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DIFFERENTIAL EQUATIONS AND PARTIAL DIFFERENTIAL EQUATIONS

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Introduction

In the study of physical phenomena, one is frequently unable to find directly the laws relating the quantities that characterize a phenomenon, whereas a relationship between the quantities and their derivatives or differentials can readily be established. One obtains equations containing the unknown functions or vector functions under the sign of the derivative.

Equations in which the unknown function or the vector function appears under the sign of the derivative are called differential equations. The following are some examples of differential equations:

$$\frac{dx}{dt} = -k \cdot x \tag{1}$$

is the equation of radioactive disintegration where: k is the disintegration constant, $x = x(t)$ is the quantity of the un-disintegrated substance at time t, and dx/dt is the rate of decay proportional to the quantity of un-disintegrating substance; (1) is a first-order differential equation.

$$m \cdot \frac{d^2\mathbf{r}}{dt^2} = \mathbf{F}\left(t, \mathbf{r}, \frac{d\mathbf{r}}{dt}\right) \tag{2}$$

is the equation of motion of the particle of mass m under the influence of a force $\mathbf{F}$ dependent on time, the position of the particle (which is determined by the radius vector $\mathbf{r} = \mathbf{r}(t)$), and its velocity $d\mathbf{r}/dt$; the force is equal to the product of the mass and the acceleration; (2) is a second-order differential equation (the highest order of the derivative).

$$\begin{cases} \dfrac{dx}{dt} = (b_1 \cdot y - d_1) \cdot x \\ \dfrac{dy}{dt} = (b_2 - d_2 \cdot x) \cdot y \end{cases} \tag{3}$$

is the Lotka-Voltera Predator-Prey model (used in biology) which calculates the predator population growth $x = x(t)$ and the pray population growth $y = y(t)$; the constants b_1 (b_2) represent the birth rate of the predator (prey) and d_1 (d_2) represent the death rate of the predator (prey); (3) is a nonlinear system of first-order differential equations.

The relation between the sought-for quantities will be found if methods are indicated for finding the unknown functions which are defined by differential equations. Finding the unknown function defined by differential equations is the principal task of the theory of differential equations.

If, in a differential equation, the unknown functions or the vector functions are functions of one variable, then the differential equation is called ordinary (ODE) (see Eqs (1)-(3) above). But, if the unknown function appearing in the differential equation is a function of two or more variables, the differential equation is called a partial differential equation (PDE), for example:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial xz^2} = 4\pi \cdot \rho(x, y, z) \tag{4}$$

is Poisson's equation, which, for example, is satisfied by the potential $u = u(x, y, z)$ of an electrostatic field, where $\rho(x, y, z)$ is the charge density.

This course covers the classes of differential equations and partial differential equations established in the curricula of the fourth semester for the students in computer science. The detailed information are given, the qualitative studies being accompanied by a rich collection of exercises and a large number of illustrations and graphs (given in Maple) to provide insight to numerical examples.

The authors

Contents

Chapter 1

First order ordinary differential equations

Definition 1.0.1. *A first order ordinary differential equation is a functional dependence of the form*

$$g(t, x, \dot{x}) = 0 \tag{1.1}$$

between the identity function $t \mapsto t$ defined on an unknown interval I, an unknown function x and its derivative $\dot{x}$ defined on the same interval I.

In equation (1.1), g is a known function, and solving the equation means determining the unknown function x which satisfies the equation.

Definition 1.0.2. *A real function x of class C^1 defined on the open interval $I \subset \mathbb{R}^1$ is called a solution of the equation(1.1) if, for any $t \in I$, $(t, x(t), \dot{x}(t))$ belongs to the domain of definition of the function g and*

$$g(t, x(t), \dot{x}(t)) = 0 \tag{1.2}$$

The graph of the solution, i.e. the set $\Gamma = \{(t, x(t))|t \in I\}$ is called the integral curve.

1.1 The primitive problem. The most elementary differential equation $\dot{x} = f(t)$

The most elementary case of a differential equation is given by

$$\dot{x} = f(t). \tag{1.3}$$

From integral calculus it is known that for a continuous real function f defined on the interval $I \subset \mathbb{R}^1$, there exists a family of real C^1-functions defined on I, whose derivative is the function f. These functions are given by:

$$x(t) = \int_{t^*}^{t} f(\tau)d\tau + C \tag{1.4}$$

in which C represents a real constant and $\int_{t^*}^{t} f(\tau)d\tau$ represents a primitive for f. For $t_0 \in I$ and $x_0 \in \mathbb{R}^1$ there is a unique solution $x = x(t)$ of the equation (1.4) which satisfies the condition $x(t_0) = x_0$ and this is given by formula:

$$x(t) = x_0 + \int_{t_0}^{t} f(s)ds. \tag{1.5}$$

The problem of determination of the solution of $\dot{x} = f(t)$ which satisfies the condition $x(t_0) = x_0$ is called initial value problem (IVP) or Cauchy problem. In this problem, t_0 and x_0 are known and are called the initial conditions. In the following, this problem will be denoted by:

$$\begin{aligned} &\dot{x} = f(t) \\ &x(t_0) = x_0 \end{aligned} \tag{1.6}$$

and its solution by $x(t; t_0, x_0)$.

Exercises

1. Compute the solutions of the following differential equations:

a) $\dot{x} = 1 + t + t^2;\ t \in \mathbb{R}^1$ **A:** $x(t) = \frac{t^3}{3} + \frac{t^2}{2} + t + C$

b) $\dot{x} = \frac{1}{t};\ t > 0$ **A:** $x(t) = \ln t + C$

c) $\dot{x} = 1 + \sin t + \cos 2t;\ t \in \mathbb{R}^1$ **A:** $x(t) = t - \cos t + \frac{1}{2}\sin 2t + C$

d) $\dot{x} = \frac{1}{1+t^2};\ t \in \mathbb{R}^1$ **A:** $x(t) = \arctan t + C$

e) $\dot{x} = \frac{1}{t^2-1};\ t \in (-1, 1)$ **A:** $x(t) = \frac{1}{2}\ln\frac{1-t}{1+t} + C$

f) $\dot{x} = \frac{1}{\sqrt{t^2-4}};\ t \in (-\infty, -2) \cup (2, \infty)$ **A:** $x(t) = \ln(t + \sqrt{t^2-4}) + C$

g) $\dot{x} = e^{2t} + \sin t;\ t \in \mathbb{R}^1$ **A:** $x(t) = \frac{1}{2}e^{2t} - \cos t + C$

2. Solve the following Cauchy problems and represent their solutions (on the computer):

a) $\dot{x} = 1 + t + t^2,\ t \in \mathbb{R}^1,\qquad x(0) = 1$

A: $x(t) = \frac{t^3}{3} + \frac{t^2}{2} + t + 1$

b) $\dot{x} = \frac{1}{t},\ t > 0,\qquad x(1) = 0$

A: $x(t) = \ln t$

c) $\dot{x}=1+\sin t+\cos 2t,\ t\in\mathbb{R}^1,$ $\quad x(-\pi) = 7$

A: $x(t)=-\cos t+\frac{1}{2}\sin 2t+t+6+\pi$

d) $\dot{x} = \frac{1}{1+t^2},\ t \in \mathbb{R}^1,$ $\quad x(-1) = -2$

A: $x(t) = \arctan t + \frac{1}{4}\pi - 2$

e) $\dot{x} = -\frac{2}{(t^2-1)^2},\ t < 1,$ $\quad x(-2) = 0$

A: $x(t)=\ln\sqrt{\frac{t-1}{t+1}}+\frac{t}{t^2-1}+\frac{2}{3}-\ln\sqrt{3}$

f) $\dot{x} = \frac{1}{\sqrt{t^2+t}},\ t > 0,$ $\quad x(1) = 1$

A: $x(t)=\ln\left(\frac{1}{2}+t+\sqrt{t^2+t}\right)+\ln 2+1$

1.2 Autonomous differential equations $\dot{x} = g(x)$

An autonomous differential equation is a functional dependence of the form

$$\dot{x} = g(x) \tag{1.7}$$

in which g is a known real and continuous function defined on the interval $J \subset \mathbb{R}^1$ such that $g(x) \neq 0\ (\forall) x \in J$.

Let $x : I \to J$ be a solution of the equation (1.7). Then for any $t \in I$ we have

$$\frac{dx}{dt} = g(x(t))$$

or

$$\frac{1}{g(x(t))} \cdot \frac{dx}{dt} = 1.$$

Using the primitives we obtain

$$\int_{t^*}^{t} \frac{1}{g(x(\tau))} \cdot \frac{dx}{d\tau}\, d\tau = \int_{t^*}^{t} d\tau$$

hence,

$$\int_{x^*}^{x} \frac{1}{g(u)}\, du = t + C. \tag{1.8}$$

Thus, a solution $x = x(t)$ of the autonomous differential equation (1.7) is a solution for the implicit equation

$$G(t, x; C) = 0 \tag{1.9}$$

in which

$$G(t, x; C) = t + C - \int_{x^*}^{x} \frac{1}{g(u)}\, du. \tag{1.10}$$

It is easy to see that, using the implicit function theorem, if $x(t; C)$ is a solution of the equation (1.9) then it is a solution for (1.7) as well.

Observation 1.2.1. If the function g is null in $x^* \in J$, then the constant function $x(t) = x^*$ is the solution of the differential equation (1.7).

Observation 1.2.2. For $t_0 \in \mathbb{R}^1$ and $x_0 \in J$, the determination of those solution of the equation (1.7) which satisfy the condition $x(t_0) = x_0$ is called initial value problem (IVP) or Cauchy problem:

$$\begin{aligned} \dot{x} &= g(x) \\ x(t_0) &= x_0 \end{aligned} \tag{1.11}$$

and its solution $x = x(t; t_0, x_0)$, is given by the implicit equation:

$$\int_{x_0}^{x} \frac{1}{g(u)}\, du = t - t_0. \tag{1.12}$$

In a Cauchy problem t_0 and x_0 are known and are called initial conditions.

Exercises

1. Solve the following differential equations:

a) $\dot{x} = 1 + x^2,\ x \in \mathbb{R}^1$ **A:** $x(t) = \tan(t + C),\ t + C \neq (2k+1) \cdot \frac{\pi}{2}$

b) $\dot{x} = e^{-x},\ x \in \mathbb{R}^1$ **A:** $x(t) = \ln(t + C),\ t + C > 0$

c) $\dot{x} = k \cdot x,\ x > 0$ **A:** $x(t) = C \cdot e^{k\,t},\ C > 0,\ t \in \mathbb{R}^1$

d) $\dot{x} = k \cdot x,\ x < 0$ **A:** $x(t) = C \cdot e^{k\,t},\ C < 0,\ t \in \mathbb{R}^1$

e) $\dot{x} = x^2,\ x > 0$ **A:** $x(t) = -\dfrac{1}{t + C},\ t + C < 1$

2. Solve the following Cauchy problems and represent their solutions (on the computer):

a) $\dot{x} = k\, x,\quad x(0) = x_0$ **A:** $x(t) = x_0\, e^{kt}$

b) $\dot{x} = -x + x^2,\quad x(0) = x_0$ **A:** $x(t) = \dfrac{x_0}{x_0 - e^t(x_0 - 1)}$

c) $\dot{x} = 1 + x^2,\quad x(0) = x_0$ **A:** $x(t) = \tan(t + \arctan x_0)$

d) $\dot{x} = x^2,\quad x(0) = x_0$ **A:** $x(t) = -\dfrac{x_0}{t\, x_0 - 1}$

1.3 Equations with separated variables

An equation with separated variables (separable equation) is a differential equation of the form:

$$\dot{x} = f(t) \cdot g(x), \tag{1.13}$$

in which f and g are known real and continuous functions $f : (a, b) \to \mathbb{R}^1$, $g : (c, d) \to \mathbb{R}^1$. If $g(x) \neq 0$, $(\forall)x \in (c, d)$, then the solutions of the equation (1.13) are determined in the same way as in the precedent section.

If $x : I \subset (a, b) \to (c, d)$ is a solution of the equation (1.13) then, for any $t \in I$, the following holds:

$$\frac{dx}{dt} = f(t) \cdot g(x(t))$$

or

$$\frac{1}{g(x(t))} \cdot \frac{dx}{dt} = f(t).$$

Using the primitive we have

$$\int_{t^*}^{t} \frac{1}{g(x(\tau))} \cdot \frac{dx}{d\tau}\, d\tau = \int_{t^*}^{t} f(\tau)\, d\tau$$

hence,

$$\int_{x^*}^{x} \frac{1}{g(u)}\, du = \int_{t^*}^{t} f(\tau)\, d\tau + C. \tag{1.14}$$

We obtain in this way that, a solution of the equation (1.13) is a solution for the implicit equation

$$G(x, t; C) = 0 \tag{1.15}$$

in which $G(x, t; C)$ is given by:

$$G(x, t; C) = \int_{t^*}^{t} f(\tau) + C - \int_{x^*}^{x} \frac{1}{g(u)}\, du. \tag{1.16}$$

It is easy to see that, using the implicit function theorem, if $x(t, C)$ is a solution for (1.15), then it is a solution for the differential equation (1.13) as well.

Example 1.3.1. Solve the following differential equation:

$$\dot{x} = \frac{1}{1 + t^2}(1 + x^2), \qquad t \in \mathbb{R}^1,\ x \in \mathbb{R}^1$$

In this case $f : \mathbb{R}^1 \to \mathbb{R}^1$, $f(t) = \frac{1}{1+t^2}$ and $g : \mathbb{R}^1 \to \mathbb{R}^1$, $g(x) = 1 + x^2$ hence,

$$G(t, x; C) = \arctan t + C - \arctan x$$

The implicit equation is:

$$\arctan t + C - \arctan x = 0$$

thus

$$x(t) = \tan(\arctan t + C)$$

Observation 1.3.1. If the function g from equation(1.13) is null in $x^* \in (c, d)$, then the constant function $x(t) = x^*$ is a solution of differential equation (1.13).

Observation 1.3.2. For $t_0 \in (a, b)$ and $x_0 \in (c, d)$, the determination of the solutions of the equation (1.13) which satisfy $x(t_0) = x_0$ is called initial value problem (IVP) or Cauchy problem and is denoted by

$$\dot{x} = f(t) \cdot g(x), \quad x(t_0) = x_0. \tag{1.17}$$

Usually, the solution of this problem is denoted by $x = x(t; t_0, x_0)$ and is given by implicit equation

$$\int_{x_0}^{x} \frac{du}{g(u)} - \int_{t_0}^{t} f(\tau)d\tau = 0. \tag{1.18}$$

In a Cauchy problem, t_0 and x_0 are known constants named initial conditions.

Observation 1.3.3. A particular class of equations with separated variables are represented by:

$$\dot{x} = A(t) \cdot x, \qquad t \in (a, b),\ x \in \mathbb{R}^1 \tag{1.19}$$

named first order homogeneous linear differential equation, in which $A(t)$ is a continuous real function defined on the interval (a, b). According to the above considerations, the solutions of the equation (1.19) are given by

$$x(t) = C \cdot e^{\int_{t^*}^{t} A(\tau)d\tau} \tag{1.20}$$

in which C is an arbitrary constant.
The solution of the Cauchy problem

$$\dot{x} = A(t) \cdot x, \quad x(t_0) = x_0 \qquad t_0 \in (a, b),\ x_0 \in \mathbb{R}^1 \tag{1.21}$$

is given by:

$$x(t; t_0, x_0) = x_0 \cdot e^{\int_{t_0}^{t} A(\tau)d\tau}. \tag{1.22}$$

Exercises

1. Compute the solutions for the following differential equations:

a) $\dot{x} = -\dfrac{t}{\sqrt{1+t^2}} \cdot \dfrac{\sqrt{1+x^2}}{x},\ x < 0,\ t \in \mathbb{R}^1$ **A:** $\sqrt{x^2+1} + \sqrt{t^2+1} = C$

b) $\dot{x} = \dfrac{t}{1+t}(1-x),\ t > -1,\ x > 1$ **A:** $\dfrac{1+t}{1-x} = C \cdot e^t$

c) $\dot{x} = \left(1 + \dfrac{1}{t}\right) \cdot \dfrac{x^2+1}{x^2+2},\ t > 0,\ x \in \mathbb{R}^1$ **A:** $x + \arctan x = \ln t + t + C$

2. Solve the following Cauchy problems and represent their solutions (on the computer):

a) $\dot{x} = \sqrt{t\,x},\ t > 0,\ x > 0, \qquad t_0 = 1,\ \ x_0 = 0$

A: $x(t) = 0$

b) $\dot{x} = -2t\dfrac{\sqrt{4-x^2}}{x},\ t \in \mathbb{R}^1,\ x \in (0,2), \quad t_0 = 0,\ \ x_0 = 1$

A: $\sqrt{4-x^2} - t^2 - \sqrt{3} = 0$

c) $\dot{x} = \dfrac{1}{1-t} \cdot \dfrac{1-x^2}{x},\ t < 1,\ x \in (0,1), \quad t_0 = 0 \quad x_0 = \dfrac{1}{2}$

A: $x(t) = \sqrt{-\dfrac{3}{4}t^2 + \dfrac{3}{2}t + \dfrac{1}{4}}$

1.4 Euler Homogeneous Equations

The Euler homogeneous equations are differential equations of the form:

$$\dot{x} = \frac{P(t,x)}{Q(t,x)} \tag{1.23}$$

in which $P(t,x)$ and $Q(t,x)$ are known homogeneous functions (in the Euler sense) of the same degree:

$$P(\lambda t, \lambda x) = \lambda^k \cdot P(t,x) \ \text{ and } \ Q(\lambda t, \lambda x) = \lambda^k \cdot Q(t,x). \tag{1.24}$$

From (1.24) we have the equality:

$$\frac{P(t,x)}{Q(t,x)} = \frac{P\left(1, \frac{x}{t}\right)}{Q\left(1, \frac{x}{t}\right)},\ (\forall) t \neq 0 \tag{1.25}$$

and thus, the homogeneous equation (1.23) has the following canonical form:

$$\dot{x} = g\left(\frac{x}{t}\right),\ (\forall) t \neq 0 \tag{1.26}$$

in which

$$g\left(\frac{x}{t}\right) = \frac{P\left(1, \frac{x}{t}\right)}{Q\left(1, \frac{x}{t}\right)}.$$

The real function g is known and continuous.

In order to find the solutions of the equation (1.26), we introduce the new unknown function $y = \dfrac{x}{t}$ which satisfies the equation:

$$\dot{y} = \frac{1}{t}[g(y) - y]. \tag{1.27}$$

The differential equations (1.23) and (1.27) are equivalent, i.e. if $x(t)$ is a solution for (1.23), then the function $y(t) = \dfrac{x(t)}{t}$ is solution for the equation (1.27) and viceversa.

In this way, to solve the homogeneous equation of Euler is equivalent to solving the differential equation with separable variables (1.27).

Exercises

1. Solve the following differential equations:

a) $\dot{x} = \dfrac{x}{t} + e^{\frac{x}{t}}$ **A:** $\ln(t) = e^{-\frac{x}{t}} + C$

b) $\dot{x} = \dfrac{x^2 + t^2}{t \cdot x}$ **A:** $x^2 = 2t^2 \ln(t) + C \cdot t^2$

c) $\dot{x} = \dfrac{t + x}{t - x}$ **A:** $\arctan\dfrac{x}{2} - \ln\sqrt{\dfrac{x^2}{t^2} + 1} = \ln t + C$

d) $\dot{x} = \dfrac{x}{t - 2\sqrt{tx}}$ **A:** $\sqrt{\dfrac{t}{x}} - \ln\dfrac{x}{t} = \ln t + C$

2. Solve the following Cauchy problems and represent their solutions (on the computer) :

a) $\dot{x} = \dfrac{4tx - x^2}{2t^2}$, $t_0 = 1$, $x_0 = 1$ **A:** $x(t) = \dfrac{2t^2}{t + 1}$

b) $\dot{x} = \dfrac{2tx}{3t^2 - x^2}$, $t_0 = 1$, $x_0 = 2$ **A:** $3(x(t))^3 - 8(x(t))^2 + 8t^4 = 0$

c) $\dot{x} = \dfrac{2t + x}{4t - x}$, $t_0 = 1$, $x_0 = 3$ **A:** $(x(t) - t)^3 \cdot (x(t) - 2t)^2 - 8t^6 = 0$

d) $\dot{x} = -\dfrac{x + t}{5x + t}$, $t_0 = 1$, $x_0 = 0$ **A:** $x(t) = -\dfrac{1}{5}t + \dfrac{1}{5}\sqrt{-4t^2 + 5}$

1.5 Generalized Homogeneous Equations

The generalized homogeneous equations are differential equations of the form:

$$\dot{x} = f\left(\frac{at + bx + c}{a_1 t + b_1 x + c_1}\right) \tag{1.28}$$

in which the real function f is known and continuous, and $c_1^2 + c^2 \neq 0$ (if $c_1 = c = 0$, the equation is Euler homogeneous). To determine the solution of this equation we take into account the following results:

Proposition 1.5.1. *If $\dfrac{a}{a_1} \neq \dfrac{b}{b_1}$ then, after a change of the independent variable t and a change of the unknown function x defined by*

$$\tau = t - t_0 \text{ and } y = x - x_0 \tag{1.29}$$

the generalized homogeneous equation (1.28) is transformed into the Euler homogeneous equation:

$$\frac{dy}{d\tau} = f\left(\frac{a\tau + by}{a_1\tau + b_1 y}\right) \tag{1.30}$$

where (t_0, x_0) *represents the solution of the algebraic system:*

$$\begin{cases} at + bx + c = 0 \\ a_1 t + b_1 x + c_1 = 0. \end{cases} \tag{1.31}$$

Proof. Direct computation. □

The change of the unknown function

$$z = \frac{y}{\tau} \tag{1.32}$$

transforms the equation (1.30) into the differential equation with separable variables:

$$\frac{dz}{d\tau} = \frac{1}{\tau}\left[f\left(\frac{a+bz}{a_1+b_1 z}\right) - z\right]. \tag{1.33}$$

Proposition 1.5.2. *If* $\dfrac{a}{a_1} = \dfrac{b}{b_1} = m$, *then after the change of the unknown function* x *defined by*

$$y(t) = a_1 t + b_1 x(t) \tag{1.34}$$

the differential equation (1.28) is transformed into the autonomous differential equation

$$\dot{y} = a_1 + b_1 \cdot f\left(\frac{my + c}{y + c_1}\right). \tag{1.35}$$

Exercises

1. Solve the following differential equation:

a) $\dot{x} = \dfrac{3t-4x+7}{4t-5x+11}$ **A:** $x(t) = -5 - \dfrac{1}{C}\left[-\dfrac{4}{5}(t+1) + \dfrac{1}{5}\sqrt{(t+9)^2 C^2 + 5}\right]$

b) $\dot{x} = -\dfrac{3t+3x-1}{t+x+1}$ **A:** $-\dfrac{1}{2}(x+t) - \ln(x+t-1) = t + C$

c) $\dot{x} = \dfrac{2(x+2)^2}{(t+x+2)^2}$ **A:** $2\arctan\dfrac{-x-2}{t} - \ln\dfrac{-x-2}{t} - \ln t - C = 0$

1.6 First Order Linear Equations

A first order linear equation is a differential equation of the form:

$$\dot{x} = A(t)x + B(t) \tag{1.36}$$

in which A and B are known real valued continuous functions $A, B : (a, b) \to \mathbb{R}^1$.

