Real Time Series analysis and modelling

Aid machine learning with dynamical insight

Hailiang Du
Department of Mathematical Sciences, Durham University
hailiang.du@durham.ac.uk
Data Science Institute, London School of Economics and Political Science
h.l.du@lse.ac.uk

All theorems are true, All models are wrong. All data are inaccurate. What are we to do?

The aim of this course is to teach you how to deal with real data, to increase your scepticism regarding reliable modelling in practice, and to expand the tool box you carry to include nonlinear techniques, both deterministic and stochastic with the aid of dynamical insight.
In short: to get you to think before you compute (and perhaps afterwards too.)

Lecture 2

Properties of deterministic nonlinear dynamical systems

Dynamical Systems Jargon

State: the configuration of the dynamical system at any time, where it is and what it is doing.

$$\mathbf{x} = (x_1, x_2, ..., x_n)$$

State space: the collection of all possible states $\mathbf{x} \in \mathbb{S}$

Dynamics: the set of rules that determine the evolution of the state

$$\mathbf{x}_t = F^t(\mathbf{x}_0)$$

$\mathbf{x}_0$ is the starting state called the initial condition

Map: discrete dynamical system takes place at regular time intervals, for example $t \in \mathbb{Z}$

Flow: continuous dynamical system whose evolution takes place continuously

$$\frac{d\mathbf{x}}{dt} = F(\mathbf{x}) \qquad t \in \mathbb{R}$$

Stochastic dynamical system evolves randomly, for example AR(1)

For deterministic dynamical system, the dynamics and initial condition define the future state unambiguously.

A simple map (from several centuries ago)

Population Growth

One pair of rabbits in a large walled garden
One month till mature
one pair born per mature pair

A Dynamical System

$$f((a, y)) = (a + y, a)$$

Mature	Young	Total
0	1	

state: (a_i, y_i)

obs: time-series of population: $x_i = a_i + y_i$

A simple map (from several centuries ago)

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$$f((a, y)) = (a + y, a)$$

Mature	Young	Total
0	1	1
1	0	1
1	1	2
2	1	3
3	2	5
5	3	8

state (a_i, y_i)

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3	2	5
5	3	8

state (a_i, y_i)
obs: time-series of population: $x_i = a_i + y_i$

1, 1, 2, 3, 5, 8, …

$$x_{i+1} = ?$$

$$x_{i+1} = x_i + x_{i-1}$$

Fibonacci (Leonardo of Pisa 1202)

1, 1, 2, 3, 5, 8, …

Question: What would the forecast error be if there had been two pairs on day one? (Measurement error). ["two" is not enough information]

Mature	Young	Total
0	2	2
2	0	2
2	2	4
4	2	6
6	4	10
10	6	16

$$s_0 = x_0 + e$$

Mature	Young	Total
0	1	1
1	0	1
1	1	2
2	1	3
3	2	5
5	3	8

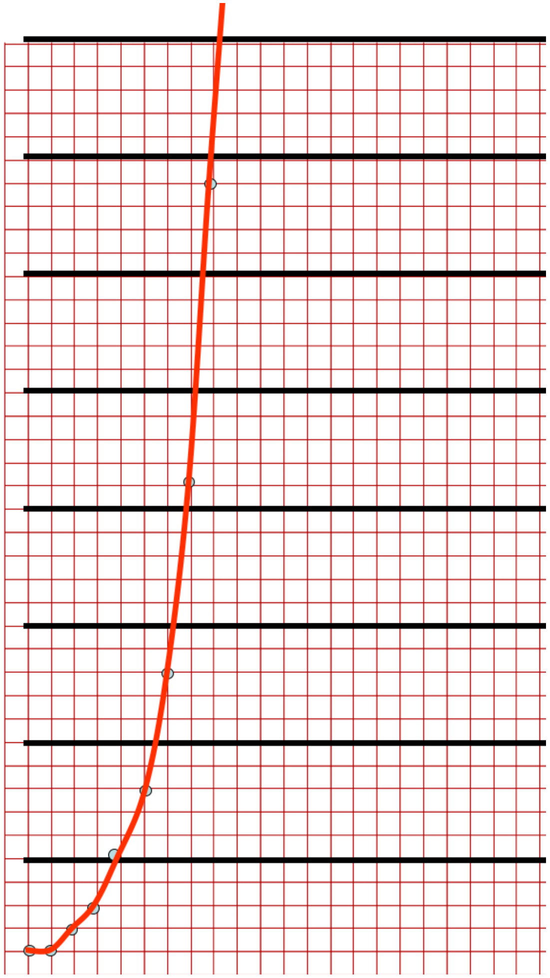
$$x_{i+1} = x_i + x_{i-1}$$

Fibonacci (Leonardo of Pisa 1202)

1, 1, 2, 3, 5, 8, …

Question: What would the forecast error be if there had been two pairs on day one? (Measurement error).

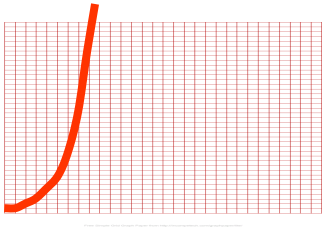
"effective exponential"



Question: If $x_i = \alpha x_{i-1}$ what is α?

$$\alpha = (1+\sqrt{5})/2$$

"The golden mean"

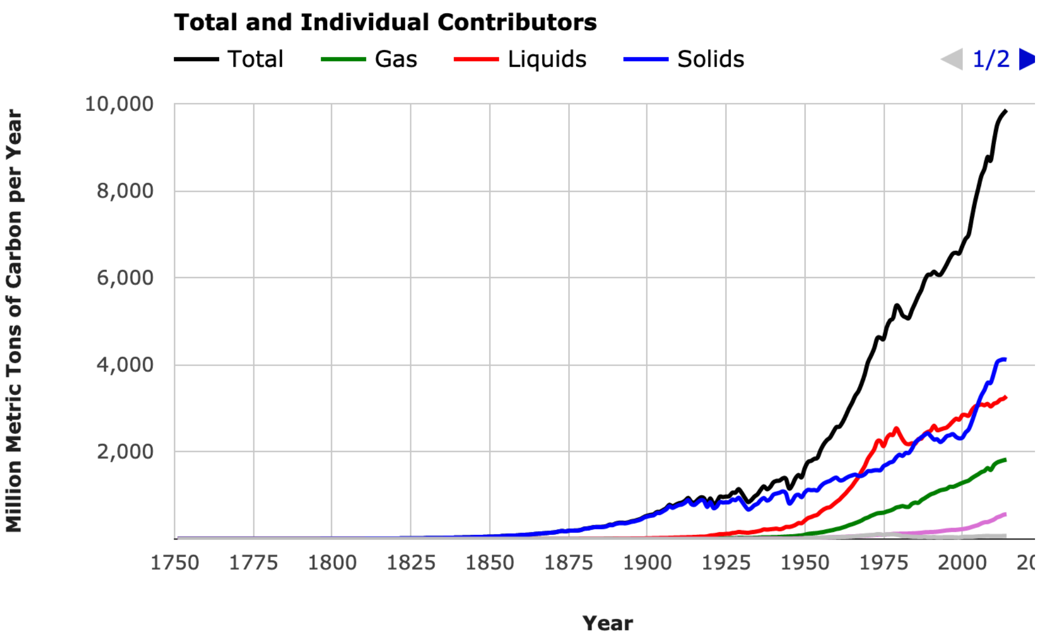


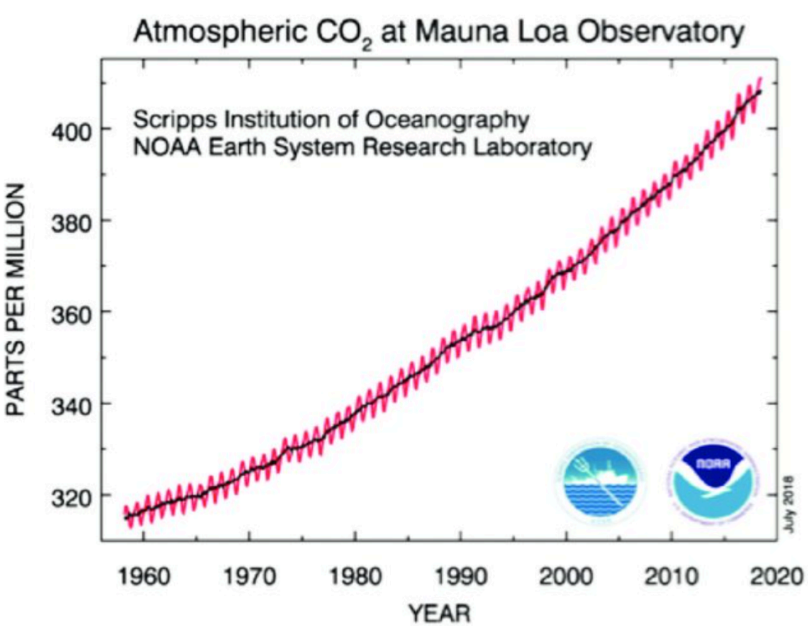
Exponential Growth

Solutions of the form $x_i = \alpha x_{i-1}$ imply exponential growth, while mathematically lovely this leads to unbounded growth which is physically unrealistic.
(unless of course, x is infinitesimal; is that physically plausible?)

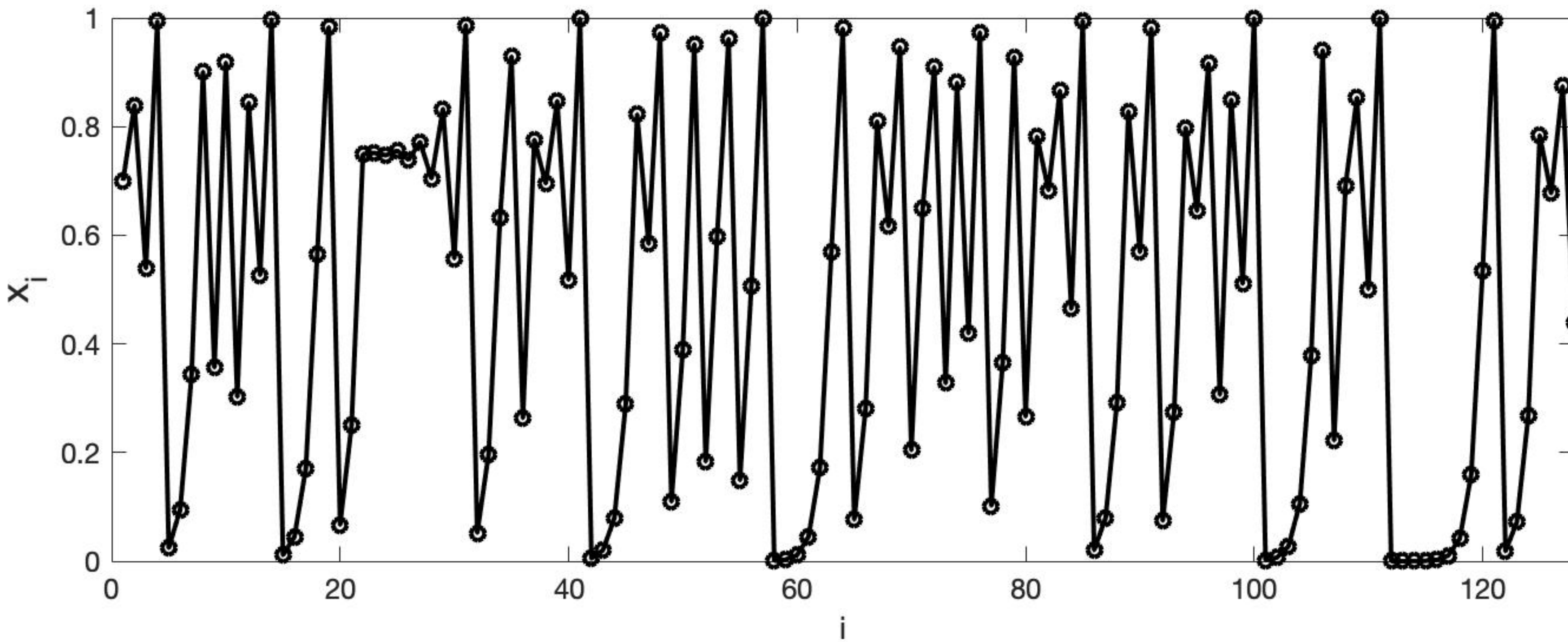
This lead us to improve the functional form of the model, including “second order” effects, like a finite food supply.

$$\boxed{x_{i+1} = a x_i(1 - x_i)}$$





Finite food supply: $x_{i+1} = 4x_i(1 - x_i)$

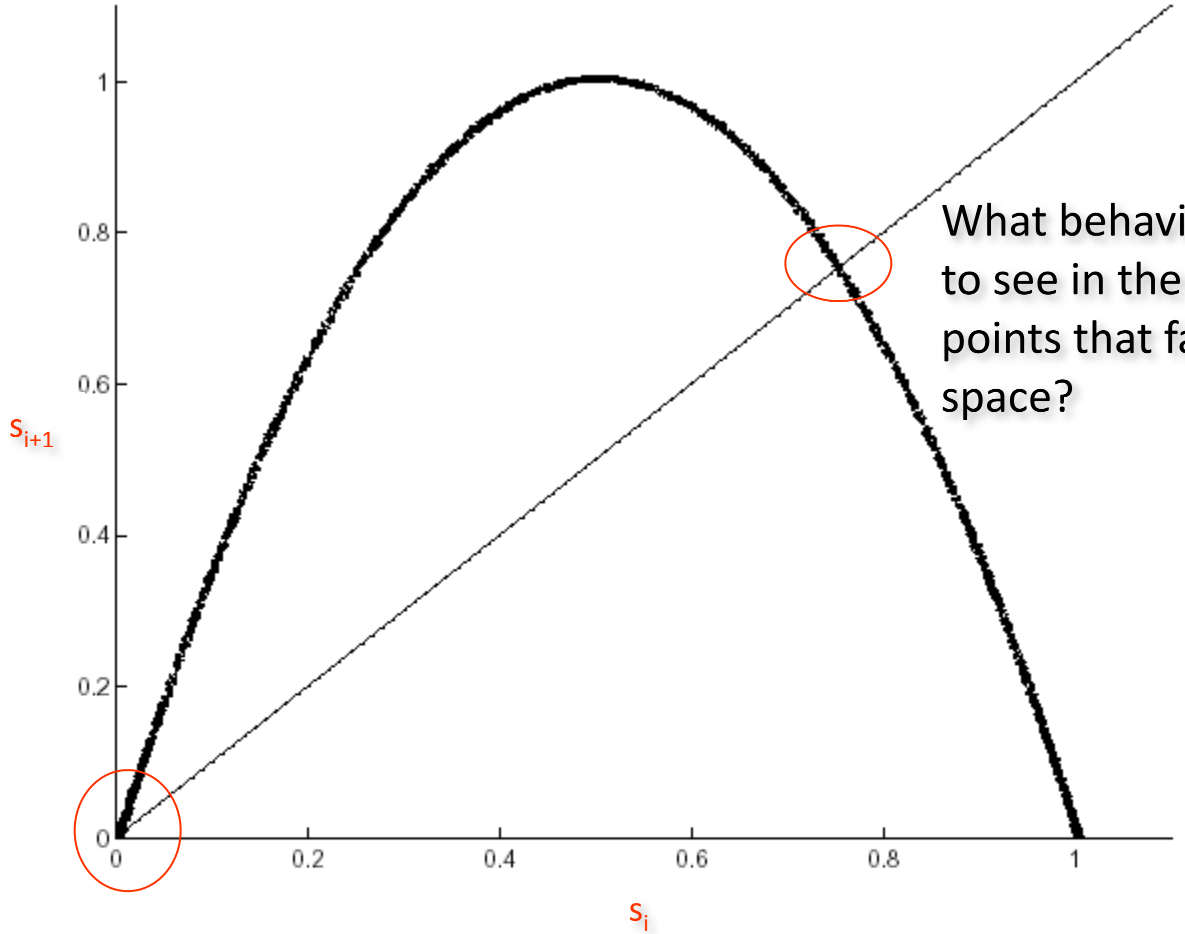


This is pretty predictable I'd say.

???Where is that Exponential Error Growth???

Could something this simple apply to anything other than rabbits?

A two dimensional delay reconstruction



What behaviour do we expect to see in the time series for points that fall here in delay space?

More definitions

Fixed point : $x^* : f(x^*) = x^*$

Iterates of f : $f : f^2(x) = f(f(x))$ and so on

An orbit or trajectory : $\{x_0, f(x_0), f^2(x_0), \ldots\} = \{x_0, x_1, x_2, \ldots\}$

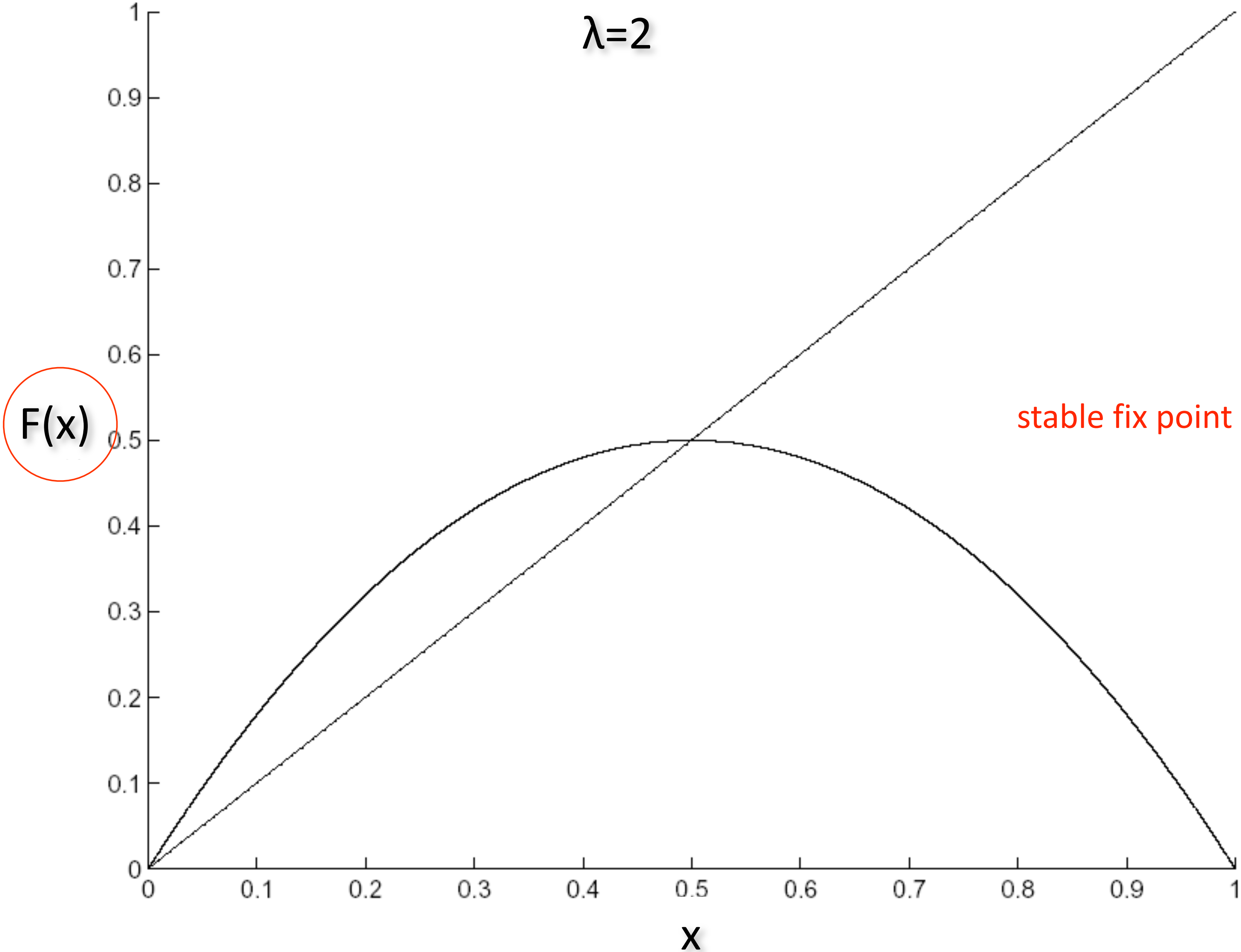
A periodic orbit of period p : $x_i : f^p(x) = x$ and $f^j(x) \neq x$ for $0 < j < p$

