BA2202 Mathematics of Finance
Handout 2

1 Preparation

Geometric series are very useful tools in evaluating a series of payments. For a finite geometric series:

$$1 + r + r^2 + r^3 + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r} \tag{1}$$

for $r \neq 1$. If $|r| < 1$, the *infinite geometric series* converges:

$$1 + r + r^2 + r^3 + \cdots = \frac{1}{1 - r} \tag{2}$$

By substituting two geometric series, formulas introduced in this handout are derived. For the finite series,

$$S = 1 + r + r^2 + r^3 + \cdots + r^n$$
$$rS = r + r^2 + r^3 + \cdots + r^{n+1}$$

Taking the difference of these two series, $S - rS$, we obtain:

$$S(1 - r) = 1 - r^{n+1} \Rightarrow S = \frac{1 - r^{n+1}}{1 - r}$$

which is the formula for the finite series.

Example 1.1. *You invest X on January 1, 2011 to receive a payment of 20 every year starting from January 1, 2012. How much you must invest if the effective annual rate of interest is 3%?*

$$20(1.03)^{-1} + 20(1.03)^{-2} + 20(1.03)^{-3} + \cdots = \frac{20(1.03)^{-1}}{1 - 1.03^{-1}} = 666.67$$

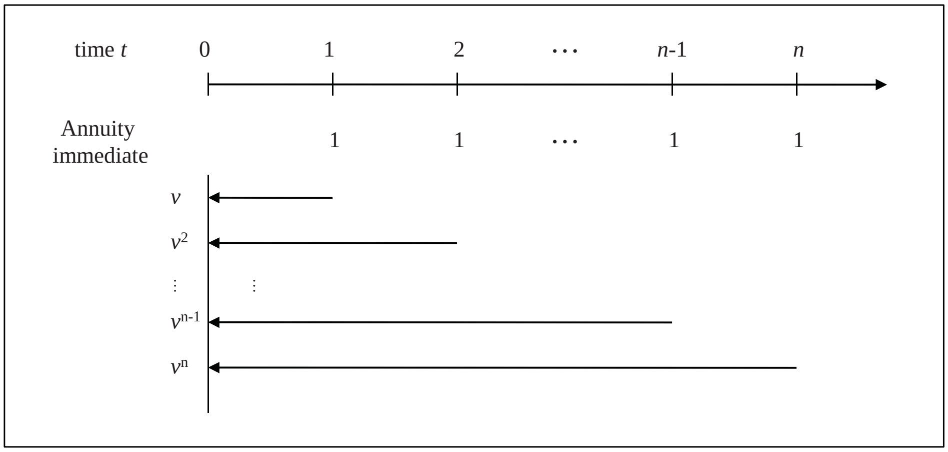
2 Annuities

In this section, we evaluate a series of periodic payments called *annuity*. The example includes regular payments from a pension plan and monthly insurance premium payments. An annuity with each regular payment of 1 is called *unit annuity*. If each payment is made at the end of the period, an annuity is *immediate* (*annuity immediate*). In contrast, if each payment is made at the beginning, an annuity is *due* (*annuity due*).

time t	0	1	2	$\cdots$	n-1	n
Annuity immediate		1	1	$\cdots$	1	1
Annuity due	1	1	1	$\cdots$	1	

2.1 Annuity Immediate

The PV of a unit annuity immediate with n payments and interest rate i is denoted by $a_{\overline{n}|i}$ (read a-angle-n-at-i) or $a_{\overline{n}|}$ when the interest rate is understood.

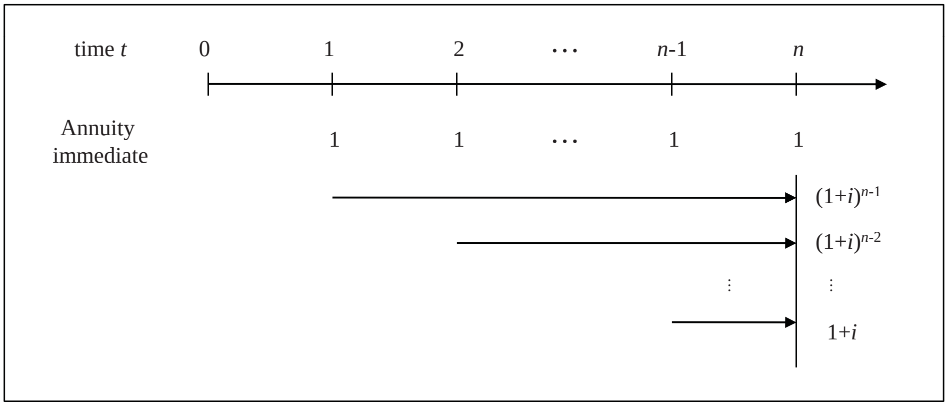


$$a_{\overline{n}|} \begin{aligned}[t] &= v + v^2 + v^3 + \cdots + v^n \\ &= v[1 + v + v^2 + \cdots + v^{n-1}] \\ &= v\tfrac{1-v^n}{1-v} \\ &= \tfrac{1-v^n}{i} \end{aligned} \tag{3}$$

