MFE230E - Empirical Methods in Finance
Week 7: Volatility

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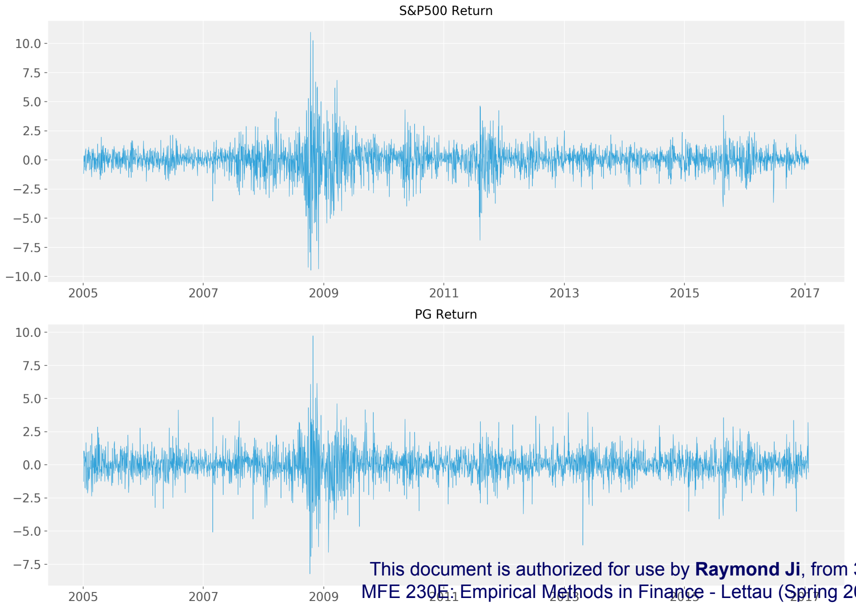
Haas School of Business

- What is volatility?
- Measures of volatility: realized, conditional, implied
- ARCH/GARCH models: Estimation and inference
- Extensions of the basic ARCH/GARCH model
- Volatility forecasting
- Realized volatility: pros and cons
- Implied volatility: VIX
- If time permits: Stochastic volatility and MCMC estimation
- Reading:
  - Ruppert ch. 18
  - Further reading: Taylor "Asset Price Dynamics, Volatility, and Prediction", chs. 8-11

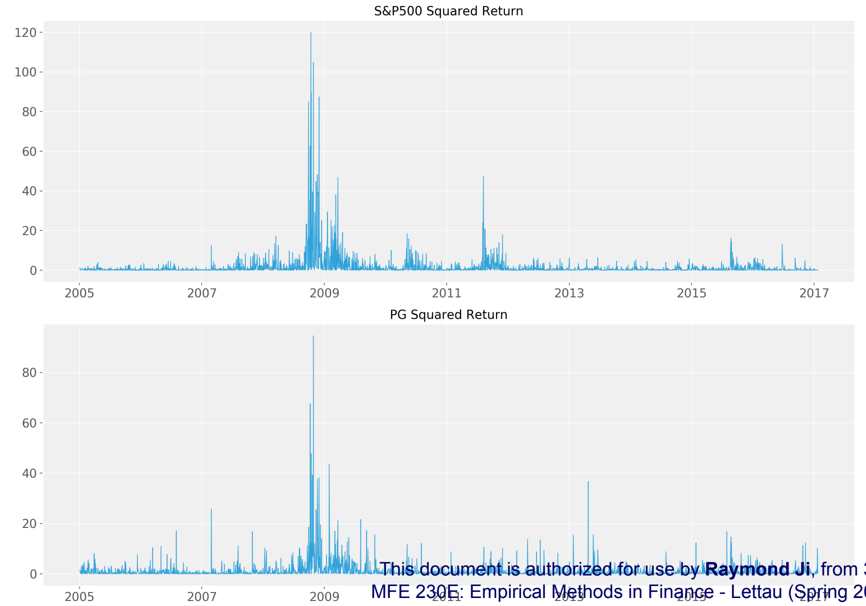
Outline

1. **Volatility is changing over time**
2. ARCH/GARCH
3. Realized volatility
4. Implied volatility
5. Stochastic Volatility and MCMC

Volatility is changing over time

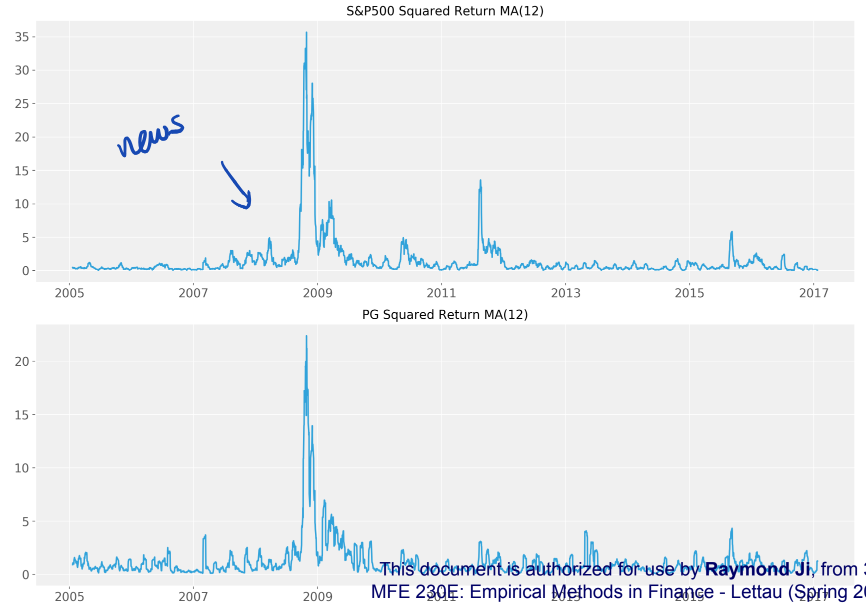


Volatility is changing over time

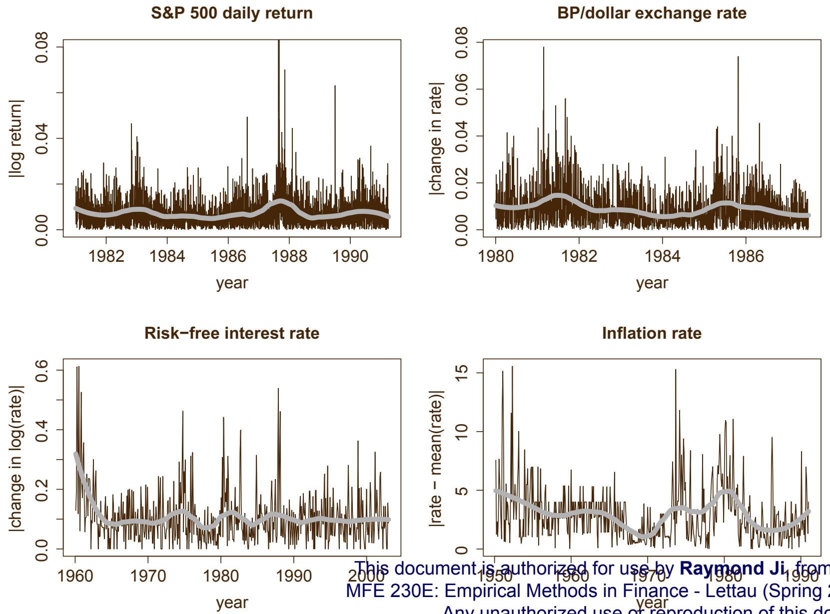


there is some persistence of volatility!

Volatility is changing over time



Volatility is changing over time in most asset markets



WHY DOES VOLATILITY CHANGE OVER TIME?

- **News announcements**: Investors update beliefs
- **Leverage**: Equity riskier as prices fall
- **Volatility feedback**: Increase in volatility raises risk premium of an asset
- **Liquidity**: Small shocks can have large effect on prices if market is illiquid
- **Economic uncertainty**: When the economic state is uncertain, changes in beliefs may cause large price changes

Definition and measures of volatility

- Volatility is usually defined as the **standard deviation** of an asset rather than the variance (Why?)
- Volatility can be measured at different frequencies: Daily, weekly, monthly, ...
- Annualized volatility: If σ is daily volatility and there are 252 trading days, annualized volatility is $\sqrt{252}\sigma$
- Question: Is annualized vol different from vol of annual data?

