Exercises IV: Drift diffusion model

Cognitive modeling lecture 2024/25, week 7

Questions marked with a star are bonus material.*

Exercise 1 - The math of the DDM

1a) *To achieve an error rate of α in a free response paradigm with 2 choice options A and B, and with evidence samples y_i, we use this decision rule:*

$$decision = \begin{cases} choose\ A, & if\ \frac{p(A|y)}{p(B|y)} > \frac{1-\alpha}{\alpha} \\ choose\ B, & if\ \frac{p(A|y)}{p(B|y)} < \frac{\alpha}{1-\alpha} \\ wait, & otherwise \end{cases} \tag{0.1}$$

Use Bayes rule to show that this equivalent to using this expression:

$$Z_T = \log \frac{p(A)}{p(B)} + \sum_{t=1}^{T} \log \frac{p(y_t|A)}{p(y_t|B)} \tag{0.2}$$

with

$$decision = \begin{cases} choose\ A, & if\ \ Z_T > \log \frac{1-\alpha}{\alpha} \\ choose\ B, & if\ \ Z_T < \log \frac{\alpha}{1-\alpha} \\ wait, & otherwise \end{cases} \tag{0.3}$$

Solution. Bayes rule:

$$p(A|y) = \frac{p(y|A)p(A)}{p(y)} \tag{0.4}$$

When we take the fraction $\frac{p(A|y)}{p(B|y)}$, the terms $p(y)$ cancel.

Considering sequential data points y_t that are i.i.d., the total probability is a product:

$$\frac{p(A|y_T)}{p(B|y_T)} = \frac{p(A)\prod_{t+1}^{T} p(y_t|A)}{p(B)\prod_{t+1}^{T} p(y_t|B)} \tag{0.5}$$

Instead of dealing with a product, we take the log to look at a sum:

$$\log \frac{p(A|y_T)}{p(B|y_T)} = \log \frac{p(A)}{p(B)} + \sum_{t=1}^{T} \log \frac{p(y_t|A)}{p(y_t|B)} = Z_T, \tag{0.6}$$

which we can now compare to the log of our accuracy criterion.

1b)* *The probability distribution of the decision variable in the DDM at time t is normally distributed as:*

$$L_T = \mathcal{N}(bT, c^2T) \tag{0.7}$$

with a time-dependent mean (constant drift μ times time T) and variance (noise parameter c^2 times time T)[1].

Show that we arrive at such a distribution starting from the following assumptions:

- *The decision variable L_T, is the sum of all updates Δ_t until time T: $L_T = \sum_{t=1}^{T} \Delta_t$*
- *Each Δ_t is the log-likelihood ratio for the current evidence sample e_t under hypothesis A versus hypothesis B: $\Delta_t = \log \frac{p(e_t|s=A)}{p(e_t|s=B)}$*
- *Measurements e_t are Gaussian-distributed with different means m_s depending on the true state s of the world (which can be A or B), and $m_A = -m_B$:*

$$p(e_t|s = \{A, B\}) = \mathcal{N}\left(m_{\{A,B\}}, \sigma^2\right) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{\left(e_t - m_{\{A,B\}}\right)^2}{2\sigma^2}\right] \tag{0.8}$$

Tip 1: Start by considering the log-likelihood ratio Δ_t for a single data point e_t, and then take the sum to derive the distribution for L_T.

Tip 2: You can rewrite $e_t = m_{\{A,B\}} + \sigma\epsilon_t$, where $\epsilon_t \sim \mathcal{N}(0, 1)$ is a standard Gaussian variable with zero mean and unit variance.

Tip 3: For independent random variables, the mean of a sum is the sum of the means. And the variance of a sum is the sum of the variances.

