BA2202 Mathematics of Finance
Handout 8

1 Stochastic Interest Rate Models

1.1 Simple Models

Instead of assuming deterministic rates of interest, we may consider a simple stochastic interest rate model. First, two notations A_n and S_n are introduced.

The accumulated value at time n of a single investment of 1 at time 0 is denoted by a random variable A_n defined by:

$$A_n = (1+i_1)(1+i_2)\cdots(1+i_n) \tag{1}$$

Similarly the accumulated value at time n of a series of annual investments, each of amount 1 at times $0, 1, 2, \ldots, n-1$ is denoted by a random variable S_n defined by:

$$\begin{aligned} S_n = (1+i_1)(1+i_2)\cdots(1+i_{n-1})(1+i_n) \\ +(1+i_2)\cdots(1+i_{n-1})(1+i_n) \\ \vdots \\ +(1+i_{n-1})(1+i_n) \\ +(1+i_n) \end{aligned} \tag{2}$$

It follows that, for $n \geq 2$:

$$S_n = (1+i_n)(1+S_{n-1}) \tag{3}$$

1.2 Interest Rate Scenario Model

Consider time interval $[0,3]$ subdivided into successive periods $[0,1], [1,2], [2,3]$. For $t = 1,2,3$, let i_t be the yield obtainable over $[t-1,t]$. We may consider the following *interest rate scenario model* in which current interest rate $i_1 = 0.03$ is known but the future interest rates are unknown. Random interest rates follow three possible scenarios with the probability of 10%, 60% and 30% respectively.

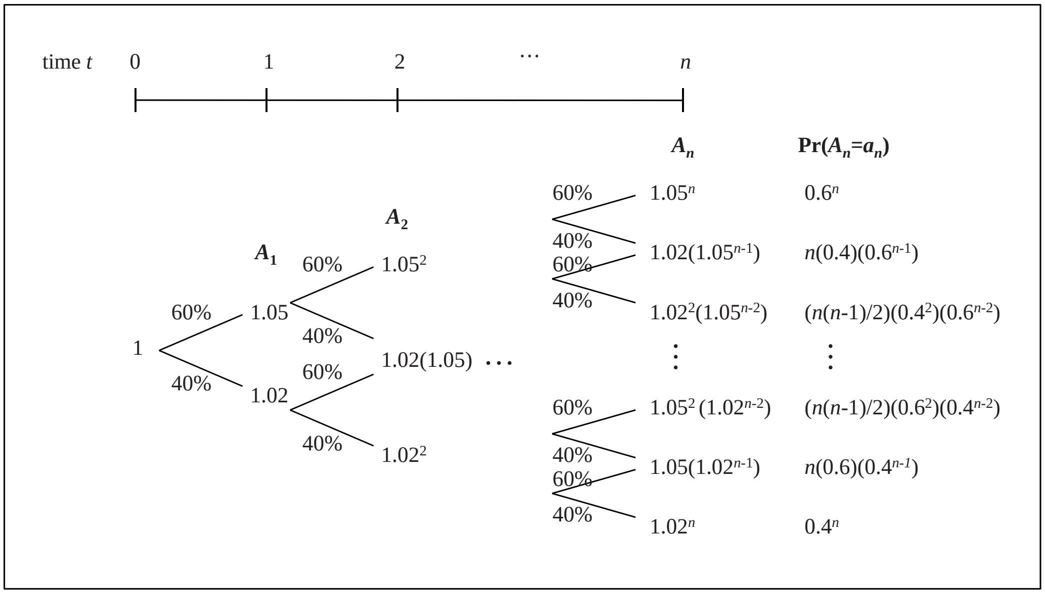
Scenario	1	2	3
Prob.	10%	60%	30%
i_1	0.03	0.03	0.03
i_2	0.02	0.03	0.04
i_3	0.02	0.03	0.06

Example 1.1. *When $a_{\overline{3|}}$ is a random variable which follows the interest rate scenario model, calculate the expected value and the standard deviation of $a_{\overline{3|}}$.*

