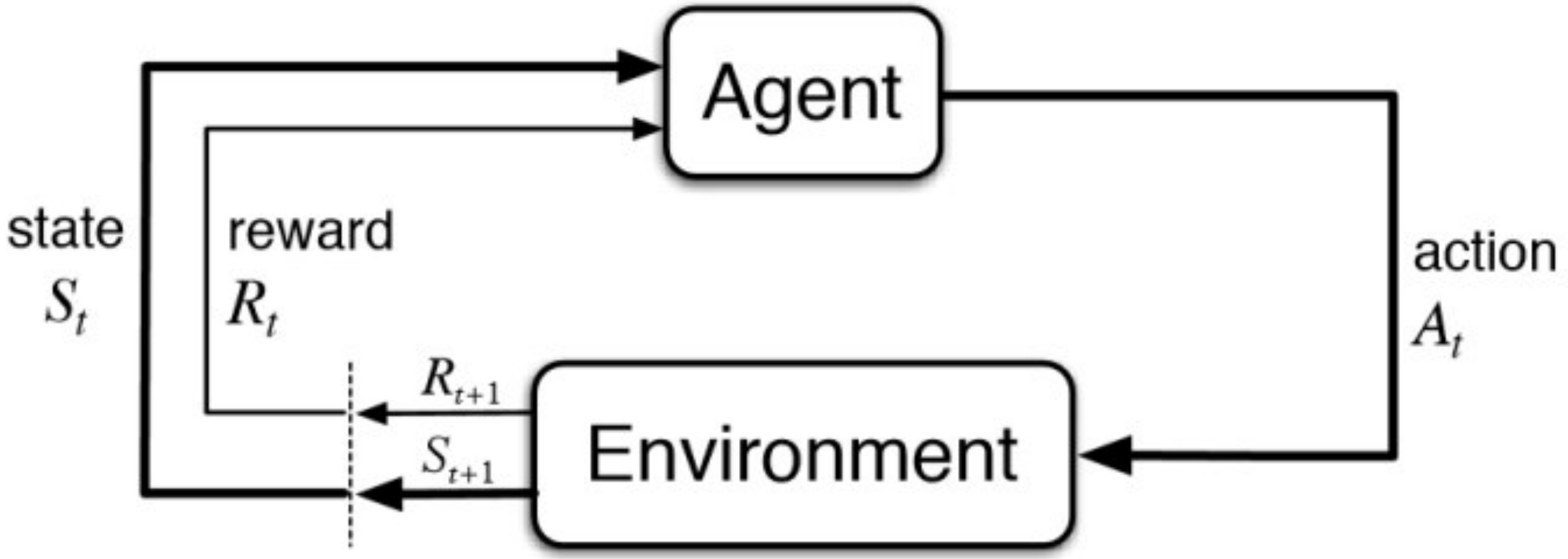
Lecture 19
Reinforcement Learning

Lei Li, **Yu-Xiang Wang**

An RL agent learns interactively through the feedbacks of an environment.



- Learning how the world works (dynamics) and how to maximize the long-term reward (control) at the same time.

Reinforcement learning problem setup

- State, Action, Reward and Observation

$$S_t \in \underline{\mathcal{S}} \quad A_t \in \underline{\mathcal{A}} \quad R_t \in \mathbb{R} \quad O_t \in \underline{\mathcal{O}}$$

- Policy: $\pi : \mathcal{S} \rightarrow \mathcal{A}$
  - When the state is observable:
  - Or when the state is not observable

$$\pi_t : (\mathcal{O} \times \mathcal{A} \times \mathbb{R})^{t-1} \rightarrow \mathcal{A}$$

- Learn the best policy that maximizes the expected reward
  - Finite horizon (episodic) RL: $\pi^* = \arg\max_{\pi \in \Pi} \mathbb{E}[\sum_{t=1}^{\cancel{T}\, H} R_t]$
  - Infinite horizon RL:

$$\pi^* = \arg\max_{\pi \in \Pi} \mathbb{E}[\sum_{t=1}^{\infty} \underline{\gamma^{t-1}} R_t]$$

~~T~~ H: horizon

$0 \leq \gamma < 1$

0.9, 0.99

γ: discount factor

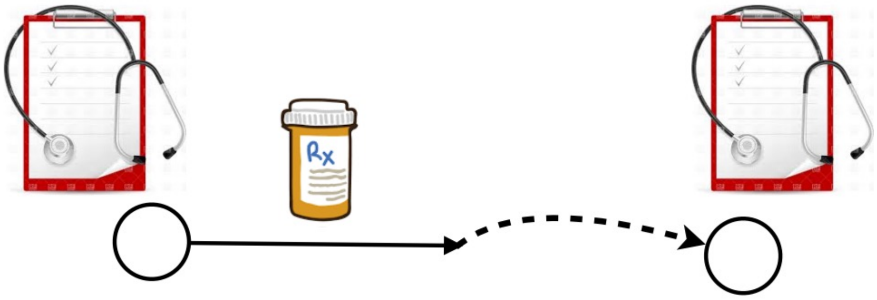
RL for robot control

- States: The physical world, e.g., location/speed/acceleration and so on.
- Observations: camera images, joint angles
- Actions: joint torques
- Rewards: stay balanced, navigate to target locations, serve and protect humans, etc.

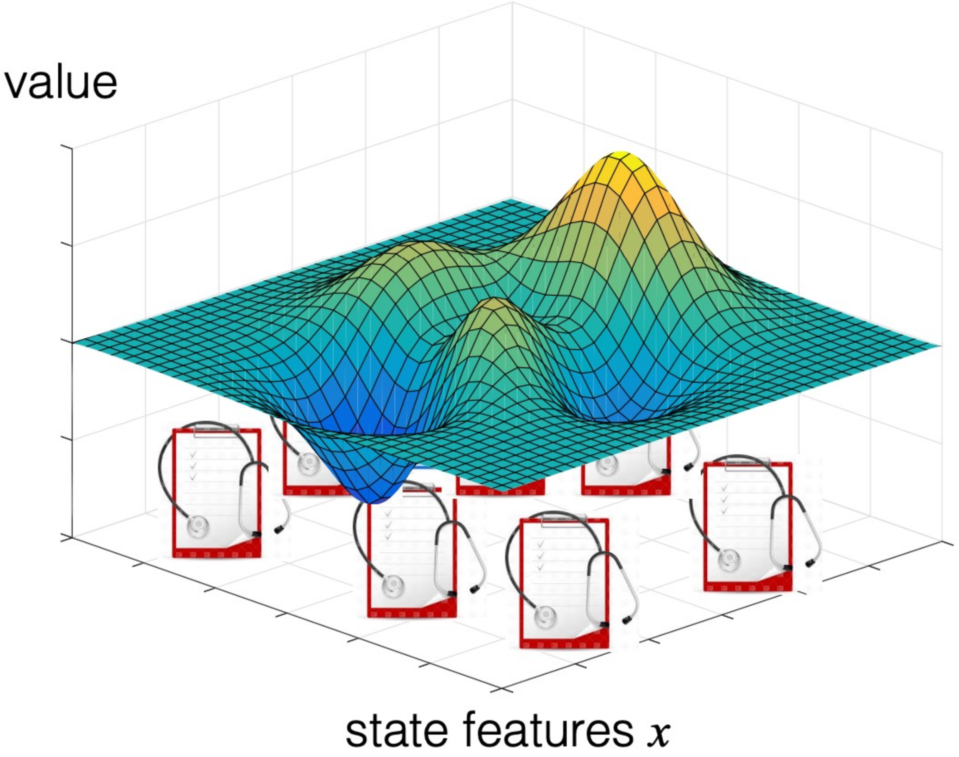
RL for Inventory Management

- State: Inventory level, customer demand, competitor’s inventory
- Observations: current inventory levels and sales history
- Actions: amount of each item to purchase
- Rewards: profit

RL for Adaptive medical treatment



- State: diagnosis
- Action: treatment
- Reward: progress in recovery



(example / illustration due to Nan Jiang)

Example: Supervised learning vs RL in movie recommendation

- Bob is described by a feature vector
  - s =[Previous movies watched / Rating / Written reviews]

- Supervised learning predicts how likely Bob will click on “aliens vs predators”

- Reinforcement learning aims at controlling Bob
  - So in the future, Bob will develop a taste for “aliens vs predators”   (e.g., from having watched “aliens” and “predators” both).

A broader view: Let’s consider a few other machine learning tasks

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- Hospitals need to decide **who to test** based on symptoms and other patient attributes

- Train a classifier on historic records to predict the test outcome.
- The accuracy is high on a holdout set!

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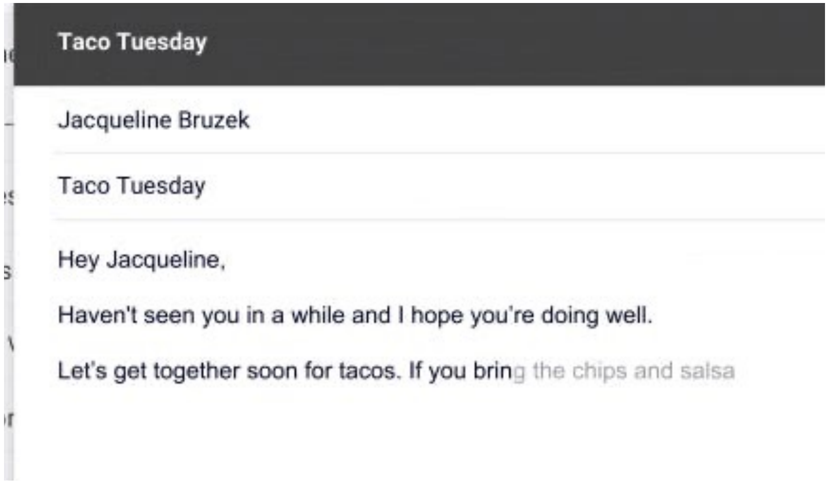
- Large tech wants to improve user experience on their popular email service



Taco Tuesday

Jacqueline Bruzek

Taco Tuesday

Hey Jacqueline,

Haven't seen you in a while and I hope you're doing well.

Let's get together soon for tacos. If you bring the chips and salsa

- Train a **large language model** with user data to **complete sentences**
- It seems to work great!

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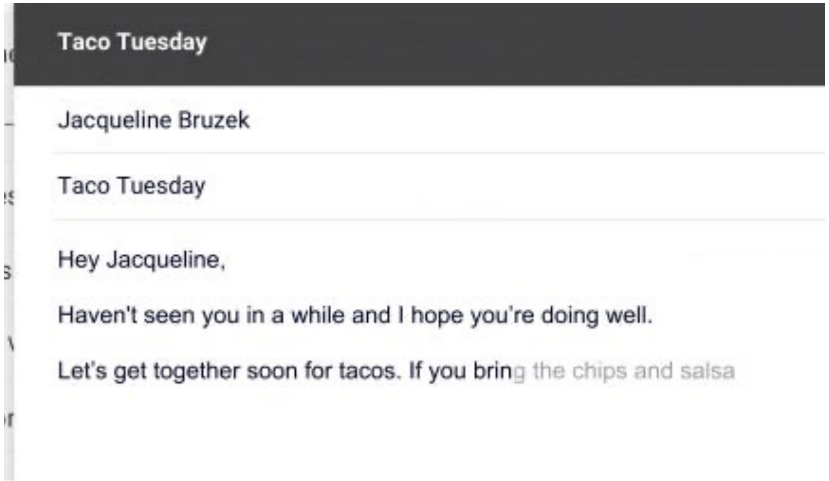

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What could go wrong?

Every machine learning problem is secretly a control (or RL) problem

- If I test patients using the new rule, the distribution of patients receiving the test will be different!

- Should I still trust my classifier?

- If I deploy the new "Guess what you will write" prompt, what users will enter may change!

- Is the model fulfilling its own prophecy?

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- Is the model fulfilling its own prophecy?

The ultimate goal is NOT prediction, but to:
minimize disease transmission / maximize user experience!

Reinforcement learning is very challenging

- The agent needs to:
  - Learn the state-transitions ----- How the world works
  - Learning the costs / rewards ----- Cost of actions
  - Learning how to search ----- Come up with a good strategy

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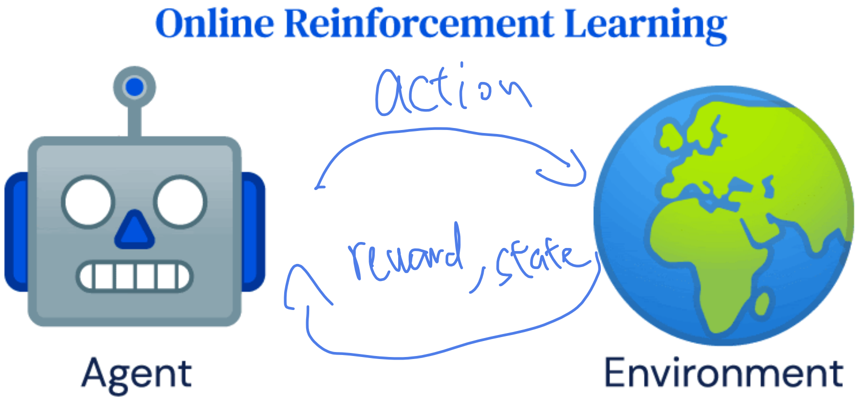
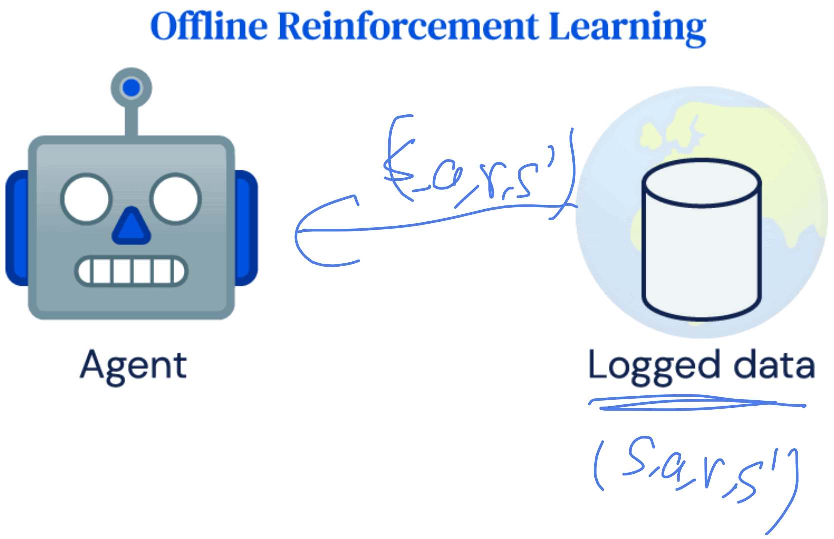
- All at the same time

Let us tackle different aspects of the RL problem one at a time

- **Markov Decision Processes: (this lecture)**
  - Dynamics are given no need to learn. planning only.
- RL algorithms (this lecture and the next)
  - Model-based RL vs Model-free RL
  - Temporal difference learning
  - Function approximation
- Exploration (final lecture if time permits)
  - Bandits: Explore-Exploit in simple settings
  - RL: Explore-Exploit in Learning MDPs

Online RL vs Offline RL

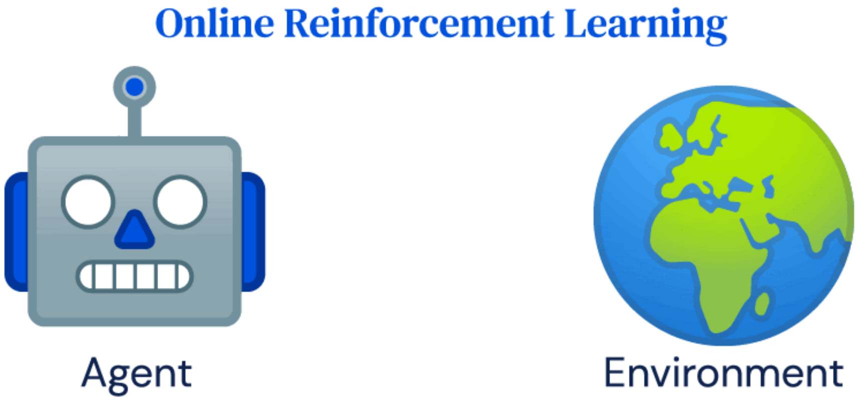




Exploration is often **expensive**, **unsafe**, **unethical** or **illegal** in practice, e.g., in self-driving cars, or in medical applications.

