BA2202 Mathematics of Finance
Handout 9

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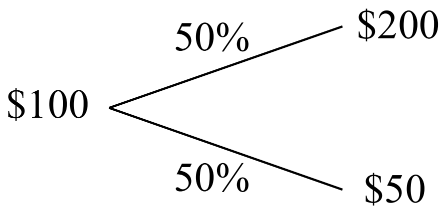
The St. Petersburg Paradox

- The St. Petersburg game is played by flipping a fair coin until it comes up tails, and the total number of flips, n, determines the prize, which equals $\$2^n$.
- If you were a "rational" gambler, how much would you pay to participate in this game?

n	P(n)	Prize	Expected Payoff
1	½	$2	$1
2	¼	$4	$1
3	1/8	$8	$1
4	1/16	$16	$1
5	1/32	$32	$1
6	1/64	$64	$1
7	1/128	$128	$1
8	1/256	$256	$1
9	1/512	$512	$1
10	1/1024	$1,024	$1

Utility Theory and Uncertainty
Utility theory

1. Introduction
   - **Utility:** the satisfaction that an individual obtains from a particular course of action.
   - **Risk (uncertainty)**: a situation in which the probabilities of the different possible outcomes are known, but it is not known which outcome will occur.



   - **Utility functions**: Utility can be assigned to each possible value of the investor's wealth by utility function
     - Log utility function: $U(w) = \log(w)$ for $w > 0$
     - Quadratic utility function: $U(w) = w + dw^2$ for $-\infty < w < -\frac{1}{2d}$
     - Power utility function: $U(w) = \frac{w^{\gamma} - 1}{\gamma}$ for $w > 0$
     - The exponential utility function: $U(w) = e^{-aw}$

Utility Theory and Uncertainty
Utility theory

2. The expected utility theorem
   - A utility function $U(w)$ can be constructed as representing an investor's utility of wealth w at some future time.
   - Decisions are made in a manner to maximize the expected value of utility given the investor's particular beliefs about the probability of different outcomes.

Example An investor who has the log utility function face the uncertainty described in the previous slide. What is the expected wealth and what is the expected utility of wealth?

- The expected wealth = $0.5(200) + 0.5(50) = \$125$
- The expected utility = $0.5\log(200) + 0.5\log(50)$ =4.6052

Utility Theory and Uncertainty
Utility theory

2. The expected utility theorem

Example An investor has an initial wealth of 100 and the log utility function. Investment Z offers a return of -18% or +20% with equal probability.

(1) What is the investor's expected utility if nothing is invested in Investment Z?
(2) What is the investor's expected utility if 100 is invested in Investment Z?
(3) What proportion *a* of wealth should be invested in Investment Z under the expected utility theorem? What is the expected utility for this investment?

Solution

(1) $\log(100) = 4.605$

(2) $E[U(w)] = 0.5\log(0.82 \times 100) + 0.5\log(1.2 \times 100) = 4.597$

(3) $E[U(w)] = 0.5\log\big(100(1 - 0.18a)\big) + 0.5\log\big(100(1 + 0.2a)\big)$

Differentiating w.r.t. *a* to find a maximum

$$\frac{dE[U(w)]}{da} = 0.5\left(\frac{-18}{100 - 18a}\right) + 0.5\left(\frac{20}{100 + 20a}\right)$$

Setting equal to zero, we find $a = 0.2777$.

$\Longrightarrow E[U(w)] = 4.6066 \;\; @a = 0.2777$

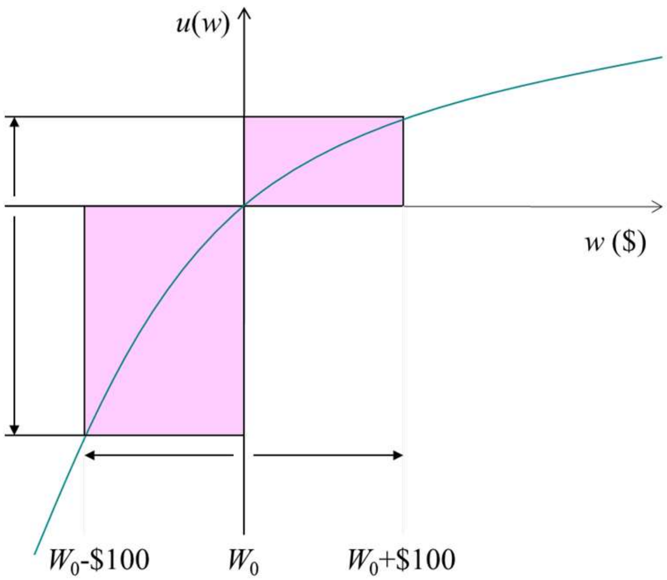
Utility Theory and Uncertainty
Economics characteristics

1. Non-satiation
   – People prefer more wealth to less (positive marginal utility of wealth):

$$U'(w) > 0$$

2. Risk Aversion
   – A risk-averse investor values an increase in wealth less highly than a decrease (diminishing marginal utility) and will reject a fair gamble:

$$U''(w) < 0$$



Utility Theory and Uncertainty
Economics characteristics

3. Risk Seeking
   – A risk-seeking investor values a marginal increase in wealth more highly than a marginal decrease and will seek a fair gamble. The utility function condition is:

$$U''(w) > 0$$

4. Risk Neutral
   – A risk-neutral investor is indifferent between a fair gamble and the status quo. The utility function condition is:

$$U''(w) = 0$$

Utility Theory and Uncertainty
Economics characteristics

Example Which statements are correct?

(1) A risk-averse person will

A. never gamble
B. accept fair gambles
C. accept fair gambles and some gambles with an expected loss

(2) A risk–neutral person will

A. always accept fair gambles
B. always accept unfair gambles
C. always accept better than fair gambles

(3) A risk–loving person will

A. always accept a gamble
B. always accept unfair gambles
C. always accept fair gambles

Utility Theory and Uncertainty

Insurance demand

1. The expected utility principle
   - When making a choice, an individual should choose the course of action that gives the highest expected utility rather than the highest expected payout.

$$E[U(w)] = \sum_{i=1}^{n} U(w_i)p(w_i)$$

   - where i = possible outcomes; w = wealth; $p(w)$ = probability of a possible outcome

2. Example
   - w_0 = \$100K
   - $U(w) = \ln(w+100)$
   - w_1 = \$0 with $p(w_1)$ = 1% and w_2 = \$100K with $p(w_2)$ = 99%
   - The expected loss (fair premium) = \$1,000
   - Insurance cost to cover the loss = \$1,200