If the function B is identically zero, then the equation (1.36) is called first order homogeneous linear differential equation and its solutions are given by the formula:

$$\tilde{x}(t) = C \cdot e^{\int_{t^*}^{t} A(\tau)d\tau} \tag{1.37}$$

in which C is an arbitrary constant.

To determine the solutions of the equation (1.36) we will take into account the fact that, the difference between two solutions of the eq. (1.36) is a solution for the corresponding homogeneous differential equation. Thus, if x is a an arbitrary solution of eq. (1.36) and $\bar{x}$ is a fixed solution of the eq.(1.36), then the difference $x - \bar{x}$ will be a solution for the corresponding homogeneous equation, i.e. $x - \bar{x}$ satisfies:

$$x(t) - \bar{x}(t) = C \cdot e^{\int_{t^*}^{t} A(\tau)d\tau}$$

or

$$x(t) = C \cdot e^{\int_{t^*}^{t} A(\tau)d\tau} + \bar{x}(t). \tag{1.38}$$

Equality (1.38) shows that an arbitrary solution $x(t)$ of eq. (1.36) is obtained by summing a particular solution $\bar{x}(t)$ of this equation with an arbitrary solution $\tilde{x}(t) = C \cdot e^{\int_{t^*}^{t} A(\tau)d\tau}$ of the corresponding homogeneous equation. Thus, finding an arbitrary solution of eq. (1.36) is reduced to finding a particular solution of this equation.

To determine a particular solution of the equation (1.36) we use "Lagrange's undetermined coefficients method". This means that, for the equation(1.36) we search a particular solution $\bar{x}(t)$ which has the form of the given function (1.37), but in this case C is a function of t ($C = C(t)$):

$$\bar{x}(t) = C(t) \cdot e^{\int_{t^*}^{t} A(\tau)d\tau}. \tag{1.39}$$

We suppose that the function $C(t)$ is derivable and thus, $\bar{x}(t)$ verifies the equation (1.36):

$$\dot{C}(t)\, e^{\int_{t^*}^{t} A(\tau)d\tau} + A(t)\, C(t)\, e^{\int_{t^*}^{t} A(\tau)d\tau} = A(t)\, C(t)\, e^{\int_{t^*}^{t} A(\tau)d\tau} + B(t)$$

or

$$\dot{C}(t) = B(t)\, e^{-\int_{t^*}^{t} A(\tau)d\tau}. \tag{1.40}$$

In §1.1 we saw that, the functions which satisfy (1.40) are given by

$$C(t) = \int_{t^*}^{t} B(u)\, e^{-\int_{t^*}^{u} A(\tau)d\tau} du + C' \tag{1.41}$$

Because we search a unique solution, we have $C' = 0$ and (1.39) becomes:

$$\bar{x}(t) = \left(\int_{t^*}^{t} B(u)\, e^{-\int_{t^*}^{u} A(\tau)d\tau} du \right) e^{\int_{t^*}^{t} A(\tau)d\tau} \tag{1.42}$$

Thus, all the solutions of the equation (1.36) are given by:

$$x(t) = C\, e^{\int_{t^*}^{t} A(\tau)d\tau} + \left(\int_{t^*}^{t} B(u)\, e^{-\int_{t^*}^{u} A(\tau)d\tau} du \right) e^{\int_{t^*}^{t} A(\tau)d\tau}. \tag{1.43}$$

For $t_0 \in (a, b)$ and $x_0 \in \mathbb{R}^1$ the equation (1.36) has a unique solution x which satisfies $x(t_0) = x_0$ and is given by the formula:

$$x(t; t_0, x_0) = x_0\, e^{\int_{t_0}^{t} A(\tau)d\tau} + \int_{t_0}^{t} B(u)\, e^{\int_{u}^{t} A(\tau)d\tau} du. \tag{1.44}$$

Exercises

1. Compute the solutions of the following differential equations:

a) $\dot{x} = \dfrac{1}{t}x - 1$ **A:** $x(t) = t\,(-\ln t + C)$

b) $\dot{x} = -\dfrac{2}{t^2-1}x + 2t + 2$ **A:** $x(t) = \dfrac{(-t^2+2t+C)\,(t+1)^2}{1-t^2}$

c) $\dot{x} = -\dfrac{2}{t^2-1}x + \dfrac{4t}{1-t^2}$ **A:** $x(t) = \left(4\ln(t+1) + \dfrac{4}{t+1} + C\right)\cdot\dfrac{(t+1)^2}{1-t^2}$

d) $\dot{x} = x - t^2$ **A:** $x(t) = t^2 + 2t + 2e^t\,C$

2. Solve the following Cauchy problems and represent their solutions (on the computer)):

a) $\dot{x} = -2tx + t^3,\quad t_0 = 0,\quad x_0 = \dfrac{e-1}{2}$ **A:** $x(t) = \dfrac{1}{2}t^2 - \dfrac{1}{2} + \dfrac{1}{2}e^{-t^2+1}$

b) $\dot{x} = \dfrac{1}{t}x - \ln t,\quad t_0 = 1,\quad x_0 = 1$ **A:** $x(t) = \left(-\dfrac{1}{2}\ln^2 t + 1\right)t$

c) $\dot{x} = -x + 2e^t,\quad t_0 = 0,\quad x_0 = 2$ **A:** $x(t) = e^t + e^{-t}$

d) $\dot{x} = -ax + be^{pt},\quad t_0 = 0,\quad x_0 = 1$ **A:** $x(t) = \left(be^{(p+a)t} - b + p + a\right)\cdot\dfrac{e^{-a\,t}}{a+p}$

1.7 Bernoulli's Equation

Bernoulli's equation has the following form:

$$\dot{x} = A(t)\,x + B(t)\,x^{\alpha} \tag{1.45}$$

in which A and B are known real valued continuous functions $A, B : (a, b) \to \mathbb{R}^1$, α is a known real number different of 0 and 1, and the unknown function $x(t)$ is positive.

To determine the solutions x (positive) of the equation (1.45) we introduce a new unknown function $y = x^{1-\alpha}$. This satisfies the equation:

$$\frac{dy}{dt} = (1-\alpha)\,A(t)\,y + (1-\alpha)\,B(t). \tag{1.46}$$

The equation (1.46) is a first order linear differential equation and its solutions are given by:

$$\begin{aligned} y(t) &= C\,e^{(1-\alpha)\int_{t^*}^{t} A(\tau)d\tau} + \\ &+ \left((1-\alpha)\int_{t^*}^{t} B(u)\,e^{-(1-\alpha)\int_{t^*}^{u} A(\tau)d\tau} du\right)\; e^{(1-\alpha)\int_{t^*}^{t} A(\tau)d\tau}. \end{aligned} \tag{1.47}$$

The positive solutions $x(t)$ of the eq. (1.45) are obtained from $y(t)$ using the formula: $x(t) = y(t)^{\frac{1}{1-\alpha}}$ and generally, are defined on the interval (a, b).

For $t_0 \in (a,b)$ and $x_0 > 0$, the equation (1.45) has a solution which verifies $x(t_0) = x_0$ and is given by

$$x(t;t_0,x_0) = y^{\frac{1}{1-\alpha}}(t;t_0,x_0) \tag{1.48}$$

where:

$$y(t;t_0,y_0) = y_0\, e^{(1-\alpha)\int_{t^*}^{t} A(\tau)d\tau} + (1-\alpha)\int_{t_0}^{t} B(u)\, e^{-(1-\alpha)\int_{u}^{t} A(\tau)d\tau} du \tag{1.49}$$

and $y_0 = x_0^{1-\alpha}$.

Exercises

1.Compute the positive solutions of the following differential equations:

a) $\dot{x} = -\dfrac{1}{t}x + \dfrac{1}{t^2}x^2$ **A:** $x(t) = \dfrac{2t}{1+2t^2\,C}$

b) $\dot{x} = \frac{4}{t}x + tx^{1/2}$ **A:** $\sqrt{x(t)} = -\left(-\dfrac{1}{2}\ln t + C\right)\cdot t^2 = 0$

c) $\dot{x} = -\dfrac{1}{t}x + tx^2$ **A:** $x(t) = -\dfrac{1}{(t-C)\,t}$

d) $\dot{x} = \dfrac{1}{t}x - 2tx^2$ **A:** $x(t) = \dfrac{3t}{2t^3+3C}$

2. Solve the following Cauchy problems:

a) $\dot{x}{=}-\dfrac{1}{t}x{+}tx^2,\quad t_0{=}1,\quad x_0{=}1$ **A:** $x(t) = -\dfrac{1}{t(t{-}2)}$

b) $\dot{x}{=}\dfrac{1}{t}x{-}2tx^2,\quad t_0{=}1,\quad x_0{=}1$ **A:** $x(t){=}\dfrac{3t}{2t^3+1}$

c) $\dot{x}{=}\dfrac{2}{t}x{+}\dfrac{1}{2t^2}x^2,\quad t_0{=}1,\quad x_0{=}1$ **A:** $x(t){=}\dfrac{2t^2}{3{-}t}$

1.8 Riccati's Equation

Riccati's equation has the form

$$\dot{x} = A(t)\,x^2 + B(t)\,x + C(t) \tag{1.50}$$

in which A, B, C are known real valued continuous functions $A, B, C : (a,b) \to \mathbb{R}^1$, $A(t) \not\equiv 0$, $C(t) \not\equiv 0$.

Proposition 1.8.1. *If $x_1(t)$ is a fixed solution of the equation (1.50) and $x(t)$ is an arbitrary solution of the same equation, the function $y(t) = x(t) - x_1(t)$ is a solution of Bernoulli's equation*

$$\dot{y} = A(t)\,y^2 + (2A(t)\,x_1 + B(t))\,y \tag{1.51}$$

Proof. Direct computation. □

Observation 1.8.1. By the change of the unknown function $y(t) = x(t) - x_1(t)$, the determination of the solution of Riccati's equation (1.50) is reduced to the determination of the solution of Bernoulli's equation, which according to §1.7 is reduced at a first order linear differential equation.

Observation 1.8.2. We can reduce directly Riccati's equation to a first order linear differential equation, making the following change of the unknown function

$$x(t) = \frac{1}{z(t)} + x_1(t)$$

Exercises

1. Compute the solutions of the following Riccati's equations:

a) $\dot{x} = -\sin t \cdot x^2 + 2\frac{\sin t}{\cos^2 t}, \qquad x_1(t) = \frac{1}{\cos t}$

A: $x(t) = \frac{1}{\cos t} + \frac{6\cos 2t + 6}{-\cos 3t - 3\cos t + 12\,C}$

b) $\dot{x} = x^2 - \frac{a}{t}x - \frac{a}{t^2}, \qquad x_1(t) = \frac{a}{t}$

A: $x(t) = \frac{a}{t} + \frac{a+1}{-t + t^{-a}(a+1)\,C}$

2. Solve the following Cauchy problems:

a) $\dot{x} = -\frac{1}{t(2t-1)}x^2 + \frac{4t+1}{t(2t-1)}x - \frac{4t}{t(2t-1)}, \quad x_1(t) = 1,\ t_0 = 2,\ x_0 = 1$

A: $x(t) = \frac{t(2t-1)}{5-t} + 1$

b) $\dot{x} = -x^2 + \frac{4}{t}x - \frac{4}{t^2}, \qquad x_1(t) = \frac{1}{t}, t_0 = 1,\ x_0 = 0$

A: $x(t) = \frac{3}{t(t+2)} + \frac{1}{t}$

1.9 Exact differential equations. Integranting factor

A differential equation of the form:

$$\dot{x} = -\frac{P(t,x)}{Q(t,x)} \tag{1.52}$$

is called exact differential equation if there is exist a C^1-function U having the property:

$$dU = P\,dt + Q\,dx. \tag{1.53}$$

This means that there is exist a C^1-function U whose differential is equal to $P\,dt + Q\,dx$, i.e. $P = \dfrac{\partial U}{\partial t}$ and $Q = \dfrac{\partial U}{\partial x}$.

Proposition 1.9.1. *If the differential equation (1.52) is an exact differential equation and $U = U(t,x)$ is a real C^1-function satisfying (1.53), then for any solution $x = x(t)$ of the equation (1.52) we have*

$$U(t, x(t)) = const.$$

Proof. To prove that the function $U(t, x(t))$ doesn't depend on t, we compute the following derivative:

$$\begin{aligned}\frac{d}{dt}U(t,x(t)) &= \frac{\partial U}{\partial t} + \frac{\partial U}{\partial x}\cdot\frac{dx}{dt} = P(t,x(t)) + Q(t,x(t))\cdot\left(-\frac{P(t,x)}{Q(t,x)}\right) = \\ &= P(t,x(t)) - P(t,x(t)) = 0\end{aligned}$$

and thus, $U(t, x(t)) = \text{const}$ □

This proposition shows that a solution $x(t)$ of an exact differential equation is a solution of the implicit equation:

$$U(t,x) = C \tag{1.54}$$

in which C is an arbitrary constant.

Thus, the determination of the solution of the exact differential equation (1.52) is reduced to the determination of the solutions of the implicit equation (1.54). This result leads to the following two problems:

1. How do we see that the differential equation (1.52) is an exact differential equation?
2. How do we determine the function $U = U(t,x)$ whose differential is $P\,dt + Q\,dx$?

An answer to these problems can be given by the following proposition.

Proposition 1.9.2. *If the functions P and Q are of C^1 class on the domain Ω and $\dfrac{\partial P}{\partial x} = \dfrac{\partial Q}{\partial t}$, then for any $(t_0, x_0) \in \Omega$ there is $r > 0$ and a real function $U = U(t,x)$ defined on the disc centered in (t_0, x_0) of radius r such that (1.53) takes place.*

Proof. For a point $(t_0, x_0) \in \Omega$ we consider $r > 0$ such that the disc centered in (t_0, x_0) of radius $r > 0$ is included in Ω. Because on the disc we must have $\dfrac{\partial U}{\partial t} = P$, we deduce that $U(t,x) = \displaystyle\int_{t_0}^{t} P(\tau, x)d\tau + \Psi(x)$ where Ψ is an unknown C^1-function. By condition $\dfrac{\partial U}{\partial x} = Q$ we obtain:

$$\int_{t_0}^{t} \frac{\partial P}{\partial x}(\tau, x)\,d\tau + \Psi'(x) = Q(t,x).$$

Taking into account the equality $\dfrac{\partial P}{\partial x} = \dfrac{\partial Q}{\partial t}$ we have:

$$\int_{t_0}^{t} \frac{\partial Q}{\partial t}(\tau, x)\, d\tau + \Psi'(x) = Q(t, x).$$

By integration we obtain the equality:

$$Q(t, x) - Q(t_0, x) + \Psi'(x) = Q(t, x)$$

from which we have:

$$\Psi'(x) = Q(t_0, x).$$

Hence, $\Psi(x)$ is given by the formula:

$$\Psi(x) = \int_{x_0}^{x} Q(t_0, y)\, dy + C \tag{1.55}$$

in which C is a real constant. Finally, we obtain for $U(t, x)$:

$$U(t, x) = \int_{t_0}^{t} P(\tau, x)\, d\tau + \int_{x_0}^{x} Q(t_0, y)\, dy + C. \tag{1.56}$$

This formula defines a set of functions $U(t, x)$ which have the property expressed by relation (1.53). $\square$

Comments: The proposition shows that the equality $\dfrac{\partial P}{\partial x} = \dfrac{\partial Q}{\partial t}$ is a sufficient condition ensuring that in the neighborhood of any point $(t_0, x_0) \in \Omega$ there exists a function $U(t, x)$ of class C^2 such that $dU = P\, dt + Q\, dx$.

Moreover, we underline that the reciprocal of this proposition is true, i.e. if there exists $r > 0$ and a function $U(t, x)$ of class C^2 on the disc centered in (t_0, x_0) and radius r such that $dU = P\, dt + Q\, dx$ for any (t, x) from this disc, then the functions P and Q are of class C^1 and $\dfrac{\partial P}{\partial x} = \dfrac{\partial Q}{\partial t}$ for any (t, x) from disc. This results is obtained on the basis of Schwartz's theorem of the inverse of the derivative's order.

Observation 1.9.1. If the functions P and Q are of class C^1 on $\Omega \subset \mathbb{R}^2$ but $\dfrac{\partial P}{\partial x} \neq \dfrac{\partial Q}{\partial t}$ then the equation (1.52) is not an exact differential equation. In this case we can observe that the equation (1.52) has the same solution as the equation

$$\dot{x} = -\frac{P(t, x) \cdot \mu(t, x)}{Q(t, x) \cdot \mu(t, x)} \tag{1.57}$$

in which $\mu(t, x)$ is a C^1-function which doesn't have a zero value.

Due to this fact, we try to determine the function $\mu(t, x)$ such that the equation (1.57) to be an exact differential equation. Imposing this condition, we have that the function $\mu(t, x)$ must satisfy:

$$\frac{\partial P}{\partial x}\, \mu + P\, \frac{\partial \mu}{\partial x} = \frac{\partial Q}{\partial t}\, \mu + Q\, \frac{\partial \mu}{\partial t}. \tag{1.58}$$

O function which satisfies (1.58) is called integrating factor, and the relation (1.58) is called the equation of the integrating factor.

Exercises

1. Solve the following differential equations:

a) $\dot{x} = \dfrac{4tx - xe^{tx}}{te^{tx} - 2t^2}$ **A:** $2t^2\,x(t) - e^{t\,x(t)} = C$

b) $\dot{x} = \dfrac{t^m + 2tx^2 + \frac{1}{t}}{x^n + 2t^2x + \frac{1}{x}}$ **A:** $\dfrac{t^{m+1}}{m+1} + \dfrac{x(t)^{n+1}}{n+1} + t^2\,x(t)^2 + \ln(t\,x(t)) = C$

c) $\dot{x} = -\dfrac{2tx - 2x^3}{t^2 - 6tx^2}$ **A:** $t^2\,x(t) - 4t\,x(t)^3 = C$

2. Solve the Cauchy problems:

a) $\dot{x} = -\dfrac{t+x}{t-x}$, $t_0 = 0,\ x_0 = 1$ **A:** $x(t) = t + \sqrt{2t^2 + 1}$

b) $\dot{x} = -\dfrac{t^2}{x^2}$, $t_0 = 1,\ x_0 = 1$ **A:** $x(t) = \sqrt[3]{-t^3 + 2}$

3. Solve the following differential equations if they have a integrating factor $\mu = \mu(t)$:

a) $\dot{x} = -\dfrac{t\,\sin x + x\,\cos x}{t\,\cos x - x\,\sin x}$ **A:** $\mu(t) = e^t$

$$e^t\left[(t-1)\sin x(t) + x(t)\cos x(t)\right] = C$$

b) $\dot{x} = -\dfrac{1 - t^2\,x}{t^2(x - t)}$ **A:** $\mu(t) = \dfrac{1}{t^2}$

$$\frac{x(t)^2}{2} - t\,x(t) - \frac{1}{t} = C$$

4. Solve the following differential equations if they have a integrating factor $\mu = \mu(x)$:

a) $\dot{x} = -\dfrac{x(1 - t\,x)}{-t}$ **A:** $\mu(x) = \dfrac{1}{x^2}$

$$\frac{t}{x(t)} - \frac{t^2}{2} = C$$

b) $\dot{x} = -\dfrac{2t\,x}{3x^2 - t^2 + 3}$ **A:** $\mu(x) = \dfrac{1}{x^2}$

$$\frac{t^2}{x(t)} + 3x(t) = C$$

1.10 Symbolic calculus of the solutions of the differential equations of the first order

The numerical examples from this course will be presented in *Maple 9* software.

For the symbolic calculus of the solution of the differential equation *Maple* uses the function *dsolve* (to solve ordinary differential equations - ODEs) which has one of the following syntaxes:

dsolve(ODE);

dsolve(ODE, x(t), extra.args);

dsolve({ODE, ICs}, x(t), extra.args);

in which:

ODE	- the differential equation which has to be solved
$x(t)$	- the unknown function which we want to determine
ICs	- the initial coonditions
$extra.args$	- the optional arguments used for the change of display mode of the solution (in explicit, implicit, or parametric form), specify the method (separable variables, Bernoulli, Riccati, etc.)

