount of money that the gambler has after i tosses, for $i \in \{0, 1, 2, 3\}$.

   (i) Construct the sample space Ω, sigma algebra $\mathcal{F}$ on Ω, probability measure $\mathbb{P}$ on $\mathcal{F}$, and the random variables M_i on $(\Omega, \mathcal{F}, \mathbb{P})$ that correspond to the described random experiment.

   (ii) Construct the filtration $(\mathcal{F}_i)_{i=0}^3$ for which the sequence of random variables $(M_i)_{i=0}^3$ is a martingale.

2. Assume that the sequence $(M_n)_{n=0}^\infty$ is a martingale with respect to the filtration $(\mathcal{F}_n)_{n=0}^\infty$. Assume that T is a bounded stopping time with respect to the same filtration. Prove that

$$\mathbb{E}[M_T] = \mathbb{E}[M_0].$$

3. You are playing the following game with a well shuffled deck of 52 cards facing down: at times $n = 1, 2, \ldots, 52$, you turn over a new card and observe its color. Just once in this game, right before turning over a card, you must say "The next card is Red!" You win the game if the next card turned over is indeed red, and lose otherwise. Let R_n be the number of red cards remaining face down after the nth card has been turned over. Show that $X_n = \frac{R_n}{52-n}$, $0 \leq n < 52$, is a martingale. Show that there is no strategy that guarantees winning with probability higher than $\frac{1}{2}$.

4. Let S_n be a simple random walk starting at $S_0 = 100$ with $S_n = S_0 + X_1 + \ldots X_n$, where $P(X_i = 1) = p$ and $P(X_1 = -1) = 1 - p = q$, with $p < 1/2$.

(i) Prove that $M_n = \left(\frac{q}{p}\right)^{S_n}$ is a martingale.

(ii) Let $\tau = \min\{n \geq 0 : S_n = 200 \text{ or } S_n = 0\}$. Prove that $\mathbb{E}[\tau] < +\infty$.

(iii) Find $P(S_\tau = 200)$.

(iv) Compute $\mathbb{E}[\tau]$.

5. What is the expected time for a symmetric random walk that starts at 0 to reach either a or b, where $a < 0 < b$ are two integers? What is the probability that it will reach b before a?

6. Let a and b be positive integers, and let W_m for $m \in \{0, 1, \dots, n\}$ be a symmetric random walk starting at a. Calculate

$$\mathbb{P}\left(\min_{0 \leq m \leq n} W_m \leq 0, W_n = b\right).$$

7. What is the probability that a symmetric random walk gets back to its starting point?

8. What is the expected number of steps for a symmetric random walk to return to its starting point?

9. There are n coats in a coat check room, belonging to n people, who make an attempt to leave by picking a coat at random. Those who pick their own coat leave, the rest return the coats and try again. Let N be the number of rounds of attempts until everyone has left. Show that $\mathbb{E}[N] = n$ and $\text{var}(N) = n$.

10. Three players A, B, and C have a, b, and c coins initially. In each turn each player tosses a fair coin. If all three outcomes are heads or all three are tails, then nothing happens; otherwise the player with an outcome different from others receives a coin from each of the other two players. The game stops when

one or more of the players end up with 0 coins. What is the average number of turns that will occur before the game ends?

11. Provide an example of a martingale that is not a Markov process and an example of a Markov process that is not a martingale.

12. A fair coin is tossed until the sequence $HTHT$ is obtained. What is the expected number of tosses?

13. Which is more likely: getting no heads when a fair coin is tossed n times, or getting fewer than n heads when a fair coin is tossed $8n$ times?

14. Assume that $X_1, X_2, \ldots, X_n$ are independent random variables such that $\mathbb{P}(X_i = 1) = p_i$ and $\mathbb{P}(X_i = 0) = 1 - p_i$, where $p_i \in (0,1)$ for every $1 \leq i \leq n$. Let $x > \mathbb{E}[X_1 + \cdots + X_n]$. Prove that

$$\mathbb{P}(X_1 + \cdots + X_n > x) \; < \; \frac{e^x (p_1 + \cdots + p_n)^x}{x^x \, e^{p_1 + \cdots + p_n}}.$$

15. A wheel of fortune has $n \geq 2$ sectors labeled by 1, 2, ..., n. In each step the wheel is rotated by one sector to the left or to the right with equal probability. The procedure is repeated until every number from the set $\{1, 2, \ldots, n\}$ appears at least once on top of the wheel. If the number 1 is on top at the beginning, determine the probability that the last number that appears on top is k, where $k \in \{2, 3, \ldots, n\}$.