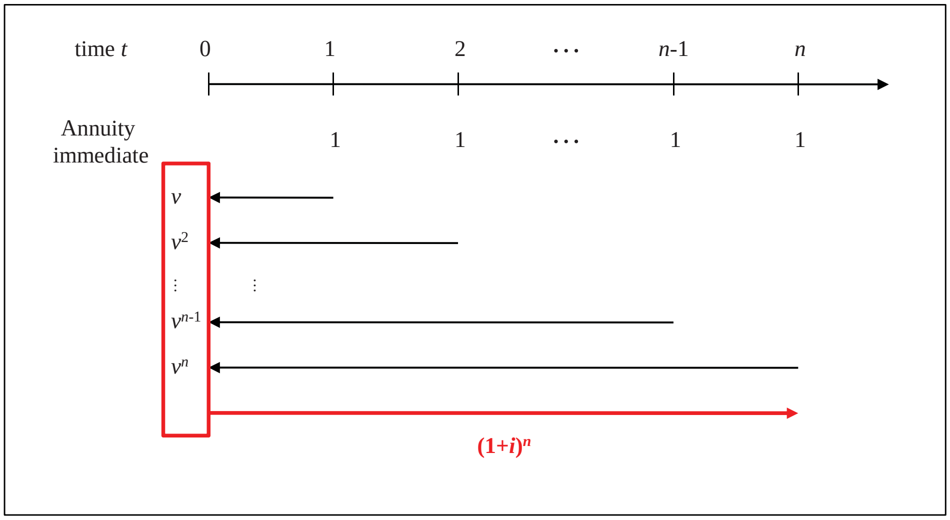
The last equality holds because $1 - v = iv$. Thus, we have the useful formula:

$$a_{\overline{n}|} = \frac{1 - v^n}{i} \tag{4}$$

The FV of a unit annuity immediate with n payments denoted by $s_{\overline{n}|}$ is defined as follows:



$$s_{\overline{n}|} \begin{aligned}[t] &= (1+i)^{n-1} + (1+i)^{n-2} + \cdots + (1+i) + 1 \\ &= 1 + (1+i) + (1+i)^2 + \cdots + (1+i)^{n-1} \\ &= \tfrac{1-(1+i)^n}{1-(1+i)} \\ &= \tfrac{(1+i)^n - 1}{i} \end{aligned} \tag{5}$$



Thus, we have

$$s_{\overline{n}|} = \frac{(1+i)^n - 1}{i} \tag{6}$$

Using $v = \frac{1}{1+i}$, we can establish the relationship between the PV and the FV of the annuity as follows:

$$(1+i)^n a_{\overline{n}|} = s_{\overline{n}|} \tag{7}$$

$$v^n s_{\overline{n}|} = a_{\overline{n}|} \tag{8}$$

If regular payments continue infinitely, an annuity is called *perpetuity.* The PV of a unit perpetuity immediate is denoted by $a_{\overline{\infty}|}$.

$$\begin{aligned} a_{\overline{\infty}|} &= v + v^2 + v^3 + \cdots \\ &= v(1 + v + v^2 + \cdots) \\ &= v\tfrac{1}{1-v} \\ &= \tfrac{1}{i} \end{aligned} \tag{9}$$

Example 2.1. *Find the PV of an annuity immediate with 100 payable monthly for 5 years when the rate of interest is 3% per annum convertible monthly.*
The effective monthly interest rate is $3/12 = 0.25\%$*. Since the annuity pays for 5 years, the number of monthly payments is 60. Thus,*

$$100a_{\overline{60}|} = \frac{1 - 1.0025^{-60}}{0.0025} = 5565.2$$

Example 2.2. *Find the FV of an annuity immediate defined in the previous example.*

$$100s_{\overline{60}|} = \frac{1.0025^{60} - 1}{0.0025} = 6464.7$$

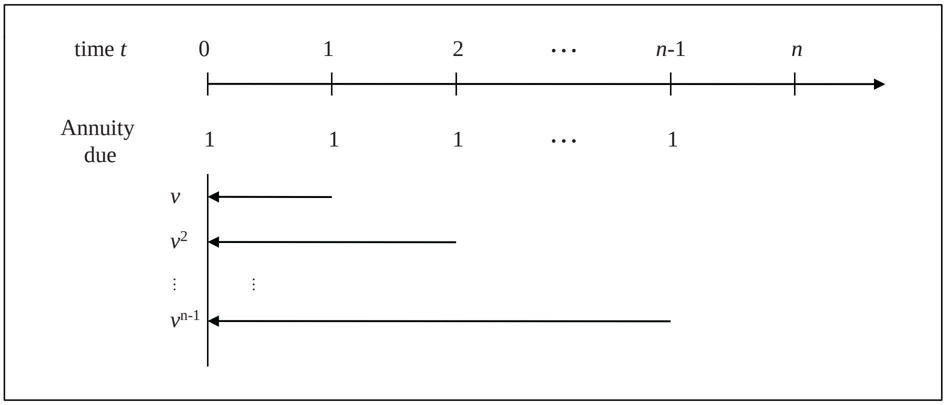
Example 2.3. *Find the PV of an annuity immediate payable 10 monthly forever if the interest rate is 3% convertible quarterly.*

$$(1+i)^{1/12} = \left[\left(1 + \frac{0.03}{4}\right)^4\right]^{1/12}$$

$$10 \times a_{\overline{\infty}|i_m} = \frac{10}{(1+\frac{0.03}{4})^{4/12}-1} = 4010.0$$

2.2 Annuity Due

The PV of a unit annuity due with n payments at interest rate i is denoted by $\ddot{a}_{\overline{n}|}$ (read a-double-dot-angle-n). It is easy to understand the value of annuity due if you know the difference in the payment schedule between annuity due and annuity immediate.