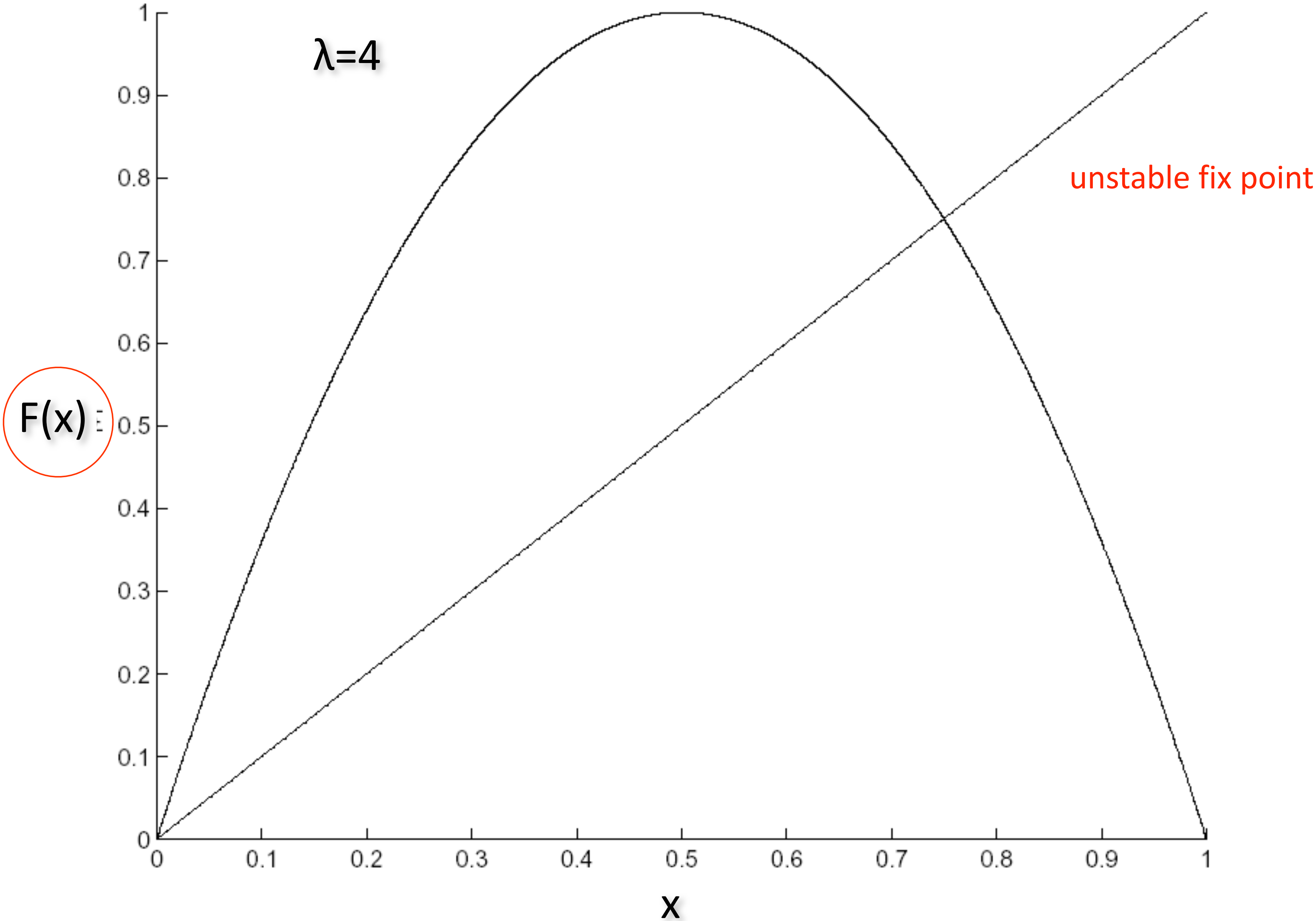
An aperiodic orbit : an orbit which does not become periodic for any p.

Basin of attraction β for a fixed point x^* :

$y \in \beta(x^*)$ iff $|f^k(y) - x^*| \to 0$ as $k \to \infty$

Note that inasmuch as a digital computer is a finite state machine, all maps iterated on a digital computer are asymptotically periodic.


λ=2
F(x)
stable fix point
x
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1



λ=4
unstable fix point
F(x)
x
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Quantify the stability of fixed points

Taylor Expansion:

$$\begin{aligned}\mathcal{F}(\mathbf{x}+\epsilon) &= \mathcal{F}(\mathbf{x}) + \epsilon\mathcal{F}^{'}(\mathbf{x}) + \frac{\epsilon^2}{2!}\mathcal{F}^{''}(\mathbf{x}) + \ldots \\ &\approx \mathcal{F}(\mathbf{x}) + \epsilon\mathcal{F}^{'}(\mathbf{x}),\end{aligned}$$

- $|\mathcal{F}^{'}(\mathbf{x}^*)| > 1$ unstable, points nearby are repelled
- $|\mathcal{F}^{'}(\mathbf{x}^*)| < 1$ stable, points nearby are attracted
- $|\mathcal{F}^{'}(\mathbf{x}^*)| = 1$ undetermined, have to investigate $\mathcal{F}^{''}$

Stability of a fixed point is determined by $F'(x^*)$

Of course, the stability will change with parameter values.

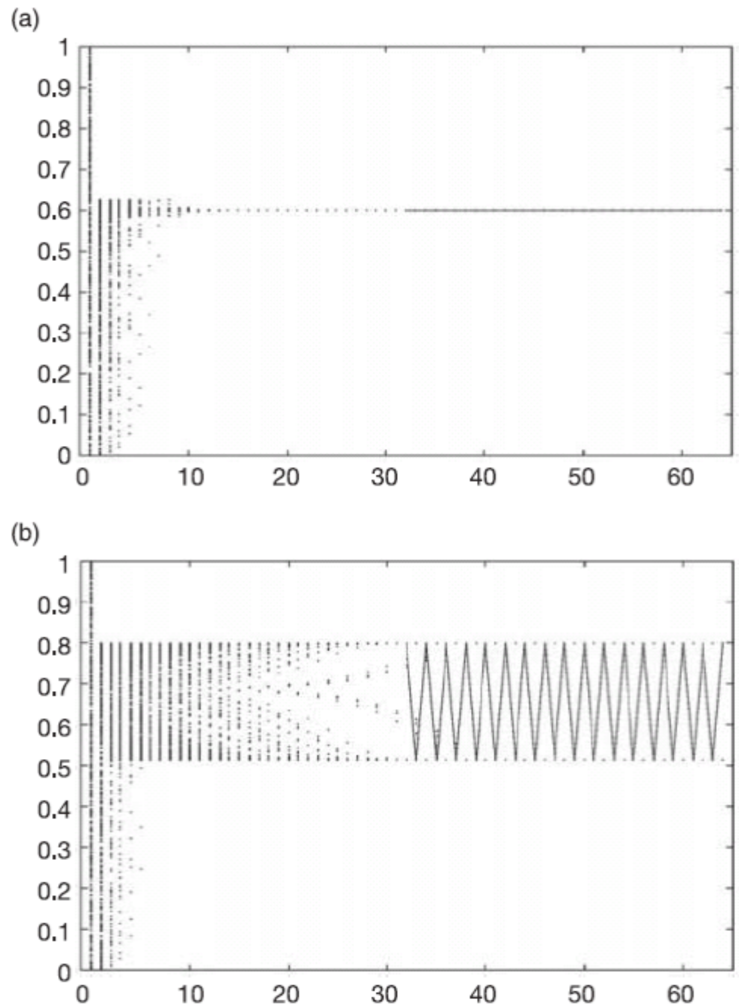
What happens as we increase λ from 2 to 4?

$$x_{i+1} = \lambda x_i(1 - x_i)$$

$$x_{i+1} = \lambda x_i(1 - x_i)$$

λ=2.4736…

λ=3.2638…

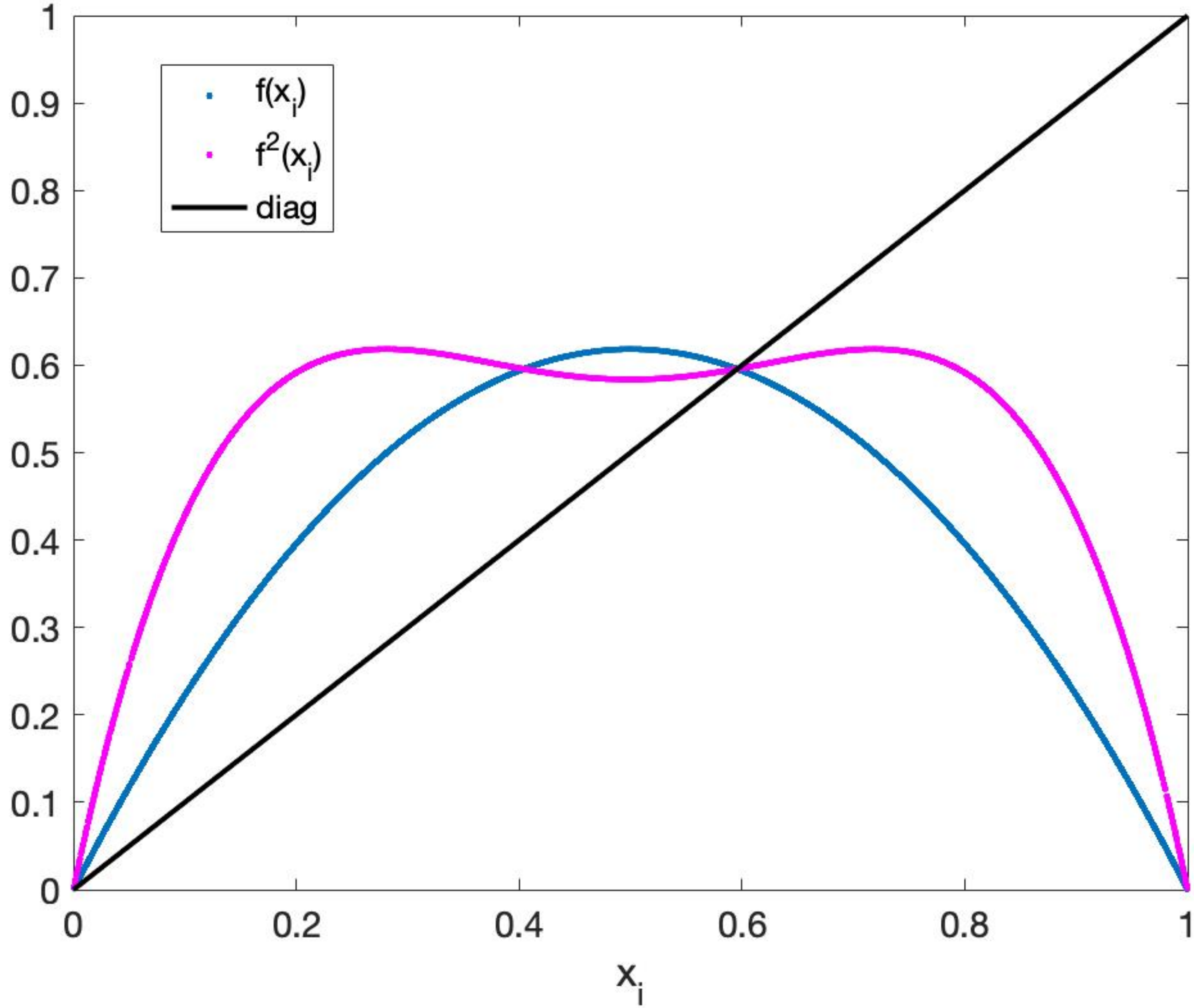


We can do this in the computer lab; everyone takes the same parameter value and their own random initial condition, what will happen?

Fixed Points

$$x_{i+1} = \lambda x_i(1 - x_i)$$

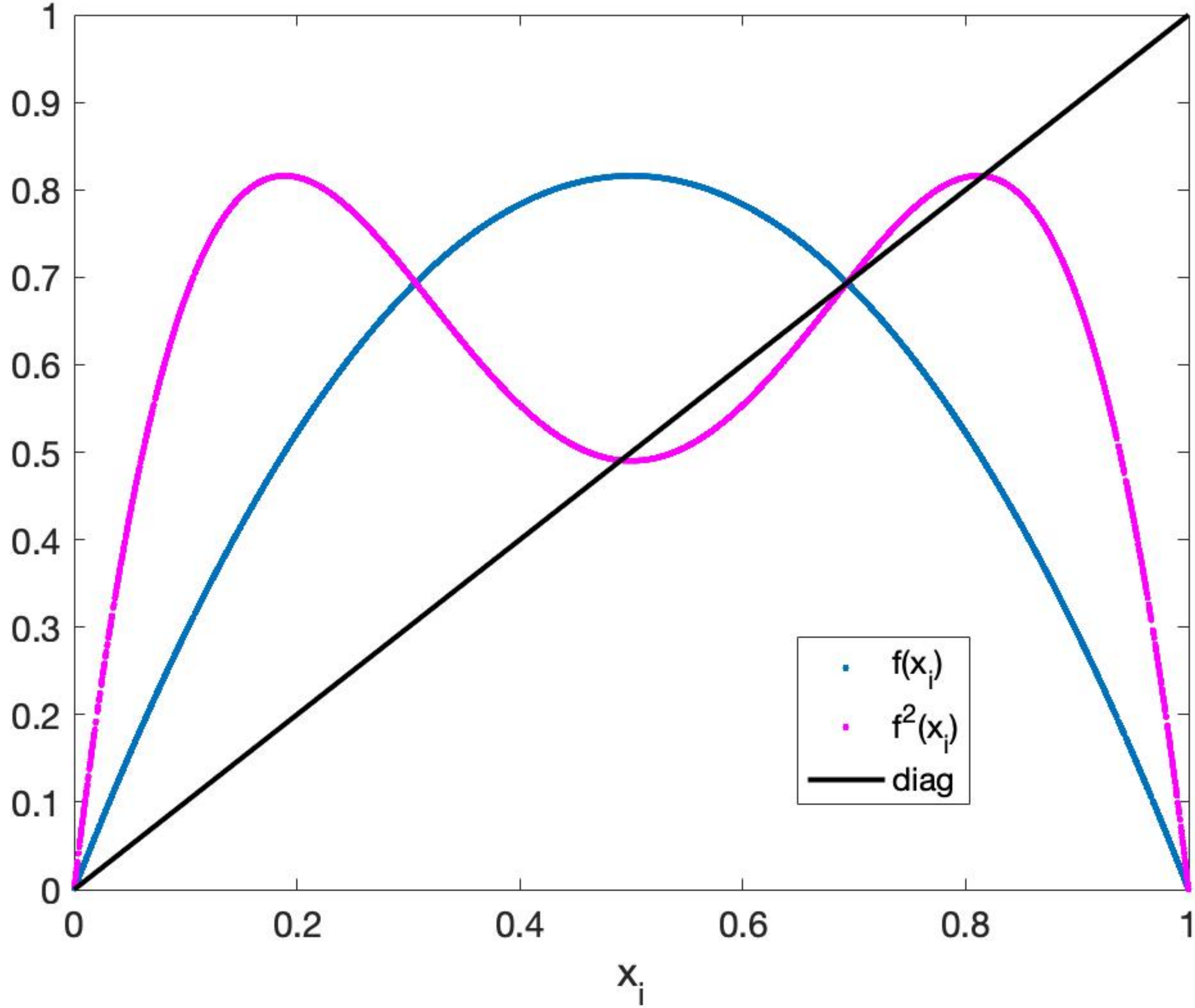
λ=2.4736...



