BA2202 Mathematics of Finance
Handout 3

1 Equations of Value

1.1 Introduction

We have learned techniques to evaluate future cashflows by finding the PV and FV, where the future cashflow and the interest rate are given. In the next several lectures, the techniques are applied to find other parameters such as the amount of regular payments and the internal rate of return of a project. For the purpose, it is necessary to set up one equation from given information to find an unknown parameter. You have already used this technique in solving exercise questions except for the cases where it is not easy to find the roots of the polynomial equation, which we focus in this section.

Consider a transaction which provides that an investor will receive payments of $b_{t_1}, b_{t_2}, \ldots, b_{t_n}$ at times $t_1, t_2, \ldots, t_n$ in return for outlays of amount $a_{t_1}, a_{t_2}, \ldots, a_{t_n}$ at these times. Let c_{t_k} be the net cash flow $(b_{t_k} - a_{t_k})$ at time t_k.

The *equation of value for the rate of interest* or the *yield equation* is defined where the sum of the discounted net cash flow is zero:

$$\sum_{k=1}^{n} c_{t_k} (1+i)^{-t_k} = 0 \tag{1}$$

When the force of interest δ is assumed, the *equation of value for the force of interest* is written by:

$$\sum_{k=1}^{n} c_{t_k} e^{-\delta t_k} = 0. \tag{2}$$

If the cash flow is described as continuous payments, where the rate of net cashflow at time t is denoted by $\rho(t)$, the equation of value is:

$$\int_0^\infty \rho(t) e^{-\delta t} dt = 0. \tag{3}$$

It is straightforward to see that, when both discrete and continuous cashflows are involved, the equation of value is defined by the sum of the PVs of those cashflows:

$$\sum_{k=1}^{n} c_{t_k} e^{-\delta t_k} + \int_0^\infty \rho(t) e^{-\delta t} dt = 0 \tag{4}$$

and the equivalent yield equation is:

$$\sum_{k=1}^{n} c_{t_k} v^{t_k} + \int_0^\infty \rho(t) v^t dt = 0. \tag{5}$$

1.2 Existence of a Unique Positive Root for Interest Rate

For any given transaction, the equation may have no roots, a unique root, or several roots for the interest rate i. If there is a unique root δ_0, it is known as the force of interest implied by the transaction, and the corresponding rate of interest $i_0 = e^{\delta_0} - 1$ is called the *yield* or the *internal rate of return*. One important class of transaction for which the yield always exists is described as follows:

Theorem 1.1 (Existence of a unique root: MS, p.38)**.** *For any transaction in which all the negative net cash flows precede all the positive net cash flows (or vice versa) the yield is well-defined.*

The next theorem shows that one class of transaction has exactly one positive root.

Theorem 1.2 (Existence of a unique positive root: MS, p.39)**.** *Let* $A_k = \sum_{j=0}^{k} c_{t_k}$*, so that* A_k *denotes the cumulative total amount received by the investor at time* t_k*. Suppose that* A_0 *and* A_n *are both non-zero and that, when any zero values are excluded, the sequence* $\{A_0, A_1, \ldots, A_n\}$ *contains precisely one change of sign. Then the yield equation has exactly one positive root.*

Example 1.1. *Consider a transaction which provides, in return for outlays of amount 4, 2, 1 at times 1, 2, 4, an investor will receive payments of 1, 3, 4, 5 at time 2, 3, 4, 5, respectively. For the investment the sequency of the net cashflow is:*

$$\{c_{t_i}\} = \{-4, -1, 3, 3, 5\}$$

and the cumulative total amount received by the investor is:

$$\{A_i\} = \{-4, -5, -2, 1, 6\}$$

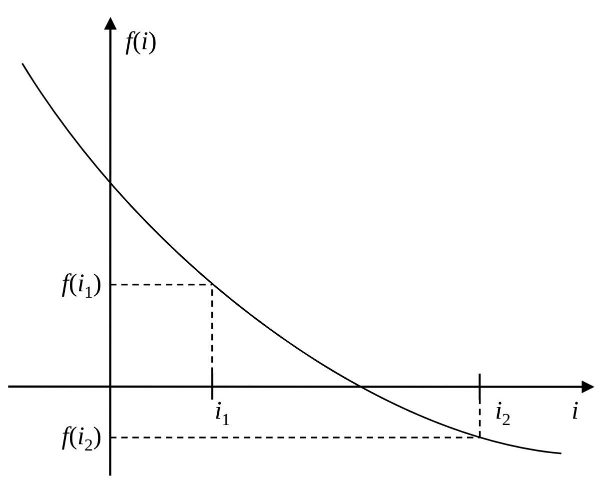
Since A_i *contains only one sign change, the yield equation has only one positive root.*