Today's lecture

1. **Conditional volatility**: Expected volatility at some future time $t+h$ based on all information available at time t: $\mathrm{E}_t[\sigma_{t+h}]$
   Note: depends on model for conditional expectations
2. **Realized vol** using high frequency data:

$$\mathrm{RV}_t = \sum_{i \in t} r_{i,t}^2$$

3. **Implied vol**: Volatility that prices and option correctly
   Example:

$$\mathrm{BS}(S_t, K, r, t, \sigma_t) = C_t$$

$$\Rightarrow \sigma_t = f(S_t, K, r, t, C_t)$$

OUTLINE

1. Volatility is changing over time
2. **ARCH/GARCH**
3. Realized volatility
4. Implied volatility
5. Stochastic Volatility and MCMC

- Engle (1982), Nobel Prize 2003
- Constant volatility:

$$r_t = \mu_t + \epsilon_t$$
$$\epsilon_t = \sigma e_t$$
$$e_t \sim N(0,1)$$

- In practice, often assume AR(1) structure for returns: $\mu_t = \mu + \phi r_{t-1}$

$$r_t = \mu + \phi r_{t-1} + \sigma e_t$$

Autoregressive Conditional Heteroskedasticity (ARCH)

- ARCH(1) model:

$$r_t = \mu_t + \epsilon_t$$

$$\epsilon_t = \sigma_t e_t$$

$$\sigma_t^2 = \omega + \alpha \epsilon_{t-1}^2$$

$$e_t \sim N(0, 1)$$

- Note: σ_t^2 is known at time $t-1$ (see stochastic vol models later)
  $\Rightarrow$ σ_t and e_t are independent

Properties of ARCH(1)

$$r_t = \mu_t + \epsilon_t$$

$$\epsilon_t = \sigma_t e_t$$

$$\sigma_t^2 = \omega + \alpha \epsilon_{t-1}^2$$

$$e_t \sim N(0, 1)$$

- Conditional variance:

$$E_{t-1}[\epsilon_t^2] = \sigma_t^2 E[e_t^2] = \sigma_t^2$$

- Unconditional variance:

$$E[\sigma_t^2] = \omega + \alpha E[\epsilon_{t-1}^2]$$
$$= \omega + \alpha E[\sigma_{t-1}^2] E[e_{t-1}^2]$$
$$\Rightarrow E[\sigma_t^2] = \frac{\omega}{1 - \alpha}$$

- Positivity: $\omega, \alpha > 0$
- Stationarity: $|\alpha| < 1$

$$\sigma_t = \sqrt{\omega + \alpha \epsilon_{t-1}^2}$$

$$r_t = \mu_t + \epsilon_t = e_t \sqrt{\omega + \alpha \epsilon_{t-1}^2} + \mu_t$$

$$E_{t-1}[\epsilon_t^2] = E_{t-1}[\sigma_t^2 e_t^2]$$
$$= E[\underset{\in F_t}{\sigma_t^2} \underset{\perp\!\!\!\perp F_t}{e_t^2} \mid e_{t-1}, \ldots, e_0]$$

- Primitive shocks of ARCH model are normal
- It can be shown that ARCH models have excess kurtosis
- For an ARCH(1):

$$K = \frac{3(1-\alpha^2)}{1-3\alpha^2}$$

- How can ARCH generate excess kurtosis even though shocks are normal?

Simulated ARCH model

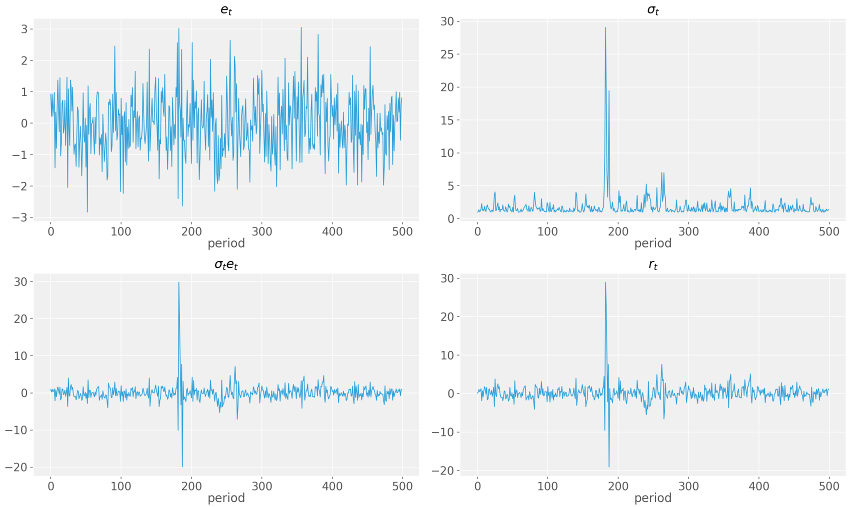
ARCH(1):

$$r_t = 0.1 + 0.1 r_{t-1} + \epsilon_t$$

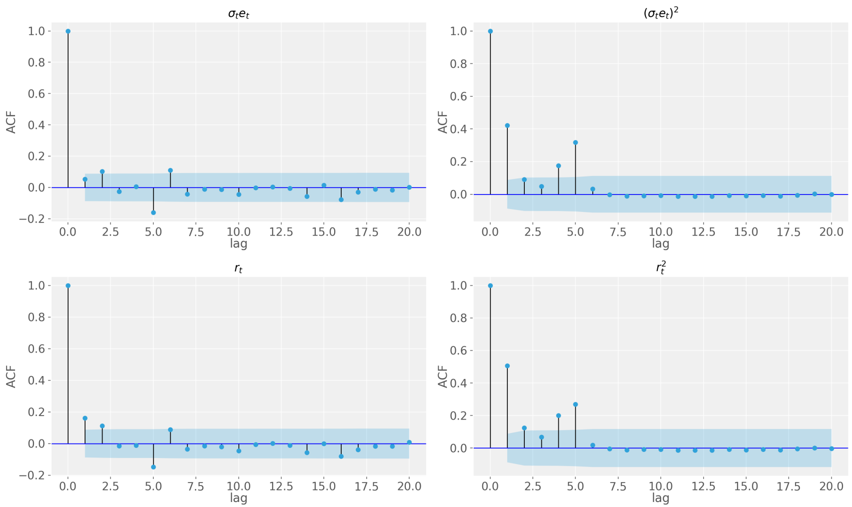
$$\epsilon_t = \sigma_t e_t$$

$$\sigma_t^2 = 1 + 0.95\,\epsilon_{t-1}^2$$

Simulated ARCH model



ARCH: AUTOCORRELATION



ARCH(p)

Easy to generalize ARCH model:

$$
\begin{aligned}
r_t &= \mu_t + \epsilon_t \\
\epsilon_t &= \sigma_t e_t \\
\sigma_t^2 &= \omega + \alpha_1 \epsilon_{t-1}^2 + \ldots + \alpha_p \epsilon_{t-p}^2 \\
e_t &\sim N(0,1)
\end{aligned}
$$

ARCH(p)

Unconditional variance:

$$E[\sigma_t^2] = \frac{\omega}{1 - \alpha_1 - \ldots - \alpha_p}$$

Results about stationarity and autocorrelation generalize to ARCH(p)

As we will see, volatility is very persistent $\Rightarrow$ require many ARCH lags

$\Rightarrow$ GARCH model

Bollerslev (1986): GARCH(1,1)

$$r_t = \mu_t + \epsilon_t$$

$$\epsilon_t = \sigma_t e_t$$

$$\sigma_t^2 = \omega + \alpha\epsilon_{t-1}^2 + \beta\sigma_{t-1}^2$$

$$e_t \sim N(0, 1)$$

- These models have been incredibly popular

- GARCH(1,1) = ARCH(∞):

$$\sigma_t^2 = \sum_{i=0}^{\infty} \beta^i \omega + \alpha \sum_{i=0}^{\infty} \beta^i \epsilon_{t-1-i}^2$$

$$(1-\beta L)\sigma_t^2 = \omega + \alpha \epsilon_{t-1}^2$$

- GARCH(1,1) = ARMA(1,1) for ϵ_t^2:

$$\epsilon_t^2 = \omega + (\alpha+\beta)\epsilon_{t-1}^2 + v_t + \beta v_{t-1}$$

$$v_t = \epsilon_t^2 - \sigma_t^2 \quad \text{noise}$$

Note that $v_t = \epsilon_t^2 - \sigma_t^2$ is the volatility "suprise".