Utility Theory and Uncertainty
Insurance demand

3. With and without insurance
   - With insurance, the expected utility is reduced to:

     $EU(w) = \ln(100K - 1{,}200 + 100) = 11.5$ --- better off
   - Without insurance, the expected utility is:

     $EU(w) = 0.99*\ln(100K + 100) + 0.01*\ln(0 + 100) = 11.4448$

4. Measures
   - **Maximum premium** (P) that an individual will be willing to pay to insure himself against a loss X is given by the solution of the equation:

     $EU(w - X) = U(w - P)$
   - E.g., the maximum premium in the previous example can be calculated by:

     $0.99*\ln(100K + 100) + 0.01*\ln(0 + 100) = \ln(100K - P + 100)$

     $\rightarrow P = 100K + 100 - e^{11.4448} = \$6{,}686$ (> the cost of coverage \$1200)
   - **Certainty equivalent:** The level of wealth $w–P$ (ie 100k–6,686)
   - **Risk premium:** a premium that an individual is willing to pay beyond a fair premium

     $\rightarrow$ Risk premium = 6,686 – 1,000 = \$5,686

Utility Theory and Uncertainty
Insurance demand

5. The minimum premium an insurer needs to charge
   - The **minimum premium** (Q) which an insurer should charge against a loss X is given by the solution of the equation:

     $EU(w + Q - X) = U(w)$

6. Example
   - Further assume that the insurer has:
     - $w_0 = \$2{,}000{,}000$
     - $U(w) = \ln(w - 1{,}000{,}000)$
   - the maximum premium in the previous example can be calculated by:

     $0.99*\ln(2m + Q - 1m) + 0.01*\ln(2m + Q - 1.1m) = \ln(2m - 1m)$

     $\rightarrow 0.99*\ln(1m + Q) + 0.01*\ln(0.9m + Q) = 13.81551$

     $\rightarrow Q \approx 1{,}053$
   - The minimum premium is much smaller than the cost of coverage (\$1,200).
   - The insure is also better off (mutually beneficial).

Utility Theory and Uncertainty
Optimal insurance

1. Simple model framework

   – Diversification is not possible
   – Risk situation is characterized by two states of the world:
   - State L: Loss occurs with probability π
   - State NL: Loss does not occur with probability (1–π)

   – No time value of money, i.e., interest rate is 0 %
   – Wealth in the two states of the world is determined by:
   - $w_L = w_0 - L$
   - $w_{NL} = w_0$

w_0: Individual's initial wealth

w_L, w_{NL}: Individual's terminal wealth

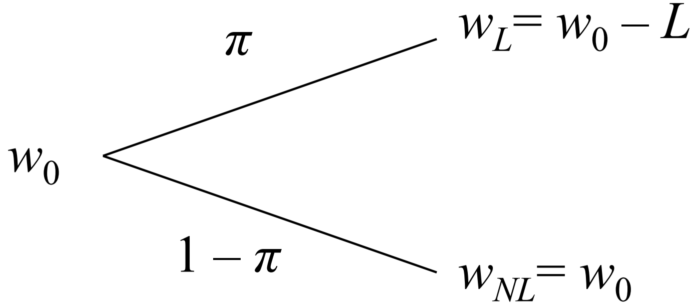
L: The amount of loss

I: Indemnity payment

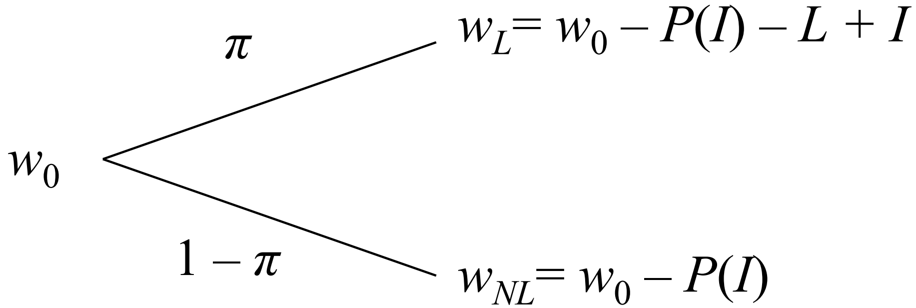
$P(I)$: Insurance Premium (πI = Fair premium)

Utility Theory and Uncertainty
Optimal insurance

Without Insurance



With Insurance



Utility Theory and Uncertainty
Optimal insurance

Individuals choose insurance coverage I which maximizes the following EU:

$$EU(w) = \pi U(w_0 - P(I) - L + I) + (1 - \pi) U(w_0 - P(I))$$

Assuming fair premium,

$$EU(w) = \pi U(w_0 - L + (1 - \pi) I) + (1 - \pi) U(w_0 - \pi I)$$

Taking a derivative w.r.t. I

$$dEU/dI = \pi(1-\pi) U'(w_0 - L + (1-\pi) I) - \pi(1-\pi) U'(w_0 - \pi I)$$

Setting $dEU/dI = 0$

$$\pi(1-\pi) U'(w_0 - L + (1-\pi) I) - \pi(1-\pi) U'(w_0 - \pi I) = 0$$

$$\Rightarrow U'(w_0 - L + (1-\pi) I) = U'(w_0 - \pi I)$$

$$\Rightarrow I^* = L$$

The optimal insurance is *full* coverage

Utility Theory and Uncertainty
Optimal insurance

2. Loading factor: λ (>0)

Premium is defined by:

$P(I) = (1+ \lambda)\pi I$

With a premium with a positive loading factor :

$w_L = w_0 - L - (1+ \lambda)\pi I + I = w_0 - L + [1 - (1+ \lambda)\pi]I$

$w_{NL} = w_0 - (1+ \lambda)\pi I$

$$EU(w) = \pi U(w_L) + (1-\pi)U(w_{NL})$$
$$= \pi U(W_0 - L + [1-(1+ \lambda)\pi]I) + (1-\pi)U(W_0 - (1+ \lambda)\pi I)$$

Utility Theory and Uncertainty
Optimal insurance

$EU(w) = \pi U(w_0 – L + [1–(1+ \lambda)\pi]I) + (1–\pi)U(w_0 –(1+ \lambda)\pi I)$

Taking a derivative w.r.t. I

$$dEU/dI = \pi[1–(1+ \lambda)\pi]U'(w_L) – \pi (1–\pi)(1+ \lambda)U'(w_{NL})$$

Setting $dEU/ dI = 0$

$[1–(1+ \lambda)\pi]U'(w_L) – (1–\pi)(1+ \lambda)U'(w_{NL}) = 0$

$\Rightarrow U'(w_L)/U'(w_{NL}) = (1–\pi)(1+ \lambda)/[1–(1+ \lambda)\pi]$

$\Rightarrow I^* < L$

The optimal insurance is *partial* coverage.

