BA2202 Mathematics of Finance
Handout 4

1 Loan Schedule

If a loan is repaid by regular repayments at a fixed rate of interest for a predetermined period, we can describe how a loan may be repaid. Consider the case where a bank lends a business owner L_0 for n years at the effective rate of interest i, in return for payment of X_t at the end of each year. The equation of value for the transaction can be described by:

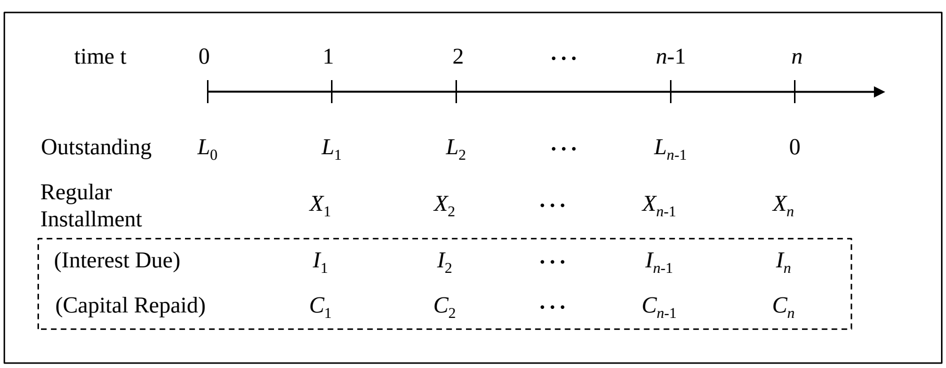
$$L_0 = vX_1 + v^2X_2 + \cdots + v^nX_n \tag{1}$$

When the regular installments are constant, the equation of value can be reduced to:

$$L_0 = Xa_{\overline{n}|} \Leftrightarrow X = \frac{L_0}{a_{\overline{n}|}} \tag{2}$$

Thus we learned how to find an unknown parameter by setting the equation of value. In this handout, more attention is paid to the dynamic aspects of such transaction. The following figure illustrates a loan schedule where a loan outstanding is reduced as periodic installments are paid.

- Loan outstanding at the beginning of the year: L_{t-1}
- Regular installment at the end of the year: X_t
- Interest due at the end of the year: I_t
- Capital repaid at the end of the year C_t
- Loan outstanding at the end of the year L_t



1.1 Loan Outstanding

Loan outstanding at time t, L_t, can be calculated both *prospectively* and *retrospectively*. The *prospective* method literally works forward and offers the equation of value where the loan outstanding at time t is the sum of discounted future repayment installments:

$$L_t = vX_{t+1} + v^2X_{t+2} + v^3X_{t+3} + \cdots + v^{n-t}X_n \tag{3}$$

To generate the equation, we may start with the fact that the loan outstanding at time n after the last repayment must be zero. Equivalently, the outstanding at time $n-1$ must be exactly the same as the repayment installment. Repeating the process backward until time t, we obtain the equation.

$$\begin{aligned}
L_n &= 0 \\
L_{n-1} &= v(X_n + L_n) = vX_n \\
L_{n-2} &= v(X_{n-1} + L_{n-1}) = vX_{n-1} + v^2 X_n \\
&\vdots \\
L_{t+1} &= v(X_{t+2} + L_{t+2}) = vX_{t+2} + v^2 X_{t+3} + \cdots + v^{n-t-1} X_n \\
L_t &= v(X_{t+1} + L_{t+1}) = vX_{t+1} + v^2 X_{t+2} + v^3 X_{t+3} + \cdots + v^{n-t} X_n
\end{aligned} \tag{4}$$

Example 1.1. *For a loan at 8% pa repayable in arrear with an initial payment of 100 and 19 further payments. The repayment amount increases by 5% each year. Calculate the loan outstanding immediately after the 4th repayment.*

The payments are: $100, 100(1.05), 100(1.05^2), 100(1.05^3), 100(1.05^4), \cdots, 100(1.05^{19})$

$$\begin{aligned}
L_4 &= vX_5 + v^2 X_6 + \cdots + v^{20-4} X_{20} \\
&= 100(1.05^4)v + 100(1.05^5)v^2 + \cdots + 100(1.05^{19})v^{20-4} \\
&= 100(1.05^4)v \left[1 + 1.05v + \cdots + (1.05v)^{15}\right] \\
&= 100(1.05^4)v \left[\tfrac{1-(1.05v)^{16}}{1-1.05v}\right]
\end{aligned} \tag{5}$$

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Example 1.2. *For a loan at 8% pa repayable at the end of each year by an initial payment of 100 and 19 further payments. The repayment amount increases by 10 each year. Calculate the loan outstanding immediately after the 4th repayment.*