Can we learn a policy from already **logged interaction data**?

Online RL vs Offline RL



Offline Reinforcement Learning
Agent
Logged data

Exploration is often **expensive**, **unsafe**, **unethical** or **illegal** in practice, e.g., in self-driving cars, or in medical applications.

Can we learn a policy from already **logged interaction data**?

***Offline RL won't be covered, but it's an important problem**

Let's start by formulating Markov Decision processes (MDP).

- Infinite horizon / discounted setting

Transition kernel $P(s'|s,a)$

expected reward $r(s,a)$

$$\mathcal{M}(\mathcal{S}, \mathcal{A}, P, r, \gamma, \mu_0)$$

Transition kernel: $P: S \times A \to \Delta(S)$ i.e. $P(s'|s,a)$

(Expected) reward function: $r: S \times A \to \mathbb{R} / [0, R_{max}]$ $\quad \mathbb{E}[R_t \mid S_t = s, A_t = a] =: r(s,a)$

Initial state distribution $\mu_0 \in \Delta(S)$

Discounting factor: $0 \leq \gamma < 1$

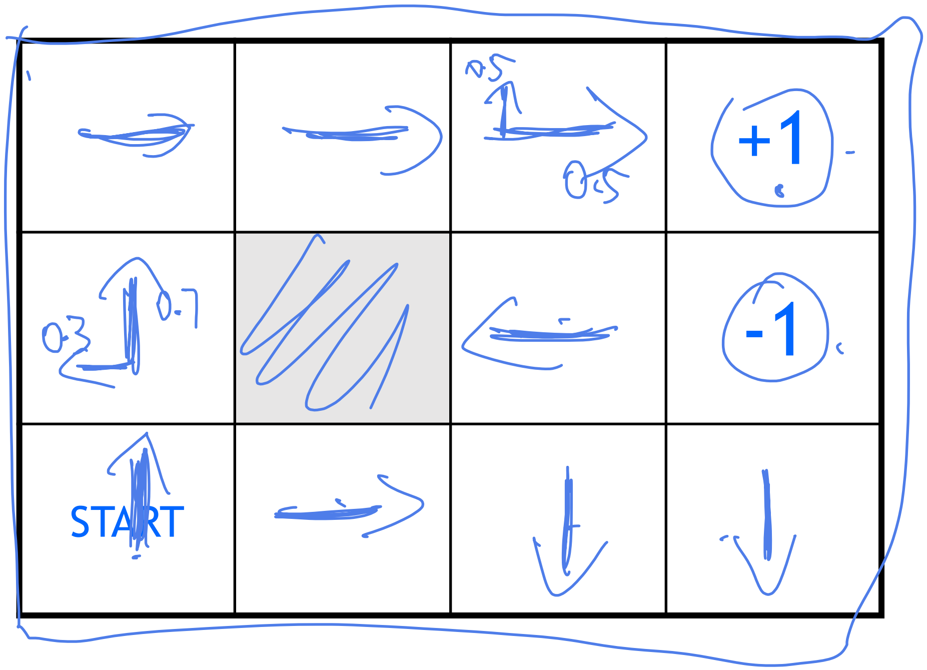
$A = |\mathcal{A}|$

$S = |\mathcal{S}|$

Example: Frozen lake.

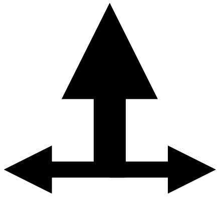
$\mu_0 = \mathbb{1}_{s_0}$

$s_0 = (1,1)$



actions: UP, DOWN, LEFT, RIGHT

UP e.g.,



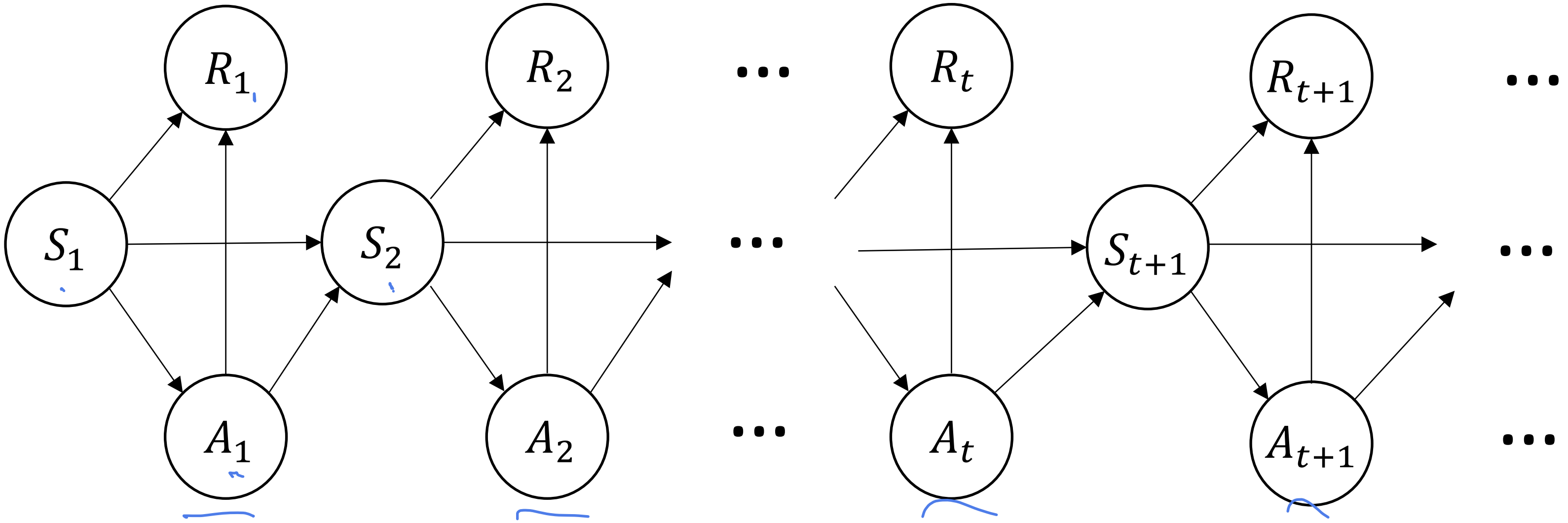
State-transitions with action **UP**:

80% move up
10% move left
10% move right

*If you bump into a wall, you stay where you are.

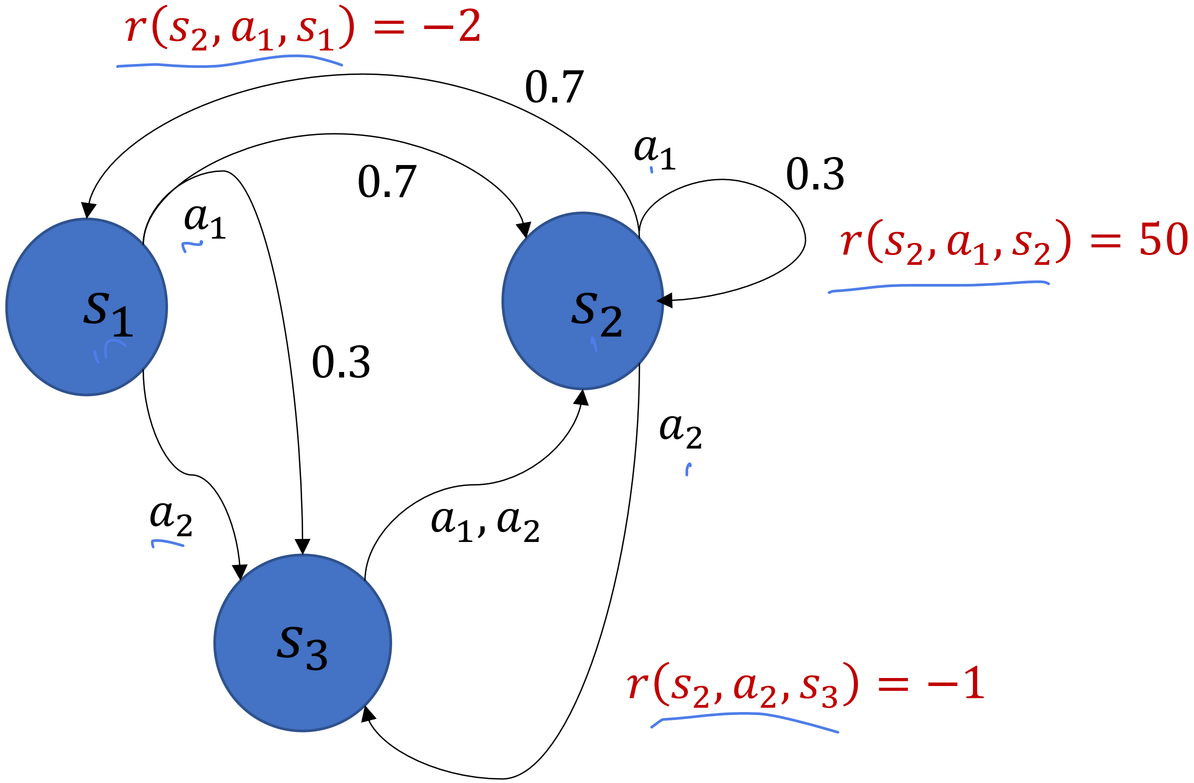
- reward +1 at [4,3], -1 at [4,2]
- reward -0.04 for each step
- Finite horizon or infinite horizon?
- What is a good policy?

Parameters of an MDP are factorizations of the joint distribution



- Initial state distribution
- Transition dynamics
- Reward distribution

State-space diagram representation of an MDP: An example with 3 states and 2 actions.



* The reward can be associated with only the state s' you transition into.
* Or the state that you transition from s and the action a you take.
* Or all three at the same time.

Reward function and Value functions

- Immediate reward function r(s,a)
  - expected **immediate** reward

$$r(s,a) = \mathbb{E}[R_1 | S_1 = s, A_1 = a]$$

$$r^\pi(s) = \mathbb{E}_{a \sim \underline{\pi(a|s)}}[R_1 | S_1 = s]$$

- state value function: $V^\pi(s)$
  - expected **long-term** return when starting in *s* and following π

$$V^\pi(s) = \mathbb{E}_\pi[R_1 + \gamma R_2 + \ldots + \gamma^{t-1} R_t + \ldots | S_1 = s]$$

- state-action value function: $Q^\pi(s,a)$
  - expected **long-term** return when starting in *s*, performing *a*, and following π

$$Q^\pi(s,a) = \mathbb{E}_\pi[R_1 + \gamma R_2 + \ldots + \gamma^{t-1} R_t + \ldots | S_1 = s, A_1 = a]$$

Optimal value function and the MDP planning problem

$$\underline{V^\star}(s) := \sup_{\pi \in \Pi} V^\pi(s)$$

$$\underline{Q^\star}(s,a) := \sup_{\pi \in \Pi} Q^\pi(s,a).$$

Goal of MDP planning:

$$\text{Find } \underline{\pi}^* \text{ such that } V^\pi(s) = \underline{V^*(s)} \quad \underline{\forall} s$$

Approximate solution:

$$\pi \text{ is } \epsilon\text{-optimal if } \underline{V^\pi(s)} \geq \underline{V^*(s)} - \epsilon \mathbf{1}$$

General policy, Stationary policy, Deterministic policy

- General policy could depend on the entire history

$$\pi : \underbrace{(\mathcal{S} \times \mathcal{A} \times \mathbb{R})^* \times \mathcal{S}}_{\text{entire history}} \to \Delta(\mathcal{A})$$

memoryless

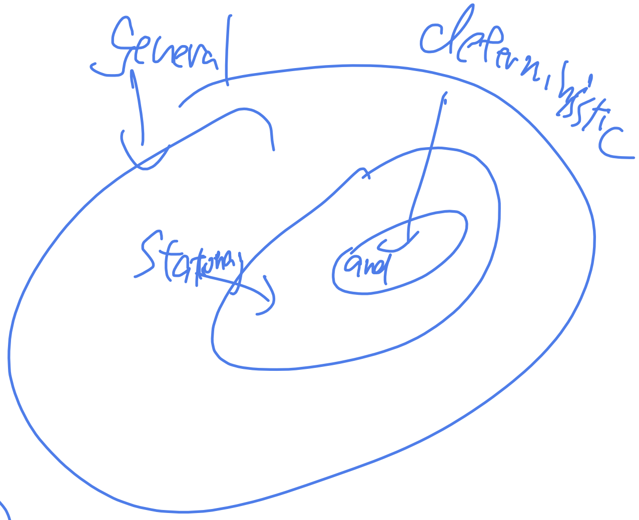
- Stationary policy

$$\pi : \mathcal{S} \to \Delta(\mathcal{A})$$

- Stationary, Deterministic policy

$$\pi : \mathcal{S} \to \mathcal{A}$$

$$|A|^{|S|}$$



Two surprising facts about MDPs

1. It suffices to consider stationary / deterministic policies.

2. There exists a stationary / deterministic policy that is optimal simultaneously for all initial state distributions.

Bellman equations – the fundamental equations of MDP and RL

- An alternative, recursive and more useful way of defining the V-function and Q function