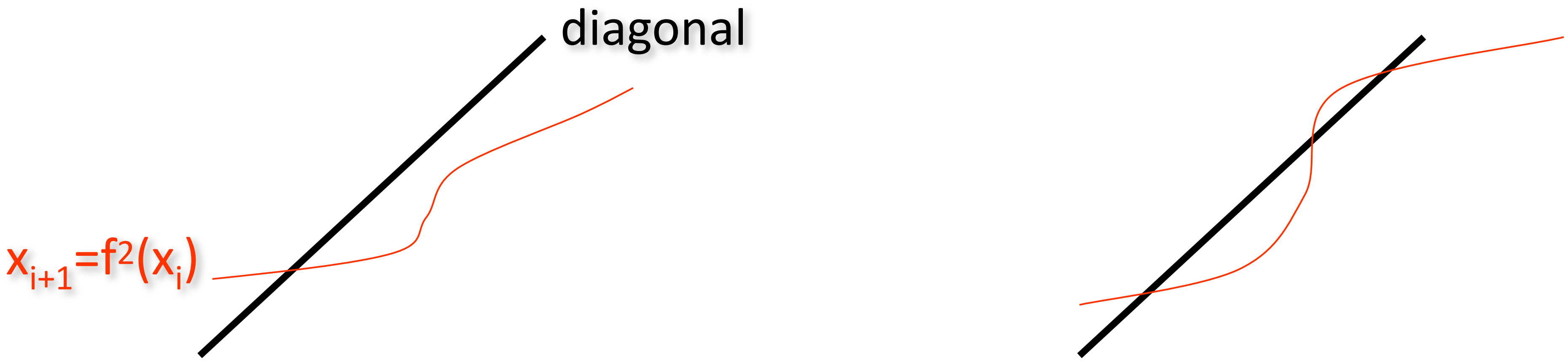
Fixed Points

$$x_{i+1} = \lambda x_i(1 - x_i)$$

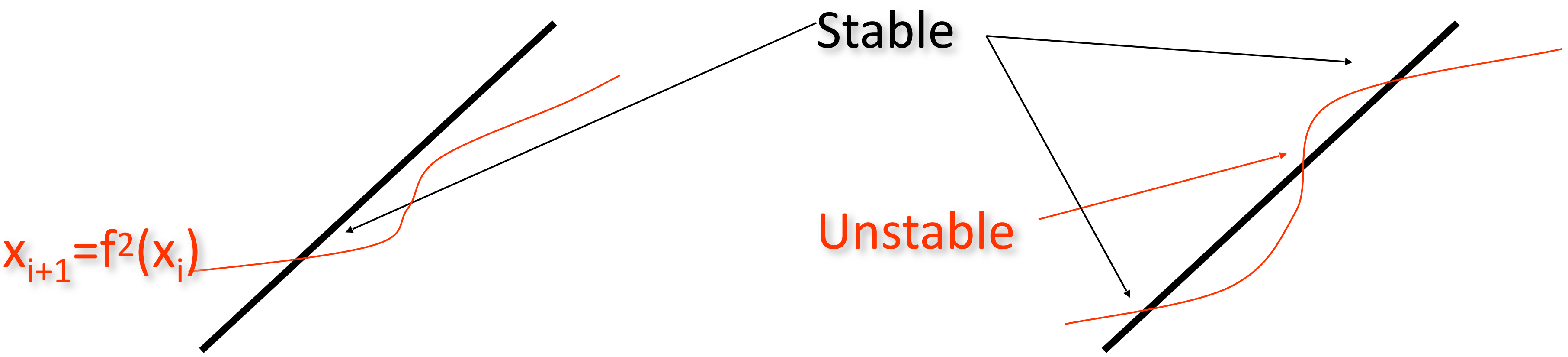
λ=3.2638…

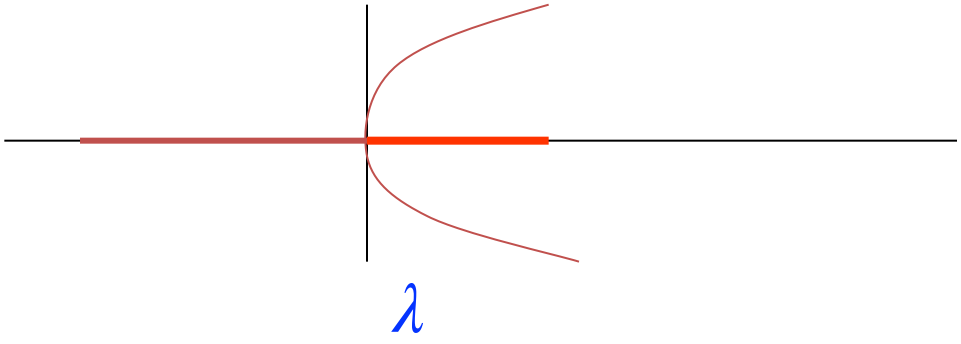


Fixed Points

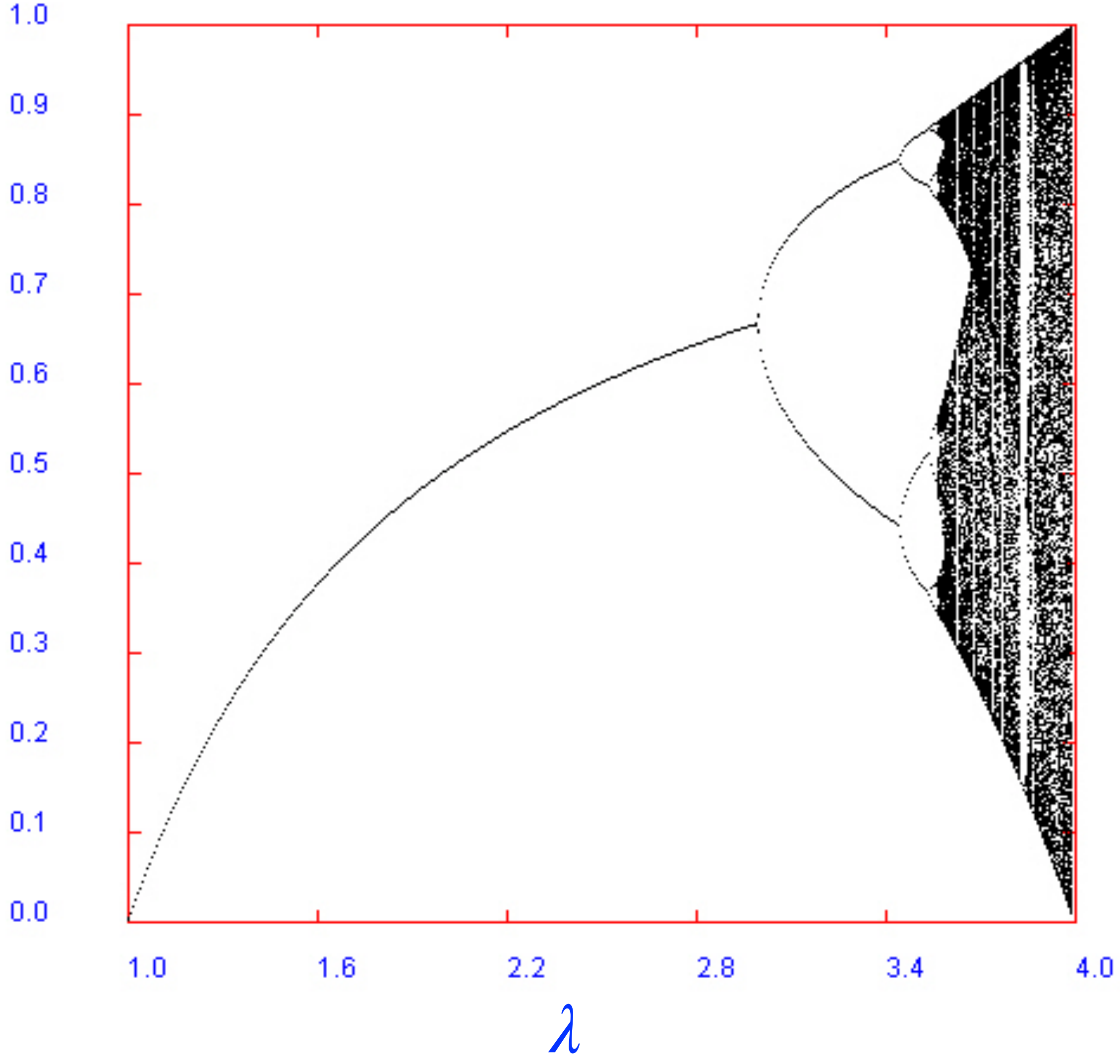


Fixed Points

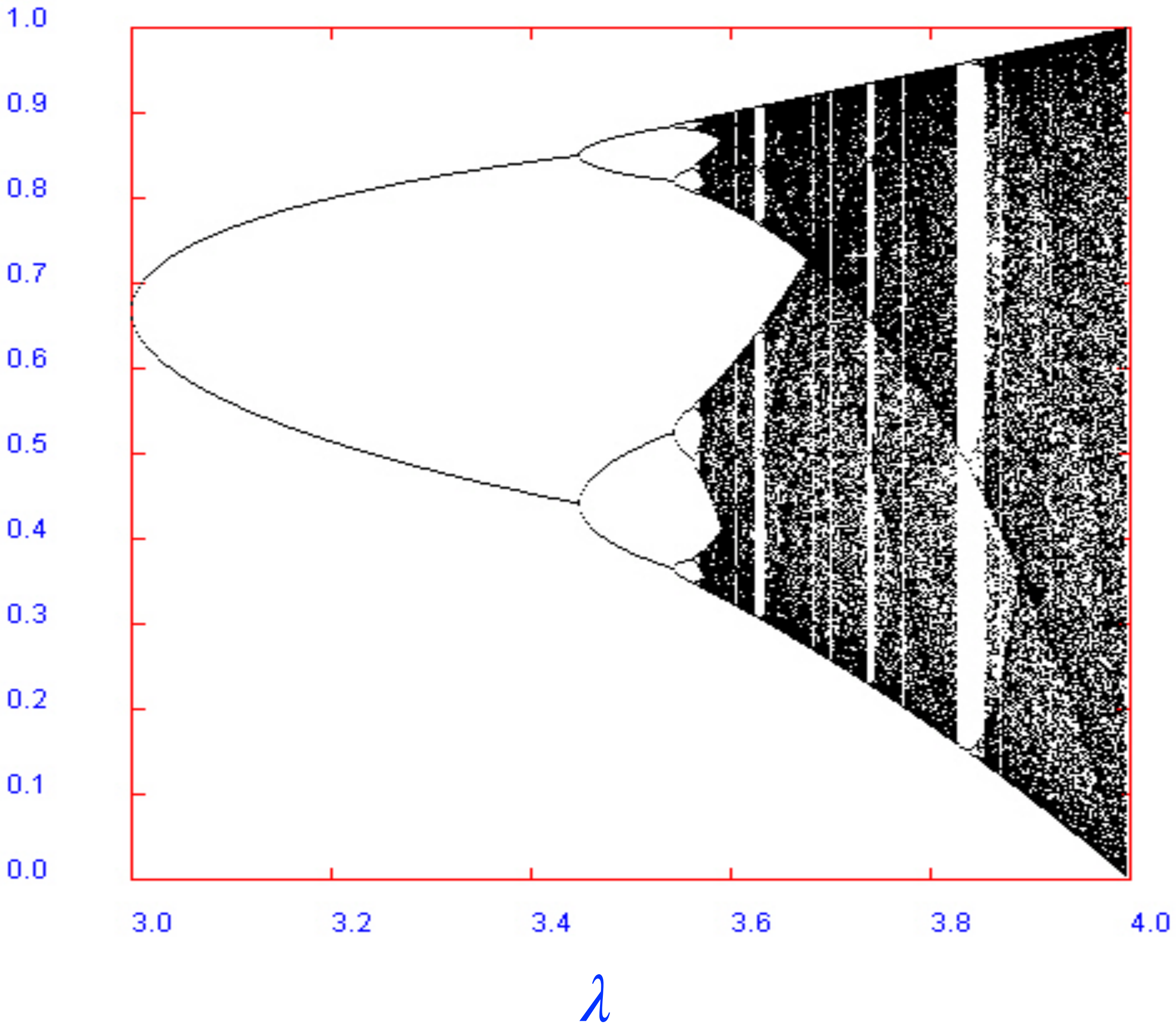




Bifurcation diagram



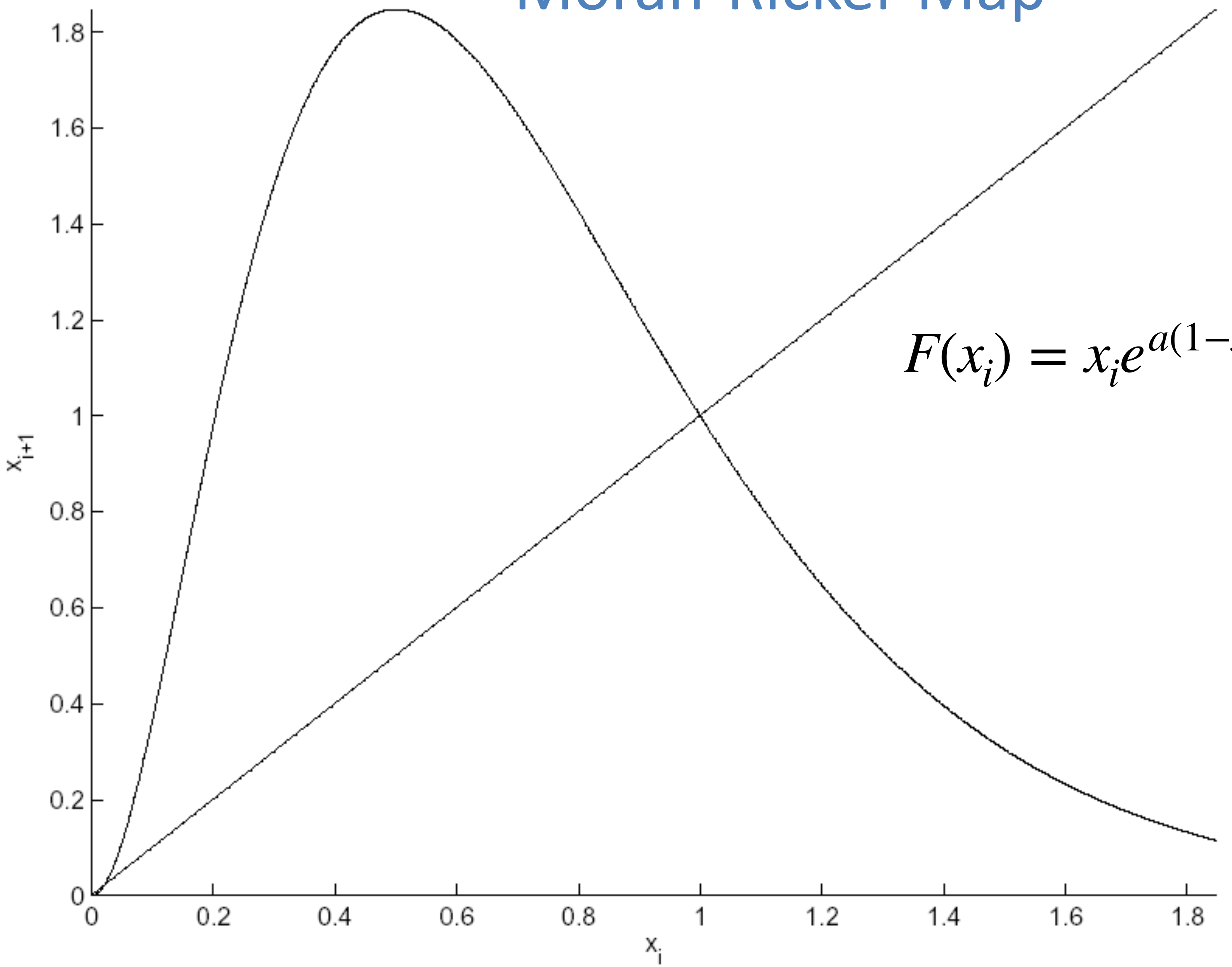
Bifurcation diagram



Unconditional distribution or climatology....

Think about climate change...

Moran-Ricker Map

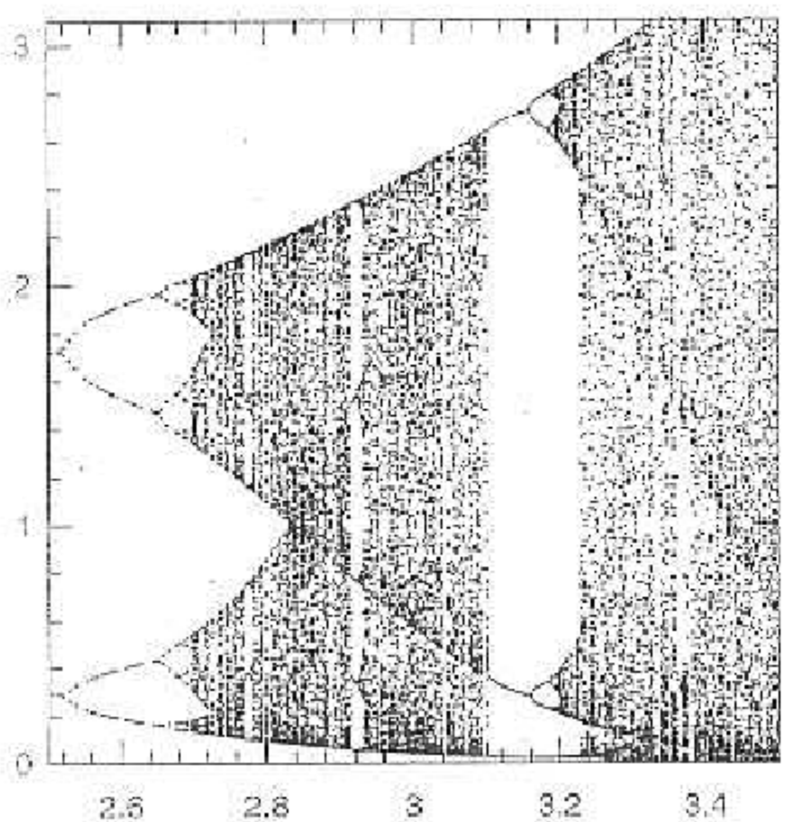


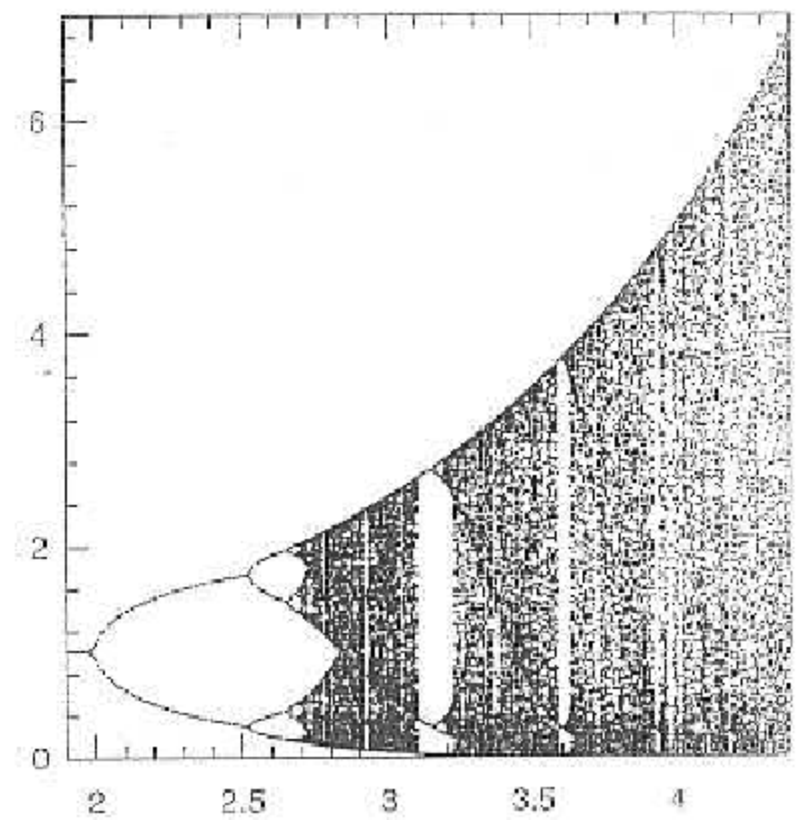
$$F(x_i) = x_i e^{a(1-x_i)}$$

Why stop graph at ~1.8?

Moran-Ricker Map

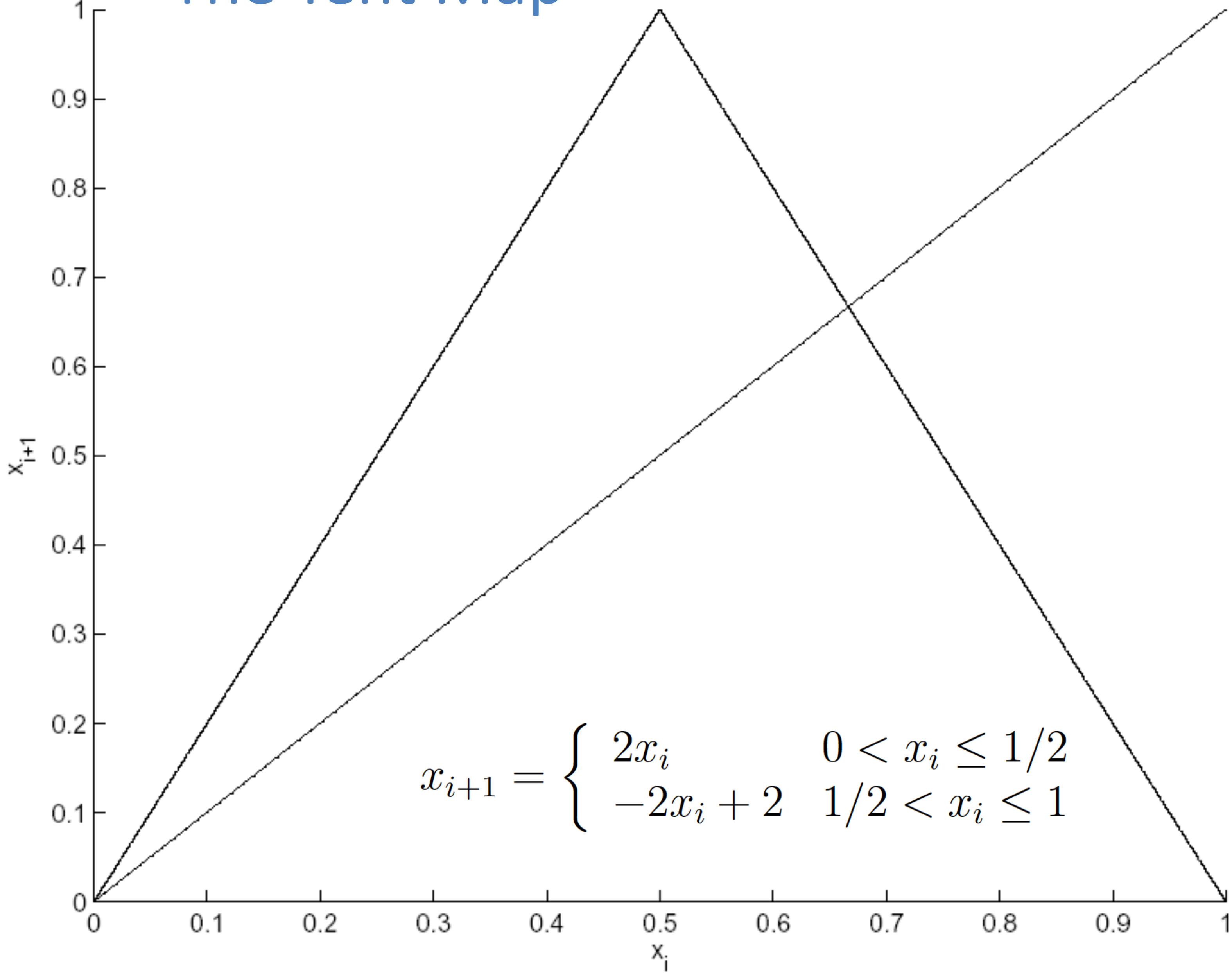
Zoom near $a = 3$



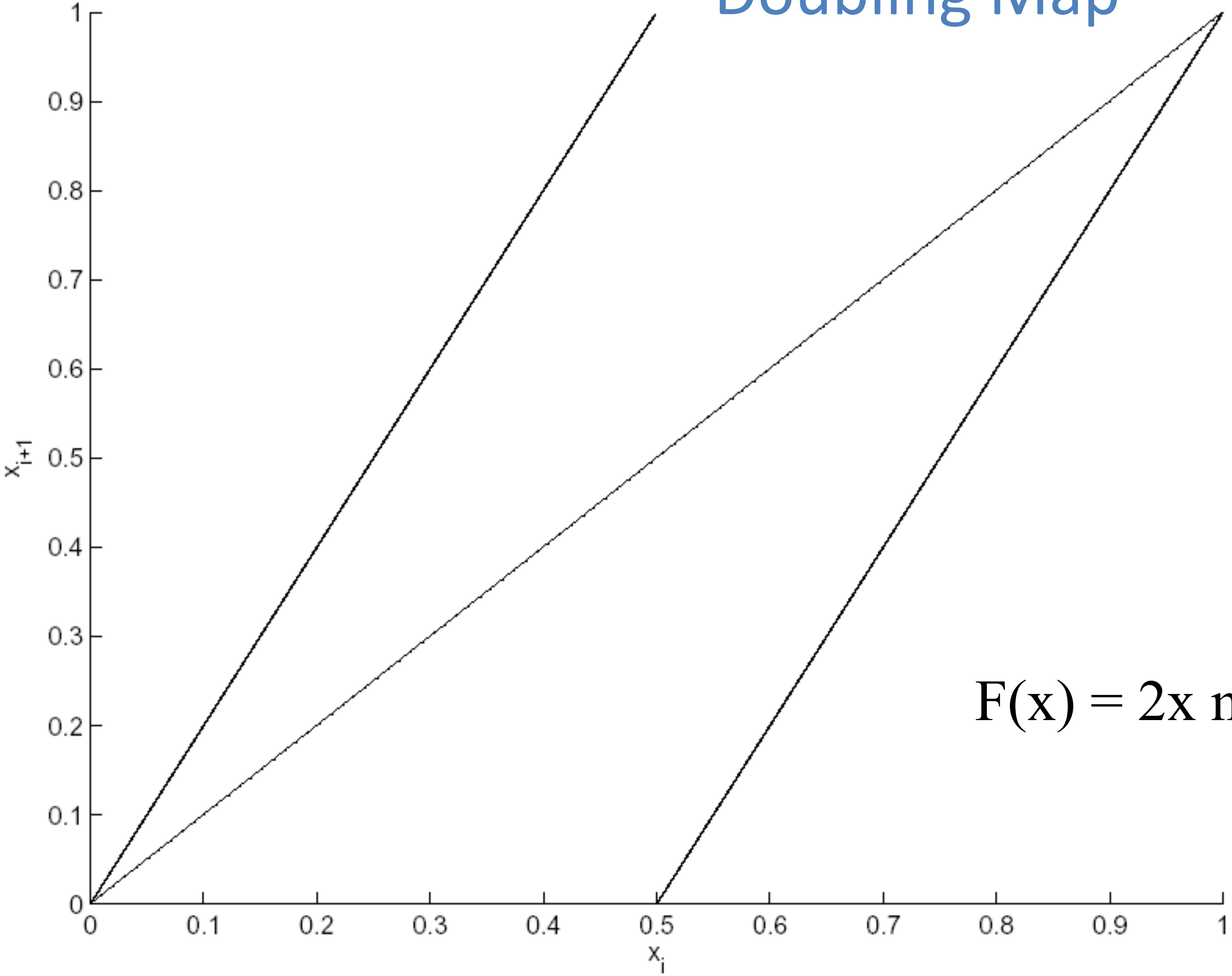


Note the ***similarity*** of this picture with that of the logistic map.