For illustration, we consider the first order differential equation:

$$\dot{x} = \frac{t}{1+t} \cdot (1-x); \quad t \in \mathbb{R} - \{-1\}, \quad x \in \mathbb{R} - \{1\}. \tag{1.59}$$

This equation is with separated variables (a particular case of first order linear differential equation).
Using the syntax $dsolve(ODE)$ we obtain the set of the solutions written in explicit form:

```
> dsolve(diff(x(t),t)=(t/(1+t))*(1-x(t)));
```

$$x(t) = \left(\frac{e^t}{1+t} + \_C1\right)(e^{-t} + e^{-t}t).$$

If we wish to display the solutions in parametric form, we use the optional argument '*parametric*' and we have:

```
> dsolve(diff(x(t),t)=(t/(1+t))*(1-x(t)),x(t),'parametric');
```

$$x(t) = 1 - \frac{e^{-t}}{\_C1} - \frac{e^{-t}t}{\_C1}.$$

Another optional argument which can be used is "the method to solve the equation".
If we want the equation to be solved as a linear differential equation, then we use the optional argument [*linear*] and we have:

```
> dsolve(diff(x(t),t)=(t/(1+t))*(1-x(t)),x(t),[linear]);
```

$$x(t) = \left(\frac{e^t}{1+t} + \_C1\right)(e^{-t} + e^{-t}t),$$

and if we want the equation to be solved as a separable equation, then we use the optional argument [*separable*] and we have:

```
> dsolve(diff(x(t),t)=(t/(1+t))*(1-x(t)),x(t),[separable]);
```

$$x(t) = \frac{(\_C1\, e^t - 1 - t)e^{-t}}{\_C1}.$$

If we do not specify the method, *Maple* chooses one of them.

In the above sequences we didn't use the initial conditions so, *Maple* used an unknown constant C in the expression of the solutions. If we will specify the initial condition, then the computer will solve an initial value problem (Cauchy's problem) an will give the corresponding solution.

For the differential equation (1.59) we will consider two Cauchy problems because the domain of the right-hand side of the equation is the reunion $(-\infty, -1) \times \mathbb{R}^1 \cup (-1, +\infty) \times \mathbb{R}^1$.

If we will consider $t > -1$ and the initial condition $x(2) = 4$, then the following solution will be obtained:

```
> dsolve({diff(x(t),t)=(t/(1+t))*(1-x(t)),x(2)=4},x(t));
```

$$x(t) = \left(\frac{e^t}{1+t} - 1/3\,\frac{e^{-2}e^2 - 4}{e^{-2}}\right)(e^{-t} + e^{-t}t),$$

and if we will consider $t < -1$ and the initial condition $x(-2) = 0$, then we will obtain the solution:

```
> dsolve({diff(x(t),t)=(t/(1+t))*(1-x(t)),x(-2)=0},x(t));
```

$$x(t) = \left(\frac{e^t}{1+t} + e^{-2}\right)(e^{-t} + e^{-t}t).$$

To represent the solution of the initial value problem, the *Maple* software uses the function *plot* (creates a two-dimensional plot of functions).
Using this function implies the following syntaxes:

plot(f,h,v);

	f	- the function which has to be representaed on graph;
in which:	h	- the domain of the function on the orizontal axis;
	v	- (optional) the domain of the function on the vertical axis.

The solution of the Cauchy problem (1.59) corresponding to the initial condition $x(2) = 4$ is represented in Figure 1.1.

```
> f1:=(exp(t)/(1+t)-1/3*(exp(-2)*exp(2)-4)/exp(-2))*
   (exp(-t)+exp(-t)*t):
> plot(f1,t=-1..infinity);
```

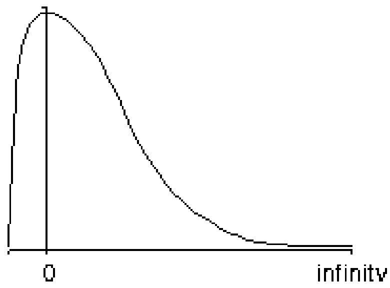


Figure 1.1:

and the solution of the Cauchy problem (1.59) corresponding to the initial condition $x(-2) = 0$ is represented in Figure 1.2.

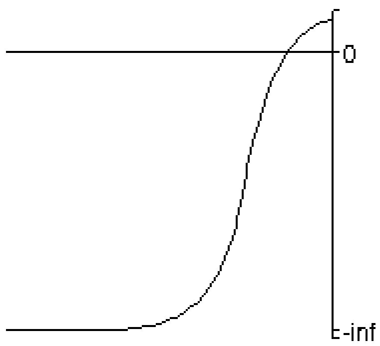


Figure 1.2:

In the followings, we solve three problems with initial conditions and we will represent the corresponding solutions:

1. The first order linear differential equation

$$\dot{x} = -x + 2e^t \tag{1.60}$$

```
> dsolve(diff(x(t),t)=-x(t)+2*exp(t),x(t),[linear]);
```

$$x(t) = e^t + e^{-t}\_C1$$

```
> dsolve({diff(x(t),t)=-x(t)+2*exp(t),x(0)=2},x(t),
[linear]);
```

$$x(t) = e^t + e^{-t}$$

```
> plot(exp(t)+exp(-t),t=-2..2);
```

or

```
> plot(exp(t)+exp(-t),t=-2..2,color=black,style=point,
axes=boxed);
```

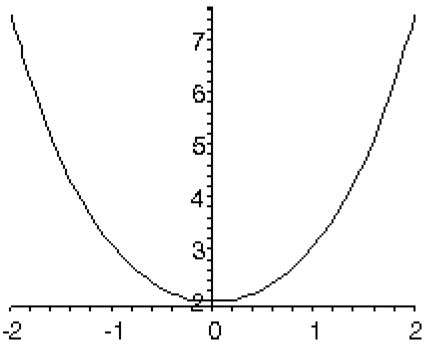


Figure 1.3:

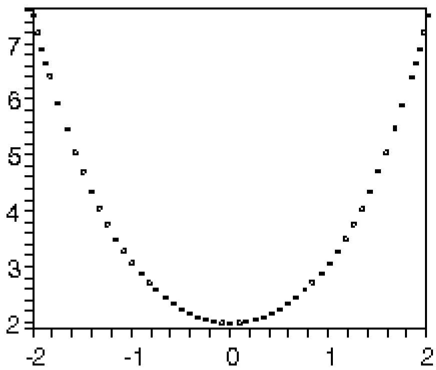


Figure 1.4:

2. Riccati's equation

$$\dot{x} = -x^2 + \frac{4}{t} \cdot x - \frac{4}{t^2}, \qquad t > 0 \tag{1.61}$$

```
> eq:=diff(x(t),t)=-x(t)^2+(4/t)*x(t)-4/t^2;
```

$$eq := \frac{d}{dt}x(t) = -(x(t))^2 + 4\,\frac{x(t)}{t} - 4\,t^{-2}$$

```
> dsolve(eq,'explicit',[Riccati]);
```

$$x(t) = (\_C1 - 1/3\,t^{-3})^{-1}\,t^{-4} + 4\,t^{-1}$$

```
> dsolve(eq,[Riccati]);
```

$$x(t) = (\_C1 - 1/3\,t^{-3})^{-1}\,t^{-4} + 4\,t^{-1}$$

```
> dsolve({eq,x(1)=2},x(t));
```

$$x(t) = \frac{4\,t^3+2}{(2+t^3)\,t}$$

```
> dsolve({eq,x(1)=2},x(t),[Riccati]);
```

$$x(t) = (-1/6 - 1/3\,t^{-3})^{-1}\,t^{-4} + 4\,t^{-1}$$

```
> sol1:=(4*t^3+2)/((2+t^3)*t):
> sol2:=1/((-1/6-1/3/t^3)*t^4)+4/t:
> plot([sol1,sol2],t=0..90,x=0..3,color=[red,blue],
  style=[point,line]);
```

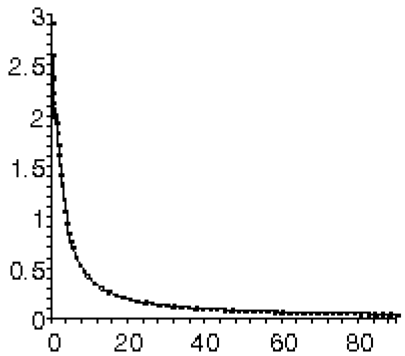


Figure 1.5:

3. The integrating factor equation

$$\dot{x} = -\frac{2 \cdot t \cdot x}{3x^2 - t^2 + 3}, \qquad 3x^2 - t^2 + 3 \neq 0 \tag{1.62}$$

```
> dsolve(diff(x(t),t)=-2*t*x(t)/(3*x(t)^2-t^2+3),'explicit');
```

$$x(t) = -1/6\,\_C1 \pm 1/6\,\sqrt{\_C1^2 - 12\,t^2 + 36}$$

```
> dsolve(diff(x(t),t)=-2*t*x(t)/(3*x(t)^2-t^2+3),'implicit');
```

$$\frac{t^2}{x(t)} + 3\,x(t) - 3\,(x(t))^{-1} + \_C1 = 0$$

```
> dsolve({diff(x(t),t)=-2*t*x(t)/(3*x(t)^2-t^2+3),x(0)=1});
```

$$x(t) = 1/6\sqrt{36-12\,t^2}$$

```
>  plot(1/6*(36-12*t^2)^(1/2), t=-1..1);
```

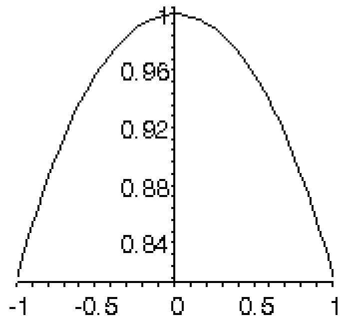


Figure 1.6:

Chapter 2

Second and higher order differential equations

Definition 2.0.1. *A differential equation of order $n \geq 2$ is a functional dependence of the form*

$$g(t, x, \dot{x}, ..., x^{(n)}) = 0 \tag{2.1}$$

between the identity function $t \mapsto t$ defined on an unknown interval $I \subset \mathbb{R}^1$, an unknown function $x(t)$ and its derivatives $\dot{x}, \ddot{x}, ..., x^{(n)}$ defined on the same interval.

In the equation (2.1), the function g is known and solving the equation means determining the function x which satisfies the given equation.

Definition 2.0.2. *A $\mathcal{C}^n$-function x defined on the open interval $I \subset \mathbb{R}^1$ is called solution of the equation (2.1) if for any $t \in I$, $(t, x(t), \dot{x}(t), ..., x^{(n)}(t))$ is in the domain of definition of g and*

$$g(t, x(t), \dot{x}(t), ..., x^{(n)}(t)) = 0. \tag{2.2}$$

2.1 Second order linear differential equations with constant coefficients

A second order linear differential equation with constant coefficients is a differential equation of the form:

$$a_2\ddot{x} + a_1\dot{x} + a_0 x = f(t) \tag{2.3}$$

in which a_0, a_1, a_2 are known real constants, $a_2 \neq 0$, $f(t)$ is a known continuous function and x is an unknown $\mathcal{C}^2$-function.

Observation 2.1.1. If $f = 0$ then the equation (2.3) is called second order homogeneous linear differential equation with constant coefficients, and if $f \neq 0$ the equation (2.3) is called second order nonhomogeneous linear differential equation with constant coefficients.

First, we will determine the solutions of the homogeneous equation and after that, we will find the solutions of the nonhomogeneous equation.

Let

$$a_2\ddot{x} + a_1\dot{x} + a_0x = 0 \tag{2.4}$$

be the homogeneous equation which corresponds to the equation (2.3).

If $a_2 = 0$, the equation (2.4) is a first order linear differential equation:

$$a_1\dot{x} + a_0x = 0$$

and its solutions are given by the formula

$$x(t) = Ce^{-\frac{a_0}{a_1}t}$$

in which C is an arbitrary real constant. We remark that the ratio $-\dfrac{a_0}{a_1}$ of the exponent, represents the solution of the algebraic equation $a_1 \cdot \lambda + a_0 = 0$, and we can find the solution $x(t) = Ce^{-\frac{a_0}{a_1}t}$ not only in the same way as in Chapter 1 § 6; we can obtain the same solution looking for solutions of the form $x(t) = Ce^{\lambda t}$. This is the idea that we use to determine the solutions of the equation (2.4).

Imposing that the function $x(t) = Ce^{\lambda t}$ satisfies the equation (2.4) we obtain that λ must verify the second order algebraic equation:

$$a_2\lambda^2 + a_1\lambda + a_0 = 0. \tag{2.5}$$

If the roots λ_1 and λ_2 of the equation (2.5) are different real numbers, then the functions

$$x_1(t) = C_1e^{\lambda_1 t} \quad \text{and} \quad x_2(t) = C_2e^{\lambda_2 t}$$

are solutions of the equation (2.4) and the function

$$x(t) = C_1e^{\lambda_1 t} + C_2e^{\lambda_2 t}$$

is a solution of (2.4), too. Moreover, for any t_0, x_0^0, $x_0^1 \in \mathbb{R}^1$ we can find the unique constants C_1 and C_2 such that the conditions

$$x(t_0) = x_0^0 \quad \text{and} \quad \dot{x}(t_0) = x_0^1 \tag{2.6}$$

are fulfilled.

Indeed, imposing the conditions (2.6) to the function $x(t) = C_1e^{\lambda_1 t} + C_2e^{\lambda_2 t}$, we obtain the following system of algebraic equations:

$$\begin{aligned} x_0^0 &= C_1e^{\lambda_1 t_0} + C_2e^{\lambda_2 t_0} \\ x_0^1 &= C_1e^{\lambda_1 t_0} + C_2e^{\lambda_2 t_0} \end{aligned}$$

in which the unknowns are C_1 and C_2.

The determinant of this system is $e^{(\lambda_1+\lambda_2)t_0} \cdot (\lambda_2 - \lambda_1)$ and it is non-zero $(\lambda_1 \neq \lambda_2)$ so, the system has a unique solution.

In particular, we have that the formula:

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \tag{2.7}$$

represents all the solutions of the equation (2.4) in the case when the equation (2.5) has two different real roots.

If the equation (2.5) has real and equal roots $\lambda_1 = \lambda_2 = \lambda$ then $x_1(t) = C_1 e^{\lambda t}$ and $x_2(t) = C_2 t \cdot e^{\lambda t}$ are the solutions of the equation (2.4). Consequently, any function $x(t)$ of the form

$$x(t) = C_1 e^{\lambda t} + C_2 t \cdot e^{\lambda t}$$

i.e.

$$x(t) = e^{\lambda t} \cdot (C_1 + C_2 t) \tag{2.8}$$

is solution of the equation (2.4).

Moreover, for any $t_0, x_0^0, x_0^1 \in \mathbb{R}^1$ we can determine the unique constants C_1 and C_2 such that the initial conditions $x(t_0) = x_0^0$ and $\dot{x}(t_0) = x_0^1$ are fulfilled.

Indeed, imposing these conditions to the function given by (2.8) we obtain the following system of algebraic equations:

$$\begin{array}{rcl} x_0^0 & = & e^{\lambda t_0}(C_1 + C_2 t_0) \\ x_0^1 & = & \lambda e^{\lambda t_0} \cdot C_1 + C_2 e^{\lambda t_0} + e^{\lambda t_0} \cdot t_0 \cdot \lambda \cdot C_2 \end{array}$$

for which the determinant is $e^{2\lambda t_0} \neq 0$.

In particular, we obtain that the formula (2.8) represents all the solutions of the equation (2.4) in the case when the equation (2.5) has real and equal roots.

We consider the last case, when the equation (2.5) has complex roots $\lambda_1 = \mu + i\nu$ and $\lambda_2 = \mu - i\nu$. For that, we will consider the functions

$$x_1(t) = C_1 e^{\mu t} \cdot \cos \nu t \quad \text{and} \quad x_2(t) = C_2 e^{\mu t} \cdot \sin \nu t$$

(C_1, C_2 real constants) and we will show that these functions are solutions of the equation (2.4).

We give the proof for the function $x_1(t)$ (which is the same as for $x_2(t)$):

$$\begin{array}{rcl} \dot{x}_1(t) & = & C_1 \mu \cdot e^{\mu t} \cdot \cos \nu t - C_1 \nu \cdot e^{\mu t} \cdot \sin \nu t \\ \ddot{x}_1(t) & = & C_1 \mu^2 \cdot e^{\mu t} \cdot \cos \nu t - 2C_1 \mu\nu \cdot e^{\mu t} \cdot \sin \nu t - C_1 \nu^2 \cdot e^{\mu t} \cdot \cos \nu t \end{array}$$

and replacing in the equation (2.5) we have:

$$\begin{array}{rcl} a_2 \ddot{x}_1 + a_1 \dot{x}_1 + a_0 x_1 & = & C_1 \cdot e^{\mu t} \cdot \cos \nu t \left[(\mu^2 - \nu^2) a_2 + \mu a_1 + a_0\right] + \\ & + & C_1 \cdot e^{\mu t} \cdot \sin \nu t \left[-2\mu\nu a_2 - \nu a_1\right]. \end{array}$$

Because

$$a_2(\mu + i\nu)^2 + a_1(\mu + i\nu) + a_0 = 0$$

we have

$$(\mu^2 - \nu^2) a_2 + \mu a_1 + a_0 + i\left[2\mu\nu a_2 + \nu a_1\right] = 0$$

and consequently:

$$(\mu^2 - \nu^2)a_2 + \mu a_1 + a_0 = 0 \quad \text{and} \quad 2\mu\nu a_2 + \nu a_1 = 0$$

taking into account these equalities we obtain

$$a_2\ddot{x}_1 + a_1\dot{x}_1 + a_0 x_1 = 0$$

which shows that the function $x_1(t) = C_1 \cdot e^{\mu t} \cdot \cos\nu t$ is a solution of the differential equation (2.4).

In the same way, $x_2(t) = C_2 \cdot e^{\mu t} \cdot \sin\nu t$ is a solution of the differential equation (2.4), as well.

We have that any function

$$x(t) = C_1 \cdot e^{\mu t} \cdot \cos\nu t + C_2 \cdot e^{\mu t} \cdot \sin\nu t \tag{2.9}$$

is a solution of the equation (2.4).

For any $t_0, x_0^0, x_0^1 \in \mathbb{R}^1$ we can determine the unique constants C_1 and C_2 such that the initial conditions $x(t_0) = x_0^0$ and $\dot{x}(t_0) = x_0^1$ are verified.

Imposing these conditions to the function (2.9) we obtain the following system of algebraic equations:

$$\begin{aligned} x_0^0 &= e^{\mu t_0} \cdot [C_1 \cdot \cos\nu t_0 + C_2 \cdot \sin\nu t_0] \\ x_0^1 &= e^{\mu t_0} \cdot [C_1 \cdot (\mu\cos\nu t_0 - \nu\sin\nu t_0) + C_2 \cdot (\mu\sin\nu t_0 + \nu\cos\nu t_0)] \end{aligned}$$

with the unknowns C_1, C_2.

The determinant of this system is $\nu \cdot e^{2\mu t_0}$ and it is different of zero.

In particular, the formula (2.9) represents all the solutions of the equation (2.4) in the case when equation (2.5) has complex roots.

Thus, we have determined all the solutions of the homogeneous linear differential equations with constant coefficients (2.4).

This fact allow us to find the general solution of the second order nonhomogeneous differential equation (2.3) i.e.:

$$a_2\ddot{x} + a_1\dot{x} + a_0 x = f(t)$$

in which f is given function.

To determine the solutions of the equation (2.3) it's important to observe that, if $\overline{x}(t)$ is a fixed solution of the equation (2.3) and $x(t)$ is an arbitrary solution of the same equation, then the difference

$$\widetilde{x}(t) = x(t) - \overline{x}(t)$$

represents a solution of the equation (2.4). Because the solutions $\widetilde{x}(t)$ of the equation (2.4) are known, to determine the solution $x(t)$ of (2.3) it is sufficient to determine a fixed (particular) solution $\overline{x}(t)$ of this equation.

A particular solution $\overline{x}(t)$ of the equation (2.3) will be determined by Lagrange's undetermined coefficients method (the same that we used in Chapter 1 § 6).