Utility Theory and Uncertainty

Optimal insurance

3. Example:
   - Probability of loss: 0.1
   - Initial wealth: 500K
   - Loss amount: 200K
   - Loading factor: $\lambda = 0.2$
   - $u(w)=\ln(w)$

Premium is defined by:

$P(I) = (1.2)(0.1)I=0.12I$

With a premium with a positive loading factor :

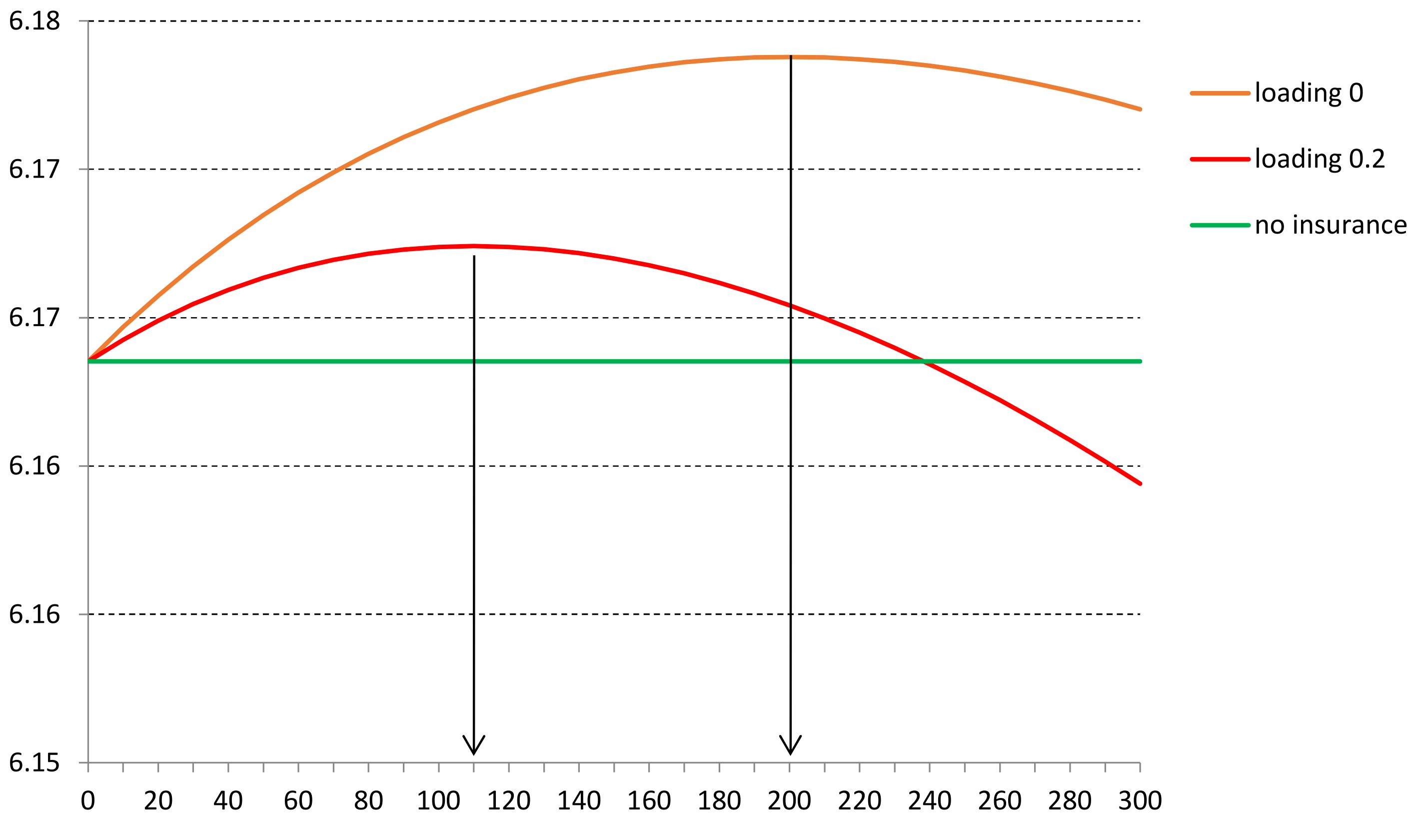
$$w_L = 500 - 200 - 0.12I + I = 300 + 0.88I$$

$$w_{NL} = 500 - 0.12I$$

$$EU(w) = \pi\, u(w_L)+(1-\pi)u(w_{NL})$$
$$= 0.1\ln(300 + 0.88I) + 0.9\ln(500 - 0.12I)$$

Utility Theory and Uncertainty
Optimal insurance

The optimal coverage for a premium loading 0.2 is about 110K



Utility Theory and Uncertainty
Optimal insurance

To find the optimal coverage, differentiate the *EU* w.r.t. *I*:

$$dEU(w)/dI = 0.1\ln(300 + 0.88I) + 0.9\ln(500 - 0.12I)$$

$$= 0.1*0.88/(300 + 0.88I) - 0.9*0.12/(500 - 0.12I)$$

Setting equal to zero:

$$0.1*0.88/(300 + 0.88I) = 0.9*0.12/(500 - 0.12I)$$

$$\rightarrow 0.1056I = 11.6$$

$$\rightarrow \boldsymbol{I^* = 109.85}$$

Check that the second derivative is negative.

