Thinning Algorithms for Simulating Point Processes

Yuanda Chen

September, 2016

Abstract

In this talk we will discuss the algorithms for simulating point processes. The simplest point process is the (homogeneous) Poisson process, which has an intensity function of a constant value λ. It can be simulated by the sum of its interarrival times. By allowing the intensity to vary, taking values given by a deterministic function $\lambda(t)$, we can extend the Poisson process to the inhomogeneous case. As proposed by (Lewis and Shedler, 1979), inhomogeneous Poisson processes can be simulated by "thinning" the points from the homogeneous versions. Further by allowing the intensity at time t to depend on all information prior to t, we get the self-exciting processes such as the Hawkes processes. The simulation of such processes can also be performed by thinning, as given by (Ogata, 1981).

1 Introduction

In this talk we will discuss the simulation of point processes. A point process can be thought of as being a sequence of points on a real number line. The line represents time and the points are the occurrence time of events. The patterns how those points are located gives us different point processes. The most commonly used and simplest point process is the homogeneous Poisson process, which assumes that the points are evenly spread over any interval. It is a perfectly reasonable process to model the arrival times of buses, assuming you are standing at a bus stop. But it will not be a good choice for modeling, say earthquakes, because there are aftershocks after the main attack, or modeling occurrences of market events, which has the clustering effect. The Hawkes process is developed for this purpose, and our goal in this talk is to introduced the simulation of Hawkes processes. Before we can talk about point processes precisely, we need to define a few concepts.

2 Fundamentals of Point Processes

Definition 2.1. *(Sigman, 1995, p.1) Let $(\Omega, \mathcal{F}, P)$ be a probability space. A point process on $\mathbf{R}^+$ is a sequence of non-negative random variables $\{T_k\}_{k=1,2,\dots}$, such that for all k, $T_k \leq T_{k+1}$.*

Here T_k represents the occurrence times of events, and such processes are called temporal point processes. When the points are distributed in, say a two dimensional space, it is called spatial point process. The elapsed time from an event to the next is called the interarrival time.

Definition 2.2. *(Ross, 1995, p.64) Consider a point process $\{T_k\}_{k=1,2,\dots}$. Define $W_k = T_k - T_{k-1}$ with the convention $T_0 = 0$. The sequence $\{W_k\}_{k=1,2,\dots}$ is called the interarrival times.*

Definition 2.3. *(Sigman, 1995, p.1, (1.2)) The process with right-continuous sample paths*

$$N(t) = \sum_{k=1,2,\dots} \mathbb{1}_{\{T_k \leq t\}}$$

is called the counting process associated with the point process $\{T_k\}_{k=1,2,\dots}$.

The counting process $N(t)$ counts the number of occurrences up to and including time t, and $N(\{t\}) = N(t) - N(t^-)$ in the following.

Definition 2.4. *(Daley and Vere-Jones, 2003, p.47, Definition 3.3.II) A point process is called simple when* $P\{N(\{t\}) = 0 \text{ or } 1 \text{ for all } t\} = 1$.

Definition 2.5. *(Last and Brandt, 1995, p.8, (1.5.1)) For a point process* $\{T_k\}_{k=1,2,\dots}$ *we define the point of explosion to be*

$$T_\infty = \lim_{k\to\infty} T_k$$

and the point process is called non-explosive if $T_k \to \infty$ *almost surely as* $k \to \infty$.

We assume the point processes discussed in this article are both simple and non-explosive.

Definition 2.6. *(Grandell, 1997, p.6) A quantity which, for small* h, *is of lower order than* h, *is denoted as* $o(h)$, *i.e.*

$$\lim_{h\to 0^+} \frac{o(h)}{h} = 0.$$

Definition 2.7. *The intensity of a point process* $N(t)$ *is defined by Khinchin (1960) as*

$$\lambda(t) = \lim_{h\to 0^+} \frac{P\{N(t, t+h] > 0\}}{h}.$$

Intuitively $\lambda(t)$ measures the rate of occurrence of points because in most cases (under certain assumptions)

$$\lim_{h\to 0^+} \frac{P\{N(t, t+h] > 0\}}{h} = \lim_{h\to 0^+} \frac{E[N(t, t+h)]}{h}$$

and on the right hand side we have number over time, or rate.

3 Homogeneous Poisson Processes

3.1 Definition

There are alternative definitions for homogeneous Poisson processes. The following definition defines a homogeneous Poisson process based on the intensity.

Definition 3.1. *(Ross, 2009, pp.314-315, Definition 5.3, Theorem 5.1) The point process* N *is a (homogeneous) Poisson process with rate* λ, $\lambda > 0$, *if and only if, for all* $t \geq 0$ *and* $h \to 0^+$,

(i) $N(0) = 0$.

(ii) The process has independent increments.

(iii) $P\{N(t+h) - N(t) = 1\} = \lambda h + o(h)$.

(iv) $P\{N(t+h) - N(t) \geq 2\} = o(h)$.

(i) says the process starts from 0 at time 0. (ii) means the increments are independent over disjoint bounded subintervals. (iii) simply says that λ is the intensity and (iv) is called orderly, which is the probability distribution property analogously to the sample path property of simpleness in Definition 2.4.

3.2 Properties

3.2.1 Number of Occurrence

The following property states that the number of points from a homogeneous Poisson process over any interval is Poisson distributed.

Proposition 3.1. *The number of events in any interval of length t is Poisson distributed with mean λt. That is, for all $s, t \geq 0$,*

$$P\{N(t+s) - N(s) = n\} = e^{-\lambda t}\frac{(\lambda t)^n}{n!}$$

for $n = 0, 1, \ldots$.

