BA2202 Mathematics of Finance
Handout 7

1 Term Structure of Interest Rate

So far we have generally assumed that the interest rate i or force of interest δ earned on an investment is independent of the term of the investment. In practice, the interest rate offered on investments does usually vary according to the term of the investment.

The *term structure of interest rates*, often called as the *yield curve*, is the relation between the interest rate and the time to maturity of the debt. For instance, the U.S. dollar interest rates paid on U.S. Treasury securities for various maturities are commonly plotted on a graph, which is informally called the *yield curve*.

1.1 Discrete Time

1.1.1 Discrete Time Spot Rates

We denote the price at issue of a unit zero coupon bond maturing in n years by P_n. The yield on a unit zero coupon bond with term n years, y_n, is called the *n-year spot rate of interest*, the average interest rate over the period from now until n years' time. By the equation of value for the zero coupon bond, we have

$$P_n = \frac{1}{(1+y_n)^n} \quad \Rightarrow \quad y_n = P_n^{-\frac{1}{n}} - 1 \tag{1}$$

Example 1.1. *The prices for zero coupon bonds of various terms are as follows: 1 year=95%, 5 years=85%, and 10 years=75%. Calculate the spot rates for these terms.*

The formula gives:

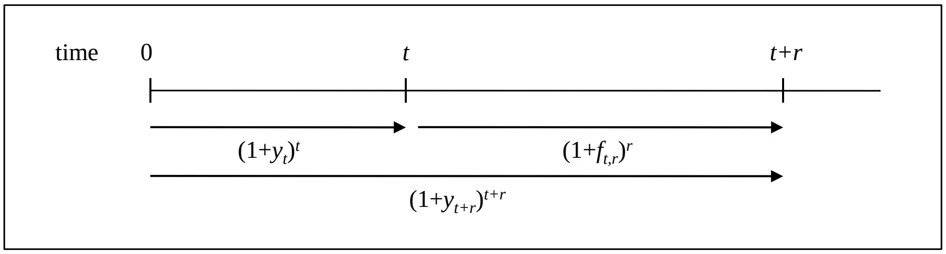
$$\begin{aligned} y_1 &= \left(\frac{95}{100}\right)^{-1} - 1 = 0.0526 \\ y_5 &= \left(\frac{85}{100}\right)^{-\frac{1}{5}} - 1 = 0.0330 \\ y_{10} &= \left(\frac{75}{100}\right)^{-\frac{1}{10}} - 1 = 0.0292 \end{aligned}$$

■

Once the term structure of interest rate is identified, we can use it to calculate the price of coupon bonds as well. A coupon bond can be regarded as a combination of zero-coupon bonds with different maturities. For instance, a bond paying coupons of D every year for n years, with a final redemption payment of R at time n can be regarded as a combined investment of n zero coupon bonds with maturity value D with terms of 1 year, 2 years, $\cdots$, n years, plus the redemption value R with term n years. This must be true under the *law of one price*. The price of the bond is:

$$\begin{aligned} A &= D(P_1 + P_1 + \cdots + P_n) + RP_n & (2) \\ &= D[(1+y_1)^{-1} + (1+y_2)^{-2} + \cdots + (1+y_n)^{-n}] + R(1+y_n)^{-n} & (3) \end{aligned}$$

Figure 1: Discrete-time Spot Rates and Forward Rates



1.1.2 Discrete Time Forward Rates

In contrast to spot rates, interest rates over a period that start at time 0, *forward rates* $f_{t,r}$ are interest rates agreed at time 0 for an investment made at time $t > 0$ for a period of r years. For instance, if an investor agrees at time 0 to invest 1 at time $t > 0$ for r years, the accumulation factor at time $t + r$ is $(1 + f_{t,r})^r$.