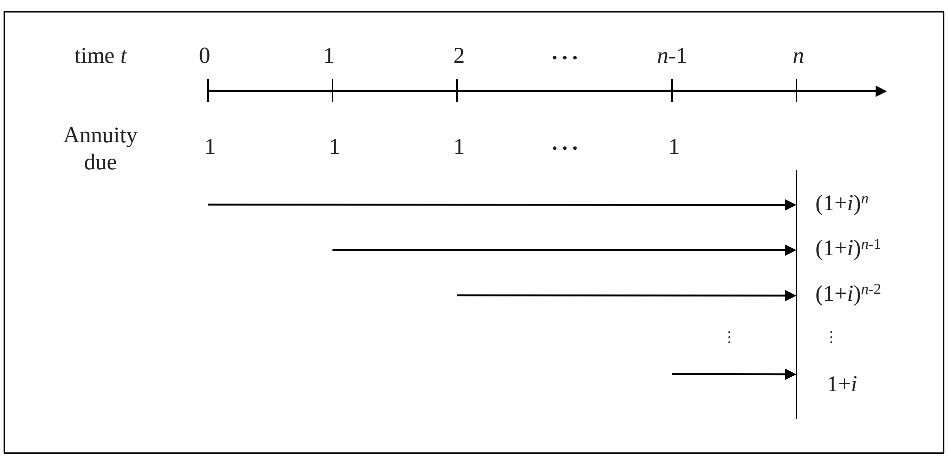


$$\begin{aligned} \ddot{a}_{\overline{n}|} &= 1 + v + v^2 + \cdots + v^{n-1} \\ &= \tfrac{1-v^n}{1-v} \\ &= \tfrac{1-v^n}{d} \end{aligned} \tag{10}$$

The last equality holds because $1 - v = d$. Thus, we have

$$\ddot{a}_{\overline{n}|} = \frac{1-v^n}{d} \tag{11}$$

The accumulated value of a unit annuity due with n payments is denoted by $\ddot{s}_{\overline{n}|}$.



$$\begin{aligned}\ddot{s}_{\overline{n|}} &= (1+i)^n + (1+i)^{n-1} + \cdots + (1+i) \\ &= (1+i)\frac{1-(1+i)^n}{1-(1+i)} \\ &= (1+i)\frac{(1+i)^n-1}{i} \\ &= \frac{(1+i)^n-1}{d}\end{aligned} \tag{12}$$

The last equality holds because $i/(1+i) = d$. Thus, we have

$$\ddot{s}_{\overline{n|}} = \frac{(1+i)^n - 1}{d} \tag{13}$$

Again, we can establish the relationship between the PV and the FV of the annuity as follows:

$$(1+i)^n \ddot{a}_{\overline{n|}} = \ddot{s}_{\overline{n|}} \tag{14}$$

$$v^n \ddot{s}_{\overline{n|}} = \ddot{a}_{\overline{n|}} \tag{15}$$

The PV of a unit perpetuity due is denoted by $\ddot{a}_{\overline{\infty|}}$.

$$\begin{aligned}\ddot{a}_{\overline{\infty|}} &= 1 + v + v^2 + \cdots \\ &= \frac{1}{1-v} \\ &= \frac{1}{d}\end{aligned} \tag{16}$$

Between annuity immediate and annuity due, we have the following relationships:

$$\ddot{a}_{\overline{n|}} = \frac{i}{d} a_{\overline{n|}} \quad or \quad \ddot{a}_{\overline{n|}} = (1+i) a_{\overline{n|}} \tag{17}$$

$$\ddot{s}_{\overline{n|}} = \frac{i}{d} s_{\overline{n|}} \quad or \quad \ddot{s}_{\overline{n|}} = (1+i) s_{\overline{n|}} \tag{18}$$

$$\ddot{a}_{\overline{\infty|}} = \frac{i}{d} a_{\overline{\infty|}} \quad or \quad \ddot{a}_{\overline{\infty|}} = (1+i) a_{\overline{\infty|}} \tag{19}$$

There are other ways to express a unit annuity due in terms of a unit annuity immediate.

$$\begin{aligned}\ddot{a}_{\overline{n|}} &= 1 + v + v^2 + \cdots + v^{n-1} = 1 + a_{\overline{n-1|}} \\ \ddot{a}_{\overline{n|}} &= \tfrac{1}{v}(v + v^2 + \cdots + v^n) = (1+i) a_{\overline{n|}}\end{aligned} \tag{20}$$

$$\begin{aligned}\ddot{s}_{\overline{n|}} &= (1+i)^n + (1+i)^{n-1} + \cdots + (1+i) = s_{\overline{n+1|}} - 1 \\ \ddot{s}_{\overline{n|}} &= (1+i)[(1+i)^{n-1} + (1+i)^{n-2} + \cdots + (1+i) + 1] = (1+i) s_{\overline{n|}}\end{aligned} \tag{21}$$