$\text{FutureValue}(s) = E_{\pi(s)} E_{s' \sim P(s,a)} \left(\text{Reward} + \gamma \cdot \text{FutureValue}_{\pi}(s') \right) = $ ~~FutureValue(s)~~

$$\underline{V^\pi}(s) = \sum_a \pi(a|s) \underbrace{\sum_{s'} P(s'|s,a)[r(s,a,s') + \gamma \underline{V^\pi(s')}]}_{\text{with } Q^\pi} = \sum_a \pi(a|s) Q^\pi(s,a)$$

b/c A

- Exercise:
  - Prove Bellman equation from the definition.
  - Write down the Bellman equation using Q function alone.

$$\underbrace{Q^\pi(s,a) = {\color{red}?} \quad \sum_{s'} P(s'|s,a) \left[r(s,a,s') + \gamma \sum_{a'} \pi(a'|s') Q^\pi(s',a') \right]}$$

Bellman optimality equations characterizes the optimal policy

$$\underline{V}^*(s) = \overset{\downarrow}{\max_{\underline{a}}} \underbrace{\sum_{s'} P(s'|s,a)[r(s,a,s') + \gamma \underline{V^*(s')}]}$$

- system of n non-linear equations
- solve for V*(s)
- easy to extract the optimal policy

- having Q*(s,a) makes it even simpler

$$\underline{\pi^*(s)} = \arg\max_a Q^*(s,a)$$

$$\underbrace{\sum_{s'} P(s'|s,a)[r(s,a,s') + \gamma V^*(s')]}$$

Bellman equations in matrix forms

- Lemma (Bellman consistency): For stationary policies, we have

$$V^\pi(s) = Q^\pi(s, \pi(s)).$$

$$Q^\pi(s,a) = r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)}\left[V^\pi(s')\right].$$

- In matrix forms:

$$\underline{V^\pi = r^\pi + \gamma P^\pi V^\pi}$$

$$\underline{Q^\pi = r + \gamma P V^\pi}$$

$$\underline{Q^\pi = r + \gamma P^\pi Q^\pi\,.}$$

$$\underline{V}^\pi = (I - \gamma P^\pi)^{-1} r^\pi$$

policy evaluation

or "prediction" problem in $\underline{\text{SuttonBarto}}$

Value iterations for MDP planning

- Recall: Bellman optimality equations

$$V^*(s) = \max_a \sum_{s'} P(s'|s,a)[r(s,a,s') + \gamma V^*(s')]$$

$$Q(s,a) = r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot|s,a)} \left[\max_{a' \in \mathcal{A}} Q(s',a') \right].$$

Bellman operator

$$\mathcal{T}Q = \underline{r + \gamma P V_Q} \quad \text{where} \quad \underline{\underline{V_Q(s)} := \max_{a \in \mathcal{A}} Q(s,a)}.$$

Theorem: Q = Q* if and only if Q satisfies the Bellman optimality equations.

Value iterations for MDP planning

- The value iteration algorithm iteratively applies the Bellman operator until it converges.
  1. Initialize Q_0 arbitrarily
  2. for i in 1,2,3,…, k, update $Q_i = \mathcal{T} Q_{i-1}$
  3. Return Q_k

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  1. Initialize Q_0 arbitrarily
  2. for i in 1,2,3,…, k, update $Q_i = \mathcal{T} Q_{i-1}$
  3. Return Q_k

- **What is the right question to ask here?**

1. does it converge? $k \to \infty$

2. How quickly does it converge?

3. Time/Space complexity

4. $\hat{\pi}(s) = \arg\max_a \hat{Q}_k(s,a)$ $\quad \|\hat{Q}_k - Q^*\|_\infty \le \epsilon \quad \Rightarrow \quad$ ~~$Q^{\hat{\pi}}(s,a) - Q^*(s,a) \le$~~ $-V^{\hat{\pi}}(s) + V^*(s) \le \frac{\epsilon}{1-\gamma}$

Convergence of value iteration for solving MDPs

- Lemma 1. The Bellman operator is a γ-contraction.

For any two vectors $Q, Q' \in \mathbb{R}^{|\mathcal{S}||\mathcal{A}|}$,

$$\underline{\|\mathcal{T}Q - \mathcal{T}Q'\|_\infty} \le \gamma \|Q - Q'\|_\infty$$

$\gamma < 1$

  - Prove this in the optional HW4.

- Fast convergence of value iterations to Q*:

$\|Q_k - Q^*\|_\infty \le \gamma^k \underline{\|Q_0 - Q^*\|_\infty}$

Bounded

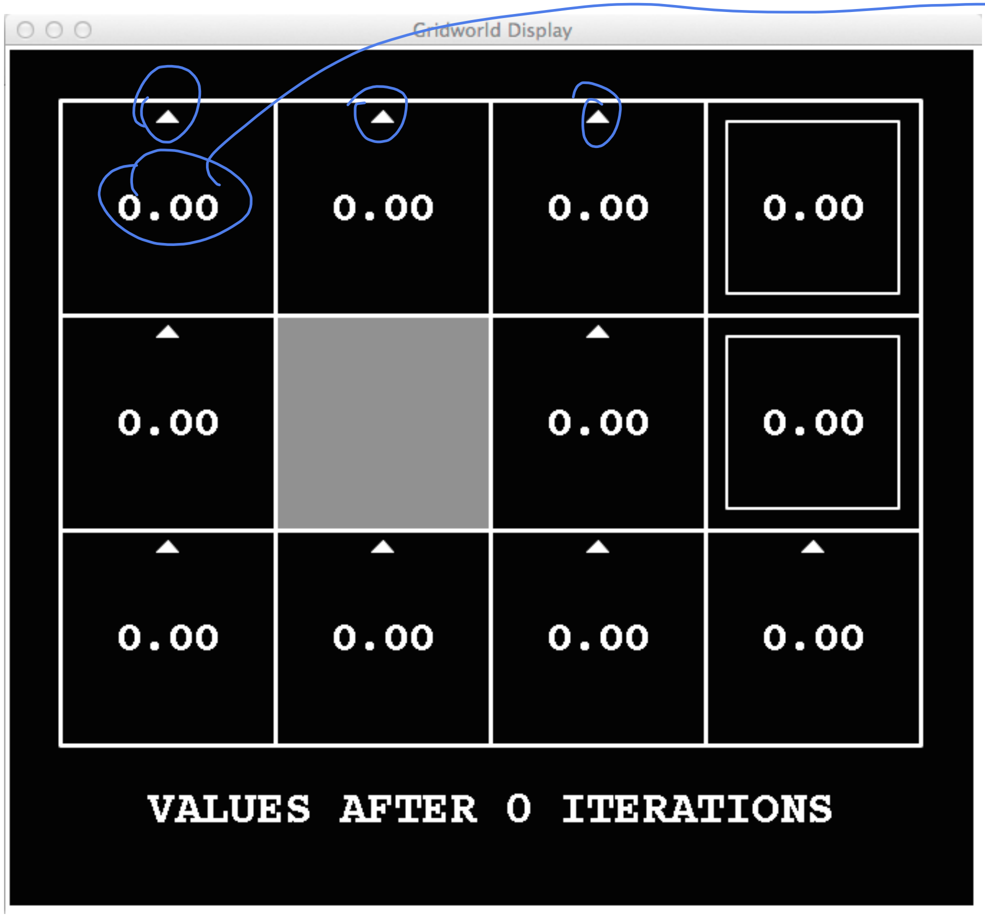
$Q' = Q^*$ $\qquad$ $\underline{\mathcal{T} \cdot Q^* = Q^*}$

$Q = Q_t$ $\qquad$ $\|\underbrace{\mathcal{T}Q_t}_{\substack{\| \\ Q_{t+1}}} - Q^*\|_\infty \le \gamma \|Q_t - Q^*\|_\infty$