16. A wheel of fortune has n sectors labeled by 1, 2, ..., n. In each step the wheel is rotated by one sector to the left or to the right with equal probability. What is the expected number of steps needed for every number from the set $\{1, 2, \ldots, n\}$ to appear at least once at the top of the wheel?

17. Assume that a and w are positive real numbers such that $w < a$. Let T be the hitting time of the set $\{-a, a\}$ by a symmetric random walk $(W_n)_{n=0}^{\infty}$ that starts at $W_0 = w$. Calculate the expected value of TW_T.

18. A random walk on the integer line moves either two steps to the right with probability $1/2$ or one step to the left with probability $1/2$. Find the proportion of unvisited sites.

19. There are n balls in a bag, colored from 1 to n. In each step, a pair of balls is chosen uniformly at random from all the pairs of differently colored balls, and then the second ball of the chosen pair is painted with the color of the first ball. Finally, both balls are placed back into the bag. What is the expected number of steps it takes for all the balls to be of the same color?

20. There are n balls in a bag, colored from 1 to n. In each step, two balls are taken from the bag, one after the other, uniformly at random, and the second ball is painted with the color of the first. Then, both balls are placed back into the bag. What is the expected number of steps it takes for all the balls to be of the same color?

1.3 Continuous Probability

1. What is the maximal possible variance of a random variable that takes values in the set $[-1, 1]$?

2. What is the maximal possible variance of a random variable that takes values in the set $[0, 1]$?

3. Assume that A, B, C are three independent identically distributed random variables. What can be said about the probability that $A < B$ and $A < C$?

4. Let X be a random variable such that $P(X \neq 0) > 0$. Suppose that for some real numbers a and b the random variables aX and bX have the same distribution. Is it true that $a = b$? What if you also assume that a and b are positive?

5. An unfair coin is tossed n times and all n tosses resulted in heads. What is the probability that the next toss will be head? Assume that the probability p of getting heads is a random variable with uniform distribution on $[0, 1]$.

6. The probability of a car passing a certain marker in a 20-minute window is 90%. What is the probability of a car passing the marker in a 5-minute window? Assume that the cars are moving independently of each other.

7. Assume that $M > 0$ is a fixed real number and that X is a random variable with uniform distribution on the interval $(0, M)$. A price of an item is exactly $\$X$. A person with a total wealth of $\$M$ has decided to buy as many of these items as possible. What is the expected amount of money that the person will have remaining?

8. How many independent random variables with uniform distribution on $[0, 1]$ have to be generated in

order to ensure with probability 95% that at least one of them is between 0.5 and 0.55?

9. Consider a random point P on the circumference of the unit circle centered at $(0, 0)$ and a random point Q inside the circle. Using PQ as diagonal, a rectangle is drawn with sides parallel to x- and y-axis. What is the probability that the rectangle is inside the circle?

10. Three points are chosen uniformly at random on a circle. What is the probability that they form an acute triangle?

11. What is the probability that two uniform random points in a square are such that the center of the square lies inside the circle formed by taking the two points as diameter?

12. Four points are chosen randomly on the unit sphere. What is the probability that the center of the sphere lies inside the tetrahedron determined by the four points?

13. Let X and Z be independent random variables with X a discrete random variable given by $P(X = 1) = P(X = -1) = 1/2$ and Z a standard normal random variable. Let $Y = XZ$. Is Y a standard normal variable? Is $Y + Z$ normal? Does (Y, Z) have joint normal density?

14. What is the expected area of the triangle generated by three points on a unit circle chosen uniformly at random and independent of each other?