- Unconditional mean of variance:

$$E[\sigma_t^2] = \frac{\omega}{1-\alpha-\beta}$$

GARCH(p, q)

GARCH(1,1) can be generalized:

$$r_t = \mu_t + \epsilon_t$$

$$\epsilon_t = \sigma_t e_t$$

$$\sigma_t^2 = \omega + \sum_{i=1}^{p} \alpha_i \epsilon_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2$$

$$e_t \sim N(0,1)$$

- Unconditional mean of variance:

$$\mathrm{E}[\sigma_t^2] = \frac{\omega}{1 - \sum \alpha_i - \sum \beta_j}$$

GARCH(p, q)

- GARCH(p, q) = ARMA($\max(p, q), q$) for ϵ_t^2
- Conditions for positivity are more complex (Nelson and Cao JEBS, 1992)
- Stationary if $\sum \alpha_p + \sum \beta_q < 1$

Simulated GARCH model

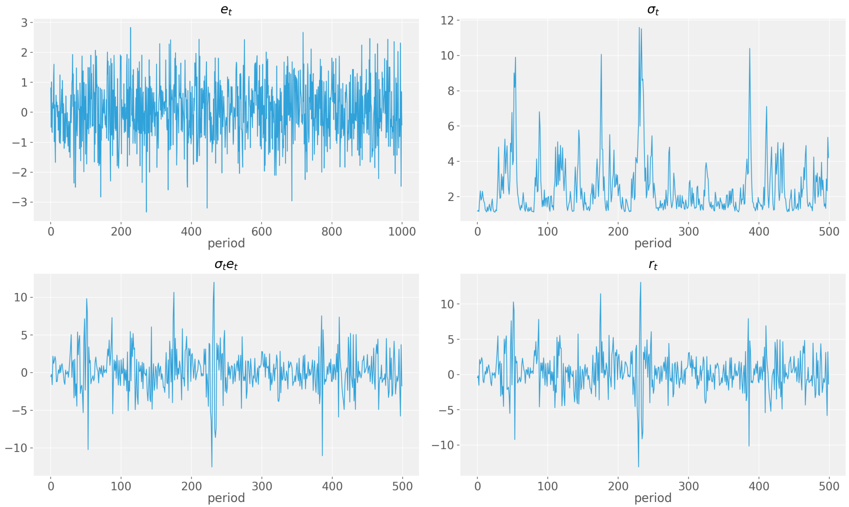
GARCH(1,1):

$$r_t = 0.1 + 0.1 r_{t-1} + \epsilon_t$$

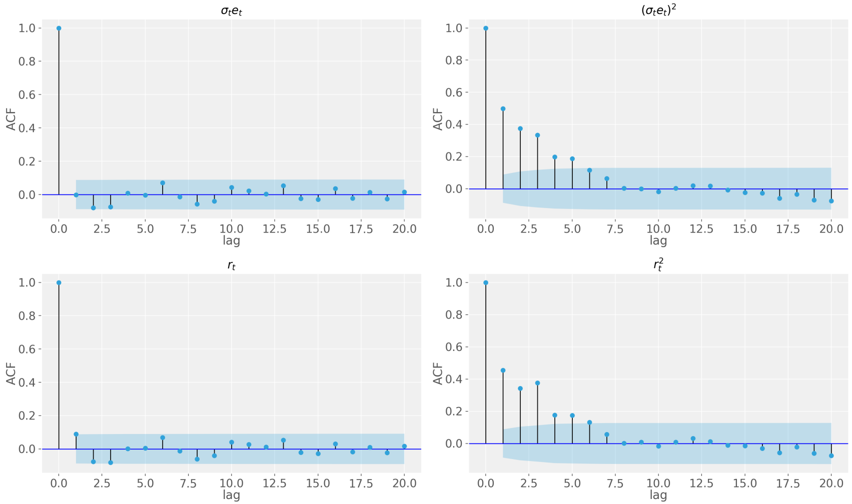
$$\epsilon_t = \sigma_t e_t$$

$$\sigma_t^2 = 1 + 0.8\epsilon_{t-1}^2 + 0.2\sigma_{t-1}^2$$

Simulated GARCH model



SIMULATED GARCH MODEL: ACF



Discussion

- GARCH model are successful because they can capture the persistence in σ_t^2, in a parsimonious way
- Relatively easy to estimate for low orders of p and q
- Trade-off: More parameters to capture the accurate dynamics, but there are more parameters to estimate.
- Those parameters have restrictions making the estimation tricky.
- For most applications, GARCH(1,1) does a good job.

Discussion

Limitations:

- In GARCH models, positive and negative shocks have symmetric effects.
- ARCH/GARCH models tend to overpredict volatility because they respond slowly to large isolated shocks.
- Volatility has complicated persistence, sometimes ARCH/GARCH models are too restrictive.
- If a structural break occurs, a recursive model will fail miserably.
- GARCH models are difficult to generalize to a multivariate setting.

Extensions

There are countless extension of the basic ARCH/GARCH model, see "Glossary to ARCH (GARCH)" by Tim Bollerslev (http://public.econ.duke.edu/~boller/Papers/glossary_arch.pdf)

- GARCH-in-mean: GARCH-M:

$$y_t = \beta x_t + \lambda \sigma_t + \epsilon_t$$
$$\epsilon_t = \sigma_t z_t$$

- Integrated GARCH: IGARCH:

$$\sum_{i=1}^{p} \beta_i + \sum_{i=1}^{q} \alpha_i = 1$$

- Glosten-Jagannathan-Runkle GARCH: GJR-GARCH

$$\sigma_t^2 = \omega + \delta\, \sigma_{t-1}^2 + \alpha\, \epsilon_{t-1}^2 + \phi\, \epsilon_{t-1}^2 I_{\epsilon_{t-1}<0}$$

- Exponential GARCH: EGARCH:

$$\log \sigma_t^2 = \omega + \alpha_1 \left(\left| \frac{\epsilon_{t-1}}{\sigma_{t-1}} \right| - \sqrt{\frac{2}{\pi}} \right) + \gamma_1 \frac{\epsilon_{t-1}}{\sigma_{t-1}} + \beta_1 \log \sigma_{t-1}^2$$

Conceptual limitations

- GARCH models are unsatisfactory from an economic perspective
- Explaining vol with past vol tells us nothing about the underlying economic factors that cause the volatility to move.

Estimation of ARCH(1): MLE

$$
\begin{aligned}
y_t &= \mu + \phi y_{t-1} + \epsilon_t \\
\epsilon_t &= \sigma_t e_t \\
\sigma_t^2 &= \omega + \alpha \, \epsilon_{t-1}^2 \\
e_t &\sim N(0, 1)
\end{aligned}
$$

Estimation of ARCH(1): MLE

Given initial conditions y_0 and ϵ_0^2:

Conditional likelihoods:

$$f_{y_1|y_0} \sim N\left(\mu + \phi y_0, \sigma_1^2\right)$$

$$f_{y_1|y_0} = \frac{1}{\sqrt{2\pi\left(\omega + \alpha\epsilon_0^2\right)}} \exp\left[\frac{-(y_1 - \mu - \phi y_0)^2}{2\left(\omega + \alpha\epsilon_0^2\right)}\right]$$

Solve forward:

$$f_{y_2|y_1} = \frac{1}{\sqrt{2\pi\left(\omega + \alpha\epsilon_1^2\right)}} \exp\left[\frac{-(y_2 - \mu - \phi y_1)^2}{2\left(\omega + \alpha\epsilon_1^2\right)}\right]$$

The unknown parameters are collected in $\theta = (\mu, \phi, \omega, \alpha)$

Then, the conditional log likelihood can be written as

$$\begin{aligned} L(\theta | y_0, \epsilon_0^2) &= \sum_{t=1}^{T} \log f_{y_t | y_{t-1}} \\ &= -\frac{T}{2} \log(2\pi) - \frac{1}{2} \sum_{t=1}^{T} \log\left(\omega + \alpha \epsilon_{t-1}^2\right) \\ &\quad - \sum_{t=1}^{T} \frac{(y_t - \mu - \phi y_{t-1})^2}{2\left(\omega + \alpha \epsilon_{t-1}^2\right)} \end{aligned}$$

We can maximize $L(\theta)$ with respect to $\theta = (\mu, \phi, \omega, \alpha)$.

Any GARCH model can be written in the following form:

Let $I_t = (y_1, y_2, \ldots, y_t)$ be the information set at t, θ a vector of parameters to be estimated.

Choose specifications for conditional mean μ_t and cond. vol σ_t^2.

Important: μ_t and h_t are known at time t, i.e. $\mu_t, \sigma_t^2 \in I_{t-1}$

$$y_t = \mu_t + \sigma_t e_t$$

$$y_t | I_{t-1} \sim N(\mu_t, \sigma_t^2)$$

$$e_t \equiv \frac{y_t - \mu_t}{\sigma_t}$$

$$e_t | I_{t-1} \sim N(0, 1)$$

Example: GARCH(1,1)

$$
\begin{aligned}
I_t &= (y_1, y_2, \ldots, y_t) \\
\mu_t &= \mu \\
\sigma_t^2 &= \omega + \alpha(y_{t-1} - \mu)^2 + \beta\sigma_{t-1}^2 \\
\theta &= (\mu, \omega, \alpha, \beta)'
\end{aligned}
$$

plus initial conditions y_0, h_0

Likelihood function

Recall that we can write the likelihood function recursively:

$$
\begin{aligned}
L(\theta) &= f(y_1, y_2, \ldots, y_T | I_0, \theta) \\
&= f(y_1 | I_0, \theta)\ f(y_2 | I_1, \theta)\ \cdots\ f(y_T | I_{T-1}, \theta)
\end{aligned}
$$

The log-likelihood is

$$\begin{aligned}
\log L(\theta) &= \sum_{t=1}^{T} f(y_t | I_{t-1}, \theta) \\
&= \sum_{t=1}^{T} \left[-\frac{1}{2}\log(2\pi) - \frac{1}{2}\log(\sigma_t^2(\theta)) - \frac{(y_t - \mu_t(\theta))^2}{2\sigma_t^2(\theta)} \right] \\
&= -\frac{1}{2}\left[T\log(2\pi) + \sum_{t=1}^{T}\Big(\log(\sigma_t^2(\theta)) + \epsilon_t^2(\theta) \Big) \right]
\end{aligned}$$

