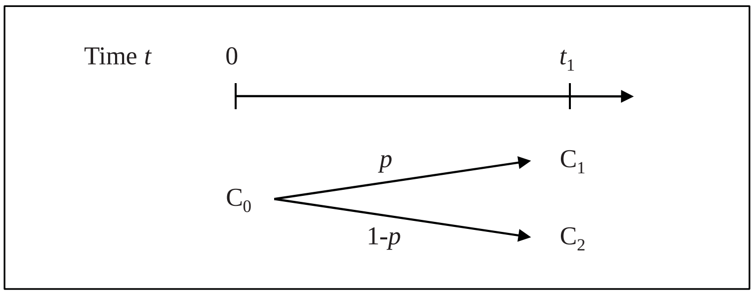
BA2202 Mathematics of Finance
Handout 1

1 Cashflow Models

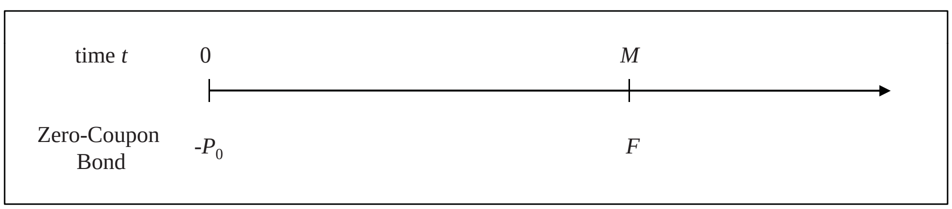
To evaluate the value of cashflows that emerge in securities and businesses, it is important to figure out two types of potential uncertainty:

- Timing of payments
- Amount of payments



The followings are examples involving a simple cashflow process.

- *Zero-coupon Bond*: A zero-coupon bond is a bond that does not make periodic interest payments. It is purchased at a price lower than its face value repaid at the time of maturity. (e.g., U.S. Treasury bills)

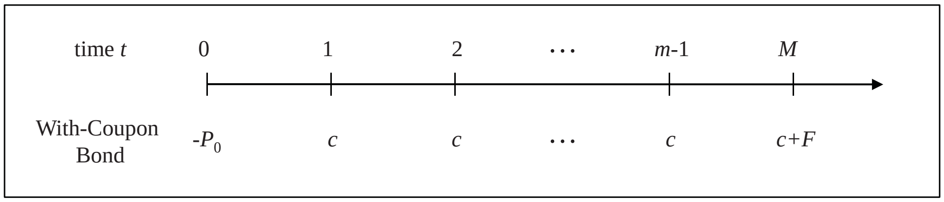


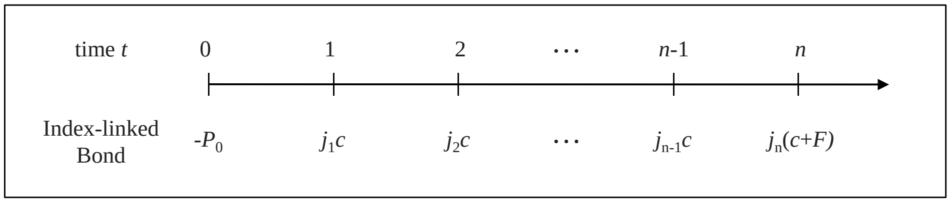
Example 1.1. *If the annual interest rate of one year zero-coupon bond is 5% and the face value is 100, what is the the price of the zero-coupon bond?*

- *Fixed Interest Security*: A type of security that pays a specified rate of interest that does not change over the life of the instrument. The face value is returned when the security matures.

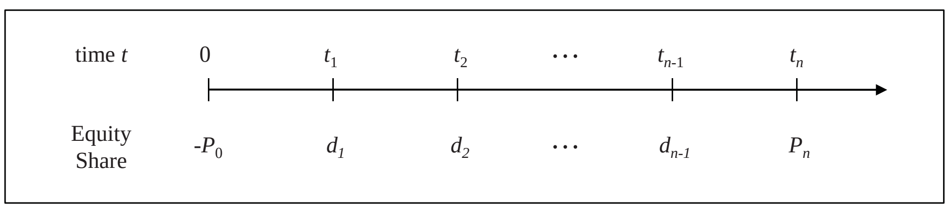
- *Index-linked Security*: A type of security that pays coupons which are initially set in line with market interest rates, but the coupons and principal payment are adjusted by the inflation Index such as Retail Prices Index.

Example 1.2. *If the inflation index is* $100 + t$*, what is the payment at maturity of 10 year coupon bond with 5% coupon rate and the face value 100?*





- *Call Deposit*: A type of bank account which allows investors instant access to their accounts without paying a charge and without informing the bank in advance. Withdrawals and deposits can be made at any time.
- *Equity Shares*: Shares which represent the right to participate in the residual assets of a business and which usually have voting rights. They will usually receive dividends, the level of which depends on firm's dividend policy.