k=0

$\pi(s) = \arg\max Q(s,\cdot)$

$V(s)$



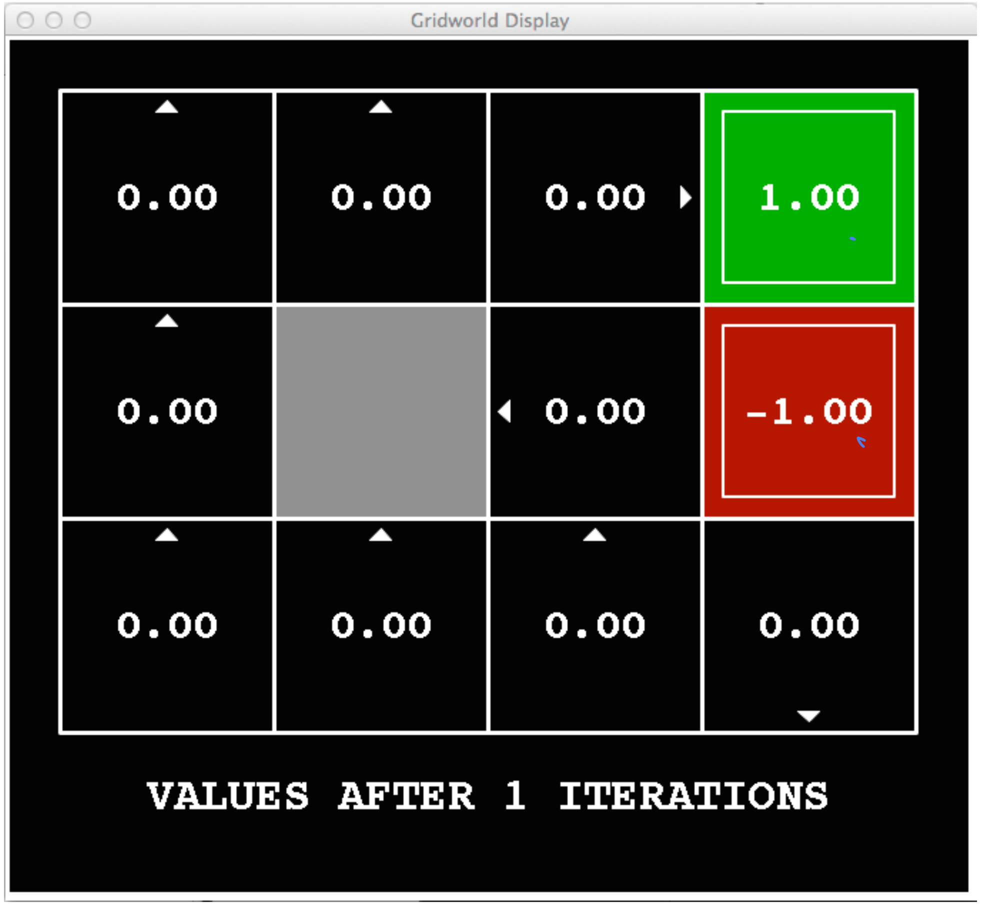
Gridworld Display

▲ 0.00	▲ 0.00	▲ 0.00	0.00
▲ 0.00		▲ 0.00	0.00
▲ 0.00	▲ 0.00	▲ 0.00	▲ 0.00

VALUES AFTER 0 ITERATIONS

Noise = 0.2
Discount = 0.9
Living reward = 0

k=1

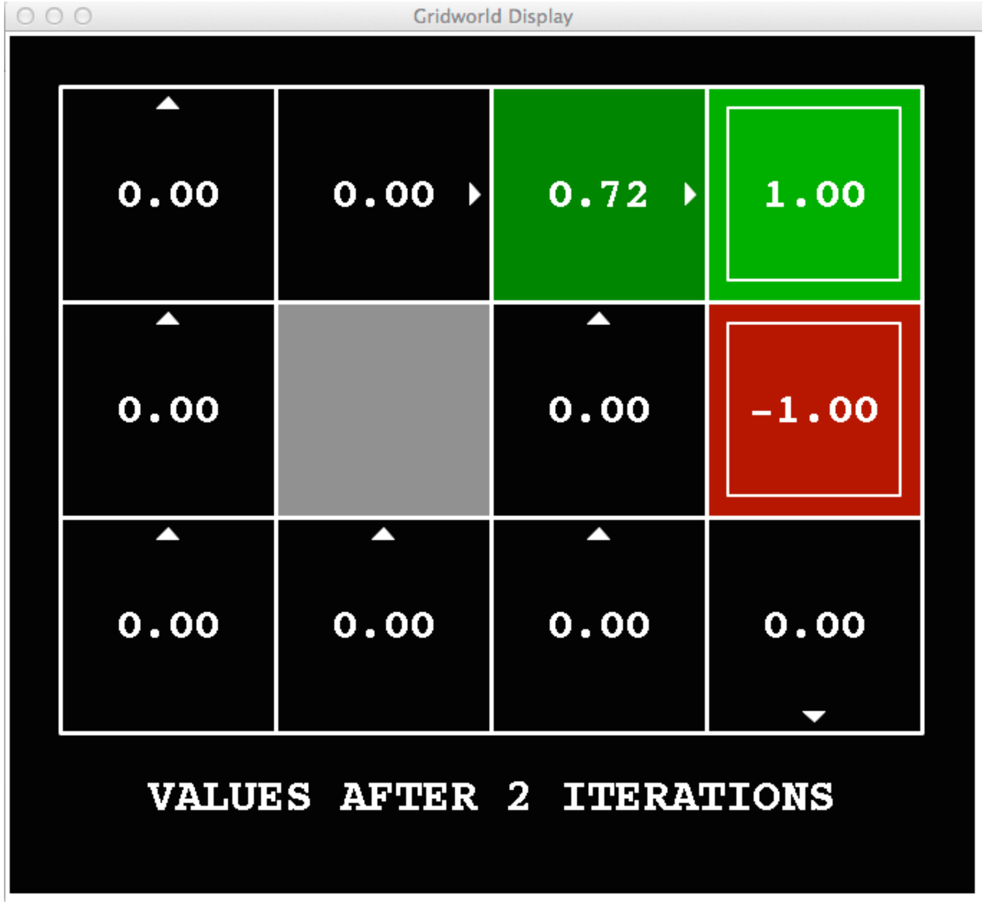


▲ 0.00	▲ 0.00	0.00 ▶	1.00
▲ 0.00		◀ 0.00	-1.00
▲ 0.00	▲ 0.00	▲ 0.00	0.00 ▼

VALUES AFTER 1 ITERATIONS

Noise = 0.2
Discount = 0.9
Living reward = 0

k=2

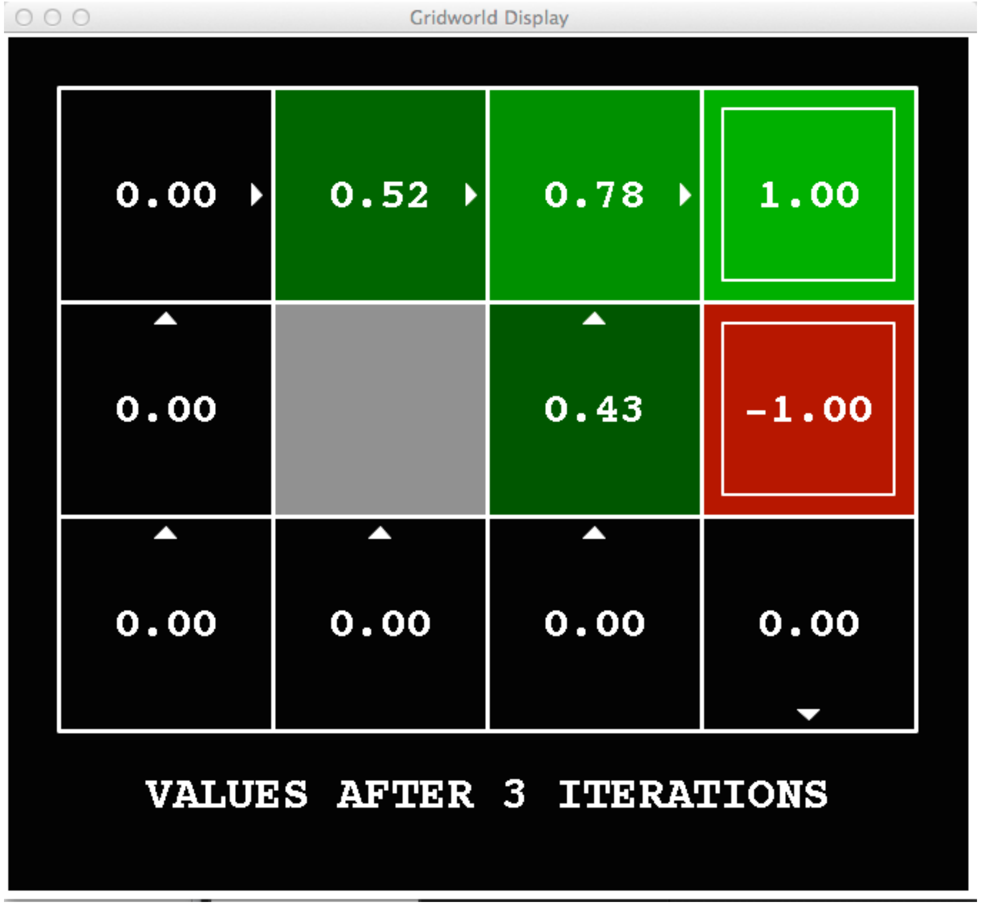


▲ 0.00	0.00 ▶	0.72 ▶	1.00
▲ 0.00		▲ 0.00	-1.00
▲ 0.00	▲ 0.00	▲ 0.00	0.00 ▼

VALUES AFTER 2 ITERATIONS

Noise = 0.2
Discount = 0.9
Living reward = 0

k=3

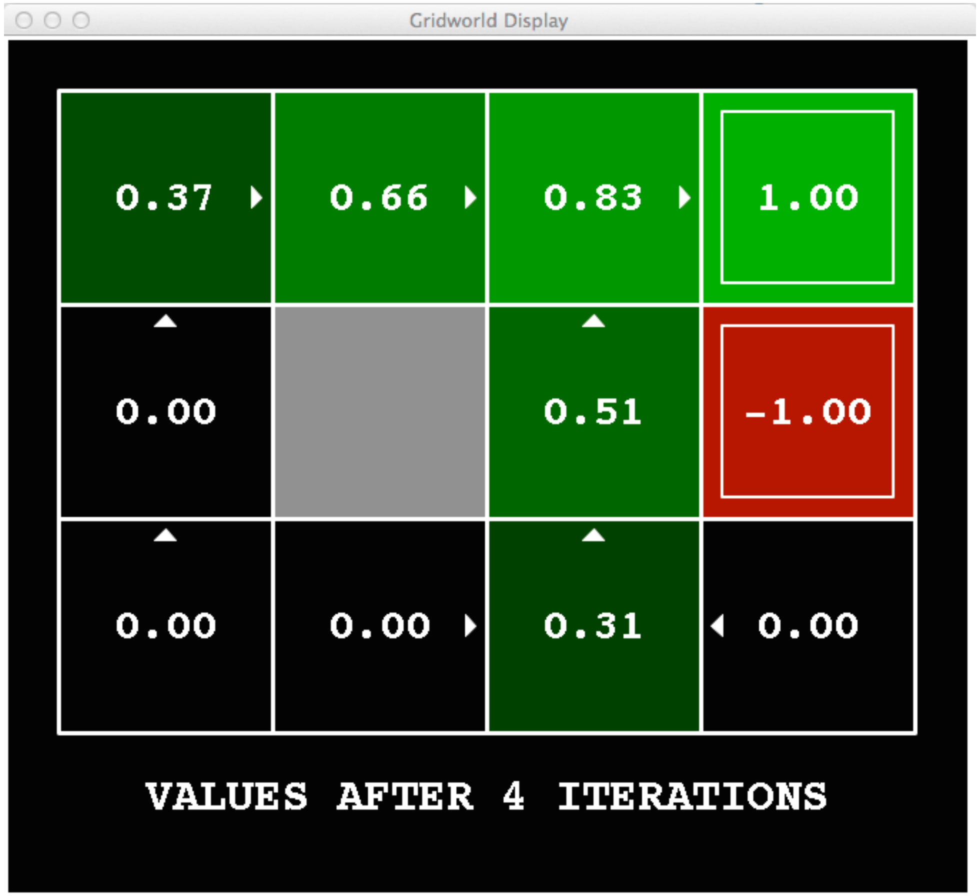


0.00 ▸	0.52 ▸	0.78 ▸	1.00
▴ 0.00		▴ 0.43	-1.00
▴ 0.00	▴ 0.00	▴ 0.00	0.00 ▾

VALUES AFTER 3 ITERATIONS

Noise = 0.2
Discount = 0.9
Living reward = 0

k=4

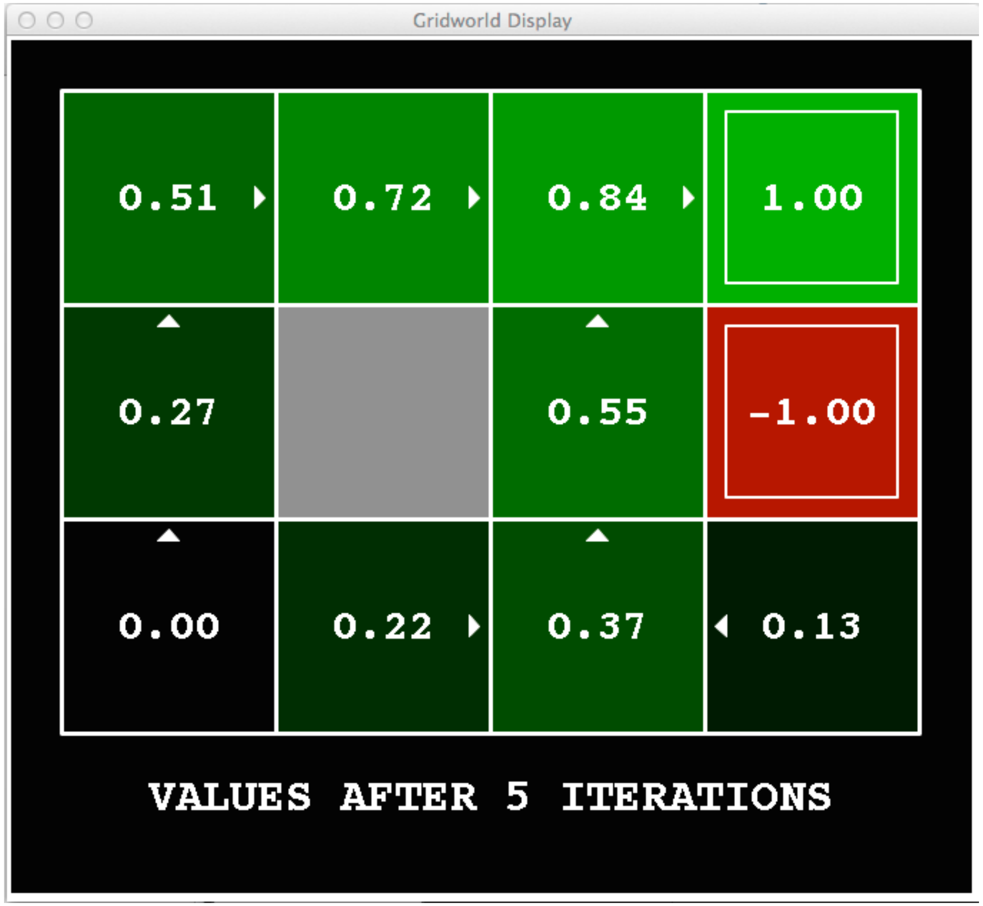


0.37 ▶	0.66 ▶	0.83 ▶	1.00
▲ 0.00		▲ 0.51	-1.00
▲ 0.00	0.00 ▶	▲ 0.31	◀ 0.00

VALUES AFTER 4 ITERATIONS

Noise = 0.2
Discount = 0.9
Living reward = 0

k=5

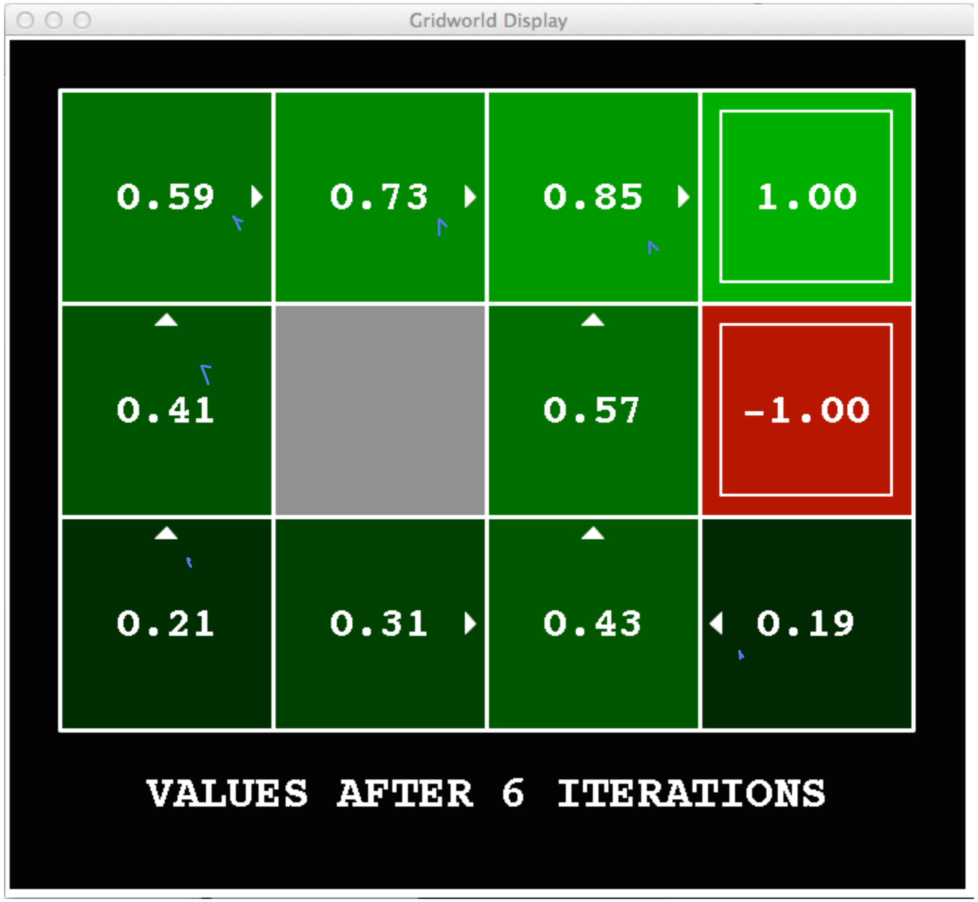


0.51 ▸	0.72 ▸	0.84 ▸	1.00
▴ 0.27		▴ 0.55	-1.00
▴ 0.00	0.22 ▸	▴ 0.37	◂ 0.13

VALUES AFTER 5 ITERATIONS

Noise = 0.2
Discount = 0.9
Living reward = 0

k=6

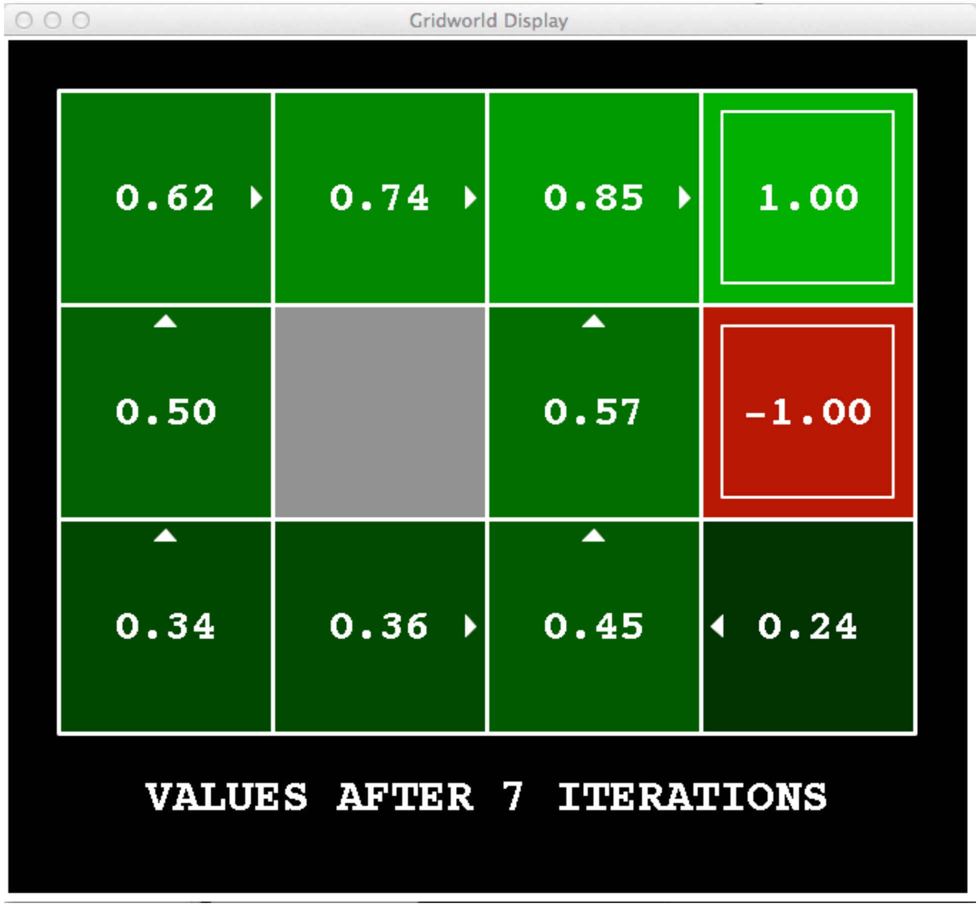


0.59 ▶	0.73 ▶	0.85 ▶	1.00
▲ 0.41		▲ 0.57	-1.00
▲ 0.21	0.31 ▶	▲ 0.43	◀ 0.19

VALUES AFTER 6 ITERATIONS

Noise = 0.2
Discount = 0.9
Living reward = 0

k=7

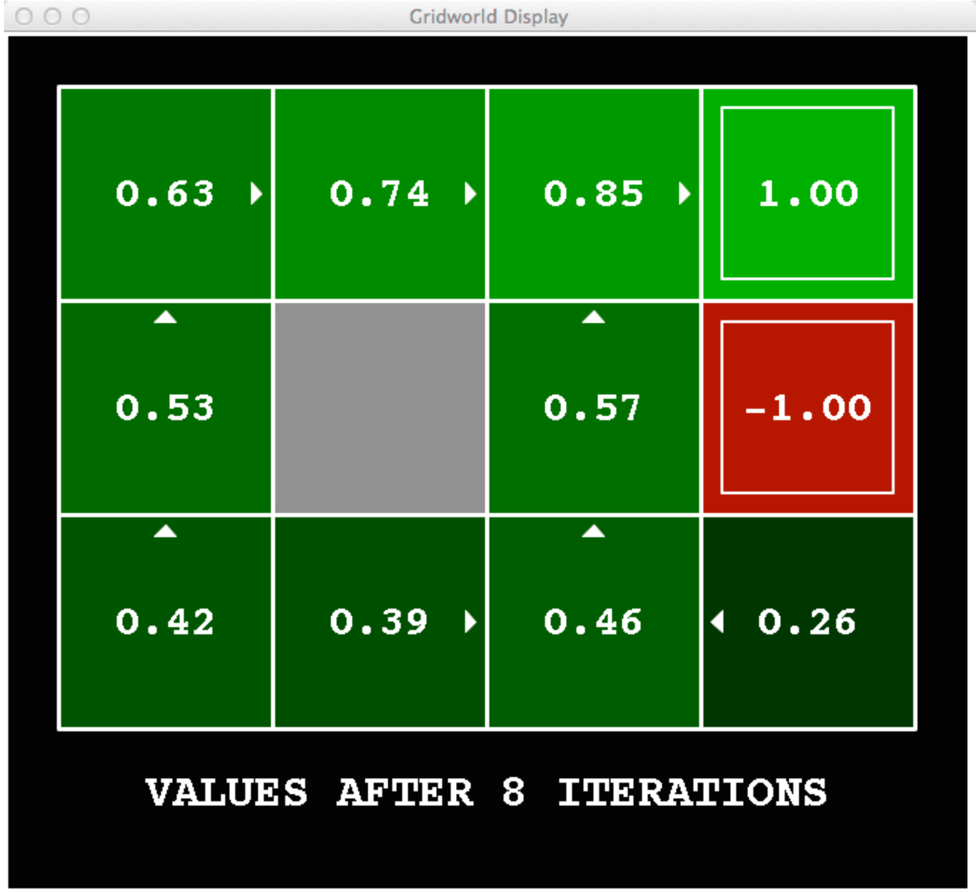


0.62 ▸	0.74 ▸	0.85 ▸	1.00
▴ 0.50		▴ 0.57	-1.00
▴ 0.34	0.36 ▸	▴ 0.45	◂ 0.24

VALUES AFTER 7 ITERATIONS

Noise = 0.2
Discount = 0.9
Living reward = 0

k=8

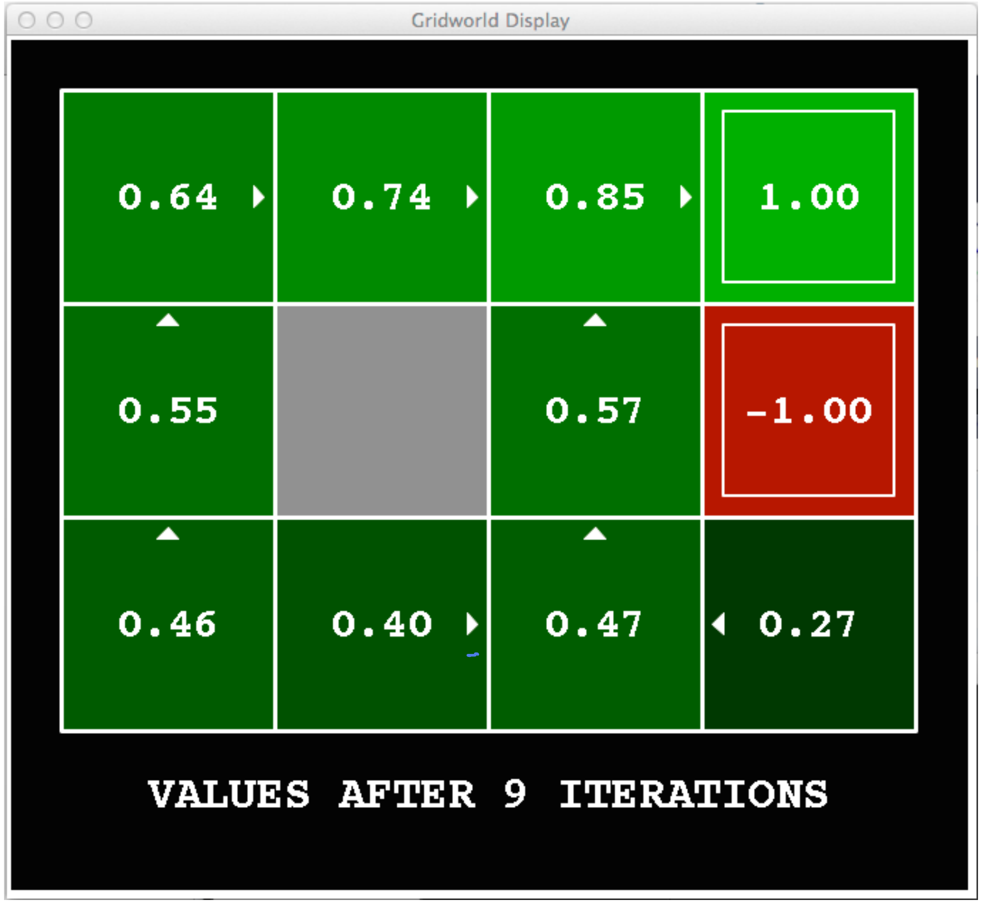


0.63 ▶	0.74 ▶	0.85 ▶	1.00
▲ 0.53		▲ 0.57	-1.00
▲ 0.42	0.39 ▶	▲ 0.46	◀ 0.26

VALUES AFTER 8 ITERATIONS

Noise = 0.2
Discount = 0.9
Living reward = 0