1.3 Approximation

Even though we figure out that a given cashflow has a unique positive root, it may be cumbersome to find the root. Consider the equation of value at time t_0:

$$\sum_{k=1}^{n} c_{t_k} (1+i)^{t_0 - t_k} = 0 \tag{6}$$

Let $f(i)$ denote the LHS of the equation as a function of interest rate i. If $f(i)$ is a monotonic function, we can find i_1 and i_2 with $f(i_1)$ and $f(i_2)$ of opposite sign. In this case, the root is unique and lies between i_1 and i_2. Once these two values are identified, we can use approximation techniques to obtain i, which satisfies $f(i) = 0$.

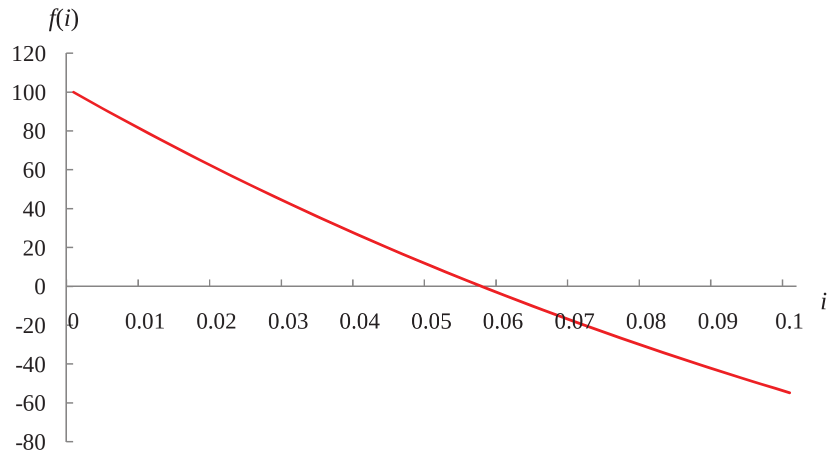


Linear interpolation is one approach to estimate an unknown interest rate by equating the proportional change in the interest rate to the proportional change in the PV:

$$\frac{i - i_1}{i_2 - i_1} = \frac{f(i) - f(i_1)}{f(i_2) - f(i_1)}. \tag{7}$$

Note that the PV corresponding to the interest rate we are looking for is zero ($f(i) = 0$) in this specific case.

Example 1.2. *A project requires each investor an immediate investment of 200 and an additional investment of 200 two years from now. An investor participates in the project will receive 500 after five years. Find the yield for the project.*



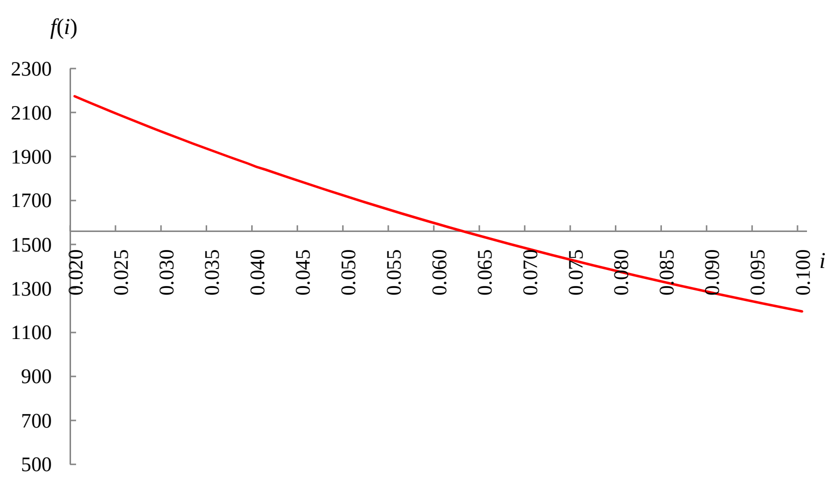
$$f(i) = -200 - 200(1+i)^{-2} + 500(1+i)^{-5} = 0$$

We can find $i_1 = 5.6\%$ *and* $i_2 = 5.8\%$ *with* $f(i_1) = 1.4089$ *and* $f(i_2) = -1.4989$ *of opposite sign. Using the linear interpolation technique, we estimate* i *by:*

$$\frac{i - 0.056}{0.058 - 0.056} = \frac{0 - 1.4089}{-1.4989 - 1.4089} \Rightarrow i \approx 5.697\%$$