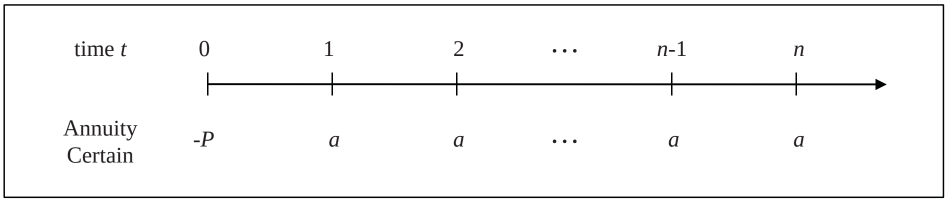
Example 1.3. *If a firm continues to pay 10 at the beginning of each year as a dividend where discount rate is 5%, what is the stock price?*

$$S = \frac{10}{1 - 0.95} = 200 \tag{1}$$

- *Annuity Certain*: An annuity that provides a fixed number of payments for a predetermined time period without exception or contingency. How does the cashflow of life annuity look like?
- *Repayment Loan*: A repayment loan is a loan that is repayable by a series of payments that include both partial repayment of the loan capital and the interest payments for the outstanding loan amount.

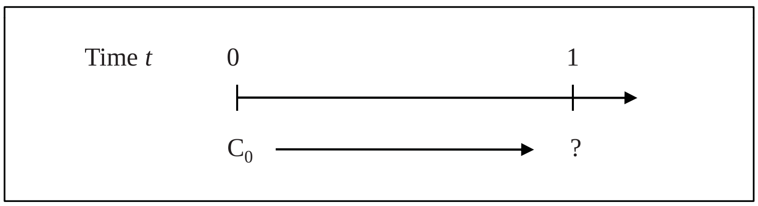
2 Time Value of Money

Today's one dollar is different from tomorrow's one dollar. In this section, we evaluate the time value of money.



2.1 Future Value

First, look forward from time t. Consider that capital C is invested at time t. How much will be the future value of the capital? How much interest will the capital earn at time $t+n$? Here we assume that cashflow is certain.



2.1.1 Simple Interest

Interest is a reward for the use of capital (or principal). Consider that capital C is invested at time t. Suppose that the investment continues for n years. Let A_{t+n} be the amount which will be received by the investor after n years. The simple interest rate per annum denoted by i is defined by the amount paid to the investor at the end of the period.

$$A_{t+1} = A_t + iC \Rightarrow A_{t+2} = C + iC + iC \Rightarrow A_{t+n} = C(1+ni) \tag{2}$$

where i is independent of time t. The key idea of simple interest rate is that interest itself does not earn further interest. The interest earned for the period is niC.

Example 2.1. *You deposit 100 in a bank account at 5% simple-interest per annum. The amount you will receive at the end of the 2nd year is* $100(1 + 2 \times 0.05) = 110$.

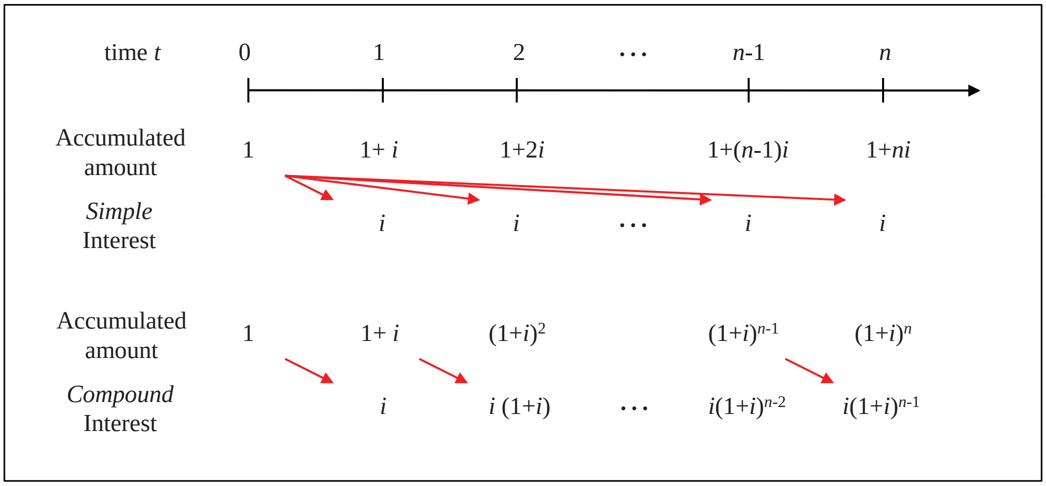
Assuming that i is simple-interest per year, you earn Ci in the bank account every year. Then, if you withdraw your money at the end of 1st year and deposit it in another account, how much you will have at the end of the 2nd year?

$$105(1 + 0.05) = 110.25$$

Should you switch your money from one bank to the other every year?