The Tent Map


x_{i+1}
x_i
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
$$x_{i+1} = \begin{cases} 2x_i & 0 < x_i \leq 1/2 \\ -2x_i + 2 & 1/2 < x_i \leq 1 \end{cases}$$

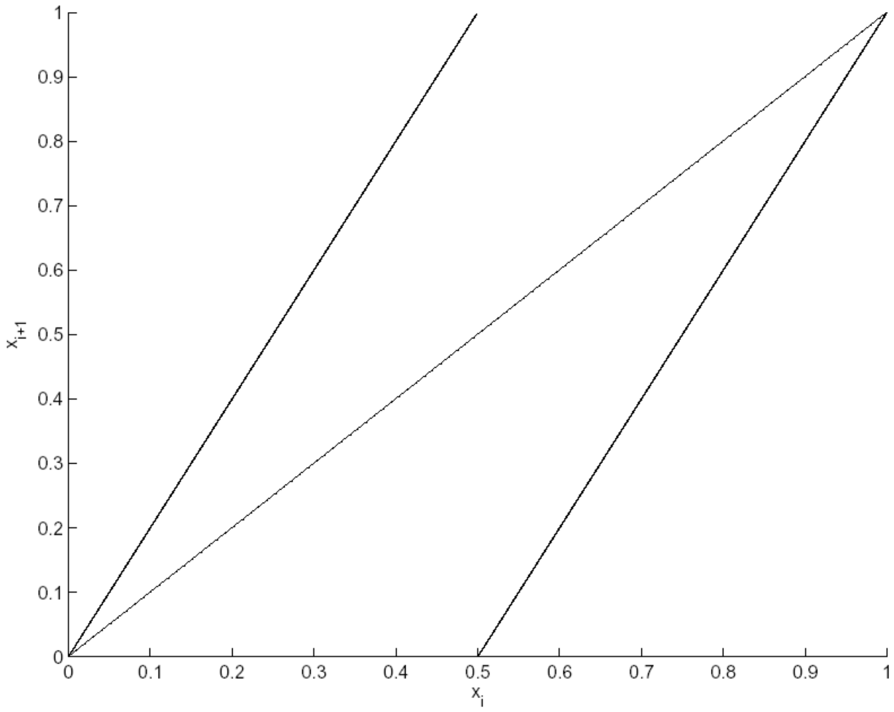
Doubling Map



$$F(x) = 2x \bmod 1$$

Doubling Map

$F(x) = 2x \bmod 1$



Doubling Map: aka the Shift Map

Consider x in binary…

Error Growth in the Doubling Map?

The Doubling Map and the Logistic Map

$$\mathrm{F(x) = 2x \bmod 1} \qquad\qquad F(x) = \lambda x(1 - x)$$

Consider $\lambda = 4$, and substitute $sin^2(\theta)$ for x...

$$F(\theta) = 2\theta \bmod 2\pi$$

Recall:

???Where is that Exponential Error Growth???

Could something this simple apply to anything other than rabbits?

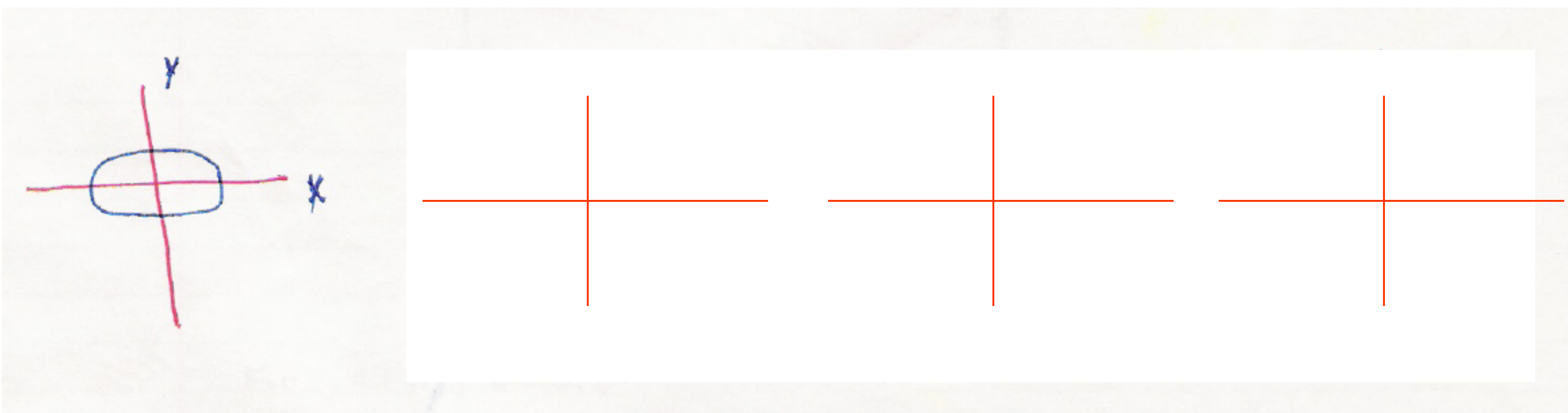
An example of 2-D map: Henon Map

$$
\begin{aligned}
x_{n+1} &= y_n + 1 - a x_n^2 \\
y_{n+1} &= b x_n
\end{aligned}
\qquad
\begin{aligned}
a &= 1.4 \\
b &= 0.3
\end{aligned}
$$

An example of 2-D map: Henon Map

$$x_{n+1} = y_n + 1 - ax_n^2 \qquad a = 1.4$$
$$y_{n+1} = bx_n \qquad b = 0.3$$

We may consider it as a composition of three maps:



$$x' = x$$
$$y' = y + 1 - ax^2$$

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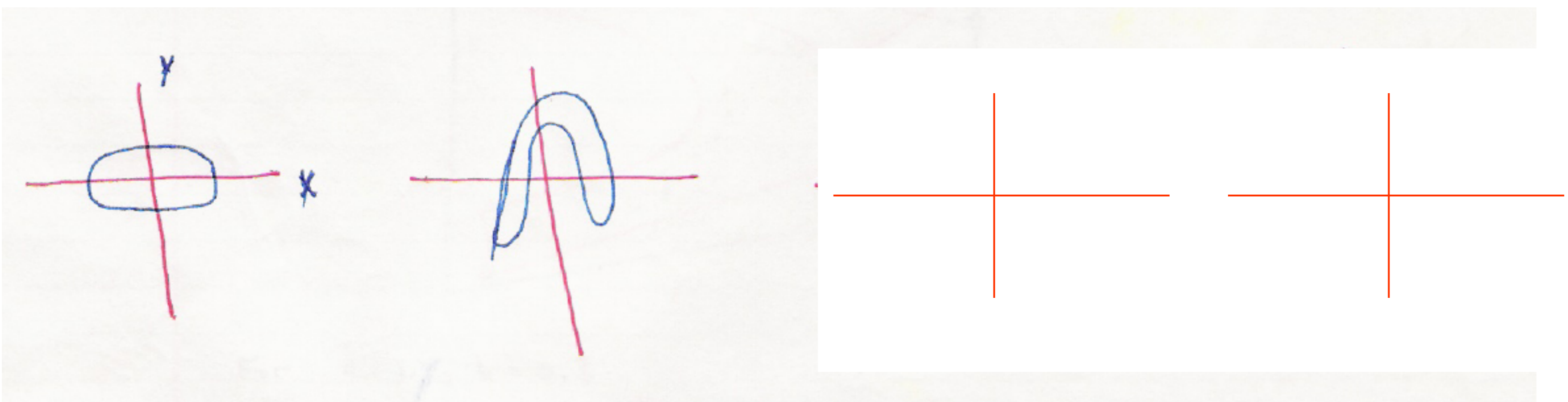

$$
\begin{aligned}
x' &= x \\
y' &= y + 1 - a x^2
\end{aligned}
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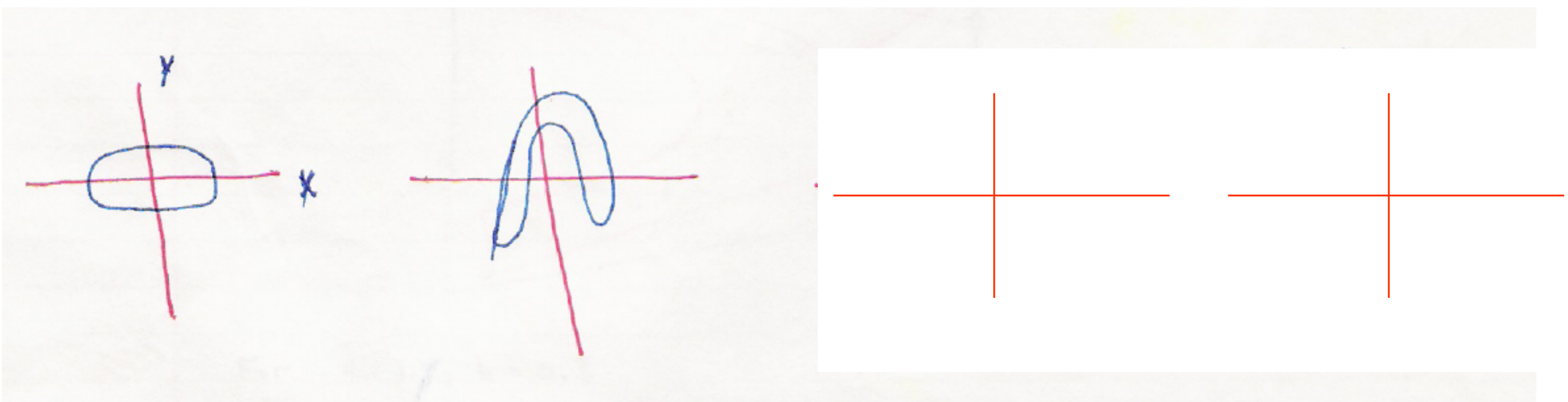

$$x' = x$$
$$y' = y + 1 - a x^2$$

$$x'' = b x'$$
$$y'' = y'$$

An example of 2-D map: Henon Map

$$
\begin{aligned}
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\end{aligned}
\qquad
\begin{aligned}
a &= 1.4 \\
b &= 0.3
\end{aligned}
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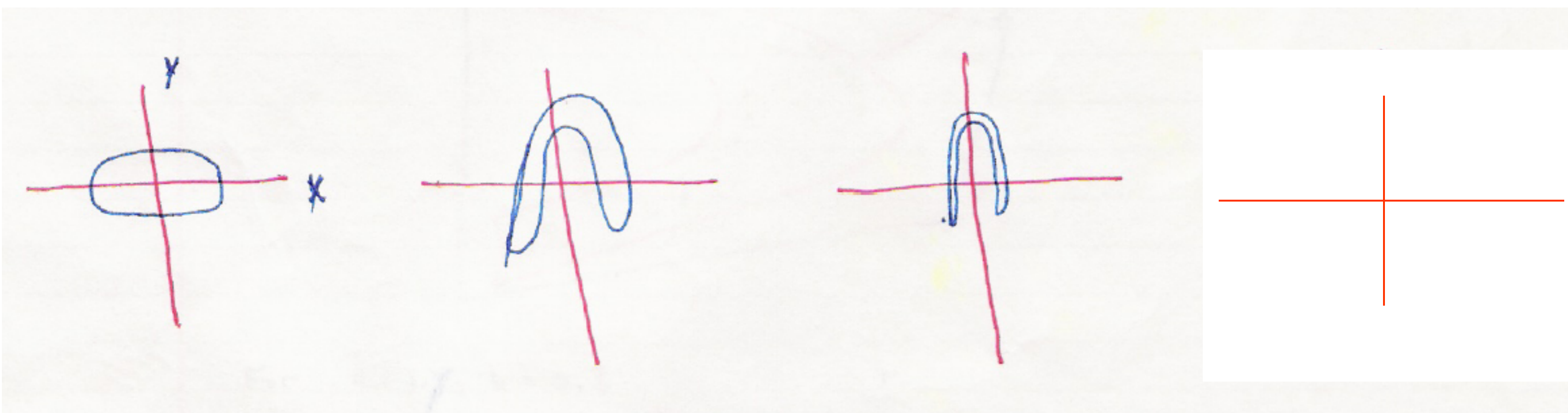
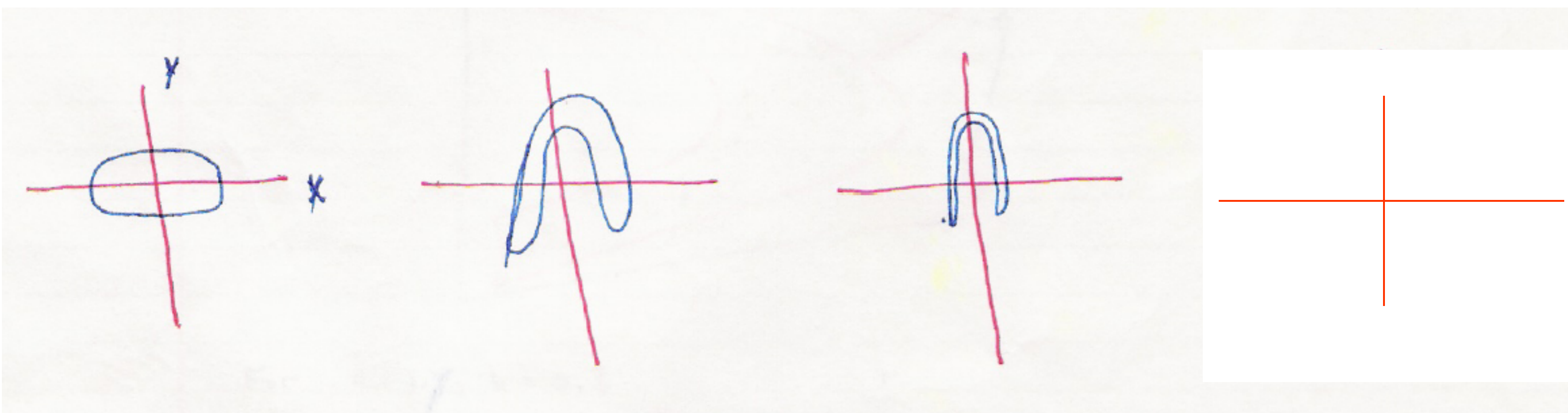


$$
\begin{aligned}
x' &= x \\
y' &= y + 1 - a x^2
\end{aligned}
\qquad
\begin{aligned}
x'' &= b x' \\
y'' &= y'
\end{aligned}
$$

An example of 2-D map: Henon Map

$$x_{n+1} = y_n + 1 - a x_n^2 \qquad a = 1.4$$
$$y_{n+1} = b x_n \qquad b = 0.3$$

We may consider it as a composition of three maps:



$$x' = x \qquad x'' = bx' \qquad x''' = y''$$
$$y' = y + 1 - ax^2 \qquad y'' = y' \qquad y''' = x''$$

An example of 2-D map: Henon Map

$$x_{n+1} = y_n + 1 - ax_n^2 \qquad a = 1.4$$
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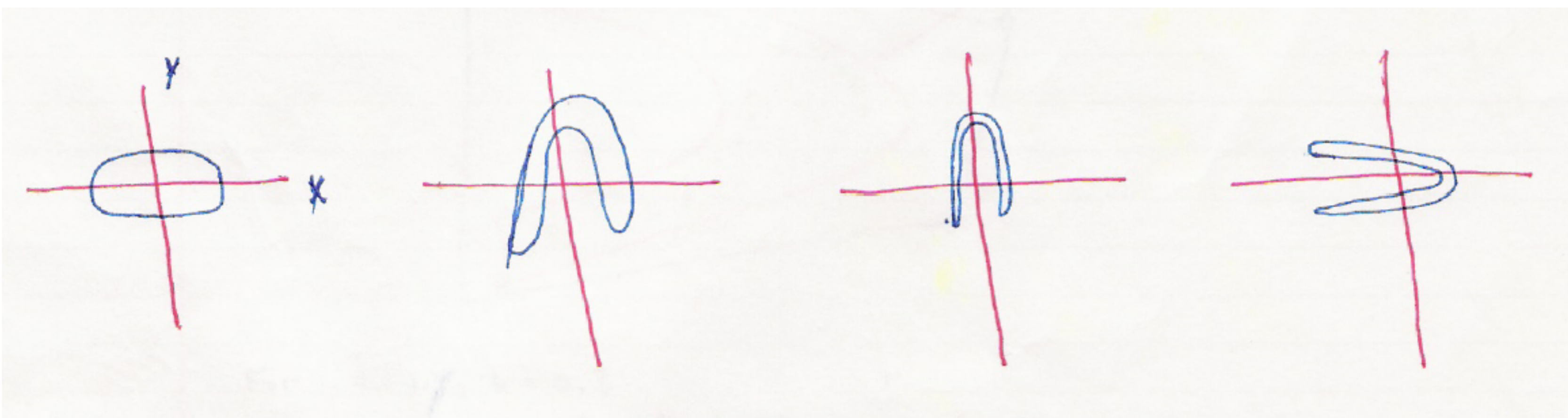
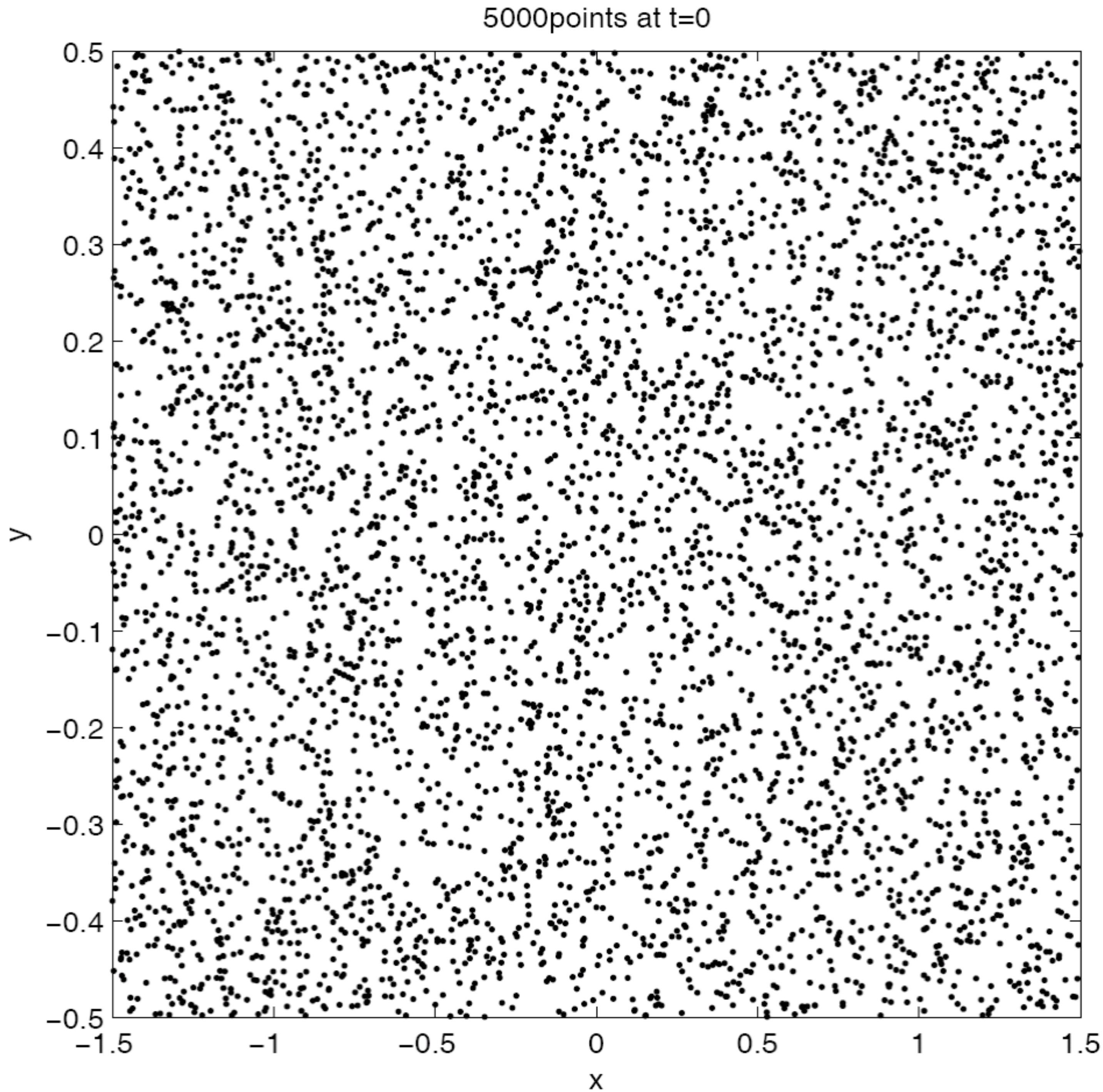