Forward rates, spot rates and zero-coupon bond prices are all connected and must be consistent. For instance, an investment of 1 at time 0 is accumulated to $(1 + y_t)^t$ in t years. If the investor agrees at time 0 to invest $(1 + y_t)^t$ at time t for r years, the accumulated value at time $t + r$ will be $(1 + y_t)^t(1 + f_{t,r})^r$. If the $t + r$ year zero-coupon bond is sold at P_{t+r} per unit nominal, which yields $(1 + y_{t+r})^{t+r}$, these two accumulation value must be the same under no arbitrage assumption. Thus,

$$(1 + y_t)^t(1 + f_{t,r})^r = (1 + y_{t+r})^{t+r} \tag{4}$$

To find $f_{t,r}$ this can be rewritten as

$$(1 + f_{t,r})^r = \frac{(1 + y_{t+r})^{t+r}}{(1 + y_t)^t} = \frac{P_t}{P_{t+r}} \tag{5}$$

Thus, knowing the prices of zero-coupon bond with various terms is equivalent to knowing the yield curve. And knowing the forward rates is also equivalent under no arbitrage assumption.

Example 1.2. *The price of zero-coupon bond with terms 2, 3 and 5-year spot rates are 95, 92, 87 per 100 nominal respectively. Calculate* $f_{2,1}$ *and* $f_{2,3}$.

The formula gives:

$$\begin{aligned} f_{2,1} &= \left(\frac{95}{92}\right) - 1 = 0.0326 \\ f_{2,3} &= \left(\frac{95}{87}\right)^{\frac{1}{3}} - 1 = 0.0298 \end{aligned}$$

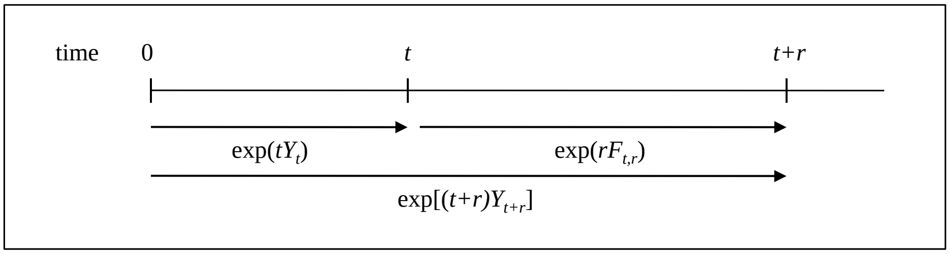
■

1.1.3 Continuous Time Spot Rates

The continuous-time spot rate, Y_t corresponds discrete annual rate y_t in the same way as δ and i. That is, $y_t = e^{Y_t} - 1$. The t year spot force of interest Y_t is found where

$$P_n = e^{-Y_n t} \quad \Rightarrow \quad Y_n = -\frac{1}{t}\log P_t \tag{6}$$

Figure 2: Continuous-time Spot Rates and Forward Rates



1.1.4 Continuous Time Forward Rates

The continuous time forward rates $F_{t,r}$ is the force of interest equivalent to the annual forward rate of interest $f_{t,r}$.

Analogously to the discrete-time example, an investment of 1 at time 0 is accumulated to e^{tY_t} in t years. If the investor agrees at time 0 to invest e^{tY_t} at time t for r years, the accumulated value at time $t+r$ will be $e^{tY_t}e^{tF_{t,r}}$. If the $t+r$ year zero-coupon bond is sold at P_{t+r} per unit nominal, which yields $e^{(t+r)Y_{t+r}}$, these two accumulation value must be the same under no arbitrage assumption.

$$e^{tY_t}e^{rF_{t,r}} = e^{(t+r)Y_{t+r}} \tag{7}$$

To find $F_{t,r}$ this can be rewritten as

$$F_{t,r} = \frac{(t+r)Y_{t+r} - tY_t}{r} = \frac{1}{r}\ln\left(\frac{P_t}{P_{t+r}}\right) \tag{8}$$

Example 1.3. *The price of zero-coupon bond with terms 2, 3 and 5-year spot rates are 95, 92, 87 per 100 nominal respectively. Calculate Y_5 and $F_{2,3}$.*

The formula gives:

$$\begin{aligned} Y_5 &= -\frac{1}{5}\ln\left(\frac{87}{100}\right) = 0.0279 \\ F_{2,3} &= \frac{1}{3}\ln\left(\frac{95}{87}\right) = 0.0293 \end{aligned}$$

■

1.2 Theories of the Term Structure of Interest Rates

The interest rates in investment markets vary depending on the time span of the investments. This variation determines the term structure of the interest rates. There are several explanations for the variation. It is possible for yield curves to be increasing (see Figure 3) or decreasing (see Figure 4).