15. Let X be a random variable. Is it always true that $\mathbb{E}[X^6] \geq \mathbb{E}[X]\,\mathbb{E}[X^5]$?

16. Let X be a random variable uniformly distributed on $[0, \pi]$. Find $E[X \mid \sin X]$.

17. Consider n points chosen uniformly at random on a unit circle, where $n \geq 3$. What is the probability that the center of the circle is inside the polygon formed by these n points?

18. Assume that $X_1, X_2, \ldots, X_n$ are independent uniform random variables on $[0, 1]$, where $n \geq 2$. What is the expected value of their minimum?

19. Assume that $X_1, X_2, \ldots, X_n$ are independent uniform random variables on $[0, 1]$, where $n \geq 2$. What is the expected value of their maximum?

20. Let $X_1 \sim U(0, 1)$, $X_2 \sim U(X_1, 1)$, $X_3 \sim U(X_2, 1)$, $\ldots$, $X_n \sim U(X_{n-1}, 1)$. What is the probability density function of X_n?

21. Let $X_1 \sim U(0, 1)$, $X_2 \sim U(X_1, 1)$, $X_3 \sim U(X_2, 1)$, $\ldots$, $X_n \sim U(X_{n-1}, 1)$. Evaluate

$$\lim_{n \to \infty} \mathbb{E}[X_1 X_2 \cdots X_n].$$

22. Let X, Y, and Z be independent random variables with uniform distribution on $[0, 1]$. Find the distribution of the random variable $(XY)^Z$.

23. Three points are chosen uniformly at random on the circle $x^2 + y^2 = 1$. What is the expected length of the arc that contains the point $(1, 0)$.

24. Let $X_1, X_2, \ldots, X_n$ be independent uniform random variables on $[0, 1]$. Let $X_{(k)}$ denote the kth order statistic. Find $\mathbb{E}[X_{(k)}]$.

25. Independent random numbers with uniform distribution on $[0, 1]$ are generated as long as they keep decreasing. The procedure stops when the obtained number is greater than the previous one. What is the average number of numbers that were picked?

What is the average value of the smallest of the chosen numbers?

26. The numbers x_1, x_2, x_3, ..., are chosen uniformly at random from $[0, 1]$ and independently from each other as long as they follow the pattern $x_1 > x_2$, $x_2 < x_3$, $x_3 > x_4$, $x_4 < x_5$, ... How many numbers on average can be chosen before the pattern is broken?

27. Let $\{X_1, X_2, \ldots, X_n\}$ be an independent sample of size n from uniform distribution on $[0, 1]$. Let p_n be the probability that $X_i + X_{i+1} \leq 1$ for all $i = 1, 2, \ldots, n-1$. Prove that $\lim_{n\to\infty} p_n^{1/n}$ exists and compute it.

28. Let X_1, X_2, X_3, ... be independent random variables with uniform distribution on $[0, 1]$. Let N be the smallest integer for which $X_1+X_2+\cdots+X_N > 1$. Evaluate $\mathbb{E}[N]$.

29. Let X_1, X_2, X_3, ... be independent random variables with uniform distribution on $[0, 1]$. Let N be the smallest integer for which $X_1+X_2+\cdots+X_N > 1$. Evaluate $\mathbb{E}[X_N]$.

30. Assume that $n \geq 2$. The line segment of length 1 is divided into n smaller segments by $n-1$ points that are chosen uniformly at random. What is the expected value of the shortest segment?

31. Two marks are placed on a stick of length 1. The positions of marks are chosen independently of each other and the distribution of each of them is uniform. After this the stick is broken randomly into n pieces. What is the probability that the two marks are on the same piece?

32. Is the product of two Gaussian random variables also a Gaussian? Is the answer any different if they are independent?

33. Let X and Y be independent standard Gaussians. Find $E[X|XY]$.

34. Let X, Y, and Z be three normal random variables. If $\rho_{X,Y} > 0$ and $\rho_{Y,Z} > 0$, do we necessarily have $\rho_{X,Z} > 0$?

35. Assume that X and Y have joint normal distribution, that each of $X, Y \sim N(0,1)$, and that their correlation is $\frac{1}{2}$. Calculate $\mathbb{P}(X \geq 0, Y \geq 0)$.