2.1.2 Compound Interest

Credited interest should earn further interest! That is the concept of compound interest. If credited interest earns further interest, the amount holds the following relationship:



$$A_{t+1} = C(1+i) \Rightarrow A_{t+2} = [C(1+i)](1+i) \Rightarrow A_{t+n} = [C(1+i)^{n-1}](1+i) \tag{3}$$

$$A_{t+n} = (1+i)A_{t+n-1} \tag{4}$$

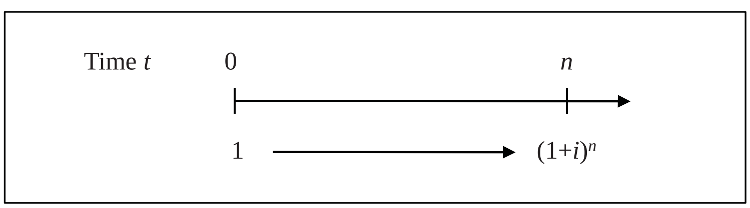
This can be rewritten as:

$$A_{t+n} = A_t(1+i)^n \tag{5}$$

Furthermore, the interest received by the investor after n periods is:

$$A_{t+n} - A_t = C(1+i)^n - C = C[(1+i)^n - 1] \tag{6}$$

Thus, $(1+i)^n$ is the factor converting the present value (PV) of money to the future value (FV) when interest rate i is applied during n periods. That is, $FV = PV(1+i)^n$.



Example 2.2. *You invest 100 at 5% compound interest for two years. The amount you will receive at the end of the period for compound interest is* $100(1+0.05)^2 = 110.25$.

When your money is accumulated by compound interest, there is no gain by switching your money from one bank to another. Thus, compound interest rate seems more reasonable.

2.2 Effective Rate of Interest

The real effect of a nominal interest rate depends on whether the interest rate is compounded or not and on the frequency of compounding period. To compare several different nominal interest rates, we use the *effective rate of interest* at time t denoted by $i(t)$ for the annual rate. Generally, the *effective rate of interest* is defined by the ratio of the amount of interest actually earned at the end of the time period to the initial investment amount.

$A(t_1, t_2)$ is defined to be the accumulated amount (also called the accumulation factor) at time t_2 when 1 is invested at time t_1. The effective rate of interest over a time period h is defined by the accumulation factors:

$$\frac{A(t, t+h) - A(t, t)}{A(t, t)} = A(t, t+h) - 1 = (1 + i(t))^h - 1 \tag{7}$$

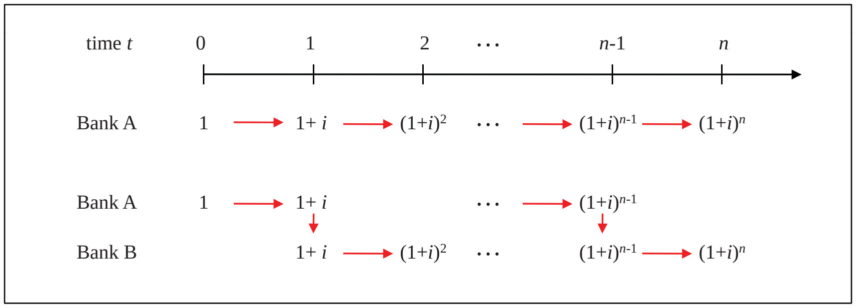
where $h > 0$.

2.2.1 Principle of Consistency

If a market is consistent, the accumulation factor does not depend on the course of action taken by investors.

$$A(t_0, t_n) = A(t_0, t_1)A(t_1, t_2) \cdots A(t_{n-1}, t_n) \tag{8}$$

for any t_k for $k = 1, 2, ..., n$.



The annual effective rate for the simple interest rate, r, is

$$i(t) = \frac{A(t, t+n+1) - A(t, t+n)}{A(t, t+n)} = \frac{(1 + r(n+1)) - (1 + nr)}{1 + nr} = \frac{r}{1 + nr} \tag{9}$$

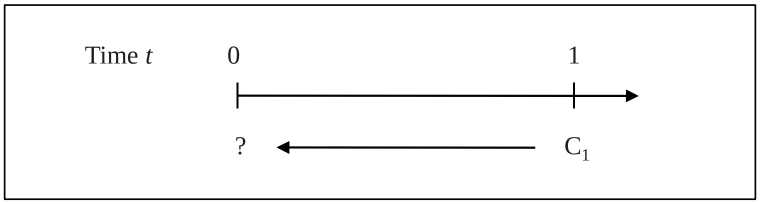
which decreases with n. In contrast, for the compound interest, r, the annual effective rate is

$$i(t) = \frac{A(t, t+n+1) - A(t, t+n)}{A(t, t+n)} = \frac{(1+r)^{n+1} - (1+r)^n}{(1+r)^n} = r \tag{10}$$

which is time-constant.