Example 1.3. *You consider purchasing one bedroom 710 sq ft unit of Marina Bay Residences at S\$1.56m for an investment purpose. The competitive rent for the first year is S\$72,000 (S\$6,000 $\times$ 12), and you expect that the rent increase by 4% pa compounded in each year and that the property will be sold after 10 years for S\$1.7m. Assuming that the unit will be always occupied and the rent is paid in the middle of each year, calculate the yield you will obtain on this investment.*

The equation of value is in 1000s:



$$1560 = 72(v^{0.5} + 1.04v^{1.5} + 1.04^2v^{2.5} + \cdots + 1.04^9v^{9.5}) + 1700v^{10}$$

This can be rewritten as:

$$1560 = 72v^{0.5}\left(\frac{1-(1.04v)^{10}}{1-1.04v}\right) + 1700v^{10}$$

By trial and error, we estimate $i \approx 6.2736\%$.

Unfortunately, the market price have not increased much and the rent went down. After 3 years from the market price was given, the current price of the unit is S$1.58m and the monthly rent is S$4,600. If the monthly rent was S$6,000 for the past 3 years and will be S$4,600 for the next 7 years, and if the unit will be sold at current price S$1.58m in 10 years, what would be the yield?

1.4 Uncertain Cashflow

If the payment or receipt of a cashflow at a particular time is uncertain, we need to incorporate the risk into the cashflow evaluation in one of the following ways:

- take the expected value of uncertain payments/receipts
- discount the cashflow at a higher rate

Consider that a transaction which provides a net cashflow at time t with only less than 100%, where the probability is denoted by p_{t_k} and p_t. Now equation (5) is revised to:

$$\sum_{k=1}^{n} p_{t_k} c_{t_k} v^{t_k} + \int_0^\infty p_t \rho(t) v^t dt = 0. \tag{8}$$

Alternatively, using expectation operator, we can rewrite it as:

$$\sum_{k=1}^{n} v^{t_k} E[\tilde{c}_{t_k}] + \int_0^\infty v^t E[\tilde{\rho}(t)] dt = 0. \tag{9}$$

where $\tilde{c}_{t_k}$ and $\tilde{\rho}(t)$ are random variables.
Furthermore, where the force of interest is constant and the probability can be described by the force of the probability μ, equation (4) is revised to:

$$\sum_{k=1}^{n} c_{t_k} e^{-\delta' t_k} + \int_0^\infty \rho(t) e^{-\delta' t} dt = 0 \tag{10}$$

where $\delta' = \delta + \mu$.

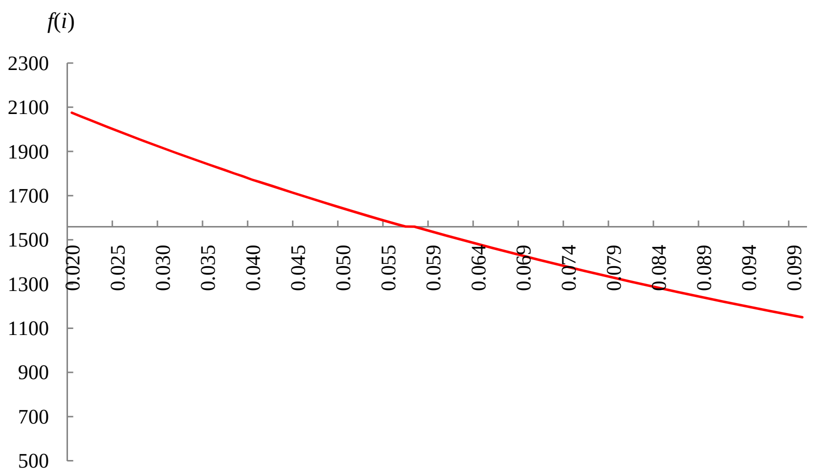
Example 1.4. *You have realized that the unit of Marina Bay Residences may not be sold after 10 years for S$1.7m due to oversupply of condo units. It may be reasonable to assume that the property will be sold for S$1.7m with 40% and S$1.5m with 60%. How is the yield for the investment affected? The equation of value is in 1000s:*

$$1560 = 72(v^{0.5} + 1.04v^{1.5} + 1.04^2v^{2.5} + \cdots + 1.04^9v^{9.5}) + [1700(0.4) + 1500(0.6)]v^{10}$$

This can be rewritten as:

$$1560 = 72v^{0.5}\left(\frac{1-(1.04v)^{10}}{1-1.04v}\right) + 1580v^{10}$$

By trial and error, we estimate $i \approx 5.6946\%$.


f(i)
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