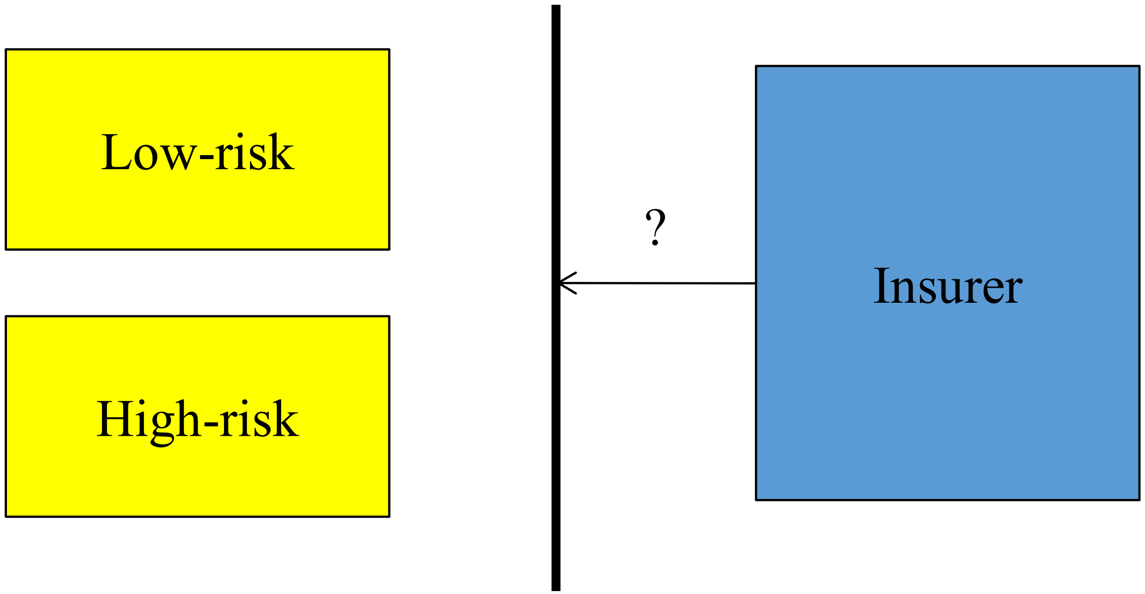
$$-0.1*0.88^2/(300 + 0.88I)^2 - 0.9*0.12^2/(500 - 0.12I)^2 < 0$$

Utility Theory and Uncertainty
Adverse selection

1. Incentives

   Full insurance is optimal for individuals if the premium is fair.

   ⇒ Is it always optimal for insurers to sell full insurance (if they can offer a fair premium)?



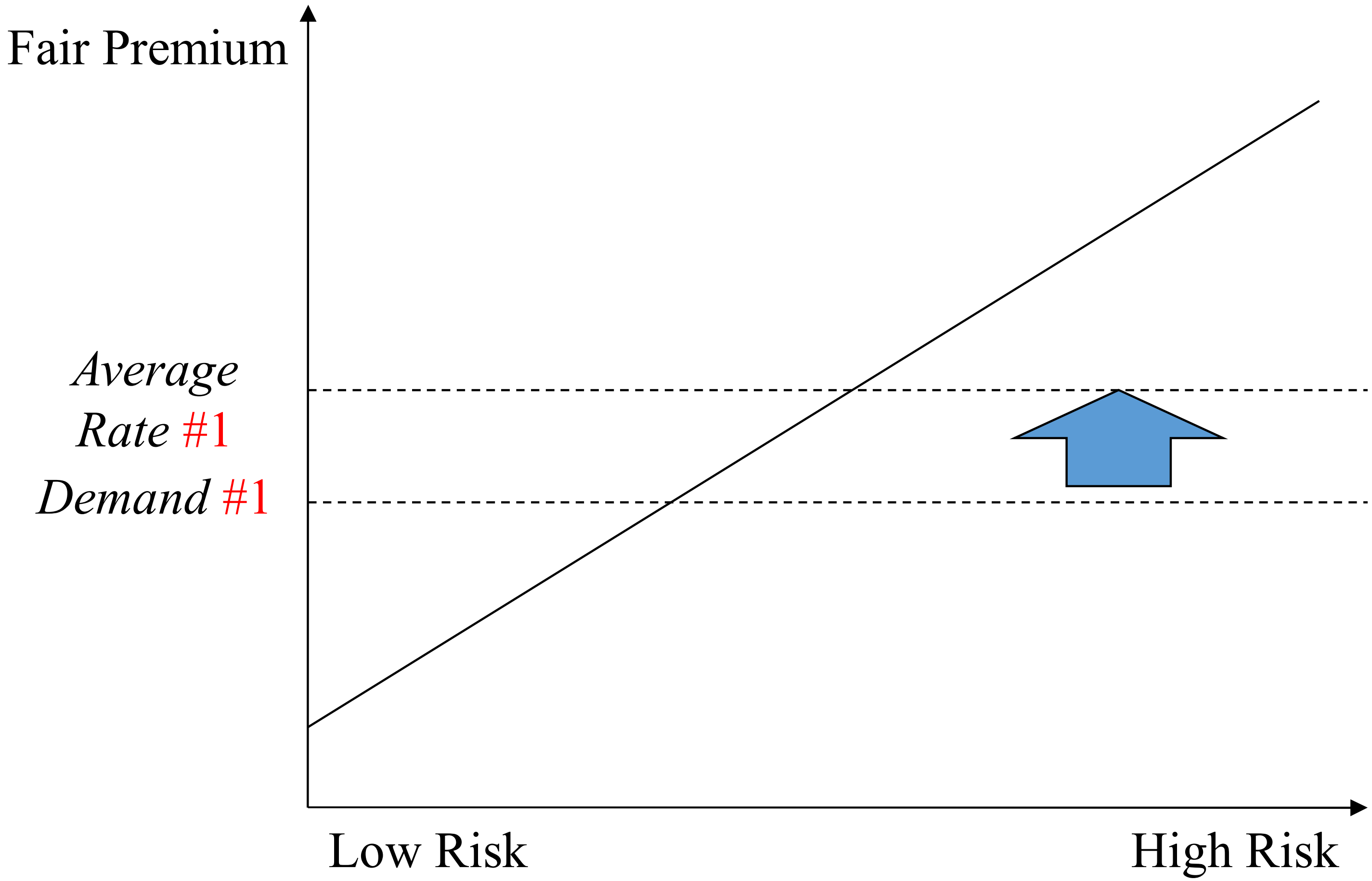
What if insurers cannot observe individuals' risk type?

Utility Theory and Uncertainty
Adverse selection

- Competitive Market
- Insurers are profit maximizers.
- The probability of loss is uniformly distributed:
  1%, 2%, 3%,…, 10%
- Loss amount is $10,000
- Insurers cannot distinguish individual's loss probability