The payments are: $100, 110, 120, \cdots, 290$

$$\begin{aligned}
L_4 &= vX_5 + v^2 X_6 + \cdots + v^{20-4} X_{20} \\
&= 140v + 150v^2 + \cdots + 290v^{20-4} \\
&= 130(v + v^2 + \cdots + v^{16}) + (10v + 20v^2 + \cdots + 160v^{16}) \\
&= 130a_{\overline{16}|} + 10(Ia)_{\overline{16}|} \\
&= 130\left(\tfrac{1-1.08^{-16}}{.08}\right) + 10\left(\frac{\frac{1-1.08^{-16}}{.08/1.08} - 16(1.08^{-16})}{.08}\right)
\end{aligned} \tag{6}$$

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For equal regular installments, the equation of value for the prospective method can be reduced to:

$$L_t = vX + v^2 X + v^3 X + \cdots + v^{n-t} X = Xa_{\overline{n-t}|} = \frac{L_0 a_{\overline{n-t}|}}{a_{\overline{n}|}} \tag{7}$$

The *retrospective* method works backward and offers the equation of value where the loan outstanding at time t is the FV at time t of the original loan outstanding less the sum of the FV at time t of paid installments:

$$L_t = (1+i)^t L_0 - \left[(1+i)^{t-1} X_1 + (1+i)^{t-2} X_2 + \cdots + (1+i) X_{t-1} + X_t\right] \tag{8}$$

To generate the equation, we consider that the loan outstanding at time t is amortized by which the repayment installment exceeds interest due at time t. Starting with time 1 and repeating the process forward until time t, we obtain the equation.

$$\begin{aligned}
L_1 &= (1+i)L_0 - X_1 \\
L_2 &= (1+i)^2 L_0 - (1+i)X_1 - X_2 \\
&\vdots \\
L_t &= (1+i)^t L_0 - \left[(1+i)^{t-1} X_1 + (1+i)^{t-2} X_2 + \cdots + (1+i) X_{t-1} + X_t\right]
\end{aligned} \tag{9}$$

For equal regular installments, the equation of value can be reduced to:

$$L_t = (1+i)^t L_0 - X\left[(1+i)^{t-1} + (1+i)^{t-2} + \cdots + (1+i) + 1\right] = (1+i)^t L_0 - X s_{\overline{t}|} \tag{10}$$

Furthermore, we can show that the outstanding is equivalent to one derived by the prospective method.

$$\begin{aligned} L_t &= (1+i)^t L_0 - \left(\tfrac{L_0}{a_{\overline{n}|}}\right)(1+i)^t a_{\overline{t}|} \\ &= (1+i)^t L_0 \left(1 - \tfrac{a_{\overline{t}|}}{a_{\overline{n}|}}\right) \\ &= (1+i)^t L_0 \left(\frac{v^t a_{\overline{n-t}|}}{a_{\overline{n}|}}\right) \\ &= \tfrac{L_0 a_{\overline{n-t}|}}{a_{\overline{n}|}} \end{aligned} \tag{11}$$

Example 1.3. *A bank offers a business owner a loan of \$10,000, which is repayable by equal annual installments at the end of each year for 5 years. Interest rate is 6% pa for the first 3 years and 8% pa thereafter. Calculate the loan outstanding immediately after the third repayment.*

We set up the following equation of value:

$$10,000 = X\left(a_{\overline{3}|6\%} + v^3 a_{\overline{2}|8\%}\right)$$

$$\Rightarrow X = \frac{10,000}{a_{\overline{3}|6\%} + v^3 a_{\overline{2}|8\%}} = \frac{10,000}{\frac{1-1.06^{-3}}{.06} + 1.06^{-3}\left(\frac{1-1.08^{-2}}{.08}\right)} = 2,397.92$$

The loan outstanding immediately after the third payment can be calculated by:

- Prospectively: $X a_{\overline{2}|8\%} = 2,397.92\left(\frac{1-1.08^{-2}}{.08}\right) = 4,276.13$
- Retrospectively: $(1+i)^3 L_0 - X s_{\overline{3}|6\%} = 10,000(1.06^3) - 2,397.92\left(\frac{1.06^3-1}{.06}\right) = 4,276.13$

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Example 1.4. *A business owner borrows a loan of \$10,000 from Bank A to be repaid by equal annual installments at the end of each year for 5 years at nominal rate of interest of 6% pa. After 2 installments Bank B offers the business owner a loan at 5% nominal pa to be repaid over the same period. If only associated cost of refinancing is a penalty of 1% of the loan outstanding to be paid to Bank A, should the business owner refinance the loan?*

