

# BA2202 Mathematics of Finance

## Handout 5

### 1 Project Appraisal

Using techniques to evaluate the PV and FV of a cashflow, we may consider whether or not an investor should finance a project and may compare alternative projects with different future cashflows. In particular, the following methods are suggested in this section:

- Net present value (NPV) and accumulated profit
- Internal rate of return (IRR)
- Discounted payback period (DPP)

### 2 Accumulated Values

Consider a project which requires an investor outlays of amount  $a_t$  followed by payments of  $b_t$  at times  $t$ . The net cashflow at time  $t$  is:

$$c_t = b_t - a_t$$

If payments are continuous where  $\rho_1(t)$  and  $\rho_2(t)$  denotes the rates of inflow and outflow at time  $t$ , respectively, the net rate of cashflow at time  $t$  is defined as:

$$\rho(t) = \rho_1(t) - \rho_2(t)$$

Assume that an investor can borrow or lend money at a fixed rate of interest  $i$  per unit time. If a net cash flow is positive, the surplus may be deposited into an account that earns an interest rate  $i$ . In the case of a negative net cashflow, the investor may borrow the deficit from a bank at an interest rate  $i$ . The accumulated value at time  $T$  of the project cashflow can be expressed as:

$$A(T) = \sum c_t (1+i)^{T-t} + \int_0^T \rho(t) (1+i)^{T-t} dt \quad (1)$$

where the summation extends over all  $t$  such that  $c_t \neq 0$ .

### 3 Net Present Value

Instead of evaluating the value of a project in a future, we may look at the PV of the net cashflow, called *net present value* denoted by  $NPV(i)$ :

$$NPV(i) = \sum c_t (1+i)^{-t} + \int_0^T \rho(t) (1+i)^{-t} dt \quad (2)$$

A benefit of discounting is that the discounted value may be defined even if the cashflow continues indefinitely. If uncertainty is involved in the cashflow, we may apply a higher interest rate called risk discount rate  $i'$

$$NPV(i) = \sum c_t (1+i')^{-t} + \int_0^T \rho(t) (1+i')^{-t} dt \quad (3)$$

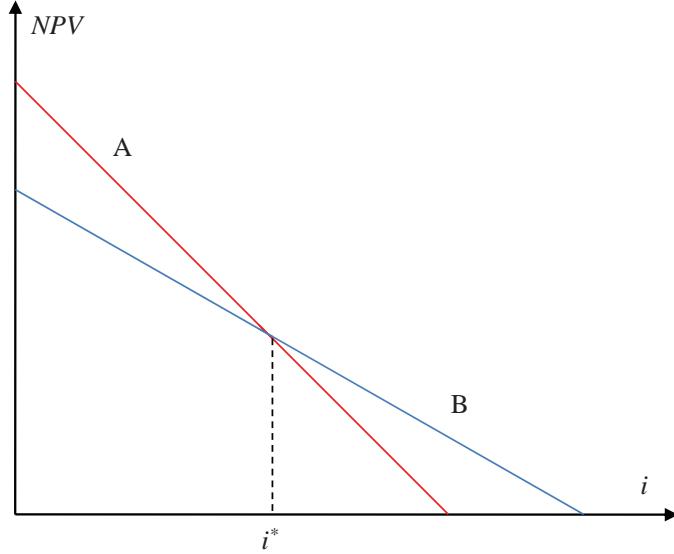
The equation leads the following statement. Suppose that an investor can borrow or lend money at a fixed rate of interest  $i$  per unit time. A project will be profitable if and only if

$$NPV(i) > 0$$

Note that the effective rate of interest  $i_0$  that satisfies the equation of value for the project:

$$NPV(i_0) = 0 \quad (4)$$

is referred to as the *internal rate of return* (IRR). A higher IRR means a more profitable project. What is the condition for the existence of a unique solution for the yield  $i_0$ ?



With regard to IRR the following statement holds. Suppose that an investor can borrow or lend money at a fixed rate of interest  $i$  per unit time. A project will be profitable if and only if

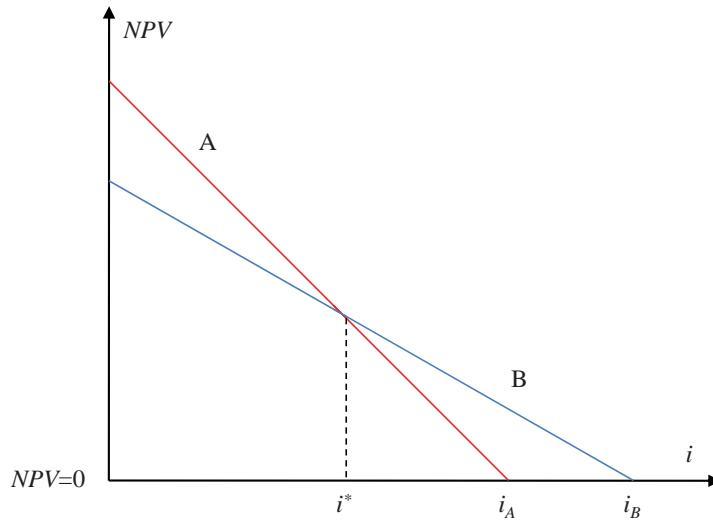
$$i_0 > i$$

Thus, IRR is used to decide if a project is profitable relative to its minimum required return or target return. Here a fixed rate of interest  $i$  is regarded as a target rate which a project must exceed.