k=9

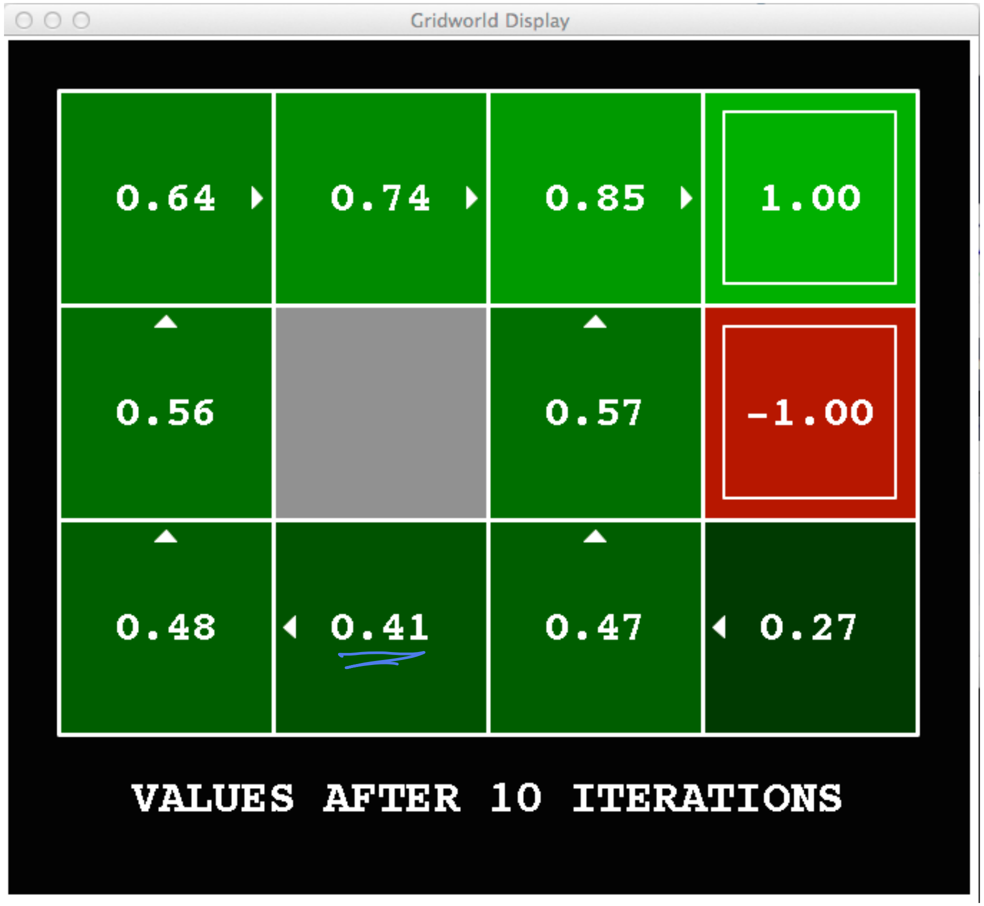


0.64 ▶	0.74 ▶	0.85 ▶	1.00
▲ 0.55		▲ 0.57	-1.00
▲ 0.46	0.40 ▶	▲ 0.47	◀ 0.27

VALUES AFTER 9 ITERATIONS

Noise = 0.2
Discount = 0.9
Living reward = 0

k=10

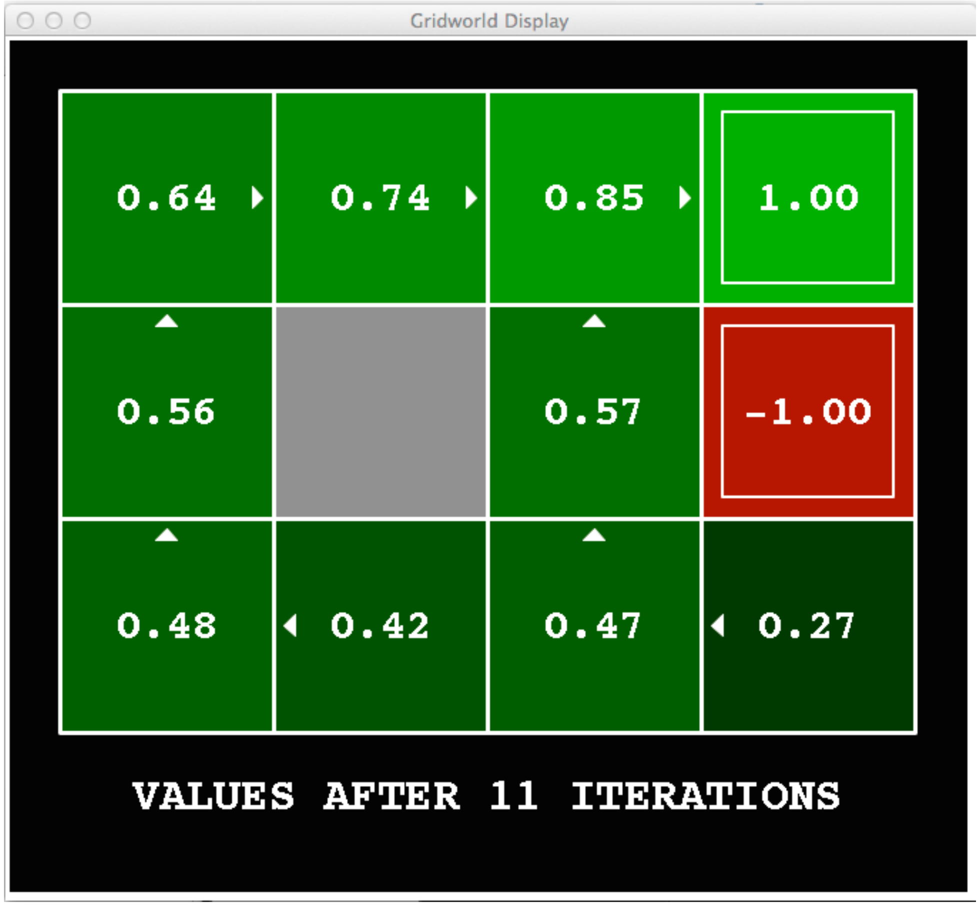


0.64 ▶	0.74 ▶	0.85 ▶	1.00
▲ 0.56		▲ 0.57	-1.00
▲ 0.48	◀ 0.41	▲ 0.47	◀ 0.27

VALUES AFTER 10 ITERATIONS

Noise = 0.2
Discount = 0.9
Living reward = 0

k=11

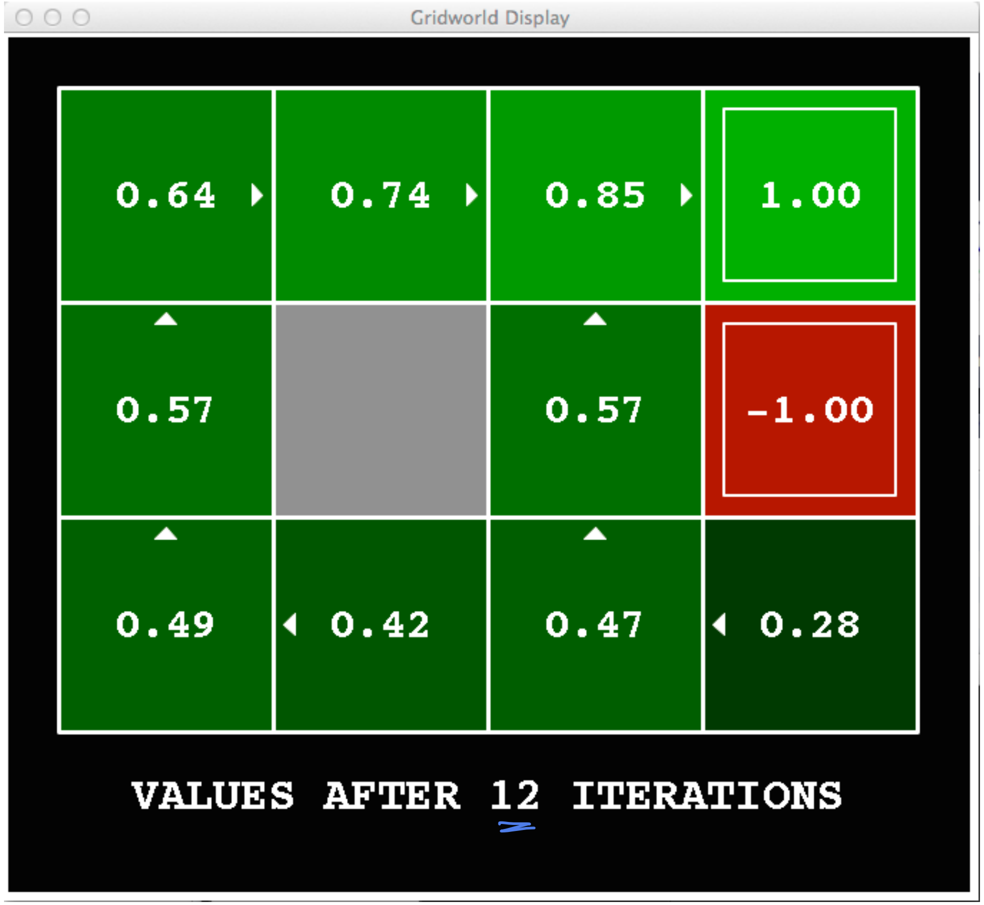


0.64 ▸	0.74 ▸	0.85 ▸	1.00
▴ 0.56		▴ 0.57	-1.00
▴ 0.48	◂ 0.42	▴ 0.47	◂ 0.27

VALUES AFTER 11 ITERATIONS

Noise = 0.2
Discount = 0.9
Living reward = 0

k=12

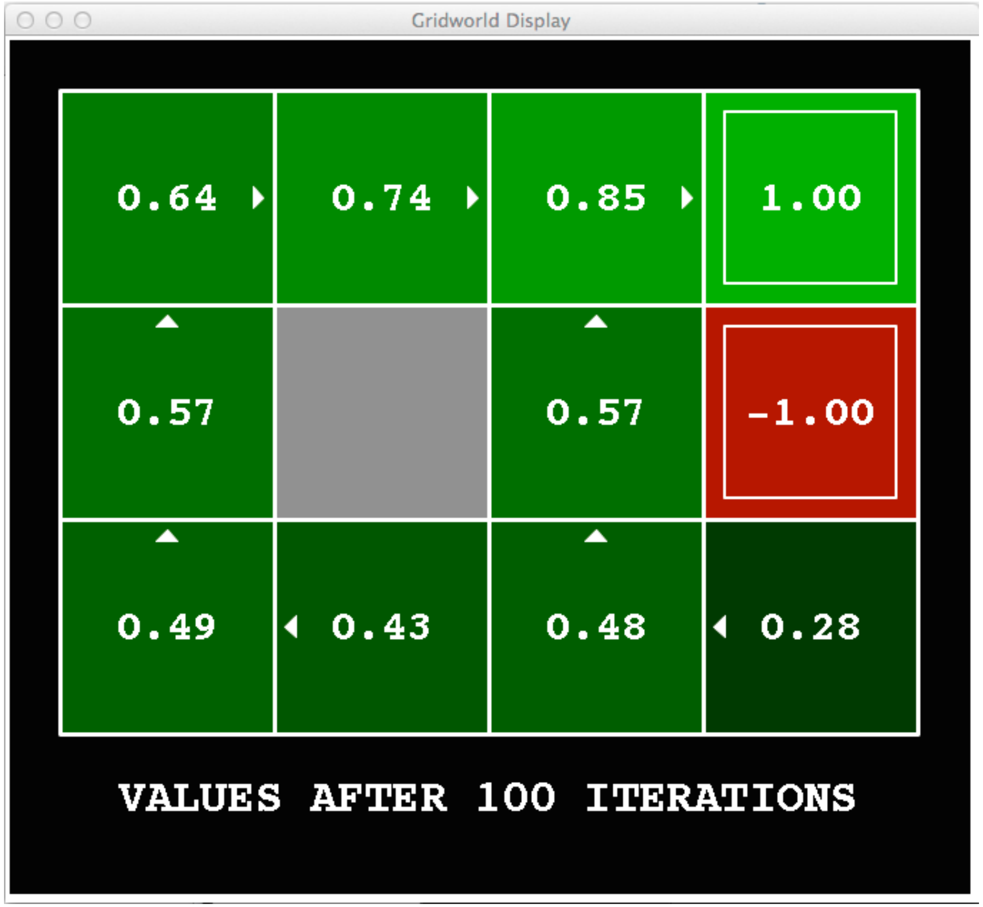


0.64 ▸	0.74 ▸	0.85 ▸	1.00
▴ 0.57		▴ 0.57	-1.00
▴ 0.49	◂ 0.42	▴ 0.47	◂ 0.28

VALUES AFTER 12 ITERATIONS

Noise = 0.2
Discount = 0.9
Living reward = 0

k=100



0.64 ▶	0.74 ▶	0.85 ▶	1.00
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▲ 0.49	◀ 0.43	▲ 0.48	◀ 0.28

VALUES AFTER 100 ITERATIONS

Noise = 0.2
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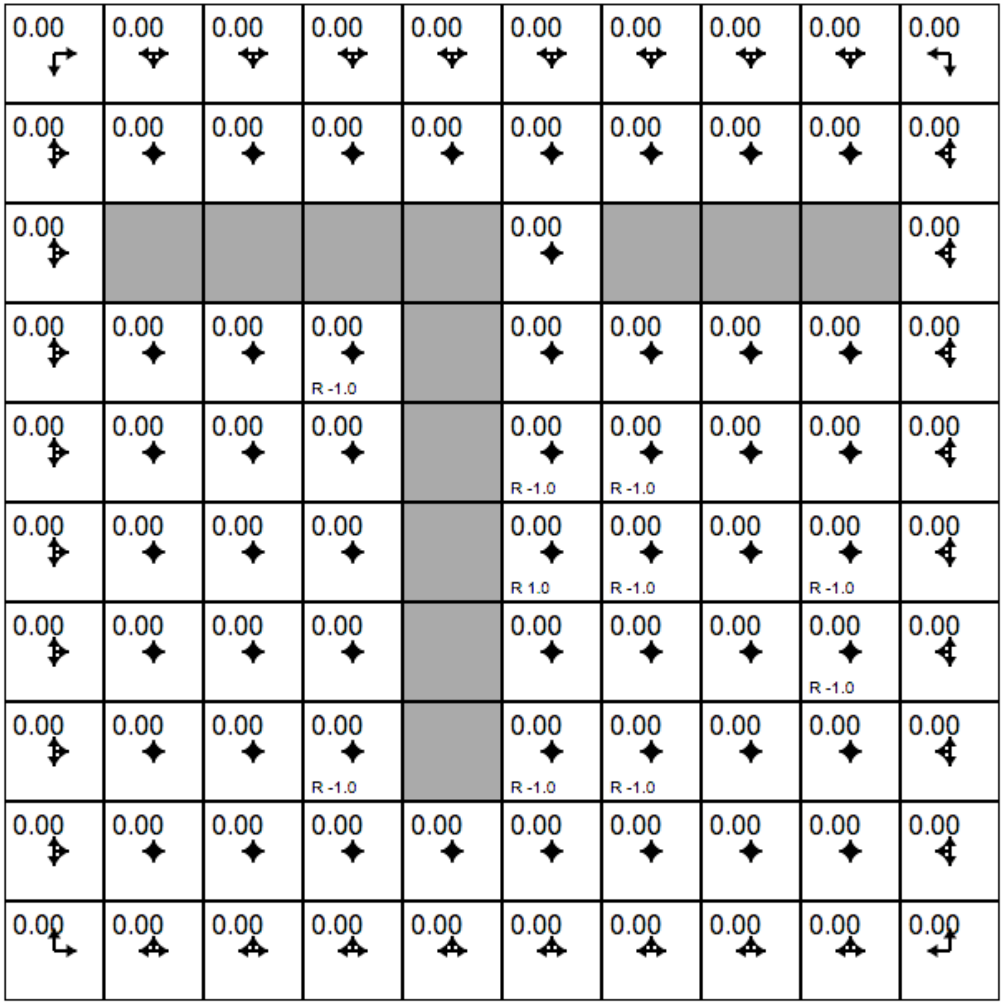
Demo: grid worlds



0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.00	▨	▨	▨	▨	0.00	▨	▨	▨	0.00
0.00	0.00	0.00	0.00 R -1.0	▨	0.00	0.00	0.00	0.00	0.00
0.00	0.00	0.00	0.00	▨	0.00 R -1.0	0.00 R -1.0	0.00	0.00	0.00
0.00	0.00	0.00	0.00	▨	0.00 R 1.0	0.00 R -1.0	0.00	0.00 R -1.0	0.00
0.00	0.00	0.00	0.00	▨	0.00	0.00	0.00	0.00 R -1.0	0.00
0.00	0.00	0.00	0.00 R -1.0	▨	0.00 R -1.0	0.00 R -1.0	0.00	0.00	0.00
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

https://cs.stanford.edu/people/karpathy/reinforcejs/gridworld_dp.html

Checkpoint

- What is RL? What are its motivating applications?

- A model of RL --- Markov Decision Processes
  - Value functions: Q functions and V functions
  - Bellman equations

- MDP planning / inference problem
  - Value iterations