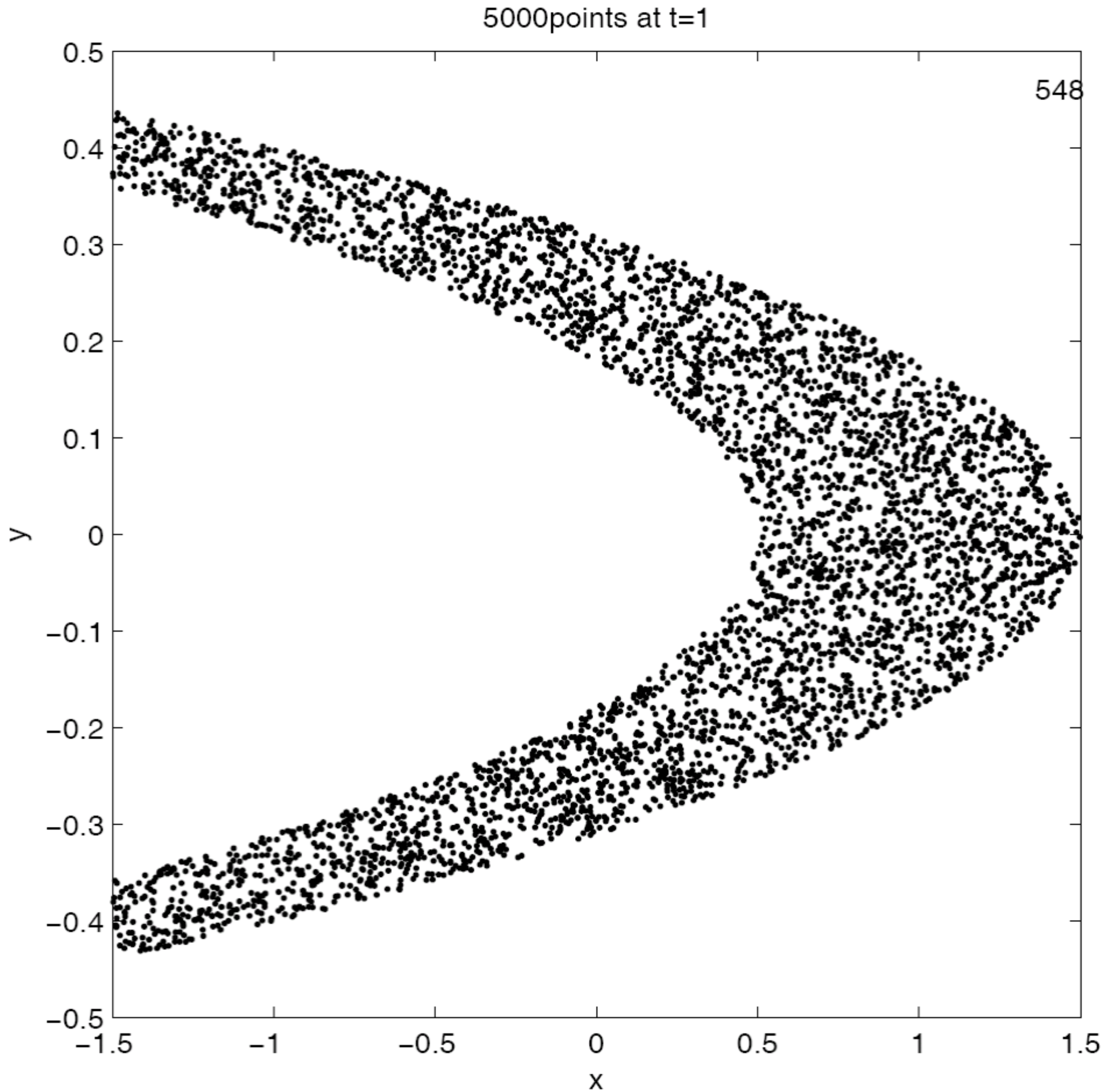
$$x' = x$$
$$y' = y + 1 - ax^2$$

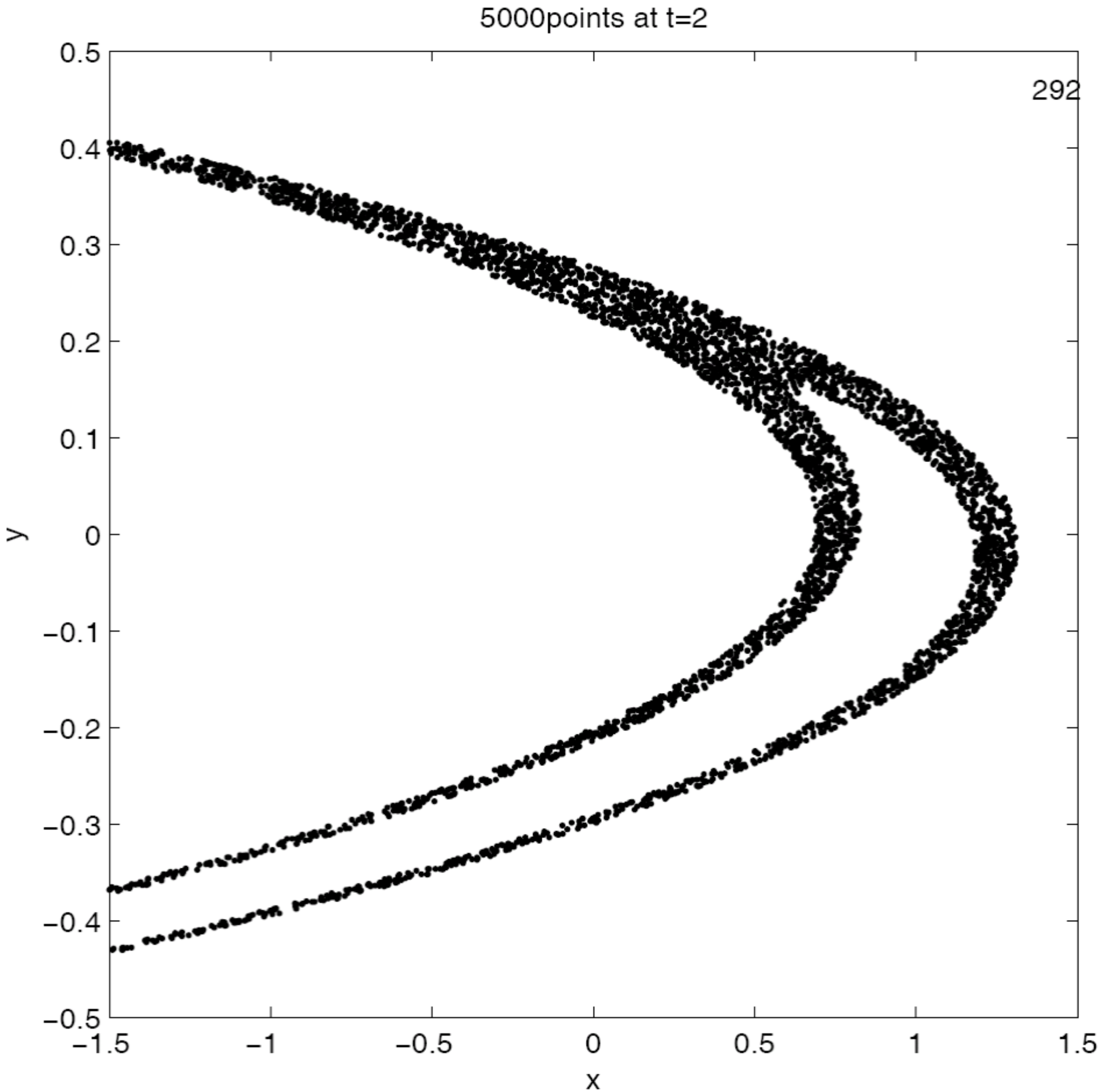
$$x'' = bx'$$
$$y'' = y'$$

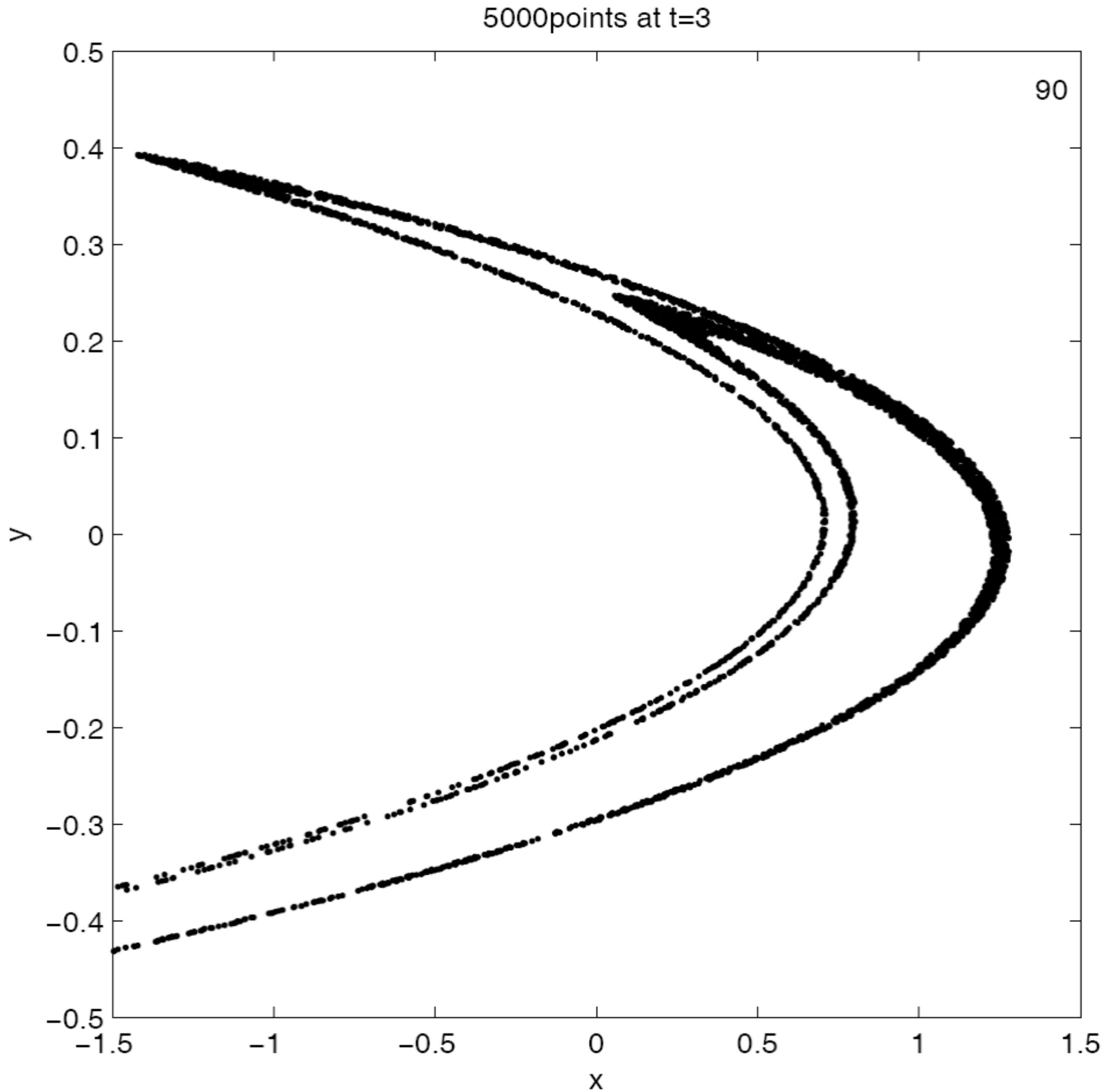
$$x''' = y''$$
$$y''' = x''$$

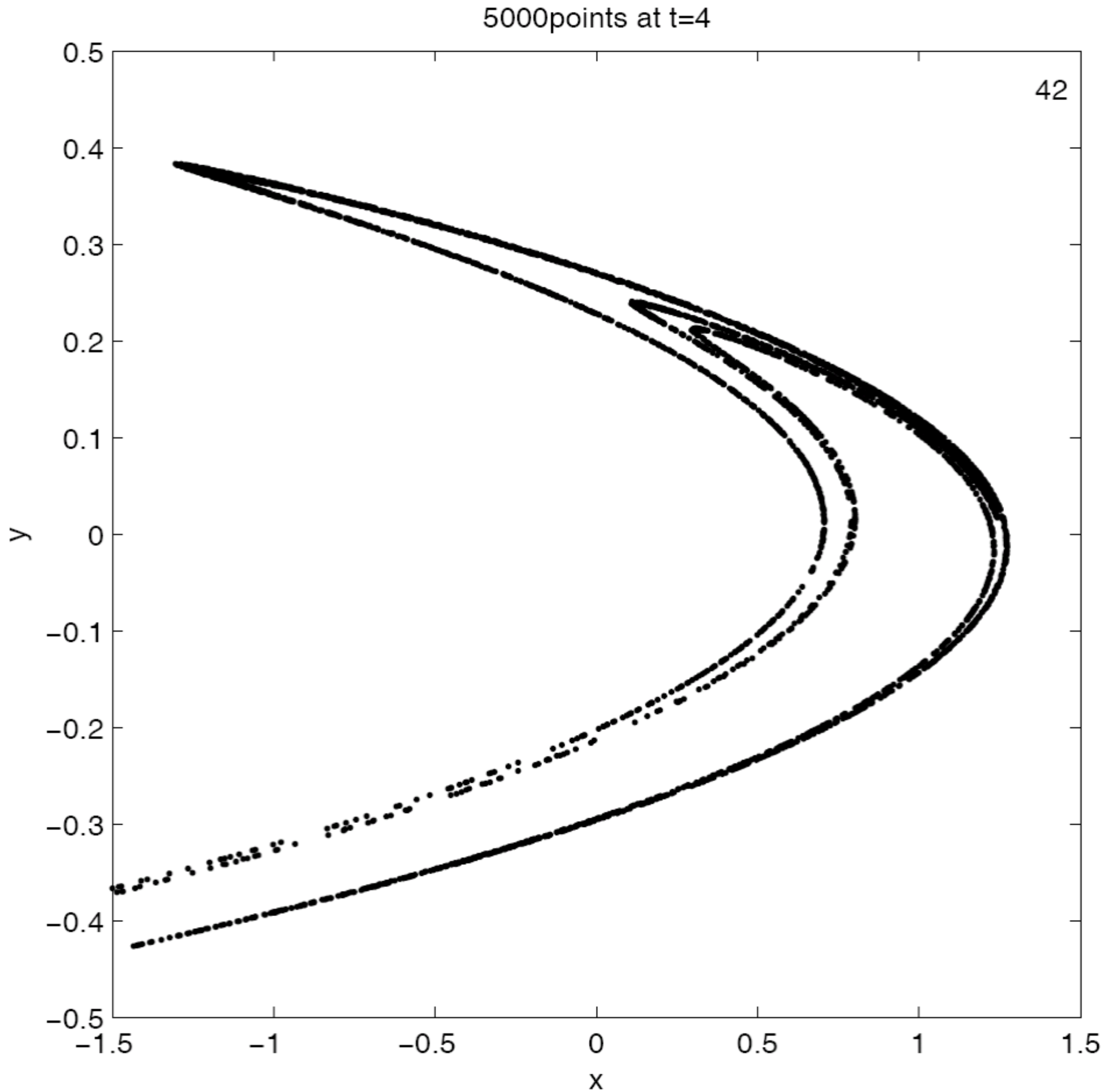
So what happens to a square of randomly selected points?