Given μ_t and σ_t^2, this function can be maximized with respect to θ

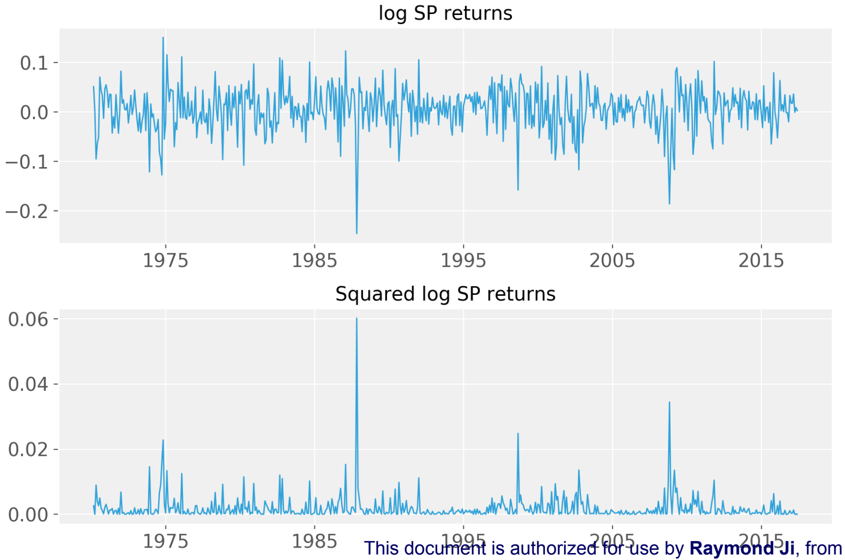
MLE Estimator

Recall:

$$\hat{\theta}_{mle} \sim N(\theta, I(\theta|y_t)^{-1})$$

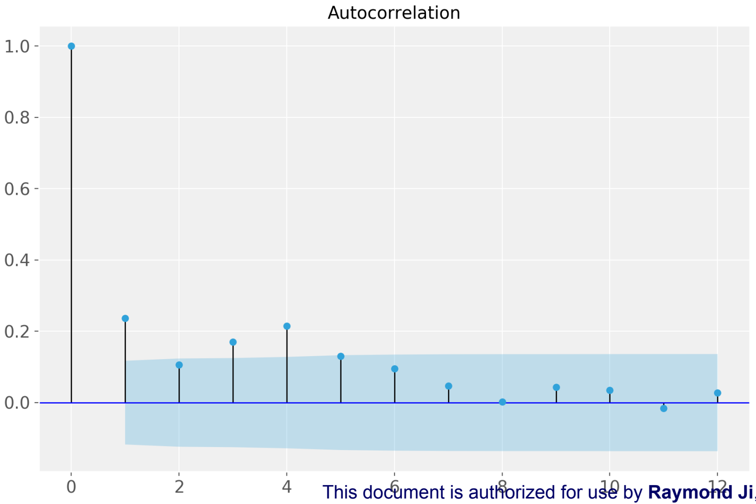
$$I(\theta|y_t) = -E\left[\frac{\partial^2 \ln L(\theta|y_t)}{\partial\theta\partial\theta'}\right]$$

Example: Monthly S&P500 returns



Example: ACF of squared residuals

Estimate AR(1) $r_t = \omega + \phi r_{t-1} + \epsilon_t$ for returns and plot $\widehat{\epsilon}_t^2$



Example: ACF of squared residuals $\hat{\epsilon}_t^2$

lag	AC	PAC	Q-stat	Prob
1	0.15	0.15	12.20	0.00
2	0.07	0.05	14.79	0.00
3	0.10	0.08	20.11	0.00
4	0.08	0.06	24.12	0.00
5	0.04	0.01	24.86	0.00
6	0.03	0.01	25.51	0.00
7	-0.03	-0.05	26.04	0.00
8	-0.01	-0.01	26.08	0.00
9	0.10	0.10	31.45	0.00
10	0.04	0.02	32.19	0.00
11	0.00	-0.01	32.19	0.00
12	0.02	0.00	32.34	0.00

ARCH(3) vs. GARCH(1,1)

ARCH(3):

$$r_t = \mu + \phi r_{t-1} + \sigma_t e_t$$

$$\sigma_t^2 = \omega + \alpha_1 \epsilon_{t-1}^2 + \alpha_2 \epsilon_{t-2}^2 + \alpha_3 \epsilon_{t-2}^2$$

GARCH(1,1):

$$r_t = \mu + \phi r_{t-1} + \sigma_t e_t$$

$$\sigma_t^2 = \omega + \alpha \epsilon_{t-1}^2 + \beta \sigma_{t-1}^2$$

EXAMPLE: ARCH(3)

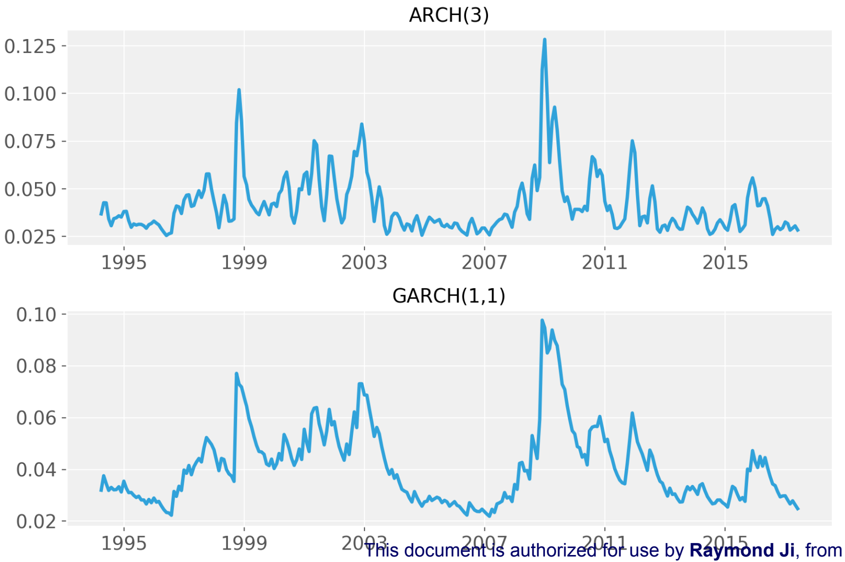
```
                         AR — ARCH Model Results
==============================================================================
Dep. Variable:                 logRet   R—squared:                    —0.003
Mean Model:                        AR   Adj. R—squared:               —0.007
Vol Model:                       ARCH   Log—Likelihood:              500.806
Distribution:                  Normal   AIC:                        —989.613
Method:            Maximum Likelihood   BIC:                        —967.869
                                        No. Observations:                277
Date:                Fri, Mar 24 2017   Df Residuals:                    271
Time:                        16:23:33   Df Model:                          6
                               Mean Model
==============================================================================
                 coef    std err          t      P>|t|       95.0% Conf. Int.
------------------------------------------------------------------------------
Const      8.4779e—03  2.425e—03      3.497  4.714e—04 [3.726e—03,1.323e—02]
logRet[1]  3.9910e—03  7.291e—02  5.474e—02      0.956    [ —0.139,  0.147]
                            Volatility Model
==============================================================================
                 coef    std err          t      P>|t|       95.0% Conf. Int.
------------------------------------------------------------------------------
omega      6.6750e—04  1.786e—04      3.737  1.860e—04 [3.174e—04,1.018e—03]
alpha[1]       0.2225  9.585e—02      2.321  2.027e—02   [3.464e—02,  0.410]
alpha[2]       0.3209      0.149      2.153  3.130e—02   [2.880e—02,  0.613]
alpha[3]       0.1690  7.601e—02      2.223  2.618e—02   [2.003e—02,  0.318]
==============================================================================
```


EXAMPLE: GARCH(1,1)

```
                           AR - GARCH Model Results
==============================================================================
Dep. Variable:                 logRet   R-squared:                      -0.002
Mean Model:                        AR   Adj. R-squared:                 -0.006
Vol Model:                      GARCH   Log-Likelihood:                503.892
Distribution:                  Normal   AIC:                          -997.785
Method:            Maximum Likelihood   BIC:                          -979.665
                                        No. Observations:                  277
Date:                Fri, Mar 24 2017   Df Residuals:                      272
Time:                        16:30:22   Df Model:                            5
                                 Mean Model
==============================================================================
                 coef    std err          t      P>|t|       95.0% Conf. Int.
------------------------------------------------------------------------------
Const      8.0032e-03  2.424e-03      3.302  9.593e-04  [3.253e-03,1.275e-02]
logRet[1]  5.9666e-04  6.770e-02  8.813e-03      0.993    [ -0.132,   0.133]
                              Volatility Model
==============================================================================
                 coef    std err          t      P>|t|       95.0% Conf. Int.
------------------------------------------------------------------------------
omega      1.1704e-04  1.357e-04      0.863      0.388 [-1.489e-04,3.830e-04]
alpha[1]       0.1878  8.474e-02      2.216  2.668e-02    [2.171e-02,  0.354]
beta[1]        0.7610      0.139      5.458  4.823e-08    [  0.488,   1.034]
==============================================================================
```

Note: Volatility is persistent!

