BA2202 Mathematics of Finance
Handout 6

1 Investments

1.1 Fixed Interest Bonds

The following terminologies are used to describe a bond transaction:

- **Issue Price (P)**: The price at which investors buy bonds when they are first issued
- **Nominal Amount (N), Principal, Face Value, and Par Value**: The amount of stock
- **Coupon Rate (D)**: The rate of interest per annum that the issuer pays to the bond holders. Under income tax at rate t_1 on interest payments, an investor's income after tax is $(1 - t_1)DN$.
- **Redemption Date and Maturity**: The date when the principal (and remaining interest) is due to be paid
- **Redemption Price (R)**: The amount of money payable at (or before) redemption date (per unit nominal). Bond is said to be redeemable:
  - *at par* if $R = 1$;
  - *at a premium* if $R > 1$;
  - *at a discount* if $R < 1$.
- **Coupon Dates**: The dates on which the issuer pays coupon to the bond holders. (typically semi-annual)

Table 1: Term Correspondence

UK		US
Stock	$\Rightarrow$	Government bonds or other kinds of marketable securities
Stockholder	$\Rightarrow$	Holders of marketable securities
Share (also equity)	$\Rightarrow$	Stock (also share and equity)
Shareholder	$\Rightarrow$	Stockholder (also shareholder)
Ordinary share	$\Rightarrow$	Common stock
Preference share	$\Rightarrow$	Preferred stock

1.1.1 Fixed Interest Government Bonds

The properties of fixed interest government bonds are summarized as follows:

- Issued at a given issue price or by tender (a type of auction)
- Investors receive coupons (typically half-yearly).
- Some redemption dates are variable (undated).
- Highly liquid and very secure
- Low volatility of return and low expected return
- Relative to inflation the income stream may be volatile.

The reasons for the rate of return is not known at outset even for a fixed stream of income:

- The coupon will be reinvested on unknown terms.
- Sale price before redemption is unknown.
- Real return is unknown
- Tax rates may change

1.1.2 Government Bills

- Short-dated securities (typically 3 months)
- Issued at a discount of $d\%$ (i.e. Reward for investors is $N - P$ if it is held until maturity.)
- Redeemed at par
- No coupon (zero-coupon bond)
- Highly liquid and secure (used as risk-free interest rate)
- Yield quoted as simple rate of discount
- Often used as a benchmark risk-free short-term investment.

Example 1.1. *91-day \$100 bills are issued by the US government. The discount rate is quoted as 8%. Calculate the price of a \$100 bill with 91 days until redemption and the effective rate of interest.*

Note that the discount rate is a simple interest rate.

$$P = 100\left(1 - 0.08 \times \frac{91}{365}\right) = 98$$

The effective interest rate solves the following:

$$98(1+i)^{91/365} = 100$$

The effective rate of interest is 8.44%:

Example 1.2. *(Continued) You are interested in investing \$10,000 in the US government bills with 91 days until redemption. Calculate the par value of the bills that could be purchased.*

$$N = \frac{10,000}{\left(1 - 0.08 \times \frac{91}{365}\right)} = 10,203.5$$

1.1.3 Fixed Interest Corporate Bonds

- Issued by a company
- Less secure, less liquid, and greater yield than government bond
- Unsecured bond and secured bond (debentures such as sinking fund provision)

1.2 Shares (Equities)

1.2.1 Ordinary Shares

- Issued by corporations
- Securities which entitle their holders to receive all the net profits of the company after interest on loans and fixed interest stocks has been paid
- Dividend paid out by managerial discretion (uncertain)
- Lowest ranked form of finance
- High risk and high expected return
- Risk of capital loss
- Shareholders get voting rights in proportion to the number of shares held.
- Normally irredeemable (undated security)
- Marketability depends on the size of the issue.

Example 1.3. *List the primary features of the income paid to shareholders.*

1.2.2 Preference Shares

- Shareholders may receive a fixed stream of income if the issuer makes sufficient profits.
- Dividends are limited to a set amount.
- Shareholders rank above ordinary shareholders.
- Voting rights are limited.
- Riskier than loan stockholders and less riskier than ordinary shareholders

Example 1.4. *Describe the difference between ordinary share and preference share.*

1.2.3 Convertibles

- Unsecured loan stocks or preference shares that can be converted into ordinary shares.
- Conversion may take place during the conversion period.
- Provides higher income than ordinary shares and lower income than loan stock or preference shares.
- Hybrid security with low risk of a debt security and the potential for large gains of an equity
- Pay interest/coupons until conversion.
- Generally less volatility than in the underlying share price before conversion.
- Security and marketability depend upon issuer.