The next is a criterion used to decide between alternatives. Again, suppose that the investor can borrow or lend money at a fixed rate of interest  $i$  per unit time. Project A is more profitable than Project B if

$$NPV_A(i) > NPV_B(i)$$

where  $NPV_A(i)$  and  $NPV_B(i)$  denote the net present value for project A and that for project B, respectively. Note that IRR is not appropriate for the purpose of comparing alternatives because  $i_A > i_B$  does not necessarily mean  $NPV_A(i) > NPV_B(i)$ . For instance, in the figure,  $i_B > i_A$  at  $NPV = 0$  but  $NPV_A(i) > NPV_B(i)$  where  $i < i^*$ .

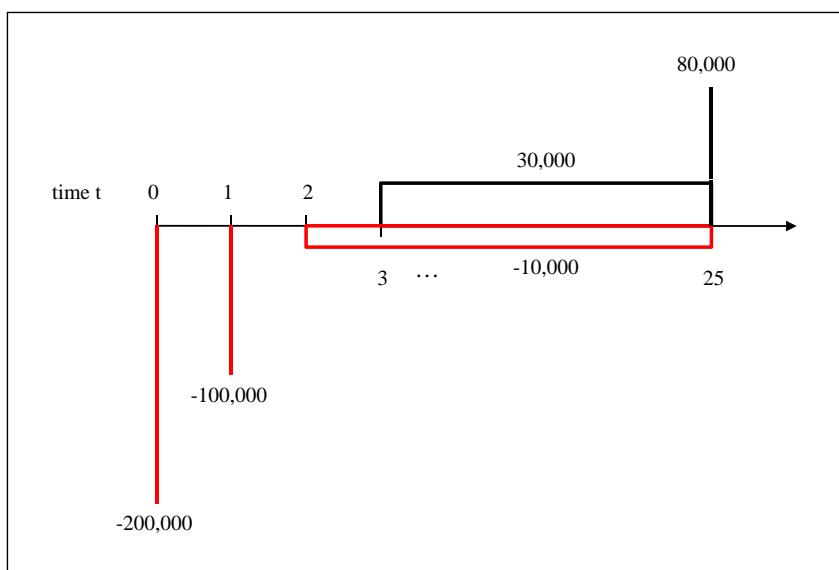


Therefore, we may follow the following criteria:

1. Calculate IRR (or see  $NPV > 0$ ) for alternatives to see if those projects will be profitable relatively to the target rate. If calculated IRR is less than the target rate, drop the project.
2. If more than one project are left, compare NPV between alternatives at the target rate. Then choose a project with the largest NPV.

**Example 3.1.** An investor is considering whether to invest in a new shop, which will have the following cashflows. Find IRR for the project.

- An initial construction cost of 200,000
- Another outlay of 100,000 at the end of the first year.
- Running cost of 10,000 per annum payable continuously from the beginning of the 3rd year to the end of 25th year.
- Revenue of 30,000 per annum payable continuously from the beginning of the 4th year to the end of 25th year.
- An income of 80,000 from selling the shop at the end of 25th year.



We can summarize the cash flows (in 1000s) as follows:

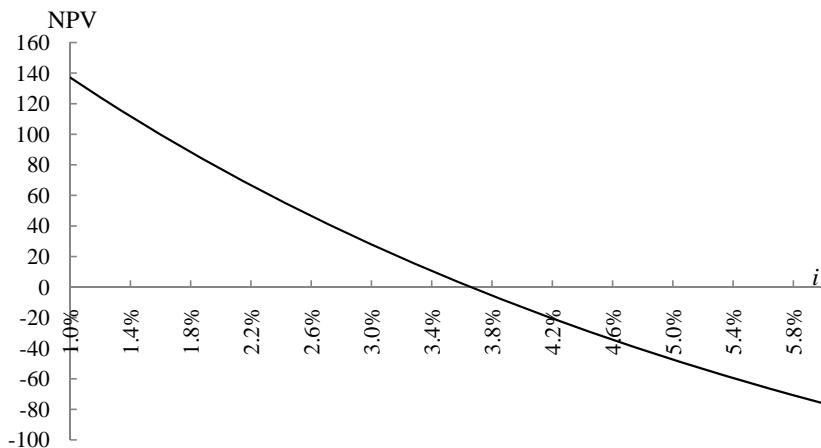
$$\begin{aligned} C_0 &= -200 \\ C_1 &= -100 \\ C_{25} &= 80 \\ \rho_1(t) &= 30 \quad \text{for } 3 \leq t \leq 25 \\ \rho_2(t) &= -10 \quad \text{for } 2 \leq t \leq 25 \end{aligned}$$