EXAMPLE: MONTHLY S&P500 RETURNS



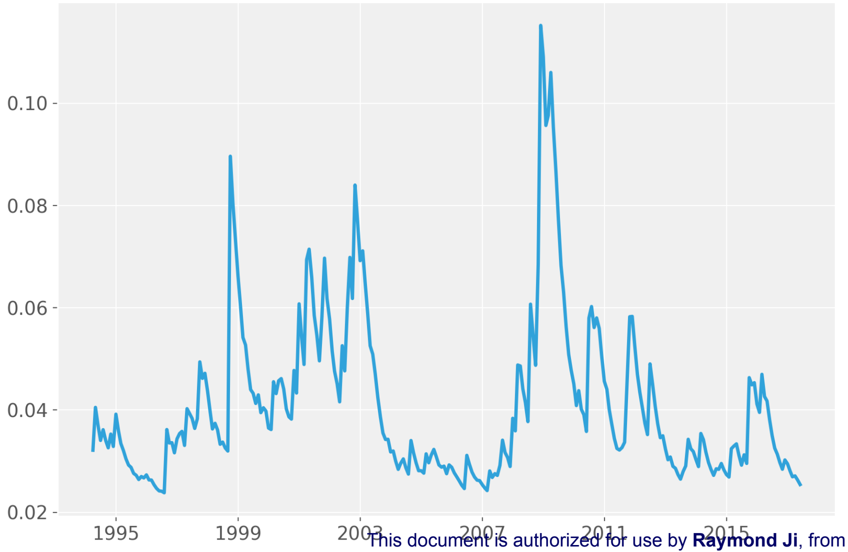
Extension: Glosten-Jagannathan-Runkle (GJR) GARCH

ARCH/GARCH: Symmetric vol response to positive/negative shocks

GJR: Asymmetric response

$$\sigma_t^2 = \omega + \beta\sigma_{t-1}^2 + \alpha\epsilon_{t-1}^2 + \gamma\epsilon_{t-1}^2\, I(\epsilon_{t-1} < 0)$$


```
                           AR — GJR—GARCH Model Results
==============================================================================
Dep. Variable:                     logRet   R—squared:                       0.003
Mean Model:                            AR   Adj. R—squared:                 —0.001
Vol Model:                       GJR—GARCH   Log—Likelihood:                506.140
Distribution:                       Normal   AIC:                          —1000.28
Method:                 Maximum Likelihood   BIC:                          —978.535
                                            No. Observations:                  277
Date:                     Fri, Mar 24 2017   Df Residuals:                      271
Time:                             16:38:40   Df Model:                            6
                                   Mean Model
==============================================================================
                 coef    std err          t       P>|t|       95.0% Conf. Int.
------------------------------------------------------------------------------
Const      6.5465e—03  2.393e—03      2.736  6.226e—03  [1.856e—03,1.124e—02]
logRet[1]      0.0227  7.578e—02      0.300      0.764     [ —0.126,  0.171]
                                Volatility Model
==============================================================================
                 coef    std err          t       P>|t|       95.0% Conf. Int.
------------------------------------------------------------------------------
omega      1.3889e—04  2.194e—04      0.633      0.527  [—2.911e—04,5.689e—04]
alpha[1]       0.0661      0.102      0.648      0.517     [ —0.134,  0.266]
gamma[1]       0.2041      0.307      0.665      0.506     [ —0.398,  0.806]
beta[1]        0.7485      0.202      3.700  2.160e—04     [  0.352,  1.145]
==============================================================================
```



0.10
0.08
0.06
0.04
0.02
1995
1999
2003
2007
2011
2015

Extension: EGARCH

$$\log \sigma_t^2 = \omega + \alpha_1 \left(\left| \frac{\epsilon_{t-1}}{\sigma_{t-1}} \right| - \sqrt{\frac{2}{\pi}} \right) + \gamma_1 \frac{\epsilon_{t-1}}{\sigma_{t-1}} + \beta_1 \log \sigma_{t-1}^2$$

- Specify process of log-vol rather then vol $\longrightarrow$ log-vol can be negative
- $\mathrm{E}\left[\left| \frac{\epsilon_{t-1}}{\sigma_{t-1}} \right| \right] = \sqrt{\frac{2}{\pi}}$
- Asymmetric response

What are the effect on log-vol of a positive/negative shock $\frac{\epsilon_{t-1}}{\sigma_{t-1}}$?

Extension: EGARCH

```
                           AR - EGARCH Model Results
==============================================================================
Dep. Variable:                   logRet   R-squared:                       0.003
Mean Model:                          AR   Adj. R-squared:                 -0.001
Vol Model:                       EGARCH   Log-Likelihood:                508.532
Distribution:                    Normal   AIC:                          -1005.06
Method:              Maximum Likelihood   BIC:                          -983.319
                                          No. Observations:                  277
Date:                   Fri, Mar 24 2017  Df Residuals:                      271
Time:                          16:40:49   Df Model:                            6
                                  Mean Model
==============================================================================
                 coef    std err          t      P>|t|        95.0% Conf. Int.
------------------------------------------------------------------------------
Const      6.0490e-03  4.501e-04     13.438  3.629e-41  [5.167e-03,6.931e-03]
logRet[1]      0.0192  2.894e-03      6.634  3.275e-11  [1.353e-02,2.487e-02]
                               Volatility Model
==============================================================================
                 coef    std err          t      P>|t|        95.0% Conf. Int.
------------------------------------------------------------------------------
omega         -0.9040      0.486     -1.858  6.316e-02    [ -1.857,4.956e-02]
alpha[1]       0.2785  5.802e-02      4.800  1.588e-06      [  0.165,  0.392]
gamma[1]      -0.2032  9.870e-02     -2.059  3.949e-02   [ -0.397,-9.784e-03]
beta[1]        0.8612  7.516e-02     11.458  2.141e-30      [  0.714,  1.008]
==============================================================================
```

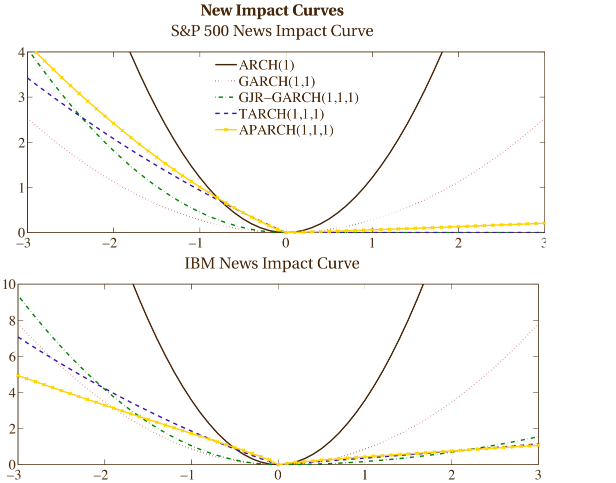
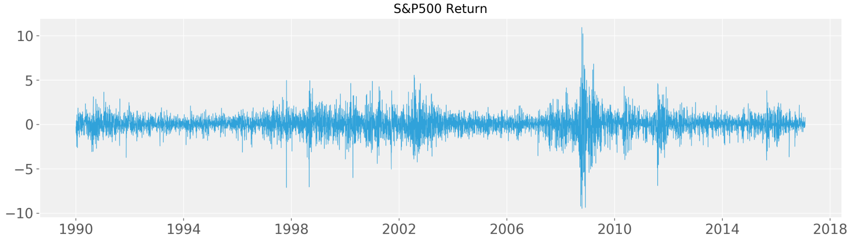
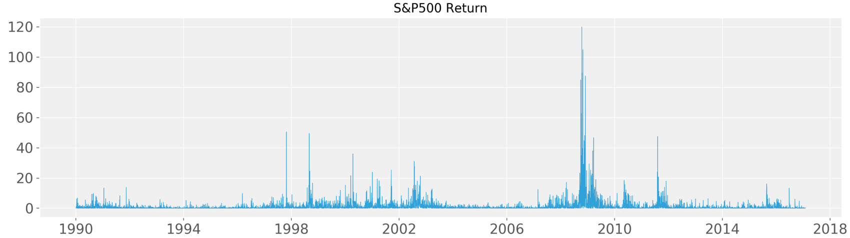



Figure 7.4: News impact curves for returns on both the S&P 500 and IBM. While the ARCH and GARCH curves are symmetric, the others show substantial asymmetries to negative news. Additionally, the fit APARCH models chose $\hat{\delta} \approx 1$ and so the NIC of the APARCH and the TARCH models appear similar.