To find the expected value, we need to calculate $a_{\overline{3|}}$ for each scenario:

$$\begin{aligned} a_{\overline{3|}}^{S1} &= \frac{1}{1.03} + \frac{1}{1.03(1.02)} + \frac{1}{1.03(1.02^2)} = 2.8559 \\ a_{\overline{3|}}^{S2} &= \frac{1}{1.03} + \frac{1}{1.03^2} + \frac{1}{1.03^3} = 2.8286 \\ a_{\overline{3|}}^{S3} &= \frac{1}{1.03} + \frac{1}{1.03(1.04)} + \frac{1}{1.03(1.04)(1.06)} = 2.7851 \end{aligned}$$

Figure 1: Varying Interest Rate Model



Therefore, the expected value is:

$$E[a_{\overline{3|}}] = 0.1(2.8559) + 0.6(2.8286) + 0.3(2.7851) = 2.8183$$

and the standard deviation is:

$$\begin{aligned}\sigma(a_{\overline{3|}}) &= \left[0.1(2.8559-2.8183)^2 + 0.6(2.8286-2.8183)^2 + 0.3(2.7851-2.8183)^2\right]^{1/2} \\ &= 0.0231\end{aligned}$$

1.3 Varying Interest Rate Models

Now consider a *varying interest rate model* where all one-period spot rates are assumed to be random and independent. In the following numerical example, the interest rates for a period $[0, t]$ are unknown. The probability that the state ω_1 and ω_2 is realized in each period is 40% and 60% respectively.

	ω_1	ω_2
i_t	0.02	0.05
$Pr(\omega)$	40%	60%

Example 1.2. *What is the probabilities that A_n will take the value $1.02(1.05^{n-1})$ and 1.02^n in the varying interest rate model above?*

There are n ways to obtain the accumulated value $1.02(1.05^{n-1})$, so the probability is:

$$Pr[A_n = 1.02(1.05^{n-1})] = n(0.4)(0.6^{n-1})$$

The probability of taking the extreme value 1.02^n is 0.4^n.

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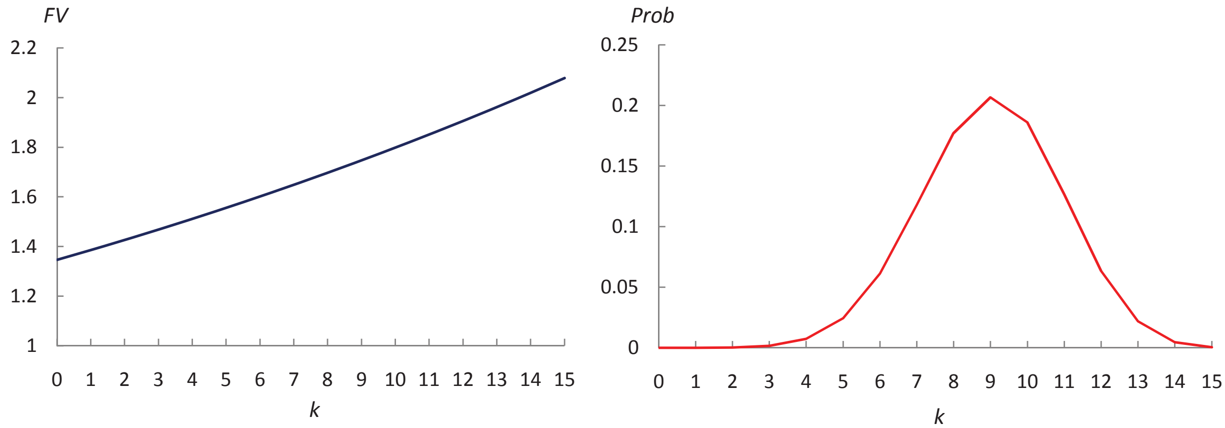
Example 1.3. *In the varying interest rate model above, what is the probability that an investment of 100 will be accumulated to more than 180 in 15 years?*

Figure 2: Example 1.3



Note that the random variable A_n has a binomial distribution, where n is the number of years and k is the number of outcomes of ω_2 with the probability of $p = 0.6$.

$$Pr(A_n = a_n) = \begin{pmatrix} n \\ k \end{pmatrix} 0.6^k 0.4^{n-k}$$

The accumulated value of 180 can be achieved by $k \geq 11$:

$$A_{15} = 1.02^4(1.05^{11}) = 1.8513$$

The probability is:

$$Pr[A_{15} \geq 1.02^4(1.05^{11})] = 0.217$$

Thus, the investment will be accumulated to at least 180 at 21.7%.