Remainder of this lecture

- RL algorithms
    - Model-based RL vs Model-free RL
    - Monte Carlo
    - Temporal Difference Learning
    - Linear function approximation

Recap: Policy Iterations and Value Iterations

- What are these algorithms for?
  - Algorithms of computing the V* and Q* functions from MDP parameters

policy evaluation

$V^{\pi_0} \to Q^{\pi_0} \xrightarrow{\max} \tilde{V}$

$\pi_1 = \arg\max Q^{\pi_0}$

policy improvement

- Policy Iterations

$$\pi_0 \longrightarrow^{E} \underline{V^{\pi_0}} \longrightarrow^{I} \underline{\pi_1} \longrightarrow^{E} V^{\pi_1} \longrightarrow^{I} \ldots \longrightarrow^{I} \pi^* \longrightarrow^{E} V^*$$

- Value iterations

$$\rightarrow \quad V_{k+1}(s) \leftarrow \max_a \sum_{s'} P(s'|s,a)[r(s,a,s') + \gamma V_k(s')]$$

- How do we make sense of them?
  - Recursively applying the Bellman equations until convergence.

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- How do we make sense of them?
  - Recursively applying the Bellman equations until convergence.

*These methods are called "Dynamic Programming" approaches in Chap 4 of Sutton and Barto.

They are no longer valid in RL

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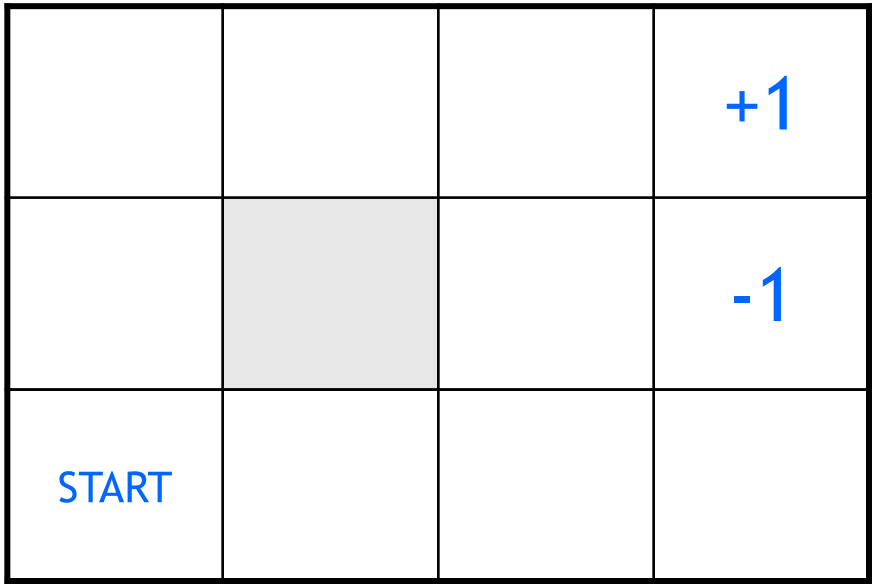
***We do not have the MDP parameters in RL!**

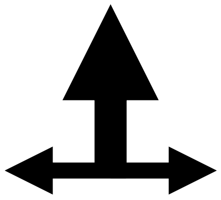
Example: Frozen lake



actions: UP, DOWN, LEFT, RIGHT

UP

80% move UP
10% move LEFT
10% move RIGHT



- reward +1 at [4,3], -1 at [4,2]
- reward -0.04 for each step
- what's the strategy to achieve max reward?

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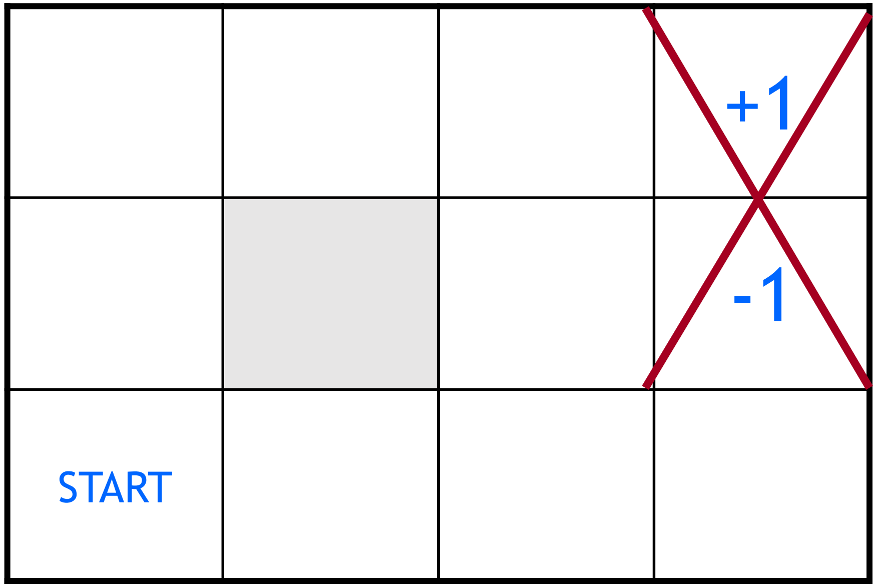

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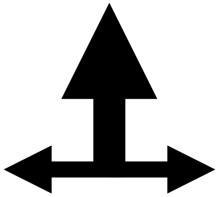

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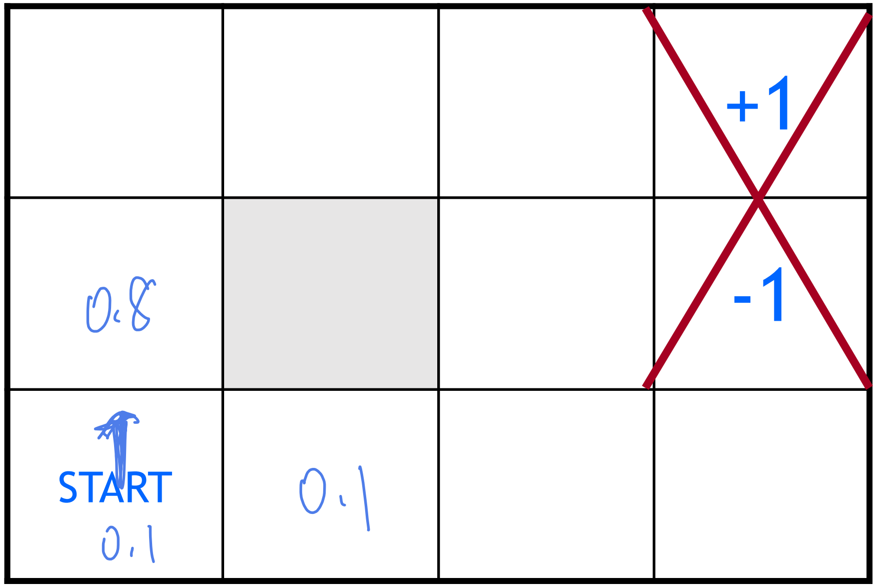