Proof. Define

$$p_n(t) = P\{N(t) = n\}$$

for $t \geq 0$ and integers $n \geq 0$. Conditions (iii), (iv) of Definition 3.1 imply that the process N is stationary. Together with condition (ii) of Definition 3.1, it means that N has both stationary (not a function of time t) and independent increments, so we have

$$\begin{aligned}
p_0(t+h) &= P\{N(t+h) = 0\} \\
&= P\{N(t) - N(0) = 0, N(t+h) - N(t) = 0\} \\
\text{[independent increments]} &= P\{N(t) - N(0) = 0\} \cdot P\{N(t+h) - N(t) = 0\} \\
\text{[stationary]} &= P\{N(t) = 0\} \cdot P\{N(h) = 0\} \\
&= p_0(t)\,(1 - \lambda h + o(h))
\end{aligned}$$

and so,

$$\frac{p_0(t+h) - p_0(t)}{h} = -p_0(t)\lambda + \frac{o(h)}{h}.$$

Let $h \to 0^+$, we have the ODE

$$\frac{dp_0(t)}{dt} = -p_0(t)\lambda$$

or equivalently,

$$\frac{dp_0(t)}{p_0(t)} = -\lambda dt$$

with the general solution

$$p_0(t) = Ce^{-\lambda t}.$$

By setting $t = 0$, we have

$$1 = P\{N(0) = 0\} = p_0(0) = C$$

and so,

$$p_0(t) = e^{-\lambda t}. \tag{1}$$

Now for $n \geq 1$,

$$\begin{aligned}
p_n(t+h) &= P\{N(t+h) = n\} \\
&= P\{N(t) = n, N(t+h) - N(t) = 0\} + P\{N(t) = n-1, N(t+h) - N(t) = 1\} \\
&\quad + P\{N(t+h) = n, N(t+h) - N(t) \geq 2\} \\
&= P\{N(t) = n\} \cdot P\{N(h) = 0\} + P\{N(t) = n-1\} \cdot P\{N(h) = 1\} + o(h) \\
&= p_n(t)(1 - \lambda h + o(h)) + p_{n-1}(t)(\lambda h + o(h)) + o(h) \\
&= (1 - \lambda h)p_n(t) + \lambda h p_{n-1}(t) + o(h)
\end{aligned} \tag{2}$$

In the derivation above we used the fact that

$$P\{N(t+h)=n, N(t+h)-N(t)\geq 2\}=o(h)$$

which is true, because,

$$\lim_{h\to 0^+}\frac{P\{N(t+h)=n, N(t+h)-N(t)\geq 2\}}{h}\leq \lim_{h\to 0^+}\frac{P\{N(t+h)-N(t)\geq 2\}}{h}=0.$$

Rearranging (2), we have

$$\frac{p_n(t+h)-p_n(t)}{h}=-\lambda p_n(t)+\lambda p_{n-1}(t)+\frac{o(h)}{h}$$

and by letting $h\to 0^+$ we have

$$\frac{dp_n(t)}{dt}=-\lambda p_n(t)+\lambda p_{n-1}(t)$$

or equivalently,

$$\frac{dp_n(t)}{dt}+\lambda p_n(t)=\lambda p_{n-1}(t) \tag{3}$$

which is a first order linear ODE and by multiplying the integrating factor $e^{\lambda t}$ to both sides, we have

$$\frac{d\left(e^{\lambda t}p_n(t)\right)}{dt}=\lambda e^{\lambda t}p_{n-1}(t) \tag{4}$$

We now use mathematical induction to show that (4) implies

$$p_n(t)=e^{-\lambda t}\frac{(\lambda t)^n}{n!} \tag{5}$$

for all integers $n\geq 1$. When $k=1$, (4) becomes

$$\frac{d\left(e^{\lambda t}p_1(t)\right)}{dt}=\lambda$$

because of (1). Solving the ODE for $p_1(t)$ we get

$$e^{\lambda t}p_1(t)=\lambda t+C$$

and by setting $t=0$ we have

$$0=P\{N(0)=1\}=p_1(0)=C$$

so

$$p_1(t)=e^{-\lambda t}\lambda t$$

and (5) holds. Assuming (5) holds for $k=n-1\geq 1$, then

$$p_{n-1}(t)=e^{-\lambda t}\frac{(\lambda t)^{n-1}}{(n-1)!}$$

and (4) becomes

$$\frac{d\left(e^{\lambda t}p_n(t)\right)}{dt}=\lambda e^{\lambda t}e^{-\lambda t}\frac{(\lambda t)^{n-1}}{(n-1)!}=\lambda\frac{(\lambda t)^{n-1}}{(n-1)!}=\frac{\lambda^n}{(n-1)!}\cdot t^{n-1}$$

which yields

$$e^{\lambda t}p_n(t)=\frac{\lambda^n}{(n-1)!}\cdot\frac{t^n}{n}+C$$

or equivalently,

$$p_n(t)=e^{-\lambda t}\frac{(\lambda t)^n}{n!}+Ce^{-\lambda t}.$$

By setting $t=0$ we have

$$0=P\{N(0)=n\}=p_n(0)=C$$

which proves that (5) holds true for all integers $n\geq 0$, and by stationarity, for any $s,t\geq 0$,

$$P\{N(t+s)-N(s)=n\}=P\{N(t)-N(0)=n\}=P\{N(t)=n\}=e^{-\lambda t}\frac{(\lambda t)^n}{n!}$$

which completes the proof. □

3.2.2 Interarrival Times

Immediately we have the following property about the interarrival times.