The yearly installment paid to Bank A is:

$$X_A = \frac{L_0}{a_{\overline{5}|6\%}} = \frac{10,000(.06)}{1-1.06^{-5}} = 2,373.96$$

The loan outstanding after 2 payments is:

$$L_2 = X_A a_{\overline{3}|} = \frac{X_A(1-1.06^{-3})}{.06} = 6,345.63$$

The original principal of the refinancing with Bank B is:

$$(1+0.01)L_2 = 6,409.09$$

Hence the new monthly installment is:

$$X_B = \frac{6,409.09}{a_{\overline{3}|5\%}} = \frac{6,409.09(.05)}{1-1.05^{-3}} = 2,353.47$$

Since the new installment is less than that paid to Bank A, the business owner should refinance its loan.

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1.2 Amortization

Note that even though regular installments are constant every period, how the amount is spent in each period changes over time. In practice, we may be interested in finding interest payment and capital repayment separately for the tax treatment, and use the following calculation order:

1. Find loan outstanding at beginning of the year: L_{t-1}
2. Calculate interest due at the end of the year: $I_t = iL_{t-1}$
3. Calculate the amount that can be used for capital repayment: $C_t = X_t - I_t$
4. Calculate loan outstanding at end of the year: $L_t = L_{t-1} - C_t$

An important principle is that each repayment must pay first for interest due on the outstanding capital. If the amount of repayment is larger than the interest due, the remainder is used to repay the capital outstanding. Otherwise, the amount of loan outstanding is increased by the interest due that was not paid (Negative amortization).

Table 1: Loan Schedule

Year t	Installment X_t	Interest Paid I_t	Capital Paid C_t	Outstanding L_t
0	-	-	-	L_0
1	X_1	iL_0	$X_1 - I_1$	$L_1 = L_0 - C_1$
2	X_2	iL_1	$X_2 - I_2$	$L_2 = L_1 - C_2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
t	X_t	iL_{t-1}	$X_t - I_t$	$L_t = L_{t-1} - C_t$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	X_n	iL_{n-1}	$X_n - I_n$	$L_n = 0$

Example 1.5. *A bank offers a mortgage loan of \$500,000 at 6% effective pa for 20 years, which is repayable by monthly payments of \$2,000 at the end of each month for the first 5 years. What is the level payment after 5 years?*

We may work on monthly time unit, where the interest rate is $i_m = 1.06^{1/12} - 1 = 0.4868\%$. The equation of value for this transaction is:

$$\begin{aligned}
L_0 &= 2,000a_{\overline{60}|} + v^{60}X_2a_{\overline{180}|} \\
500,000 &= 2,000\left(\tfrac{1-0.004868^{-60}}{0.004868}\right) + 1.004868^{-60}X_2\left(\tfrac{1-0.004868^{-180}}{0.004868}\right) \\
\Rightarrow X_2 &= \frac{500,000-2,000\left(\frac{1-0.004868^{-60}}{0.004868}\right)}{1.004868^{-60}\left(\frac{1-0.004868^{-180}}{0.004868}\right)} = 4,428.24
\end{aligned} \tag{12}$$

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For level repayments, the equation of value can be reduced to $L_0 = Xa_{\overline{n}|}$. Consider a standardized transaction such that the loan principal is $L_0 = a_{\overline{n}|}$, where the level payment is one.

- Loan outstanding at beginning of the year: $L_{t-1} = a_{\overline{n-t+1}|}$
- Installment at the end of the year: $X = 1$

- Interest due at the end of the year: $I_t = ia_{\overline{n-t+1}|} = 1 - v^{n-t+1}$
- Capital repayment at the end of the year: $C_t = v^{n-t+1}$
- Loan outstanding at end of the year: $L_t = a_{\overline{n-t+1}|} - v^{n-t+1} = a_{\overline{n-t}|}$

Table 2: Loan Schedule

Year t	Installment X_t	Interest Paid I_t	Capital Paid C_t	Outstanding L_t
0	-	-	-	$a_{\overline{n}|}$
1	1	$1 - v^n$	v^n	$a_{\overline{n}|} - v^n = a_{\overline{n-1}|}$
2	1	$1 - v^{n-1}$	v^{n-1}	$a_{\overline{n-1}|} - v^{n-1} = a_{\overline{n-2}|}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
t	1	$1 - v^{n-t+1}$	v^{n-t+1}	$a_{\overline{n-t+1}|} - v^{n-t+1} = a_{\overline{n-t}|}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	1	$1 - v$	v	$a_{\overline{1}|} - v = 0$