The continuous cash flows can be rewritten as the net cash flows as well:

$$\begin{aligned} \rho(t) &= -10 \quad \text{for } 2 \leq t \leq 3 \\ \rho(t) &= 20 \quad \text{for } 3 < t \leq 25 \end{aligned}$$

By the NPV equation,

$$NPV(i) = -200 - 100v - 10v^2\bar{a}_{23|} + 30v^3\bar{a}_{22|} + 80v^{25} \quad (5)$$

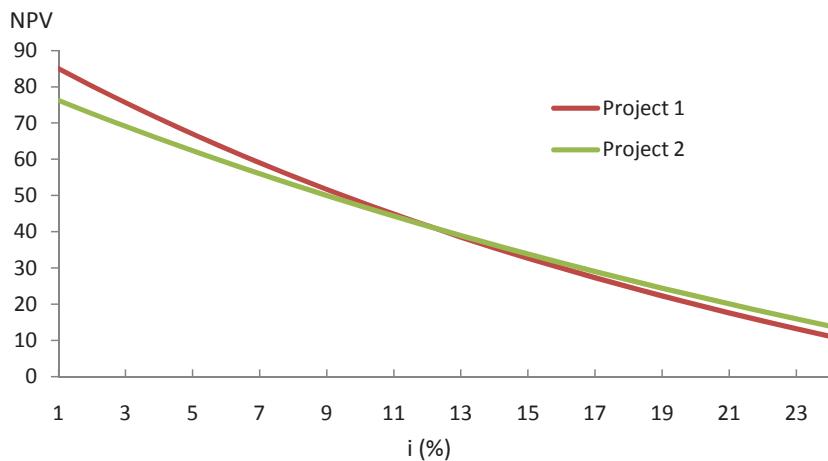


We obtain IRR  $i_0 = 3.66\%$  by linear interpolation.

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**Example 3.2.** There are two projects with the following cash flow streams. Which project represents line A and B in the figure?

- Project 1:  $C_0 = -50, C_1 = -30, C_2 = -30, C_3 = +200$
- Project 2:  $C_0 = -100, C_1 = -20, C_2 = +200$



**Example 3.3.** An investor is considering whether to invest in either or both of the following loans. Suppose that the investor can lend or borrow money at 4% per annum. What is the optimal decision?

- Loan 1: The investor receives  $1000a_{\overline{15}}^{(4)}$  in return for the investment of 10000.
- Loan 2: In return for the investment of 11000, the investor receives  $605a_{\overline{18}}$  and a return of its outlay at the end of this period.

The NPV for loan 1 is defined by:

$$NPV_1(i) = -10000 + 1000a_{\overline{15}}^{(4)}$$

By setting  $NPV_1(i) = 0$  and solving the equation, we have

$$10 = a_{\overline{15}}^{(4)} \Rightarrow i \approx 5.88\%$$

For loan 2 we have:

$$NPV_2(i) = -11000 + 605a_{\overline{18}} + 11000v^{18}$$

By setting  $NPV_2(i) = 0$  and solving the equation, we have  $i \approx 5.5\%$ . Thus, both loans are viable in terms of 4% required rate.

To compare those loans, we calculate  $NPV(4\%)$ .

$$\begin{aligned} NPV_1(.04) &= -10000 + 1000a_{\overline{15}.04}^{(4)} = 1284 \\ NPV_2(.04) &= -11000 + 605a_{\overline{18}.04} + 11000v^{18} = 2089 \end{aligned}$$

Loan 2 generate a larger profit. Therefore, although both loans are profitable, loan 2 should be chosen if only one investment is allowed.

## 4 Different Interest Rate for Lending and Borrowing

We relax the assumption that an investor can borrow or lend money at the same rate of interest, and assume an higher interest rate  $i$  on borrowings and a lower interest rate  $j$  on investments. In this case, we must calculate the accumulation of net cashflows. Interest rate applied in each period depends on whether the investor's account is in credit. If it is negative, the investor must borrow the deficit at interest rate  $i$ . Otherwise, the account earn interest at rate  $j$ . The calculation also depends on the investor's repayment option.

**Example 4.1.** Consider a project with the following cashflow at  $i = 6.25\%$  and  $j = 4\%$ , which occurs at the end of year. Calculate the accumulated value at the end of 5 years if capital repayment is not allowed until the end of Year 3.