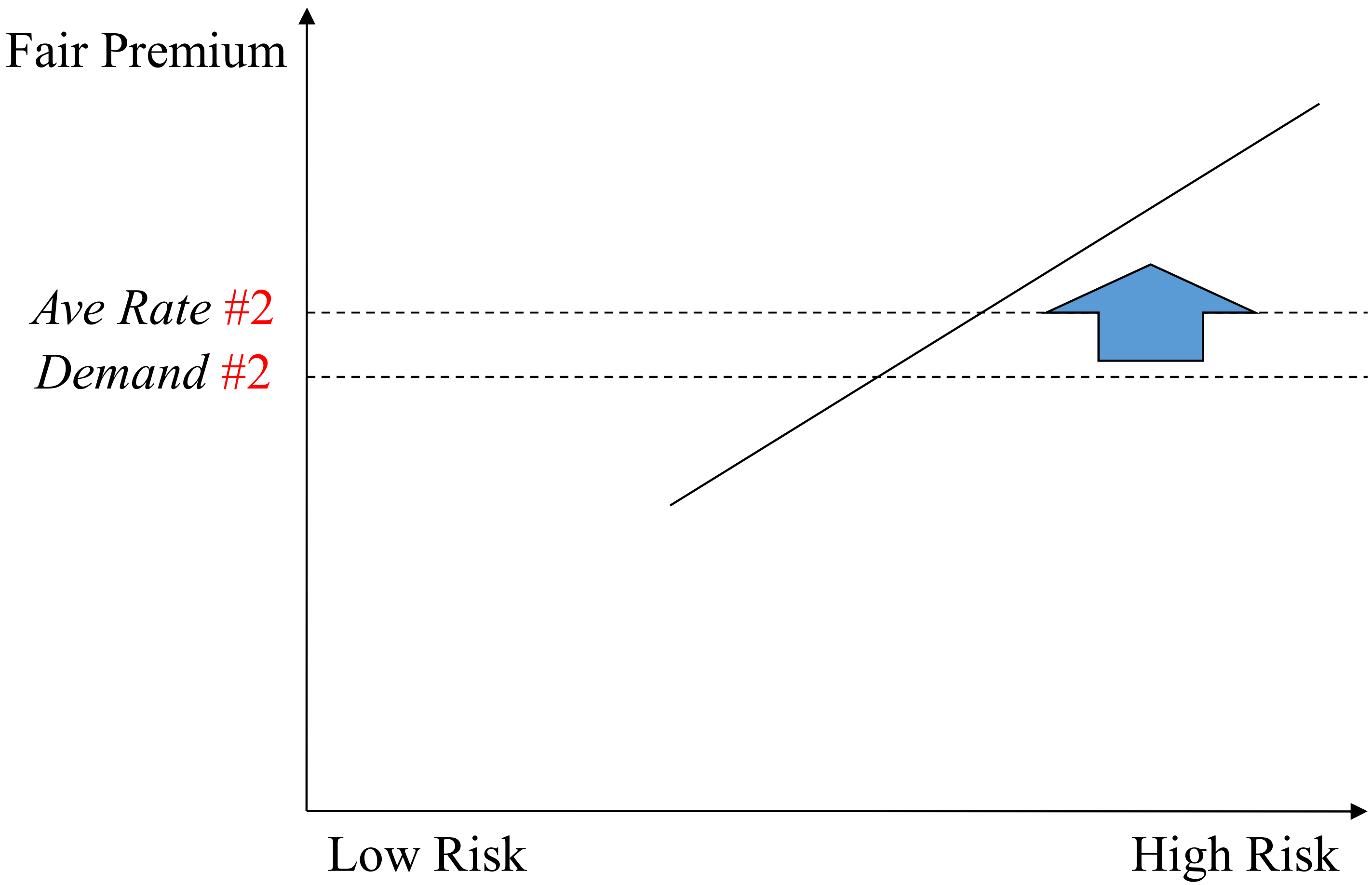
1. Insurers offer a full coverage policy (offer their premium, e.g., a fair premium=$550) first.
2. Consumers purchase a policy.
3. Insurer's loss realize as expected.
4. Revise the premium according to the loss experience.
5. Consumers purchase a policy.
6. Repeat the process….