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As you know, if $(1 + i_t)$ is binomially distributed, the distribution of A_n can be approximated by $N(np, np(1-p))$ when n increases. However, normal distribution is hardly assumed in practice, and lognormal distribution, as introduced below, is considered better for a random variable A_n. Why?

1.4 Moments of A_n

Let's consider the kth moment of A_n. We have:

$$E[A_n^k] = E\left[\prod_{t=1}^{n}(1+i_t)^k\right] = \prod_{t=1}^{n} E\left[(1+i_t)^k\right] \tag{4}$$

The second equality holds when i_t for $t = 1, 2, \ldots, n$ are independent. Thus, if interest rates are independent and the yield each year has mean μ_i and variance σ_i^2, the first moment ($k = 1$) is:

$$\begin{aligned} E[A_n] &= \prod_{t=1}^{n}(1 + E[i_t]) && (5) \\ &= (1+\mu_i)^n && (6) \end{aligned}$$

The second moment ($k = 2$) is:

$$E[A_n^2] = \prod_{t=1}^{n} E\left[(1+i_t)^2\right] \quad (7)$$

$$= \prod_{t=1}^{n} E\left[1 + 2i_t + i_t^2\right] \quad (8)$$

$$= (1 + 2\mu_i + E[i^2])^n \quad (9)$$

Therefore, the variance of A_n is:

$$\sigma_{A_n}^2 = (1 + 2\mu_i + E[i^2])^n - (1+\mu_i)^{2n} \quad (10)$$

Example 1.4. *Calculate the mean and standard deviation of the accumulated value at time $t = 10$ of an investment of 10,000 at time $t = 0$ in the following varying interest rate model.*

	ω_1	ω_2	ω_3
i_t	0.02	0.03	0.05
$Pr(\omega)$	20%	50%	30%

The first and second moment of the interest rate is:

$$\mu_i = 0.2(.02) + 0.5(0.03) + 0.3(0.05) = 0.034$$

$$E[i_t^2] = 0.2(.02^2) + 0.5(0.03^2) + 0.3(0.05^2) = 0.00128$$

Therefore, The mean of the accumulated value in 10 years is:

$$E[10,000A_{10}] = 10,000(1+0.034)^{10} \quad (11)$$

$$= 13,970 \quad (12)$$

The standard deviation of the accumulated value in 10 years is:

$$\sigma(10,000A_n) = 10,000\left[(1+2\mu_i+E[i^2])^n - (1+\mu_i)^{2n}\right]^{1/2}$$

$$= 10,000\left[(1+2(0.034)+0.00128)^{10} - (1+0.034)^{2(10)}\right]^{1/2}$$

$$= 475.89$$

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1.5 Moments of S_n

Let's consider the kth moment of S_n. We have:

$$E[S_n^k] = E\left[((1+i_n)(1+S_{n-1}))^k\right] \quad (13)$$

If interest rates are independent and the yield each year has mean μ_i and variance σ_i^2, the first moment ($k = 1$) is:

$$E[S_n] = E[(1+i_n)(1+S_{n-1})] \quad (14)$$

$$= (1+\mu_i)(1+E[S_{n-1}]) \quad (15)$$

The second moment ($k = 2$) is:

$$E[S_n^2] = E\left[((1+i_n)(1+S_{n-1}))^2\right] \quad (16)$$

$$= (1+2\mu_i+E[i^2])(1+2E[S_{n-1}]+E[S_{n-1}^2]) \quad (17)$$

The variance of S_n is:

$$\sigma_{S_n}^2 = E[S_n^2] - E[S_n]^2 \quad (18)$$

Example 1.5. *Calculate the standard deviation of the accumulated value at time $t = 3$ of an investment of 1 at time $t = 0, 1, 2$ in the 3-state varying interest rate model above.*

The first moment of the accumulated value is:

$$\begin{aligned} E[S_1] &= (1 + \mu_i) = 1.034 \\ E[S_2] &= (1 + \mu_i)(1 + E[S_1]) = 1.034(1 + 1.034) = 2.103 \\ E[S_3] &= (1 + \mu_i)(1 + E[S_2]) = 1.034(1 + 2.103) = 3.209 \end{aligned}$$