Solution. In the log likelihood ratio for a single data point e_i, the normalizations cancel to give

$$\Delta_t = \log \frac{p(e_t|s = +1)}{p(e_t|s = -1)} = \frac{1}{2\sigma^2}\left[-\left(e_t - m_A\right)^2 + \left(e_t - m_B\right)^2\right] \tag{5}$$

Applying the tip above, we can rewrite this as

$$\Delta_t = \frac{1}{2\sigma^2}\left(-((m_{\{A,B\}} + \sigma\epsilon) - m_A)^2 + ((m_{\{A,B\}} + \sigma\epsilon) - m_B)^2\right) \tag{0.9}$$

Let's assume that $s = A$ so $m_{\{A,B\}} = m_A$ (if $s = B$ then the result is the same with a reversed sign). In that case, the means in the first term $e_t - m_A$ cancel, leaving

[1] ignoring the effect of bounds

$$\Delta_t = \frac{\delta^2 m^2}{2\sigma^2} + \frac{\delta m}{\sigma}\epsilon_t \tag{0.10}$$

where $\delta m = m_A - m_B$. Taking into account our assumption that $m_A = -m_B = m$, then $\delta m = 2m$, and

$$\Delta_t = 2\frac{m^2}{\sigma^2} + 2\frac{m}{\sigma}\epsilon_t \tag{0.11}$$

The first term is a constant **drift**, and the second term is a random **diffusion**.

The SPRT says that we should add up these evidences, $L_T = \sum_{t=1}^{T} \Delta_t$. Noting that the Δ_t are independent, and using Tip 3, we can find that the mean of the sum will be $2\frac{\mu^2}{\sigma^2}T$ and the variance will be $4\frac{\mu^2}{\sigma^2}T$. Thus, we have Gaussian-distributed log-likelihood ratio L_T:

$$L_T \sim \mathcal{N}\left(2\frac{\mu^2}{\sigma^2}T,\ 4\frac{\mu^2}{\sigma^2}T\right) = \mathcal{N}(bT, c^2T), \tag{0.12}$$

as claimed.

Exercise 2 - DDM parameters

2a) *Explore the impact of the DDM parameters on reaction time distributions. Use PyDDM model GUI as set up in Part 1 of the Python notebook for this exercise sheet. Note down your observations for the effects of changing the noise parameter and the non-decision time. Can you compensate for changes in noise by also changing the drift rate and bound?*

Solution. Noise, drift and bound parameters can be scaled by the same number without changing the model predictions. Non-decision time only changes the mean RT, but does not affect error rates. These conclusions can also be drawn from the analytical expressions for error rate and mean RT for the simple DDM which were presented as part of the lecture:

$$\mathrm{ER} = \frac{1}{1 + e^{2\mu B/c^2}} \tag{0.13}$$

$$\mathrm{RT} = \frac{B}{\mu}\tanh\frac{\mu B}{c^2} + t_{nd} \tag{0.14}$$

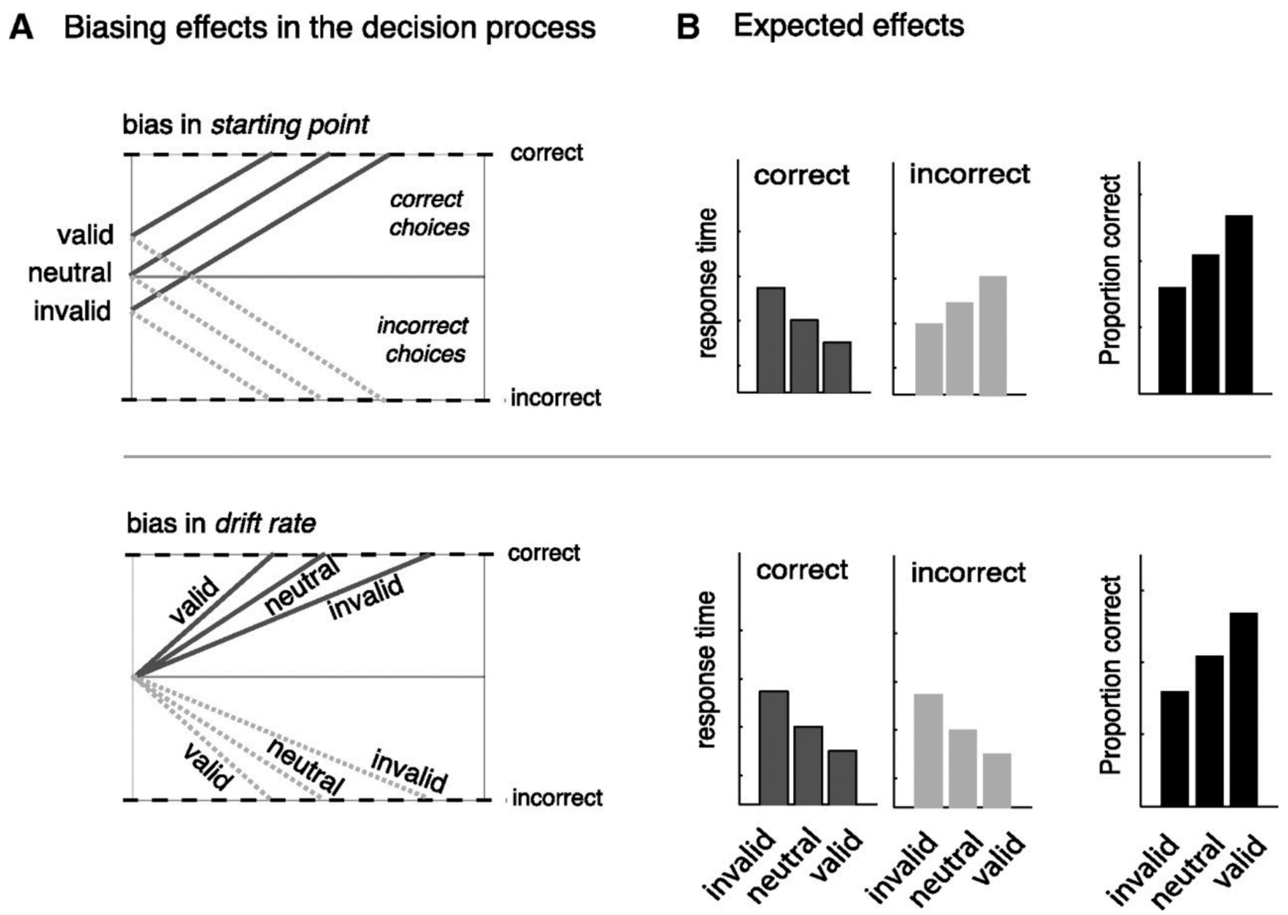
2b) *People often show decision biases, even if both choice options are equally likely. Such biases can also be elicited experimentally, for example by providing cues that inform participants about which decision is more likely to be correct in the upcoming trial. In the DDM, such biases could be implemented by a change in starting point, or by a bias in drift rate (higher drift rate for one option than the other). How can we experimentally distinguish between these options?*

Try to come up with a prediction about how the RT distributions for correct responses and errors would look in each case. You can also use the model GUI from above to help you. Tip: Consider correct responses and errors separately for the preferred (biased) option and the other option.

Solution. If we change the starting point (move it towards the preferred option), correct responses for this option will be faster, and errors will slower (and less likely). In contrast, the RTs for the other option will show the opposite pattern: errors (choosing the preferred option when the unpreferred option was true) will be fast, and correct responses will take longer.

If we use a higher drift rate for the preferred option, RTs both errors and correct responses in this option will be faster. RTs for both errors and correct responses for the unpreferred option will be slower.

Such predictions are discussed in detail in Mulder et al. 2012 Journal of Neuroscience, and Urai et al. 2019 eLife.



Exercise 3* - Fitting data using PyDDM

This task is optional, but recommended if you want to use the DDM in your project work.

In the online repository for this exercise sheet, you will find a Python notebook that walks you through using the PyDDM toolbox to simulate and fit DDM models.
Bonus question: You have analysed two datasets with choices and RTs. One dataset is from a group of middle-aged, healthy participants. The other dataset is from a group of older (but also healthy) people. You are interested in changes in decision strategies with aging. You have fit simple DDMs and have found a lower drift rate in the group of older people. How do you interpret your results?

Solution. See notebook for the first part.
Before you can go ahead and interpret your group differences, you should perform several steps to confirm your model results are valid (see Lecture 'Model fitting and model comparison' and Tutorial 'Step-by-step guide to building cognitive models'. You should:

- make sure your model parameters are recoverable (simulate choices under different parameter settings and try to recover the parameters
- compare the predictions that your final fitted model makes about behaviour and RT with the actual data. Can the model recover the main empirical findings?
- if you've used a simple DDM, it is likely that your fit will not be great. Check whether you can improve your fit by allowing for collapsing bounds, variable starting points across trials, variable drift rates across trials, evidence-dependent noise, etc.
- if you have only fitted one model, it is highly likely that there will be a better model for your data. Compare models that propose different mechanisms for generating the data and select the one that explains your data best.