How to choose the best model? Daily S&P 500 returns





ARCH

$$\sigma_t^2 = \omega + \alpha \epsilon_{t-1}^2$$

GARCH

$$\sigma_t^2 = \omega + \alpha \epsilon_{t-1}^2 + \beta \sigma_{t-1}^2$$

GJR-GARCH

$$\sigma_t^2 = \omega + \ \delta \ \sigma_{t-1}^2 + \ \alpha \ \epsilon_{t-1}^2 + \ \phi \ \epsilon_{t-1}^2 I_{\epsilon_{t-1}<0}$$

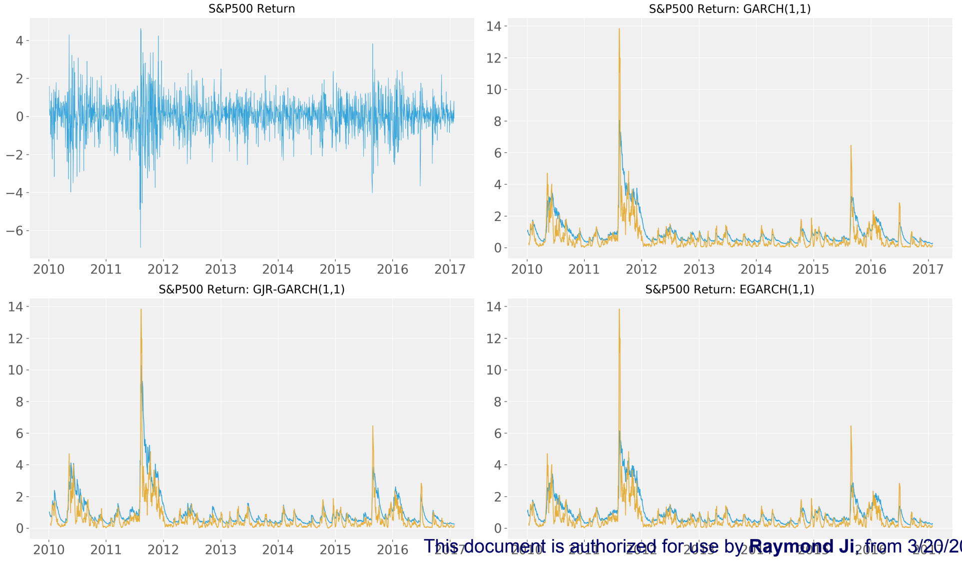
EGARCH:

$$\log \sigma_t^2 = \omega + \alpha_1 \left(\left| \frac{\epsilon_{t-1}}{\sigma_{t-1}} \right| - \sqrt{\frac{2}{\pi}} \right) + \gamma_1 \frac{\epsilon_{t-1}}{\sigma_{t-1}} + \beta_1 \log \sigma_{t-1}^2$$

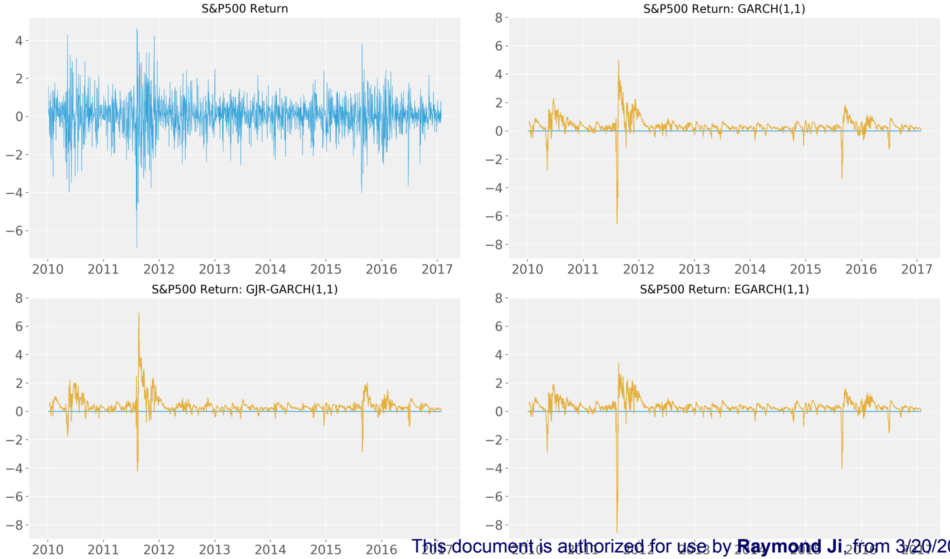
Example: Daily S&P returns

- Sample: 1/3/1990 – 12/31/2017
- Estimate models using data from 1/3/1990 to 12/31/2009
- Forecast volatility out-of-sample using data from 1/2/2010 to 12/31/2017
- Compare forecast errors

CONDITIONAL VOLATILITY



Forecast errors



Example: S&P returns

Out-of-sample forecasting:

Daily data

Models	GARCH(1,1)	GARCH(2,2)	GJR(1,1)	EGARCH(1,1)
BIC	13694.83	13698.69	13566.22	13706.67
RMSE	0.58	0.58	0.63	0.57
MAE	0.42	0.41	0.42	0.41

Monthly data

Models	GARCH(1,1)	GARCH(2,2)	GJR(1,1)	EGARCH(1,1)
BIC	1359.93	1369.68	1364.39	1361.34
RMSE	25.36	25.90	25.17	25.17
MAE	12.10	12.73	11.73	12.11

TODAY'S LECTURE

1. **Conditional volatility**: Expected volatility at some future time $t+h$ based on all information available at time t: $\mathrm{E}_t[\sigma_{t+h}]$
   Note: depends on model for conditional expectations
2. **Realized vol using high frequency data:**

$$\mathbf{RV}_t = \sum_{i \in t} r_{i,t}^2$$

3. **Implied vol**: Volatility that prices and option correctly
   Example:

$$\mathrm{BS}(S_t, K, r, t, \sigma_t) = C_t$$

$$\Rightarrow \sigma_t = f(S_t, K, r, t, C_t)$$

Outline

Alternative to ARCH/GARCH models: Realized volatility

- Use high-frequency dats to estimate volatility
- Example: log-price follows a continuous time Wiener process:

$$dp_t = \mu\, dt + \sigma\, dW_t$$

  where the coefficients have been normalized ($p_0, p_1, \ldots$ represent days: $p_1 - p_0$ is a daily return)
- For the S&P500: $\mu \approx .00031, \sigma \approx .0125$

Alternative to ARCH/GARCH models: Realized volatility

- Suppose each day, we observe $m+1$ observations: $p_{0,t}, p_{1,t}, \ldots, p_{m,t}$
- Realized variance is defined as

$$\mathrm{RV}_t^{(m)} = \sum_{i=1}^{m} (p_{i,t} - p_{i-1,t})^2 = \sum_i r_{i,t}^2$$

- Since p_t follows a Brownian motion:

$$r_{i,t} = N(\mu/m, \sigma^2/m)$$

$$\Rightarrow \mathrm{E}[\mathrm{RV}_t^{(m)}] = \mathrm{E}\sum_i r_{i,t}^2 = \mathrm{E}\sum_i \left(\frac{\mu}{m} + \frac{\sigma}{\sqrt{m}}\epsilon_{i,t}\right)^2$$

$$\epsilon_{i,t} \sim N(0,1)$$

The moments of RV can be derived in closed form (algebra is tedious):

$$\mathrm{E}[\mathrm{RV}_t^{(m)}] = \mathrm{E}\sum_i r_{i,t}^2 = \mathrm{E}\sum_i \left(\frac{\mu}{m} + \frac{\sigma}{\sqrt{m}}\epsilon_{i,t}\right)^2$$
$$= \frac{\mu^2}{m} + \sigma^2$$

$$\mathrm{Var}(\mathrm{RV}_t^{(m)}) = 4\frac{\mu^2\sigma^2}{m^2} + 2\frac{\sigma^4}{m}$$

$$\Rightarrow \quad \lim_{m\to\infty} \mathrm{E}[\mathrm{RV}_t^{(m)}] = \sigma^2$$
$$\lim_{m\to\infty} \mathrm{Var}(\mathrm{RV}_t^{(m)}) = 0$$

Hence $\mathrm{RV}_t^{(m)}$ is a consistent estimator of σ^2.