Example 2.4. *Calculate $\ddot{a}_{\overline{20|}}$ and $\ddot{s}_{\overline{20|}}$ at 5% effective per annum.*

$$\ddot{a}_{\overline{20|}} = \frac{1 - 1.05^{-20}}{0.05/1.05} = 13.085$$

$$\ddot{s}_{\overline{20|}} = \frac{1.05^{20} - 1}{0.05/1.05} = 34.719$$

Example 2.5. *Find the PV of an annuity due with 100 payable for 10 years when the rate of interest is 3% per annum convertible semiannually. Also find the PV, assuming interest rate of 3% convertible monthly.*
When interest is convertible semiannually, the effective semiannual interest rate is $3/2 = 1.5\%$

$$1 + i = \left(1 + \frac{0.03}{2}\right)^2 = 1.015^2$$

$$100\ddot{a}_{\overline{10}|} = 100\left(\frac{1-1.015^{-20}}{\frac{1.015^2-1}{1.015^2}}\right) = 877.79$$

The effective monthly interest rate rate is $3/12 = 0.25\%$

$$1+i = \left(1+\frac{0.03}{12}\right)^{12} = 1.0025^{12}$$

$$100\ddot{a}_{\overline{10}|} = 100\left(\frac{1-1.0025^{-120}}{\frac{1.0025^{12}-1}{1.015^{12}}}\right) = 877.10$$

2.3 Continuous Annuities

A unit continuous annuity still pays 1 per period but spread the payment out continuously by paying $1dt$ in each time interval of dt. The PV of a unit continuous annuity from time 0 to n is denoted by $\bar{a}_{\overline{n}|}$.

$$\begin{aligned}\bar{a}_{\overline{n}|} &= \int_0^n v^t dt \\ &= \int_0^n e^{-\delta t} dt \\ &= [\tfrac{-e^{-\delta t}}{\delta}]_0^n \\ &= \tfrac{1-e^{-\delta n}}{\delta} \\ &= \tfrac{1-v^n}{\delta}\end{aligned} \tag{22}$$

The last equality holds because $\delta = ln(1+i)$. Thus, we have

$$\bar{a}_{\overline{n}|} = \frac{1-v^n}{\delta} = \frac{i}{\delta}a_{\overline{n}|} \tag{23}$$

Similarly, you can easily show that:

$$\bar{s}_{\overline{n}|} = \frac{(1+i)^n-1}{\delta} = \frac{i}{\delta}s_{\overline{n}|} \tag{24}$$

and

$$\bar{a}_{\overline{\infty}|} = \frac{1}{\delta} = \frac{i}{\delta}a_{\overline{\infty}|} \tag{25}$$

Example 2.6. *Find the PV of an annuity with 100 continuously payable for 10 years when the rate of interest is 3% per annum.*

$$100\bar{a}_{\overline{10}|} = 100\left(\frac{1-1.03^{-10}}{\ln(1.03)}\right) = 865.75$$

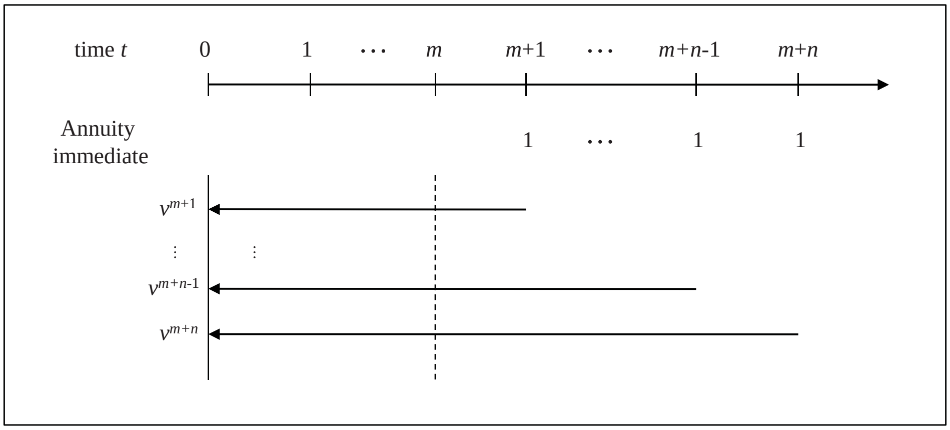
2.4 Deferred Annuities

A unit *deferred annuity* pays 1 per period but starts the payment in the future. The PV of a unit m-period deferred annuity immediate for which the first payment is to be made at time $m+1$ can be expressed by ${}_{m|}a_{\overline{n}|}$.

$$\begin{aligned}{}_{m|}a_{\overline{n}|} &= v^{m+1} + v^{m+2} + \cdots + v^{m+n} \\ &= v^m[v + v^2 + \cdots + v^n] \\ &= v^m a_{\overline{n}|}\end{aligned} \tag{26}$$

Also,

$$\begin{aligned}{}_{m|}a_{\overline{n}|} &= v^m \tfrac{1-v^n}{i} \\ &= \tfrac{(1-v^{m+n})-(1-v^m)}{i} \\ &= a_{\overline{m+n}|} - a_{\overline{m}|}\end{aligned} \tag{27}$$



The last equality illustrates that a deferred annuity can be evaluated by the difference of two immediate annuities. The relationship also implies:

$$a_{\overline{m+n}|} = a_{\overline{m}|} + v^m a_{\overline{n}|} \tag{28}$$

An annuity immediate with $m+n$ payments can be decomposed into a m-period annuity immediate and a m-period deferred immediate annuity with n payments.