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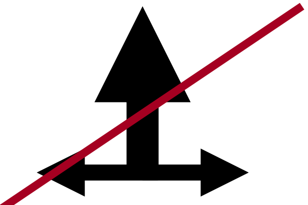

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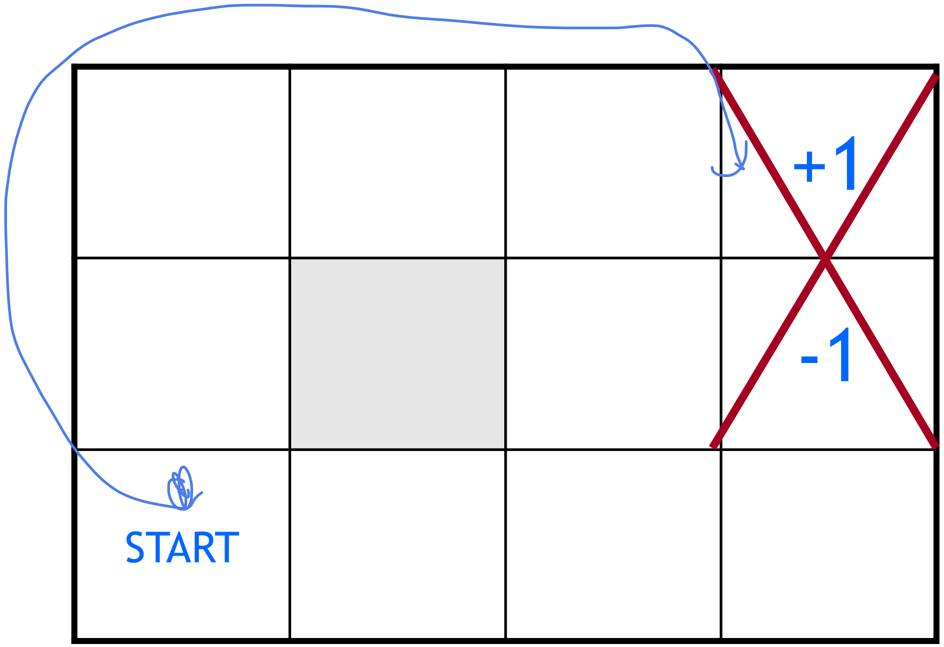
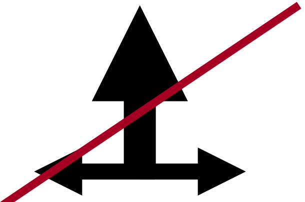


Action 1, Action 2, Action 3, Action 4

actions: ~~UP, DOWN, LEFT, RIGHT~~

~~**UP**~~

~~80% move UP~~
~~10% move LEFT~~
~~10% move RIGHT~~



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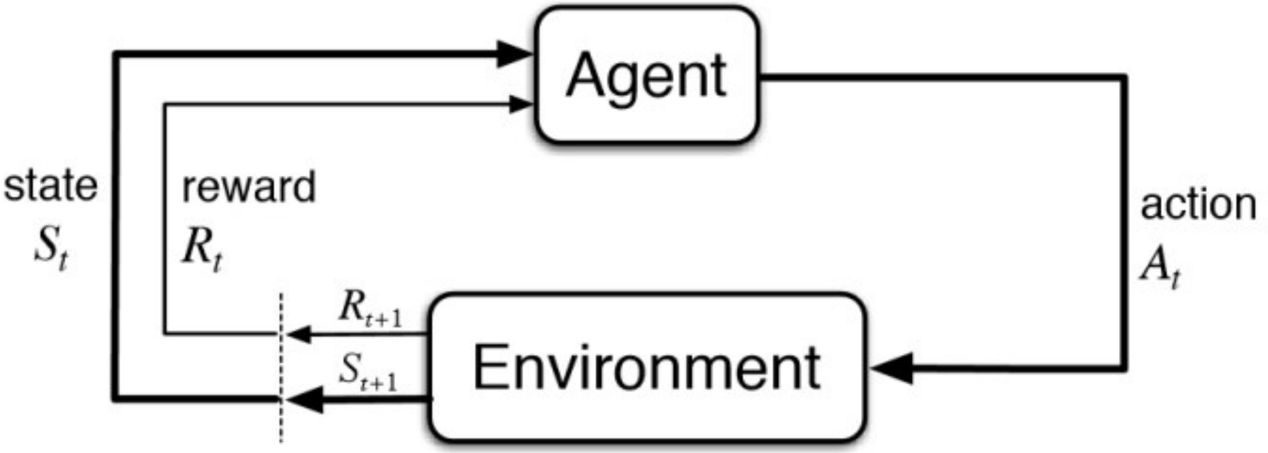
Instead, reinforcement learning agents have “online” access to an environment

- State, Action, Reward
- Unknown reward function, unknown state-transitions.
- Agents can “act” and “experiment”, rather than only doing offline planning.



Idea 1: **Model-based** Reinforcement Learning

- Model-based idea
  - Let’s approximate the model based on experiences
  - Then solve for the values as if the learned model were correct

- Step 1: Get data by running the agent to explore
  - Many data points of the form: $\{(\underline{s_1, a_1, s_2,} r_1), \ldots, (\underline{s_N, a_N, s_{N+1}, r_N})\}$

- Step 2: Estimate the model parameters
  - $\underline{\hat{P}(s'|s,a)}$ --- plug-in / MLE. We need to observe the transition many times for each s, a
  - $\hat{r}(s', \underline{a,} s)$ --- this is an estimate of the empirical rewards.

Then we can plug in these estimates and then use dynamic programming for policy evaluation / improvements.

$$V_{k+1}^{\pi}(s) \leftarrow \sum_a \pi(a|s) \sum_{s'} \hat{P}(s'|s,a)[\hat{r}(s,a,s') + \gamma V_k^{\pi}(s')]$$

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* As usual, **“hat”** indicates empirical estimates.

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* These iterations will produce $\underline{\hat{V}}^*$ and $\underline{\hat{Q}}^*$ functions, and then $\underline{\hat{\pi}}^*$

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  - Often we need to take a carefully chosen sequence of actions to reach a state
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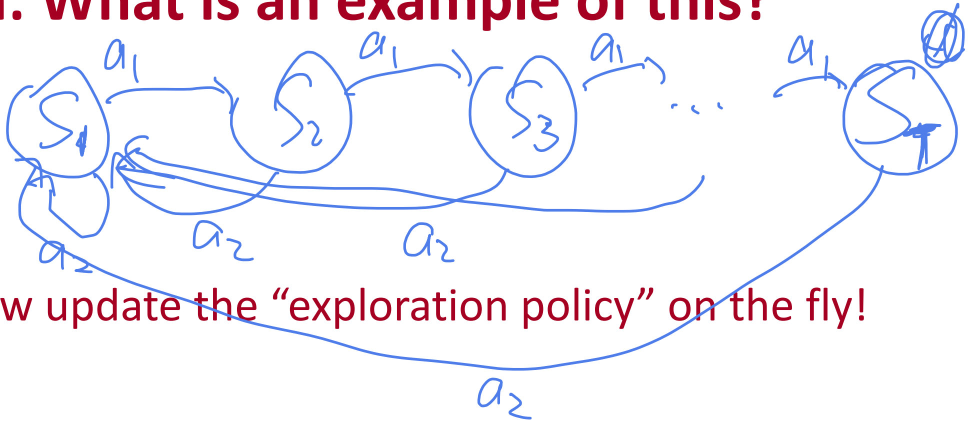
  - **Question: What is an example of this?**

$\pi(a|s) = 0.5$ for $a = a_1$
for $a = a_2$

$r(s_T, a_1) = 1$



*Need to somehow update the “exploration policy” on the fly!

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- How does the error in the estimated Q* function affect the suboptimality of the policy that maximizes \hat{Q}*?
- Answered by “Simulation Lemma” (Kearns and Singh, 2002)
    - Resurgence of research on this more recently: Yin and W. (2020), Yin, Bai and W. (2020)

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$Q^*: S \times A \to \mathbb{R}$

$O(|S||A|)$

$P: S \times S \times A \to [0,1]$

$O(|S|^2|A|)$

- Recall: Policy iterations

$$\pi_0 \rightarrow^E V^{\pi_0} \rightarrow^I \pi_1 \rightarrow^E V^{\pi_1} \rightarrow^I \ldots \rightarrow^I \pi^* \rightarrow^E V^*$$

  - Maybe we can do policy evaluation / value iterations without estimating the model?

Monte Carlo Policy Evaluation (Prediction)

- want to estimate $V^{\pi}(s)$

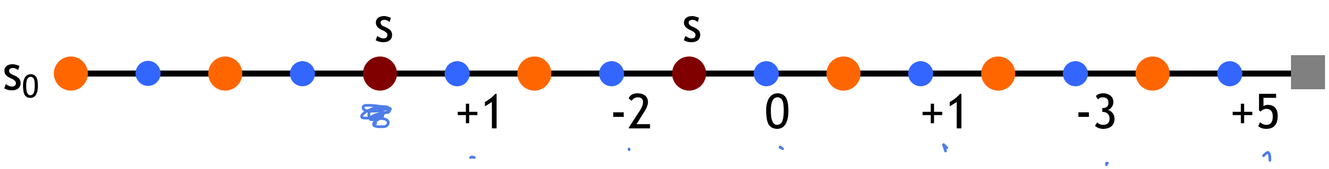
  = expected return starting from s and following π

  - estimate as average of observed returns in state s

- We can execute the policy π

- first-visit MC

  - average returns following the first visit to state s



$1 + (-2) 1 \cdot \gamma^2 + 1 \cdot \gamma^6 + (-3) \gamma^7 \dots$

$G_1(s) = +2$

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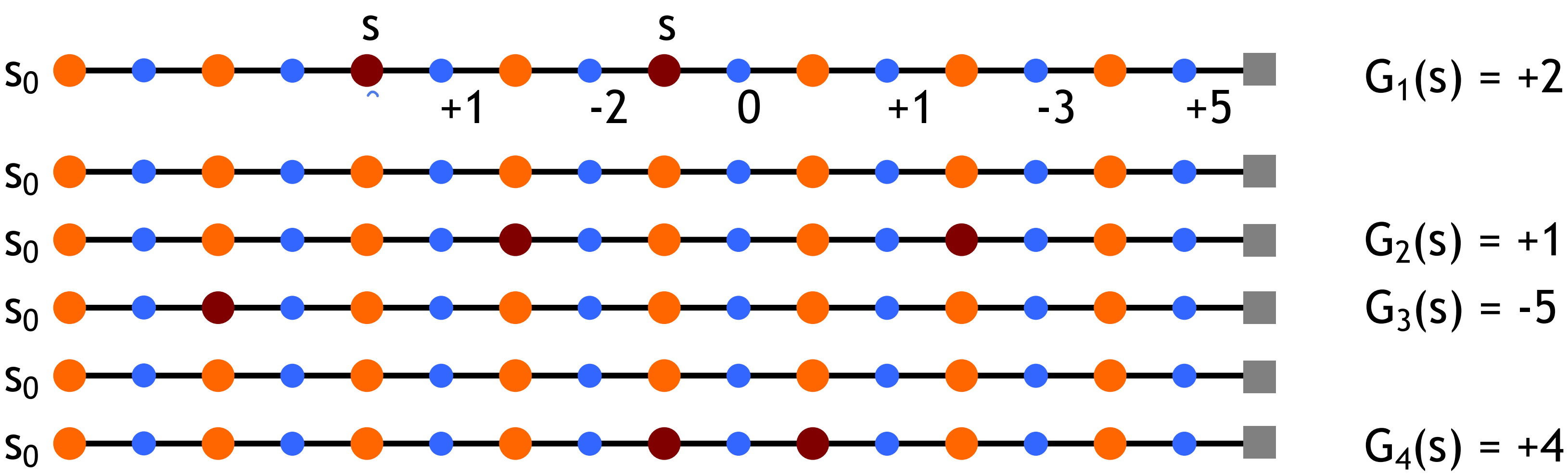


$G_1(s) = +2$

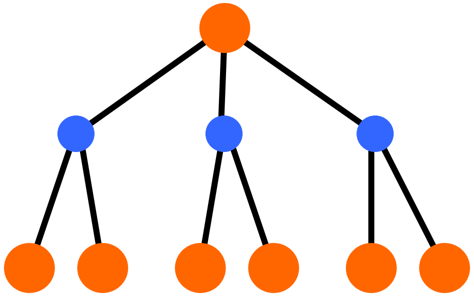
$G_2(s) = +1$

$G_3(s) = -5$

$G_4(s) = +4$

$$\underline{V^{\pi}(s)} \approx \underline{(2 + 1 - 5 + 4)/4} = 0.5$$

Monte Carlo Policy Optimization (Control)



- V^π not enough for policy improvement
  - need exact model of environment
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$$\pi'(s) = \arg\max_a Q^\pi(s,a)$$

- MC control

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  - update after each episode

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  - greedy policy won’t explore all actions
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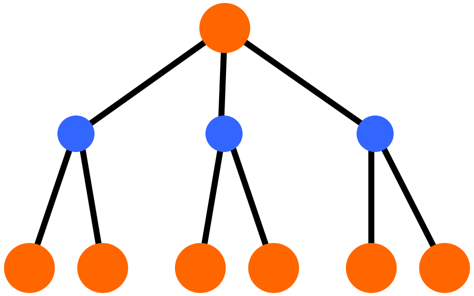

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  - greedy policy won't explore all actions eps-greedy, or bonus design.
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Improved Monte-Carlo Q-function estimate using Bellman equations

- Recall:

$$\underline{Q^\pi(s,a)} = \sum_{s'} P(s'|s,a)[r(s,a,s') + \gamma \sum_{a'} \pi(a'|s') Q^\pi(s',a')]$$

