MH4511 Sampling & Survey

Tutorial 8 Solution AY2025/26 Semester 1

Problem 8.1 (Solution)

a) Primary sampling units (psu): blocks of the housing estate.
Secondary sampling units (ssu): households of the housing estate.

b) For each $psu = i, ssu = j$, the weight $w_{ij} = \frac{N}{n}\frac{M_i}{m_i}$, so

$$w_{1j} = \frac{20}{5} \times \frac{60}{12} = 20.0 \ , \quad j = 1, 2, \dots, 12$$
$$w_{2j} = \frac{20}{5} \times \frac{60}{12} = 20.0 \ , \quad j = 1, 2, \dots, 12$$
$$w_{3j} = \frac{20}{5} \times \frac{52}{10} = 20.8 \ , \quad j = 1, 2, \dots, 10$$
$$w_{4j} = \frac{20}{5} \times \frac{78}{15} = 20.8 \ , \quad j = 1, 2, \dots, 15$$
$$w_{5j} = \frac{20}{5} \times \frac{55}{11} = 20.0 \ , \quad j = 1, 2, \dots, 11$$

c) We have

$$\hat{t} = \frac{N}{n}\sum \hat{t}_i = \frac{20}{5} \times 33744 = 134976$$

$$\widehat{Var}(\hat{t}) = N^2\left(1 - \frac{n}{N}\right)\frac{s_t^2}{n} + \sum \frac{N}{n}\left(1 - \frac{m_i}{M_i}\right)M_i^2\frac{s_i^2}{m_i}$$
$$= 20^2\left(1 - \frac{5}{20}\right) \times \frac{2 \times 10^6}{5} + [24000 + 17208 + 22713.6 + 26280 + 25520]$$
$$= 1.2 \times 10^8 + 115721.6 = 120{,}115{,}721.6 = (10{,}959.7)^2$$

$$SE(\hat{t}) = 10{,}959.7$$

d) We have

$$\hat{\bar{y}}_r = \frac{\sum \hat{t}_i}{\sum M_i} = \frac{33744}{305} = 110.64$$

$$\widehat{Var}(\hat{\bar{y}}_r) = \frac{1}{\bar{M}^2}\left(1 - \frac{n}{N}\right)\frac{s_r^2}{n} + \frac{1}{nN\bar{M}^2}\sum\left(1 - \frac{m_i}{M_i}\right)M_i^2\frac{s_i^2}{m_i} \ , \quad \text{where } \bar{M} = \frac{1}{n}\sum M_i$$
$$= \frac{1}{\left(\frac{305}{5}\right)^2}\left(1 - \frac{5}{20}\right)\frac{213000}{5} + [0.016 + 0.012 + 0.015 + 0.018 + 0.017]$$
$$= 8.586 + 0.078 = 8.664 = (2.94)^2$$

$$SE(\hat{\bar{y}}_r) = 2.94$$

Block i	(M_i)	(m_i)	$(\bar{y}_i)$	$(\hat{t}_i)$	(s_i^2)	$\frac{N}{n}\left(1-\frac{m_i}{M_i}\right)M_i^2\frac{s_i^2}{m_i}$
1	60	12	101	6,060	25	24,000
2	60	12	72	4,320	18	17,280
3	52	10	121	6,292	26	22,713.6
4	78	15	99	7,722	20	26,208
5	55	11	170	9,350	29	25,520
Total	**305**	**60**		**33,744**		**115,721.6**

Block i	(M_i)	(m_i)	$(\bar{y}_i)$	$(\hat{t}_i)$	(s_i^2)	$\frac{1}{nN\bar{M}^2}\left(1-\frac{m_i}{M_i}\right)M_i^2\frac{s_i^2}{m_i}$
1	60	12	101	6,060	25	0.016
2	60	12	72	4,320	18	0.012
3	52	10	121	6,292	26	0.015
4	78	15	99	7,722	20	0.018
5	55	11	170	9,350	29	0.017
Total	**305**	**60**		**33,744**		**0.078**

Problem 8.2 (Solution)

a) Primary sampling units (psu): the 90 plants own by the manufacturer.
Secondary sampling units (ssu): all the machines in the 90 plants.

b) This is a 2-stage sampling question. We know $N = 90, n = 6$.

$$\hat{t}_{unb} = \frac{N}{n}\sum_{i=1}^{n}\sum_{j=1}^{m_i}\frac{M_i}{m_i}y_{ij} = \frac{N}{n}\sum_{i=1}^{n}\hat{t}_i = \frac{N}{n}\sum_{i=1}^{n}M_i\bar{y}_i = \frac{90}{6}\times 1304.9 = 19573.5$$

$$SE(\hat{t}_{unb}) = \sqrt{N^2\left(1-\frac{n}{N}\right)\frac{s_t^2}{n} + \frac{N}{n}\sum_{i=1}^{n}\left(1-\frac{m_i}{M_i}\right)M_i^2\frac{s_i^2}{m_i}}$$

$$= \sqrt{90^2\times\left(1-\frac{6}{90}\right)\times\frac{768}{6} + \frac{90}{6}\times 16293} \approx \sqrt{967680 + 244395} \approx 1101$$