We also define the corresponding deferred annuity due by ${}_{m|}\ddot{a}_{\overline{n}|}$.

$$\begin{aligned} {}_{m|}\ddot{a}_{\overline{n}|} &= v^m + v^{m+1} + \cdots + v^{m+n-1} \\ &= v^m \ddot{a}_{\overline{n}|} \end{aligned} \tag{29}$$

Also,

$$\begin{aligned} {}_{m|}\ddot{a}_{\overline{n}|} &= v^m \ddot{a}_{\overline{n}|} \\ &= v^m (1+i) a_{\overline{n}|} \\ &= v^{m-1} a_{\overline{n}|} \end{aligned} \tag{30}$$

It is also straightforward to evaluate the corresponding deferred annuity payable continuously denoted by ${}_{m|}\bar{a}_{\overline{n}|}$.

$$\begin{aligned} {}_{m|}\bar{a}_{\overline{n}|} &= \int_m^{m+n} e^{-\delta t} dt \\ &= e^{-\delta m} \int_0^n e^{-\delta t} dt \\ &= v^m \bar{a}_{\overline{n}|} \end{aligned} \tag{31}$$

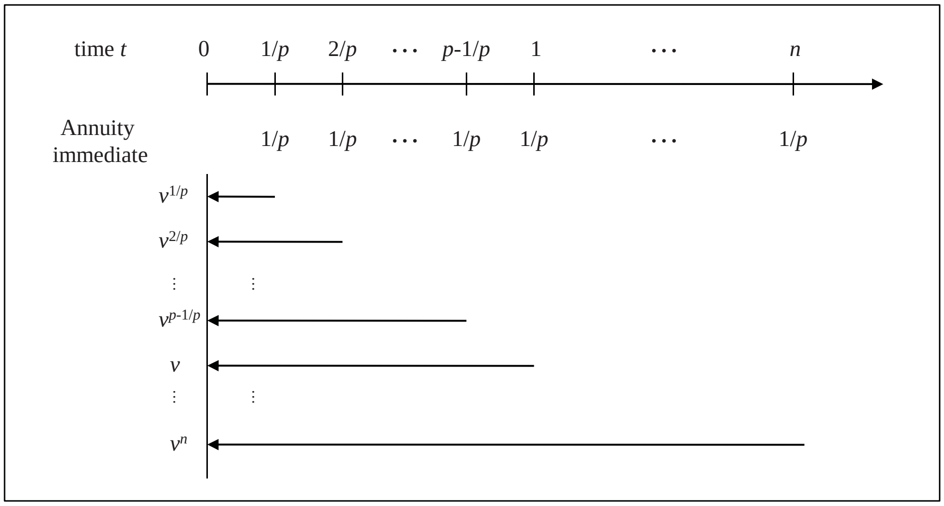
Also,

$$\begin{aligned} {}_{m|}\bar{a}_{\overline{n}|} &= \int_m^{m+n} e^{-\delta t} dt \\ &= \int_0^{m+n} e^{-\delta t} dt - \int_0^m e^{-\delta t} dt \\ &= \bar{a}_{\overline{m+n}|} - \bar{a}_{\overline{m}|} \end{aligned} \tag{32}$$

2.5 Annuities Payable $p$thly

2.5.1 The Present Value

There are cases where annuity payments are made more than once a year. We consider a n-period annuity payable $p$thly, which still pays a total amount of one in each period. Thus, each payment $1/p$ is made at times $1/p, 2/p, 3/p, \ldots, n$. The annuity is denoted by $a^{(p)}_{\overline{n}|}$.



$$a_{\overline{n}|}^{(p)} = \begin{aligned} &= \tfrac{1}{p}\textstyle\sum_{t=1}^{np} v^{t/p} \\ &= \tfrac{1}{p}\textstyle\sum_{t=1}^{np} (v^{1/p})^t \\ &= \tfrac{1}{p}[\tfrac{1-(v^{1/p})^{np+1}}{1-v^{1/p}} - 1] \\ &= \tfrac{1}{p}\tfrac{v^{1/p}(1-v^n)}{1-v^{1/p}} \end{aligned} \tag{33}$$

Note a finite geometric series:

$$r + r^2 + r^3 + \cdots + r^n = \frac{1-r^{n+1}}{1-r} - 1$$

The PV can be further simplified.

$$a_{\overline{n}|}^{(p)} \begin{aligned} &= \tfrac{1}{p}\tfrac{v^{1/p}(1-v^n)}{1-v^{1/p}} \\ &= \tfrac{1-v^n}{p[(1+i)^{1/p}-1]} \\ &= \tfrac{1-v^n}{i^{(p)}} \end{aligned} \tag{34}$$

It is beneficial to identify the relationship between a regular annuity immediate and an annuity payable $p$thly. Remember that the relationship between these annuities must hold:

$$a_{\overline{n}|}^{(p)} = \frac{1-v^n}{i^{(p)}} = \frac{i}{i^{(p)}} a_{\overline{n}|} \tag{35}$$