36. Assume that X and Y are bivariate normal random variables with mean 0 and covariance matrix $\Sigma = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$. Evaluate $\mathbb{E}\left[e^X \middle| Y = y_0\right]$.

37. What is the smallest value that $\text{corr}(X, Y)$ can have if it is known that there exists a random variable Z such that $\text{corr}(X, Z) = \alpha$ and $\text{corr}(Y, Z) = \beta$?

38. If X and Y are two uniform random variables on $[0, 1]$ with correlation ρ, what is the distribution of $X + Y$?

39. Let U and V be uniformly distributed random variables in $[0, 1]$ with correlation ρ. Calculate the conditional expectation $E[U|V = v]$.

40. The correlation matrix of n random variables has all the off–diagonal entries equal to ρ. What are the possible values for ρ?

1.4 Brownian Motion

1. If W is a Brownian motion and $s < t$, find
$$\operatorname{var}(W_s + W_t).$$

2. If W is a Brownian motion and $s < t < u$, evaluate
$$\mathbb{E}[W_s W_t W_u].$$

3. If $B(t) = (B_1(t), B_2(t))$ is a two-dimensional Brownian motion, prove that $X(t) = B_1(t)B_2(t)$ is a martingale.

4. If W is a Brownian motion and $a, T > 0$, evaluate
$$\mathbb{P}\left(W_t \leq a\sqrt{t} \text{ for all } t \leq T\right).$$

5. If X_t and Y_t are independent Brownian motions, what is the distribution of $\frac{X_t}{Y_t}$?

6. Assume that B_t is a Brownian motion. Determine whether B_t^2 is a martingale, a submartingale, or a supermartingale.

7. If $a < 0 < b$, what is the probability that the Brownian motion hits level a before level b?

8. What is the expected time for Brownian motion to hit a or b, where a and b are two real numbers that satisfy $a < 0 < b$?

9. The price of an asset has lognormal distribution given by $S(t) = S_0 e^{\mu t + \sigma B_t}$, where B_t is a Brownian motion. The interest rate is r. Determine the price of a derivative security that pays \$1 if $S(5) < S(3)$ and $S(3) > S(2)$, and 0 otherwise.

10. The price of an asset has lognormal distribution given by $S(t) = S_0 e^{\mu t + \sigma B_t}$, where B_t is a Brownian motion. The interest rate r satisfies $r = \frac{\sigma^2}{2}$. Determine the price of a derivative security that pays \$1 if $S(5) < S(2)$ and $S(3) > S(2)$, and 0 otherwise.

11. If W_t is a Brownian motion, denote by m_T the minimum of W_t on the interval $[0, T]$. Find the joint cumulative distribution function for the random vector (W_T, m_T).

12. If $B(t) = (X(t), Y(t))$ is a two-dimensional Brownian motion and a and b two positive real numbers, what is the probability that $B(t)$ hits the line $x = a$ before the line $y = b$?

13. Determine the probability that the Brownian motion hits zero in the time interval $[t_1, t_2]$ for $t_1 < t_2$.

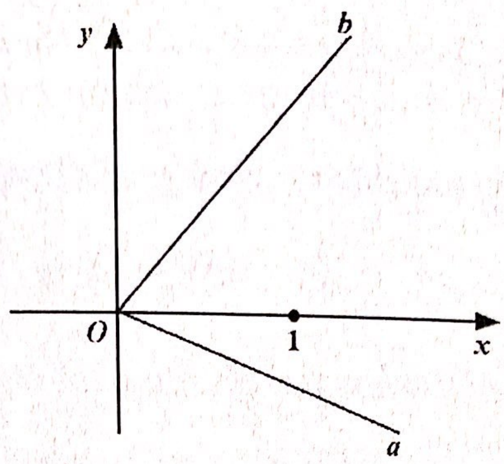
14. Let L_t be the time of the last zero of the Brownian motion before time t. Find the distribution of L_t.

15. What is the expected total time that the Brownian motion $\{B_t\}_{0 \le t < +\infty}$ spends above the line $y = t$?