In the followings, we will present this method in the case when the algebraic equation (2.5) has real and different roots λ_1, λ_2. In this case, the solutions of the homogeneous linear differential equation with constant coefficients (2.4) will be written in the form (2.7):

$$\widetilde{x}(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}.$$

We search for a particular solution $\overline{x}(t)$ of the homogeneous equation (2.3) in the same form as (2.7) but in this case C_1, C_2 will be $\mathcal{C}^1$-functions of variable t:

$$\overline{x}(t) = C_1(t)e^{\lambda_1 t} + C_2(t)e^{\lambda_2 t} \tag{2.10}$$

To impose that the function $\overline{x}(t)$ satisfies the equation (2.3) we first compute its derivative:

$$\dot{\overline{x}}(t) = \dot{C}_1(t)e^{\lambda_1 t} + \dot{C}_2(t)e^{\lambda_2 t} + C_1(t)\lambda_1 e^{\lambda_1 t} + C_2(t)\lambda_2 e^{\lambda_2 t} \tag{2.11}$$

In the followings, we will compute the second order derivative $\ddot{\overline{x}}$, by differentiating with respect to t in the expression (2.11). This will introduce the second order derivatives of the functions $C_1(t), C_2(t)$. In order to avoid this, we impose the condition:

$$\dot{C}_1(t)e^{\lambda_1 t} + \dot{C}_2(t)e^{\lambda_2 t} = 0 \tag{2.12}$$

Using this condition, (2.11) becomes:

$$\dot{\overline{x}}(t) = C_1(t)\lambda_1 e^{\lambda_1 t} + C_2(t)\lambda_2 e^{\lambda_2 t} \tag{2.13}$$

and by derivation we have:

$$\ddot{\overline{x}}(t) = \dot{C}_1(t)\lambda_1 e^{\lambda_1 t} + \dot{C}_2(t)\lambda_2 e^{\lambda_2 t} + C_1(t)\lambda_1^2 e^{\lambda_1 t} + C_2(t)\lambda_2^2 e^{\lambda_2 t}. \tag{2.14}$$

Replacing (2.13) and (2.14) in (2.3) we have:

$$\begin{aligned} C_1(t)(a_2\lambda_1^2 + a_1\lambda_1 + a_0)e^{\lambda_1 t} &+ C_2(t)(a_2\lambda_2^2 + a_1\lambda_2 + a_0)e^{\lambda_2 t} + \\ &+ \dot{C}_1(t)a_2\lambda_1 e^{\lambda_1 t} + \dot{C}_2(t)a_2\lambda_2 e^{\lambda_2 t} = f(t) \end{aligned}$$

or

$$\dot{C}_1(t)\lambda_1 e^{\lambda_1 t} + \dot{C}_2(t)\lambda_2 e^{\lambda_2 t} = \frac{1}{a_2} f(t) \tag{2.15}$$

Thus, the algebraic system defined by the equations (2.12) and (2.15):

$$\begin{aligned} \dot{C}_1(t)e^{\lambda_1 t} \quad &+ \quad \dot{C}_2(t)e^{\lambda_2 t} \quad = \quad 0 \\ \dot{C}_1(t)\lambda_1 e^{\lambda_1 t} \quad &+ \quad \dot{C}_2(t)\lambda_2 e^{\lambda_2 t} \quad = \quad \frac{1}{a_2} f(t) \end{aligned} \tag{2.16}$$

of unknown functions $\dot{C}_1(t), \dot{C}_2(t)$ (the derivatives of the functions $C_1(t)$ and $C_2(t)$), has the determinant $(\lambda_2 - \lambda_1)e^{(\lambda_1+\lambda_2)t} \neq 0$ and, permits to determine the functions $\dot{C}_1(t)$ and $\dot{C}_2(t)$:

$$\begin{aligned} \dot{C}_1(t) &= -\frac{1}{a_2(\lambda_2 - \lambda_1)} \cdot e^{-(\lambda_1+\lambda_2)t} \cdot e^{\lambda_2 t} \cdot f(t) \\ \dot{C}_2(t) &= \frac{1}{a_2(\lambda_2 - \lambda_1)} \cdot e^{-(\lambda_1+\lambda_2)t} \cdot e^{\lambda_1 t} \cdot f(t) \end{aligned} \tag{2.17}$$

So, the functions $C_1(t)$ and $C_2(t)$ are given by:

$$\begin{aligned} C_1(t) &= -\frac{1}{a_2(\lambda_2-\lambda_1)}\int_{t^*}^{t} e^{-\lambda_1\tau}\cdot f(\tau)d\tau \\ C_2(t) &= \frac{1}{a_2(\lambda_2-\lambda_1)}\int_{t^*}^{t} e^{-\lambda_2\tau}\cdot f(\tau)d\tau \end{aligned} \tag{2.18}$$

and a particular solution of the nonhomogeneous equation (2.3) is:

$$\begin{aligned} \overline{x}(t) &= -\frac{1}{a_2(\lambda_2-\lambda_1)}\cdot e^{\lambda_1 t}\int_{t^*}^{t} e^{-\lambda_1\tau}\cdot f(\tau)d\tau + \\ &+ \frac{1}{a_2(\lambda_2-\lambda_1)}\cdot e^{\lambda_2 t}\int_{t^*}^{t} e^{-\lambda_2\tau}\cdot f(\tau)d\tau. \end{aligned} \tag{2.19}$$

It follows that any solution of the equation (2.3) is given by

$$x(t) = \widetilde{x}(t) + \overline{x}(t)$$

which is equivalent to:

$$\begin{aligned} x(t) &= C_1e^{\lambda_1 t} + C_2e^{\lambda_2 t} - \frac{1}{a_2(\lambda_2-\lambda_1)}\cdot e^{\lambda_1 t}\int_{t^*}^{t} e^{-\lambda_1\tau}\cdot f(\tau)d\tau + \\ &+ \frac{1}{a_2(\lambda_2-\lambda_1)}\cdot e^{\lambda_2 t}\int_{t^*}^{t} e^{-\lambda_2\tau}\cdot f(\tau)d\tau \end{aligned} \tag{2.20}$$

By similar arguments, in the case when the algebraic equation (2.5) has real and equal roots $\lambda_1 = \lambda_2 = \lambda$, we find the following particular solution for the equation (2.3):

$$\overline{x}(t) = e^{\lambda t}\left[-\frac{1}{a_2}\int_{t^*}^{t} e^{-\lambda\tau}\cdot\tau\cdot f(\tau)d\tau + \frac{t}{a_2}\int_{t^*}^{t} e^{-\lambda\tau}\cdot f(\tau)d\tau\right]$$

and the following general solution

$$\begin{aligned} x(t) &= e^{\lambda_1 t}(C_1 + C_2 t) + \\ &+ e^{\lambda t}\left[-\frac{1}{a_2}\int_{t^*}^{t} e^{-\lambda\tau}\cdot\tau\cdot f(\tau)d\tau + \frac{t}{a_2}\int_{t^*}^{t} e^{-\lambda\tau}\cdot f(\tau)d\tau\right] \end{aligned} \tag{2.21}$$

In the case when the algebraic equation (2.5) has complex roots $\lambda_1 = \mu + i\nu$ and $\lambda_1 = \mu - i\nu$, using Lagrange's undetermined coefficients method we find the particular solution:

$$\begin{aligned} \overline{x}(t) &= -\frac{1}{a_2\nu}\cdot e^{\mu t}\cdot\cos\nu t\int_{t^*}^{t} e^{-\mu\tau}\cdot\sin\nu\tau\cdot f(\tau)d\tau + \\ &+ \frac{1}{a_2\nu}\cdot e^{\mu t}\cdot\sin\nu t\int_{t^*}^{t} e^{-\mu\tau}\cdot\cos\nu\tau\cdot f(\tau)d\tau \end{aligned}$$

and the general solution

$$\begin{aligned} x(t) &= C_1 e^{\mu t} \cdot \cos \nu t + C_2 e^{\mu t} \cdot \sin \nu t - \\ &- \frac{1}{a_2 \nu} \cdot e^{\mu t} \cdot \cos \nu t \int_{t^*}^{t} e^{-\mu \tau} \cdot \sin \nu \tau \cdot f(\tau) d\tau + \\ &+ \frac{1}{a_2 \nu} \cdot e^{\mu t} \cdot \sin \nu t \int_{t^*}^{t} e^{-\mu t} \cdot \cos \nu \tau \cdot f(\tau) d\tau \end{aligned} \tag{2.22}$$

Generally, for any $t_0, x_0^0, x_0^1 \in \mathbb{R}^1$ we can determine the constants C_1 and C_2 which appear in the formula of $x(t)$ for the nonhomogeneous equations ((2.20), (2.21), (2.22)) such that the initial conditions $x(t_0) = x_0^0$ and $\dot{x}(t_0) = x_0^1$ are fulfilled.

Exercises

1. Solve the following problems with the initial data:

$a)$ $\ddot{x} - x = 0 \qquad x(0) = 2, \quad \dot{x}(0) = 0$

A: $x(t) = e^t + e^{-t}$

$b)$ $\ddot{x} + 2\dot{x} + x = 0 \quad x(0) = 0, \quad \dot{x}(0) = 1$

A: $x(t) = t \cdot e^{-t}$

$c)$ $\ddot{x} - 4\dot{x} + 4x = 0 \quad x(1) = 1, \quad \dot{x}(1) = 0$

A: $x(t) = 3e^{2t-2} - 2t \cdot e^{2t-2}$

$d)$ $\ddot{x} + x = 0 \qquad x\left(\frac{\pi}{2}\right) = 1, \quad \dot{x}\left(\frac{\pi}{2}\right) = 0$

A: $x(t) = \sin t$

$e)$ $\ddot{x} + \dot{x} + x = 0 \qquad x(0) = 0, \quad \dot{x}(0) = 1$

A: $x(t) = \frac{2}{3}\sqrt{3} \cdot e^{-\frac{1}{2}t} \cdot \sin\left(\frac{2}{3}\sqrt{3}t\right)$

2. Solve the following differential equations:

$a)$ $\ddot{x} + 3\dot{x} + 2x = \frac{1}{1+e^t}$

A: $x(t) = e^{-t} \cdot \ln(1+e^t) + e^{-2t} \cdot \ln(1+e^t) - e^{-2t} \cdot C_1 + e^{-t} \cdot C_2$

$b)$ $\ddot{x} - 6\dot{x} + 9x = \frac{9t^2 + 6t + 2}{t^3}$

A: $x(t) = e^{3t} \cdot C_1 + t \cdot e^{3t} \cdot C_2 + \dfrac{1}{t}$

$c)$ $\ddot{x} + x = \dfrac{e^t}{2} + \dfrac{e^{-t}}{2}$

A: $x(t) = C_1 \cdot \sin t + C_2 \cdot \cos t + \dfrac{1}{4}(e^{2t} + 1) \cdot e{-}t$

$d)$ $\ddot{x} - 3\dot{x} + 2x = 2e^{2t}$

A: $x(t) = (2te^t - 2e^t + C_1 e^t + C_2)e^t$

$e)$ $\ddot{x} - 4\dot{x} + 4x = 1 + e^t + e^{2t}$

A: $x(t) = C_1 \cdot e^{2t} + C_2 t \cdot e^{2t} + \dfrac{1}{4} + \dfrac{1}{2}t^2 e^{2t} + e^t$

$f)$ $\ddot{x} + x = \sin t + \cos 2t$

A: $x(t) = C_1 \sin t + C_2 \cos t - \dfrac{2}{3}\cos t^2 + \dfrac{1}{3} - \dfrac{1}{2} t \cos t$

$g)$ $\ddot{x} - 2(1+m)\dot{x} + (m^2 + 2m)x = e^t + e^{-t}, \qquad m \in \mathbb{R}^1$

A: $x(t) = C_1 \cdot e^{mt} + C_2 \cdot e^{(m+2)t} + \dfrac{((m+3)e^{2t} + m - 1) \cdot e^{-t}}{m^3 + 3m^2 - m - 3}$

$h)$ $\ddot{x} - 5\dot{x} + 6x = 6t^2 - 10t + 2$

A: $x(t) = C_1 \cdot e^{3t} + C_2 \cdot e^{2t} + t^2$

$i)$ $\ddot{x} - 5\dot{x} = -5t^2 + 2t$

A: $x(t) = \dfrac{1}{3}t^3 + \dfrac{1}{5}e^{5t} \cdot C_1 + C_2$

$j)$ $\ddot{x} + x = te^{-t}$

A: $x(t) = C_1 \sin t + C_2 \cos t + \dfrac{1}{2}(-1 + t) \cdot e^t$

$k)$ $\ddot{x} - x = te^t + t + t^3 e^{-t}$

$$\textbf{A:}\ x(t) = C_1e^{-t} + C_2e^{t} + \frac{1}{16}(-4te^{2t} + 2e^{2t} - 16te^{t} - 2t^4 + \\ +4t^2e^{2t} - 4t^3 - 6t^2 - 6t - 3) \cdot e^{-t}$$

l) $\ddot{x} - 7\dot{x} + 6x = \sin t$

$$\textbf{A:}\ x(t) = C_1 \cdot e^{t} + C_2 \cdot e^{6t} + \frac{7}{74}\cos t + \frac{5}{74}\sin t$$

m) $\ddot{x} - 4\dot{x} + 4x = \sin t \cdot \cos 2t$

$$\textbf{A:}\ x(t) = C_1e^{2t} + C_2te^{2t} - \frac{10}{169}\sin t \cdot \cos t^2 - \frac{191}{4225}\sin t + \\ +\frac{24}{169}\cos t^3 - \frac{788}{4225}\cos t$$

n) $\ddot{x} + x = \cos t - \cos 3t$

$$\textbf{A:}\ x(t) = C_1 \sin t + C_2 \cos t + \frac{1}{2} \cdot t \sin t + \frac{1}{2}\cos t^3 - \frac{1}{8}\cos t$$

2.2 Linear differential equations of order n with constant coefficients

A linear differential equation of order n with constant coefficients is a differential equation of the form

$$a_n x^{(n)} + a_{n-1}x^{(n-1)} + \ldots + a_1\dot{x} + a_0x = f(t) \tag{2.23}$$

in which $a_0, a_1, \ldots, a_{n-1}, a_n$ are known real constants, $a_n \neq 0$, $f(t)$ is a known continuous function and x is a real valued unknown $\mathcal{C}^n$-function.

Observation 2.2.1. If $f = 0$, then the equation (2.23) is called homogeneous linear differential equation of order n with constant coefficients, and if $f \neq 0$ the equation (2.23) is called nonhomogeneous linear differential equation of order n with constant coefficients.

First, we will solve the homogeneous equation corresponding to the equation (2.23), i.e.

$$a_n x^{(n)} + a_{n-1}x^{(n-1)} + \ldots + a_1\dot{x} + a_0x = 0 \tag{2.24}$$

To determine the solutions of the equation (2.24) we search for solutions of the form $x(t) = C \cdot e^{\lambda t}$. Imposing that these functions satisfy the equation (2.24) we obtain that λ must satisfy the algebraic equation

$$a_n\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0 = 0 \tag{2.25}$$

called characteristic equation.

If all the roots of the equation (2.25) are different real numbers $\lambda_1, \lambda_2, \ldots \lambda_n$ then the functions $x_i(t) = C_i \cdot e^{\lambda_i t}, i = \overline{1, n}$ are solutions of the equation (2.24) and any function $x(t)$ given by:

$$x(t) = C_1 \cdot e^{\lambda_1 t} + C_2 \cdot e^{\lambda_2 t} + \ldots + C_n \cdot e^{\lambda_n t} \tag{2.26}$$

is a solution of the equation (2.24) ($C_1, C_2, \ldots, C_n$ are arbitrary real constants).

Moreover, for any $t_0, x_0^0, x_0^1, ..., x_0^{n-1} \in \mathbb{R}^1$ we can determine the unique constants $C_1, C_2, \ldots, C_n$ such that

$$x(t_0) = x_0^0, \ \ \dot{x}(t_0) = x_0^1, \ \ldots, x^{(n-1)}(t_0) = x_0^{n-1}$$

In particular, we obtain that the formula (2.26) represents all the solutions of the equation (2.24) in this case.

If the characteristic equation (2.25) has a simple pair of conjugate complex roots $\lambda = \mu + i\nu$ and $\overline{\lambda} = \mu - i\nu$, then the following solutions correspond to this pair of roots:

$$x_\lambda^1(t) = C_\lambda^1 \cdot e^{\mu t} \cdot \cos \nu t \quad \text{and} \quad x_\lambda^2(t) = C_\lambda^2 \cdot e^{\mu t} \cdot \sin \nu t$$

For $\mu = 0$ these solutions become:

$$x_\lambda^1(t) = C_\lambda^1 \cdot \cos \nu t \quad \text{and} \quad x_\lambda^2(t) = C_\lambda^2 \cdot \sin \nu t$$

Thus, if the characteristic equation has *2k* simple complex roots $\lambda_j = \mu_j + i\nu_j$ and $\overline{\lambda_j} = \mu_j - i\nu_j$, $j = \overline{1, k}$, and $n - 2k$ simple real roots $\lambda_{2k+1}, \ldots, \lambda_n$, then any function $x(t)$ given by:

$$x(t) = \sum_{j=1}^{k} C_j^1 \cdot e^{\mu_j t} \cdot \cos \nu_j t + \sum_{j=1}^{k} C_j^2 \cdot e^{\mu_j t} \cdot \sin \nu_j t + \sum_{j=2k+1}^{n} C_j \cdot e^{\lambda_j t} \tag{2.27}$$

is a solution of the equation (2.24) ($C_j^1, C_j^2, j = \overline{1, k}$ and $C_j, j = \overline{2k+1, n}$ are arbitrary real constants).

Moreover, for any $t_0, x_0^0, x_0^1, ..., x_0^{n-1} \in \mathbb{R}^1$ we can determine the unique constants $C_j^1, C_j^2, \ j = \overline{1, k}$ and $C_j, j = \overline{2k+1, n}$ such that the initial conditions $x(t_0) = x_0^0, \ \dot{x}(t_0) = x_0^1, ..., x^{(n-1)}(t_0) = x_0^{n-1}$ are satisfied. In particular, the formula (2.27) represents all the solutions of the equation (2.24) in this case.

If the characteristic equation (2.25) has k real roots $\lambda_1, \ldots, \lambda_k$ having the multiplicity orders $q_1, \ldots, q_k$ and l complex conjugate roots $\mu_1 \pm i\nu_1, \ldots, \mu_l \pm i\nu_l$ having the multiplicity orders $r_1, \ldots, r_l$, then any function $x(t)$ given by the formula:

$$x(t) = \sum_{j=1}^{k} e^{\lambda_j t} \cdot P_{q_j - 1}(t) + \sum_{j=1}^{l} e^{\mu_j t} \cdot \left[Q_{r_j - 1}(t) \cdot \cos \nu_j t + R_{r_j - 1}(t) \cdot \sin \nu_j t \right] \tag{2.28}$$

is a solution of the equation (2.24), where $P_{q_j - 1}(t)$ are polynomial functions of degrees $q_j - 1$ with real unknown coefficients and $Q_{r_j - 1}$, $R_{r_j - 1}$ are polynomial functions of degrees $r_j - 1$ with real unknown coefficients.

Moreover, for any $t_0, x_0^1, x_0^2, ..., x_0^{n-1} \in \mathbb{R}^1$ we can determine the unique coefficients of the polynomial functions $P_{q_j-1}, Q_{q_j-1}, R_{q_j-1}$ such that the initial conditions $x(t_0) = x_0^1, \dot{x}(t_0) = x_0^2, ..., x^{(n-1)}(t_0) = x_0^{n-1}$ are satisfied.

In particular, the formula (2.28) represents all the solutions of the equation (2.24) in this case.

We would like to remember that, the objective of this section is to solve the nonhomogeneous linear differential equation of order n, with constant coefficients (2.23):

$$a_n x^{(n)} + a_{n-1}x^{(n-1)} + \ldots + a_1\dot{x} + a_0 x = f(t)$$

in which $a_0, a_1, \ldots, a_{n-1}, a_n$ are given real constants, $a_n \neq 0$, f is an known continuous function and x is a real unknown $\mathcal{C}^n$-function.

To determine the solution of the equation (2.23) it is important to observe that, if $\overline{x}(t)$ is a fixed solution of the equation (2.23) and $x(t)$ is an arbitrary solution of the same equation, then the difference $\overline{x}(t) - x(t) = \widetilde{x}(t)$ is an arbitrary solution of the homogeneous linear differential equation with constant coefficients (2.24). Because the solutions $\widetilde{x}(t)$ of the homogeneous equation (2.24) are given by (2.28), in order to determine the solution $x(t)$ of the equation (2.23), we have to determine a particular solution $\overline{x}(t)$ of this equation.

A particular solution $\overline{x}(t)$ of the equation (2.23) will be found using Lagrange's undetermined coefficients method.

We illustrate this by an example ($n = 3$):

Example 2.2.1. To determine the solutions of the following equation:

$$\dddot{x} + 4\ddot{x} + 5\dot{x} = 4e^t$$

We consider the corresponding homogeneous equation

$$\dddot{x} + 4\ddot{x} + 5\dot{x} = 0$$

The characteristic equation is:

$$\lambda^3 + 4\lambda^2 + 5\lambda = 0$$

with the roots:

$$\lambda_0 = 0, \lambda_1 = -2 - i, \lambda_2 = -2 + i.$$

The solutions of the homogeneous equation are given by:

$$y(t) = C_1 + C_2 \cdot e^{-2t} \cdot \cos t + C_3 \cdot e^{-2t} \cdot \sin t.$$

We search for $\overline{x}(t)$ a particular solution of the nonhomogeneous equation, of the following form

$$\overline{x}(t) = C_1(t) + C_2(t) \cdot e^{-2t} \cdot \cos t + C_3(t) \cdot e^{-2t} \cdot \sin t$$

in which $C_1(t), C_2(t), C_3(t)$ are $\mathcal{C}^1$-functions which must be determined.