- *Expectation Theory*: The relative attraction of short and longer-term investments will vary according to expectations of future movements in interest rates. For instance, an expectation of a fall in interest rates will make short-term investments less attractive and longer-term investments more attractive.

- *Liquidity Preference*: Longer dated bonds are more sensitive to interest rate movements than short-dated bonds. Risk averse investors will require compensation for the greater risk of loss on longer bonds (→ explains increasing yield curve).

Figure 3: Daily Treasury Yield Curve Rates

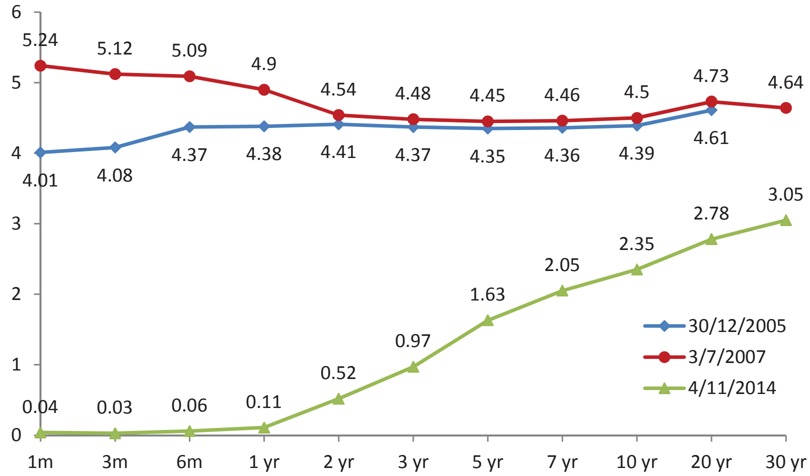
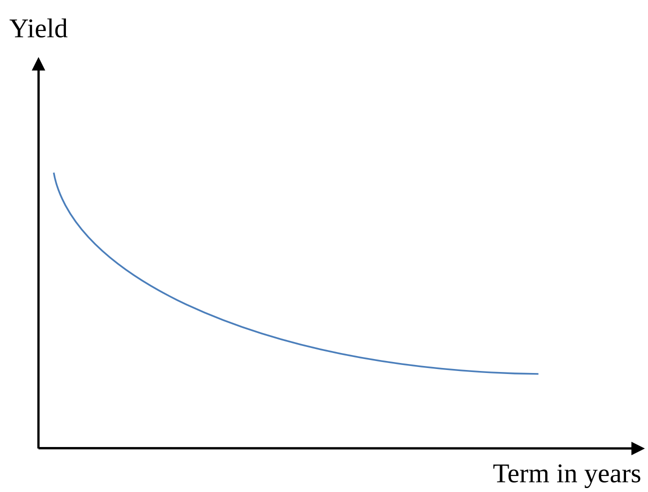


Figure 4: Decreasing Yield Curve



- *Market Segmentation*: The term structure emerges from these different forces of supply and demand. For instance, banks invest in very short-term bonds, while pension funds tend to invest in long-dated bonds. For the supply side, governments' and firms' demand for short-term and long-term bond may vary and may not correspond to the investors' demand.

1.3 Yields to Maturity and Par Yield

The *yield to maturity* for a coupon paying bond, also called the *redemption yield*, is defined as the effective rate of interest at which the discounted value of the future cashflow of a bond equal the price. For a n-year bond with annual coupon payments D redeemable at R per nominal, the yield to maturity is found by solving the following equation of value in terms of the effective rate of interest i:

$$P = Da_{\overline{n}|i} + Rv^n \tag{9}$$

In contrast, the n-year *par yield* is the coupon per nominal that would be payable on a bond with term n years, which would give the price equal to its nominal value (or par value) under the current term structure, assuming the bond is redeemed at par. That is, the n-year *par yield* is D_n such that:

$$1 = D_n(v_{y_1} + v_{y_2}^2 + v_{y_3}^3 + \cdots + v_{y_n}^n) + 1v_{y_n}^n \tag{10}$$

where $v_{y_t} = (1 + y_t)^{-1}$.