2.3 Present Values

Next, look backward from time $t + n$ to time t. Consider that an investor wants capital C at time $t + n$. How much capital must be invested at time t if interest rate is i during the period? From:



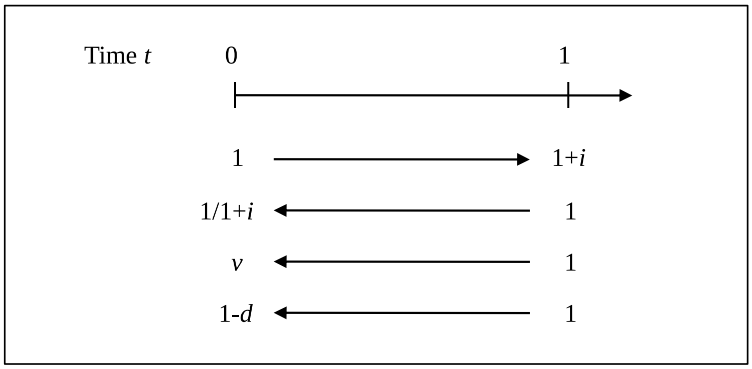
$$C = A_{t+n} = (1+i)^n A_t \tag{11}$$

This is rearranged as:

$$A_t = \frac{C}{(1+i)^n} = v^n C \tag{12}$$

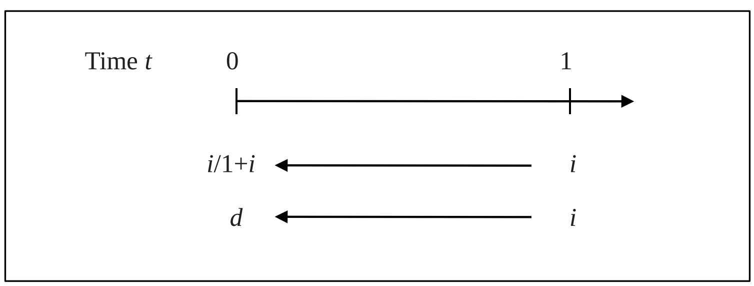
where we define the discount factor as $v = 1/(1+i)$. Thus, $v^n = 1/(1+i)^n$ is the factor with which the FV of money is converted into the PV. Also, the *rate of discount* d is defined in the following form:

$$A_t = C(1-d)^n \tag{13}$$



The rate of discount that can be defined by interest rate i is interpreted as the discounted present value of interest earned at the end of period:

$$1 - d = \frac{1}{1+i} \Leftrightarrow d = \frac{i}{1+i} = 1 - v = iv \tag{14}$$



Example 2.3. *For an interest rate of 2% compounded annually, the PV of 100 payable in 3 years is*

$$\frac{100}{(1+0.02)^3} = 94.232$$

Example 2.4. *If an investor can earn 5% interest for one year, the rate of discount d is*

$$d = \frac{i}{1+i} = \frac{0.05}{1+0.05} = 0.0476.$$

Example 2.5. *Two types of bonds are available in the market: one-year zero-coupon bond at the discount of 5% and bond with 5% coupon payable in one year. Which bond do you prefer?*

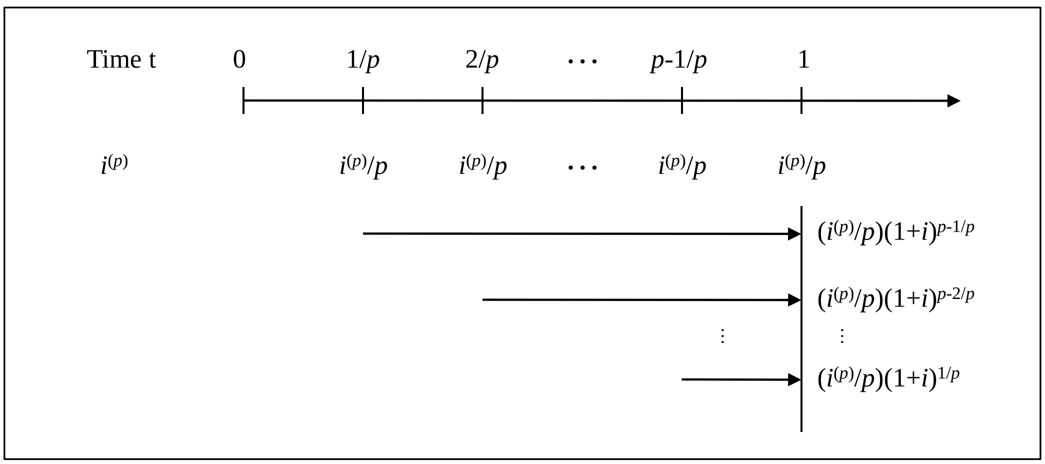
2.4 Frequency of Compounding

So far we implicitly assume that interest is paid at the end of a period or is deducted up-front. However, interest may be credited more frequently. Assuming that investment period is one year, we define the *nominal rate of interest payable p times a year*, denoted by $i^{(p)}$. For instance,