Example 1.5. *Do convertibles give lower income than conventional loan stock? If so, why?*

2 The Valuation of Securities

2.1 Fixed Interest Securities

2.1.1 Price

A question to be discussed here is what price A should be paid by an investor to secure a net yield of i per annum? For a fixed interest securities with the nominal amount N, the price per unit nominal is $P = A/N$.

Assuming no taxes, the price per 100 nominal to be paid for a fixed interest stock at interest at $D\%$ per annum payable half-yearly and redeemable at par in n years is calculated as follows. At an interest rate i:

$$A = Da_{\overline{n}|}^{(2)} + 100v^n \tag{1}$$

Alternatively, the price can be solved by using the effective semiannual rate $i_2 = (1+i)^{1/2} - 1$:

$$A = \left(\frac{D}{2}\right) a_{\overline{2n}|i_2} + 100v_{i_2}^{2n} \tag{2}$$

Example 2.1. *Suppose that an investor requires a yield of 5% per annum ignoring tax. Find price per 100 nominal of 10-year stock with half-yearly coupon payments of 3% pa. The next coupon is due in 6 months' time and the stock is redeemable at par.*

$$\begin{aligned} A &= 3a_{\overline{10}|i}^{(2)} + 100v^{10} \\ &= 3\left[\frac{1-v^{10}}{2(1.05^{1/2}-1)}\right] + 100\left(1.05^{-10}\right) \\ &= 3 \times 7.817 + 100 \times 0.614 \\ &= 84.84 \end{aligned}$$

Thus the price that satisfies the investor's required return is below the nominal amount. In other words, the investor pays for the fixed-interest bond up to 84.84. If the price is more than that, the investor does not invest on the bond.

Example 2.2. *How would the price change if the investor's required yield is lowered to 2%?*

$$\begin{aligned} A &= 3\left[\frac{1-v^{10}}{2(1.02^{1/2}-1)}\right] + 100\left(1.02^{-10}\right) \\ &= 3 \times 9.027 + 100 \times 0.820 \\ &= 109.12 \end{aligned}$$

It is intuitive that the price that satisfies the investor's required return is above the nominal amount because the required yield is lower than the coupon rate.

Here we allow for income tax, which is negative cashflows for an investor. Thus, it is expected that price that satisfies an investor's required yield should be decreased by the tax payment. Let income tax rate be t_1. The net present value of payments received is:

$$A = Da_{\overline{n}|}^{(2)} + 100v^n - t_1 Da_{\overline{n}|}^{(2)} \tag{3}$$

$$= (1-t_1)Da_{\overline{n}|}^{(2)} + 100v^n \tag{4}$$

We modify the question above by imposing income tax to an investor.

Example 2.3. *Suppose that an investor requires a yield of 5% per annum after paying income tax at 20%. Find price per 100 nominal of 10-year stock with half-yearly coupon payments of 3% pa. The next coupon is due in 6 months' time and the stock is redeemable at par.*

$$\begin{aligned}
A &= 3(1-0.2)a^{(2)}_{\overline{10}|i} + 100v^{10} \\
&= 2.4\left[\frac{1-v^{10}}{2(1.05^{1/2}-1)}\right] + 100\left(1.05^{-10}\right) \\
&= 2.4 \times 7.817 + 100 \times 0.614 \\
&= 80.15
\end{aligned}$$

Thus the price that satisfies the investor's required return decreases by 4.69 due to the tax payment.

2.1.2 Yield

We may be interested in calculating net yield per annum if a bond is held until maturity, given that an investor pays price A. The calculation can be done by the equation we define above. The yield may be found by trial and error.

- *Gross Redemption Yield*: a redemption yield without any allowance for tax
- *Net Redemption Yield*: a redemption yield after tax payment

Example 2.4. *Suppose that an investor pays 90 for 100 nominal of 5-year stock with half-yearly coupon payments of 6% pa. The stock is redeemable at par. Find a gross redemption yield.*

Using the equation above, we have:

$$\begin{aligned}
A &= 6a^{(2)}_{\overline{5}|i} + 100v^{5} \\
90 &= 6\left[\frac{1-v^{5}}{2((1+i)^{1/2}-1)}\right] + 100(1+i)^{-5}
\end{aligned}$$

By trial and error and linear interpolation, we obtain $i \approx 8.7\%$.