We compute the first derivative of the function $\overline{x}(t)$ and we obtain:

$$\begin{aligned}\dot{\overline{x}} &= \dot{C}_1 + \dot{C}_2 \cdot e^{-2t} \cdot \cos t + \dot{C}_3 \cdot e^{-2t} \cdot \sin t - 2C_2 \cdot e^{-2t} \cdot \cos t - \\ &- 2C_3 \cdot e^{-2t} \cdot \sin t - C_2 \cdot e^{-2t} \cdot \sin t + C_3 \cdot e^{-2t} \cdot \cos t\end{aligned}$$

Imposing that $\dot{C}_1, \dot{C}_2, \dot{C}_3$ verify:

$$\dot{C}_1 + \dot{C}_2 \cdot e^{-2t} \cdot \cos t + \dot{C}_3 \cdot e^{-2t} \cdot \sin t = 0$$

we obtain:

$$\dot{\overline{x}} = -C_2 \cdot e^{-2t} \cdot (2\cos t + \sin t) + C_3 \cdot e^{-2t} \cdot (\cos t - 2\sin t)$$

We compute the second order derivative of the function $\overline{x}(t)$ and we obtain:

$$\begin{aligned}
\ddot{\overline{x}} &= -\dot{C}_2 \cdot e^{-2t} \cdot (2\cos t + \sin t) + \dot{C}_3 \cdot e^{-2t} \cdot (\cos t - 2\sin t) + \\
&\quad +2C_2 \cdot e^{-2t} \cdot (2\cos t + \sin t) - 2C_3 \cdot e^{-2t} \cdot (\cos t - 2\sin t) - \\
&\quad -C_2 \cdot e^{-2t} \cdot (-2\sin t + \cos t) + C_3 \cdot e^{-2t} \cdot (-\sin t - 2\cos t) = \\
&= -\dot{C}_2 \cdot e^{-2t} \cdot (2\cos t + \sin t) + \dot{C}_3 \cdot e^{-2t} \cdot (\cos t - 2\sin t) + \\
&\quad +C_2 \cdot e^{-2t} \cdot (3\cos t + 4\sin t) + C_3 \cdot e^{-2t} \cdot (3\sin t - 4\cos t).
\end{aligned}$$

Imposing that $\dot{C}_2, \dot{C}_3$ verify:

$$-\dot{C}_2 \cdot e^{-2t} \cdot (2\cos t + \sin t) + \dot{C}_3 \cdot e^{-2t} \cdot (\cos t - 2\sin t) = 0$$

we have

$$\ddot{\overline{x}} = C_2 \cdot e^{-2t} \cdot (3\cos t + 4\sin t) + C_3 \cdot e^{-2t} \cdot (3\sin t - 4\cos t)$$

For the third order derivative of $\overline{x}(t)$ we obtain:

$$\begin{aligned}
\dddot{\overline{x}} &= \dot{C}_2 \cdot e^{-2t} \cdot (3\cos t + 4\sin t) + \dot{C}_3 \cdot e^{-2t} \cdot (3\sin t - 4\cos t) - \\
&\quad -2C_2 \cdot e^{-2t} \cdot (3\cos t + 4\sin t) - 2C_3 \cdot e^{-2t} \cdot (3\sin t - 4\cos t) + \\
&\quad +C_2 \cdot e^{-2t} \cdot (-3\sin t + 4\cos t) + C_3 \cdot e^{-2t} \cdot (3\cos t + 4\sin t) = \\
&= \dot{C}_2 \cdot e^{-2t} \cdot (3\cos t + 4\sin t) + \dot{C}_3 \cdot e^{-2t} \cdot (3\sin t - 4\cos t) + \\
&\quad +C_2 \cdot e^{-2t} \cdot (-2\cos t - 11\sin t) + C_3 \cdot e^{-2t} \cdot (-2\sin t + 11\cos t).
\end{aligned}$$

Replacing all these derivative in the given equation we have:

$$\begin{array}{llll}
\dot{C}_2 \cdot e^{-2t} \cdot (3\cos t + 4\sin t) & +\dot{C}_3 \cdot e^{-2t} \cdot (3\sin t - 4\cos t) & + & \\
+C_2 \cdot e^{-2t} \cdot (-2\cos t - 11\sin t) & +C_3 \cdot e^{-2t} \cdot (-2\sin t + 11\cos t) & + & \\
+C_2 \cdot e^{-2t} \cdot (12\cos t + 16\sin t) & +C_3 \cdot e^{-2t} \cdot (12\sin t - 16\cos t) & - & \\
-C_2 \cdot e^{-2t} \cdot (10\cos t + 5\sin t) & +C_3 \cdot e^{-2t} \cdot (5\cos t - 10\sin t) & = & 4e^t
\end{array}$$

or

$$\dot{C}_2 \cdot e^{-2t} \cdot (3\cos t + 4\sin t) + \dot{C}_3 \cdot e^{-2t} \cdot (3\sin t - 4\cos t) = 4e^t$$

This equality, together with the imposed conditions on the functions $\dot{C}_1, \dot{C}_2, \dot{C}_3$ conduct us to the following linear system of algebraic equations in the unknowns $\dot{C}_1, \dot{C}_2, \dot{C}_3$:

$$\left\{\begin{array}{llll}
\dot{C}_1 & +\dot{C}_2 \cdot e^{-2t} \cdot \cos t & +\dot{C}_3 \cdot e^{-2t} \cdot \sin t & = 0 \\
 & -\dot{C}_2 \cdot e^{-2t} \cdot (2\cos t + \sin t) & +\dot{C}_3 \cdot e^{-2t} \cdot (\cos t - 2\sin t) & = 0 \\
 & \dot{C}_2 \cdot e^{-2t} \cdot (3\cos t + 4\sin t) & +\dot{C}_3 \cdot e^{-2t} \cdot (-4\cos t + 3\sin t) & = 4e^t
\end{array}\right.$$

The last two equations lead us to:

$$\begin{cases} \dot{C}_2 \cdot (-2\cos t - \sin t) & +\dot{C}_3 \cdot (\cos t - 2\sin t) & = 0 \\ \dot{C}_2 \cdot (3\cos t + 4\sin t) & +\dot{C}_3 \cdot (-4\cos t + 3\sin t) & = 4e^{-t} \end{cases}$$

The determinant of the system is:

$$\begin{aligned} \Delta &= (-2\cos t - \sin t)(-4\cos t + 3\sin t) \\ &- (\cos t - 2\sin t)(3\cos t + 4\sin t) = \\ &= 8\cos^2 t - 6\sin t\cos t + 4\sin t\cos t - 3\sin^2 t - 3\cos^2 t - \\ &- 4\sin t\cos t + 6\sin t\cos t + 8\sin^2 t = 8 - 3 = 5 \end{aligned}$$

and the solutions are given by:

$$\dot{C}_2 = -\frac{4}{5} \cdot e^t(\cos t - 2\sin t) \qquad \dot{C}_3 = \frac{4}{5} \cdot e^t(-2\cos t - \sin t).$$

Replacing $\dot{C}_2, \dot{C}_3$ in the first equation, we obtain $\dot{C}_1$:

$$\begin{aligned} \dot{C}_1 &= \frac{4}{5} \cdot e^{-t}\cos t(\cos t - 2\sin t) + \frac{4}{5} \cdot e^{-t}\sin t(2\cos t + \sin t) = \\ &= \frac{4}{5} \cdot e^{-t}[\cos^2 t + \sin^2 t] = \\ &= \frac{4}{5} \cdot e^{-t} \end{aligned}$$

Thus, the derivatives of $\dot{C}_1, \dot{C}_2, \dot{C}_3$ are found:

$$\begin{aligned} \dot{C}_1 &= \frac{4}{5} \cdot e^{-t} \\ \dot{C}_2 &= -\frac{4}{5} \cdot e^t[\cos t - 2\sin t] \\ \dot{C}_3 &= -\frac{4}{5} \cdot e^t[2\cos t + \sin t] \end{aligned}$$

from which we have:

$$\begin{aligned} C_1 &= -\frac{4}{5} \cdot e^{-t} \\ C_2 &= \frac{1}{10} \cdot e^t[-12\cos t + 4\sin t] \\ C_3 &= \frac{1}{10} \cdot e^t[4\cos t + 12\sin t] \end{aligned}$$

Consequently, we obtain:

$$\begin{aligned} \overline{x}(t) &= -\frac{4}{5} \cdot e^{-t} + \frac{1}{10} \cdot e^{-t}[-12\cos t + 4\sin t] \cdot \cos t + \\ &+ \frac{1}{10} \cdot e^{-t}[4\cos t + 12\sin t] \cdot \sin t \end{aligned}$$

from where we find that, the general solution of the homogeneous equation

$$x(t) = \widetilde{x}(t) + \overline{x}(t)$$

is:

$$\begin{aligned} x(t) &= C_1 + c_2 e^{-2t}\cos t + C_3 e^{-2t}\sin t - \frac{4}{5}\cdot e^{-t} + \\ &+ \frac{1}{10}\cdot e^{-t}[-12\cos t + 4\sin t]\cdot \cos t + \\ &+ \frac{1}{10}\cdot e^{-t}[4\cos t + 12\sin t]\cdot \sin t \end{aligned}$$

Exercises

1. Solve the following differential equations (on the computer):

a) $\dddot{x} - 2\ddot{x} - \dot{x} + 2x = 0$

A: $x(t) = C_1 \cdot e^{t} + C_2 \cdot e^{2t} + C_3 \cdot e^{-t}$

b) $x^{(4)} - 5\ddot{x} + 4x = 0$

A: $x(t) = C_1 \cdot e^{t} + C_2 \cdot e^{2t} + C_3 \cdot e^{-t} + C_4 \cdot e^{-2t}$

c) $\dddot{x} - 6\ddot{x} + 12\dot{x} - 8x = 0$

A: $x(t) = C_1 \cdot e^{2t} + C_2 \cdot t \cdot e^{2t} + C_3 \cdot t^2 \cdot e^{2t}$

d) $x^{(7)} + 3x^{(6)} + 3x^{(5)} + x^{(4)} = 0$

A: $x(t) = C_1 \cdot e^{-t} + C_2 \cdot t \cdot e^{-t} + C_3 \cdot t^2 \cdot e^{-t} + C_4 + C_5 \cdot t + C_6 \cdot t^2 + C_7 \cdot t^3$

e) $\dddot{x} - \ddot{x} + \dot{x} - x = 0$

A: $x(t) = C_1 \cdot e^{t} + C_2 \sin t + C_3 \cos t$

f) $x^{(4)} + 2\ddot{x} + x = 0$

A: $x(t) = C_1 \sin t + C_2 \cos t + C_3 t \cdot \sin t + C_4 t \cdot \cos t$

g) $x^{(4)} - 3\dddot{x} + 5\ddot{x} - 3\dot{x} + 4x = 0$

A: $x(t) = C_1 \sin t + C_2 \cos t + C_3 e^{\frac{3}{2}t} \cdot \sin\left(\frac{\sqrt{7}}{2}t\right) + C_4 e^{\frac{3}{2}t} \cdot \cos\left(\frac{\sqrt{7}}{2}t\right)$

2. Determine the solutions of the following initial value problems:

$$a)\ \dddot{x} - 2\ddot{x} - \dot{x} + 2x = 0 \quad x(0) = 0, \quad \dot{x}(0) = 1 \quad \ddot{x}(0) = 2$$

$$\textbf{A:}\ x(t) = -\frac{1}{2}e^{t} + \frac{2}{3}e^{2t} - \frac{1}{6}e^{-t}$$

$$b)\ \dddot{x} - \ddot{x} + \dot{x} - x = 0 \quad x(1) = 0, \quad \dot{x}(1) = 1 \quad \ddot{x}(1) = 2$$

$$\textbf{A:}\ x(t) = e^{t-1} - (\sin 1) \cdot \sin t - (\cos 1) \cdot \cos t$$

$$c)\ x^{(4)} - 5\ddot{x} + 4x = 0 \quad x(0) = 0, \quad \dot{x}(0) = 1 \quad \ddot{x}(0) = 2, \dddot{x}(0) = 3$$

$$\textbf{A:}\ x(t) = -\frac{1}{6} \cdot e^{t} + \frac{1}{6} \cdot e^{-2t} - \frac{1}{2} \cdot e^{-t} + \frac{1}{2} \cdot e^{2t}$$

3. Solve the following differential equations:

$a)$ $\dddot{x} - 2\ddot{x} - \dot{x} + 2x = t + 1$

$$\textbf{A:}\ x(t) = \frac{3}{4} + \frac{1}{2} \cdot t + C_1 e^{t} + C_2 e^{2t} + C_3 e^{-t}$$

$b)$ $\dddot{x} - 6\ddot{x} + 12\dot{x} - 8x = \sin t$

$$\textbf{A:}\ x(t) = -\frac{11}{125}\cos t - \frac{2}{125}\sin t + C_1 e^{2t} + C_2 t^2 \cdot e^{2t} + C_3 t^3 \cdot e^{2t}$$

2.3 Euler linear differential equations of order n

Definition 2.3.1. *The linear differential equation of order n of the form:*

$$a_n \cdot t^n \cdot x^{(n)} + a_{n-1} \cdot t^{n-1} \cdot x^{(n-1)} + \ldots + a_1 \cdot t \cdot \dot{x} + a_0 \cdot x = 0 \tag{2.29}$$

in which $a_0, a_1, \ldots, a_n$ are real constants, is called Euler linear differential equation of order n.

Proposition 2.3.1. *Using the change of the variable $|t| = e^{\tau}$, the differential equation (2.29) is reduced to the linear differential equation of order n with constant coefficients.*

Proof. Let $x = x(t)$ be a solution of the equation (2.29) and y the function $y(\tau) = x(e^{\tau})$.

For $t > 0$, we have $x(t) = y(\ln t)$, and by successive derivation we have:

$$\begin{aligned}
\frac{dx}{dt} &= \frac{dy}{d\tau} \cdot \frac{1}{t} \\
\frac{d^2x}{dt^2} &= \frac{d^2y}{d\tau^2} \cdot \frac{1}{t^2} - \frac{dy}{d\tau} \cdot \frac{1}{t^2} = \frac{1}{t^2} \cdot \left(\frac{d^2y}{d\tau^2} - \frac{dy}{d\tau} \right) \\
\frac{d^3x}{dt^3} &= -\frac{2}{t^3} \cdot \left(\frac{d^2y}{d\tau^2} - \frac{dy}{d\tau} \right) + \frac{1}{t^3} \cdot \left(\frac{d^3y}{d\tau^3} - \frac{d^2y}{d\tau^2} \right) = \\
&= \frac{1}{t^3} \cdot \left(\frac{d^3y}{d\tau^3} - 3\frac{d^2y}{d\tau^2} + 2\frac{dy}{d\tau} \right)
\end{aligned}$$

If we suppose that, for $1 \leq k < n$ we have

$$\frac{d^kx}{dt^k} = \frac{1}{t^k} \cdot \sum_{i=1}^{k} c_i^k \cdot \frac{d^iy}{d\tau^i}$$

then by a new derivation we have the equality

$$\frac{d^{k+1}x}{dt^{k+1}} = \frac{1}{t^{k+1}} \cdot \sum_{i=1}^{k+1} c_i^{k+1} \cdot \frac{d^iy}{d\tau^i}$$

In this way, we have that the derivative of any order $(1 \leq k \leq n)$ of the function x is expressed as a product between $\dfrac{1}{t^{k+1}}$ and a linear combination of the derivatives of order $i \leq k+1$ of the function y.

Replacing in (2.29), we obtain that the function y verifies a linear differential equation of order n with constant coefficients.

We determine the solutions $y = y(\tau)$ of this equation and then the solutions $x(t)$ of equation (2.29) for $t > 0$:

$$x(t) = y(\ln t).$$

For $t < 0$ in the same way, we obtain:

$$x(t) = y(\ln |t|).$$

$\square$

Exercises:

Solve the following differential equations:

1. $t^2\ddot{x} + t\dot{x} - x = 0$

$$\textbf{A:}\ \ x(t) = C_1 \cdot t + C_2 \cdot \frac{1}{t}$$

2. $12t^3\dddot{x} - 25t^2\ddot{x} + 28t\dot{x} - 6x = 0$

$$\textbf{A:}\ \ x(t) = C_1 \cdot t^2 + C_2 \cdot t^{\frac{1}{12}} + C_3 \cdot t^3$$

3. $t^2\ddot{x} + t\dot{x} = 0$

$$\textbf{A:}\ \ x(t) = C_1 + C_2 \cdot \ln t$$

4. $t^2\ddot{x} - t\dot{x} + x = 0$

$$\textbf{A:}\ \ x(t) = C_1 \cdot t + C_2 \cdot t \cdot \ln t$$

2.4 Symbolic calculus of the solutions of the differential equations of order n

To solve the differential equations of higher order ($n \geq 2$) *Maple* uses the same function *dsolve* (solve ordinary differential equations - ODEs) which has been presented in the previous chapter.

The novelty which appears in this case consists of writing the syntax corresponding to the higher order. For example, the derivative of the second order of the function $x(t)$ can be written in one of the following ways:

```
diff(x(t),t,t)

diff(x(t),t$2)

(D@@2)(x)(t)
```

For exemplification, we will solve some exercises and initial value problems:

1. The second order homogeneous linear differential equation:

$$\ddot{x} - 4\dot{x} + 4x = 0; \tag{2.30}$$

```
> eq1:=diff(x(t),t,t)-4*diff(x(t),t)+4*x(t)=0;
```

$$eq1 \;:=\; \frac{d^2}{dt^2}x\,(t) - 4\,\frac{d}{dt}x\,(t) + 4\,x\,(t) = 0$$

```
> dsolve(eq1,x(t));
```

$$x\,(t) = \_C1\; e^{2\,t} + \_C2\; e^{2\,t}t$$

```
> dsolve({eq1,x(1)=1,D(x)(1)=0},x(t));
```

$$x\,(t) = 3\,\frac{e^{2\,t}}{e^2} - 2\,\frac{e^{2\,t}t}{e^2}$$

```
> sol1:=3*exp(2*t)/exp(2)-2*exp(2*t)*t/exp(2):
> plot(sol1,t=-infinity..infinity);
```

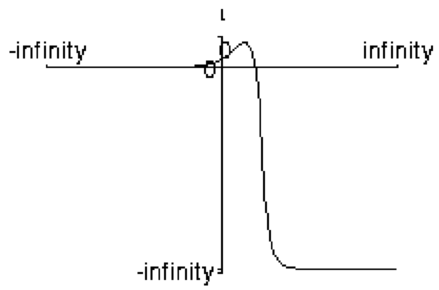


Figure 2.1:

We observe that, if we don't have initial data the general solution is given with respect to two constants. More precisely, the number of constants is the same as the order of the equation. In this case, the general solutions are being expressed with respect to two constants.

In order that *Maple* displays the solution of an initial value problem, we must give n initial conditions:

$$x(t_0) = x_0^0,\ \ \dot{x}(t_0) = x_0^1,\ \dots, x^{(n-1)}(t_0) = x_0^{n-1}.$$

2. The fourth order homogeneous linear differential equation:

$$x^{(4)} - 5\ddot{x} + 4x = 0; \tag{2.31}$$

```
> eq2:=diff(x(t),t,t,t,t)-5*diff(x(t),t,t)+4*x(t)=0;
```

$$eq2 := \frac{d^4}{dt^4}x(t) - 5\frac{d^2}{dt^2}x(t) + 4\,x(t) = 0$$

```
> eq2:=diff(x(t),t$4)-5*diff(x(t),t$2)+4*x(t)=0;
```

$$eq2 := \frac{d^4}{dt^4}x(t) - 5\frac{d^2}{dt^2}x(t) + 4\,x(t) = 0$$

```
> eq2:=(D@@4)(x)(t)-5*(D@@2)(x)(t)+4*x(t)=0;
```

$$eq2 := \left(D^{(4)}\right)(x)(t) - 5\left(D^{(2)}\right)(x)(t) + 4\,x(t) = 0$$

```
> dsolve(eq2,x(t));
```

$$x(t) = \_C1\, e^{-2\,t} + \_C2\, e^{-t} + \_C3\, e^{2\,t} + \_C4\, e^{t}$$

```
> dsolve({eq2,x(0)=0,D(x)(0)=1,(D@@2)(x)(0)=2,
  (D@@3)(x)(0)=3},x(t));
```

$$x(t) = -1/2\,e^{-t} + 1/6\,e^{-2\,t} + 1/2\,e^{2\,t} - 1/6\,e^{t}$$

```
> sol2:=-1/2*exp(-t)+1/6*exp(-2*t)+1/2*exp(2*t)-
      1/6*exp(t):
> plot(sol2,t=-2..2);
```

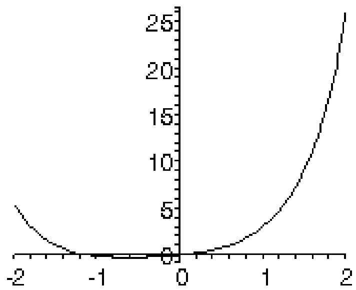


Figure 2.2:

3. The third order nonhomogeneous linear differential equation with constant coefficients:

$$\dddot{x} - 6\ddot{x} + 12\dot{x} - 8x = \sin t; \tag{2.32}$$

```
> eq3:=diff(x(t),t,t,t)-6*diff(x(t),t,t)+12*diff(x(t),t)
```

```
-8*x(t)=sin(t);
```

$$eq3 := \frac{d^3}{dt^3}x(t) - 6\frac{d^2}{dt^2}x(t) + 12\frac{d}{dt}x(t) - 8x(t) = \sin(t)$$

```
> dsolve(eq3);
```

$$x(t) = -\frac{11}{125}\cos(t) - \frac{2}{125}\sin(t) + \_C1\, e^{2t} + \_C2\, e^{2t}t + \_C3\, e^{2t}t^2$$

```
> dsolve({eq3,x(0)=0,D(x)(0)=2,(D@@2)(x)(0)=4});
```

$$x(t) = -\frac{11}{125}\cos(t) - \frac{2}{125}\sin(t) + \frac{11}{125}e^{2t} + \frac{46}{25}e^{2t}t - \frac{19}{10}e^{2t}t^2$$

```
> sol3:=-11/125*cos(t)-2/125*sin(t)+11/125*exp(2*t)+

      46/25*exp(2*t)*t-19/10*exp(2*t)*t^2:

> plot(sol3,t=-4..1);
```

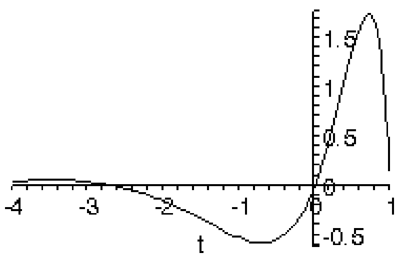


Figure 2.3:

In the followings, we will present another function for plotting *DEtools [DEplot]* (plot solutions of an equation or a system of DEs) to visualize the solution of this Cauchy problem. In this syntax, the differential equation is included in the instruction so, we don't have to solve the equation before. The syntax of this function can have one of the following forms:

with(DEtools):DEplot(deqns, vars, trange, options);

with(DEtools):DEplot(deqns, vars, trange, inits, options);

with(DEtools):DEplot(deqns, vars, trange, xrange, yrange, options);

with(DEtools):DEplot(deqns, vars, trange, inits, xrange, yrange, options);

in which:

deqns	- the differential equation of any order which we want to solve or the list of first order differential equations (in the case of the systems of differential equations)
vars	- the independent variable or the list of the independent variables
trange	- the domain of definition of the independent variable
inits	- the list of the initial conditions
xrange	- the domain of variation of the first dependent variable
yrange	- the domain of variation of the second dependent variable
options	- different options: the display mode of the solution, the method which we want to use in solving the differential equation, etc.

```
> with(DEtools):DEplot(eq3,x(t),t=-4..1,[[x(0)=0,D(x)(0)=2,
  (D@@2)(x)(0)=4]],x=-0.6..1.8,stepsize=.05,title=‘Solution
   of the Cauchy Problem‘);
```

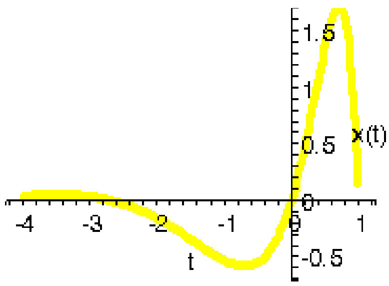


Figure 2.4:

4. The third order Euler linear differential equation:

$$t^2\dddot{x} + 5t\ddot{x} + 4\dot{x} = \ln t, \qquad t > 0 \tag{2.33}$$

```
> eq4:=t^2*diff(x(t),t,t,t)+5*t*diff(x(t),t,t)+
        4*diff(x(t),t)=ln(t);
```

$$eq4 := t^2\frac{d^3}{dt^3}x(t) + 5\,t\frac{d^2}{dt^2}x(t) + 4\,\frac{d}{dt}x(t) = \ln(t)$$

```
> dsolve({eq4,x(2)=2,D(x)(2)=1/2,(D@@2)(x)(2)=3});
```

$$x(t) = -2\ln(2) + 19 + \frac{(-29+2\ln(2))\cdot\ln(t)}{t} + \frac{32\ln(2)-2(\ln(2))^2-32}{t} + 1/4t(\ln(t)) - 2$$

```
> sol4:=-2*ln(2)+19+(-29+2*ln(2))*ln(t)/t+(32*ln(2)-
          2*ln(2)^2-32)/t+1/4*t*(ln(t)-2):
> plot(sol4,t=0.1..infinity,axes=boxed);
```

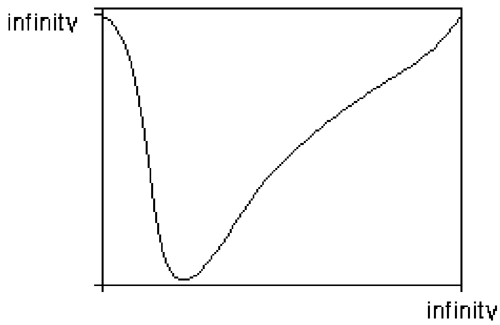


Figure 2.5:

Chapter 3

First order systems of linear differential equations with constant coefficients

3.1 First order systems of homogeneous linear differential equations with constant coefficients

Definition 3.1.1. *A first order system of homogeneous linear differential equations with constant coefficients is a system of differential equation of the form:*

$$\begin{aligned} \dot{x}_1 &= a_{11} \cdot x_1 + a_{12} \cdot x_2 + \ldots + a_{1n} \cdot x_n \\ \dot{x}_2 &= a_{21} \cdot x_1 + a_{22} \cdot x_2 + \ldots + a_{2n} \cdot x_n \\ &\vdots \\ \dot{x}_n &= a_{n1} \cdot x_1 + a_{n2} \cdot x_2 + \ldots + a_{nn} \cdot x_n \end{aligned} \tag{3.1}$$

in which $x_1, x_2, \ldots, x_n$ are n unknown functions and $\dot{x_1}, \dot{x_2}, \ldots, \dot{x_n}$ are their derivatives. In the system (3.1) the coefficients a_{ij} are known constants.