Likewise we define the PV of a n-period annuity due payable $p$thly. Each payment $1/p$ remains the same but the payments are made at times $0, 1/p, 2/p, 3/p, \ldots, (np-1)/p$. The annuity denoted by $\ddot{a}_{\overline{n}|}^{(p)}$ is

$$\ddot{a}_{\overline{n}|}^{(p)} = \frac{1-v^n}{d^{(p)}} = \frac{i^{(p)}}{d^{(p)}} a_{\overline{n}|}^{(p)} \tag{36}$$

The last equality implies:

$$\begin{aligned} a_{\overline{n}|}^{(p)} &= v^{1/p}\ddot{a}_{\overline{n}|}^{(p)} \\ \ddot{a}_{\overline{n}|}^{(p)} &= (1+i)^{1/p} a_{\overline{n}|}^{(p)} \end{aligned} \tag{37}$$

Example 2.7. *Find the PV of an annuity immediate payable quarterly for 5 years. The rate of interest is 3% per annum effective.*
By simply using the formula, we obtain

$$a^{(4)}_{\overline{5}|} = \frac{1-1.03^{-5}}{4(1.03^{0.25}-1)} = 4.6309$$

Example 2.8. *(Continued) If the rate of interest is 3% per annum convertible quarterly, find the PV of an annuity immediate payable quarterly for 5 years.*
By simply using the formula, we obtain

$$a^{(4)}_{\overline{5}|} = \frac{1-(1+0.03/4)^{-20}}{0.03} = 4.627$$

Example 2.9. *Find the PV of an annuity immediate payable monthly for 6 months (each payment is 1), assuming interest rate is 8% per annum convertible quarterly.*
From the effective quarterly interest rate $8/4 = 2\%$, first calculate $i^{(3)}$.

$$1+i = \left(1+\frac{i^{(p)}}{p}\right)^p \Rightarrow i^{(3)} = 3(1.02^{1/3}-1) = 0.019868$$

Since 6 monthly payments of one is converted to 2 quarterly payments of 3,

$$3a^{(3)}_{\overline{2}|} = 3\left(\frac{1-1.02^{-2}}{0.019868}\right) = 5.8633$$

Alternatively, you may work in monthly rates. Since interest rate of 8% per annum convertible quarterly is equivalent to the effective quarterly interest rate of 2%, corresponding monthly interest rate effective is:

$$1.02^{1/3}-1 = 0.00662271.$$

The PV of an annuity payable monthly for 6 month is:

$$a_{\overline{6}|} = \frac{1-1.02^{-2}}{1.02^{1/3}-1} = 5.8633$$

Example 2.10. *Find the PV of an annuity due with payable semiannually for 5 years (each payment is 1/2), assuming interest rate is 8% per annum convertible quarterly.*
An effective semiannual interest rate is:

$$1+i = \left(1+\frac{i^{(p)}}{p}\right)^p = 1.02^2.$$

The PV of an annuity due semiannually payable for 5 years is:

$$\frac{1}{2}\ddot{a}_{\overline{10}|} = \frac{1}{2}\left(\frac{1-1.02^{-20}}{\frac{1.02^2-1}{1.02^2}}\right)$$

2.5.2 The Future Value

The FV and the PV of an annuity hold the following relationship:

$$\begin{aligned} s_{\overline{n}|}^{(p)} &= (1+i)^n a_{\overline{n}|}^{(p)} \\ &= (1+i)^n \frac{i}{i^{(p)}} a_{\overline{n}|} \\ &= \frac{i}{i^{(p)}} s_{\overline{n}|} \\ &= \frac{(1+i)^n - 1}{i^{(p)}} \end{aligned} \tag{38}$$

Also,

$$\begin{aligned} \ddot{s}_{\overline{n}|}^{(p)} &= (1+i)^n \ddot{a}_{\overline{n}|}^{(p)} \\ &= (1+i)^n \frac{i}{d^{(p)}} a_{\overline{n}|} \\ &= \frac{i}{d^{(p)}} s_{\overline{n}|} \\ &= \frac{(1+i)^n - 1}{d^{(p)}} \end{aligned} \tag{39}$$

2.5.3 Deferred Annuities Payable $p$thly

It is straightforward to evaluate the corresponding deferred annuity. The PV of a unit m-period deferred annuity payable $p$thly denoted by ${}_{m|}a_{\overline{n}|}^{(p)}$ is

$$\begin{aligned} {}_{m|}a_{\overline{n}|}^{(p)} &= v^m a_{\overline{n}|}^{(p)} \\ {}_{m|}\ddot{a}_{\overline{n}|}^{(p)} &= v^m \ddot{a}_{\overline{n}|}^{(p)} \end{aligned} \tag{40}$$