- $i^{(12)}$: Nominal rate payable monthly
- $i^{(4)}$: Nominal rate payable quarterly

More generally, suppose that a person borrows 1 at time 0 for repayment at time 1. He wants to pay the interest on the loan in p equal installments over the period.

- $i^{(p)}$: The total amount of interest, payable in equal installments $i^{(p)}/p$ at the *end* of each pth subperiod (i.e., $1/p, 2/p, ..., (p-1)/p, 1$)
- $d^{(p)}$: The total amount of interest, payable in equal installments $d^{(p)}/p$ at the *beginning* of each pth subperiod (i.e., $0, 1/p, 2/p, ..., (p-1)/p$)



In terms of $i^{(p)}$, the total amount of interest must hold the following relationship with the effective rate of interest:

$$i = \sum_{t=1}^{p} \frac{i^{(p)}}{p}(1+i)^{(p-t)/p}$$

This can be rearrange to:

$$1 + i = \left(1 + \frac{i^{(p)}}{p}\right)^p \quad \Leftrightarrow \quad i^{(p)} = p[(1+i)^{\frac{1}{p}} - 1]. \tag{15}$$

From the relationship between interest rate and discount rate: $v = 1 - d = 1/(1+i)$, discount rate is also expressed in terms of $i^{(p)}$:

$$\begin{aligned} v &= \left(1 + \tfrac{i^{(p)}}{p}\right)^{-p} \quad \Leftrightarrow \quad i^{(p)} = p\left(v^{-\frac{1}{p}} - 1\right) \\ 1 - d &= \left(1 + \tfrac{i^{(p)}}{p}\right)^{-p} \quad \Leftrightarrow \quad i^{(p)} = p\left[(1-d)^{-\frac{1}{p}} - 1\right] \end{aligned} \tag{16}$$

If you are interested in finding the accumulated amount after tp period (e.g., two years, $p = 12$ and $t = 2$), it can be calculated by:

$$(1+i)^t = \left[1 + \frac{i^{(p)}}{p}\right]^{tp}. \tag{17}$$

Analogously, we find the following relationship in terms of $d^{(p)}$.

$$d = \sum_{t=1}^{p} \frac{d^{(p)}}{p}(1-d)^{(t-1)/p}$$

which can be rearranged to:

$$\begin{aligned} 1-d &= \left(1 - \tfrac{d^{(p)}}{p}\right)^p \quad \Leftrightarrow \quad d^{(p)} = p\left[1-(1-d)^{\frac{1}{p}}\right] \\ v &= \left(1 - \tfrac{d^{(p)}}{p}\right)^p \quad \Leftrightarrow \quad d^{(p)} = p\left(1 - v^{\frac{1}{p}}\right) \end{aligned} \tag{18}$$

From the relationship between interest rate and discount rate: $1 - d = v = 1/(1 + i)$, interest rate is also expressed in terms of $d^{(p)}$:

$$1 + i = \left(1 - \frac{d^{(p)}}{p}\right)^{-p} \quad \Leftrightarrow \quad d^{(p)} = p[1-(1+i)^{-\frac{1}{p}}]. \tag{19}$$

From the formulas above, we obtain nominal equivalence:

$$\left(1 + \frac{i^{(p)}}{p}\right)^p = \left(1 - \frac{d^{(p)}}{p}\right)^{-p}. \tag{20}$$

2.4.1 Non-integral Compound Period

There are cases where accumulation period divided by compounding period is not an integer. For instance, you may want to know the FV of an investment of 100 after 20 months at the nominal rate of interest of 2% compounded semiannually (i.e., $20/6 = 3.333$). There are several different ways to handle the residual of time period. First, we may consider that no interest can be earned after first 18 months. Alternatively, interest may be credited for the last 2 months in simple-interest method or compound-interest method. If the simple-interest method is applied for the residual period, the FV is calculated by:

$$100\left(1 + \frac{0.02}{2}\right)^3\left(1 + 0.02 \times \frac{2}{12}\right) = 103.3735.$$

If the compound-interest method is applied for the residual period, the FV is:

$$100\left(1 + \frac{0.02}{2}\right)^3\left(1 + \frac{0.02}{2}\right)^{0.333} = 103.3724.$$

Unless stated otherwise, we assume the compound-interest method for calculating the interest over a non-integral compounding period.