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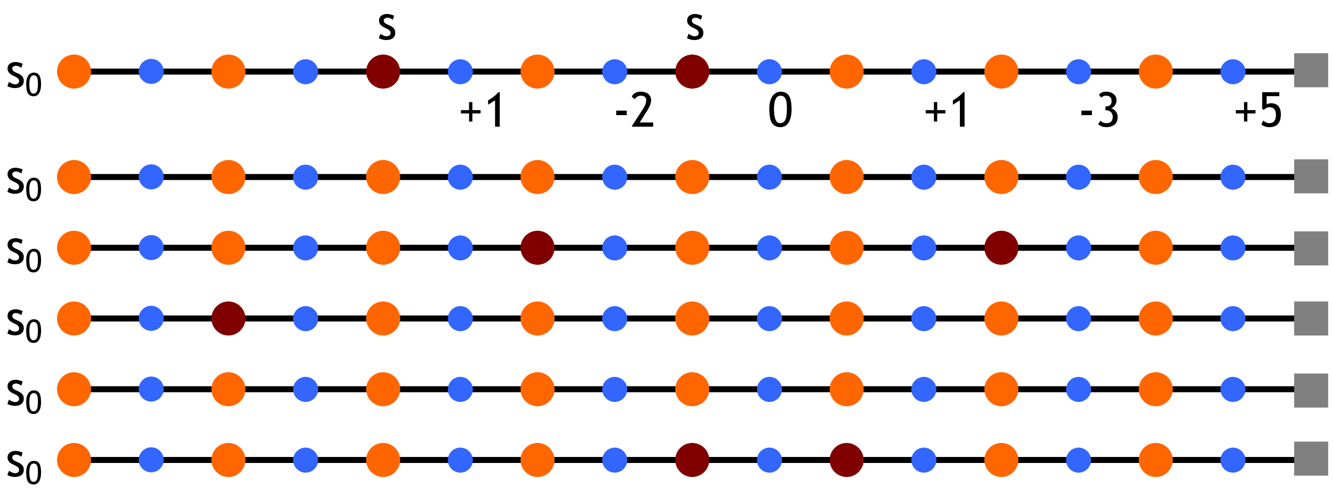
*No need to estimate P(s’ | s,a) or r(s,a,s’) as intermediate steps.
*Require only O(SA) space, rather than O(S^2A)

Online averaging representation of MC



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G3(s) = -5

G4(s) = +4

$V^\pi(s) \approx (2 + 1 - 5 + 4)/4 = 0.5$

- Alternative, *online averaging* update

$$V(S_t) \leftarrow \underline{V(S_t)} + \boxed{\alpha\left[G_t - V(S_t)\right],} \quad \text{where } \alpha = 1/\underline{N_{S_t}}$$

$$V(S_t) = \frac{\sum_{i=1}^{N_{S_t}} G_t^i}{N_{S_t}}$$

DP + MC = Temporal Difference Learning

- **Monte Carlo** $V(S_t) \leftarrow V(S_t) + \alpha\Big[G_t - V(S_t)\Big],$

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$$\underline{\mathbb{E}_\pi[G_t] = \mathbb{E}_\pi[R_t | S_t] + \gamma V^\pi(S_{t+1})}$$

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- TD-Policy evaluation

$$V(S_t) \leftarrow V(S_t) + \alpha\left[R_{t+1} + \underline{\gamma \overset{\downarrow}{V}(S_{t+1})} - V(S_t)\right]$$

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- TD-Policy evaluation

Bootstrapping!

$$V(S_t) \leftarrow V(S_t) + \alpha\Big[\underbrace{R_{t+1} + \gamma V(S_{t+1})}_{\substack{\pi \\ \text{updated expected gain} \\ \text{after seeing } R_{t+1}}} - \underbrace{V(S_t)}_{\substack{\pi \\ \text{expected gain} \\ \text{before seeing } R_{t+1}}}\Big]$$

Bootstrap’s origin

- “The Surprising Adventures of Baron Münchausen”
    - Rudolf Erich Raspe, 1785

- In statistics: Brad Efron’s resampling methods
- In computing: Booting…
- In RL: It simply means TD learning

TD policy optimization (TD-control)

- SARSA (On-Policy TD-control)
  - Update the Q function by bootstrapping Bellman Equation

$$Q(S,A) \leftarrow Q(S,A) + \alpha\left[\underbrace{R + \gamma Q(S',A') - Q(S,A)}\right]$$

  - Choose the next A’ using Q, e.g., eps-greedy.

- Q-Learning (Off-policy TD-control)
  - Update the Q function by bootstrapping Bellman Optimality Eq.

$$Q(S,A) \leftarrow Q(S,A) + \alpha\left[R + \underline{\gamma \max_a} Q(S',a) - Q(S,A)\right]$$

  - Choose the next A’ using Q, e.g., eps-greedy, or any other policy.

Remarks:

- These are **proven to converge** asymptotically.
- Much more data-efficient in practice, than MC.
- Regret analysis is still active area of research.

Advantage of TD over Monte Carlo

- Given a trajectory, a roll-out, of T steps.
    - MC updates the Q function only once
    - TD updates the Q function (and the policy) T times!

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Remark: This is the same kind of improvement from Gradient Descent to Stochastic Gradient Descent (SGD).

Model-free vs Model-based RL algorithms

- Different function approximations

approximate P, r vs. Q^*

neural nets

~~no~~ neural nets

- Different space efficiency

$O(S^2A)$ $O(SA)$

- Which one is more statistically efficient?
  - More or less equivalent in the tabular case.
  - Different challenges in their analysis.

The problem of large state-space is still there

- We need to represent and learn SA parameters in Q-learning and SARSA.

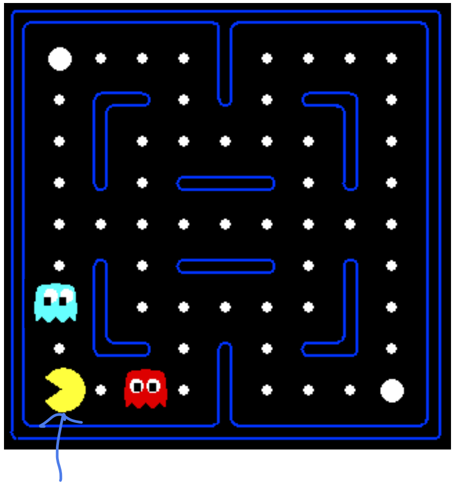
- S is often large
    - 9-puzzle, Tic-Tac-Toe: 9! = 362,800, S^2 = 1.3*10^11
    - PACMAN with 20 by 20 grid. S = O(2^400), S^2 = O(2^800)

- O(S) is not acceptable in some cases.

- Need to think of ways to “generalize”/share information across states.

Example: Pacman

Let's say we discover through experience that this state is bad:



(From Dan Klein and Pieter Abbeel)

Example: Pacman

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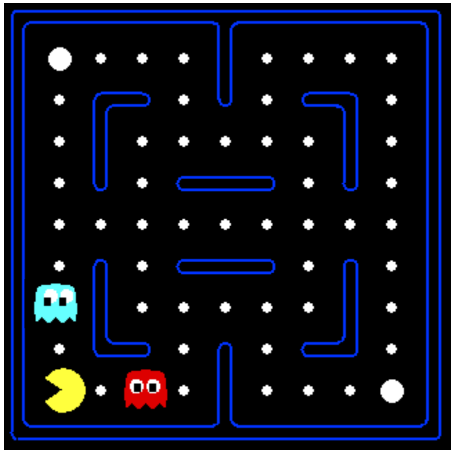
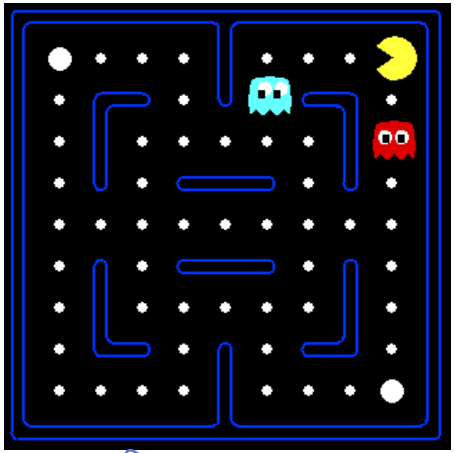


In naïve q-learning, we know nothing about this state:



(From Dan Klein and Pieter Abbeel)

Example: Pacman

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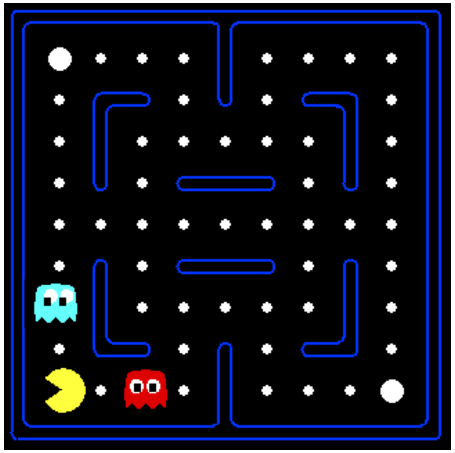

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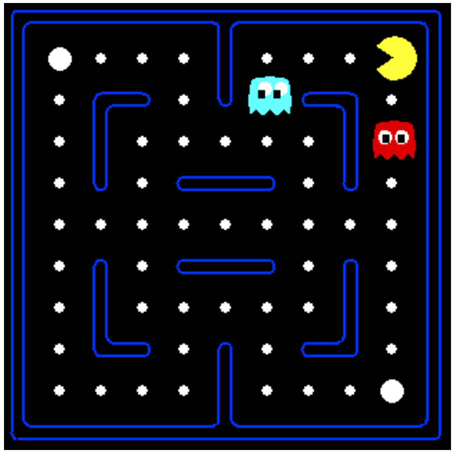
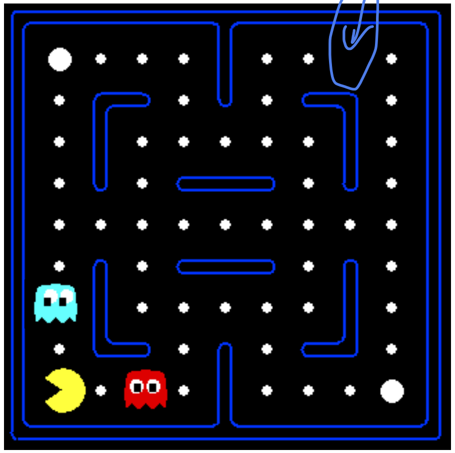


Or even this one!



(From Dan Klein and Pieter Abbeel)

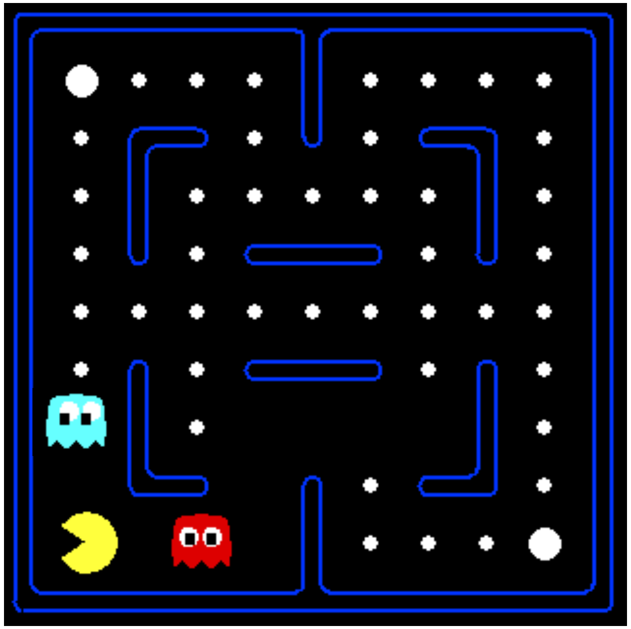
Video of Demo Q-Learning Pacman – Tiny – Watch All

Video of Demo Q-Learning Pacman – Tiny – Silent Train

Video of Demo Q-Learning Pacman – Tricky – Watch All

Why not use an evaluation function? A Feature-Based Representations

- Solution: describe a state using a vector of features (properties)
  - Features are functions from states to real numbers (often 0/1) that capture important properties of the state
  - Example features:
    - Distance to closest ghost
    - Distance to closest dot
    - Number of ghosts
    - 1 / (dist to dot)^2
    - Is Pacman in a tunnel? (0/1)
    - …… etc.
    - Is it the exact state on this slide?
  - Can also describe a q-state (s, a) with features (e.g. action moves closer to food)



Linear Value Functions

- Using a feature representation, we can write a q function (or value function) for any state using a few weights:
  - $V_{\mathbf{w}}(s) = w_1f_1(s) + w_2f_2(s) + \ldots + w_nf_n(s)$
  - $Q_{\mathbf{w}}(s,a) = w_1f_1(s,a) + w_2f_2(s,a) + \ldots + w_nf_n(s,a)$
- Advantage: our experience is summed up in a few powerful numbers
- Disadvantage: states may share features but actually be very different in value!

Updating a linear value function

Updating a linear value function

- Original Q learning rule tries to reduce prediction error at s, a:

$$Q(s,a) \text{ ⍰ } Q(s,a) + \alpha \cdot [R(s,a,s') + \gamma \max_{a'} Q(s',a') - Q(s,a)]$$

Updating a linear value function

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$$Q(s,a) \boxed{?} Q(s,a) + \alpha \cdot [R(s,a,s') + \gamma \max_{a'} Q(s',a') - Q(s,a)]$$

- Instead, we update the weights to try to reduce the error at s, a:

Updating a linear value function

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- Instead, we update the weights to try to reduce the error at s, a:

$$\begin{aligned} w_i &\leftarrow w_i + \alpha \cdot [R(s,a,s') + \gamma \max_{a'} Q(s',a') - Q(s,a)] \, \partial Q_{\mathbf{w}}(s,a)/\partial w_i \\ &= w_i + \alpha \cdot [R(s,a,s') + \gamma \max_{a'} Q(s',a') - Q(s,a)] \, f_i(s,a) \end{aligned}$$

Updating a linear value function

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$$Q(s,a) \leftarrow Q(s,a) + \alpha \cdot [R(s,a,s') + \gamma \max_{a'} Q(s',a') - Q(s,a)]$$

- Instead, we update the weights to try to reduce the error at s, a:

$$w_i \longleftarrow w_i + \alpha \cdot [R(s,a,s') + \gamma \max_{a'} Q(s',a') - Q(s,a)] \, \partial Q_{\mathbf{w}}(s,a)/\partial w_i$$
$$= w_i + \alpha \cdot [R(s,a,s') + \gamma \max_{a'} Q(s',a') - Q(s,a)] \, f_i(s,a)$$

- Qualitative justification:
  - Pleasant surprise: increase weights on positive features, decrease on negative ones
  - Unpleasant surprise: decrease weights on positive features, increase on negative ones

PACMAN Q-Learning (Linear function approx.)

Deriving the TD via incremental optimization that minimizes Bellman errors

- Mean Square Error and Mean Square Bellman error

So far, in RL algorithms

- Model-based approaches
  - Estimate the MDP parameters.
  - Then use policy-iterations, value iterations.

- Monte Carlo methods:
  - estimating the rewards by empirical averages

- Temporal Difference methods:
  - Combine Monte Carlo methods with Dynamic Programming

- Linear function approximation in Q-learning
  - Similar to SGD
  - Learning heuristic function

Final lecture

- Wrap up RL algorithm

- Exploration