Example 1.6. *A bank offers a loan of at 6% effective pa for 10 years, which is repayable by yearly payments of 1,000 at the end of each year. What is the interest payment and the capital repayment in the 6th installment?*

You may use:

- $I_t = ia_{\overline{n-t+1}|} = 1 - v^{n-t+1}$
- $C_t = v^{n-t+1}$

$$I_6 = 1000\left(1 - 1.06^{-(10-6+1)}\right) = 252.74$$

$$C_6 = 1000v^{10-6+1} = 1000(1.06^{-(10-6+1)}) = 747.26$$

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Example 1.7. *For a 8% loan repayable by yearly level payments at the end of each year, the capital repayment in the 2nd payment is 5,522.79. Find the capital repayment in the 4th payment.*

You may use: $C_t = v^{n-t+1}$

$$C_2 = Xv^{n-2+1} = 5321.89 \Rightarrow Xv^{n-1} = 5,522.79$$

$$C_4 = Xv^{n-4+1} = Xv^{n-1}v^{-2} = 5522.79(1.08^2) = 6,441.78$$

Or more simply, we may work with:

$$C_{t+k} = v^{n-(t+k)+1} = v^{n-t+1}v^{-k} = (1+i)^k C_t$$

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Example 1.8. *For a 6% loan repayable by yearly level payments at the end of each year, the capital repayment in the first installment is 5,321.89. Find the capital repayment in the 4th payment.*

You may use: $C_t = v^{n-t+1}$

$$C_1 = Xv^n = 5,321.89$$

$$C_4 = Xv^{n-4+1} = Xv^n v^{-3} = 5,321.89(1.06^3) = 6,338.46$$

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Example 1.9. *(A bank offers a business owner a loan of \$10,000, which is repayable by equal annual installments at the end of each year for 5 years. Interest rate is 6% pa for the first 3 years and 8% pa thereafter. Calculate the loan outstanding immediately after the third repayment.)*

We found that the equal annual installment is \$2,397.92 in the loan example. The loan schedule is illustrated in the tablE.

Table 3: Loan Schedule Example

Year t	Outstanding L_{t-1}	Installment X_t	Interest Paid I_t	Capital Paid C_t	Outstanding L_t
1	10,000	2,397.92	600.00	1,797.92	8,202.08
2	8,202.08	2,397.92	492.12	1,905.80	6,296.28
3	6,296.28	2,397.92	377.78	2,020.15	4,276.13
4	4,276.13	2,397.92	342.09	2,055.83	2,220.30
5	2,220.30	2,397.92	177.62	2,220.30	0

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It is easy to calculate interest paid and capital repaid over several installments. For instance, we may calculate those which correspond to a series of repayments from t to $t+n$ as follows:

1. Calculate L_{t-1} and L_{t+n}
2. Calculate $\sum_{k=t}^{t+n} C_k = L_{t-1} - L_{t+n}$, and
3. Calculate $\sum_{k=t}^{t+n} I_k = (n+1)X - (L_{t-1} - L_{t+n})$.

Note that the sum of installments must equal to the sum of the total capital repayment and the total interest payment.

Example 1.10. *(A bank offers a business owner a loan of \$10,000, which is repayable by equal annual installments at the end of each year for 5 years. Interest rate is 6% pa for the first 3 years and 8% pa thereafter. Calculate the loan outstanding immediately after the third repayment.)*

We found that the equal annual installment is \$2,397.92 and the loan outstanding immediately after the third payment is \$4,276.13. Find the total capital repaid and interest paid by the last two payments

The total capital repayment is:

$$\sum_{k=4}^{5} C_k = L_3 - L_5 = \$4,276.13$$

which is the same as the loan outstanding at the end of the third period. The total interest payment is:

$$\sum_{k=4}^{5} I_k = X_4 + X_5 - (L_3 - L_5) = \$2,397.92 \times 2 - \$4,276.13 = 519.71.$$

These results can be checked in the loan schedule table above.

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1.3 Installments Payable More Frequently Than Annually

In practice, many loans are repaid in quarterly, monthly, or even weekly installments. If repayment installments are payable $p$thly at an interest rate i pa effective, the equation of value for the loan transaction is modified to:

$$L_0 = v^{1/p} X_{1/p} + v^{2/p} X_{2/p} + v^{3/p} X_{3/p} + \cdots + v^n X_n$$

For level installments, the equation of value can be reduced to:

$$L_0 = pX_p a^{(p)}_{\overline{n}|i}$$

where X_p is the amount of instalment payable $p$thly. No new concept is introduced in this change of payment frequency.