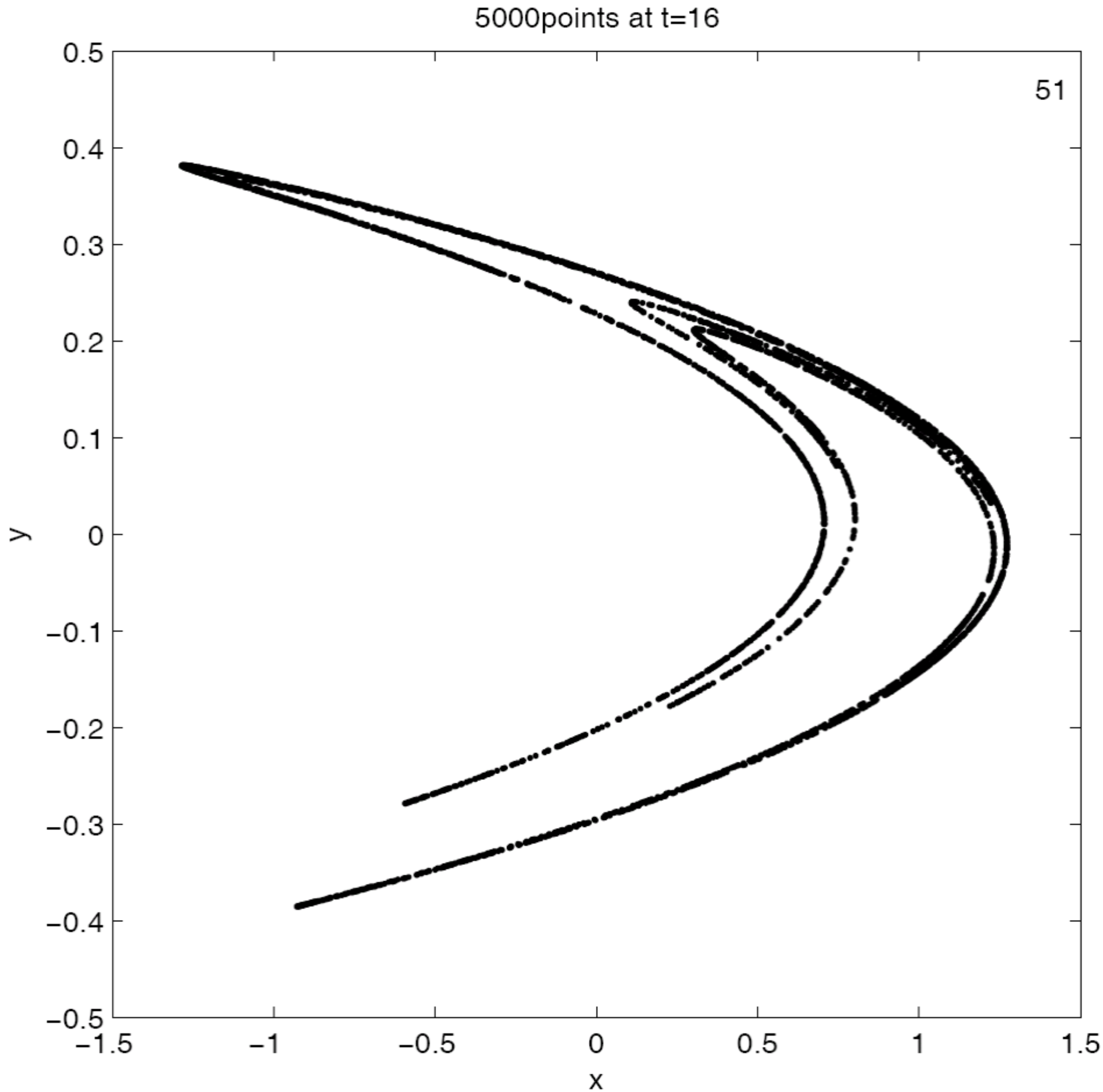

5000points at t=0
y
x
0.5
0.4
0.3
0.2
0.1
0
−0.1
−0.2
−0.3
−0.4
−0.5
−1.5
−1
−0.5
0
0.5
1
1.5


5000points at t=1
548
y
x
0.5
0.4
0.3
0.2
0.1
0
−0.1
−0.2
−0.3
−0.4
−0.5
−1.5
−1
−0.5
0
0.5
1
1.5

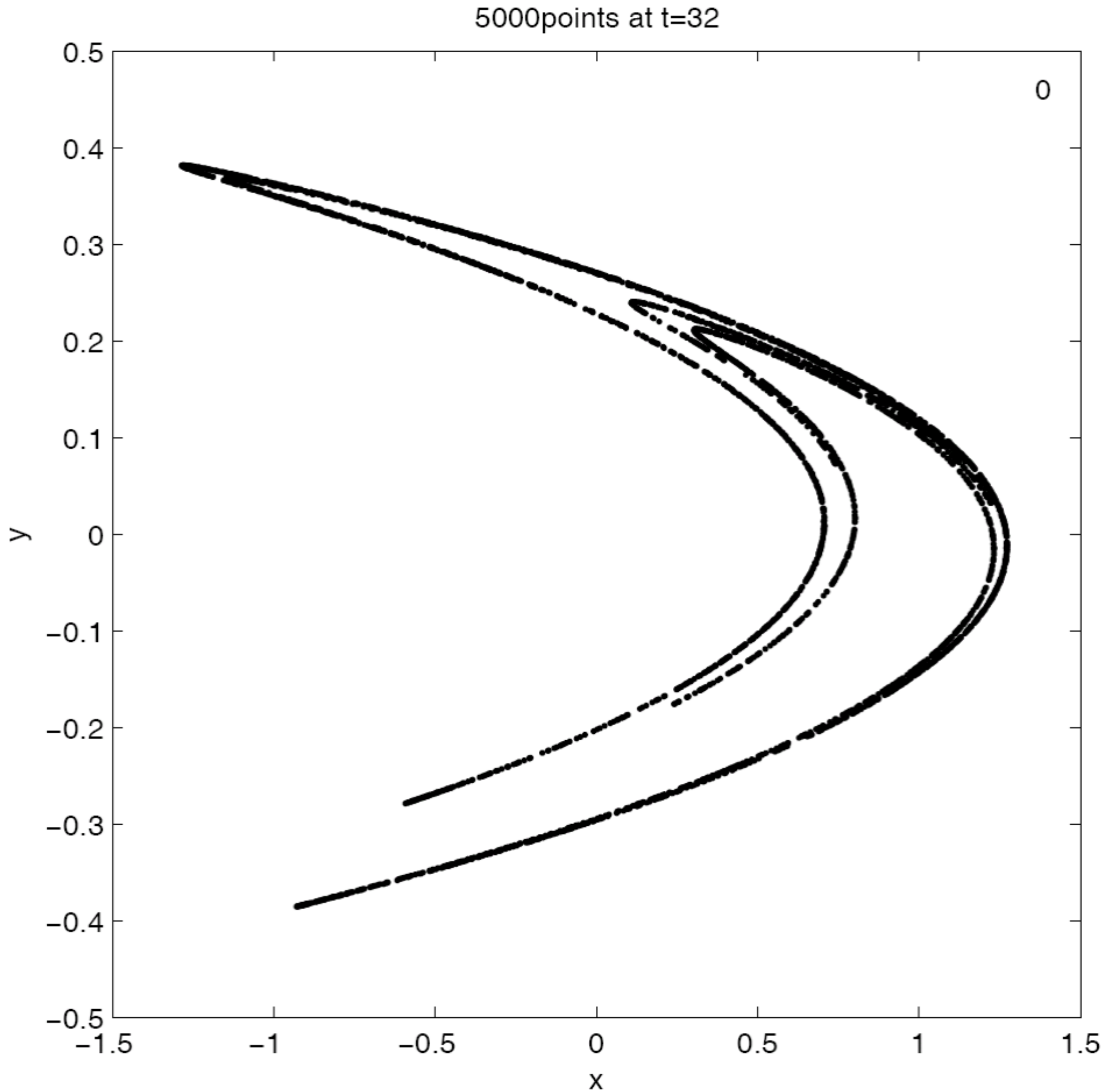

5000points at t=2
292
y
x
0.5
0.4
0.3
0.2
0.1
0
−0.1
−0.2
−0.3
−0.4
−0.5
−1.5
−1
−0.5
0
0.5
1
1.5

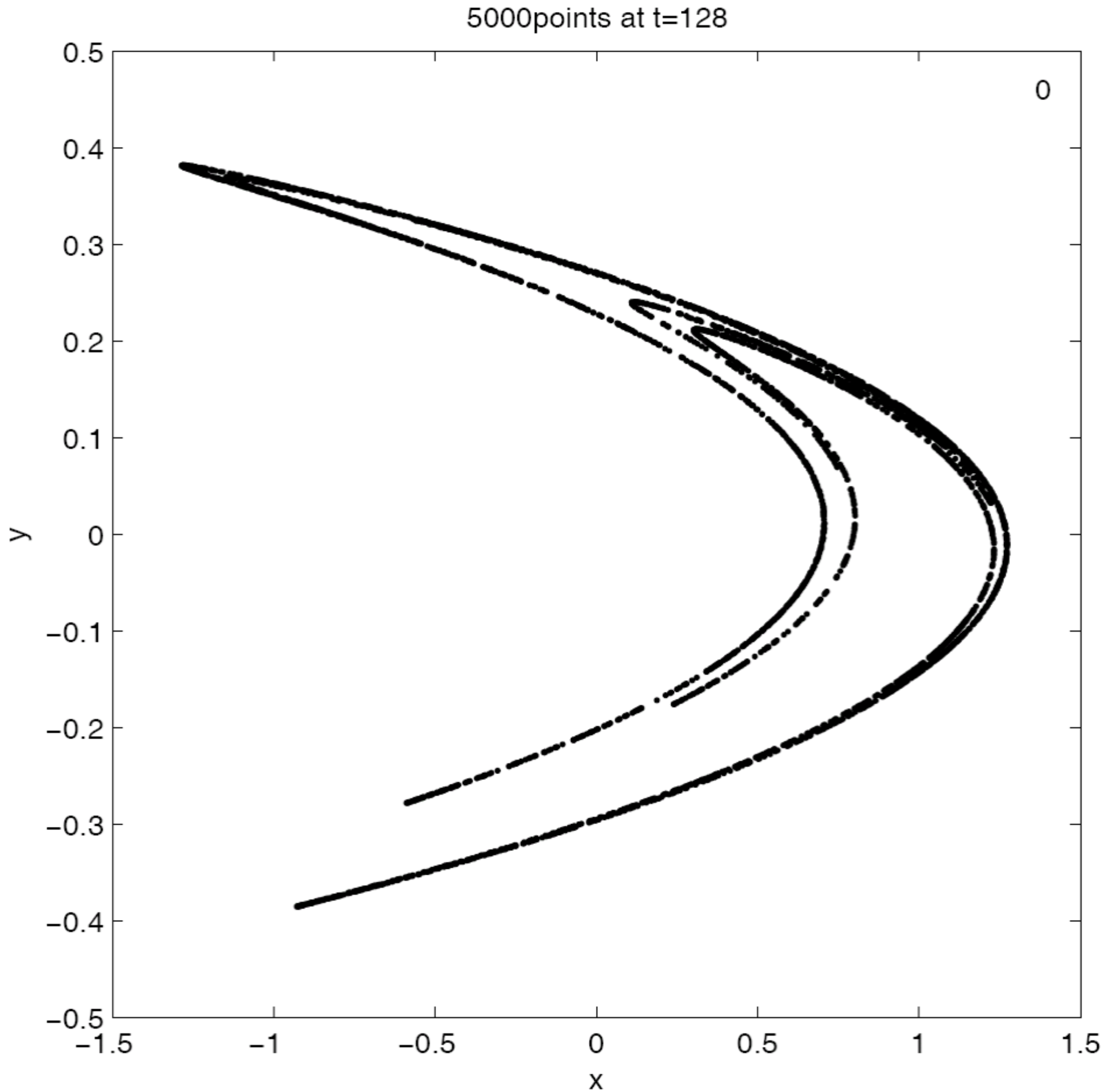

5000points at t=3
90
y
x
0.5
0.4
0.3
0.2
0.1
0
−0.1
−0.2
−0.3
−0.4
−0.5
−1.5
−1
−0.5
0
0.5
1
1.5


5000points at t=4
42
0.5
0.4
0.3
0.2
0.1
0
−0.1
−0.2
−0.3
−0.4
−0.5
y
−1.5
−1
−0.5
0
0.5
1
1.5
x



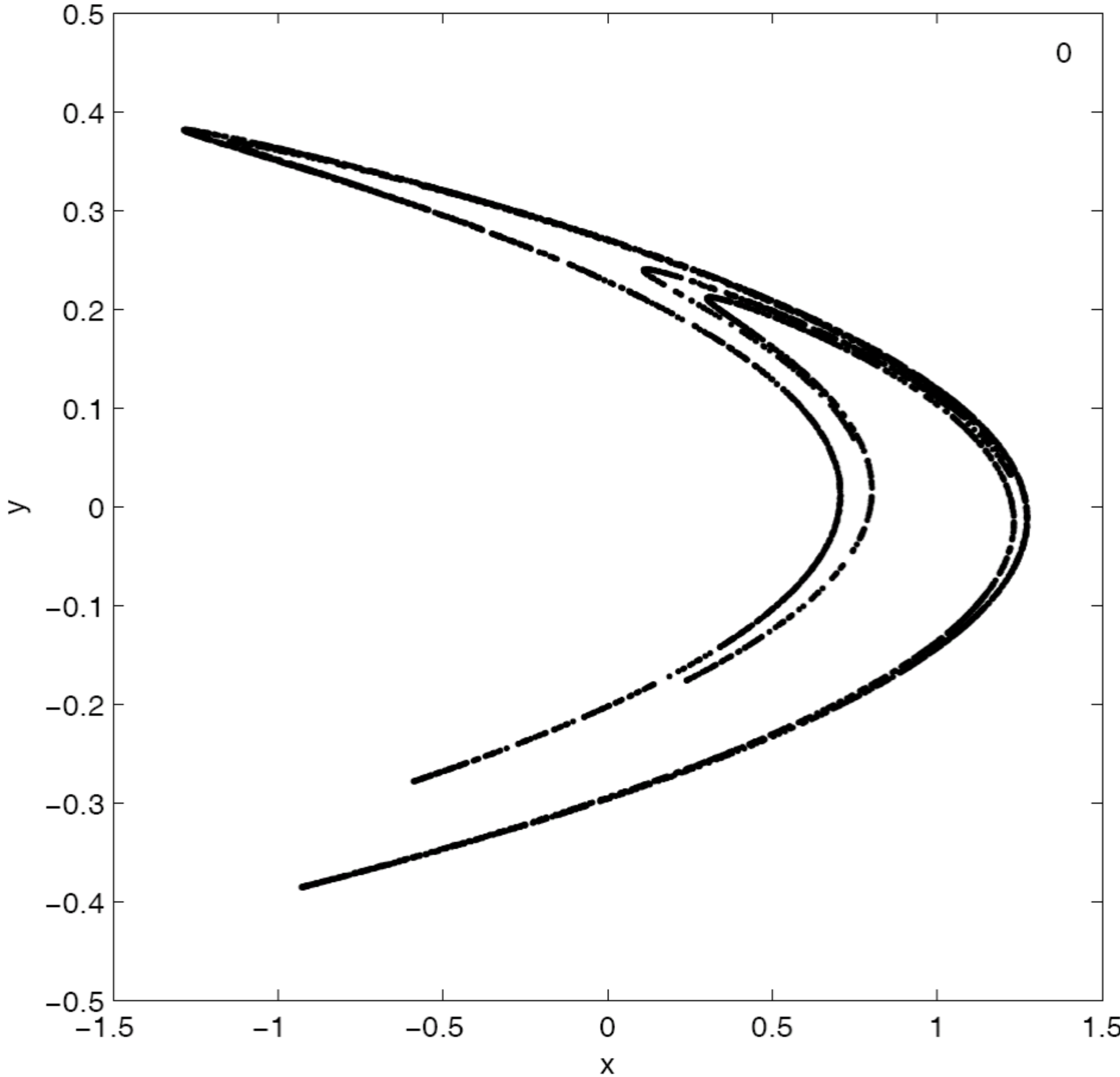
5000points at t=16
51
0.5
0.4
0.3
0.2
0.1
0
−0.1
−0.2
−0.3
−0.4
−0.5
y
−1.5
−1
−0.5
0
0.5
1
1.5
x


5000points at t=32
0
0.5
0.4
0.3
0.2
0.1
0
−0.1
−0.2
−0.3
−0.4
−0.5
y
−1.5
−1
−0.5
0
0.5
1
1.5
x



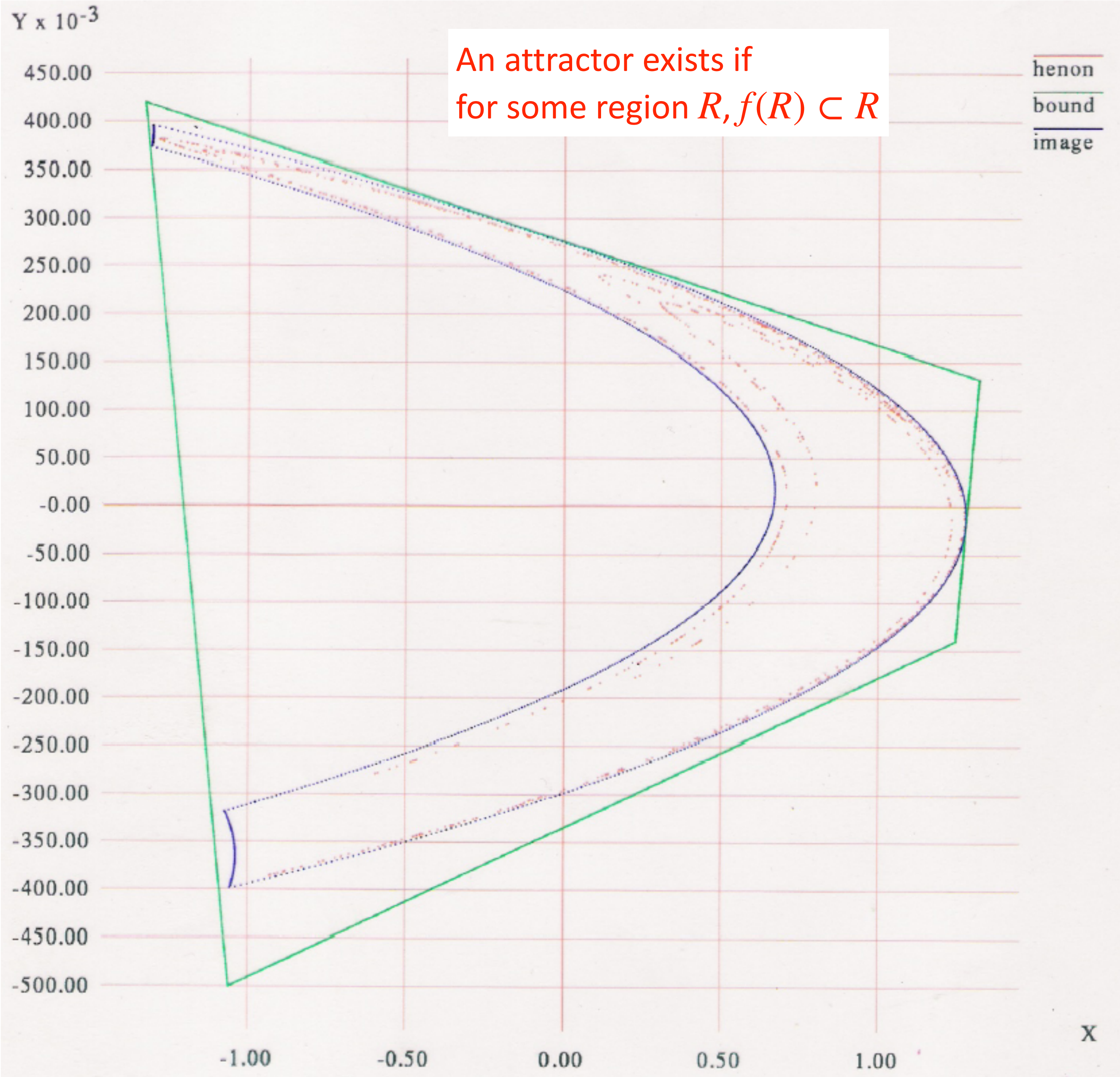
5000points at t=128
0
0.5
0.4
0.3
0.2
0.1
0
−0.1
−0.2
−0.3
−0.4
−0.5
y
−1.5
−1
−0.5
0
0.5
1
1.5
x

Attractor



An attractor exists if
for some region $R, f(R) \subset R$

An attractor exists if
for some region R, $f(R) \subset R$



Jacobian Matrix

If **p** is a point in $\mathbf{R}^n$ and F is differentiable at **p**, then its derivative is given by $J_F(\mathbf{p})$. In this case, the linear map described by $J_F(\mathbf{p})$ is the best linear approximation of F near the point **p**, in the sense that

$$F(\mathbf{x}) = F(\mathbf{p}) + J_F(\mathbf{p})(\mathbf{x}-\mathbf{p}) + o(\|\mathbf{x}-\mathbf{p}\|)$$

for **x** close to **p** and where o is the little o-notation (for $x \to p$, not $x \to \infty$) and $\|\mathbf{x}-\mathbf{p}\|$ is the distance between **x** and **p**.