2.1.3 Term to Redemption

Consider a loan of nominal N which has n-year $p$thly payable annual interest at $D\%$ per unit nominal. The redeemable price is $R\%$ per unit nominal. An investor is subject to income tax at t_1. In summary,

- DN represents annual interest payment (e.g., $5\% \times 100 = 5$)
- RN represents redemption payment (e.g., $110\% \times 100 = 110$)
- D/R represents interest rate per redeemable price (e.g., $5\% \times 110\% = 4.55\%$)

We investigate how price for the loan is affected by the term to redemption n and a net effective annual yield of i. Let $g = D/R$. The price for this loan is defined by:

$$\begin{aligned}
A(n,i) &= (1-t_1)DNa^{(p)}_{\overline{n}|} + RNv^{n} \\
&= (1-t_1)DNa^{(p)}_{\overline{n}|} + RN\left(1 - i^{(p)}a^{(p)}_{\overline{n}|}\right) \\
&= RN + a^{(p)}_{\overline{n}|}\left[(1-t_1)DN - i^{(p)}RN\right] \\
&= RN + RNa^{(p)}_{\overline{n}|}\left[(1-t_1)\left(\frac{D}{R}\right) - i^{(p)}\right] \\
&= RN + RNa^{(p)}_{\overline{n}|}\left[(1-t_1)g - i^{(p)}\right]
\end{aligned}$$

A term within a bracket on the RHS changes its sign according the relationship between $(1-t_1)g$ and $i^{(p)}$. Hence, we get the following results:

- If $i^{(p)} = (1-t_1)g$, $A(n,i) = RN\ \forall n$.
- If $i^{(p)} < (1-t_1)g$, $A(n,i)$ is an increasing function of n
- If $i^{(p)} > (1-t_1)g$, $A(n,i)$ is a decreasing function of n

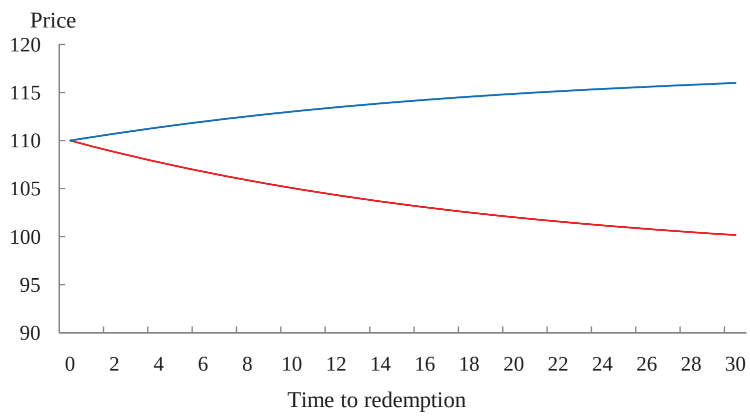
Example 2.5. *An investor requires a yield of 5% and is subject to income tax at 20%. The investor considers investing on a loan of 100 nominal with interest payable half-yearly at 6% pa. The stock is redeemable at 110%. Find price function of the term to redemption.*

Since $i = 5\%$, $i^{(2)} = 2(1.05^{1/2} - 1) = 4.94\%$. And since $(1-t_1)g = 0.8(0.06/1.1) = 4.36$, then $i^{(p)} > (1-t_1)g$. Thus, we know that price that satisfies the investor's required yield is a decreasing function of n.

Next example illustrates a case where price is an increasing function of the term to redemption.

Example 2.6. *If interest rate is 8% instead of 6%, how would this change affect the price function?*

The required yield still remains the same $i^{(2)} = 2(1.05^{1/2} - 1) = 4.94\%$. $(1-t_1)g = 0.8(0.08/1.1) = 5.82$, then $i^{(p)} < (1-t_1)g$. Thus, the price that satisfies the investor's required yield is an increasing function of n.



These results are intuitive. When net interest payment (relative to redemption value) is larger than an investor's required yield, it should be willing to pay more for a longer term to redemption.

