Q1: For $f_{3,1}$,

$$(1+y_3)(1+f_{3,1})(1+f_{4,3})^3 = (1+y_7)^7$$

$$\Rightarrow 1.06^3(1+f_{3,1})(1.052)^3 = 1.05^7$$

$$\Rightarrow f_{3,1} = \frac{1.05^7}{1.06^3(1.052)^3} - 1 = \underline{0.0148}.$$

For $f_{5,2}$

$$(1+y_5)^5(1+f_{5,2})^2 = (1+y_7)^7$$

$$\Rightarrow 1.057^5(1+f_{5,2})^2 = 1.05^7$$

$$\Rightarrow f_{5,2} = \left(\frac{1.05^7}{1.057^5}\right)^{1/2} - 1 = \underline{0.0327}$$

For y_4

$$(1+y_4)^4(1+f_{4,3})^3 = (1+y_7)^7$$

$$\Rightarrow y_4 = \left(\frac{1.05^7}{1.052^3}\right)^{1/4} - 1 = \underline{0.0485}$$

For $f_{3,4}$

$$(1+y_3)^3(1+f_{3,4})^4 = (1+y_7)^7$$

$$\Rightarrow f_{3,4} = \left(\frac{1.05^7}{1.06^3}\right)^{1/4} - 1 = \underline{0.0426}$$

Q2:

Using the par yield:

$$100 = \frac{4.15}{1+y_1} + \frac{104.15}{(1+y_2)^2} \quad \sim ①$$

Using the two-year fixed income stock.

$$105.4 = \frac{8}{1+y_1} + \frac{98+8}{(1+y_2)^2} \quad \sim ②$$

Let $x = \frac{1}{1+y_1}$, $y = \frac{1}{1+y_2}$, from ① and ②

$$\begin{cases} 100 = 4.15x + 104.15y & \sim ①' \\ 105.4 = 8x + 106y & \sim ②' \end{cases}$$

From ①'

$$y = \frac{100 - 4.15x}{104.15}$$

To ②'

$$105.4 = 8x + 106\left(\frac{100 - 4.15x}{104.15}\right)$$

$$\Rightarrow x = 0.95959$$

To ②'

$$105.4 = 8(0.95959) + 106y$$

$$\Rightarrow y = 0.921917$$

Then

$$y_1 = \frac{1}{x} - 1 = \underline{0.0421}$$

$$y_2 = \left(\frac{1}{y}\right)^{1/2} - 1 = \underline{0.0415}$$

Q3

Asset ~~B~~ A : $V_A = (1+i)^{-11}$

$$C_A = \frac{V_A''(i)}{V_A} = \frac{(-11)(-12)(1+i)^{-13}}{(1+i)^{-11}} = 132(1+i)^{-2} = \underline{109.1} \quad @\ i=0.1$$

Asset B : $V_B = 9663(1+i)^{-5} + 26.910(1+i)^{-20}$

$$C_B = \frac{9663(-5)(-6)(1+i)^{-7} + 26910(-20)(-21)(1+i)^{-22}}{9663(1+i)^{-5} + 26910(1+i)^{-20}} = \underline{153.7} \quad @\ i=0.1$$

Asset C : $V_C = \frac{1-(1+i)^{-50}}{i}$

$$\left\{ \begin{aligned} V_C' &= \frac{i(50(1+i)^{-51}) - (1-(1+i)^{-50})}{i^2} \\ V_C'' &= \Big[i^2(50i(-51)(1+i)^{-52} + 50(1+i)^{-51} - 50(1+i)^{-51}) \\ &\quad - 2i(50i(1+i)^{-51} - (1-(1+i)^{-50})) \Big] / i^4 \end{aligned} \right.$$

$$C_C = \frac{50(-51)i^3(1+i)^{-52} - 2i[50i(1+i)^{-51} - (1-(1+i)^{-50})]}{i^3(1-(1+i)^{-50})}$$

$$= \underline{174.1} \quad @\ i=0.1$$

Asset C has the largest convexity b/c the payments are spread out over a long time period.

Q4:

Define x = the maturity value of 5-year bond.

y = the maturity value of 10-year bond.

The PV of the asset and liability is:

$$V_A(0.07) = x v^5 + y v^{10}$$

$$V_L(0.07) = 50000(v^6 + v^8) = 62418 \quad @\ 7\%$$

From Redington's first condition

$$V_A(0.07) = V_L(0.07) \Rightarrow x v^5 + y v^{10} = 62418. \quad \sim ①$$

By the second condition

$$V_A'(0.07) = V_L'(0.07) \Rightarrow 5x v^6 + 10 y v^{11} = 50000(6v^7 + 8v^9)$$

$$= 404398 \quad \sim ②$$

Solving ① and ②. we find

$$x = 53710, \quad y = 47457$$

Check if the third condition is satisfied.

$$V_A''(0.07) = 30 x v^7 + 110 y v^{12} = 3321,152$$

$$V_L''(0.07) = 50000(42v^8 + 72v^{10}) = 3,052,250.$$

Thus,

$$V_A'' > V_L''$$

Since all three conditions are satisfied, the fund is immunized against small change in interest rate around 7%.