Example 1.4. *Calculate the 3 year par yield if the annual term structure of interest rates is given by (2%, 2.5%, 3%).*

By the formula:

$$1 = D_3(1.02^{-1} + 1.025^{-2} + 1.03^{-3}) + 1.03^{-3} \quad \Rightarrow \quad D_3 = 0.0298$$

■

2 Duration, Convexity and Immunization

How the investment of a fixed interest portfolio would be affected by a change in interest rates? For risk management purpose, an investment manager should be able to answer the question, because interest rates generally fluctuate quite significantly. In this section, we introduce several measures of vulnerability to interest rate movements and a method of mitigating the interest rate risk.
For simplicity, we assume a flat yield curve, and that when interest rate change, all change by the same amount, so that the curve stays flat.

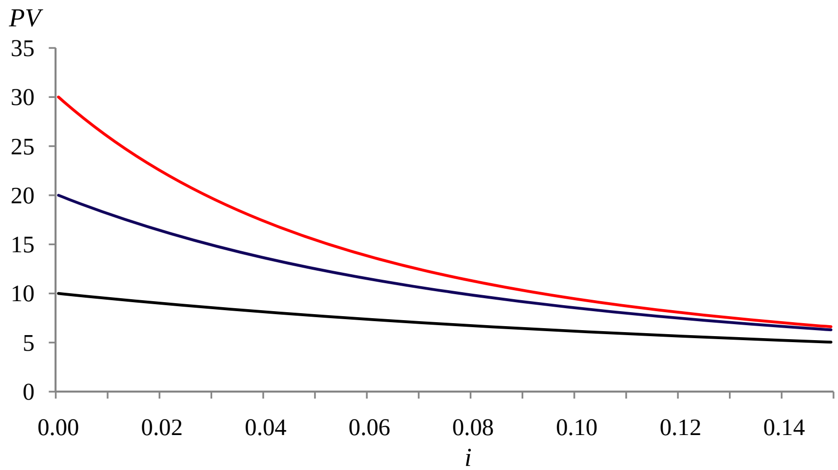
Suppose that a firm holds the PV of assets $V_A(i)$ to meet the PV of liability $V_L(i)$ at effective rate of interest i. It is natural that the firm is required to maintain $V_A \geq V_L$. Since both $V_A(i)$ and $V_L(i)$ represent the PV of asset cashflows $\{A_{t_k}\}$ and the PV of liability cashflows $\{L_{t_k}\}$, both are sensitive to the rate of interest. We know that both the value of the assets and the value of the liability fluctuate in a same direction such that:

- both decrease if interest rates rise, and
- both increase if interest rates fall.

Therefore, the interest rate risk for the firm is considered as cases such that:

- V_A decreases by more than V_L when interest rates rise, and

Figure 5: Interest Rate Sensitivity



- V_A increases by less than V_L when interest rates fall.

It may be beneficial to illustrate the interest rate sensitivity of annuities. See Figure 5. The all of three curves have a downward slope. The steepest, middle, and flattest curves represent the interest rate sensitivity of $a_{\overline{30}|}$, $a_{\overline{20}|}$, and $a_{\overline{10}|}$. Thus, it is illustrated that an investment with longer duration has a greater sensitivity to a change in the interest rates.

2.1 Effective Duration

One measure of the sensitivity of a series of cashflows, to movements in the interest rates, is the *effective duration* (also called *volatility*). Consider a series of cashflows $\{C_{t_k}\}$ for $k = 1, 2, \cdots, n$. Let A be the PV of the payments at rate i, so that:

$$A(i) = \sum_{k=1}^{n} C_{t_k} v_i^{t_k}$$

The *effective duration* is defined by:

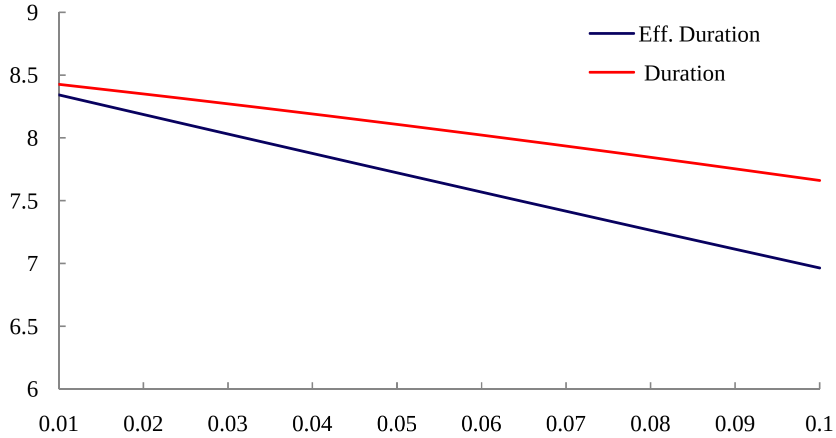
$$\begin{aligned} \nu(i) &= -\frac{1}{A(i)}\frac{d}{di}A(i) && (11) \\ &= -\frac{A'(i)}{A(i)} && (12) \\ &= \frac{\sum_{k=1}^{n} t_k C_{t_k} v_i^{t_k+1}}{\sum_{k=1}^{n} C_{t_k} v_i^{t_k}} && (13) \end{aligned}$$