In vector calculus, the **Jacobian matrix** is the matrix of all first-order partial derivatives of a vector-valued function. Suppose $F : \mathbf{R}^n \to \mathbf{R}^m$ is a function from Euclidean n-space to Euclidean m-space. Such a function is given by m real-valued component functions, $y_1(x_1,\ldots,x_n)$, ..., $y_m(x_1,\ldots,x_n)$. The partial derivatives of all these functions (if they exist) can be organized in an m-by-n matrix, the Jacobian matrix J of F, as follows:

$$J = \begin{bmatrix} \dfrac{\partial y_1}{\partial x_1} & \cdots & \dfrac{\partial y_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial y_m}{\partial x_1} & \cdots & \dfrac{\partial y_m}{\partial x_n} \end{bmatrix}.$$

http://en.wikipedia.org/wiki/Jacobian_matrix_and_determinant

Henon Map

$$x_{n+1} = y_n + 1 - ax_n^2 \qquad a = 1.4$$
$$y_{n+1} = bx_n \qquad b = 0.3$$

Has Jacobian $J = \begin{pmatrix} -2ax & 1 \\ b & 0 \end{pmatrix}$

It is one-to-one with an inverse:

$$x_i = b^{-1} y_{i+1}$$
$$y_i = x_{i+1} - 1 + ab^{-2} y_{i+1}^2$$

Is the most general quadratic map with constant det(J) = -b

Has fixed points:

$$x^* = \frac{1}{2a}[-(1-b) \pm \sqrt{(1-b)^2 + 4a}]$$
$$y^* = bx^*$$

Henon Map

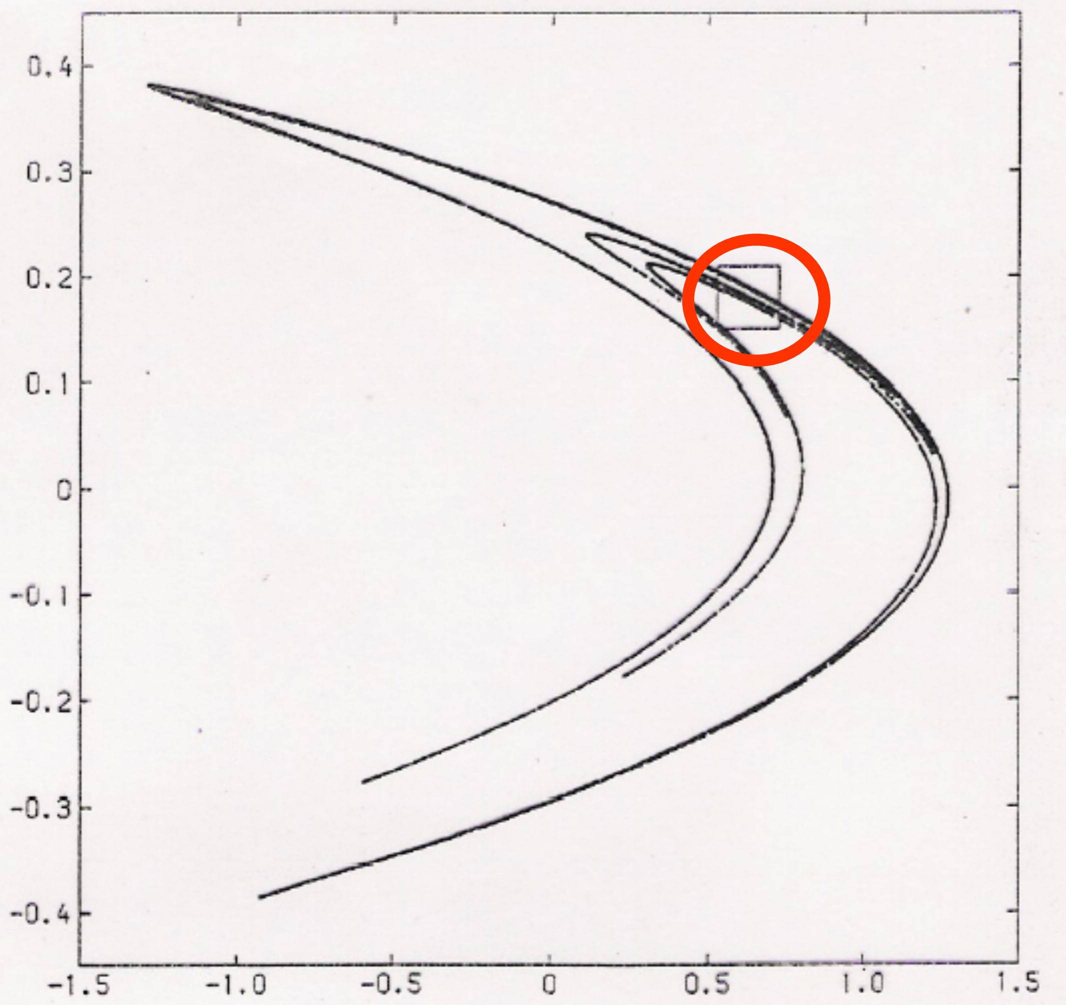


Fig. 3. Same as Figure 2, but starting from $x_0=0.63135448$, $y_0=0.18940634$

Henon Map

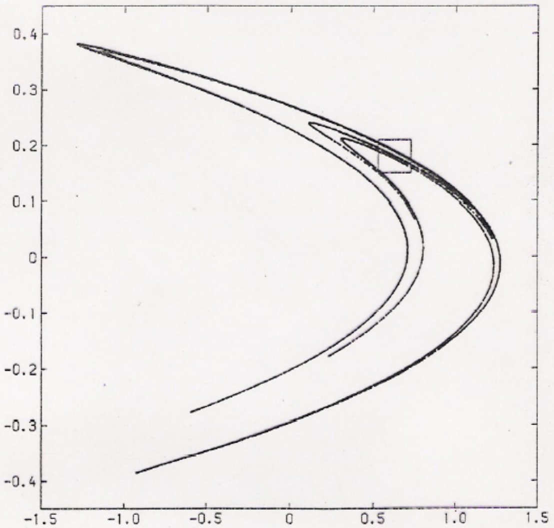


Fig. 3. Same as Figure 2, but starting from $x_0 = 0.63135448$, $y_0 = 0.18940634$

Henon Map

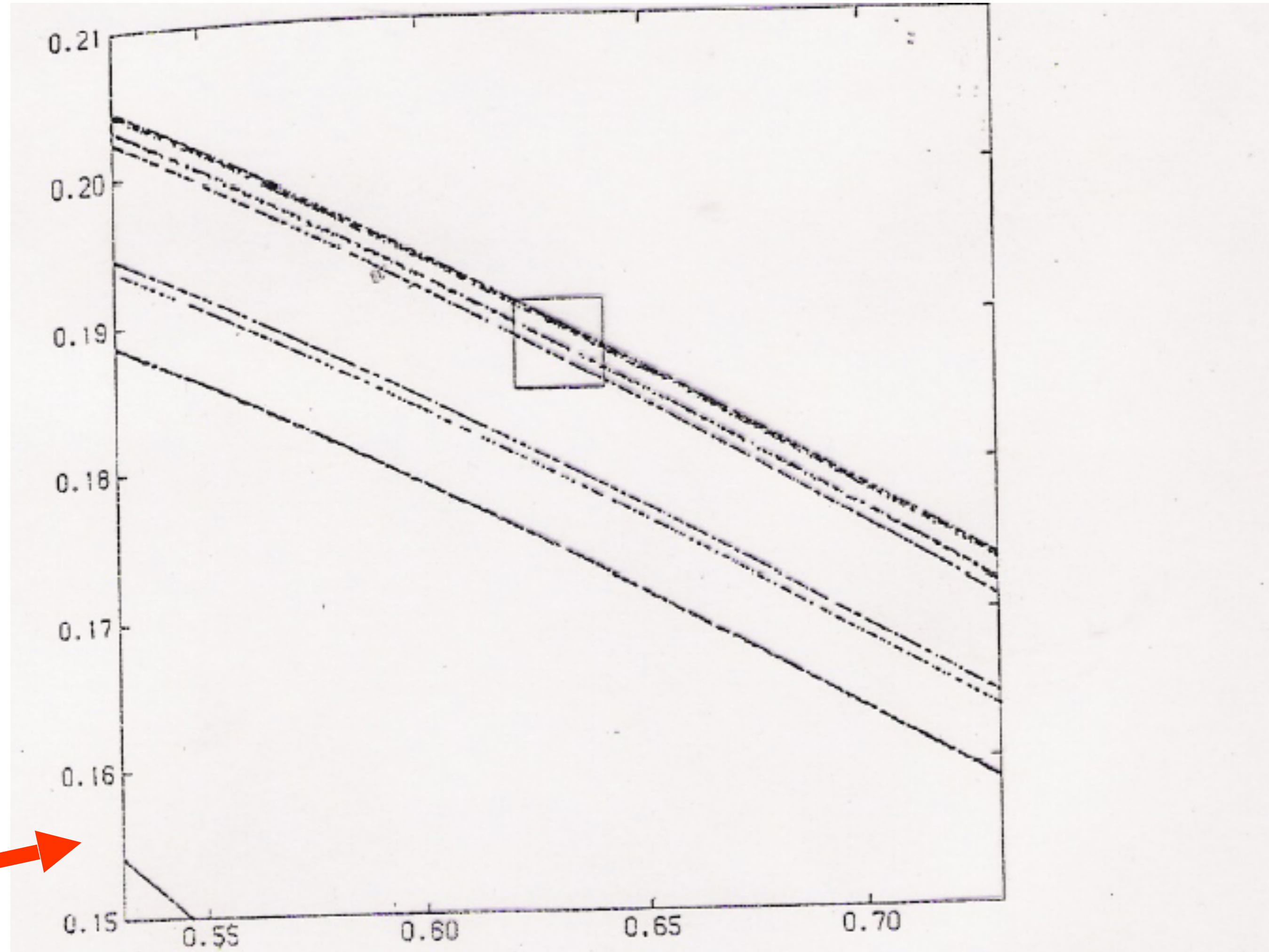
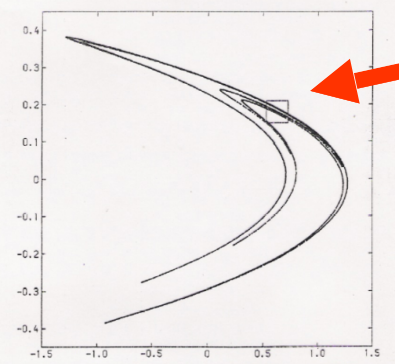


Fig. 4. Enlargement of the squared region of Figure 3. The number of computed points is increased to n 10^5



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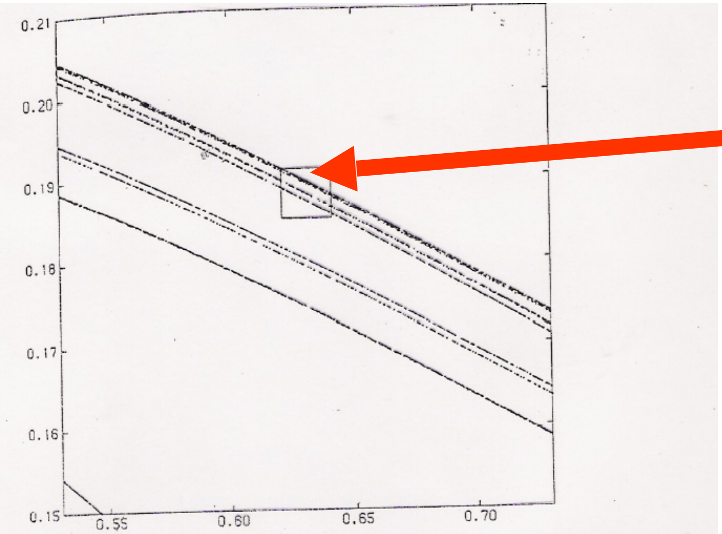


Fig. 4. Enlargement of the squared region of Figure 3. The number of computed points is increased to $n = 10^5$

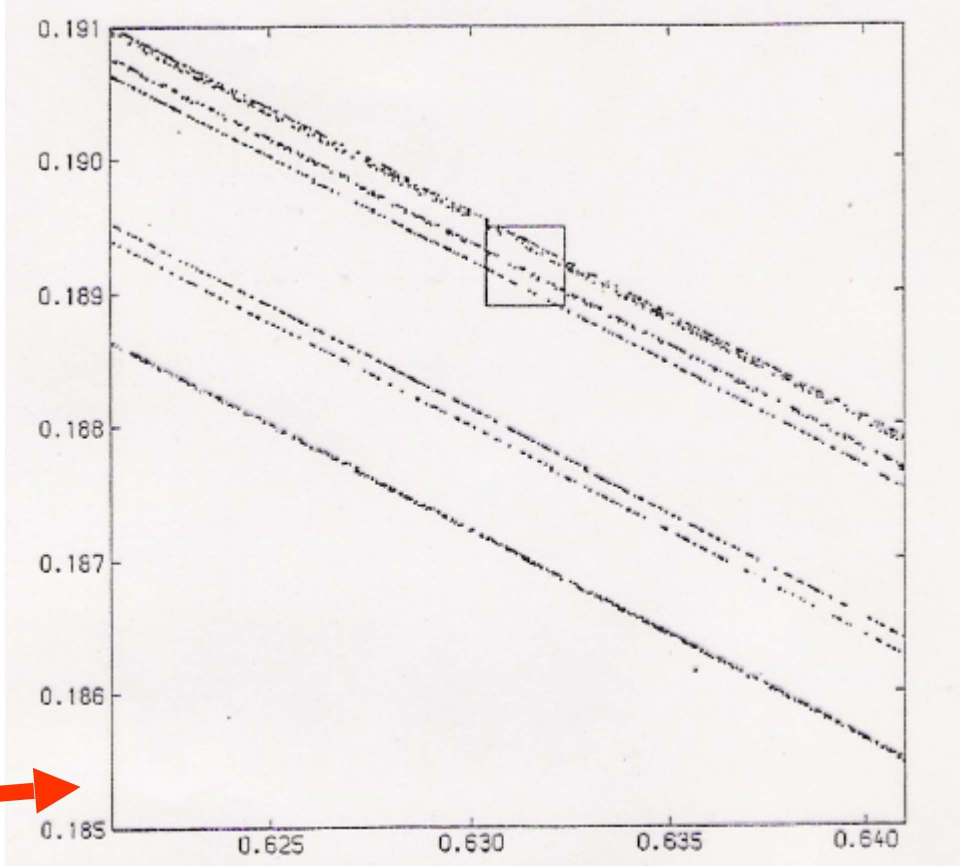


Fig. 5. Enlargement of the squared region of Figure 4; $n = 10^6$

Henon Map

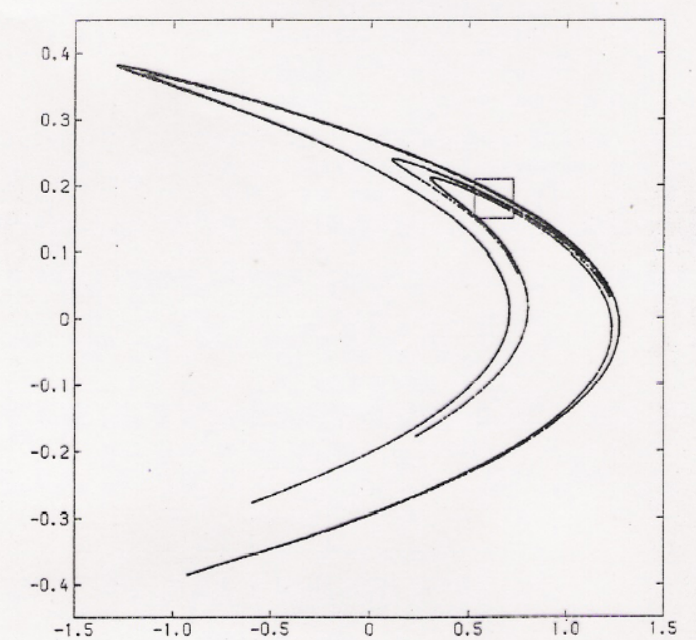
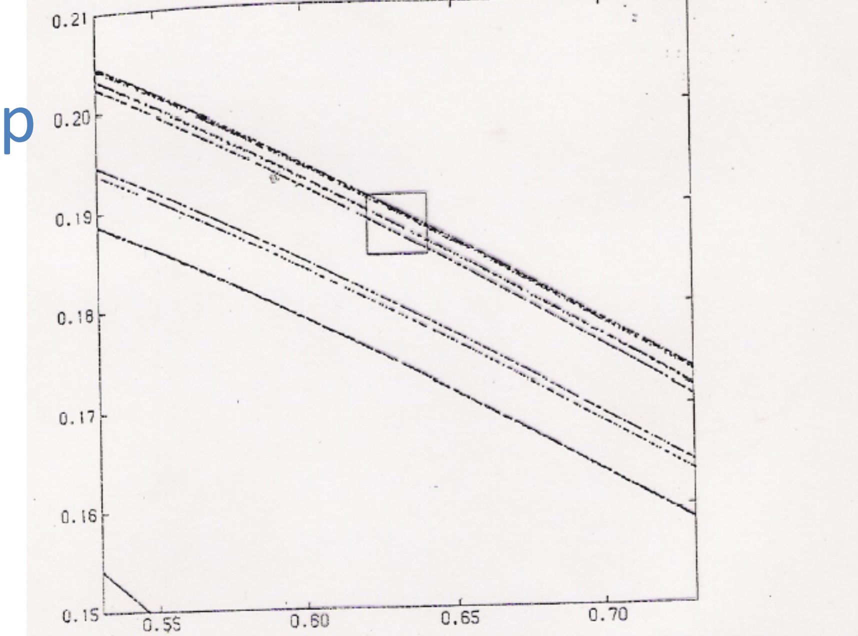


Fig. 3. Same as Figure 2, but starting from $x_0=0.63135448$, $y_0=0.18940634$



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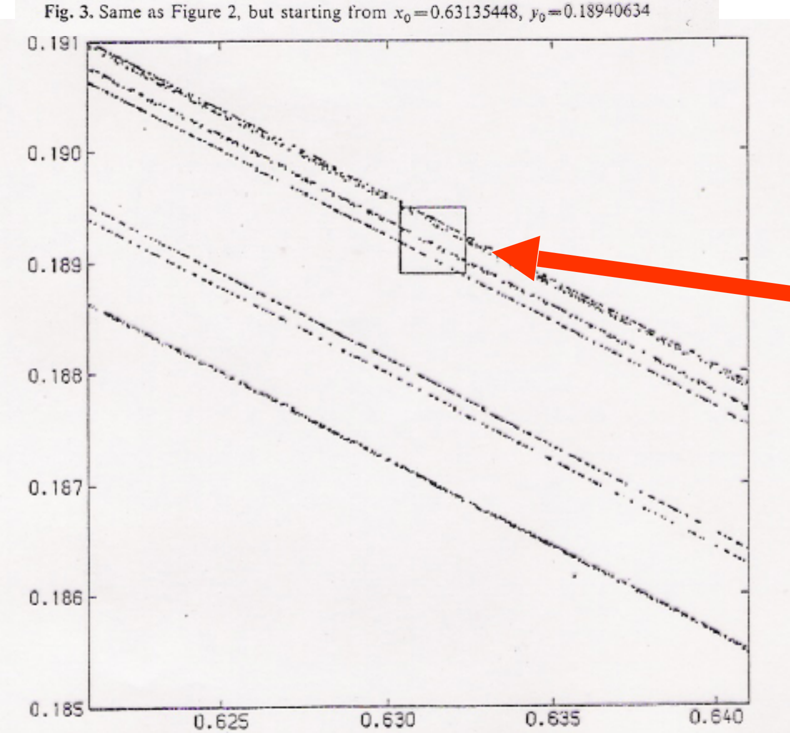


Fig. 5. Enlargement of the squared region of Figure 4: $n=10^6$

fractal

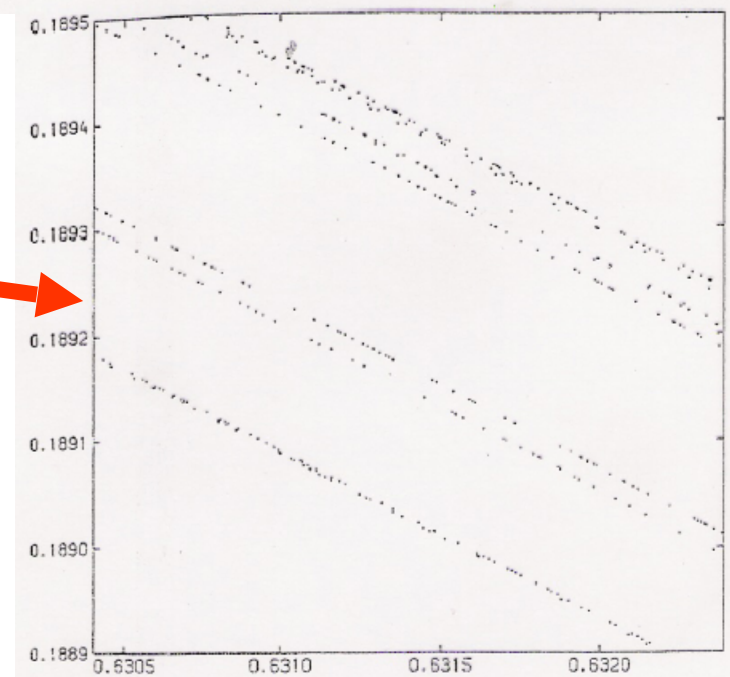


Fig. 6. Enlargement of the squared region of Figure 5; $n=5\times10^6$

Henon Map

$$\begin{aligned} x_{n+1} &= y_n + 1 - a x_n^2 \\ y_{n+1} &= b x_n \end{aligned}$$

$$J = \begin{pmatrix} -2ax & 1 \\ b & 0 \end{pmatrix}$$

Guess where the fixed point is…

Near the fixed point, the Jacobian determines the dynamics of the map.