Definition 3.1.2. *An ordered system of n real valued functions $x_1, x_2, \ldots, x_n$ of class $\mathcal{C}^1$ is a solution of the system (3.1) if, for any $t \in \mathbb{R}^1$, it verifies:*

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij} \cdot x_j(t).$$

Definition 3.1.3. *Being given $t_0 \in \mathbb{R}^1$ and $(x_1^0, x_2^0, \ldots, x_n^0) \in \mathbb{R}^n$, the determination of the solution $(x_1(t), x_2(t), \ldots, x_n(t))$ of the system (3.1) which verifies $x_i(t_0) = x_i^0$ $i = \overline{1,n}$ is called initial value problem (IVP) or Cauchy problem.*

If we denote by A the matrix with the elements the constants a_{ij}: $A = (a_{ij})_{i,j=\overline{1,n}}$, and by X the one-column matrix $X = (x_1, x_2, \ldots x_n)^T$, then the system (3.1) can be rewritten in the following form:

$$\dot{X} = A \cdot X. \tag{3.2}$$

The Cauchy problem (the initial value problem) can be written as:

$$\dot{X} = A \cdot X, \quad X(t_0) = X^0 \tag{3.3}$$

and consists of the determination of the matrix function $X = X(t)$ which verifies the equation (3.2) and the initial condition $X(t_0) = X^0$.

Theorem 3.1.1 (The existence of the solution of the Cauchy problem). *For any $t_0 \in \mathbb{R}^1$ and $X^0 = (x_1^0, x_2^0, \ldots x_n^0)^T$ the Cauchy problem (3.3) has a solution defined on $\mathbb{R}^1$.*

Proof. Let be the sequence of column matrix functions defined as follows:

$$\begin{aligned}
X^0(t) &= X^0 = I \cdot X^0 \\
X^1(t) &= X^0 + \int_{t_0}^{t} A \cdot X^0(\tau)d\tau = [I + \frac{(t-t_0)\cdot A}{1!}] \cdot X^0 \\
X^2(t) &= X^0 + \int_{t_0}^{t} A \cdot X^1(\tau)d\tau = [I + \frac{(t-t_0)\cdot A}{1!} + \frac{(t-t_0)^2\cdot A^2}{2!}] \cdot X^0 \\
X^3(t) &= X^0 + \int_{t_0}^{t} A \cdot X^2(\tau)d\tau = \\
&= [I + \frac{(t-t_0)\cdot A}{1!} + \frac{(t-t_0)^2\cdot A^2}{2!} + \frac{(t-t_0)^3\cdot A^3}{3!}] \cdot X^0 \\
&\ldots \\
X^m(t) &= X^0 + \int_{t_0}^{t} A \cdot X^{m-1}(\tau)d\tau = \\
&= [I + \frac{(t-t_0)\cdot A}{1!} + \frac{(t-t_0)^2\cdot A^2}{2!} + \ldots + \frac{(t-t_0)^m\cdot A^m}{m!}] \cdot X^0 \\
&\ldots
\end{aligned}$$

The matrix functions from this sequence verify the inequality:

$$\|X^{m+p}(t) - X^m(t)\| \leq \sum_{k=m+1}^{m+p} \frac{|t-t_0|^k \cdot \|A\|^k}{k!} \cdot \|X^0\|, (\forall)\, m, p \in I\!N, \ (\forall)\, t \in \mathbb{R}^1$$

so, the sequence of matrix functions $\{X^m(t)\}_{m \in n}$ is uniformly fundamental on any compact $K \subset \mathbb{R}^1$. Thus, we obtain that the sequence is uniformly convergent on any compact $K \subset \mathbb{R}^1$ and we can compute the limit of the equality

$$X^m(t) = X^0 + \int_{t_0}^{t} A \cdot X^{m-1}(\tau)d\tau.$$

for $m \to \infty$.

Thus, the limit $X(t)$ of the sequence $X^m(t)$

$$X(t) = \lim_{m \to \infty} X^m(t)$$

verifies the equality

$$X(t) = X^0 + \int_{t_0}^{t} A \cdot X(\tau)d\tau$$

or

$$X(t) = X^0 + A \cdot \int_{t_0}^{t} X(\tau)d\tau.$$

So, the function $X(t)$ is of class $\mathcal{C}^1$ and its derivative satisfies $\dot{X}(t) = A \cdot X(t)$, i.e. $X(t)$ is a solution of the system (3.2). Taking $t = t_0$ in the equality

$$X(t) = X^0 + A \cdot \int_{t_0}^{t} X(\tau)d\tau$$

we obtain $X(t_0) = X^0$ which shows that the function $X(t)$ is a solution of the Cauchy problem (3.3). In this way, we have shown that the Cauchy problem (3.3) has a solution. □

Observation 3.1.1. In this proof we have considered the norm of the matrix A being given by $\|A\| = \sup\limits_{\|X\| \leq 1} \| A \cdot X \|$.

Observation 3.1.2. The sequence of $n \times n$ quadratic matrices used in this proof:

$$U^m(t; t_0) = I + \frac{(t - t_0) \cdot A}{1!} + \frac{(t - t_0)^2 \cdot A^2}{2!} + \ldots + \frac{(t - t_0)^m \cdot A^m}{m!}$$

is fundamental, too.
Indeed:

$$\| U^{m+p}(t; t_0) - U^m(t; t_0) \| \leq \sum_{k=m+1}^{m+p} \frac{| t - t_0 |^k \cdot \| A \|^k}{k!}, \ (\forall) m, p \in I\!N, \ t \in \mathbb{R}^1$$

and the sequence of $n \times n$ matrix functions $\{U^m(t; t_0)\}_{m \in I\!N}$ is uniformly convergent on any compact $K \subset \mathbb{R}^1$. The limit of this sequence is the sum of the matrix series

$$\sum_{m=0}^{\infty} \frac{(t - t_0)^m \cdot A^m}{m!}$$

which is denoted by $e^{(t-t_0)\cdot A}$:

$$e^{(t-t_0)\cdot A} = \sum_{m=0}^{\infty} \frac{(t - t_0)^m \cdot A^m}{m!}$$

Observation 3.1.3. The matrix function $e^{(t-t_0)\cdot A}$ is called the resolvent matrix of the system (3.2). A solution of the Cauchy problem (3.3) is obtained by multiplying the matrix $e^{(t-t_0)\cdot A}$ by the column matrix X^0:

$$X(t; t_0, X^0) = e^{(t-t_0)\cdot A} \cdot X^0.$$

This solution is defined on $\mathbb{R}^1$.

Theorem 3.1.2 (The uniqueness of the solution of the Cauchy problem). *The Cauchy problem (3.3) has an unique solution.*

Proof. We suppose the contrary, that the Cauchy problem (3.3) has two solutions: $X(t;t_0,X^0)$ and $\widetilde{X}(t)$. Being I an interval on which are defined both solutions, $(I \ni t_0)$, we have:

$$X(t;t_0,X^0) = X^0 + A \cdot \int_{t_0}^{t} X(\tau;t_0,X^0)d\tau, \quad (\forall)t \in I$$

and

$$\widetilde{X}(t) = X^0 + A \cdot \int_{t_0}^{t} \widetilde{X}(\tau)d\tau, \quad (\forall)t \in I$$

from which we deduce:

$$\begin{aligned} X(t;t_0,X^0) - \widetilde{X}(t) \quad &= A \cdot \int_{t_0}^{t} [X(\tau;t_0,X^0) - \widetilde{X}(\tau)]d\tau \\ \|X(t;t_0,X^0) - \widetilde{X}(t)\| \quad &\leq \|A\| \cdot \left| \int_{t_0}^{t} \|X(\tau;t_0,X^0) - \widetilde{X}(\tau)\| d\tau \right| < \\ &< \varepsilon + \|A\| \cdot \left| \int_{t_0}^{t} \|X(\tau;t_0,X^0) - \widetilde{X}(\tau)\| d\tau \right| \\ &\quad (\forall)\varepsilon > 0 \ (\forall)t \in I. \end{aligned}$$

For $t > t_0$ we have

$$\frac{\|X(t;t_0,X^0) - \widetilde{X}(t)\|}{\varepsilon + \displaystyle\int_{t_0}^{t} \|A\| \cdot \|X(\tau;t_0,X^0) - \widetilde{X}(\tau)\| d\tau} \leq 1 \ \Leftrightarrow$$

$$\frac{\|A\| \cdot \|X(t;t_0,X^0) - \widetilde{X}(t)\|}{\varepsilon + \displaystyle\int_{t_0}^{t} \|A\| \cdot \|X(\tau;t_0,X^0) - \widetilde{X}(\tau)\| d\tau} \leq \|A\| \ \Leftrightarrow$$

$$\frac{\dfrac{d}{dt}\left(\varepsilon + \displaystyle\int_{t_0}^{t} \|A\| \cdot \|X(\tau;t_0,X^0) - \widetilde{X}(\tau) \parallel d\tau\right)}{\varepsilon + \displaystyle\int_{t_0}^{t} \|A\| \cdot \|X(\tau;t_0,X^0) - \widetilde{X}(\tau)\| d\tau} \leq \|A\| \ \Leftrightarrow$$

$$\ln\left(\varepsilon + \int_{t_0}^{t} \|A\| \cdot \|X(\tau;t_0,X^0) - \widetilde{X}(\tau) \parallel d\tau\right) - ln(\varepsilon) \leq \parallel A \parallel (t - t_0) \ \Leftrightarrow$$

$$\ln \frac{\varepsilon + \displaystyle\int_{t_0}^{t} \|A\| \cdot \|X(\tau;t_0,X^0) - \widetilde{X}(\tau) \parallel d\tau}{\varepsilon}) \leq \parallel A \parallel \cdot (t - t_0) \ \Leftrightarrow$$

$$\varepsilon + \int_{t_0}^{t} \|A\| \cdot \|X(\tau;t_0,X^0) - \widetilde{X}(\tau) \parallel d\tau \leq \varepsilon \cdot e^{\|A\| \cdot (t-t_0)}, \ (\forall)t \geq t_0, \ \varepsilon > 0.$$

$$\Longrightarrow \|X(t;t_0,X^0) - \widetilde{X}(t)\| < \varepsilon e^{\|A\|(t-t_0)}, \ (\forall)t \geq t_0, \ (\forall)\varepsilon > 0$$

For t fixed and $\varepsilon \to 0$ we have

$$\|X(t;t_0,X^0) - \widetilde{X}(t)\| = 0.$$

Thus, we have shown that, for any $t \geq t_0$ and $t \in I$ the equality $\widetilde{X}(t) = X(t; t_0, X^0)$ is fulfilled.

In the same way, for $t \leq t_0$, $t \in I$ we obtain $\widetilde{X}(t) = X(t; t_0, X^0)$. Finally, the equality

$$\widetilde{X}(t) = X(t; t_0, X^0)$$

shows that for any $t \in I$, $\widetilde{X}(t)$ coincides with the solution $X(t; t_0, X^0)$ given in the existence theorem. $\square$

Observation 3.1.4. From the existence and uniqueness theorems we obtain that, any solution of the system (3.2) is defined on $\mathbb{R}^1$ and is given by the formula $X(t) = e^{(t-t_0)\cdot A} \cdot X^0$.
Indeed, let X(t) be an arbitrary solution of the system (3.2) defined on the interval I , $t_0 \in I$ and $X(t_0) = X^0$. According to the uniqueness theorem, the considered solution $X(t)$ coincides with the function $X(t; t_0, X^0) = e^{(t-t_0)\cdot A} \cdot X^0$ which represents the solution of the Cauchy problem (3.3):

$$X(t) \equiv X(t; t_0, X^0) \equiv e^{(t-t_0)\cdot A} \cdot X^0.$$

Theorem 3.1.3. *The set S of the solutions of the first order system of homogeneous linear differential equations with constant coefficients is a vector space of dimension n.*

Proof. Let $X^1(t)$ and $X^2(t)$ be two solutions of the system (3.2) and α, β two real constants. Taking into account the equality:

$$\alpha \cdot X^1(t) + \beta \cdot X^2(t) = e^{(t-t_0)\cdot A} \cdot [\alpha \cdot X^1(t_0) + \beta \cdot X^2(t_0)]$$

we obtain that the function $\alpha \cdot X^1(t) + \beta \cdot X^2(t)$ is a solution of the system (3.2). Consequently, the set S of the solutions of the system (3.2) is a vector space.

To prove that the dimension of the vector space S is n, we will consider the canonical basis $b^1, b^2, \dots, b^n$ in $\mathbb{R}^n$:

$$b^1 = (1, 0, 0, \dots, 0)^T,\ b^2 = (0, 1, 0, \dots, 0)^T,\ \dots,\ b^n = (0, 0, 0, \dots, 1)^T$$

and the system of solutions:

$$X^i(t) = e^{t\cdot A} \cdot b^i,\ i = \overline{1, n}.$$

We will show that the system of solutions $X^1(t), X^2(t), \dots, X^n(t)$ is a basis in the space S. For this, let $c_1, c_2, \dots, c_n$, be n real constants such that

$$\sum_{k=1}^{n} c_k \cdot e^{t\cdot A} \cdot b^k = 0 \qquad (\forall) t \in \mathbb{R}^1$$

In particular, for $t = 0$ we have

$$\sum_{k=1}^{n} c_k \cdot b^k = 0$$

so, $c_1 = c_2 = \dots = c_n = 0$. Thus, we have that the system of functions $X^i(t) = e^{t\cdot A} \cdot b^i$, $i = \overline{1, n}$ is linearly independent.

Let $X(t)$ be an arbitrary solution of the system (3.2). For the vector $X(0) \in \mathbb{R}^n$, there exist n real constants $c_1, c_2, \ldots, c_n$ such that

$$X(0) = \sum_{k=1}^{n} c_k \cdot b^k.$$

We construct the function

$$\widetilde{X}(t) = \sum_{k=1}^{n} c_k \cdot X^k(t)$$

and we see that this is a solution of the system (3.2) which verifies:

$$\widetilde{X}(0) = \sum_{k=1}^{n} c_k \cdot X^k(0) = \sum_{k=1}^{n} c_k \cdot b^k = X(0).$$

Using the existence and uniqueness theorems we have that:

$$\widetilde{X}(t) = X(t),\ (\forall)t.$$

Consequently, any arbitrary solution $X(t)$ is a linear combination

$$X(t) = \sum_{k=1}^{n} c_k \cdot X^k(t)$$

of the solutions $X^k(t)$. □

Definition 3.1.4. *A system of n solutions $\{X^k(t)_{k=\overline{1,n}}\}$ of the equation (3.2) is called fundamental if the system of functions $\{X^k(t)_{k=\overline{1,n}}\}$ is linearly independent.*

Theorem 3.1.4. *A system of n solutions $\{X^k(t)_{k=\overline{1,n}}\}$ of the equation (3.2) is fundamental if and only if the real valued function:*

$$W(X^1(t), \ldots, X^n(t)) = det(x^i_j(t)),$$

called the system's Wronskian cannot be null. We denote:

$$X^i(t) = (x^i_1(t), x^i_2(t), \ldots, x^i_n(t))^T.$$

Proof. First, we proof the necessity. We suppose the contrary, that the system of solutions $X^1(t), X^2(t), \ldots, X^n(t)$ is fundamental and there is a point $t_0 \in \mathbb{R}^1$ in which the Wronskian is null: $det(x^i_j(t_0)) = 0$. In these conditions, the homogeneous algebraic system of n equations and n unknowns $c_1, c_2, \ldots, c_n$

$$\sum_{i=1}^{n} c_i \cdot x^i_j(t_0) = 0 \ \ j = \overline{1,n}$$

has a non-null solution $c_i = c^0_i,\ i = \overline{1,n}$. Using this solution $c_i = c^0_i,\ i = \overline{1,n}$ (not all c^0_i are null) we construct the function:

$$X(t) = \sum_{i=1}^{n} c^0_i \cdot X^i(t) \ \ t \in \mathbb{R}^1.$$

This function is a solution of the system (3.2) and

$$X(t_0) = \sum_{i=1}^{n} c_i^0 \cdot X^i(t_0) = 0.$$

According to the uniqueness theorem, $X(t)$ is identically null:

$$\sum_{i=1}^{n} c_i^0 \cdot X^i(t) = 0, \quad (\forall) t \in \mathbb{R}^1.$$

This fact is an absurdity because the system of n solutions $\{X^i(t)_{i=\overline{1,n}}\}$ is linearly independent.

For the sufficiency, we suppose that the Wronskian $W(X^1(t), \ldots, X^n(t)) = det(x_j^i(t))$ can't be null and we show that the system of solutions $\{X^i(t)_{i=\overline{1,n}}\}$ is fundamental.

We suppose contrary, that the system of solutions $\{X^i(t)_{i=\overline{1,n}}\}$ isn't fundamental (isn't linearly independent). Thus, there is a system of constants $\{c_i^0\}$, $i = \overline{1,n}$, not all null, such that

$$\sum_{i=1}^{n} c_i^0 \cdot X^i(t) = 0$$

for any $t \in \mathbb{R}^1$. This equality implies

$$\sum_{i=1}^{n} c_i^0 \cdot x_j^i(t) = 0 \ (\forall) t \in \mathbb{R}^1 \ j = \overline{1,n}$$

i.e. $det(x_j^i(t)) = 0$ $(\forall) t \in \mathbb{R}^1$; absurd. $\square$

Theorem 3.1.5 (Liouville). *A system of n solutions $\{X^i(t)_{i=\overline{1,n}}\}$ of the system (3.2) is fundamental if and only if there is a point $t_0 \in \mathbb{R}^1$ in which the Wronskian*

$$W(X^1(t), \ldots, X^n(t)) = det(x_j^i(t))$$

is non-null.

Proof. Taking into account the previous theorem, it is sufficient to show that, if there exists $t_0 \in \mathbb{R}^1$ such that

$$W(X^1(t_0), \ldots, X^n(t_0)) \neq 0$$

then, for any $t \in \mathbb{R}^1$, $W(X^1(t), \ldots, X^n(t)) \neq 0$. We compute the Wronskian of the system of solutions $\{X^i(t)_{i=\overline{1,n}}\}$ and we obtain

$$\frac{d}{dt} W(X^1(t), \ldots, X^n(t)) = \left(\sum_{i=1}^{n} a_{ii} \right) \cdot W(X^1(t), \ldots, X^n(t)$$

From here, the following equality results:

$$W(X^1(t), \ldots, X^n(t)) = W(X^1(t_0), \ldots, X^n(t_0)) \cdot \exp \left[\left(\sum_{i=1}^{n} a_{ii} \right) \cdot (t - t_0) \right]$$

which shows that for any $t \in \mathbb{R}^1$ we have $W(X^1(t), \ldots, X^n(t)) \neq 0$. $\square$

Observation 3.1.5. In the proof of the theorem which asserts that the solutions of the system (3.2) forms a vector space of dimension n, we have seen that if

$$b^1 = (1, 0, 0, \dots, 0)^T,\ b^2 = (0, 1, 0, \dots, 0)^T,\ \dots,\ b^n = (0, 0, 0, \dots, 1)^T$$

then the system of functions

$$X^1(t) = e^{t\cdot A} \cdot b^1,\quad X^2(t) = e^{t\cdot A} \cdot b^2,\quad \dots,\quad X^n(t) = e^{t\cdot A} \cdot b^n$$

is a fundamental system of solutions. Taking into account that the solution $X^i(t)$ represents the ith column of the quadratic matrix $e^{t\cdot A}$, we deduce that we can construct the solutions of the equations (3.2) if we know the elements of the matrix $e^{t\cdot A}$.

To determine the elements of the matrix $e^{t\cdot A}$ we use the following results of linear algebra:

Proposition 3.1.1. *If the matrix A is similar with the matrix A_0, i.e.*

$$A = S \cdot A_0 \cdot S^{-1}$$

then the matrix $e^{t\cdot A}$ is similar with matrix $e^{t\cdot A_0}$ i.e.

$$e^{t\cdot A} = S \cdot e^{t\cdot A_0} \cdot S^{-1}.$$

This is true because for any $k \in I\!N$ we have $A^k = S \cdot A_0^k \cdot S^{-1}$.