Proposition 3.2. *(Ross, 1995, p.64, Proposition 2.2.1) The interarrival times $\{W_k\}_{k=1,2,\ldots}$ of a (homogeneous) Poisson process with rate $\lambda > 0$, are independent identically distributed exponential random variables having mean $1/\lambda$.*

Proof. From Proposition 3.1,

$$P\{W_1 > t\} = P\{N(t) = 0\} = e^{-\lambda t}$$

so W_1 is exponentially distributed with parameter λ. For W_2, by independent increments, we have

$$\begin{aligned} P\{W_2 > t | W_1 = s\} &= P\{N(t+s) - N(s) = 0 | N(s) - N(s^-) = 1\} \\ &= P\{N(t+s) - N(s) = 0\} \\ &= P\{N(t) = 0\} \\ &= e^{-\lambda t} \end{aligned}$$

and

$$P\{W_2 > t\} = E\left[P\{W_2 > t | W_1\}\right] = e^{-\lambda t}$$

so W_2 is exponentially distributed with parameter λ. Also the fact that

$$P\{W_2 > t\} = P\{W_2 > t | W_1 = s\}$$

implies the independence between W_2 and W_1. Repeating the same argument proves the proposition. □

Naturally we have the following algorithm for generating the points t_k in a homogeneous Poisson process by generating the interarrival times w_k and taking the sum $t_k = \sum_{i=1}^{k} w_i$. This is sometimes called the "interarrival scheduling". Assuming there is a random number generator `uniform(0,1)` that generates uniformly distributed random variables on $(0, 1)$.

Algorithm 1: Simulation of a Homogeneous Poisson Process with Rate λ, on $[0, T]$.

Input: λ, T
Initialize $n = 0$, $t_0 = 0$;
while *True* **do**
  Generate $u \sim$ `uniform(0,1)`;
  Let $w = -\ln u/\lambda$; `// so that` $w \sim$ `exponential(`λ`)`
  Set $t_{n+1} = t_n + w$;
  if $t_{n+1} > T$ **then**
    return $\{t_k\}_{k=1,2,\ldots,n}$
  else
    Set $n = n + 1$;
  end
end

3.2.3 Joint Density

From Proposition 3.1 we can easily derive the joint density of the occurrence times.

Theorem 3.3. *The joint density that exactly n events occur in the interval $[0, T]$ for a Poisson process with rate λ taking values $0 < t_1 < t_2 < \cdots < t_n \leq T$ is*

$$f_{T_1,\ldots,T_n}(t_1, \ldots, t_n; T_{n+1} > T) = \lambda^n e^{-\lambda T}.$$

Proof. For $k = 1, 2, \ldots, n+1$, let $W_k = T_k - T_{k-1}$ be the interarrival times and $w_k = t_k - t_{k-1}$ be the realization, with the convention $T_0 = t_0 = 0$. By Proposition 3.2, $\{W_k\}_{k=1,2,\ldots,n+1}$ are independent identically distributed exponential random variables with parameter λ and density

$$f_W(w) = \lambda e^{-\lambda w}$$

and tail CDF

$$P\{W > t\} = \bar{F}(t) = 1 - F(t) = e^{-\lambda t}$$

so by the change of variables formula (see Casella and Berger, 2002, p.158, (4.3.2) and p.185, (4.6.7)), the joint density function of $T_1, \ldots, T_n$ is

$$\begin{aligned} f_{T_1,\ldots,T_n}(t_1, \ldots, t_n; T_{n+1} > T) &= f_{W_1,\ldots,W_n}(w_1, \ldots, w_n)|J|^{-1} \cdot P\{W_{n+1} > T - t_n\} \\ &= \prod_{k=1}^{n} f_W(t_k - t_{k-1}) \cdot e^{-\lambda(T-t_n)} \\ &= \lambda^n e^{-\lambda \sum_{k=1}^{n}(t_k - t_{k-1})} \cdot e^{-\lambda(T-t_n)} \\ &= \lambda^n e^{-\lambda T} \end{aligned}$$

where

$$J_{ij} = \frac{dT_i}{dW_j}$$

and $|J| = 1$ because J is lower triangular matrix with all non-zero elements being 1. □

4 Inhomogeneous Poisson Processes

4.1 Definition

By allowing the intensity λ in Definition 3.1 to vary according to a deterministic function of t, we have the following definition for inhomogeneous Poisson process.

Definition 4.1. *(Ross, 2009, p.339, Definition 5.4) The point process N is said to be an inhomogeneous Poisson process with intensity function $\lambda(t) \geq 0$, $t \geq 0$, if*

(i) $N(0) = 0$.

(ii) The process has independent increments.

(iii) $P\{N(t+h) - N(h) = 1\} = \lambda(t)h + o(h)$.

(iv) $P\{N(t+h) - N(h) \geq 2\} = o(h)$.

Analogously, we have a similar property for inhomogeneous Poisson processes.

Proposition 4.1. *Let the point process N be an inhomogeneous Poisson process with intensity function $\lambda(t)$, then $N(t)$ follows a Poisson distribution with parameter $\int_0^t \lambda(s)ds$, i.e.*

$$P\{N(t) = n\} = \frac{e^{-\int_0^t \lambda(s)ds} \left(\int_0^t \lambda(s)ds\right)^n}{n!} \tag{6}$$

Proof. The proof is similar to the second part of the proof for Proposition 3.1. Analogously to (1) for the homogeneous case, we now have

$$p_0(t) = e^{-\int_0^t \lambda(s)ds}$$

by replacing λ with $\lambda(t)$ in the derivation. Also (3) becomes

$$\frac{dp_n(t)}{dt} + \lambda(t)p_n(t) = \lambda(t)p_{n-1}(t)$$

and by multiplying $e^{\int_0^t \lambda(s)ds}$ to both sides, we now have the ODE

$$\frac{d\left(e^{\int_0^t \lambda(s)ds}p_n(t)\right)}{dt} = \lambda(t)e^{\int_0^t \lambda(s)ds}p_{n-1}(t).$$

We again use mathematical induction by assuming

$$p_{n-1}(t) = \frac{e^{-\int_0^t \lambda(s)ds}\left(\int_0^t \lambda(s)ds\right)^{n-1}}{(n-1)!}$$

so that

$$\frac{d\left(e^{\int_0^t \lambda(s)ds}p_n(t)\right)}{dt} = \lambda(t)\frac{\left(\int_0^t \lambda(s)ds\right)^{n-1}}{(n-1)!}$$

solving which yields

$$p_n(t) = e^{-\int_0^t \lambda(s)ds}\frac{\left(\int_0^t \lambda(s)ds\right)^{n}}{n!}.$$