This is the rate of change of value of A with i, which is independent of the size of the PV. Note that the payments do not depend on the rate of interest.

2.2 Duration

Another measure of interest rate sensitivity is the *duration*, also called *Macaulay Duration* or *discounted mean term* (DMT). This measure is the mean term of a series of cashflows $\{C_{t_k}\}$, weighted by present

Figure 6: Effective Duration and the Macaulay Duration



value. The duration measure of the cashflows is defined by:

$$\tau(i) = \frac{\sum_{k=1}^{n} t_k C_{t_k} v_i^{t_k}}{\sum_{k=1}^{n} C_{t_k} v_i^{t_k}} \tag{14}$$

$$= (1+i)\nu(i) \tag{15}$$

Note that the duration for a continuously payable payment stream is calculated by integrals for the continuous payments.

Example 2.1. *Evaluate the effective duration and the Macaulay duration of a bond of 100 nominal redeemable at par in 10 year's time with annual coupon of 5% at various interest rates.*

The effective duration is:

$$\begin{aligned}
\nu(i) &= \frac{\sum_{t=1}^{n} tCv^{t+1}}{\sum_{t=1}^{n} Cv^t} \\
&= \frac{\sum_{t=1}^{10} 5tv^{t+1} + 10(100)v^{11}}{\sum_{t=1}^{10} 5v^t + 100v^{10}}
\end{aligned}$$

The Macaulay duration is:

$$\begin{aligned}
\tau(i) &= (1+i)\nu(i) \\
&= \frac{\sum_{t=1}^{n} tCv^{t}}{\sum_{t=1}^{n} Cv^t} \\
&= \frac{\sum_{t=1}^{10} 5tv^{t} + 10(100)v^{10}}{\sum_{t=1}^{10} 5v^t + 100v^{10}} \\
&= \frac{5(Ia)_{\overline{10|}} + 10(100)v^{10}}{5a_{\overline{10|}} + 100v^{10}}
\end{aligned}$$

See Figure 6 for the durations in terms of interest rates.

■

As observed in the example above, the duration of an n year coupon bond payable D annually, redeemed at R is:

$$\tau(i) = \frac{D(Ia)_{\overline{n}|} + Rnv^n}{Da_{\overline{n}|} + Rv^n} \tag{16}$$

It is obvious that the duration of an n year zero-coupon bond of nominal amount 100 is simply:

$$\tau(i) = \frac{100nv^n}{100v^n} = n \tag{17}$$

2.3 Convexity

To find more accurately the effect of a change in the interest rate, we employ another measure called the *convexity*. Remember that Figure 5 show that the curve for an annuity with longer duration is more curved than the others. Thus, just looking at the tangent line does not provide the precise effect. It can be also shown by Taylor series by assuming a small change ε:

$$\begin{aligned} A(i+\varepsilon) &\approx A(i) + \varepsilon A'(i) + \frac{\varepsilon^2}{2}A''(i) && (18) \\ \frac{A(i+\varepsilon) - A(i)}{A(i)} &= \varepsilon\frac{A'(i)}{A(i)} + \frac{\varepsilon^2}{2}\frac{A''(i)}{A(i)} && (19) \\ &= -\varepsilon\nu(i) + \frac{\varepsilon^2}{2}\frac{A''(i)}{A(i)} && (20) \end{aligned}$$

Thus, the second component of the RHS improves the accuracy of the interest rate sensitivity. The *convexity* of the cashflow series $\{C_{t_k}\}$ is defined by:

$$\begin{aligned} c(i) &= \frac{A''(i)}{A(i)} && (21) \\ &= \frac{\sum_{k=1}^{n} t_k(t_k+1)C_{t_k}v_i^{t_k+2}}{\sum_{k=1}^{n} C_{t_k}v_i^{t_k}} && (22) \end{aligned}$$

It is straightforward to see that $c(i)$ is always positive.