The second moment of the accumulated value is:

$$\begin{aligned} E[S_1^2] &= (1 + 2\mu_i + E[i^2]) = 1 + 2(0.034) + 0.00128 = 1.06928 \\ E[S_2^2] &= (1 + 2\mu_i + E[i^2])(1 + 2E[S_1] + E[S_1^2]) = 1.06928(1 + 2(1.034) + 1.06928) = 4.42391 \\ E[S_3^2] &= (1 + 2\mu_i + E[i^2])(1 + 2E[S_2] + E[S_2^2]) = 1.06928(1 + 2(2.103) + 4.42391) = 10.2971 \end{aligned}$$

Therefore, the standard deviation of the accumulated value in 3 years is:

$$\begin{aligned} \sigma_{S_n} &= \left\{E[S_3^2] - E[S_3]^2\right\}^{1/2} \\ &= \left(10.2971 - 3.209^2\right)^{1/2} \\ &= 0.04341 \end{aligned}$$

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2 Independent Log-normal Model

The independent lognormal model assumes that $(1 + i_t)$ for $t = 1, 2, \ldots, n$ are independently log-normally distributed with parameters μ and σ^2. This implies that the random variable $\ln(1 + i_t) \sim N(\mu, \sigma^2)$. The mean and variance of the lognormal random variable $(1 + i_t)$ are:

$$E[1 + i_t] = e^{\mu + \sigma^2/2} \tag{19}$$

$$Var(1 + i_t) = e^{2\mu + \sigma^2}(e^{\sigma^2} - 1) \tag{20}$$

which can be derived by the moment generating function:

$$M_X(k) = E\left[e^{kX}\right] = E\left[e^{k\log(1+i_t)}\right] = E\left[(1 + i_t)^k\right] \tag{21}$$

The MGF of the normal distribution is:

$$M_X(k) = E\left[(1 + i_t)^k\right] = e^{\mu k + (\sigma k)^2/2} \tag{22}$$

For $k = 1$ and $k = 2$,

$$M_X(1) = E\left[(1 + i_t)\right] = e^{\mu + \sigma^2/2} \tag{23}$$

$$M_X(2) = E\left[(1 + i_t)^2\right] = e^{2\mu + 2\sigma^2} \tag{24}$$

Since $Var(1 + i_t) = E\left[(1 + i_t)^2\right] - E\left[1 + i_t\right]^2$,

$$Var(1 + i_t) = e^{2\mu + 2\sigma^2} - e^{2\mu + \sigma^2} = e^{2\mu + \sigma^2}(e^{\sigma^2} - 1) \tag{25}$$

Similarly, the mean and variance of the one-period discount factor are:

$$E\left[\frac{1}{1+i_t}\right] = e^{-\mu+\sigma^2/2} \tag{26}$$

$$Var\left(\frac{1}{1+i_t}\right) = e^{-2\mu+\sigma^2}(e^{\sigma^2}-1) \tag{27}$$

which can be derived by using $k=-1$ and $k=-2$,

$$M_X(-1) = E\left[(1+i_t)^{-1}\right] = e^{-\mu+\sigma^2/2} \tag{28}$$

$$M_X(-2) = E\left[(1+i_t)^{-2}\right] = e^{-2\mu+2\sigma^2} \tag{29}$$

Hence

$$Var((1+i_t)^{-1}) = e^{-2\mu+2\sigma^2} - e^{-2\mu+\sigma^2} = e^{-2\mu+\sigma^2}(e^{\sigma^2}-1). \tag{30}$$

For multiple periods, we have

$$A_n = \prod_{t=1}^{n}(1+i_t) \quad \Rightarrow \quad \ln A_n = \sum_{t=1}^{n}\ln(1+i_t) \tag{31}$$

The sum of a set of independent normal random variables is a normal random variable. Hence, when the random variable $(1+i_t)$ are independent and each is log-normally distributed with parameters μ and σ^2, the random variable A_n is log-normally distributed with parameters $n\mu$ and $n\sigma^2$, where:

$$E[A_n] = e^{n\mu+n\sigma^2/2} \tag{32}$$

$$Var(A_n) = e^{2n\mu+n\sigma^2}(e^{n\sigma^2}-1) \tag{33}$$

The mean and variance for the discount factor are:

$$E\left[\frac{1}{A_n}\right] = e^{-n\mu+n\sigma^2/2} \tag{34}$$

$$Var\left[\frac{1}{A_n}\right] = e^{-2n\mu+n\sigma^2}(e^{n\sigma^2}-1) \tag{35}$$