□

Apparently, the number of points in the interval $[a, b]$ follows a Poisson distribution with parameter $\int_a^b \lambda(s)ds$, or more precisely,

$$P\{N(b) - N(a) = n\} = \frac{e^{-\int_a^b \lambda(s)ds}\left(\int_a^b \lambda(s)ds\right)^{n}}{n!}.$$

Letting $n = 0$ we get the probability that there is no point in the interval $[a, b]$ to be

$$P\{N(a, b] = 0\} = e^{-\int_a^b \lambda(s)ds} \tag{7}$$

which determines the law of occurrence for the next point. For simple point process, points occur one by one and by the checking this condition we can make sure whether the point process is an inhomogeneous Poisson process.

4.2 Simulation

To simulate the points in an inhomogeneous Poisson process, we can simulate the points sequentially. We only need to guarantee that (7) is satisfied and the following is one such algorithm, sometimes called the Lewis' thinning algorithm.

Figure 1 shows an illustrative example of a simulated inhomogeneous Poisson process on the interval $[0, 2\pi]$, with the intensity function $\lambda(t) = 1 + \sin(t)$ shown as the solid curve. The rate of the homogeneous Poisson process which dominates $\lambda(t)$ is set to be $\bar{\lambda} = \sup_{0\le t\le 2\pi} \lambda(t) = 2$ and is shown as the horizontal dashed line. The simulated points for the homogeneous Poisson process are shown as circles, while those accepted as the points for the inhomogeneous Poisson process are labeled with cross marks. Vertically above each point on the x-axis, there is another point plotted with a height of $D\bar{\lambda}$, where D is the uniform random variable used in line 7 of Algorithm 2. The points above $\lambda(t)$ are those rejected and are marked with circles; the ones below accepted and labeled with cross marks.

The following theorem describes the algorithm mathematically and it makes sure the resulting point process satisfies (7).

Algorithm 2: (Lewis and Shedler, 1979, p.7, Algorithm 1) Simulation of an Inhomogeneous Poisson Process with Bounded Intensity Function $\lambda(t)$, on $[0, T]$.

Input: $\lambda(t)$, T
Initialize $n = m = 0$, $t_0 = s_0 = 0$, $\bar{\lambda} = \sup_{0 \le t \le T} \lambda(t)$;
while $s_m < T$ **do**
  Generate $u \sim$ `uniform(0,1)`;
  Let $w = -\ln u/\bar{\lambda}$; `// so that` $w \sim$ `exponential(`$\bar{\lambda}$`)`
  Set $s_{m+1} = s_m + w$; `//` $\{s_m\}$ `are points in the homo. Poisson process`
  Generate $D \sim$ `uniform(0,1)`;
  if $D \le \lambda(s_{m+1})/\bar{\lambda}$ **then** `// accepting with probability` $\lambda(s_{m+1})/\bar{\lambda}$
    $t_{n+1} = s_{m+1}$; `//` $\{t_n\}$ `are points in the inhomo. Poisson process`
    $n = n + 1$; `// updating` n `to the index of last point in` $\{t_n\}$
  end
  $m = m + 1$; `// updating` m `to the index of last point in` $\{s_m\}$
end
if $t_n \le T$ **then**
  return $\{t_k\}_{k=1,2,\ldots,n}$
else
  return $\{t_k\}_{k=1,2,\ldots,n-1}$
end

Theorem 4.2. *Consider a homogeneous Poisson process $\bar{N}(t)$ with intensity function $\bar{\lambda}$. Let $\bar{t}_1, \bar{t}_2, \ldots, \bar{t}_{\bar{N}(T)}$ be the points of the process in the interval $(0, T]$. Suppose that for $0 \le t \le T$, $0 \le \lambda(t) \le \bar{\lambda}$. For $k = 1, 2, \ldots, \bar{N}(T)$, delete the point $\bar{t}_k$ with probability $1 - \lambda(\bar{t}_k)/\bar{\lambda}$; then the remaining points form a point process $N(t)$ satisfying (7).*

Proof. Consider the probability where there are n points in the point process $\bar{N}(t)$ and each of them is deleted and denote it as p_n. From Theorem 3.3 the joint density of n points taking values $a \le t_1 < \cdots < t_n \le b$ is $\bar{\lambda}^n e^{-\bar{\lambda}(b-a)}$. Consider the unordered values $s_1, \ldots, s_n$ such that $\{s_1, \ldots, s_n\} = \{t_1, \ldots, t_n\}$, where each s_k takes value in $[a, b]$. The probability

$$\begin{aligned}
p_n &= \int_a^b \int_{t_1}^b \cdots \int_{t_{n-1}}^b \bar{\lambda}^n e^{-\bar{\lambda}(b-a)} \prod_{k=1}^n \left(1 - \frac{\lambda(t_k)}{\bar{\lambda}}\right) dt_1 dt_2 \cdots dt_n \\
&= \frac{e^{-\bar{\lambda}(b-a)}}{n!} \int_a^b \int_a^b \cdots \int_a^b \prod_{k=1}^n \left(\bar{\lambda} - \lambda(t_k)\right) ds_1 ds_2 \cdots ds_n \\
&= \frac{e^{-\bar{\lambda}(b-a)}}{n!} \left(\int_a^b \left(\bar{\lambda} - \lambda(t_k)\right) ds\right)^n \\
&= \frac{e^{-\bar{\lambda}(b-a)}}{n!} \left(\bar{\lambda}(b-a) - \int_a^b \lambda(s) ds\right)^n
\end{aligned}$$