Log price with time-varying drift and stochastic volatility:

$\mathrm{RV}_t^{(m)}$ is a consistent estimator of **integrated variance:**

$$\lim_{m\to\infty} \mathrm{RV}_t^{(m)} \xrightarrow{p} \int_t^{t+1} \sigma_s^2 \, ds$$

If in addition the log price contains jumps:

$\mathrm{RV}_t^{(m)}$ is a consistent estimator of **quadratic variation:**

$$\lim_{m\to\infty} \mathrm{RV}_t^{(m)} \xrightarrow{p} \int_t^{t+1} \sigma_s^2 \, ds + \sum_{0<s\leq t} \Delta J_s^2$$

Conclusion: **Realized volatility is a robust method to estimate the integrated variance on day t.**

Trade-offs:

- Too frequent: market microstructure effects
- Too infrequent: not using all the information, how reliable are asymptotics?
- In practice: 1 to 5 minute prices are a good compromise
- Issues:
  - bid-ask bounce (What is that?)
  - price interpolation required since prices observed at irregular intervals

Much more: MFE230X – High Frequency Finance

MODELING RV

- RV_t is a consistent **estimator** of σ_t^2
- Measurement errors can lead to errors in variables
  $\Rightarrow$ Regression estimates using RV_t can be biased and even inconsistent
- Either need to model measurement error explicitly (later) or use ad-hoc approach
- Adjusting for bid-ask bounce:

$$\mathrm{RV}_t^{AC1,m} = \sum_i r_{i,t}^2 + 2\sum_i \cancel{r_{i,t} r_{i,t-1}}$$

$r_{i,t}\, r_{i-1,t}$

RV FOR S&P500

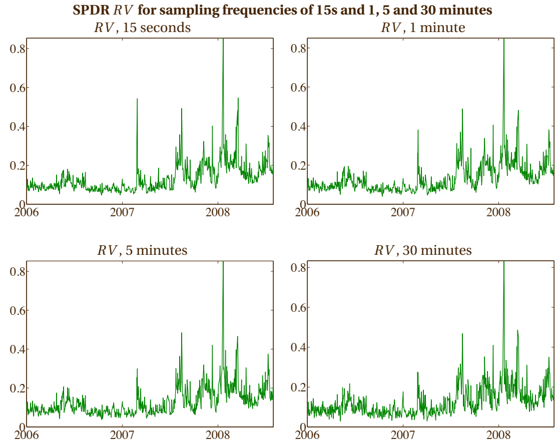


Figure 7.9: The four panels of this figure contain the Realized Variance for every day the market was open from January 3, 2006 until July 31, 2008. The estimated RV have been transformed into annualized volatility ($\sqrt{252 \cdot RV_t^{(m)}}$). While these plots appear superficially similar, the 1- and 5-minute RV are the most precise and the 15-second RV is biased upward.

$RV^{AC1,m}$ FOR S&P500

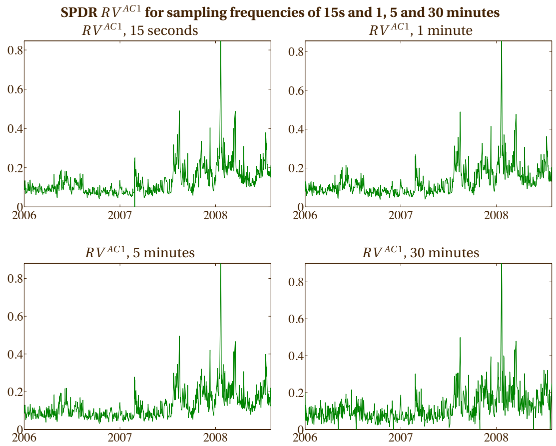


Figure 7.10: The four panels of this figure contain a noise robust version Realized Variance, RV^{AC1}, for every day the market was open from January 3, 2006 until July 31, 2008 transformed into annualized volatility. The 15-second RV^{AC1} is much better behaved than the 15-second RV although it still exhibits some strange behavior. In particular the negative spikes are likely due to errors in the data, a common occurrence in the TAQ database.

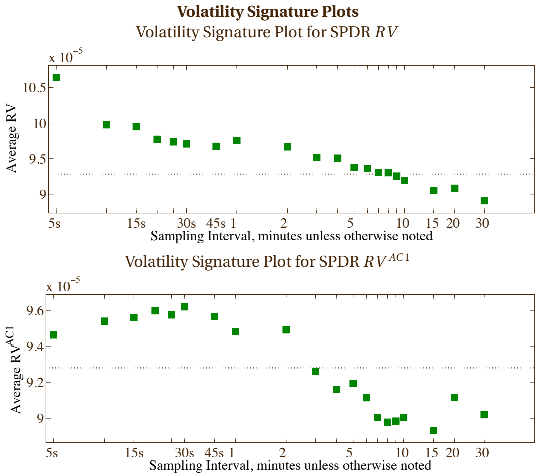


Figure 7.11: The volatility signature plot for the RV shows a clear trend. Based on visual inspection, it would be difficult to justify sampling more frequently than 3 minutes. Unlike the volatility signature plot of the RV, the signature plot of RV^{AC1} does not monotonically increase with the sampling frequency, and the range of the values is considerably smaller than in the RV signature plot. The decreases for the highest frequencies may be due to errors in the data which are more likely to show up when prices are sampled frequently.

Outline

A THIRD WAY TO MEASURE VOLATILITY: IMPLIED VOL (IV)

- Stock price S_t follows a geometric Brownian motion plus drift:

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t$$

- The Black-Scholes value of a call option with strike price K, time to maturity T with interest rate r is $C(T, K, S_t, \sigma, r)$
- Since the value of a call is monotonic in σ, the call price formula can be inverted to yield the volatility as a fct. of T, K, S_t, σ, r and the observed call price C_t:

$$\sigma_t^{\text{implied}} = g(C_t, T, K, S_t, r)$$

- Note: This measure of implied volatility depends on the Black-Scholes formula and its assumptions

The smile: Higher IV for out of the money options:

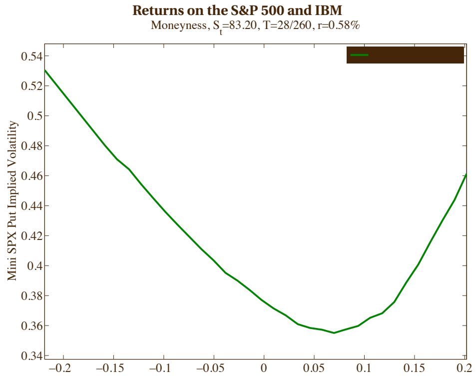
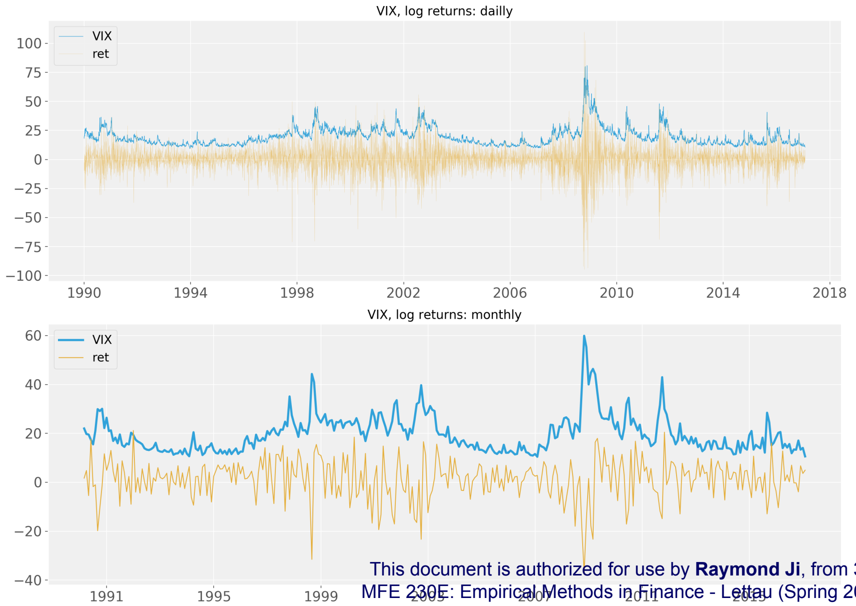


Figure 7.12: Plot of the Black-Scholes implied volatility "smile" on January 23, 2009 based on mini S&P 500 puts.