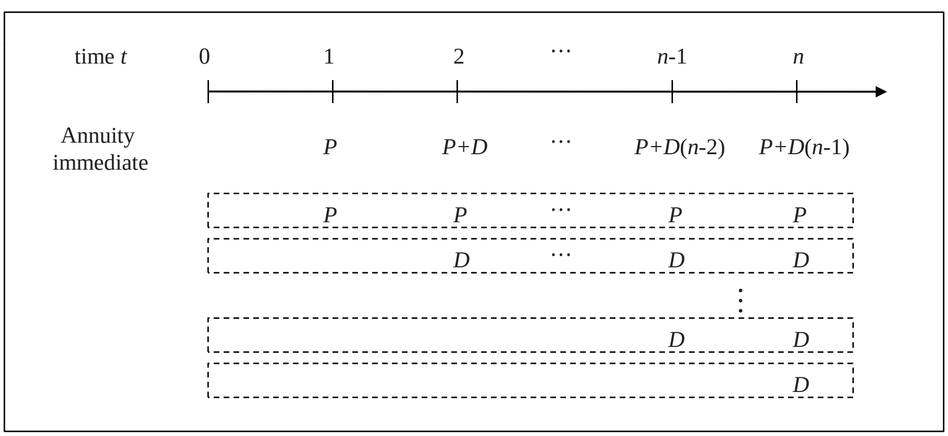
Also,

$$\begin{aligned} {}_{m|}a_{\overline{n}|}^{(p)} &= a_{\overline{m+n}|}^{(p)} - a_{\overline{m}|}^{(p)} \\ {}_{m|}\ddot{a}_{\overline{n}|}^{(p)} &= \ddot{a}_{\overline{m+n}|}^{(p)} - \ddot{a}_{\overline{m}|}^{(p)} \end{aligned} \tag{41}$$

2.6 Variable Annuities

2.6.1 Increasing Annuities with Terms in Arithmetic Progression

Now we allow that the amount of each payment varies over time. The first type is an annuity that n payments consist of the initial payment P and the jth payment $P + (j-1)D$ for $j = 1, 2, \ldots, n$. The PV of the variable annuity immediate is



$$\begin{aligned}
&vP + v^2(P+D) + v^3(P+2D) + \cdots + v^n[P+(n-1)D] \\
&= P(v + v^2 + v^3 + \cdots + v^n) + D[v^2 + 2v^3 + \cdots + (n-1)v^n] \\
&= Pa_{\overline{n}|} + D[v^2 + v^3 + \cdots + v^n] + D[v^3 + v^4 + \cdots + v^n] + \cdots + Dv^n \\
&= Pa_{\overline{n}|} + D(va_{\overline{n-1}|} + v^2 a_{\overline{n-2}|} + \cdots + v^n) \\
&= Pa_{\overline{n}|} + D\textstyle\sum_{j=1}^{n-1} v^j a_{\overline{n-j}|}
\end{aligned} \tag{42}$$

Further, the second term can be rewritten as:

$$\begin{aligned}
&Pa_{\overline{n}|} + D\textstyle\sum_{j=1}^{n-1} v^j a_{\overline{n-j}|} \\
&= Pa_{\overline{n}|} + D\textstyle\sum_{j=1}^{n-1} v^j \frac{1-v^{n-j}}{i} \\
&= Pa_{\overline{n}|} + D\frac{\sum_{j=1}^{n-1} v^j - (n-1)v^n}{i} \\
&= Pa_{\overline{n}|} + D\frac{\sum_{j=1}^{n} v^j - nv^n}{i} \\
&= Pa_{\overline{n}|} + D\frac{a_{\overline{n}|} - nv^n}{i}
\end{aligned} \tag{43}$$

A special case of $P = D = 1$ is called a n-period unit *increasing annuity.* The PV denoted by $(Ia)_{\overline{n}|}$ is defined as:

$$\begin{aligned}
(Ia)_{\overline{n}|} &= a_{\overline{n}|} + \frac{a_{\overline{n}|} - nv^n}{i} \\
&= \frac{(1+i)a_{\overline{n}|} - nv^n}{i} \\
&= \frac{\ddot{a}_{\overline{n}|} - nv^n}{i}
\end{aligned} \tag{44}$$

Similarly, it can be shown that the FV denoted by $(Is)_{\overline{n}|}$ is

$$(Is)_{\overline{n}|} = (1+i)^n (Ia)_{\overline{n}|} = \frac{\ddot{s}_{\overline{n}|} - n}{i} \tag{45}$$

With regard to the annuity due, we have the following relationship with corresponding annuity immediate. It is recommended to show the relationships.

$$(I\ddot{a})_{\overline{n}|} = \frac{i}{d}(Ia)_{\overline{n}|} = (1+i)(Ia)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{d} \tag{46}$$

and

$$(I\ddot{s})_{\overline{n}|} = \frac{i}{d}(Is)_{\overline{n}|} = (1+i)^n (I\ddot{a})_{\overline{n}|} = \frac{\ddot{s}_{\overline{n}|} - n}{d} \tag{47}$$

The following fact may help you to show the relationship.