Now, we know that there is a total of 4500 machines, hence,

$$\hat{\bar{y}}_{unb} = \hat{\bar{y}} = \frac{\hat{t}_{unb}}{M_0} = \frac{19573.5}{4500} = 4.35$$

$$SE(\hat{\bar{y}}_{unb}) = \frac{SE(\hat{t}_{unb})}{M_0} = \frac{1101}{4500} = 0.245$$

Plant Index (i)	Number of Machines (M_i)	Sampled Number of Machines (m_i)	Sample Mean $\bar{y}_i$	Sample Variance s_i^2	$\hat{t}_i = M_i \times \bar{y}_i$	$\left(1-\frac{m_i}{M_i}\right) \times M_i^2\frac{s_i^2}{m_i}$
1	50	10	5.40	11.40	270	2280
2	65	13	4.00	10.70	260	2782
3	45	9	5.70	16.80	256.5	3024
4	48	10	4.80	13.30	230.4	2425.92
5	52	10	1.30	11.20	67.6	2446.08
6	58	12	3.80	15.00	220.4	3335
Total					1304.9	16293

Problem 8.3 (Solution)

a) Using SRS

$$\hat{t}_{unb} = \frac{N}{n}\sum_{i=1}^{n} t_i = \frac{8}{3} \times [4320 + 4160 + 5790] = 38053.3$$

$$\hat{\bar{y}}_{unb} = \frac{\hat{t}_{unb}}{M_0} = \frac{38053.3}{12950} = 2.94$$

$$\widehat{Var}(\hat{t}_{unb}) = N^2\left(1 - \frac{n}{N}\right)\frac{s_t^2}{n} = 8^2\left(1 - \frac{3}{8}\right)\frac{898.46^2}{3} = 10{,}763{,}111.1 \approx 3281^2$$

$$\text{Hence,} \qquad SE(\hat{\bar{y}}_{unb}) = \frac{\sqrt{Var(\hat{t}_{unb})}}{M_0} = \frac{3281}{12950} = 0.2533$$

b) Using ratio estimator

$$\hat{t}_r = \frac{\sum_{i=1}^{n} t_i}{\sum_{i=1}^{n} M_i} \times M_0 = \frac{14270}{7210} \times 12950 = 25630.6$$

$$\hat{\bar{y}}_r = \frac{\hat{t}_r}{M_0} = \frac{25630.6}{12950} = 1.98$$

$$\widehat{Var}(\hat{t}_r) = N^2\left(1 - \frac{n}{N}\right)\frac{s_r^2}{n} = 8^2\left(1 - \frac{3}{8}\right)\frac{482.86^2}{3} = 3{,}108{,}703.9 \approx 1763^2$$

$$\left(\text{where } s_r^2 = \frac{1}{n-1}\sum_{i=1}^{n}\left[t_i - \frac{\sum_{i=1}^{n} t_i}{\sum_{i=1}^{n} M_i} M_i\right]^2\right)$$

$$\text{Hence,} \qquad SE(\hat{\bar{y}}_r) = \frac{\sqrt{Var(\hat{t}_r)}}{M_0} = \frac{1763}{12950} = 0.1362$$

c) Using PPS

$$\hat{\bar{y}}_\psi = \frac{\hat{t}_\psi}{M_0} = \frac{1}{M_0} \times \frac{1}{n}\sum_{i=1}^{n} \frac{t_i}{\psi_i} = \frac{1}{M_0} \times \frac{1}{n}\sum_{i=1}^{n} \frac{t_i}{\frac{M_i}{M_0}} = \frac{1}{n}\sum_{i=1}^{n} \frac{t_i}{M_i} \qquad \left(\text{where } \psi_i = \frac{M_i}{M_0}\right)$$

$$= \frac{1}{3} \times \left[\frac{4320}{2100} + \frac{4160}{1910} + \frac{5790}{3200}\right] = \frac{1}{3} \times [2.06 + 2.18 + 1.81] = 2.01$$

$$\hat{t}_\psi = M_0 \times \hat{\bar{y}}_\psi = \frac{M_0}{n}\sum_{i=1}^{n} \frac{t_i}{M_i} = \frac{12950}{3} \times \left[\frac{4320}{2100} + \frac{4160}{1910} + \frac{5790}{3200}\right] = 26092.2$$

$$\widehat{Var}(\hat{t}_\psi) = \frac{1}{n(n-1)}\sum_{i=1}^{n}\left[\frac{t_i}{\psi_i} - \hat{t}_\psi\right]^2 = \frac{2433.6^2}{3} = 1{,}974{,}137.9 \approx 1405^2$$

$$\text{Hence,} \qquad SE(\hat{\bar{y}}_\psi) = \frac{\sqrt{Var(\hat{t}_\psi)}}{M_0} = \frac{1405}{12950} = 0.1085$$

Division	Number of Employees (M_i)	Total Number of Sick Leave (t_i)	$\psi_i = \frac{M_i}{M_0}$	$\frac{t_i}{\psi_i}$	$t_i - \frac{\sum_{i=1}^{n} t_i}{\sum_{i=1}^{n} M_i} M_i$
1	1200	--			
2	450	--			
3	2100	4320	0.1622	26640	163.69
4	860	--			
5	2840	--			
6	1910	4160	0.1475	28205	379.74
7	390	--			
8	3200	5790	0.2471	23431	-543.43
Total	12,950	14,270		78276	0
Std Dev		898.46		2433.60	482.86