Q1

$(1+i_t) \overset{ind}{\sim}$ lognormally distributed

$$\begin{cases} \mu_1 = 0.04 \ , \ \sigma_1^2 = 0.02 & \text{for } t = 1,2,3 \\ \mu_2 = 0.03 \ , \ \sigma_2^2 = 0.03 & \text{for } t = 4,5 \end{cases}$$

1a) FV of 1 at $t=5$

$$A_5 = \prod_{t=1}^{5} (1+i_t) \Rightarrow \log A_5 = \log\left[\prod_{t=1}^{5}(1+i_t)\right] = \sum_{t=1}^{5} \log(1+it) \sim N(\mu, \sigma^2).$$

Since $(1+i_t)$'s are ind.

$$E[\log A_5] = \sum_{t=1}^{5} E[\log(1+i_t)] = 3\mu_1 + 2\mu_2 = 0.18$$

$$Var(\log A_5) = \sum_{t=1}^{5} Var(\log(1+i_t)) = 3\sigma_1^2 + 2\sigma_2^2 = 0.12$$

Thus. $A_5 \sim \log N(0.18, 0.12)$

$$E[A_5] = e^{0.18 + 0.12/2} = \underline{1.2713}$$

$$Var(A_5) = e^{2(0.18)+0.12}(e^{0.12}-1) = \underline{0.2064}$$

(b)

$$E[\log(\tfrac{1}{A_5})] = -E[\log A_5] = -0.18$$

$$Var(\log(\tfrac{1}{A_5})) = Var(\log A_5) = 0.12$$

Thus.

$$E[\tfrac{1}{A_5}] = e^{-0.18 + 0.12/2} = \underline{0.8869}$$

$$Var(\tfrac{1}{A_5}) = e^{-2(0.18)+0.12}(e^{0.12}-1) = \underline{0.1003}$$

Q2.

$(1+i_t) \sim \log N(0.05, 0.005) \Rightarrow \log(1+i_t) \sim N(0.05, 0.005)$

To find the mean and variance of $\frac{3}{A_4}$

$$\frac{3}{A_4} = \frac{3}{\prod_{t=1}^{4}(1+i_t)} = 3\prod_{t=1}^{4}(1+i_t)^{-1}$$

$$\Rightarrow \log\left(\frac{3}{A_4}\right) = \log 3 - \sum_{t=1}^{4}\log(1+i_t)$$

$$\Rightarrow E\left[\log\left(\frac{3}{A_4}\right)\right] = \log 3 - \sum_{t=1}^{4}E[\log(1+i_t)] = \log - 4(0.05) = 0.8986$$

$$Var\left(\log\left(\frac{3}{A_4}\right)\right) = \sum_{t=1}^{4}Var(1+i_t) = 4(0.005) = 0.02$$

Thus,

$$\log\left(\frac{3}{A_4}\right) \sim N(0.8986, 0.02)$$

The 95% CI of the $\log\left(\frac{3}{A_4}\right)$ is

$$(0.8986 \pm 1.96(0.02^{1/2})) = (0.6214, 1.1758)$$

The 95% CI of $\left(\frac{3}{A_4}\right)$ is

$$(e^{0.6214}, e^{1.1758}) = (1.8616, 3.2407)$$

Q3.

$$FV = 4000\left[(1+i_4)(1+i_5)(1+i_6) + (1+i_5)(1+i_6) + (1+i_6)\right] + 109000$$

$$E[FV] = 4000\left[1.055(1.06)(1.045) + 1.06(1.045) + 1.045\right] + 10900$$

$$= \underline{122,285.3}$$

Q4.

$(1+i_t) \sim \log N(\mu, \sigma^2) \Rightarrow \log(1+i_t) \sim N(\mu, \sigma^2)$

From given info.

$$\begin{cases} e^{\mu+\sigma^2/2} = 1.05 \\ e^{2\mu+\sigma^2}(e^{\sigma^2}-1) = 0.007 \end{cases}$$

By solving these, $\sigma^2 = 0.006329$, $\mu = 0.045626$

Note interest rate is constant.

$\log A_5 \sim N(5\mu, 25\sigma^2)$ $\because Var(\log A_T) = Var(5\log(1+i_T)) = 25\sigma^2$

$$\begin{aligned} Pr(10000 A_5 > 15000) &= Pr(A_5 > 1.5) \\ &= Pr(\log A_5 > \log 1.5) \\ &= Pr\left(Z > \frac{\log 1.5 - 5\mu}{5\sigma}\right) \\ &= Pr(Z > 0.4458) \\ &= \underline{32.79\%} \end{aligned}$$