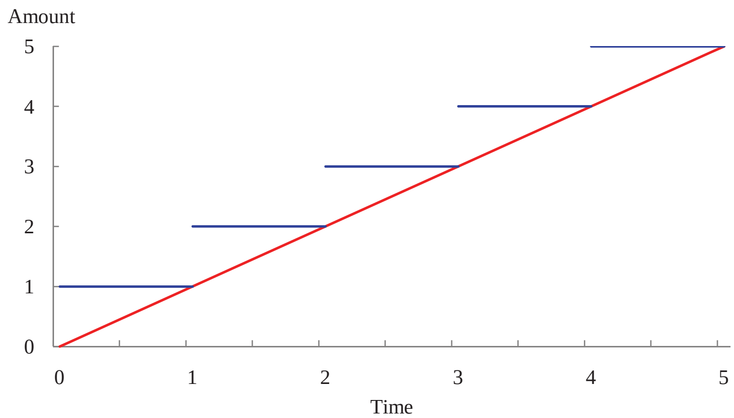
$$\begin{aligned}
(Ia)_{\overline{n}|} &= v + 2v^2 + 3v^3 + \cdots + nv^n \\
\Rightarrow (1+i)(Ia)_{\overline{n}|} &= 1 + 2v + 3v^2 + \cdots + nv^{n-1}
\end{aligned} \tag{48}$$

$(1+i)(Ia)_{\overline{n}|} - (Ia)_{\overline{n}|}$ implies

$$\begin{aligned}
i(Ia)_{\overline{n}|} &= 1 + v + v^2 + v^3 + \cdots + v^{n-1} - nv^n \\
&= \ddot{a}_{\overline{n}|} - nv^n
\end{aligned} \tag{49}$$

2.6.2 Continuous Payments

Now we consider two types of increasing annuities payable continuously. Those annuities are distinguished by the difference of the rate of payment. One type has a constant rate of payment t during tth period and another type pays a rate of payment t at time t. Thus, the payment of the former type is defined as a step function of time, and the payment of the latter type can be described by a linear function of time. Their PVs are denoted by $(I\bar{a})_{\overline{n}|}$ and $(\bar{I}a)_{\overline{n}|}$, respectively.



$$
\begin{aligned}
(I\bar{a})_{\overline{n}|} &= \textstyle\int_0^1 v^s ds + \int_1^2 2v^s ds + \cdots + \int_{n-1}^n nv^s ds \\
&= \textstyle\sum_{t=1}^n (\int_{t-1}^t tv^s ds) \\
&= \frac{\ddot{a}_{\overline{n}|} - nv^n}{\delta}
\end{aligned} \tag{50}
$$

Similarly, it can be shown that

$$(\bar{I}\bar{a})_{\overline{n}|} = \int_0^n tv^t dt = \frac{\bar{a}_{\overline{n}|} - nv^n}{\delta} \tag{51}$$

2.6.3 Decreasing Annuities with Terms in Arithmetic Progression

Next type is an annuity whose n payments consist of the initial payment P and the jth payment $P + (j-1)D$ for $j = 1, 2, \ldots, n$. Here $P = n$ and $D = -1$ is negative, though each payment $P + (j-1)D$ remains positive. The PV of the variable annuity immediate denoted by $(Da)_{\overline{n}|}$ is defined as:

$$(Da)_{\overline{n}|} = na_{\overline{n}|} - \frac{a_{\overline{n}|} - nv^n}{i} = \frac{n - a_{\overline{n}|}}{i} \tag{52}$$

Similarly, it can be shown that the FV denoted by $(Ds)_{\overline{n}|}$ is

$$(Ds)_{\overline{n}|} = \tfrac{n(1+i)^n - s_{\overline{n}|}}{i} \tag{53}$$

With regard to the annuity due, we have the following relationship with corresponding annuity immediate. No additional memorization is required, though it is recommended to show the deviations.

$$(D\ddot{a})_{\overline{n}|} = \frac{n - a_{\overline{n}|}}{d} = (1+i)(Da)_{\overline{n}|} \tag{54}$$

and

$$(Ds)_{\overline{n}|} = (1+i)^n (Da)_{\overline{n}|} \tag{55}$$

$$(D\ddot{s})_{\overline{n}|} = (1+i)^n (D\ddot{a})_{\overline{n}|} \tag{56}$$

As $n \to \infty$, the annuity formula converges to:

$$\lim_{n\to\infty} Pa_{\overline{n}|} + D\left(\frac{a_{\overline{n}|} - nv^n}{i}\right) = \frac{P}{i} + \frac{D}{i^2} \tag{57}$$

2.6.4 Increasing Annuities with Terms in Geometric Progression

Now annuity payments increases with a rate of growth of g. A geometrically increasing sequence with a rate of growth g for n terms is:

$$1, (1+g), (1+g)^2, \ldots, (1+g)^{n-1}$$

The PV of the variable annuity immediate is:

$$\begin{aligned} PV &= \tfrac{1}{1+i} + \tfrac{1+g}{(1+i)^2} + \tfrac{(1+g)^2}{(1+i)^3} + \cdots + \tfrac{(1+g)^{n-1}}{(1+i)^n} \\ &= \tfrac{1}{1+i}[1 + (\tfrac{1+g}{1+i}) + (\tfrac{1+g}{1+i})^2 + \cdots + (\tfrac{1+g}{1+i})^{n-1} \\ &= \tfrac{1}{1+i}\tfrac{1-(\frac{1+g}{1+i})^n}{1-\frac{1+g}{1+i}} \\ &= \tfrac{1-(\frac{1+g}{1+i})^n}{i-g} \end{aligned} \tag{58}$$

If $g < i$, the PV of the perpetuity is simply

$$\frac{1}{i-g} \tag{59}$$