Solution for Question 1

(a) The expected return and variance of a portfolio is defined by a proportion of wealth invested in Asset X, w_X.

$$E[r_p] = w_X E[r_X] + (1 - w_X)E[r_Y] = 0.06w_X + 0.08(1 - w_X) = 0.08 - 0.02w_X$$

$$\begin{aligned} V(r_p) &= w_X^2\sigma_X^2 + w_Y^2\sigma_Y^2 + 2w_X\, w_Y\, \sigma_X\, \sigma_Y\, \rho \\ &= .0004w_X^2 + .0025(1 - w_X)^2 + .001w_X\,(1 - w_X) \\ &= .0004w_X^2 + .0025 - .005w_X + .0025w_X^2 + .001w_X\; - .001w_X^2 \\ &= .0025 - .004w_X + .0019w_X^2 \end{aligned}$$

Substituting those into the utility function:

$$E[U] = E[r_i] - \alpha Var(r_i) = 0.08 - 0.02w_X - \alpha(.0025 - .004w_X + .0019w_X^2)$$

Differentiating and setting to zero:

$$\frac{dE[U]}{dw_X} = -0.02 - \alpha(-.004 + .0038w_X\,) = 0$$

$$\Rightarrow -.0038\alpha w_X \;= 0.02 - .004\alpha$$

$$\Rightarrow w_X \;= \frac{.004\alpha - 0.02}{.0038\alpha} = \frac{20\alpha - 100}{19\alpha} = \frac{20}{19} - \frac{100}{19\alpha}$$

Check that the second derivative is negative.

(b) Differentiating w_X w.r.t α:

$$\Rightarrow \frac{dw_X}{d\alpha} = \frac{100}{19\alpha^2} > 0$$

This shows that as α increases, the proportion of wealth invested in Asset X increases too. This relationship is explained by the fact that the investor is risk averse in that his expected utility is reduced by the variance of returns and the size of α determines the degree of risk aversion.

Solution for Question 2

(a) Non-satiation requires $U'(w)>0$. So $0<w<3/2$.

(b) Differentiating gives $U''(w)=-2$. Hence:

- $A(w) = \frac{2}{3-2w} > 0 \implies A'(w) = \frac{4}{(3-2w)^2} > 0$
- $R(w) = \frac{2w}{3-2w} > 0 \implies R'^{(w)} = \frac{2}{3-2w} + \frac{4w}{(3-2w)^2} = \frac{6}{(3-2w)^2} > 0$

Therefore, as the investor's wealth decreases:

- The absolute amount of his investment in risky assets will increase (as his absolute risk aversion decreases as his wealth decreases).
- The proportion of his wealth invested in risky assets will increase (as his relative risk aversion decreases as his wealth decreases).

(c) Integration gives $U(w) = a + 3w - w^2$ where a is a constant which can be ignored. Comparing the expected utilities of investments:

- $EU_X = 0.3U(0.1) + 0.4U(0.3) + 0.3U(0.5) = 0.786$
- $EU_Y = 0.3U(0) + 0.2U(0.2) + 0.5U(0.9) = 1.057$
- $EU_Z = 0.45U(0.2) + 0.1U(0.3) + 0.45U(0.4) = 0.801$

Therefore the investor should choose Investment Y which has the largest expected utility.

Solution for Question 3

(a) The maximum premium P is found by solving:

$$U(140000 - P) = E[U(140000 - X)]$$

$$\implies \sqrt{140000 - P} = .99\sqrt{140000} + .01\sqrt{40000} = 372.424$$

$$\implies P = 140000 - 372.424^2 = 1300$$

Thus, the maximum premium is \$1300, which exceeds the expected loss of \$1000.

(b) The minimum premium Q is found by solving

$$U(100m) = E[U(100m + Q - X)]$$

$$\implies 100m = .99(100m + Q) + .01(99{,}900{,}000 + Q) = 99{,}999{,}000 + Q$$

$$\implies Q = 1{,}000$$

Thus, the minimum premium required by the insurer is less than the maximum premium the investor is willing to pay, meaning that the contract is feasible.

Solution for Question 4

(a) Non-satiation requires $U'(w)>0$. So $w>0$.

(b) Differentiating gives $U''(w)=-1/w^2$. Hence:

- $A(w) = \frac{1/w^2}{1/w} = \frac{1}{w} \Rightarrow A'(w) = -\frac{1}{w^2} < 0$
- $R(w) = 1$

Therefore, as the investor's wealth decreases:

- The absolute amount of his investment in risky assets will decrease (as his absolute risk aversion increases as his wealth decreases).
- The proportion of his wealth invested in risky assets does not change (as his relative risk aversion is constant).