We may be interested in investigating the effect of the term to redemption on the yield. From the equation:

$$A(n,i) = RN + RNa_{\overline{n}|}^{(p)}\left[(1-t_1)g - i^{(p)}\right]$$

we formulate the *capital gain test*:

$$\begin{cases} A(n,i) = RN & & \text{if } i^{(p)} = (1-t_1)g \\ A(n,i) > RN & \text{(capital loss)} & \text{if } i^{(p)} < (1-t_1)g \\ A(n,i) < RN & \text{(capital gain)} & \text{if } i^{(p)} > (1-t_1)g \end{cases} \tag{5}$$

This test implies that:

- If $i^{(p)} = (1-t_1)g$, indifferent to n

- If $i^{(p)} < (1-t_1)g$, a higher yield for a large n
- If $i^{(p)} > (1-t_1)g$, a higher yield for a small n

The results are also intuitive because an investor's required yield determines the price of a loan. If an investor has a high (low) required yield, it only pays up to relatively low (high) price, which may result in a capital gain (loss). The sooner (later) the capital gain (loss) is received the better.

2.1.4 Optional Redemption Dates

In the case where a security is issued without a fixed redemption date, the terms of issue may provide the right to redeem the security at the borrower's option at any interest date on or after some specified date (or between two specified dates). In such cases, the borrower will take an advantage of the option and minimize interest payment. Although the borrower's choice depends on the future market rate, an investor can determine either:

- The maximum price to be paid
- The minimum net yield the investor will obtain

The capital test implies:

- If $i^{(p)} < (1-t_1)g$ (a capital loss), the minimum yield occurs for the smallest n
- If $i^{(p)} > (1-t_1)g$ (a capital gain), the minimum yield occurs for the largest n

Suppose that the investor needs to achieve a net annual yield of at least i. The price must be determined by n such that the minimum yield will occur. If the borrower has a time interval to redeem the loan during $[n_1, n_2]$, the price is calculated at:

$$\begin{cases} n_1 & \text{if} \quad i^{(p)} < (1-t_1)g \\ n_2 & \text{if} \quad i^{(p)} > (1-t_1)g \end{cases} \tag{6}$$

If $i^{(p)} = (1-t_1)g$, the net annual yield will be i regardless of the borrower's choice of the redemption date.

2.1.5 Deferred Income Tax

Suppose it is possible for an investor to defer its income tax payment. The annual tax is paid in a single installment due k years after the second half-yearly coupon payment each year. Due to the delay of tax payment, the net present value of payments received will be increased. Price that satisfies an investor's required yield is:

$$A = DNa_{\overline{n}|}^{(2)} + RNv^n - t_1 DNv^k a_{\overline{n}|} \tag{7}$$

If tax payments are due k years after each coupon is received, the price is:

$$A = DNa_{\overline{n}|}^{(2)} + RNv^n - t_1 DNv^k a_{\overline{n}|}^{(2)} \tag{8}$$

2.2 Equities

Let the value of an equity just after a dividend payment be A, and let D be the amount of this dividend payment. The value of an equity without further uncertainty is:

$$A = \sum_{t=1}^{\infty} D_t v_i^t \tag{9}$$

where i is the return on the share, given price A. If dividends are assumed to grow in $D_t = (1+g)^t D$ where g is a constant dividend growth rate, the equation is:

$$A = D\left[\frac{1+g}{1+i} + \frac{(1+g)^2}{(1+i)^2} + \frac{(1+g)^3}{(1+i)^3} \cdots\right] \tag{10}$$

$$= Da_{\overline{\infty}|i'} \tag{11}$$

$$= \frac{D(1+g)}{i-g} \tag{12}$$

where $i' = (1+i)/(1+g) - 1$. Nothing is new.

2.3 Real Rates of Interest

2.3.1 Real Yields Using an Inflation Index

Assume that an inflation index Q_{t_k} at time t_k is given. A series of nominal payments {-100, 5, 5, 105} can be converted to the inflation-adjusted value of the payments by $X'_t = X_t/(Q_t/Q_0)$. Note that this is not a calculation to inflate payments.