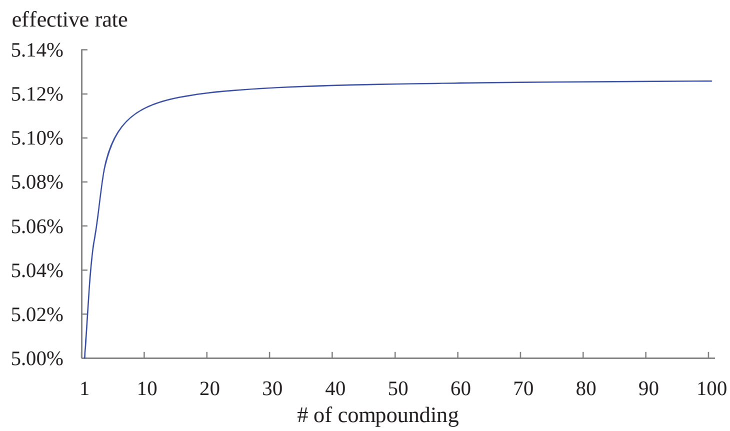
Example 2.6. *For 5% interest rate payable at the end of each quarter is compounded quarterly, the effective rate is*

$$i = \left(1 + \frac{i^{(p)}}{p}\right)^p - 1 = \left(1 + \frac{0.05}{4}\right)^4 - 1 = 0.0509.$$

If the 5% interest rate is converted monthly, the effective rate is

$$i = \left(1 + \frac{0.05}{12}\right)^{12} - 1 = 0.0512.$$

The relationship between the frequency of compounding and the effective rate is illustrated:



Example 2.7. *If 5% interest rate payable at the beginning of each quarter is compounded quarterly, the effective rate of discount is:*

$$d = 1 - \left(1 - \frac{d^{(p)}}{p}\right)^p = 1 - \left(1 - \frac{0.05}{4}\right)^4 = 0.04907.$$

Note that this effective rate is smaller than corresponding i.

Example 2.8. *In order to earn the effective rate of 5%, we want to find the nominal rate of interest payable semiannually. Using the formula, we obtain:*

$$i^{(p)} = p[(1+i)^{\frac{1}{p}} - 1] = 2[(1+0.05)^{\frac{1}{2}} - 1] = 0.0494$$

And we check if the nominal rate actually earns the effective rate of 5%.

$$i = \left(1 + \frac{0.0494}{2}\right)^2 - 1 \approx 0.05$$

Example 2.9. *A 3-month Treasury Bill is offered at a discount of 4%. The effective 3-month interest rate is:*

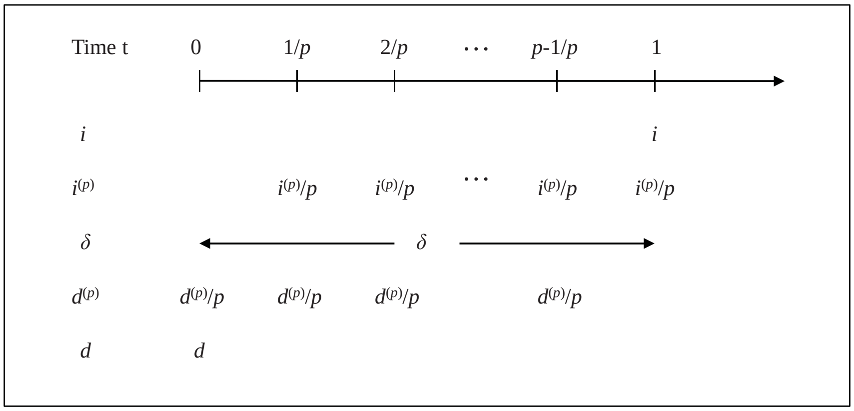
$$\frac{d^{(p)}/p}{1 - d^{(p)}/p} = \frac{0.04/4}{1 - 0.04/4} = 0.0101$$

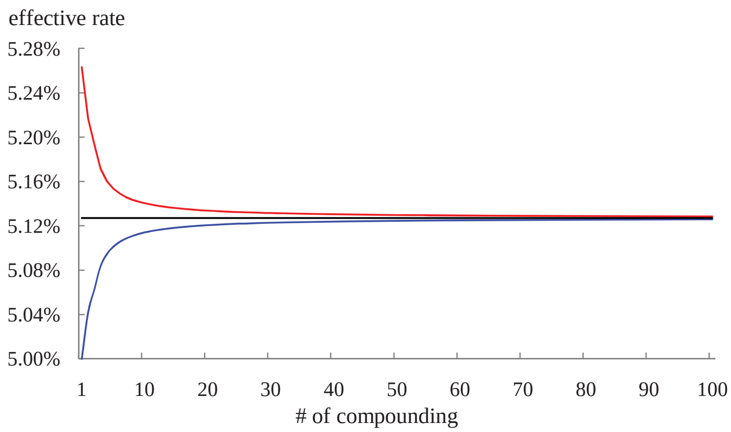
And the effective rate of interest per annum is:

