Solution for Question 5

From given information, we know $(1+i) \sim logN(\mu, \sigma^2)$, and

$$E[1+i] = e^{\mu+\frac{1}{2}\sigma^2} = 1.06$$

$$Var(1+i) = Var(i) = e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right) = 0.2^2$$

Solving these two equations, we have parameter values: $\mu = 0.04078$ and $\sigma^2 = 0.03498$.
The probability we want to know is

$$\Pr(700A(0,6) > 1000) = \Pr\left(A(0,6) > \frac{10}{7}\right)$$

Since $A(0,6) \sim logN(6\mu, 6\sigma^2)$ under independent rate assumption,

$$\Pr\left(A(0,6) > \frac{10}{7}\right) = Pr\left(\frac{lnA(0,6) - 6\mu}{\sqrt{6}\sigma} > \frac{ln\left(\frac{10}{7}\right) - 6(0.04078)}{\sqrt{6}(0.03498^{1/2})}\right) = 1 - \Phi(0.24448)$$

$$= 1 - 0.59657 = 0.40343$$

Solution for Question 6

It may be convenient to calculate expected values first:

$$E[i_1] = \frac{1}{3}(0.04 + 0.06 + 0.08) = 0.06$$
$$E[i_2] = 0.75(0.08) + 0.25(0.04) = 0.07$$
$$E[i_3] = 0.60(0.08) + 0.40(0.04) = 0.064$$
$$E[i_1^2] = \frac{1}{3}(0.04^2 + 0.06^2 + 0.08^2) = 0.003867$$
$$E[i_2^2] = 0.75(0.08^2) + 0.25(0.04^2) = 0.00520$$
$$E[i_3^2] = 0.60(0.08^2) + 0.40(0.04^2) = 0.00448$$
$$Var(i_1) = 0.003867 - 0.06^2 = 0.00027$$
$$Var(i_2) = 0.00520 - 0.07^2 = 0.00030$$
$$Var(i_3) = 0.00448 - 0.064^2 = 0.00038$$

(a) The expected accumulated value is calculated by the recursive relationship

$$E[A_0] = 0$$
$$E[A_k] = (1 + E[i_k])(V_k + E[A_{k-1}]) = (1 + E[i_k])(V_k + E[A_{k-1}])$$

Using these

$$E[A_1] = (1 + E[i_1])V_1 = 1.06(10{,}000) = 10{,}600$$
$$E[A_2] = (1 + E[i_2])(V_2 + E[A_1]) = 1.07(20{,}000 + 10{,}600) = 32{,}742$$
$$E[A_3] = (1 + E[i_3])(V_3 + E[A_2]) = 1.064(15{,}000 + 32{,}742) = 50{,}797.49$$

Alternatively, the expected accumulated value is:

$$\begin{aligned} EV &= E[10000A(0,3) + 20000A(1,3) + 15000A(2,3)] \\ &= 10000E[A(0,3)] + 20000E[A(1,3)] + 15000E[A(2,3)] \\ &= 10000(1.20678) + 20000(1.13848) + 15000(1.064) \\ &= 50{,}797.49 \end{aligned}$$

(b) The variance is calculated by using the following recursive relationship:

$$E[A_0] = 0$$
$$E[A_0^2] = 0$$
$$E[A_k^2] = ((1 + E[i_k])^2 + Var(i_k))(V_k^2 + 2V_kE[A_{k-1}] + E[A_{k-1}^2])$$

Using these

$$E[A_1^2] = (1.06^2 + 0.00027)(10{,}000^2 + 0) = 112{,}386{,}667$$
$$E[A_2^2] = (1.07^2 + 0.00030)(20{,}000^2 + 2(20{,}000)(10{,}600) + E[A_1^2]) = 1{,}072{,}350{,}011$$
$$E[A_3^2] = (1.064^2 + 0.00038)(15{,}000^2 + 2(15{,}000)(32{,}742) + E[A_2^2]) = 2{,}581{,}612{,}745$$

Using these, we find the s.d. is

$$\sqrt{Var(A_3)} = \sqrt{E[A_3^2] - E[A_3]^2} = \sqrt{2{,}581{,}612{,}745 - 50{,}797.49^2} = \sqrt{1{,}227{,}958}$$
$$= 1{,}108.13$$

(c) There are only 3 ways for the accumulated value to exceed 51,000 where Year 2 and Year 3 interest rate must be the highest and Year 1 interest rate can be any one of three. So the probability is.

$$0.75 \times 0.6 = 0.45$$

Solution for Question 7

(a) $j = 0.07$

$\Rightarrow E[A] = 10000(1 + j)^{15} = 10000(1.07)^{15} = 27{,}590.32$

(b) $s^2 = 0.04^2 \times 0.2 + 0.07^2 \times 0.6 + 0.10^2 \times 0.2 - 0.07^2$

$= 0.00526 - 0.00490 = 0.00036$

$Var(A) = 10000^2((1 + 2j + j^2 + s^2)^{15} - (1 + j)^{30})$

$= 10000^2(1.14526^{15} - 1.07^{30})$

$= 10000^2(0.0359828)$

$SD(A) = 1896.9$

(c)

(i) By $j = 0.07$, mean accumulated value will be the same as in (a). However, the spread of the yields around the mean is lower than in (a). Hence, the standard deviation of the accumulation will be lower than 0.18969.

(ii) Investing over a longer term does increase the SD of the investment return, i.e., not a risk reducer, while the SD of the annualized rate of return, calculated by $s/n^{0.5}$, is reduced. Thus, there is time diversification effect, but yet the risk of total return becomes more uncertain for a longer investment.

(d) The accumulated value will be at least 15,500 only if the yield is 7% at most for 1 year and 10% for the rest. Since the probability of having 10% for all 5 years is 0.2^5 and the probability of having 7% for 1 year and 10% for 4 years is $5 \times 0.6 \times 0.2^4$, then:

Prob = $1 - 0.2^5 - 5 \times 0.2^4 \times 0.6 = 99.488\%$.