Now the number of points in the point process $\bar{N}(t)$, or n, can be any non-negative integer and the sum

$$\begin{aligned}
\sum_{n=0}^{\infty} p_n &= e^{-\bar{\lambda}(b-a)} \sum_{n=0}^{\infty} \left(\bar{\lambda}(b-a) - \int_a^b \lambda(s) ds\right)^n / n! \\
&= e^{-\bar{\lambda}(b-a)} \cdot e^{\bar{\lambda}(b-a) - \int_a^b \lambda(s) ds} \\
&= e^{-\int_a^b \lambda(s) ds}
\end{aligned}$$

equals the probability that there is no point in the interval $[a, b]$, and (7) is satisfied. □

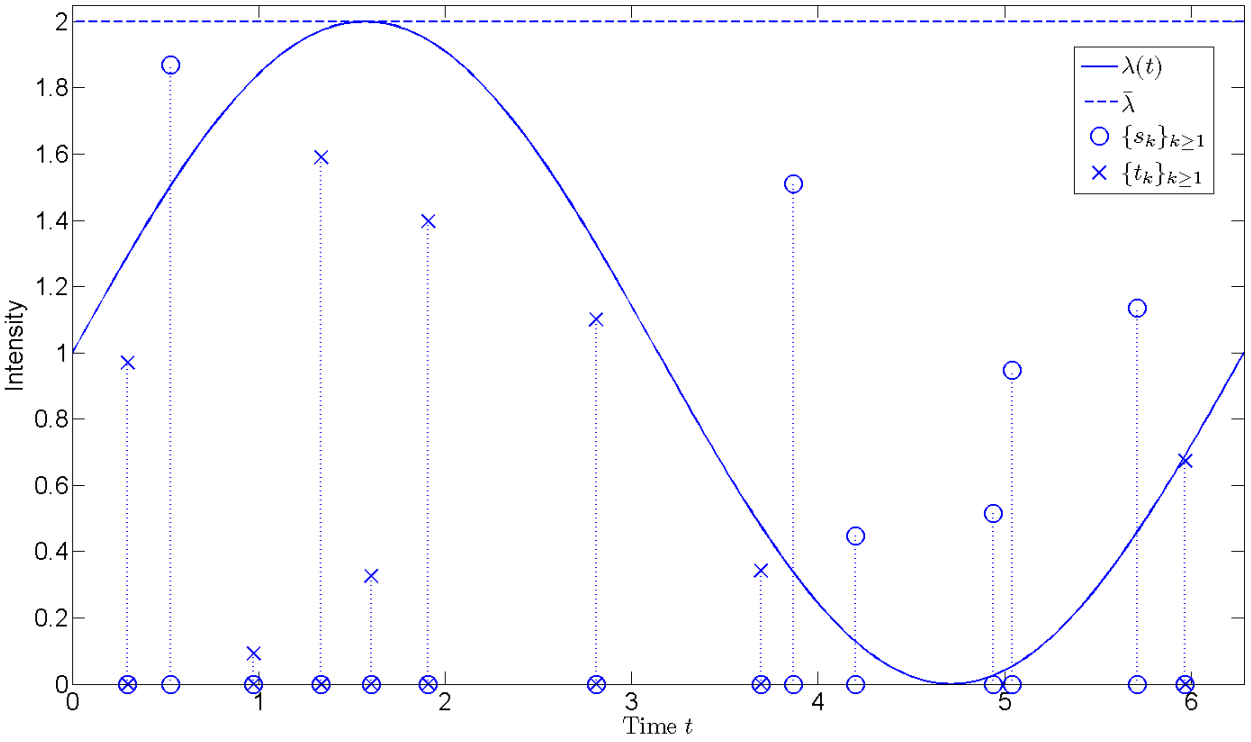

Figure 1: An illustrative example of simulating an inhomogeneous Poisson process with intensity function $\lambda(t) = 1 + \sin(t)$ on the interval $[0, 2\pi]$ using Algorithm 2. The intensity functions for the homogeneous and inhomogeneous Poisson processes are shown as the horizontal dashed line and the solid curve, respectively. On the x-axis, the simulated points for the homogeneous Poisson process are marked by circles, and the points labeled as cross marks, are the simulated points for the inhomogeneous Poisson process. Vertically above each point on the x-axis, a point with a height of $D\bar{\lambda}$ is also plotted, with D being the uniform random variable generated during the simulation. The points above $\lambda(t)$ are rejected and marked with circles, while the points below are accepted as the points for the inhomogeneous Poisson process and are marked with crosses.

Remark 4.1. *Notice that in Theorem 4.2 the only constraint on the constant $\bar{\lambda}$ is that $\lambda(t) \leq \bar{\lambda}$ for all t in the domain of interest, $[0, T]$. Because $\bar{\lambda}$ is the intensity of the homogeneous Poisson process $\bar{N}$, so a larger value of $\bar{\lambda}$ will lead to more generated points in $\bar{N}$. At the meantime, $\bar{\lambda}$ determines the probability at which a point is accepted, and a larger $\bar{\lambda}$ corresponds to a smaller probability of point acceptance. As a result, a smaller value of $\bar{\lambda}$ is preferred in practice when using Algorithm 2, for a higher efficiency (less iterations of the while-loop) and hence $\bar{\lambda} = \sup_{t\in[0,T]} \lambda(t)$ is often the choice.*

Because a Poisson process has independent increments, i.e.

$$\begin{aligned} P\{N(s,t] = 0 | N(0,s] = 0\} &= \frac{P\{N(0,t] = 0\}}{P\{N(0,s] = 0\}} \\ &= \frac{e^{-\int_0^t \lambda(u)du}}{e^{-\int_0^s \lambda(u)du}} \\ &= e^{-\int_s^t \lambda(u)du} \\ &= P\{N(s,t] = 0\} \end{aligned}$$

for any $0 < s < t \leq T$, and the generation of points is sequential (the variable s_m is increasing through the iterations of the while-loop), the choice of $\bar{\lambda}$ can be modified as $\bar{\lambda} = \sup_{t \in [s_m, T]} \lambda(t)$, to allow adjustment of $\bar{\lambda}$ when generating the next point in $\bar{N}$.