Utility Theory and Uncertainty
Adverse selection

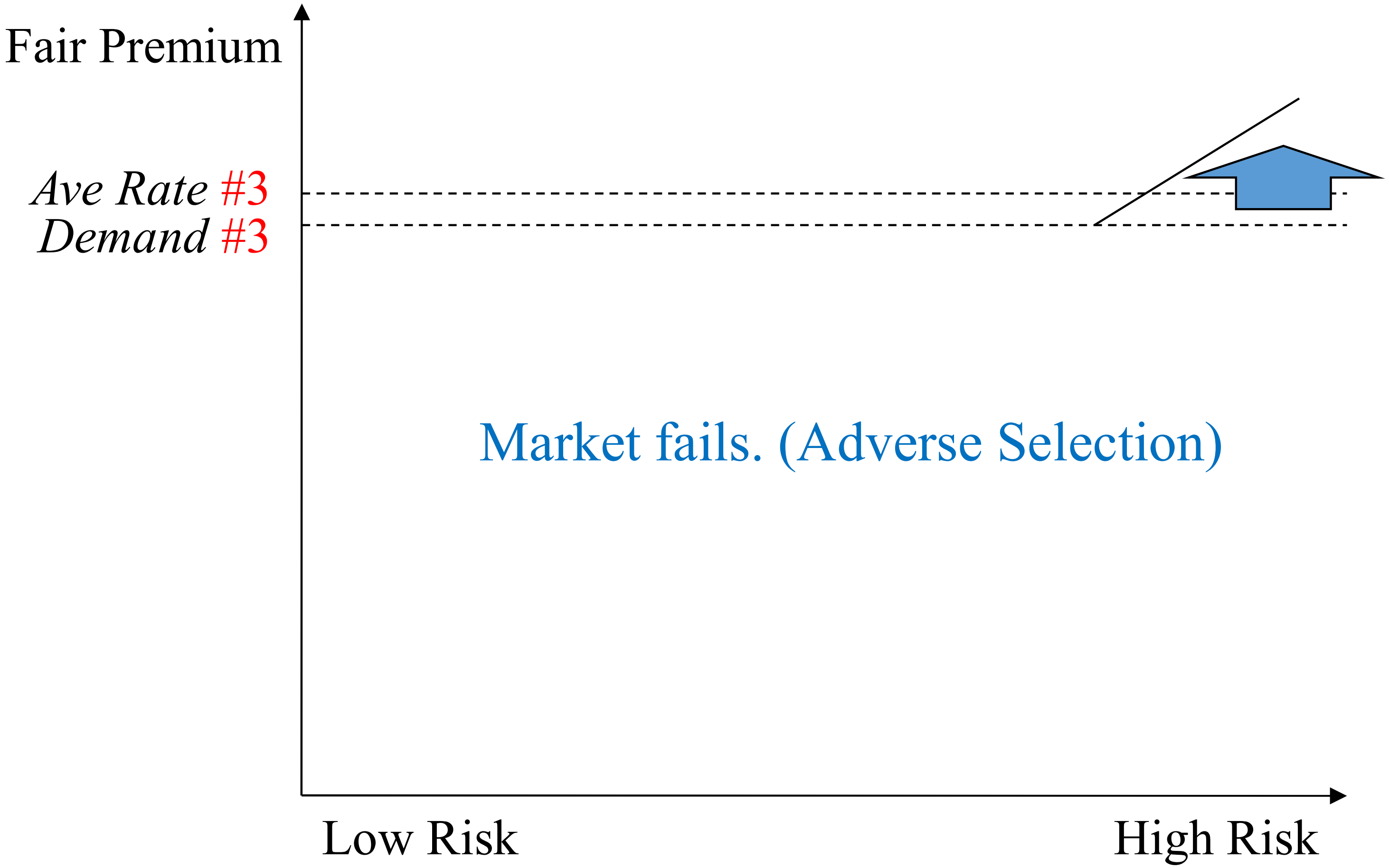


Utility Theory and Uncertainty

Adverse selection

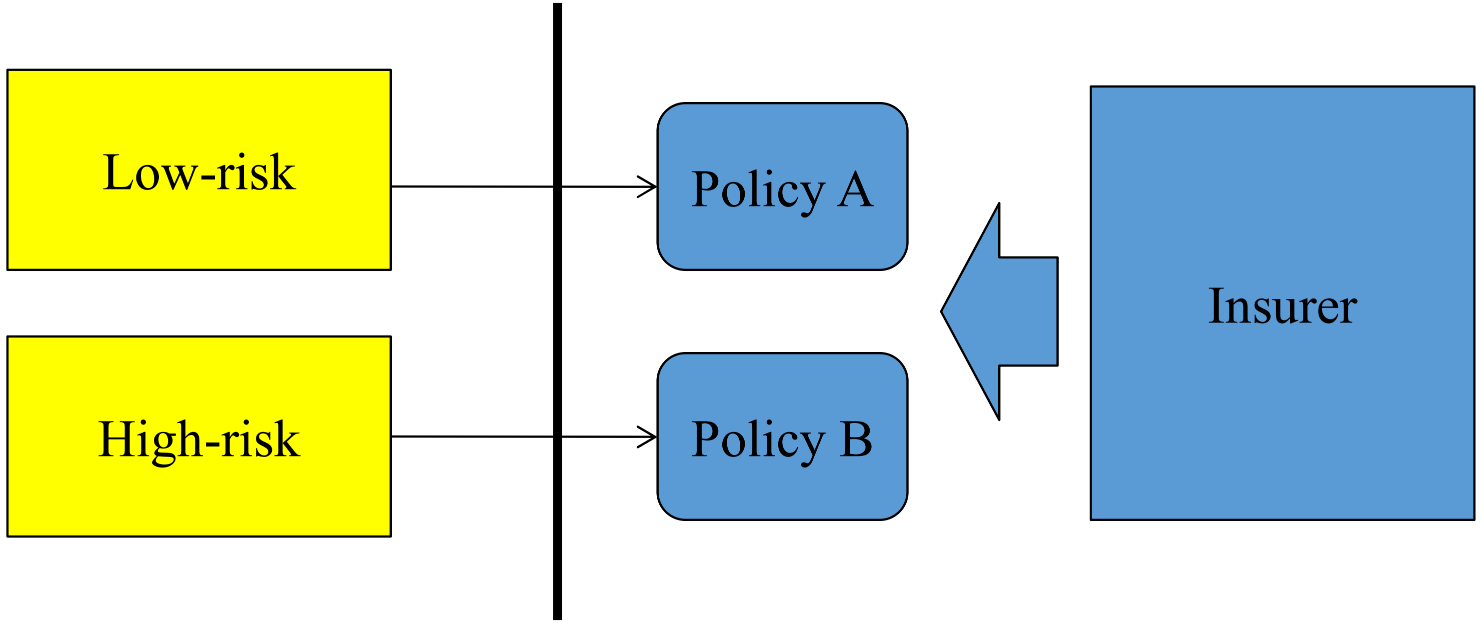


Utility Theory and Uncertainty
Adverse selection



Utility Theory and Uncertainty
Adverse selection

2. Optimal contract
   Insurers rather offer several different contracts such that low-risk chooses Policy A and high-risk chooses Policy B.



For example:

- Policy A is priced at low-risk fair rate but has a higher deductible
- Policy B is priced at high-risk fair rate but has a lower (or no) deductible

Utility Theory and Uncertainty
Adverse selection

Example:

- High-risk individual‘s Probability of loss: 0.1
- Low-risk individual‘s Probability of loss: 0.05
- Initial wealth: 500K
- Loss amount: 200K
- $U(w)=\ln(w)$

Insurers offer two policys:

- Policy A: Coverage 50K (partial coverage) at premium rate 0.05
- Policy B: Coverage 200K (full coverage) at premium rate 0.1

Utility Theory and Uncertainty
Adverse selection

Low-risk Individuals:

For Policy A

$EU(w) = 0.05\ln(300 + 0.95(50K)) + 0.95\ln(500 – 0.05(50K)) = 6.1917$

For Policy B

$EU(w) = 0.05\ln(300 + 0.9(200K)) + 0.95\ln(500 – 0.1(200K)) = 6.1738$

⇒ L-type chooses Policy A (Its fair premium with a high deductible)

High-risk Individuals:

For Policy A

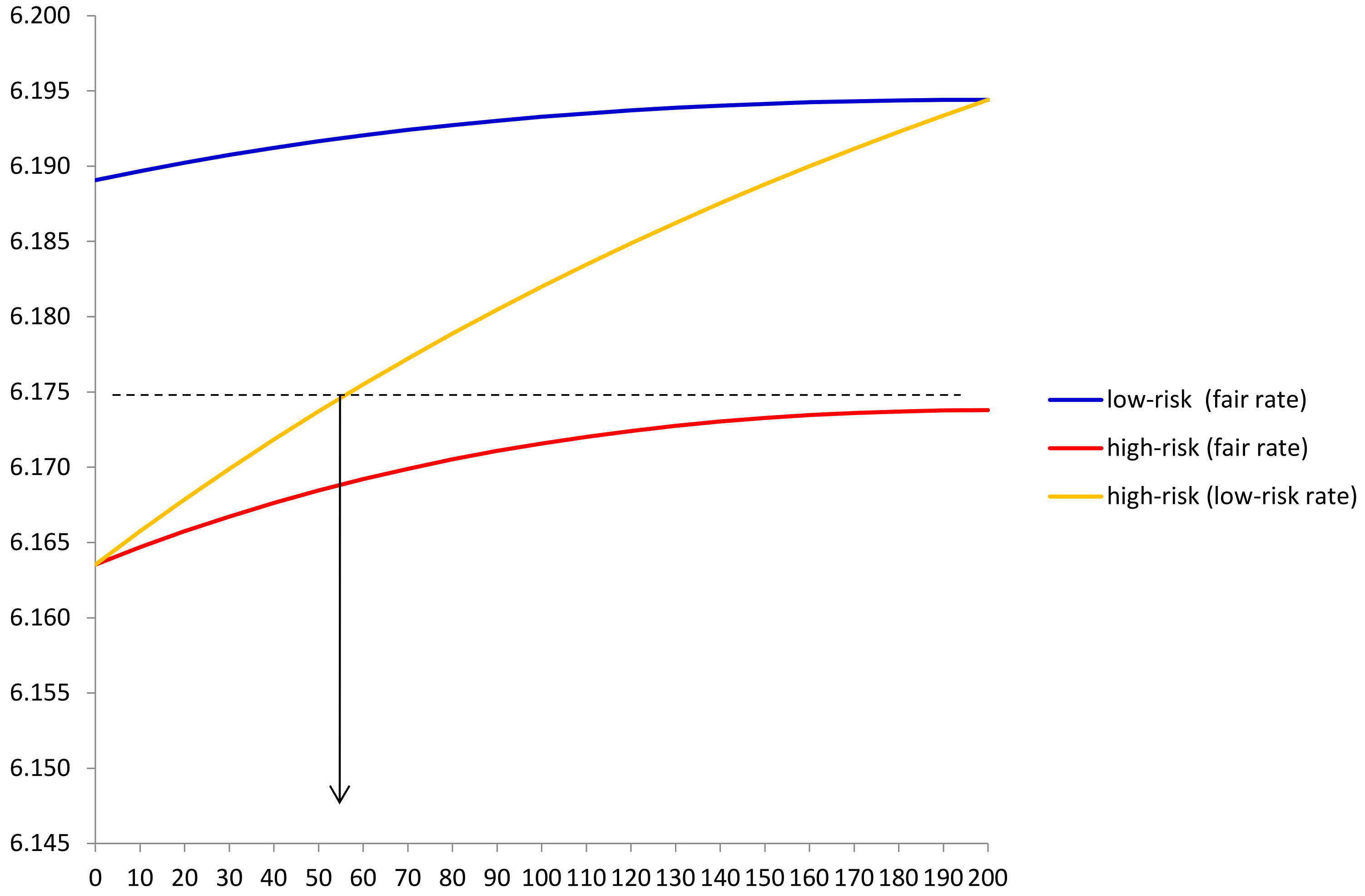
$EU(w) = 0.1\ln(300 + 0.95(50K)) + 0.9\ln(500 – 0.05(50K)) = 6.1737$

For Policy B

$EU(w) = 0.1\ln(300 + 0.9(200K)) + 0.9\ln(500 – 0.1(200K)) = 6.1738$

⇒ H-type chooses Policy B (Full coverage at its fair premium)

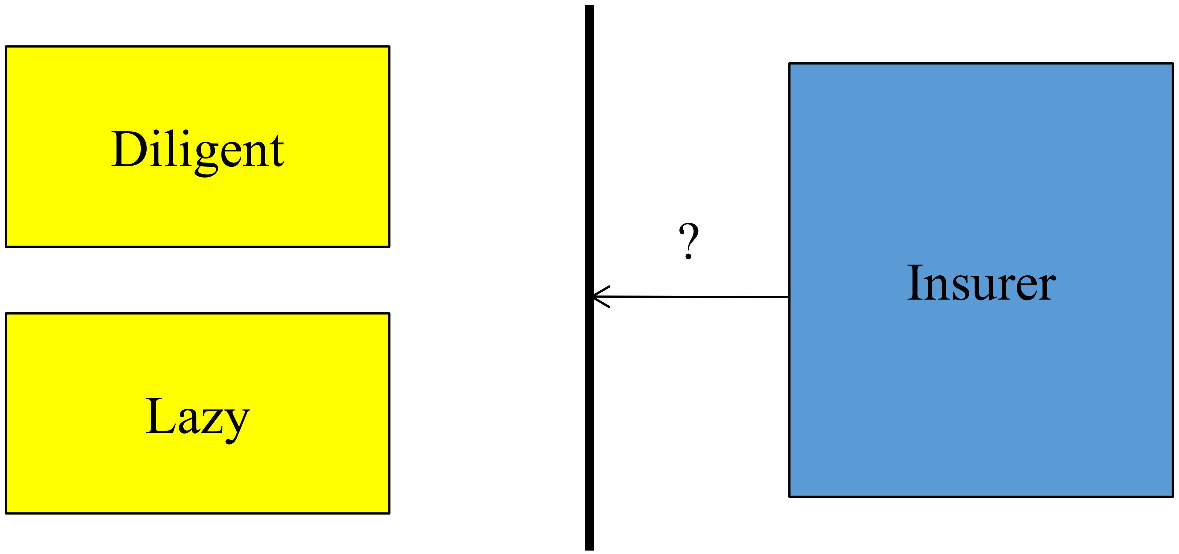
Basic Model of Insurance Demand
Adverse selection



Utility Theory and Uncertainty
Moral Hazard

1. Incentives

Individuals are different in their costs of exerting effort to avoid losses.



What if insurers cannot observe individuals' level of effort?

Utility Theory and Uncertainty
Moral Hazard

Example:

- Individual‘s probability of loss: 0.1
- Diligent individuals’ cost of effort: 100 for10% reduction of loss prob.
- Lazy individuals never exert effort to reduce loss prob.
- Loss amount: 200K
- Initial wealth: 500K
- $U(w)=\ln(w)$

What would happen if loss is fully covered?