(c) Integrating U' gives:

$$U(w) = \ln w \quad \text{for } w > 0$$

where a constant is irrelevant here.

Her expected utility is

$$.75U(2) + .25U(1) = .75\ln(2) + .25\ln(1) = 0.51986$$

The fixed gain is defined where:

$$U(x) = EU = 0.51986$$
$$\Rightarrow \ln(w) = 0.51986 \Rightarrow w = 1.6818$$

Therefore, the minimum gain is 1.68.

Solution for Question 5

(a) The risk premium can be determined by finding certainty equivalent where $U(CE) = EU(W)$. From given information,

$$\sqrt{CE} = 0.1\sqrt{100K - 50K} + 0.9\sqrt{100K}$$

Solving this in terms of CE, we have $CE = 94{,}228$. Thus the maximum premium is 5772. Since the expected loss is 5000, the risk premium is 772.

(b) The optional insurance coverage is determined where the individual's expected utility is maximized. The objective function is:

$$EU = 0.1\sqrt{50K + 0.89I} + 0.9\sqrt{100K - 0.11I}$$

Solving the FOC in terms of I,

$$\frac{\partial EU}{\partial I} = 0.089(50K + 0.89I)^{-1/2} - 0.099(100K - 0.11I)^{-\frac{1}{2}} = 0$$

$$\Longrightarrow \frac{100K - 0.11I}{50K + 0.89I} = \left(\frac{0.099}{0.089}\right)^2 = 1.23734$$

$$\Longrightarrow 100K - 0.11I = 1.23734(50K + 0.89I) \Longrightarrow 1.21123I = 38{,}133$$

Therefore,

$$I^* = 31{,}483$$

Check that the SOC is satisfied:

$$\frac{\partial^2 EU}{\partial I^2} = -0.5 \times 0.089 \times 0.89(50K + 0.89I)^{-\frac{3}{2}} - 0.5 \times 0.099 \times 0.11(100K - 0.11I)^{-\frac{3}{2}} < 0$$

(c) The absolute and relative risk aversion measures are defined as follows.

$$A(w) = \frac{-u''(w)}{u'(w)} = \frac{\frac{1}{4}w^{-3/2}}{\frac{1}{2}w^{-1/2}} = \frac{1}{2w} \Longrightarrow A'(w) = -\frac{1}{2w^2} < 0$$

$$R(w) = \frac{-wu''(w)}{u'(w)} = \frac{\frac{1}{4}w^{-3/2}}{\frac{1}{2}w^{-1/2}} = \frac{1}{2} \Longrightarrow R'(w) = 0$$

The ARA is decreasing, meaning that the individual prefers to hold an increasing amount of wealth in risky assets as wealth increases. In contrast, the RRA is constant, meaning that the individual prefers to hold a constant proportion of wealth in risky assets as wealth increases.

<u>Solution for Question 6</u>

(a) Her expected utility is

$$.75U(40000) + .25U(30000)$$
$$= .75(40000 - 10^{-5}(40000)^2) + .25(30000 - 10^{-5}(30000)^2)$$
$$= 23{,}250$$

The fixed income x is defined by:

$$U(x) = EU = 23{,}250$$
$$\Rightarrow x - 10^{-5}x^2 = 23{,}250 \Rightarrow 10^{-5}x^2 - x + 23{,}250 = 0$$

Solving the quadratic equation:

$$\frac{1 \pm \sqrt{1 - 4 \times 10^{-5} \times 23{,}250}}{2 \times 10^{-5}} = 63229 \textit{ or } 36771$$

Therefore, the minimum income is 36,771.

(b)

RRA is defined by: $R(w) = wA(w)$. Differentiating w.r.t w gives:

$$\frac{\partial R}{\partial w} = A + w\frac{\partial A}{\partial w}$$

This implies that if $\frac{\partial R}{\partial w} < 0$ then $\frac{\partial A}{\partial w}$ must be negative given w and $A(w)$ are both positive for risk-averse individual. Therefore Statement (a) holds.

In contrast, even if $\frac{\partial A}{\partial w} < 0$, it does not necessarily mean $\frac{\partial R}{\partial w} < 0$. It depends on the relative relationships of A, w, and $\frac{\partial A}{\partial w}$. E.g., if A is extremely large, $A + w\frac{\partial A}{\partial w}$ could be positive. Therefore, Statement (b) does not hold.