5 Hawkes Processes

5.1 Definition

Definition 5.1. *(Donald L. Snyder, 1991, p.287) A point process is called self-exciting if the intensity $\lambda(\cdot)$ depends not only on time t but also the entire past of the point process.*

The entire past, or history of a point process, is defined mathematically by the definition given below.

Definition 5.2. *(Grandell, 1997, p.51) For any process $N(t)$, the natural filtration $F^N = \left(\mathcal{F}_t^N; t \geq 0\right)$ is defined by*

$$\mathcal{F}_t^N = \sigma\left\{N(s); s \leq t\right\}.$$

In other words, $\mathcal{F}_t^N$ is the σ-algebra generated by N up to time t, and represents the internal history of N up to time t.

With this definition, we can define the intensity $\lambda(\cdot)$ for a self-exciting point process analogously as (2.7). However, more appropriately here, it should be called the intensity process, because $\lambda(\cdot)$ is itself a random process whose sample path depends on the realization of $N(\cdot)$.

Definition 5.3. *(Last and Brandt, 1995, p.10, (1.6.4)) Let $N(t)$ be a point process with natural filtration $\mathcal{F}_t^N$. The left-continuous process defined by*

$$\lambda(t|\mathcal{F}_{t^-}^N) = \lim_{h \to 0^+} \frac{P\{N(t+h) - N(t) > 0 | \mathcal{F}_{t^-}^N\}}{h}$$

is called the stochastic intensity function of the point process.

The reason for using left continuity is connected with predictability: if the conditional intensity has a discontinuity at a point of the process, then its value at that point should be defined by the history before that point, not by what happens at the point itself (Daley and Vere-Jones, 2003, p.232).

The Hawkes processes introduced below is an example of self-exciting point processes.

Definition 5.4. *(Hawkes, 1971, p.84, (7) and (8)) A univariate simple point process $N(t)$ satisfying*

(i) $N(t) = 0$.

(ii) $\lambda(t)$ *is a left-continuous stochastic process given by the Stieltjes integral*

$$\lambda(t) = \mu + \int_0^t \alpha e^{-\beta(t-s)} dN(s) = \mu + \sum_{\{k: t_k < t\}} \alpha e^{-\beta(t-t_k)} \tag{8}$$

where $\mu > 0$ and $0 < \alpha < \beta$.

(iii) $\lambda(t)$ *is the stochastic intensity of the point process*

$$P\{N(t+h) - N(t) = 1 | \mathcal{F}_{t^-}^N\} = \lambda(t)h + o(h)$$

(iv) The point process is orderly

$$P\{N(t+h) - N(t) \geq 2 | \mathcal{F}_{t^-}^N\} = o(h)$$

is called a univariate Hawkes process with exponential decay on $[0, \infty)$.

From Definition 5.4, it is easy to see that for a given realization, $\lambda(t)$ is piece-wisely non-increasing with jumps of size α at occurrences of points. More properties are shown in Figure 2 which illustrates the sample paths of the conditional intensity $\lambda(t)$ and the associated counting process $N(t)$ for a given univariate Hawkes process with exponential decay that has parameters $\mu = 1.2$, $\alpha = 0.6$ and $\beta = 0.8$. The left-continuous conditional intensity, which starts from and keeps at the base intensity $\mu = 1.2$ until the occurrence of the first point, is shown in the top panel. After each occurrence, the intensity jumps with a size of $\alpha = 0.6$ and then immediately starts decaying at a rate determined by $\beta = 0.8$. On the other hand, the associated right-continuous counting process, which is a step function that jump by 1 at each occurrence, is shown in the bottom panel.

5.2 Simulation

From (8) it is obvious that given the information of the first k points, $t_1, \ldots, t_k$, the intensity $\lambda(t)$ is deterministic on $[t_k, t_{k+1}]$, where t_{k+1} is the location of the next point which is stochastic. As a result, the generation of the next point in a Hawkes process can be considered as generating the first point in an inhomogeneous Poisson process. Ogata modified Algorithm 2 to simulate Hawkes processes. This is sometimes called Ogata's modified thinning algorithm. In his modification, $\bar{\lambda}$ is allowed to be adjusted as described in Remark 4.1.

Algorithm 3: (Ogata, 1981, p.25, Algorithm 2) Simulation of a Univariate Hawkes Poisson with Exponential Kernel $\gamma(u) = \alpha e^{-\beta u}$, on $[0, T]$.

Input: μ, α, β, T
1 Initialize $\mathcal{T} = \emptyset$, $s = 0$, $n = 0$;
2 **while** $s < T$ **do**
3 Set $\bar{\lambda} = \lambda(s^+) = \mu + \sum_{\tau \in \mathcal{T}} \alpha e^{-\beta(s-\tau)}$;
4 Generate $u \sim$ `uniform(0,1)`;
5 Let $w = -\ln u / \bar{\lambda}$; `// so that` $w \sim$ `exponential(`$\bar{\lambda}$`)`
6 Set $s = s + w$; `// so that` s `is the next candidate point`
7 Generate $D \sim$ `uniform(0,1)`;
8 **if** $D\bar{\lambda} \leq \lambda(s) = \mu + \sum_{\tau \in \mathcal{T}} \alpha e^{-\beta(s-\tau)}$ **then** `// accepting with prob.` $\lambda(s)/\bar{\lambda}$
9 $n = n + 1$; `// updating the number of points accepted`
10 $t_n = s$; `// naming it` t_n
11 $\mathcal{T} = \mathcal{T} \bigcup \{t_n\}$; `// adding` t_n `to the ordered set` $\mathcal{T}$
12 **end**
13 **end**
14 **if** $t_n \leq T$ **then**
15 **return** $\{t_k\}_{k=1,2,\ldots,n}$
16 **else**
17 **return** $\{t_k\}_{k=1,2,\ldots,n-1}$
18 **end**