16. If B_t is a standard Brownian motion, evaluate
$$\mathbb{P}\left[B_1 > B_2 \middle| B_3 = \sqrt{3} \right].$$

17. Let $P_t = (Z_t, W_t)$ be a two-dimensional stochastic process that starts at $(0, 0)$. The processes Z_t and W_t are independent Brownian motions. What is the expected time for P_t to exit the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where a and b are fixed constants?

18. Let Oa and Ob be two rays in the two dimensional plane such that $\angle aOb$ contains the point $(1, 0)$ as shown in the picture below. What is the probability that the two dimensional Brownian motion starting at $(1, 0)$ hits the ray Oa before hitting the ray Ob?



19. Evaluate the variance of $\int_0^T W_s\, ds$.

20. Compute $\mathbb{E}\left[W_t^4 \middle| \mathcal{F}_s\right]$, for $0 < s < t$.

21. Let $\tau = \inf\{t > 0 : |B_t| = 1\}$. Find $\mathrm{var}(\tau)$.

22. Let τ denote the exit time of W_t from the interval $[-a, b]$ for a, $b > 0$. Evaluate $\mathbb{E}\left[e^{-\lambda\tau}\right]$, where λ is a positive real number.

23. Let $X_t = \int_0^t \frac{dW_s}{\sqrt{1+s}}$. Compute $\mathbb{P}\left(\sup_{0\le t\le 2} X_t \ge 1\right)$.

24. Let $X_t = \mu t + W_t$ be a Brownian motion with drift $\mu > 0$. Let $a, b > 0$ and let τ be the first time when X_t hits a or $-b$. Find $\mathbb{P}(X_\tau = -b)$, $\mathbb{E}[X_\tau]$, $\mathbb{E}[\tau]$, $\mathrm{var}(X_\tau)$, and $\mathrm{var}(\tau)$.

25. Calculate $\mathbb{E}\left[W_t \middle| \int_0^t W_s\, ds = x\right]$.

1.5 Stochastic Differential Equations

1. Suppose that $dS_t = \mu S_t\, dt + \sigma S_t\, dW_t$. What is the stochastic differential equation satisfied by $\log S_t$?

2. Solve the stochastic differential equation

$$dX_t = e^{2t}(1 + 2W_t^2)\, dt + 2e^{2t} W_t\, dW_t.$$

3. Solve the stochastic differential equation

$$dX_t = e^t(1 + W_t^2)\, dt + (1 + 2e^t W_t)\, dW_t,$$

with $X_0 = 0$.

4. Solve the stochastic differential equation

$$dX_t = t^2\, dt + e^{\frac{t}{2}} \cos W_t\, dW_t,$$

with $X_0 = 0$. Compute $\mathbb{E}[X_t]$ and $\mathrm{var}(X_t)$.

5. Solve the stochastic differential equation

$$dX_t = (2 - X_t)\, dt + e^{-t} W_t\, dW_t.$$

6. For $t < 1$, solve the stochastic differential equation

$$dX_t = -\frac{X_t}{1-t} dt + dW_t,$$

with $X_0 = 0$. Compute the mean and covariance functions of X_t. Is X_t a Brownian bridge?

7. Determine the stochastic process X_t such that

$$\begin{aligned} dX_t &= (X_t + a)(X_t + b)\left(X_t + \frac{a+b}{2}\right) dt \\ &\quad + (X_t + a)(X_t + b)\, dW_t, \end{aligned}$$

with $X_0 = \xi$, where a and b are positive numbers. Consider the cases $a = b$ and $a \neq b$ separately.