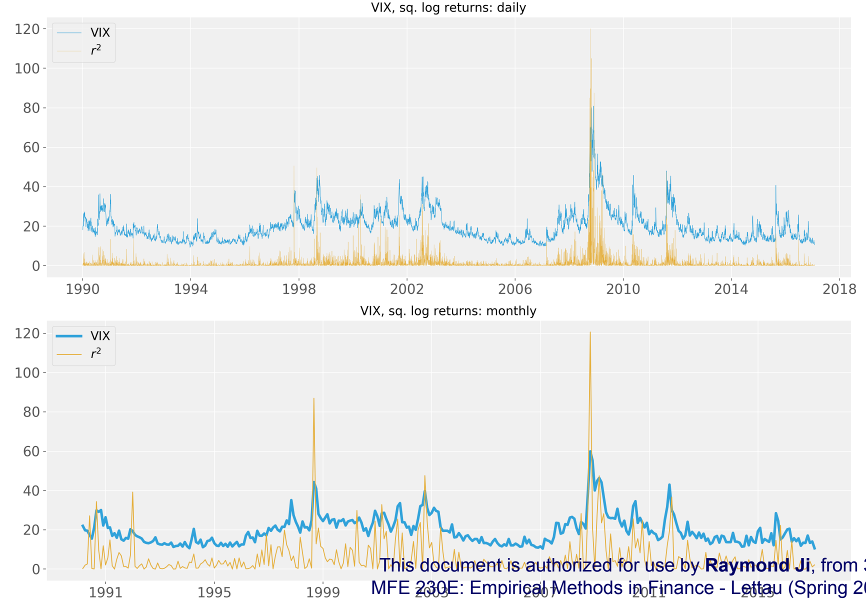
This pattern reflects the misspecification of the BS model (normality, no jumps, constant volatility and interest rates)

- It is possible to construct implied vol measures that do not rely on a specific model for options
- Relies on properties of risk-neutral and physical measures, see derivatives course
- Motivated by these theoretical results, the CBOE constructs the **VIX** volatility index based on a basket of put and call options
- The VIX is quoted in percentage points and translates to the (annualized) expected movement in the S&P 500 index over the next 30-day period
- For example, if the VIX is 15, this represents an expected annualized change of 15% over the next 30 days

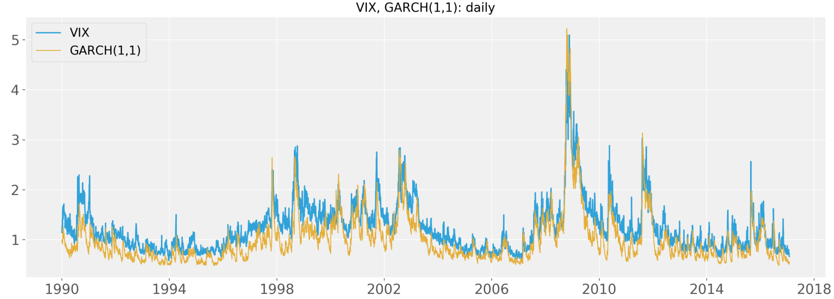
VIX AND THE LOG RETURNS

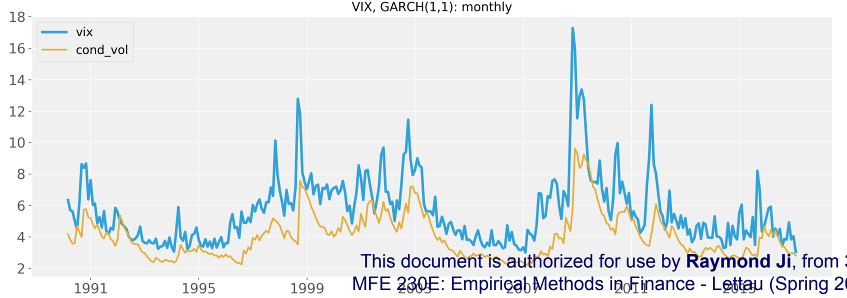


VIX AND THE SQUARED RETURN



VIX vs. GARCH(1,1)





VIX vs. GARCH(1,1): VAR

```
Summary of Regression Results
==================================
Model:                         VAR
Method:                        OLS
Date:            Mon, 08, May, 2017
Time:                     11:48:50
--------------------------------------------------------------------
No. of Equations:         2.00000    BIC:                   -1.60173
Nobs:                     323.000    HQIC:                  -1.64389
Log likelihood:          -640.623    FPE:                   0.187890
AIC:                     -1.67190    Det(Omega_mle):        0.184448
--------------------------------------------------------------------
Results for equation vix
===========================================================================
              coefficient       std. error           t-stat            prob
---------------------------------------------------------------------------
const            0.808573         0.204453            3.955           0.000
L1.vix           0.813855         0.046096           17.656           0.000
L1.GARCH         0.059040         0.070751            0.834           0.405
===========================================================================

Results for equation GARCH
===========================================================================
              coefficient       std. error           t-stat            prob
---------------------------------------------------------------------------
const            0.017453         0.064159            0.272           0.786
L1.vix           0.243710         0.014465           16.848           0.000
L1.GARCH         0.652777         0.022202           29.401           0.000
===========================================================================
```


Compare forecasts of

- GARCH(1,1)
- VIX

Regressions:

$$\text{RV}_{t+j} = \alpha + \beta \, \text{vol-measure(s)}_t + v_{t+j}$$

VIX	GARCH	Rsq
0.19		0.27
(0.01)		
	0.81	0.24
	(0.04)	
0.15	0.21	0.27
(0.01)	(0.08)	

Daily data, horizon: 30 days

VIX	GARCH	Rsq
7.62		0.48
(0.13)		
	34.32	0.48
	(0.71)	
3.99	18.24	0.51
(0.27)	(1.54)	

Monthly data, horizon: 1 month

VIX	GARCH	Rsq
0.60		0.23
(0.09)		
	0.60	0.10
	(0.11)	
0.71	-0.22	0.23
(0.14)	(0.18)	

Summary

- Volatility is changing over time
- ARCH/GARCH class: Properties, extensions, MLE estimation
- Realized volatility: High frequency data, properties
- Implied volatility: VIX
- Comparison of different models

Outline

1. Volatility is changing over time
2. ARCH/GARCH
3. Realized volatility
4. Implied volatility
5. **Stochastic Volatility and MCMC**

Stochastic Volatility and MCMC

A more sophisticated GARCH model

- Ruppert ch. 20
- Excellent treatment of MCMC methods: Tsay ch. 12

GARCH models are easy to estimate by MLE:

$$r_t = \mu_t + \sigma_t \epsilon_t$$

$$r_t | I_t \sim N(\mu_t, \sigma_t^2)$$

$$\text{standard. residuals: } \epsilon_t \equiv \frac{r_t - \mu_t}{\sigma_t} \sim N(0, 1)$$

$$\log L(\theta) = \sum_{t=1}^{T} f(r_t | I_{t-1}, \theta)$$

$$= \sum_{t=1}^{T} -\frac{1}{2} \log(2\pi) - \frac{1}{2} \log(\sigma_t^2(\theta)) - \frac{(r_t - \mu_t(\theta))^2}{2\sigma_t^2(\theta)}$$

Limitations:

- Difficult to extend to multivariate settings
- Todays' volatility is known since it is a function of past shocks only
- Does not allow for unexpected shocks to volatility
- Limited use for risk management and volatility hedging

General SV model:

$$r_t - \mu_t = \sigma_t \, u_t$$

with

$$\text{Var}(\sigma_t | r_{t-1}, r_{t-2} \ldots) > 0$$

Question: How can we model σ_t and still keep the model manageable?

Let α be the unconditional mean of $\log \sigma_t$.

$$r_t = \sigma_t \, u_t$$

$$\log \sigma_t - \alpha = \phi(\log \sigma_{t-1} - \alpha)) + \eta_t$$

$$u_t \sim N(0, 1)$$

$$\eta_t \sim N(0, \sigma_\eta^2)$$

Question: How can we estimate the SV model?

RECALL: ARCH MODEL

MLE for ARCH(1)

$$r_t = \mu + u_t$$

$$u_t = \sigma_t v_t$$

$$\sigma_t^2 = \omega + a\, u_{t-1}^2$$

$$v_t \sim N(0,1)$$

Conditional likelihoods:

$$f_{r_2|r_1} \sim N(\mu, h_2) = \frac{1}{\sqrt{2\pi\left(\omega + a\, u_1^2\right)}} \exp\left[\frac{-(r_2-\mu)^2}{2\left(\omega + a\, u_1^2\right)}\right]$$

Can we use MLE to estimate the SV model?

$$r_t = \sigma_t \, u_t$$
$$\log \sigma_t - \alpha = \phi(\log \sigma_{t-1} - \alpha)) + \eta_t$$
$$u_t \sim N(0, 1)$$
$$\eta_t \sim N(0, \sigma_\eta^2)$$

Is the conditional likelihood $f_{r_2|r_1}$ normal?

Candidates:

- OLS?
- GMM?

Moments of r_t:

$$\begin{aligned}
\mathrm{E}\left[|r_t|\right] &= \sqrt{\frac{2}{\pi}}\mathrm{E}\left[\sigma_t\right] \\
\mathrm{E}\left[r_t^2\right] &= \mathrm{E}\left[\sigma_t^2\right] \\
\mathrm{E}\left[|r_t^3|\right] &= 2\sqrt{\frac{2}{\pi}}\mathrm{E}\left[\sigma_t^3\right] \\
\mathrm{E}\left[|r_t^4|\right] &= 3\mathrm{E}\left[\sigma_t^4\right]
\end{aligned}$$

$$\begin{aligned}
\mathrm{E}\left[|r_t r_{t-j}|\right] &= \frac{2}{\pi}\mathrm{E}\left[\sigma_t \sigma_{t-j}\right] \\
\mathrm{E}\left[|r_t^2 r_{t-j}^2|\right] &= \mathrm{E}\left[\sigma_t^2 \sigma_{t-j}^2\right]
\end{aligned}$$

GMM has very poor small sample properties. Alternatives?

Candidates:

- Kalman filter (next week)
- Markov Chain Monte Carlo (last week)