Figure 3 illustrates the use of Algorithm 3 for a univariate Hawkes process with exponential decay. The parameters are $\mu = 1.2$, $\alpha = 0.6$ and $\beta = 0.8$. All the candidate points are marked on the x-axis where the vertical dotted lines are, among which the ones accepted to be the points for the Hawkes process are label with t_k where $k = 1, 2, \ldots, 8$. The conditional intensity $\lambda(t)$ is plotted as the solid curve. The left-continuous piece-wisely constant $\bar{\lambda}$ used in each iteration inside the "while" loop at line 2 of Algorithm 3 is plotted as the dashed lines. On each vertical dotted line, there are two points plotted. The lower ones are marked with triangles and represent the values of the uniform random variable D (ranges from 0 to 1) generated at line 7 of Algorithm 3; the upper ones represent the values of $D\bar{\lambda}$ (ranges from 0 to $\bar{\lambda}$) at line 8 and are either labeled with cross marks for those accepted (when $D\bar{\lambda} \leq \lambda(\cdot)$) or with circles for those rejected.

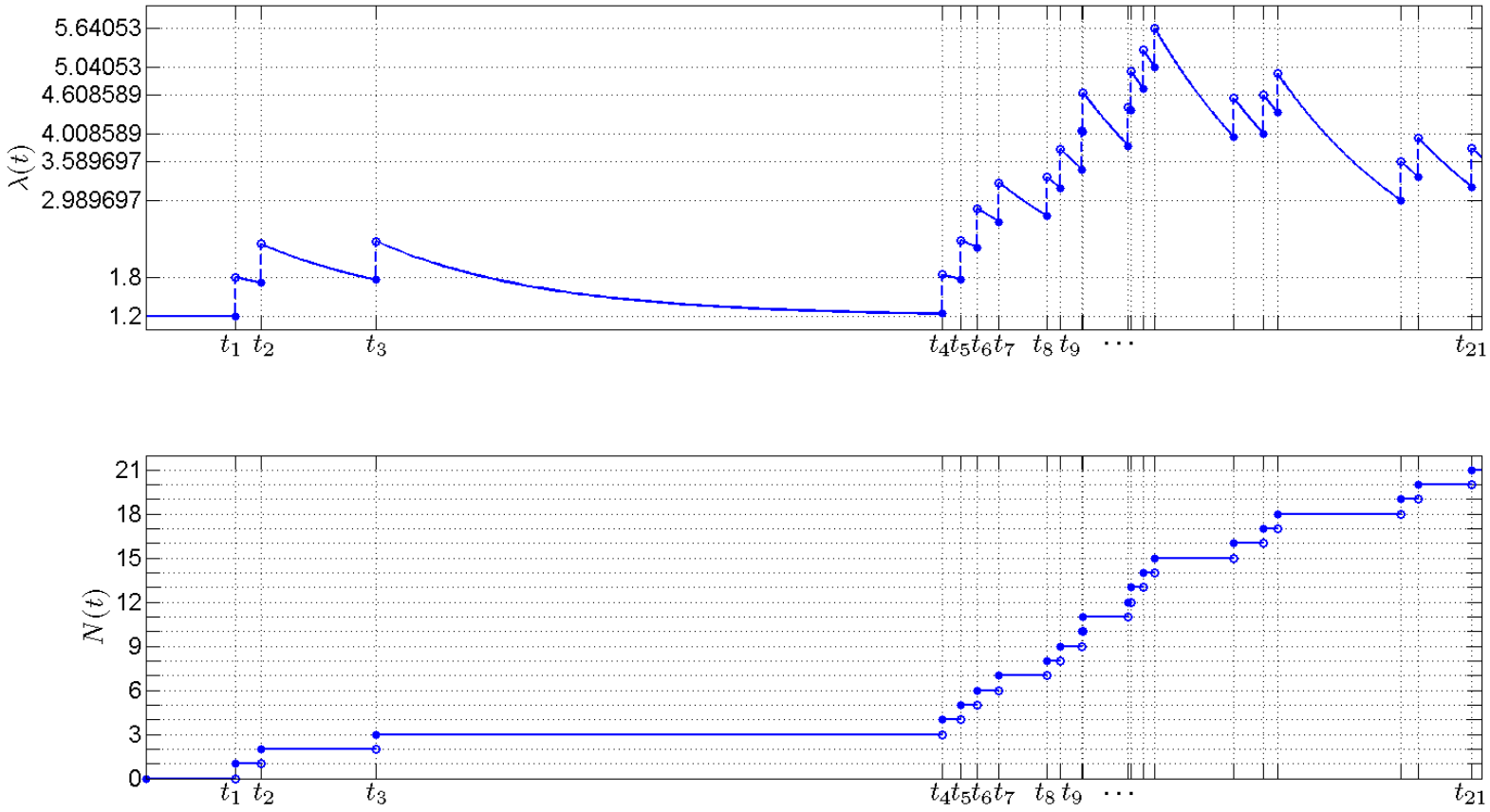

Figure 2: An illustrative example of the left-continuous conditional intensity $\lambda(t)$ (top panel) and the associated right-continuous counting process $N(t)$ (bottom panel) for a univariate Hawkes process with exponential decay that has parameters $\mu = 1.2$, $\alpha = 0.6$ and $\beta = 0.8$. The conditional intensity starts from and keeps at the base intensity $\mu = 1.2$ until the occurrence of the first point. After each occurrence, the intensity jumps with a size of $\alpha = 0.6$ and then immediately starts decaying at a rate determined by $\beta = 0.8$. On the other hand, the associated counting process is a step function that jump by 1 at each occurrence.