Table 2: Real Yield Calculation

Year t	Payment X_t	Inflation Index Q_t	Inflation-Adj Payment X'_t
0	-100	120	-100(120/120)
1	5	123	5(120/123)
2	5	127	5(120/127)
3	105	132	105(120/132)

The equation of value gives the *real* yield:

$$\begin{aligned} 100 &= +5\left(\frac{120}{123}\right)v_{i'} + 5\left(\frac{120}{127}\right)v_{i'}^2 + 105\left(\frac{120}{132}\right)v_{i'}^3 \\ i' &= 1.74\% \end{aligned}$$

Note that the numerators of the fractions are all Q_0. In general, the real yield equation is defined by:

$$\sum_{k=1}^{n} C_{t_k}\left(\frac{Q_0}{Q_{t_k}}\right)v_{i'}^{t_k} = 0 \quad \Leftrightarrow \quad \sum_{k=1}^{n} \frac{C_{t_k}}{Q_{t_k}}v_{i'}^{t_k} = 0 \tag{13}$$

2.3.2 Real Yields Given Constant Inflation Assumption

Future inflation is unknown. Therefore, considering the real value of future cashflows requires some assumption about future inflation. The simplest assumption is a constant rate of inflation of j per annum. Remember the relationship between a rate of inflation and real rate of interest i':

$$1 + i' = \frac{1+i}{1+j}$$

This implies that:

$$\frac{1}{1+i'}\frac{1}{1+j} = \frac{1}{1+i} \quad \Rightarrow \quad v_{i'}v_j = v_i$$

where i is often referred to as *money rate of interest (return)* to distinguish it from real rate of interest. For a fixed net cashflow series $\{C_{t_k}\}$, $k = 1, 2, \ldots, n$, the real yield i' is the solution of the real yield equation:

$$\sum_{k=1}^{n} C_{t_k} v_j^{t_k} v_{i'}^{t_k} = 0 \tag{14}$$

2.3.3 Payments Related to the Rate of Inflation

The index-linked government security is an example that cashflows are adjusted by future inflation. The actual cashflows will be unknown until the inflation index becomes available. Let the nominal cashflows at time t be c_t. Then the actual cashflow at time t can be expressed as:

$$C_t = c_t \frac{Q_t}{Q_0} \tag{15}$$

$$C_t = c_t \prod_{k=0}^{t} (1 + j_k) \tag{16}$$

The first equation is used when the inflation index is available and the second equation is used when inflation rate j_t for time t is available to calculate the inflation-adjusted cashflows.

If the real yield i' is calculated by the same inflation index, the solution can be derived from the yield equation of the nominal amount:

$$\sum_{k=1}^{n} C_{t_k} v_j^{t_k} v_{i'}^{t_k} = 0 \tag{17}$$

$$\Rightarrow \sum_{k=1}^{n} c_t \frac{Q_t}{Q_0} \frac{Q_0}{Q_t} v_{i'}^{t_k} = 0 \tag{18}$$

$$\Rightarrow \sum_{k=1}^{n} c_t v_{i'}^{t_k} = 0 \tag{19}$$

However, note that the index used to inflate the cashflows may be different from one used to calculated the real yield. The important example is the index-linked government security, which coupon payments are inflated by the inflation index three month before the payment is made.

2.4 Index-linked Bonds

Index-linked bond has coupon and redemption payments which are increased according to an index of inflation. To evaluate the cashflows, we consider n-year index-lined bond with nominal coupon D per unit nominal face value payable half-yearly. The nominal redemption price is R per unit nominal. We assume that payment at time t is inflated according to an index with value Q_t where the base value is Q_0. Given an effective interest rate i per annum, the equation of value is defined by:

$$A = \sum_{k=1}^{2n} \left(\frac{D}{2} \frac{Q_{k/2}}{Q_0} \right) v_i^{k/2} + \left(R \frac{Q_n}{Q_0} \right) v_i^n \tag{20}$$

If annual rate of inflation j_t (not a constant) is available instead of inflation index, each cashflow is adjusted as follows:

$$\begin{aligned}
C_{1/2} &= c_{1/2}(1 + j_1)^{1/2} \\
C_1 &= c_1(1 + j_1) \\
C_{3/2} &= c_{3/2}(1 + j_1)(1 + j_2)^{1/2} \\
C_2 &= c_2(1 + j_1)(1 + j_2)
\end{aligned}$$

Example 2.7. *An index-linked bond pays half-yearly coupons in arrear and is redeemable at par in 10 years. The nominal coupon paid is 4 for 100 nominal. Assuming that a rate of future inflation is 3% pa. For the bond, how much should this non-taxpayer pay to attain a money rate of return of 5% pa?*