Example 1.11. *A bank offers a business owner a loan of \$10,000, which is repayable by equal monthly installments at the end of each year for 5 years. Interest rate is 7% pa effective.*

(a) *Calculate the interest paid and the capital repaid in the fourth year.*

(b) *Calculate the interest paid and the capital repaid in the 20th installment.*

First, we set up the equation of value to find the monthly repayment:

$$10,000 = 12X_m a^{(12)}_{\overline{5}|}$$

$$\Rightarrow X_m = \frac{10,000}{12\left(\frac{1-1.07^{-5}}{12(1.07^{1/12}-1)}\right)} = 197.00$$

For (a), the loan outstanding immediately after the third payment can be calculated prospectively:

$$L_3 = 12X_m a^{(12)}_{\overline{2}|} = 12 \times 197 \times \frac{1-1.07^{-2}}{12(1.07^{1/12}-1)} = 4,409.59$$

The loan outstanding immediately after the fourth payment can be calculated prospectively:

$$L_4 = 12X_m a^{(12)}_{\overline{1}|} = 12 \times 197 \times \frac{1-1.07^{-1}}{12(1.07^{1/12}-1)} = 2,279.35$$

So the capital repaid in the fourth year is:

$$C_4 = L_3 - L_4 = 4,409.59 - 2,279.35 = 2,130.24$$

The interest paid in the fourth year is:

$$I_4 = 12X_m - C_4 = 12(197) - 2,130.24 = 233.75$$

Note that iL_3 does not provide the interest paid in the 4th year because the loan outstanding decreases after each monthly repayment.

For (b), we need to find the interest paid and the capital repaid for a single installment. The calculation is simple but it may be convenient to change a time unit to a monthly basis, where the monthly effective interest rate is $i_m = 1.07^{1/12} - 1$. Since we want to calculate I_{20} and C_{20}, L_{19} is necessary. Hence,

$$L_{19} = X_m a_{\overline{60-19}|} = X_m a_{\overline{41}|} = 197\left(\frac{1-1.07^{-41/12}}{1.07^{1/12}-1}\right) = 7,191.05$$

The interest paid in the 20th payment is:

$$I_{20} = i_m L_{19} = \left(1.07^{1/12} - 1\right) \times 7,191.05 = 40.66$$

The capital repaid in the 20th payment is:

$$C_{20} = X_m - I_{20} = 197 - 40.66 = 156.34$$

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1.4 Flat Rate and APR

Suppose a loan L_0 is repaid over n years by equal $p$thly repayments of X_p, then the total amount of installments is npX_p. The *flat rate of interest pa* is defined by:

$$F = \frac{npX_p - L_0}{nL_0} = \frac{(npX_p - L_0)/n}{L_0} \tag{13}$$

which means the average interest payment per year over a loan amount. In contrast, Annual Percentage Rate of charge (APR) is defined as the effective annual rate of interest, which is determined by solving the equation of value in terms of $i^{(p)}$:

$$L_0 = pX_p a_{\overline{n}|}^{(p)} \Rightarrow i = \left(1 + \frac{i^{(p)}}{p}\right)^p - 1$$

Example 1.12. *An individual borrows 1000 from a bank. If a loan is payable, in arrear, of 24 monthly installments. The flat rate of interest charged on the loan is 6% pa. Find the monthly installment and the APR on this loan.*

Using the definition of the flat rate of interest,

$$0.06 = \frac{24X_{12} - L_0}{2L_0} \Rightarrow X_{12} = \frac{2 \times 0.06 \times 1000 + 1000}{24} = 46.667$$

In order to find corresponding APR, we need to solve the equation of value in terms of $i^{(12)}$.

$$1000 = 12X_{12} a_{\overline{2}|}^{(12)} \Rightarrow a_{\overline{2}|}^{(12)} = \frac{1000}{12X_{12}} = 1.7857$$

By trial and error,

$$i \approx 0.1171$$

We may start estimating the effective rate of interest at $i = 2F$, which shows a rough relationship between the effective rate of interest and the flat rate of interest.

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Example 1.13. *A bank offers a mortgage loan of 200,000 at 7.2% nominal rate convertible monthly for 15 years payable of level monthly installments at the end of each month. The bank charges a service fee of 2% for the original principal. Find the APR on this loan.*

The monthly effective rate is 0.6%.

$$X_{12} = \frac{200,000(1.02)}{a_{\overline{180}|}} = \frac{200,000(1.02)(0.006)}{1 - 1.006^{-180}} = 1,856.5$$

In order to find corresponding APR, we need to solve the equation of value.

$$200,000 = 1,856.5 a_{\overline{180}|}$$

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