2.4 Immunization

As discussed at the beginning of this section, an investment manager may be interested in reducing the interest rate risk to avoid a deficit. Here we consider an approach to reduce the risk by using interest rate sensitivity measures. Let $\nu_A(i)$ and $\nu_L(i)$ be the volatility measure of the asset and liability cashflows respectively, and let $c_A(i)$ and $c_L(i)$ be the convexity measure of the asset and liability cashflows respectively.

A fund is said to be *immunized* against small movements in the rate of interest of ε if and only if $V_A(i_0) = V_L(i_0)$ and $V_A(i_0+\varepsilon) \geq V_L(i_0+\varepsilon)$, where i_0 is the current interest rate.

Consider the surplus $S(i) \equiv V_A(i) - V_L(i)$. By Taylor series, we can expand the surplus as:

$$S(i_0+\varepsilon) = S(i_0) + \varepsilon S'(i_0) + \frac{\varepsilon^2}{2}S''(i_0) + \cdots \tag{23}$$

where ε represents a small movement of interest rate and can be either positive or negative. We investigate conditions required for the surplus $S(i_0+\varepsilon)$ to be positive by looking at each term on the RHS of the equation.

1. For the first term, the definition of the immunization requires that $V_A(i_0) = V_L(i_0)$.

2. For the second term, note that ε can be either positive or negative. Thus, to reduce the impact of a change in interest rate, we may require a condition that $S'(i_0) = 0$. This is equivalent to $V'_A(i_0) = V'_L(i_0)$. Combining this requirement with $V_A(i_0) = V_L(i_0)$, we also have:

$$\begin{aligned} -\frac{V'_A(i_0)}{V_A(i_0)} &= -\frac{V'_L(i_0)}{V_L(i_0)} \qquad (24) \\ \nu_A(i) &= \nu_L(i) \qquad (25) \end{aligned}$$

   Thus, the second condition is that the volatilities or the durations of the two cashflows are equal.

3. For the third term, ε^2 is always positive. Thus, if $S''(i_0) > 0$, then the third term is always positive. This condition is equivalent to:

$$\begin{aligned} V''_A(i_0) &> V''_L(i_0) \qquad (26) \\ c_A(i) &> c_L(i) \qquad (27) \end{aligned}$$

   Thus, the third condition is that the convexity of the asset cashflows is greater than that of liability cashflows.

If these three conditions are satisfied, the fund is protected against small movements in interest rates, which is known as *Redington's immunization.*

Example 2.2. *Suppose that a fund needs to make payments of 10,000 at the end of fourth and seventh years. Find a combination of holding a 3-year zero-coupon bond and a 10-year zero-coupon bond so that immunization to small changes in interest rates can be achieved where interest rates are currently 7% pa at all durations.*

Let X_3 and X_{10} denote the nominal values of those zero-coupon bonds. By the Redington's first condition,

$$\begin{aligned} V_A(.07) &= V_L(.07) \\ X_3v^3 + X_{10}v^{10} &= 10,000(v^4 + v^7) \\ X_3v^3 + X_{10}v^{10} &= 13,856 \end{aligned}$$

the second condition gives,

$$\begin{aligned} -V'_A(.07) &= -V'_L(.07) \\ 3X_3v^4 + 10X_{10}v^{11} &= 10,000(4v^5 + 7v^8) \\ 3X_3v^4 + 10X_{10}v^{11} &= 69,260 \end{aligned}$$

Solving these two equations, we have $X_3 = 11,280$ and $X_{10} = 9,144$. To see if this combination satisfies the third condition,

$$\begin{aligned} V''_A(.07) &= 12X_3v^5 + 110X_{10}v^{12} = 12(11,280)v^5 + 110(9,144)v^{12} = 543,124 \\ V''_L(.07) &= 10,000(20v^6 + 56v^9) = 437,871 \end{aligned}$$

Thus, we have $V''_A(i_0) > V''_L(i_0)$, which satisfies the third condition. In other words, it is possible to immunize a fund against small movements of interest rates by holding the identified amounts of zero-coupon bonds.

■