Let $1+i' = \frac{1+i}{1+j}$. The price can be identified by:

$$\begin{aligned}
A &= 2(1+j)^{1/2}v_i^{1/2} + 2(1+j)v_i + 2(1+j)^{3/2}v_i^{3/2} + \cdots + 102(1+j)^{10}v_i^{10} \\
&= 2\left[v_{i'}^{1/2} + v_{i'} + \cdots + v_{i'}^{10}\right] + 100v_{i'}^{10} \\
&= 2a_{\overline{20}|i'_2} + 100v_{i'}^{10} \\
&= 2\left[\frac{1-v_{i'}^{20/2}}{(1+i')^{1/2}-1}\right] + 100v_{i'}^{10} \\
&= 118.72
\end{aligned}$$

Equivalently,

$$\begin{aligned}
A &= 4a^{(2)}_{\overline{10}|i'} + 100v_{i'}^{10} \\
&= 4\left\{\frac{1-v_{i'}^{10}}{2[(1+i')^{1/2}-1]}\right\} + 100v_{i'}^{10}
\end{aligned}$$

When coupon payments are adjusted by lagged inflation index, calculation becomes slightly complicated. Suppose the following inflation index is given.

Table 3: Inflation Index Example

Date	Index	Date	Index	Date	Index
1/1/08	$Q_{1/1/08}$	1/1/09	$Q_{1/1/09}$	1/1/10	$Q_{1/1/10}$
1/4/08	$Q_{1/4/08}$	1/4/09	$Q_{1/4/09}$	1/4/10	$Q_{1/4/10}$
1/7/08	$Q_{1/7/08}$	1/7/09	$Q_{1/7/09}$	1/7/10	$Q_{1/7/10}$
1/10/08	$Q_{1/10/08}$	1/10/09	$Q_{1/10/09}$	1/10/10	$Q_{1/10/10}$

An index-linked bond is issued on 1/4/08 for a price of P per N nominal redeemed at $R\%$ in two years' time. All coupon and redemption payments are linked to the inflation index three months prior to the payment date. The coupons on the bond are of "nominal" amount D per unit nominal payable semiannually in arrears on the 1st day of April and October every year. To obtain the real yield, we need to calculate the inflation-adjusted payments as shown in the table.

Table 4: Inflation-adjusted Cashflow

Date	Index	Payment X'_t	Date	Index	Payment X'_t
1/1/08	$Q_{1/1/08}$	-	1/4/09	$Q_{1/4/09}$	$\frac{DN}{2}\left(\frac{Q_{1/1/09}}{Q_{1/1/08}}\right)$
1/4/08	$Q_{1/4/08}$	-	1/7/09	$Q_{1/7/09}$	-
1/7/08	$Q_{1/7/08}$	-	1/10/09	$Q_{1/10/09}$	$\frac{DN}{2}\left(\frac{Q_{1/7/09}}{Q_{1/1/08}}\right)$
1/10/08	$Q_{1/10/08}$	$\frac{DN}{2}\left(\frac{Q_{1/7/08}}{Q_{1/1/08}}\right)$	1/1/10	$Q_{1/1/10}$	-
1/1/09	$Q_{1/1/09}$	-	1/4/10	$Q_{1/4/10}$	$(\frac{DN}{2}+RN)\left(\frac{Q_{1/1/10}}{Q_{1/1/08}}\right)$

Using the inflation adjusted payments, we set up the equation of value as follow:

$$\begin{aligned}
P = \quad & \left(\frac{DN}{2}\frac{Q_{1/7/08}}{Q_{1/1/08}}\right)v^{1/2}\left(\frac{Q_{1/4/08}}{Q_{1/10/08}}\right) + \left(\frac{DN}{2}\frac{Q_{1/1/09}}{Q_{1/1/08}}\right)v^{1}\left(\frac{Q_{1/4/08}}{Q_{1/4/09}}\right) \\
+ \quad & \left(\frac{DN}{2}\frac{Q_{1/7/09}}{Q_{1/1/08}}\right)v^{3/2}\left(\frac{Q_{1/4/08}}{Q_{1/10/09}}\right) + \left[(\frac{DN}{2}+RN)\frac{Q_{1/1/10}}{Q_{1/1/08}}\right]v^{2}\left(\frac{Q_{1/4/08}}{Q_{1/4/10}}\right)
\end{aligned}$$

Note that each inflation adjusted payment is discounted by both real yield and actual inflation rate without a lag. This is the relationship we observed earlier. The money rate of discount equals to the product of the real rate of discount and the inflation rate of discount:

$$\frac{1}{1+i} = \frac{1}{1+i'}\frac{1}{1+j} = v_{i'}v_j$$

By solving the equation of value in terms of i', we obtain the real yield of the index-linked bond.