$$i = 1.0101^4 - 1 = 0.0410$$

The nominal rate of interest payable quarterly is:

$$i^{(4)} = 4 \times 0.0101 = 0.0404.$$





2.5 The Force of Interest/Discount

Define the length of interest compounding period $h = 1/p$ and consider the case when a nominal interest rate is payable very frequently ($p \to +\infty$). The *force of interest* at time t, denoted by $\delta(t)$, is defined by:

$$\delta(t) = \lim_{h \to 0+} \left[\frac{A(t, t+h) - 1}{h} \right] \tag{21}$$

Since $A(t, t+h) = A(t_0, t+h)/A(t_0, t)$ from the principle of consistency,

$$\begin{aligned} \delta(t) &= \lim_{h \to 0+} \left[\tfrac{A(t,t+h)-1}{h} \right] \\ &= \lim_{h \to 0+} \left[\tfrac{A(t_0,t+h)-A(t_0,t)}{A(t_0,t)} \tfrac{1}{h} \right] \\ &= \tfrac{A'(t_0,t)}{A(t_0,t)} \\ &= \tfrac{d}{dt} \ln A(t_0, t) \end{aligned} \tag{22}$$

Thus, the accumulation factor can be defined in terms of the force of interest rate:

$$A(t_0, t) = \exp \left[\int_{t_0}^{t} \delta(u) du \right] \tag{23}$$

More generally for $t_1 < t_2$, the *accumulation factor* is defined as:

$$A(t_1, t_2) = \exp \left[\int_{t_1}^{t_2} \delta(t) dt \right] \tag{24}$$

And since $A(t_2, t_1) = A(t_0, t_1)/A(t_0, t_2) = 1/A(t_1, t_2)$, the corresponding *discount factor* is

$$A(t_2, t_1) = \exp\left[-\int_{t_1}^{t_2} \delta(t)dt\right] \tag{25}$$

If $\delta(t) = \delta$, both the accumulation factor and discount factor can be simplified to:

$$A(t_1, t_2) = e^{\delta(t_2 - t_1)} \tag{26}$$

$$A(t_2, t_1) = e^{-\delta(t_2 - t_1)} \tag{27}$$

Define the discounted present value of 1 at time t:

$$v(t) = \exp\left[-\int_0^t \delta(s)ds\right] \tag{28}$$

If $\delta(t) = \delta$, the *discounted present value* is reduced to:

$$v(t) = v^t = e^{-t\delta} \tag{29}$$

The *rate of interest* per time unit is defined in terms of the force of interest δ:

$$i = e^{\delta} - 1 \Leftrightarrow \delta = \ln(1 + i) \tag{30}$$

The *rate of discount* d is defined by:

$$1 - d = v = e^{-\delta} \tag{31}$$

Example 2.10. *Given* $\delta(t) = \frac{1}{t+1}$. *A(0,n) can be obtained through integration:*

$$\begin{aligned} A(0,t) &= \exp[\textstyle\int_0^n \delta(t)dt] \\ &= \exp[\textstyle\int_0^n \frac{1}{t+1}dt] \\ &= \exp[\ln(n+1)] \\ &= n+1 \end{aligned} \tag{32}$$

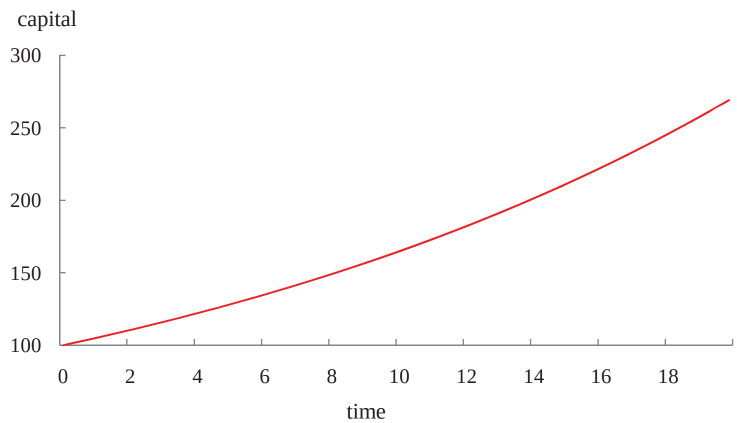
Example 2.11. *You invest 100 in a project earning a time-constant force of interest of 5%. To know how long it will take to accumulate 200:*

$$200 = 100e^{0.05t} \Rightarrow 0.05t = ln2$$

$$\therefore t = 13.863$$

Example 2.12. *Given* $i > 0$ *and* $p > 1$, *the relationship between* d, i, δ, $d^{(p)}$, *and* $i^{(p)}$ *is:*

$$d < d^{(p)} < \delta < i^{(p)} < i$$



2.6 Approximations of interest rates

When interest rate i is not large, we can approximate δ and d by i. Recall the relationship between δ and i:

$$\begin{aligned} \delta &= \ln(1+i) \\ &= \scriptstyle 1 \textstyle - \tfrac{1}{2}i^2 + \tfrac{1}{3}i^3 - \tfrac{1}{4}i^4 + \cdots \end{aligned} \tag{33}$$