Utility Theory and Uncertainty
Moral Hazard

2. Optimal contract

If full coverage is offered, **diligent Individuals:**

Without Effort

$EU(w) = 0.1\ln(300 + 0.9(200K)) + 0.9\ln(500 - 0.1(200K)) = 6.1738$

With Effort

$EU(w) = (0.9*0.1)\ln(300-0.1+0.9(200K))$
$+ (1-0.9*0.1)\ln(500 - 0.1 - 0.1(200K)) = 6.1736$

$\Rightarrow$ **Diligent individuals choose not exerting effort**

If coverage is \$180K, **diligent Individuals:**

Without Effort

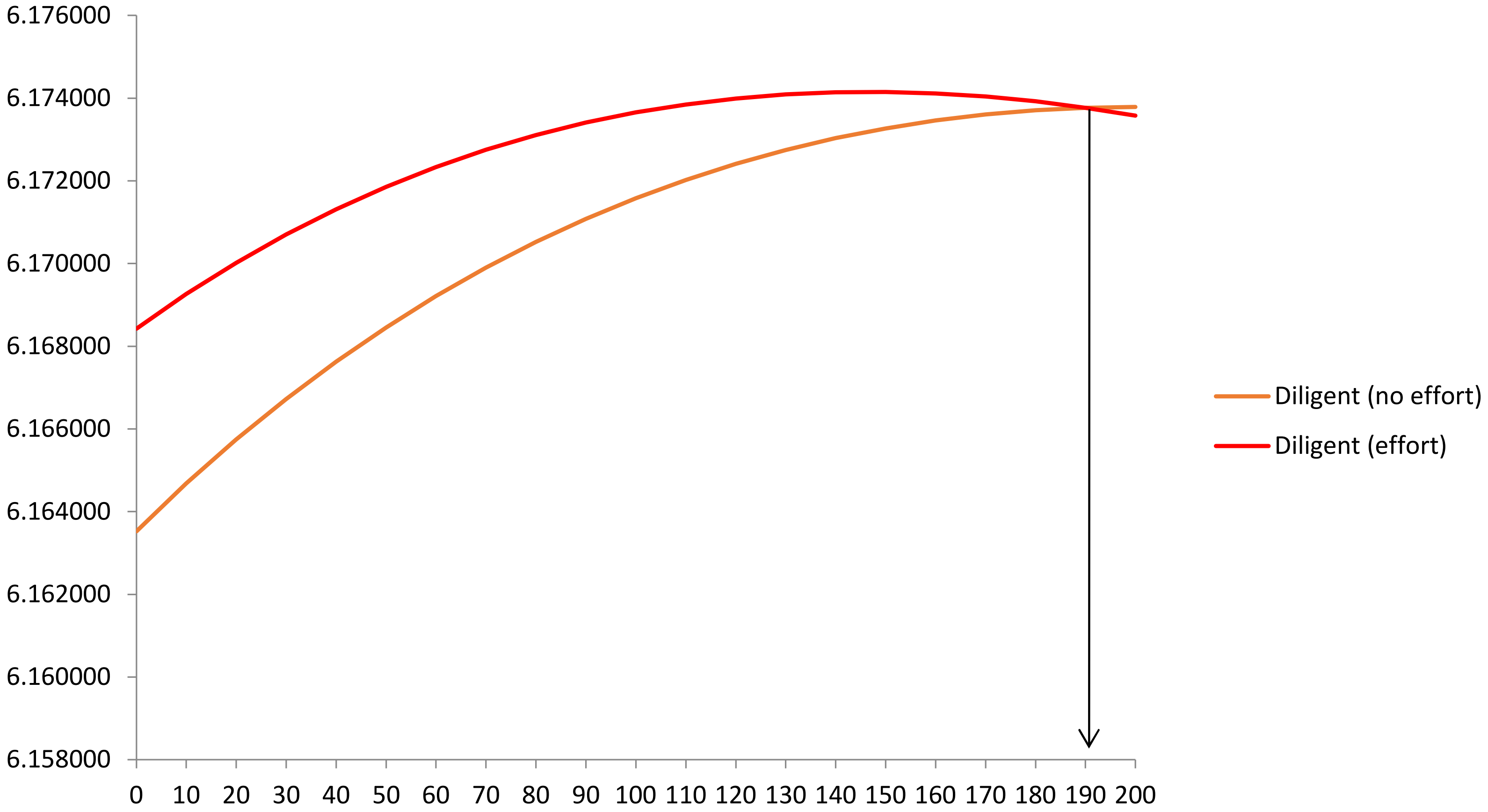
$EU(w) = 0.1\ln(300 + 0.9(180K)) + 0.9\ln(500 - 0.1(180K)) = 6.1737$

With Effort

$EU(w) = (0.9*0.1)\ln(300-0.1+0.9(180K))$
$+ (1-0.9*0.1)\ln(500 - 0.1 - 0.1(180K)) = 6.1739$

$\Rightarrow$ **Diligent individuals choose exerting effort**

Utility Theory and Uncertainty
Moral Hazard



Utility Theory and Uncertainty
Moral Hazard

Change in one party's incentives after a contract agreement is made.

- Insurance (private insurance, deposit insurance, health insurance etc)
- Bank bailout
- Too big to fail
- Subprime mortgage crisis

Utility Theory and Uncertainty
Measuring risk aversion

How does risk aversion change with wealth?

- Absolute risk aversion (ARA)

$$A(w) = \frac{-U''(w)}{U'(w)}$$

→Investors who hold an increasing (decreasing) absolute amount of wealth in risky assets as they get wealthier exhibit declining (increasing) absolute risk aversion. In practice, decreasing absolute risk aversion is often assumed.

- Relative risk aversion (RRA)

$$R(w) = -w\frac{U''(w)}{U'(w)}$$

→Investors who hold an increasing (decreasing) proportion of their wealth in risky assets as they get wealthier exhibit declining (increasing) relative risk aversion.

	Absolute risk aversion	Relative risk aversion
Increasing	$A'(w) > 0$	$R'(w) > 0$
Constant	$A'(w) = 0$	$R'(w) = 0$
Decreasing	$A'(w) < 0$	$R'(w) < 0$

Utility Theory and Uncertainty

Measuring risk aversion

- The quadratic utility function: $U(w) = w + dw^2$

$$U'(w) = 1 + 2dw; \;\; U"(w) = 2d$$

Therefore, non-satiation condition imposes $-\infty < w < -\frac{1}{2d}$ and risk averse utility requires $d < 0$.

$$A(w) = \frac{-2d}{1+2dw} \Rightarrow A'(w) = \frac{4d^2}{(1+2dw)^2} > 0$$

$$R(w) = \frac{-2dw}{1+2dw} \Rightarrow R'(w) = \frac{-2d}{(1+2dw)^2} > 0$$

- The log utility function: $U(w) = \log w$

$$U'(w) = \frac{1}{w}; \;\; U"(w) = -\frac{1}{w^2}$$

Both non-satiation and risk averse utility conditions are satisfied for $w>0$.

$$A(w) = \frac{1}{w} \Rightarrow A'(w) = -\frac{1}{w^2} < 0$$

$$R(w) = 1 \Rightarrow R'(w) = 0$$

→ This is consistent with an investor who keeps a constant proportion of wealth invested in risky assets as they get richer.

Utility Theory and Uncertainty
Limitations

- Need to know the precise form and shape of the individual's utility function.
- Cannot be applied separately to each of several sets of risky choices.
- May not be possible to consider a utility function for the firm.