2.5 Capital Gains Tax

2.5.1 Loan with Allowance for Capital Gains Tax

Assuming no taxes, price per unit nominal to be paid for a fixed interest stock at interest at $D\%$ per annum payable $p$thly and redeemable at $R\%$ in n years is calculated as follows. At an interest rate i pa:

$$A = Da_{\overline{n|}}^{(p)} + Rv_i^n$$

Obviously, the first component of the RHS is the PV of the coupon payments and the second component represents the PV of the redemption payment. We denote the first component by $I = Da_{\overline{n|}}^{(p)}$ and the second component by $K = Rv_i^n$. Thus, the equation is rewritten by:

$$A = I + K$$

If an investor is subject to income tax, we know the price A' is defined by:

$$A' = (1-t_1)I + K$$

Further, if an investor is subject to capital gains tax as well, we denote the price by A''. According to capital gain test we learned earlier, we know that the amount of capital gain is:

$$\begin{cases} 0 & \text{if } i^{(p)} \le (1-t_1)g \\ R - A'' & \text{if } i^{(p)} > (1-t_1)g \end{cases} \tag{21}$$

Thus, the price of the security depends on the sign of the capital test. For $i^{(p)} > (1-t_1)g$, the price is defined by:

$$\begin{aligned} A'' &= (1-t_1)Da_{\overline{n|}}^{(p)} + Rv^n - t_2(R - A'')v^n & (22)\\ &= (1-t_1)I + (1-t_2)K + t_2A''v^n & (23)\\ &= \frac{(1-t_1)I + (1-t_2)K}{1-t_2v^n} & (24) \end{aligned}$$

Then the price is summarized by:

$$A'' = \begin{cases} (1-t_1)I + K & \text{if } i^{(p)} \le (1-t_1)g \\ \frac{(1-t_1)I+(1-t_2)K}{1-t_2v^n} & \text{if } i^{(p)} > (1-t_1)g \end{cases} \tag{25}$$

Example 2.8. *An investor considers purchasing a 5-year stock with semiannual coupon payments of 6% pa. The investor is subject to 33% tax on income and capital gain. What price would the investor, who requires a yield of 5% pa, pay for 100 nominal redeemable at par?*

First, we take the capital test to see if the investor will have a capital gain.

$$(1-t_1)g = 0.67 \times \frac{0.06}{1.00} = 0.0402$$

And

$$i^{(2)} = 2(1.05^{1/2} - 1) = 0.0494$$

Thus, there is a capital gain. The price is:

$$
\begin{aligned}
P &= (1-t_1)Da_{\overline{n}|}^{(p)} + Rv^n - t_2(R-A'')v^n && (26)\\
&= 0.67 \times 6\left[\frac{1-v^5}{2(1.05^{1/2}-1)}\right] + 100v^5 - 0.33(100-P)v^5 && (27)\\
&= 17.619 + 78.353 - 0.259(100-P) && (28)\\
&= 94.567 && (29)
\end{aligned}
$$

Example 2.9. *Using the price obtained above, check if the price attains an annual effective yield of 5%.*

The equation of value is set up as follows:

$$
\begin{aligned}
94.567 &= 0.67 \times 6\left[\frac{1-v^5}{2((1+i)^{1/2}-1)}\right] + 100v^5 - 0.33(100-94.567)v^5 && (30)\\
& && (31)
\end{aligned}
$$

It can be seen that the equation holds for $i = 5\%$.

2.5.2 Other Related Topics for Capital Gains Tax

It may be possible to reduce the amount of capital gain by reference to an adjusted purchase price, which is increased according to an approved index. This adjustment is referred to as the *indexation of gains*. For instance, if the index increased by 20% over the period during which the investor owned an asset. Suppose that the asset is purchased at 100 and sold at 150. Without approved index, the capital gain is 50 if no other offsetting capital losses exist. In contrast, the approved index of 20% over the period reduces the capital gain to 30 (= $150 - 100 \times 1.2$).