Proposition 3.1.2. *(Jordan's theorem)*

For any matrix A there is a block diagonal matrix

$$A_0 = diag(A_{01}, A_{02}, \dots A_{0m})$$

and a nonsingular matrix S having the following properties:

i) *A_{0j} is a quadratic matrix of order q_j, $j = \overline{1,m}$ and $\displaystyle\sum_{j=1}^{m} q_j = n$;*

ii) *A_{0j} is a matrix of the form $A_{0j} = \lambda_j \cdot I_j + N_j$, $j = \overline{1,m}$, where λ_j is an eigenvalue of the matrix A, I_j is the unity matrix of order q_j, N_j is a nilpotent matrix: $N_j = \left(b^j_{kl}\right), k, l = \overline{1, q_j}$ with $b^j_{k,k+1} = 1$ and $b^j_{kl} = 0$, for $l \neq k+1$, and q_j is maximal and equal to the multiplicity order of the eigenvalue λ_j;*

iii) *$A = S \cdot A_0 \cdot S^{-1}$.*

Proposition 3.1.3. *The matrix $e^{t\cdot A_0}$ has the form:*

$$e^{t\cdot A_0} = diag\left(e^{t\cdot A_{01}}, e^{t\cdot A_{02}}, \dots, e^{t\cdot A_{0m}}\right)$$

and the matrix $e^{t\cdot A_{0j}}$ has the form:

$$e^{t\cdot A_{0j}} = e^{\lambda_j\cdot t} \cdot \begin{bmatrix} 1 & \frac{t}{1!} & \frac{t^2}{2!} & \cdots & \frac{t^{q_j-1}}{(q_j-1)!} \\ 0 & 1 & \frac{t}{1!} & \cdots & \frac{t^{q_j-2}}{(q_j-2)!} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Theorem 3.1.6. *The elements of the matrix $e^{t\cdot A} = S \cdot e^{t\cdot A_0} \cdot S^{-1}$ are functions of the form:*

$$u_{ij}(t)=\sum_{k=1}^{p} e^{\lambda_k t} P_{q_k-1}^{ij}(t)+\sum_{k=1}^{l} e^{\mu_k t}\left[Q_{r_k-1}^{ij}(t)\cos\nu_k t+R_{r_k-1}^{ij}(t)\sin\nu_k t\right], \quad i,j=\overline{1,n}$$

where $\lambda_1,\dots,\lambda_p$ are real eigenvalues of A with multiplicity orders $q_1,\dots,q_p$; $\mu_k + i\nu_k$, $k=\overline{1,l}$ are complex eigenvalues of A with multiplicity orders r_k; P_{q_k-1}, Q_{r_k-1} and R_{r_k-1} are polynomial functions with real coefficients of degrees q_k-1 and r_k-1, respectively.

The results is immediately obtained on the basis of Propositions (3.1.1, 3.1.2, 3.1.3).

Theorem 3.1.7. *The solutions of the system (3.2) are functions of the form:*

$$e^{\lambda_k t} P_{q_k-1}(t) + \sum_{k=1}^{l} e^{\mu_k t}\left[Q_{r_k-1}(t)\cos\nu_k t + R_{r_k-1}(t)\sin\nu_k t\right],$$

where $\lambda_1,\dots,\lambda_p$ are real eigenvalues of A with multiplicity orders $q_1,\dots,q_p$; $\mu_k + i\nu_k$, $k=\overline{1,l}$ are complex eigenvalues of A with multiplicity orders r_k; P_{q_k-1}, Q_{r_k-1} and R_{r_k-1} are column vectors having polynomial elements of degrees q_k-1 and r_k-1, respectively.

Exercises

1. Solve the following systems:

a) $\begin{cases} \dot{x_1} = -x_1 + 8x_2 \\ \dot{x_2} = x_1 + x_2 \end{cases}$ **A:** $\begin{cases} x_1(t) = c_1 \cdot e^{3t} + c_2 \cdot e^{-3t} \\ x_2(t) = \frac{1}{2} \cdot c_1 \cdot e^{3t} - \frac{1}{4} \cdot c_2 \cdot e^{-3t} \end{cases}$

b) $\begin{cases} \dot{x_1} = -3x_1 + 2x_2 \\ \dot{x_2} = -2x_1 + x_2 \end{cases}$ **A:** $\begin{cases} x_1(t) = c_1 \cdot e^{-t} + c_2 \cdot t \cdot e^{-t} \\ x_2(t) = c_1 \cdot e^{-t} + \frac{2t+1}{2} \cdot c_2 \cdot e^{-t} \end{cases}$

c) $\begin{cases} \dot{x_1} = 2x_1 - x_2 \\ \dot{x_2} = x_1 + 2x_2 \end{cases}$ **A:** $\begin{cases} x_1(t) = c_1 \cdot \cos t \cdot e^{2t} + c_2 \cdot \sin t \cdot e^{2t} \\ x_2(t) = c_1 \cdot \sin t \cdot e^{2t} - c_2 \cdot \cos t \cdot e^{2t} \end{cases}$

d) $\begin{cases} \dot{x_1} = 3x_1 + 12x_2 - 4x_3 \\ \dot{x_2} = -x_1 - 3x_2 + x_3 \\ \dot{x_3} = -x_1 - 12x_2 + 6x_3 \end{cases}$ **A:** $\begin{cases} x_1(t) = c_1 \cdot e^{2t} + c_2 \cdot e^{t} + c_3 \cdot e^{3t} \\ x_2(t) = -\frac{3}{8} c_1 \cdot e^{2t} - \frac{1}{2} c_2 \cdot e^{t} - \frac{1}{3} c_3 \cdot e^{3t} \\ x_3(t) = -\frac{7}{8} c_1 \cdot e^{2t} - c_2 \cdot e^{t} - c_3 \cdot e^{3t} \end{cases}$

e) $\begin{cases} \dot{x_1} = x_1 + x_2 - 2x_3 \\ \dot{x_2} = 4x_1 + x_2 \\ \dot{x_3} = 2x_1 + x_2 - x_3 \end{cases}$ **A:** $\begin{cases} x_1(t) = -\frac{1}{2} c_1 \cdot e^{-t} + \quad + \frac{1}{4} c_3 \cdot e^{t} \\ x_2(t) = c_1 \cdot e^{-t} + c_2 \cdot e^{t} + c_3 \cdot t \cdot e^{t} \\ x_3(t) = + \frac{1}{2} c_2 \cdot e^{t} + \frac{1}{2} c_3 \cdot t \cdot e^{t} \end{cases}$

f) $\begin{cases} \dot{x_1} = 2x_1 - x_2 - x_3 \\ \dot{x_2} = 3x_1 - 2x_2 - 3x_3 \\ \dot{x_3} = -x_1 + x_2 + 2x_3 \end{cases}$ **A:** $\begin{cases} x_1(t) = c_2 + c_3 \cdot e^{t} \\ x_2(t) = c_1 \cdot e^{t} + 3\, c_2 \\ x_3(t) = -c_1 \cdot e^{t} - c_2 + c_3 \cdot e^{t} \end{cases}$

g) $\begin{cases} \dot{x_1} = -x_1 - x_2 \\ \dot{x_2} = \quad\quad -x_2 - x_3 \\ \dot{x_3} = \quad\quad\quad\quad -x_3 \end{cases}$ **A:** $\begin{cases} x_1(t) = c_1 \cdot e^{-t} - c_2 \cdot t \cdot e^{-t} - \frac{1}{2} c_3 \cdot t^2 \cdot e^{-t} \\ x_2(t) = \quad\quad + c_2 \cdot e^{-t} \quad - \quad c_3 \cdot t \cdot e^{-t} \\ x_3(t) = \quad\quad\quad\quad c_3 \cdot e^{-t} \end{cases}$

2. Solve the following Cauchy problem:

a) $\begin{cases} \dot{x_1} = x_2 \\ \dot{x_2} = x_1 \end{cases}$ $\begin{matrix} x_1(0) = 1 \\ x_2(0) = 0 \end{matrix}$ **A:** $\begin{cases} x_1(t) = \frac{1}{2} \, e^{-t} + \frac{1}{2} e^t \\ x_2(t) = -\frac{1}{2} e^{-t} + \frac{1}{2} e^t \end{cases}$

b) $\begin{cases} \dot{x_1} = 11x_1 + 16x_2 \\ \dot{x_2} = -2x_1 - x_2 \end{cases}$ $\begin{matrix} x_1(1) = 0 \\ x_2(1) = 1 \end{matrix}$ **A:** $\begin{cases} x_1(t) = 4e^{3t-3} + 4e^{7t-7} \\ x_2(t) = 2e^{3t-3} - e^{7t-7} \end{cases}$

c) $\begin{cases} \dot{x_1} = x_1 - x_2 \\ \dot{x_2} = -4x_1 - 2x_2 \end{cases}$ $\begin{matrix} x_1(1) = 1 \\ x_2(1) = 1 \end{matrix}$ **A:** $\begin{cases} x_1(t) = \frac{3}{5} e^{2t-2} + \frac{2}{5} e^{-3t+3} \\ x_2(t) = -\frac{3}{5} e^{2t-2} + \frac{8}{5} e^{-3t+3} \end{cases}$

3. Let be the following system of differential equations

$$\begin{cases} \dot{x_1} = a \cdot x_1 + b \cdot x_2 \\ \dot{x_2} = c \cdot x_1 + d \cdot x_2 \end{cases}$$

with $a \cdot d - b \cdot c \neq 0$. Prove that:

i) if $(a-d)^2 + 4 \cdot b \cdot c \geq 0$, $a + d < 0$ and $a \cdot d - b \cdot c > 0$, then any non-null solution of the system tends to $(0, 0)$.

ii) if $(a-d)^2 + 4 \cdot b \cdot c \geq 0$, $a + d > 0$ and $a \cdot d - b \cdot c > 0$, then any non-null solution of the system tends in norm to $+\infty$.

iii) if $(a-d)^2 + 4 \cdot b \cdot c < 0$ and $a + d < 0$, then all the non-null solutions of the system tend to $(0, 0)$.

iv) if $(a-d)^2 + 4 \cdot b \cdot c < 0$ and $a + d > 0$, then all the non-null solutions of the system tend in norm to $+\infty$.

v) if $(a-d)^2 + 4 \cdot b \cdot c < 0$ and $a + d = 0$, all the non-null solutions of the system are periodic.

3.2 First order systems of nonhomogeneous linear differential equations with constant coefficients

Definition 3.2.1. *A first order system of nonhomogeneous linear differential equations with constant coefficients is a system of differential equations of the form:*

$$\begin{aligned} \dot{x}_1 &= a_{11} \cdot x_1 + a_{12} \cdot x_2 + \ldots + a_{1n} \cdot x_n \quad + f_1(t) \\ \dot{x}_2 &= a_{21} \cdot x_1 + a_{22} \cdot x_2 + \ldots + a_{2n} \cdot x_n \quad + f_2(t) \\ &\vdots \\ \dot{x}_n &= a_{n1} \cdot x_1 + a_{n2} \cdot x_2 + \ldots + a_{nn} \cdot x_n \quad + f_n(t) \end{aligned} \tag{3.4}$$

where $x_1, x_2, \dots, x_n$ are unknown real functions and $\dot{x_1}, \dot{x_2}, \dots, \dot{x_n}$ are their derivatives.

In the system (3.4) the coefficients a_{ij} are known constants, and $f_i : \mathbb{R}^1 \to \mathbb{R}$ are known real and continuous functions.

Definition 3.2.2. *An ordered system of n real functions $x_1, x_2, \dots, x_n$ of class $\mathcal{C}^1$ is a solution of the system (3.4) if it verifies:*

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij} \cdot x_j + f_i(t) \ (\forall) t \in \mathbb{R}$$

Definition 3.2.3. *Being given $t_0 \in \mathbb{R}^1$ and $(x_1^0, x_2^0, \dots, x_n^0) \in \mathbb{R}^n$, the problem of determination of the solution $(x_1(t), x_2(t), \dots, x_n(t))$ of the system (3.4) which verifies $x_i(t_0) = x_i^0$ $i = \overline{1,n}$, is called initial value problem or Cauchy problem.*

Denoting by A the $n \times n$ quadratic matrix which has the elements a_{ij}: $A = (a_{ij})_{i,j=\overline{1,n}}$, by $F(t)$ the column matrix $F(t) = (f_1(t), f_2(t), \dots, f_n(t))$ and by $X(t)$ the column matrix $X = (x_1, x_2, \dots, x_n)^T$, the system (3.4) becomes:

$$\dot{X} = A \cdot X + F(t) \tag{3.5}$$

and the Cauchy problem has the form

$$\dot{X} = A \cdot X + F(t), \qquad X(t_0) = X^0 \tag{3.6}$$

Theorem 3.2.1 (Existence, uniqueness and representation of the solution of the initial value problem). *If the function $F(t)$ is continuous on $\mathbb{R}^1$, then for any $t_0 \in \mathbb{R}^1$ and $X^0 \in \mathbb{R}^n$ the initial value problem (3.6) has a unique solution defined on $\mathbb{R}^1$ and this is given by the following formula:*

$$X(t; t_0, X^0) = e^{(t-t_0) \cdot A} \cdot X^0 + \int_{t_0}^{t} e^{(t-s) \cdot A} \cdot F(s) ds \tag{3.7}$$

Proof. To prove that the Cauchy problem (3.6) has at most one solution, we suppose the contrary, that $X^1(t)$ and $X^2(t)$ are two solutions of the problem (3.6), and we consider the function $X^3(t) = X^1(t) - X^2(t)$.

It easy to see that $X^3(t)$ is a solution of the Cauchy problem

$$\dot{X}^3 = A \cdot X^3, \quad X^3(t_0) = 0.$$

According to the theorem of uniqueness of the solution of the Cauchy problem for homogeneous systems we obtain:

$$X^3(t) = 0, \ (\forall) t.$$

So,

$$X^1(t) - X^2(t) \equiv 0,$$

which is contrary to the hypothesis $X^1(t) \neq X^2(t)$.

We still have to prove that the function $Z(t)$ defined by:

$$Z(t) = e^{(t-t_0)\cdot A} \cdot X^0 + \int_{t_0}^{t} e^{(t-s)\cdot A} \cdot F(s)ds$$

for any $t \in \mathbb{R}^1$ verifies (3.6).

The function $Z(t)$ is correctly defined, it is of class $\mathcal{C}^1$ on $\mathbb{R}^1$ and its derivative verifies:

$$\dot{Z}(t) = A \cdot e^{(t-t_0)\cdot A} \cdot X^0 + F(t) + \int_{t_0}^{t} A \cdot e^{(t-s)\cdot A} \cdot F(s)ds = A \cdot Z + F(t).$$

It follows that $Z(t)$ is a solution of the nonhomogeneous equation (3.5). Moreover, computing $Z(t_0)$ we find that $Z(t_0) = X^0$. $\square$

Problem 3.2.1. A substance A is decomposed into two other substances B and C. The velocity of the reaction for every substance is proportional with the quantity of substance un-decomposed. Determine the variation of the quantities x and y, as functions of time. The initial quantity of substance a is given, and the quantities of substances B and C formed after one our are given, too: $\dfrac{a}{8}$ and $\dfrac{3a}{8}$.

Answer: At the moment t, the quantity of substance A is $a - x - y$. So, the reaction velocities of B and C are:

$$\begin{cases} \dot{x} = k_1 \cdot (a - x - y) \\ \dot{y} = k_2 \cdot (a - x - y) \end{cases}$$

or

$$\begin{cases} \dot{x} = -k_1 \cdot x - k_1 \cdot y + k_1 \cdot a \\ \dot{y} = -k_2 \cdot x - k_2 \cdot y + k_2 \cdot a \end{cases}$$

The matrix A is in this case $A = \begin{bmatrix} -k_1 & -k_1 \\ -k_2 & -k_2 \end{bmatrix}$.

The eigenvalues of A are roots of the equation

$$(k_1 + \lambda) \cdot (k_2 + \lambda) - k_1 \cdot k_2 = 0.$$

These roots are $\lambda_1 = -(k_1 + k_2)$ and $\lambda_2 = 0$, so, the matrix $e^{t\cdot A}$ is:

$$e^{t\cdot A} = \frac{1}{k_1 + k_2} \begin{bmatrix} -k_1 \cdot e^{-(k_1+k_2)\cdot t} + k_2 & e^{-(k_1+k_2)\cdot t} - 1 \\ k_1 \cdot k_2 \cdot e^{-(k_1+k_2)\cdot t} - k_1 \cdot k_2 & k_1 + k_2 \cdot e^{-(k_1+k_2)\cdot t} \end{bmatrix}$$

According to formula (3.7) we have:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{(t-1)\cdot A} \cdot \begin{pmatrix} a/8 \\ 3/8 \end{pmatrix} + \int_{1}^{t} e^{(t-s)\cdot A} \begin{pmatrix} k_1 \cdot a \\ k_2 \cdot a \end{pmatrix} ds$$

Exercises

1. Solve the following systems of nonhomogeneous differential equations:

a) $\begin{cases} \dot{x_1} = \quad\quad x_2 \\ \dot{x_2} = x_1 \quad\quad + e^t + e^{-t} \end{cases}$

$$\textbf{A:}\begin{cases} x_1(t) = c_1\cdot e^{-t}+c_2\cdot e^{t}+(\frac{1}{2}t-\frac{1}{4})e^{t}-(\frac{1}{2}t+\frac{1}{4})e^{-t} \\ \\ x_2(t) = -c_1\cdot e^{-t}+c_2\cdot e^{t}+(\frac{1}{2}t+\frac{1}{4})e^{t}+(\frac{1}{2}t-\frac{1}{4})e^{-t} \end{cases}$$

b) $\begin{cases} \dot{x_1} = 11x_1+16x_2+t \\ \dot{x_2} = -2x_1-x_2+1-t \end{cases}$

$$\textbf{A:}\begin{cases} x_1(t) = c_1\cdot e^{3t}+c_2\cdot e^{7t}+\frac{23}{49}-\frac{5}{7}t \\ \\ x_2(t) = -\frac{1}{2}c_1\cdot e^{3t}-\frac{1}{4}c_2\cdot e^{7t}-\frac{18}{49}+\frac{3}{7}t \end{cases}$$

c) $\begin{cases} \dot{x_1} = x_1-x_2+3t^2 \\ \dot{x_2} = -4x_1-2x_2+2+8t \end{cases}$

$$\textbf{A:}\begin{cases} x_1(t) = c_1\cdot e^{2t}+c_2\cdot e^{-3t}-t^2 \\ \\ x_2(t) = -c_1\cdot e^{2t}+4c_2\cdot e^{-3t}+2t+2t^2 \end{cases}$$

3.3 Solving the linear differential equation of order n with constant coefficients by reducing it to first order system of linear differential equation with constant coefficients

We consider the homogeneous linear differential equation of order n with constant coefficients:

$$a_n\cdot x^{(n)}+a_{n-1}\cdot x^{(n-1)}+\ldots+a_1\cdot\dot{x}+a_0\cdot x=0 \tag{3.8}$$

in which a_n is supposed to be different fom zero.

The differential equation (3.8) has the same solutions as the differential equation:

$$x^{(n)}+b_{n-1}\cdot x^{(n-1)}+\ldots+b_1\cdot\dot{x}+b_0\cdot x=0 \tag{3.9}$$

in which $b_i=\dfrac{a_i}{a_n}$. The equation (3.9) is equivalent to the first order system of differential equation with constant coefficients:

$$\dot{Y}=A\cdot Y \tag{3.10}$$

in which the column matrix Y is $Y=(x,u^1,u^2,\ldots,u^{n-1})$, and the quadratic matrix A is:

$$A=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 1 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & 0 & \ldots & 1 \\ -\dfrac{a_0}{a_n} & -\dfrac{a_1}{a_n} & \ldots & \ldots & \ldots & \ldots & -\dfrac{a_{n-1}}{a_n} \end{bmatrix}$$

The eigenvalues of this matrix represent the roots of the algebraic equation

$$a_n \cdot \lambda^n + a_{n-1} \cdot \lambda^{n-1} + \ldots + a_1 \cdot \lambda + a_0 = 0. \tag{3.11}$$

In this way, we obtain that the solutions of the linear differential equation of order n (3.8) are functions of the form:

$$x(t) = \sum_{j=1}^{p} e^{\lambda_j t} P_{q_j-1}(t) + \sum_{j=1}^{l} e^{\mu_j t} \left[Q_{r_j-1}(t) \cos \nu_j t + R_{r_j-1}(t) \sin \nu_j t \right]$$

in which $\lambda_j, j = \overline{1,k}$ are real roots of the equation (3.11) with multiplicity orders $q_1, \ldots, q_p$; $\mu_j + i\nu_k$, $k = \overline{1,l}$ are complex roots of the equation (3.11) having multiplicity orders r_j; P_{q_j-1}, Q_{r_j-1} and R_{r_j-1} are polynomial functions of degrees $q_j - 1$ and $r_j - 1$, respectively.

If the linear differential equation of order n with constant coefficients is nonhomogeneous:

$$a_n \cdot x^{(n)} + a_{n-1} \cdot x^{(n-1)} + \ldots + a_1 \cdot \dot{x} + a_0 \cdot x = f(t) \tag{3.12}$$

and $a_n \neq 0$, $f(t)$ is a continuous real function, then any solution $x = x(t)$ of this equation is of the form:

$$x(t) = \sum_{j=1}^{p} e^{\lambda_j t} P_{q_j-1}(t) + \sum_{j=1}^{l} e^{\mu_j t} \left[Q_{r_j-1}(t) \cos \nu_j t + R_{r_j-1}(t) \sin \nu_j t \right] + \bar{x}(t)$$

where $\overline{x}(t)$ is a fixed solution of the equation (3.12).

If the equation (3.12) doesn't have pure imaginary roots and f is a periodical function of period T, then the equation (3.12) has one periodical solution of period T.