The negative net cash flows in the first 2 years must be financed at the interest rate  $i$ . And the positive net cash flows earns interest at  $j$ . The following table shows the accumulated value of the project.

$$\begin{aligned} A(3) &= 87 - 95(1 + 0.0625) + 19.063(1.04) = 5.888 \\ A(5) &= A(3)(1.04^2) = 6.368 \end{aligned}$$

Year	Inflow	Outflow	Net Cashflow
0	-	80	-80
1	10	20	-10
2	30	5	25
3	87	-	87

Year	Net Cashflow	Interest Paid	Loan Outstanding	Deposit Balance
0	-80	-	80	0
1	-10	5.000	95	0
2	25	5.937	95	19.063
3	87	5.937	0	5.888

If repayment is allowed at the end of each year, the investor can use its positive net cash flows to repay a fraction of loan balance. In the table below, you see the difference in year 2 and year 3.

$$A(5) = 6.317(1.04^2) = 6.832$$

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Thus, repayment reduces the total interest payment and lead the higher accumulated profit. From a lender's point of view, repayment reduces its profit due to loss of its lending. This is one of reasons that lenders are likely to impose penalties for early repayment of loan outstanding.

## 5 Payback Periods

Another measure to evaluate an investment project financed by outside borrowing is the *discounted payback period* (DPP). Consider a project in which a net cashflow balance changes sign only once at time  $t^*$ , the change being from negative to positive. If the time  $t^*$  exists, it is referred to as the DPP defined by the smallest  $t$  such that  $A(t) \geq 0$ , where:

$$A(t) = \sum_{s \leq t} c_s (1+i)^{t-s} + \int_0^t \rho(s) (1+i)^{t-s} ds \quad (6)$$

Note that finding DPP does not depend on interest rate on investment  $j$  because the net cashflow balance is negative during the discounted payback period.

Since the net cashflow generated after the DPP is accumulated as profit, we can define the accumulated profit of the project after  $n$  years in terms of the time  $t^*$ :

$$P(T) = A(t^*) (1+j)^{T-t^*} + \sum_{s > t^*} c_s (1+j)^{T-s} + \int_{t^*}^T \rho(s) (1+j)^{T-s} ds \quad (7)$$

For an investment of  $C_0$  in return for a regular payments of  $R$  payable at the end of each year for  $n$  years. The discounted payback period is the smallest integer  $t$  such that:

$$A(t) = -C(1+i)^t + R s_{\bar{i}} \quad (8)$$

Year	Net Cashflow	Interest Paid	Capital Repaid	Accumulated Balance
0	-80	-	-	-80
1	-10	5.000	-	-95
2	25	5.937	19.063	-75.937
3	87	4.746	75.937	6.317

Equivalently,

$$Ra_{\bar{t}i} \geq C_0 \quad (9)$$

The project is viable if  $t^* \leq n$ . And the accumulated profit after  $n$  years is:

$$P(n) = A(t^*)(1+j)^{n-t^*} + Rs_{\overline{n-t^*}|j} \quad (10)$$

**Example 5.1.** An investor is considering whether to invest in either or both of the following loans. Suppose that the investor can lend or borrow money at 4% per annum. Find the discounted payback period.

- Loan 1: The investor receives  $1000a_{\overline{15}}^{(4)}$  in return for the investment of 10000.
- Loan 2: In return for the investment of 11000, the investor receives  $605a_{\overline{18}}$  and a return of its outlay at the end of this period.

For loan 1:

$$A_1(t) = -10000(1.04)^t + 1000s_{\overline{t}|0.04}^{(4)} \Rightarrow t_1^* = 13$$

Or equivalently, the same result can be obtained by finding the smallest integer  $t$  such that:

$$10000 \leq 1000a_{\overline{15}|0.04}^{(4)}$$

For loan 2, it is obvious that the accumulated value is negative before the return of the outlay. Therefore, loan 1 has a shorter discounted payback period. ■

**Example 5.2.** An investor will receive an annuity of 1000 payable for 20 years at the end of each year in return for an investment of 10000. Find the discounted payback period if the interest rate on borrowings is 5% pa and that on investing is 3% pa.

$$A(t) = -10000(1.05)^t + 1000s_{\overline{t}|0.05}$$

Setting  $A(t) = 0$ , we get

$$t^* = 15$$

Thus, the discounted payback period is 15 years. And,

$$A(15) = -10000(1.05^{15}) + 1000s_{\overline{15}|0.05} = 789.28$$

Therefore, the accumulated profit is

$$\begin{aligned} P(n) &= A(t^*)(1+j)^{n-t^*} + Rs_{\overline{n-t^*}|j} \\ &= 789.28(1.03)^{20-15} + 1000s_{\overline{20-15}|0.03} \\ &= 6224.13 \end{aligned}$$
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