What happens close to the fixed point?
$(x^* + \epsilon, y^* + \gamma), \epsilon, \gamma \to 0$

$$x^* = 1 - a{x^*}^2 + y^*$$
$$y^* = bx^*$$

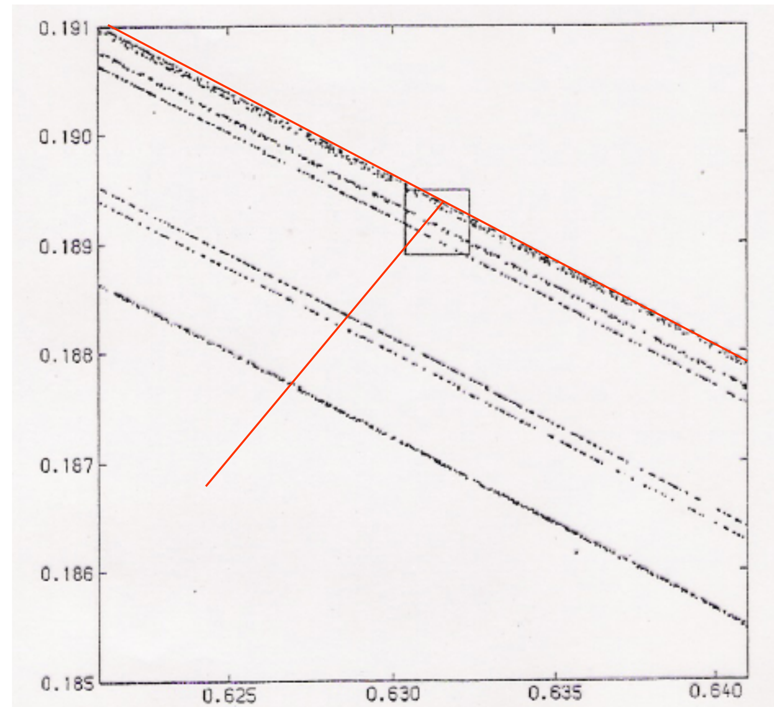


Fig. 5. Enlargement of the squared region of Figure 4; $n = 10^6$

$$x^* + \epsilon' = 1 - a(x^* + \epsilon)^2 + (y^* + \gamma)$$
$$y^* + \gamma' = b(x^* + \epsilon)$$

$$\epsilon' = -2ax^*\epsilon + \gamma$$
$$\gamma' = b\epsilon$$

$$\begin{pmatrix} \epsilon' \\ \gamma' \end{pmatrix} = J(x^*, y^*) \begin{pmatrix} \epsilon \\ \gamma \end{pmatrix}$$

fractal indicates chaos….

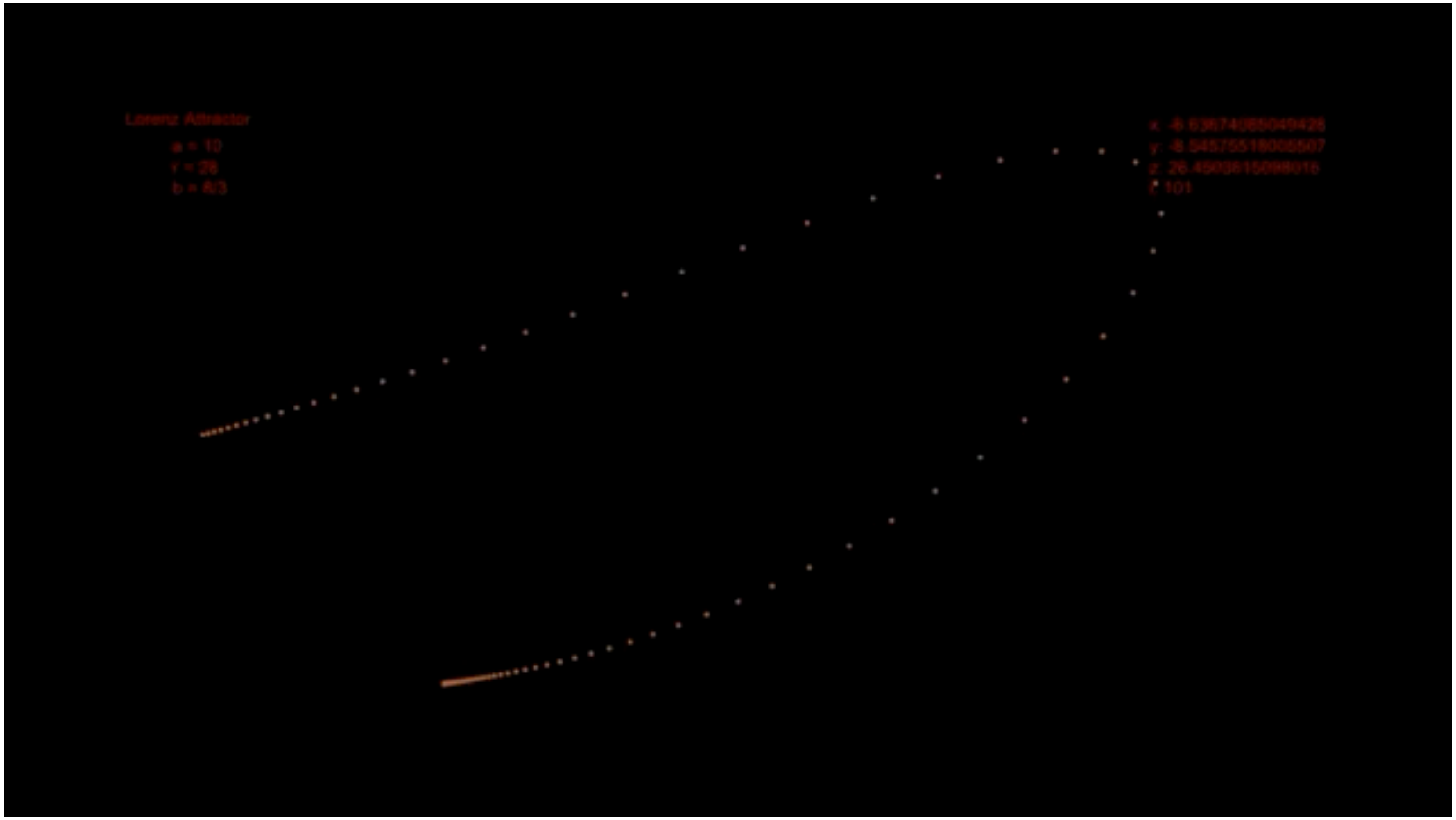
Classical example of Chaos: Double Pendulum

“Random”?

“Unpredictable”?

Aperiodic/Nonperiodic

Classical example of Chaos: Lorenz 63



Continuous in time, never repeat itself, so called nonperiodic flow.

Simple mathematical model (for atmospheric convection) based on three ordinary differential equations:

$$
\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= \sigma(y - x), \\
\frac{\mathrm{d}y}{\mathrm{d}t} &= x(\rho - z) - y, \\
\frac{\mathrm{d}z}{\mathrm{d}t} &= xy - \beta z.
\end{aligned}
$$

Lorenz, E. N. (1963). Deterministic Nonperiodic Flow, *Journal of Atmospheric Sciences*, *20*(2), 130-141.

Defining Chaos

Deterministic dynamics

Sensitive Dependence on Initial Condition

Recurrent Dynamics

Chaotic systems, look randomly, in fact are all deterministic. If we know the system and the initial condition, the future is certain.

If things are deterministic, what makes things unpredictable?

Defining Chaos

Deterministic dynamics

Sensitive Dependence on Initial Condition

Recurrent Dynamics

Butterflies and Seagulls, Causation and Explanation

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, 139th MEETING

Subject........................Predictability; Does the Flap of a Butterfly's wings in Brazil Set Off a Tornado in Texas?

Author........................Edward N. Lorenz, Sc.D.
Professor of Meteorology

Address........................Massachusetts Institute of Technology
Cambridge, Mass. 02139

Time............................10:00 a.m., December 29, 1972

Place...........................Sheraton Park Hotel, Wilmington Room

Program........................AAAS Section on Environmental Sciences
New Approaches to Global Weather: GARP
(The Global Atmospheric Research Program)

Convention Address...............Sheraton Park Hotel

RELEASE TIME
10:00 a.m., December 29

Lest I appear frivolous in even posing the title question, let alone suggesting that it might have an affirmative answer, let me try to place it in proper perspective by offering two propositions.

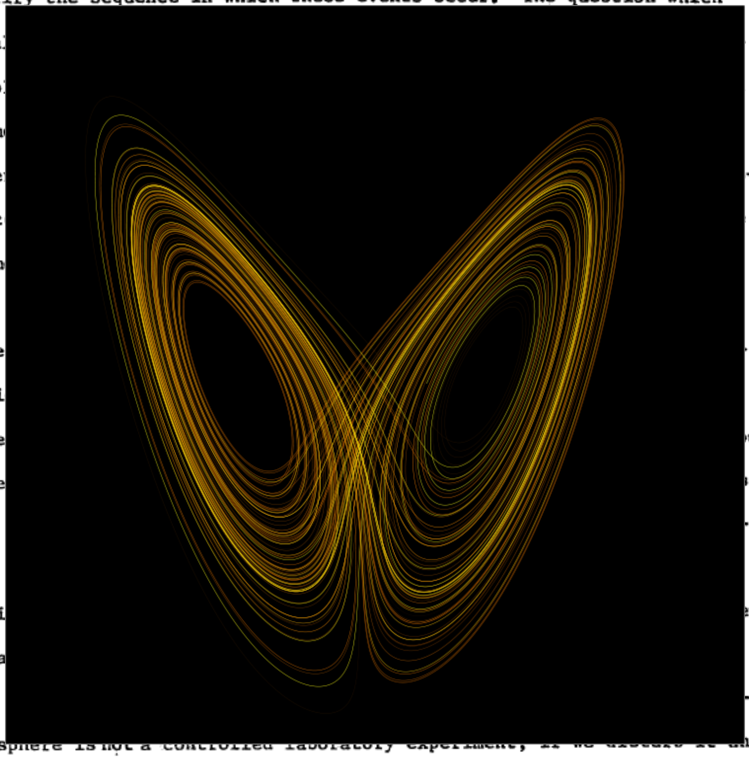
1. If a single flap of a butterfly's wings can be instrumental in generating a tornado, so also can all the previous and subsequent flaps of its wings, as can the flaps of the wings of millions of other butterflies, not to mention the activities of innumerabl... tures, including our own species.

2. If the flap of a butterfly's wings can be instrumental in generating a tornado, it can equally well be instrumental in preventing a tornado.

More gen... bances neithe... ious weather

modify the sequence in which these events occur. The question which

rea... amp... imm... cie... enc... atm... the... fli... are... que... at... al... sli... era...



mosphere is not a controlled laboratory experiment; if we disturb it and then observe what happens, we shall never know what would have happened

...y claim that we can learn what would ...he weather forecast would imply that the question whose answer we seek has already been answered in the negative.

The bulk of our conclusions are based upon computer simulation of the atmosphere. The equations to be solved represent our best attempts

The butterfly effect

In order to forecast hurricane, one have to take account all the butterflies all over the world. Really????

Sensitive Dependence on Initial Condition

For want of a nail the shoe was lost,

for want of a shoe the horse was lost,

and for want of a horse the rider was lost,

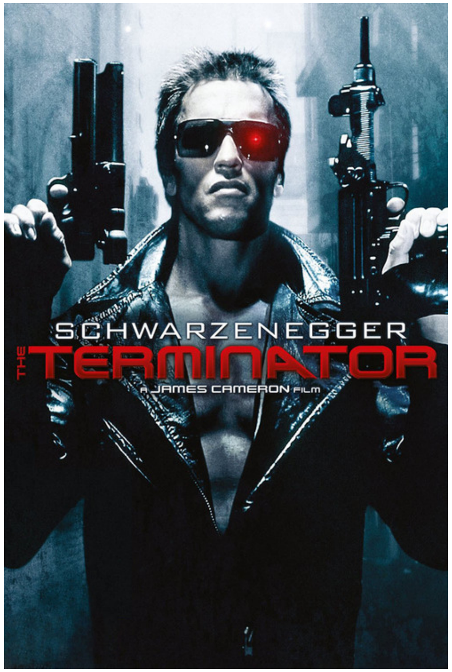
being overtaken and slain by the enemy,

all for the want of a horse-shoe nail.

The idea is not new

Poor Richard's Almanack in 1758 by Benjamin Franklin

Sensitive Dependence on Initial Condition



The idea also in famous movies

Sensitive Dependence on Initial Condition

$$\text{For all } x,\ \exists\ \delta > 0 \text{ such that}$$
$$\forall\ \epsilon > 0\ \exists\ y,\ n > 0 \text{ such that}$$

$$| x - y | < \epsilon$$

$$| f^n(x) - f^n(y) | > \delta$$

Note that δ is independent of x.

Assuming our model is perfect

With or without butterfly does make big difference, but it would usually require a large *n*

Defining Chaos

Deterministic dynamics

Sensitive Dependence on Initial Condition

Recurrent Dynamics

Recurrent Dynamics

A system is recurrent if the state of the system returns to itself.

For any initial condition we require that

$$| \mathbf{x}_0 - f^{\tau}(\mathbf{x}_0) | < \epsilon$$

“History Doesn’t Repeat Itself, but It Often Rhymes” – Mark Twain ponent must be unstable, in the sense that solutions temporarily approximating it do not continue to do so. A nonperiodic solution with a transient component is sometimes stable, but in this case its stability is one of its transient properties, which tends to die out.

To verify the existence of deterministic nonperiodic flow, we have obtained numerical solutions of a system of three ordinary differential equations designed to represent a convective process. These equations possess three steady-state solutions and a denumerably infinite set of periodic solutions. All solutions, and in particular the periodic solutions, are found to be unstable. The remaining solutions therefore cannot in general approach the periodic solutions asymptotically, and so are nonperiodic.

When our results concerning the instability of nonperiodic flow are applied to the atmosphere, which is ostensibly nonperiodic, they indicate that prediction of the sufficiently distant future is impossible by any method, unless the present conditions are known exactly. In view of the inevitable inaccuracy and incompleteness of weather observations, precise very-long-range forecasting would seem to be non-existent.

There remains the question as to whether our results really apply to the atmosphere. One does not usually regard the atmosphere as either deterministic or finite, and the lack of periodicity is not a mathematical certainty, since the atmosphere has not been observed forever.

The foundation of our principal result is the eventual necessity for any bounded system of finite dimensionality to come arbitrarily close to acquiring a state which it has previously assumed. If the system is stable, its future development will then remain arbitrarily close to its past history, and it will be quasi-periodic.

In the case of the atmosphere, the crucial point is then whether analogues must have occurred since the state of the atmosphere was first observed. By analogues, we mean specifically two or more states of the atmosphere, together with its environment, which resemble each other so closely that the differences may be ascribed to errors in observation. Thus, to be analo-

analogues have not occurred during this period, some accurate very-long-range prediction scheme, using observations at present available, may exist. But, if it does exist, the atmosphere will acquire a quasi-periodic behavior, never to be lost, once an analogue occurs. This quasi-periodic behavior need not be established, though, even if very-long-range forecasting is feasible, if the variety of possible atmospheric states is so immense that analogues need never occur. It should be noted that these conclusions do not depend upon whether or not the atmosphere is deterministic.

There remains the very important question as to how long is "very-long-range." Our results do not give the answer for the atmosphere; conceivably it could be a few days or a few centuries. In an idealized system, whether it be the simple convective model described here, or a complicated system designed to resemble the atmosphere as closely as possible, the answer may be obtained by comparing pairs of numerical solutions having nearly identical initial conditions. In the case of the real atmosphere, if all other methods fail, we can wait for an analogue.

Acknowledgments. The writer is indebted to Dr. Barry Saltzman for bringing to his attention the existence of nonperiodic solutions of the convection equations. Special thanks are due to Miss Ellen Fetter for handling the many numerical computations and preparing the graphical presentations of the numerical material.

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Extra case: so how do small uncertainties grow with time? (Henon Map)

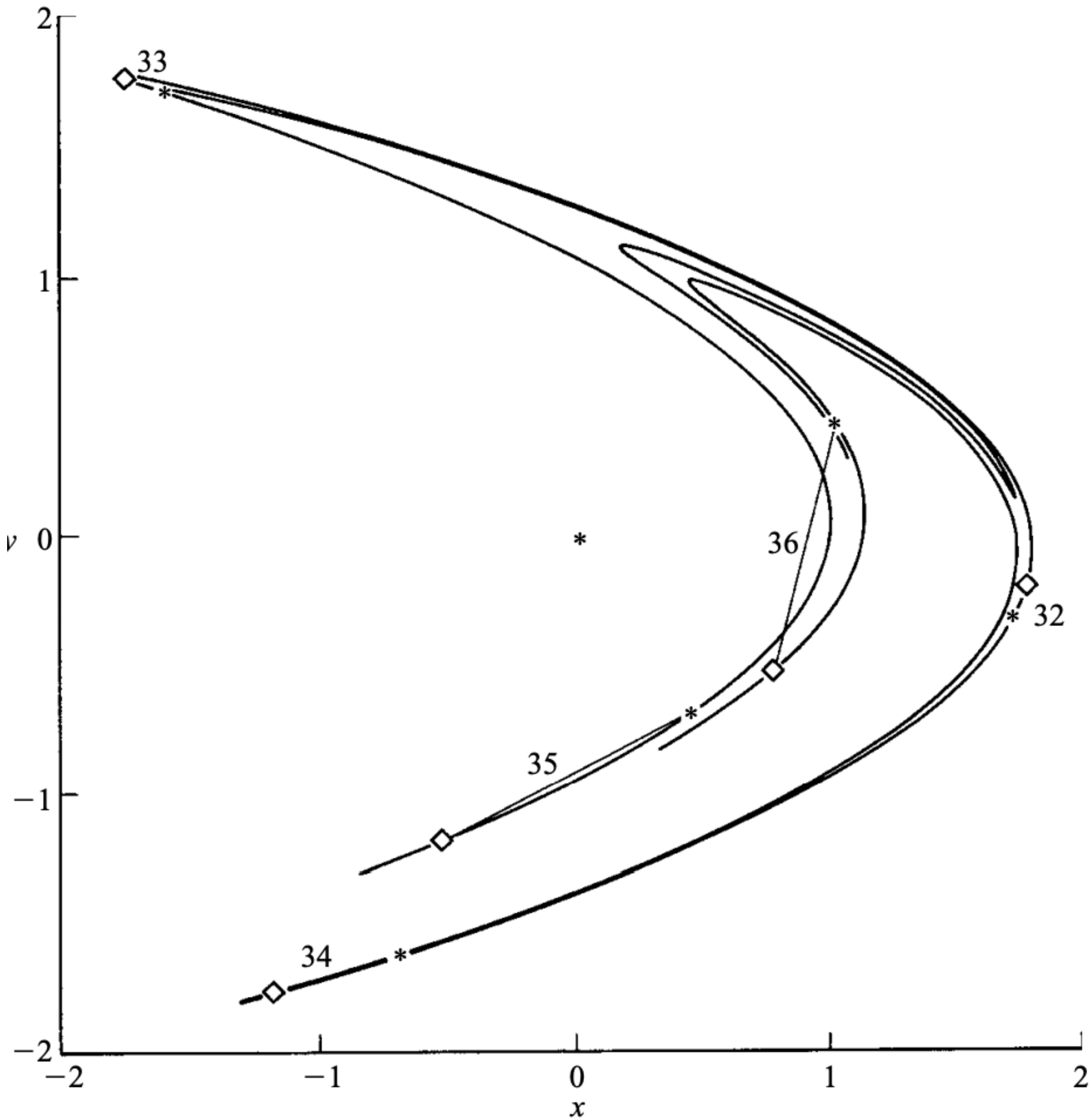

Figure 1.15 After a relatively small number of iterates, two trajectories, one computed using single precision, the other computed using double precision, both originating from the same initial condition, are far apart. (This figure courtesy of Y. Du.)

Due to???

E. Ott, "Chaos in Dynamical Systems," 2nd ed, Cambridge University Press, New York, 2002.