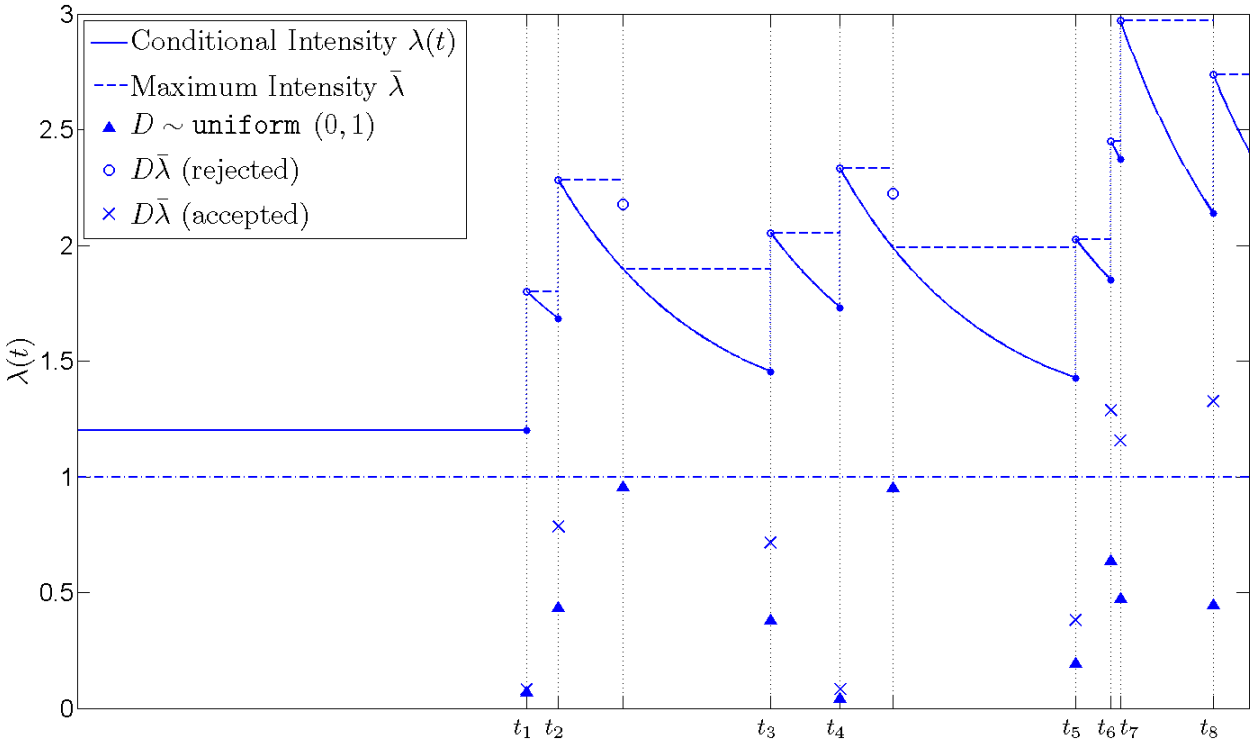

Figure 3: An illustrative example of simulating a univariate Hawkes process with exponential kernel $\gamma(u) = \alpha e^{-\beta u}$ where $\alpha = 0.6$, $\beta = 0.8$ and base intensity $u = 1.2$, using Algorithm 3. All the candidate points are marked on the x-axis where the vertical dotted lines are, among which the ones accepted to be the points for the Hawkes process are label with t_k where $k = 1, 2, \ldots, 8$. The conditional intensity $\lambda(t)$ is plotted as the solid curve. The left-continuous piece-wisely constant $\bar{\lambda}$ used in each iteration inside the "while" loop at line 2 of Algorithm 3 is plotted as the dashed lines. On each vertical dotted line, there are two points plotted. The lower ones are marked with triangles and represent the values of the uniform random variable D (ranges from 0 to 1) generated at line 7 of Algorithm 3; the upper ones represent the values of $D\bar{\lambda}$ (ranges from 0 to $\bar{\lambda}$) at line 8 and are either labeled with cross marks for those accepted (when $D\bar{\lambda} \leq \lambda(\cdot)$) or with circles for those rejected.

References

Casella, G. and R. Berger
2002. *Statistical Inference*, Duxbury advanced series in statistics and decision sciences. Thomson Learning.

Daley, D. and D. Vere-Jones
2003. *An Introduction to the Theory of Point Processes, Volume 1.* Springer.

Donald L. Snyder, M. I. M. a.
1991. *Random Point Processes in Time and Space*, Springer Texts in Electrical Engineering, 2 edition. Springer-Verlag New York.

Grandell, J.
1997. *Mixed Poisson Processes (Chapman & Hall/CRC Monographs on Statistics & Applied Probability).* Chapman and Hall/CRC.

Hawkes, A. G.
1971. Spectra of some self-exciting and mutually exciting point processes. *Biometrika*, 58(1):83–90.

Khinchin, A.
1960. *Mathematical Methods in the Theory of Queueing*, Griffin's statistical monographs & courses no. 7. Hafner.

Last, G. and A. Brandt
1995. *Marked Point Processes on the Real Line: The Dynamical Approach (Probability and Its Applications)*. Springer.

Lewis, P. A. W. and G. S. Shedler
1979. Simulation of nonhomogeneous Poisson processes by thinning. *Naval Res. Logistics Quart*, 26:403–413.

Ogata, Y.
1981. On lewis' simulation method for point processes. *IEEE Trans. Inform. Theory*, 27(1):23–31.

Ross, S. M.
1995. *Stochastic Processes*. Wiley.

Ross, S. M.
2009. *Introduction to Probability Models, Tenth Edition*. Academic Press.

Sigman, K.
1995. *Stationary Marked Point Processes: An Intuitive Approach (Stochastic Modeling Series)*. Chapman and Hall/CRC.