This implies that:

$$\ln A_n \sim N(n\mu, n\sigma^2) \quad \textit{and} \quad \frac{\ln A_n - n\mu}{\sigma\sqrt{n}} \sim N(0,1) \tag{36}$$

$$\ln\left(\frac{1}{A_n}\right) \sim N(-n\mu, n\sigma^2) \quad \textit{and} \quad \frac{\ln(1/A_n) + n\mu}{\sigma\sqrt{n}} \sim N(0,1) \tag{37}$$

Example 2.1. *Calculate the mean of $a_{\overline{10}|}$ when the random variable $(1+i_t)$ is log-normally distributed with parameters $\mu = 0.03$ and $\sigma^2 = 0.016$.*

First, let us define r_a such that

$$\frac{1}{1+r_a} = E\left[\frac{1}{1+i_t}\right] = e^{-\mu+\sigma^2/2}$$

Thus, $r_a = e^{\mu - \sigma^2/2} - 1$. The mean is

$$
\begin{aligned}
E[a_{\overline{10}|}] &= E\left[\sum_{t=1}^{10}\prod_{j=1}^{t}\frac{1}{1+i_j}\right] \\
&= \sum_{t=1}^{10}\prod_{j=1}^{t}E\left[\frac{1}{1+i_j}\right] \\
&= \sum_{t=1}^{10}\left[\frac{1}{(1+r_a)^t}\right] \\
&= a_{\overline{10}|r_a}
\end{aligned}
$$

Thus, $E[a_{\overline{n}|}] = a_{\overline{n}|r_a}$ Since $\mu = 0.03$ and $\sigma^2 = 0.016$, $r_a = e^{0.03-.016/2} - 1 = 0.0222$. The mean is:

$$
\begin{aligned}
E[a_{\overline{10}|}] &= a_{\overline{10}|2.22\%} \\
&= \frac{1 - 1.0222^{-10}}{0.0222} \\
&= 8.8780
\end{aligned}
$$

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We can show that the similar relationship holds for the FV of the annuity. First, let us define r_s such that

$$1 + r_s = E\left[1 + i_t\right] = e^{\mu + \sigma^2/2}$$

Thus, $r_s = e^{\mu + \sigma^2/2} - 1$. The mean of the FV of an annuity due is

$$
\begin{aligned}
E[\ddot{s}_{\overline{n}|}] &= E\left[\sum_{t=1}^{n}\prod_{j=1}^{t}(1 + i_{n-j+1})\right] \\
&= \sum_{t=1}^{n}\prod_{j=1}^{t}E\left[1 + i_{n-j+1}\right] \\
&= \sum_{t=1}^{n}(1 + r_s)^t \\
&= \ddot{s}_{\overline{n}|r_s}
\end{aligned}
$$

Thus, $E[\ddot{s}_{\overline{n}|}] = \ddot{s}_{\overline{n}|r_s}$

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Example 2.2. *A lump sum of 10,000 will be invested at time 0 for 5 years.* $(1+i_t)$ *for* $t = 0, 1, \ldots, 5$ *is an independent log-normal random variable with mean 1.05 and variance 0.007. What is the probability that the investment will be accumulated to more than 15,000 in 5 years' time?*

From the mean and variance, we need to find μ and σ^2. Since $e^{\mu + \frac{\sigma^2}{2}} = 1.05$ and $e^{2\mu + \sigma^2}(e^{\sigma^2} - 1) = 0.007$, then:

$$
\begin{aligned}
1.05^2(e^{\sigma^2} - 1) &= 0.007 \\
\sigma^2 &= \ln\left(\frac{0.007}{1.05^2} + 1\right) = 0.006329
\end{aligned}
$$

and $\mu = 0.04563$. Since we know $\ln A_n \sim N(n\mu, n\sigma^2)$, the probability is:

$$
\begin{aligned}
Pr(10,000A_5 > 15,000) &= Pr(A_5 > 1.5) \\
&= Pr(\ln A_5 > \ln 1.5) \\
&= Pr\left(Z > \frac{\ln 1.5 - 5(0.04563)}{\sqrt{5(0.006329)}}\right) \\
&= Pr\left(Z > 0.9967\right) \\
&= 0.159
\end{aligned}
$$

$\blacksquare$