For small interest rate i, we obtain:

$$\delta \approx i - \frac{1}{2}i^2 \tag{34}$$

Similarly, d is represented as:

$$\begin{aligned} d &= i(1+i)^{-1} \\ &= i(1 - i + i^2 - i^3 + \cdots) \\ &= i - i^2 + i^3 - i^4 + \cdots \end{aligned} \tag{35}$$

For small interest rate i, we obtain:

$$d \approx i - i^2 \tag{36}$$

From these results we confirm $d < \delta < i$. It is also easy to approximate d and i in terms of δ. Using the following power series:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} \cdots \tag{37}$$

if δ is small,

$$\begin{aligned} i &= e^{\delta} - 1 = \left(1 + \delta + \tfrac{\delta^2}{2} + \tfrac{\delta^3}{3} \cdots\right) - 1 \\ &\approx \delta + \tfrac{\delta^2}{2} \end{aligned} \tag{38}$$

Similarly,

$$\begin{aligned} d &= 1 - e^{-\delta} = 1 - \left(1 - \delta + \tfrac{\delta^2}{2} - \tfrac{\delta^3}{3} \cdots\right) \\ &\approx \delta - \tfrac{\delta^2}{2} \end{aligned} \tag{39}$$

You may memorize the following relationships:

$$\begin{aligned} \ln(1+x) &= \textstyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots && \text{for } x \in (-1, 1] \\ e^x &= \textstyle\sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \cdots && \text{for } x \in R \\ (1-x)^{-1} &= \textstyle\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots && \text{for } x \in (-1, 1) \end{aligned} \tag{40}$$

3 Real and Money Interest Rates

Let $A(0,t)$ be the accumulated amount which ignores the effect of inflation (*money rate of interest*) and $A^*(0,t)$ the accumulated amount reduced for the effect of inflation (*real rate of interest*).

- Inflation: $A(0,t) > A^*(0,t)$
- Deflation: $A(0,t) < A^*(0,t)$

It is important to consider whether future value is adjusted by inflation or whether you are interested in real amount.

We denote r_I as the inflation rate, and r_N as the nominal rate of interest defined by $r_N = A(0,1) - 1$. Note that the term "nominal" used in the previous sentence simply means the rate of interest that does reflect the effect of inflation. The real rate of interest r_R is defined by:

$$1 + r_R = \frac{1 + r_N}{1 + r_I} \tag{41}$$

Example 3.1. *An investment offers an effective annual rate of 5%. Inflation is 3% per year. The real rate of return on the investment is:*

$$\frac{1.05}{1.03} - 1 = 0.0194$$

4 Discounting and Accumulating

In this section we study discounting and accumulating a series of payments.

4.1 Present Values of Discrete Cashflows

The present value of a series of payments of c_{t_1}, c_{t_2},..., c_{t_n} due at times t_1, t_2,..., t_n is given by:

$$PV = c_{t_1}v(t_1) + c_{t_2}v(t_2) + \cdots + c_{t_n}v(t_n) = \sum_{j=1}^{n} c_{t_j}v(t_j) \tag{42}$$

where $v(t)$ is the present value of 1 due at time t. If $v(t) = v^t$ (If the effective rate of interest is constant over the period), the present value is:

$$PV = \sum_{j=1}^{n} c_{t_j}v^{t_j} \tag{43}$$

4.2 Continuous Cashflows

An investor will be paid interests continuously between times 0 and T. Let the *rate of payment* at time t be $\rho(t)$. The present value of the entire cashflow is obtained by:

$$PV = \int_0^T v(t)\rho(t)dt \tag{44}$$

Now continuous payments is received at the rate of payment $\rho(t)$ from time a to time b. The force of interest during the time is $\delta(t)$. The present value at time a of this payment stream is:

$$PV = \int_a^b \rho(t) \exp\left[-\int_a^t \delta(s)ds\right] dt \tag{45}$$

Similarly, the accumulated value at time b of this cashflow is:

$$FV = \int_a^b \rho(t) \exp\left[\int_t^b \delta(s)ds\right] dt \tag{46}$$