8. Solve the stochastic differential equation

$$dX_t = (\alpha + \beta X_t)\, dt + \sigma\, dW_t.$$

9. Solve the stochastic differential equation

$$dX_t = (\alpha + \beta X_t)\, dW_t.$$

10. Evaluate the integral

$$\int_0^T W_s^2\, dW_s.$$

11. The prices M_t, N_t, and P_t of three assets have the following dynamics:

$$\begin{aligned} dM_t &= \mu_M M_t\, dt + \sigma_M M_t\, dW_M(t); \\ dN_t &= \mu_N N_t\, dt + \sigma_N N_t\, dW_N(t); \\ dP_t &= \mu_P P_t\, dt + \sigma_P P_t\, dW_P(t). \end{aligned}$$

Assume that

$$\begin{aligned} dW_M(t)\, dW_N(t) &= \rho_{MN}\, dt; \\ dW_N(t)\, dW_P(t) &= \rho_{NP}\, dt; \\ dW_P(t)\, dW_M(t) &= \rho_{MP}\, dt. \end{aligned}$$

What is the mean of $M_t N_t P_t$? What is the correlation between M_t and $N_t P_t$?

12. Let $X_t = \mathbb{E}\left[W_T^3 \,|\, \mathcal{F}_t\right]$, $0 \le t \le T$, where W_t is a Brownian motion. Find a stochastic process Y_t such that $X_t = \int_0^t Y_u\, dW_u$, $0 \le t \le T$.

13. Suppose that $dS_t = \mu S_t\, dt + \sigma S_t\, dW_t$. For which α is the process S_t^α a martingale?

14. Let X_t be the solution of the stochastic differential equation

$$dX_t = \mu X_t \, dt + \sigma X_t \, dW_t$$

with $X_0 = 1$. Let τ be the exit time of X_t from the interval $\left(\frac{1}{2}, 2\right)$. Compute $P(X_\tau = 2)$.

15. Given a standard Brownian motion W_t, let $X_t = e^{W_t}$. What is $\mathbb{E}\left[X_t\right]$ for $t = 2$?

16. What is the correlation between W_t and $\int_0^t W_s \, ds$?

17. Consider the process defined by $X_0 = x_0 > 0$ and $dX_t = X_t^{\frac{3}{2}} \, dW_t$. What are the mean and variance of X_t for fixed t?

18. Assume that $X_t = 2^{B_t}$, where B_t is a Brownian motion. Is X_t a martingale? If not, how can you make X_t a martingale by a change of measure?

19. Calculate

$$\mathbb{E}\left[W(t) e^{-\lambda \int_0^t s \, dW(s)}\right]$$

in two different ways: using Girsanov's theorem and without Girsanov's theorem.

20. Let X_t be such that $dX_t = \theta(t) \, dt + dW_t$, where $\theta(s)$ is a bounded function. Let

$$D_t = e^{-\int_0^t \theta(s) \, dW_s - \frac{1}{2} \int_0^t \theta^2(s) \, ds}.$$

Prove that D_t and $Z_t = X_t D_t$ are martingales. Use this to show that $(W_t + t) e^{-W_t - \frac{1}{2} t}$ is a martingale.

21. If W_t is a standard Brownian motion and b and α are positive real numbers, calculate

$$\mathbb{P}\left(\max_{0 \le t \le T} \{W_t + bt\} \le \alpha\right).$$

22. Assume that $dr_t = (\mu - \alpha r_t)\,dt + \sigma\,dW_t$. If $r_0 = 0$, evaluate $\mathbb{E}\left[r_t^3\right]$.

23. Assume that $r(t)$ satisfies $r(0) = r_0$ and

$$dr(t) = \left(\frac{d\mu(t)}{dt} + \lambda\,(\mu(t) - r(t))\right) t + \sigma(t)\,dW(t).$$

Determine $r(t)$, its expectation, and variance.

24. Find a solution of the partial differential equation

$$2\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} + x = 0,$$

$0 \le t < T$, with terminal condition $u(T, x) = x^2$.

25. The asset prices $X(t)$ and $Y(t)$ satisfy the stochastic differential equations

$$\begin{aligned} dX(t) &= \mu_x X(t)\,dt + \sigma_x X(t)\,dW_x(t); \\ dY(t) &= \mu_y Y(t)\,dt + \sigma_y Y(t)\,dW_y(t), \end{aligned}$$

with $dW_x(t)\,dW_y(t) = \rho_{xy}\,dt$ and $\rho_{xy} \in (-1, 1)$. Let r denote the risk-free interest rate. Find the price at time 0 of the exchange option with payoff

$$V(T) = \max\{X(T) - Y(T), 0\}.$$