Exercises

Solve the following linear differential equations of higher order with constant coefficients by reducing them to first order systems of linear differential equation with constant coefficients:

a) $\ddot{x} - x = 0$ $\quad x(0) = 2$ $\quad \dot{x}(0) = 0$ $\quad$ **A:** $x(t) = e^t + e^{-t}$

b) $\ddot{x} + 2\dot{x} + x = 0$ $\quad x(0) = 0$ $\quad \dot{x}(0) = 1$ $\quad$ **A:** $x(t) = t \cdot e^{-t}$

c) $\ddot{x} - 4\dot{x} + 4x = 0$ $\quad x(1) = 1$ $\quad \dot{x}(1) = 0$ $\quad$ **A:** $x(t) = 3e^{2t-2} - 2te^{2t-2}$

d) $\ddot{x} + x = 0$ $\quad x\left(\frac{\pi}{2}\right) = 1$ $\quad \dot{x}\left(\frac{\pi}{2}\right) = 0$ $\quad$ **A:** $x(t) = \sin t$

e) $\ddot{x} + \dot{x} + x = 0$ $\quad x(0) = 0$ $\quad \dot{x}(1) = 1$ $\quad$ **A:** $x(t) = \frac{2\sqrt{3}}{3} e^{-\frac{1}{2}t} \sin\left(\frac{\sqrt{3}}{2}\right)$

f) $\dddot{x} - 2\ddot{x} - \dot{x} + 2x = 0$ $x(0) = 0$ $\dot{x}(0) = 1$ $\ddot{x}(0) = 2$

A: $x(t) = \frac{-1}{2} \cdot e^t + \frac{2}{3} \cdot e^{2t} - \frac{1}{6} \cdot e^{-t}$

g) $\dddot{x} - \ddot{x} + \dot{x} - x = 0$ $x(1) = 0$ $\dot{x}(1) = 1$ $\ddot{x}(1) = 2$

A: $x(t) = e^{t-1} + (\sin 1) \cdot \sin t - (\cos 1) \cdot \cos t$

h) $x^{(4)} - 5\ddot{x} + 4x = 0$ $x(0) = 0$ $\dot{x}(0) = 1$ $\ddot{x}(0) = 2$ $\dddot{x}(0) = 3$

A: $x(t) = -\frac{1}{6} \cdot e^t + \frac{1}{6} \cdot e^{-2t} - \frac{1}{2} \cdot e^{-t} + \frac{1}{2} \cdot e^{2t}$

3.4 Symbolic calculus of the solutions of the first order systems of linear differential equations with constant coefficients

For solving the systems of differential equations, *Maple* uses the function *dsolve* (solve ordinary differential equations - ODEs) which has been presented in the previous chapters.

In the syntax, the differential equation will be replaced by a list of differential equations which form the system, and the initial condition will be replaced by a list of initial conditions $x_i(t_0) = x_i^0$, respectively, which correspond to the every unknown function $x_i(t)$, $i = \overline{1,n}$:

$dsolve(\{ODE1, ODE2, ..., ODEn\});$

$dsolve(\{ODE1, ODE2, ..., ODEn\}, \{x_1(t), x_2(t), ..., x_n(t)\}, extra.args);$

$dsolve(\{ODE1, ODE2, ..., ODEn, x_1(t_0){=}x_1^0, x_2(t_0){=}x_2^0, ..., x_n(t_0){=}x_n^0\},$
$\{x_1(t), x_2(t), ..., x_n(t)\}, extra.args);$

For exemplification, we will solve the following systems of linear differential equations with constant coefficients.

1. Solve the system of two homogeneous linear differential equations with constant coefficients:

$$\begin{cases} \dot{x_1} = -x_1 + 8x_2 \\ \dot{x_2} = x_1 + x_2 \end{cases} \tag{3.13}$$

using the initial conditions $x_1(0) = 0$, $x_2(0) = 1$ and represent the solution:

```
> sys1_Eq1:=diff(x1(t),t)=-x1(t)+8*x2(t);
```

$$sys1\_Eq1 := \frac{d}{dt}x1(t) = -x1(t) + 8\,x2(t)$$

```
> sys1_Eq2:=diff(x2(t),t)=x1(t)+x2(t);
```

$$sys1\_Eq2 := \frac{d}{dt}x2(t) = x1(t) + x2(t)$$

```
> dsolve(sys1_Eq1,sys1_Eq2,x1(0)=1,x2(0)=1,x1(t),x2(t));
```

$$\{x2(t) = 5/6\,e^{3\,t} + 1/6\,e^{-3\,t}, x1(t) = 5/3\,e^{3\,t} - 2/3\,e^{-3\,t}\}$$

```
> sol_x1:=5/3*exp(3*t)-2/3*exp(-3*t):
> sol_x2:=5/6*exp(3*t)+1/6*exp(-3*t):
> plot([sol_x1,sol_x2],t=0..1,color=[red,blue],
  style=[line,point]);
```

Here is another way to solve the given initial value problem, in which we obtain the phase portrait or we represent the solution in 3D:

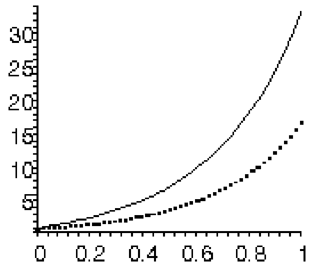


Figure 3.1:

```
>  with(DEtools):
>  DEplot(sys1_Eq1,sys1_Eq2,x1(t),x2(t),t=0..1,
>  [[x1(0)=1,x2(0)=1]],x1=0..40,x2=0..20,scene=
    [x1(t),x2(t)]);
```

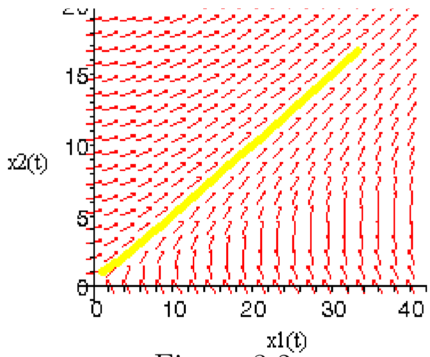


Figure 3.2:

```
>  with(DEtools):
>  DEplot3d(sys1_Eq1,sys1_Eq2,x1(t),x2(t),t=0..1,
>  [[x1(0)=1,x2(0)=1]],x1=0..40,x2=0..20,scene=
    [t,x1(t),x2(t)]);
```

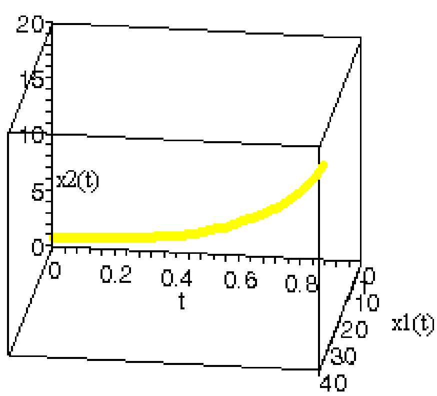


Figure 3.3:

2. Solve the system of three homogeneous linear differential equations with constant coefficients:

$$\begin{cases} \dot{x_1} = -x_1 - x_2 \\ \dot{x_2} = \quad\quad -x_2 - x_3 \\ \dot{x_3} = \quad\quad\quad\quad -x_3 \end{cases} \tag{3.14}$$

using the initial conditions $x_1(0) = 1$, $x_2(0) = 1$, $x_3(0) = 2$, and represent its solution:

```
> sys2_Eq1:=diff(x1(t),t)=-x1(t)-x2(t);
```

$$sys2\_Eq1 := \frac{d}{dt}x1(t) = -x1(t) - x2(t)$$

```
> sys2_Eq2:=diff(x2(t),t)=-x2(t)-x3(t);
```

$$sys2\_Eq2 := \frac{d}{dt}x2(t) = -x2(t) - x3(t)$$

```
> sys2_Eq3:=diff(x3(t),t)=-x3(t);
```

$$sys2\_Eq3 := \frac{d}{dt}x3(t) = -x3(t)$$

```
> dsolve({sys2_Eq1,sys2_Eq2,sys2_Eq3},{x1(t),x2(t),x3(t)});
```

$$\{ x1(t) = 1/2\,(\_C3\,t^2 - 2\,\_C2\,t + 2\,\_C1)\,e^{-t},$$
$$x2(t) = -(\_C3\,t - \_C2)\,e^{-t},$$
$$x3(t) = \_C3\,e^{-t} \}$$

```
> dsolve({sys2_Eq1,sys2_Eq2,sys2_Eq3,x1(0)=1,x2(0)=0,
  x3(0)=2});
```

$$\{ x1(t) = 1/2\,(2\,t^2 + 2)\,e^{-t}$$
$$x2(t) = -2\,te^{-t},$$
$$x3(t) = 2\,e^{-t}\}$$

```
> plot([1/2*(2*t^2+2)*exp(-t),-2*t*exp(-t),2*exp(-t)],
  t=-1.3..8,colour=[green,black,blue],thickness=[3,4,1],
  style=[line,point,line]);
```

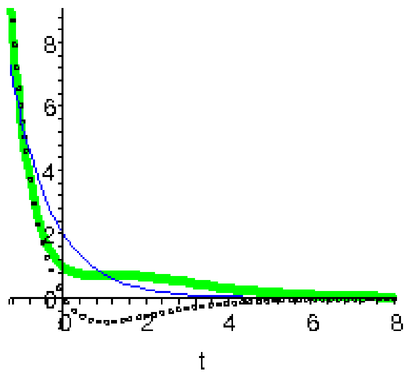


Figure 3.4:

3. Solve the system of two nonhomogeneous linear differential equations with constant coefficients:

$$\begin{cases} \dot{x_1} = \quad x_1 - \; x_2 + \quad 3t^2 \\ \dot{x_2} = -4x_1 - 2x_2 + 2 + 8t \end{cases} \tag{3.15}$$

using the initial conditions $x_1(1) = 1$, $x_2(1) = 0$, and represent its solution:

```
> sys3_Eq1:=diff(x1(t),t)=x1(t)-x2(t)+3*t^2;
```

$$sys3\_Eq1 := \frac{d}{dt}x1(t) = x1(t) - x2(t) + 3t^2$$

```
> sys3_Eq2:=diff(x2(t),t)=-4*x1(t)-2*x2(t)+2+8*t;
```

$$sys3\_Eq2 := \frac{d}{dt}x2(t) = -4\,x1(t) - 2\,x2(t) + 2 + 8\,t$$

```
> dsolve({sys3_Eq1,sys3_Eq2});
```

$$\{\, x1(t) = e^{-3\,t}\_C2 + e^{2\,t}\_C1 - t^2$$
$$x2(t) = 4\,e^{-3\,t}\_C2 - e^{2\,t}\_C1 + 2\,t + 2\,t^2,\ \}$$

```
> dsolve({sys3_Eq1,sys3_Eq2,x1(1)=1,x2(1)=0});
```

$$\{\, x1(t) = \frac{12}{5}\,e^{-2}e^{2\,t} - 2/5\,e^{3}e^{-3\,t} - t^2,$$
$$x2(t) = -\frac{12}{5}\,e^{-2}e^{2\,t} - 8/5\,e^{3}e^{-3\,t} + 2\,t + 2\,t^2\ \}$$

```
> x1:=12/5*exp(-2)*exp(2*t)-2/5*exp(3)*exp(-3*t)-t^2:
> x2:=-12/5*exp(-2)*exp(2*t)-8/5*exp(3)*exp(-3*t)+2*t+
    2*t^2:
> plot([x1,x2],t=-0.1..2,color=[red,green],style=
  [line,point]);
```

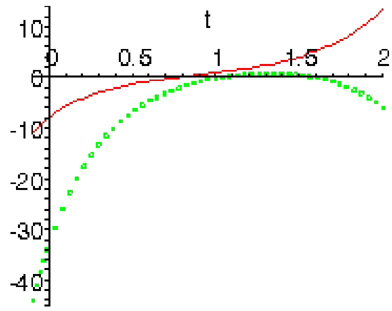


Figure 3.5:

Chapter 4

Theorems of existence and uniqueness. Qualitative properties of the solutions. Numerical methods. Prime integrals

4.1 Theorems of existence and uniqueness of the initial value problem for the first order nonlinear differential equation

Let be

$$\dot{x} = f(t, x); \quad x(t_0) = x_0 \tag{4.1}$$

an initial value problem (IVP) where $f : \Omega \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^1$ and $(t_0, x_0) \in \Omega$.

In the followings, we will give a theorem concerning the existence of the (local) solution of the initial value problem (4.1). Let $a > 0$ and $b > 0$ be positive numbers such that the rectangle

$$\Delta = \{\,(t, x)|\ \ |t - t_0| \leq a \text{ and } |x - x_0| \leq b\}$$

is included in Ω ; $\Delta \subset \Omega$.

Theorem 4.1.1 (The Cauchy- Lipschitz theorem of existence of a local solution). *If f is continuous on the rectangle Δ and it is a Lipschitz function with respect to variable x on Δ, then the initial value problem (4.1) has a local solution defined on the interval $I_h = [t_0 - h,\ t_0 + h]$, where $h = \min\left\{a, \dfrac{b}{M}, \dfrac{1}{K+1}\right\}$, $M = \max\limits_{(t,x)\in\triangle} |f(t, x)|$ and K is a Lipschitz constant on Δ:*

$$|f(t, x) - f(t, y)| \leq K|x - y|, \quad \forall (t, x), (t, y) \in \triangle$$

Proof. Let $x_0(t) \equiv x_0$ be a constant function. Starting from this function, we will construct the sequence of functions $\{x_n(t)\}_{n\in\mathbb{N}}$ defined as follows:

$$\begin{array}{rcll} x_1(t) & = & x_0 + \int_{t_0}^{t} f(\tau, x_0(\tau))d\tau & \\ x_2(t) & = & x_0 + \int_{t_0}^{t} f(\tau, x_1(\tau))d\tau & \\ x_3(t) & = & x_0 + \int_{t_0}^{t} f(\tau, x_2(\tau))d\tau & (\forall)t \in I_h \\ & & \cdots\cdots\cdots & \\ x_n(t) & = & x_0 + \int_{t_0}^{t} f(\tau, x_{n-1}(\tau))d\tau & \\ & & \cdots\cdots\cdots & \end{array}$$

First, we will show that the functions of this sequence are well defined. This means to proof that, for any $n \geq 1$ and $t \in I_h$ we have $(t, x_n(t)) \in \Delta$. Using mathematical induction, we show that for any $t \in I_h$ we have $(t, x_n(t)) \in \Delta$.

Step I (verification):

For $n = 1$ we have

$$x_1(t) = x_0 + \int_{t_0}^{t} f(\tau, x_0(\tau))d\tau$$

and so, $|x_1(t) - x_0| \leq M|t - t_0| \leq Mh \leq b,\ (\forall)t \in I_h$.

This fact implies that $(t, x_1(t)) \in \Delta$ for any $t \in I_h$.

For $n = 2$ we have

$$x_2(t) = x_0 + \int_{t_0}^{t} f(\tau, x_1(\tau))d\tau$$

and so, $|x_2(t) - x_0| \leq M|t - t_0| \leq Mh \leq b,\ (\forall)t \in I_h$. From this, we have that $(t, x_2(t)) \in \Delta$ for any $t \in I_h$.

Step II (implication):

We suppose that $(t, x_n(t)) \in \Delta$, $(\forall)\ t \in I_h$ and we show that $(t, x_{n+1}(t)) \in \Delta$, $(\forall)\ t \in I_h$.

For this, we compute $x_{n+1}(t)$ and we find

$$x_{n+1}(t) = x_0 + \int_{t_0}^{t} f(\tau, x_n(\tau))d\tau$$

from which $|x_{n+1}(t) - x_0| \leq M|t - t_0| \leq Mh \leq b,\ (\forall)\ t \in I_h$. Hence, $(t, x_{n+1}(t)) \in \Delta$ for any $t \in I_h$.

We have shown that $(\forall)n \geq 1$ and $(\forall)t \in I_h$, $(t, x_n(t)) \in \Delta$.

Let us evaluate the maximum of the absolute values $|x_{n+1}(t) - x_n(t)|$ on I_h. For this assertion, we will take into account the equalities:

$$x_{n+1}(t) = x_0 + \int_{t_0}^{t} f(\tau, x_n(\tau))d\tau$$

$$x_n(t) = x_0 + \int_{t_0}^{t} f(\tau, x_{n-1}(\tau))d\tau$$

so we have:

$$|x_{n+1}(t) - x_n(t)| \leq |K \int_{t_0}^{t} |x_n(\tau) - x_{n-1}(\tau)|d\tau|$$

It follows that the inequality holds:

$$\max_{t\in I_h} |x_{n+1}(t) - x_n(t)| \leq \frac{K}{K+1} \max_{\tau \in I_h} |x_n(\tau) - x_{n-1}(\tau)|$$

from which:

$$\begin{aligned}\max_{t\in I_h} |x_{n+1}(t) - x_n(t)| &\leq \left(\frac{K}{K+1}\right)^n \cdot \max_{t\in I_h} |x_1(t) - x_0| \leq \\ &\leq \left(\frac{K}{K+1}\right)^n \cdot M \cdot h \leq \left(\frac{K}{K+1}\right)^n \cdot b\end{aligned}$$

Writing the function $x_n = x_n(t)$ in the form:

$$x_n(t) = x_0(t) + \sum_{i=0}^{n-1} (x_{i+1}(t) - x_i(t))$$

we see that the sequence $x_n(t)$ is the sequence of partial sums of the function series

$$x_0(t) + \sum_{n=0}^{\infty} (x_{n+1}(t) - x_n(t))$$

As $|x_{n+1}(t) - x_n(t)| \leq \left(\frac{K}{K+1}\right)^n \cdot b$, $(\forall)$ $t \in I_h$, the convergence of the series $b \cdot \sum_{n=0}^{\infty} \left(\frac{K}{K+1}\right)^n$ and the Weierstrass theorem implies that the function series $x_0(t) + \sum_{n=0}^{\infty} (x_{n+1}(t) - x_n(t))$ is absolutely and uniformly convergent on the interval I_h, to a function $x = x(t)$. It follows that the sequence of partial sums, i.e. the sequence $x_n(t)$ is uniformly convergent to the function $x(t)$.

In the followings, we take into account that $(\forall)$ $t \in I_h$ and we have that

$$\left| \int_{t_0}^{t} [f(s, x_n(s)) - f(s, x(s))]ds \right| \leq K \cdot h \cdot \max_{s\in I_h} |x_n(s) - x(s)|$$

and

$$\lim_{n\to\infty} \int_{t_0}^{t} f(s, x_n(s))ds = \int_{t_0}^{t} f(s, x(s))ds, \ (\forall) t \in I_h$$

Passing to the limit in the equality

$$x_{n+1}(t) = x_0 + \int_{t_0}^{t} f(\tau, x_n(\tau))d\tau$$

we obtain

$$x(t) = x_0 + \int_{t_0}^{t} f(\tau, x(\tau))d\tau$$

This shows that the function $x(t)$ is of class C^1 and verifies $\dot{x}(t) = f(t, x(t))$; $x(t_0) = x_0$.

In this way, we have proved that the initial value problem (4.1) has a solution defined on the interval I_h. The problem (4.1) might have a solution defined on a larger interval than I_h. This is the reason for which the solution that has been constructed in this theorem is called *local solution.* □

Theorem 4.1.2 (The Cauchy- Lipschitz theorem of uniqueness of the local solution). *If the conditions from the Cauchy-Lipschitz theorem of the existence of a local solution are satisfied for the initial value* (t_0, x_0)*, then the problem* (4.1) *cannot have two different solutions on the interval* J, $I_h \supset J \ni t_0$.

Proof. We suppose the contrary, i.e. that there are two functions $x, y : J \subset I_h \to \mathbb{R}^1$ which are solutions of the initial value problem (4.1). These solutions verify:

$$x(t) = x_0 + \int_{t_0}^{t} f(\tau, x(\tau))d\tau, \qquad y(t) = y_0 + \int_{t_0}^{t} f(\tau, y(\tau))d\tau$$

Hence, functions $x(t)$ and $y(t)$ verify the inequality:

$$|x(t) - y(t)| < \varepsilon + K \left| \int_{t_0}^{t} |x(\tau) - y(\tau)| d\tau \right| \quad (\forall) t \in J, \quad (\forall) \varepsilon > 0$$

From here, using the Gronwall lemma, we have:

$$|x(t) - y(t)| \leq \varepsilon \cdot e^{K|t-t_0|} \quad (\forall) t \in J, \quad (\forall) \varepsilon > 0$$

For a fixed $t \in J$, passing to the limit for $\varepsilon \to 0$, we obtain that $x(t) = y(t)$, $(\forall) t \in J$ □

Problem 4.1.1. It is known that the radioactive materia disintegrates and its disintegration velocity is proportional, at any moment, with the remaining quantity of radioactive materia. If $x(t)$ represents the remaining quantity of radioactive materia, then

$$\dot{x} = -a \cdot x$$

where a is a positive constant.
In the case of the disintegration of the radioactive carbon C^{14}, we have that $a = \dfrac{1}{8000}$ and hence, the equation becomes $\dot{x} = -\dfrac{1}{8000} \cdot x$.
Show that, if at the moment t_0, we know the quantity of the radioactive carbon C^{14} obtained from an animal or plant sample found in a geologic strait, then we can reconstitute the age of this sample.

Answer Let x_0 be a quantity of radioactive carbon C^{14} from a sample at the moment t_0.

The initial value problem:

$$\begin{cases} \dot{x} = -\dfrac{1}{8000} \cdot x \\ x(t_0) = x_0 \end{cases}$$

has a unique solution and this is given by

$$x(t; t_0, x_0) = x_0 e^{-\frac{1}{8000}(t-t_0)}$$

If x_1 is a normal value of the quantity of radioactive carbon C^{14} in a living animal or plant, then equating

$$x_1 = x_0 e^{-\frac{1}{8000}(t-t_0)}$$

we find an equation in t, which gives us the time period t during which the animal or plant was alive, and the difference $t - t_0$ shows the age of the sample.

Problem 4.1.2. A cylindrical tank has a circular hole at the base, from which the liquid from the tank can be drained. A question as in the Problem 4.1.1 is the following: if at a certain moment we see that the tank is empty, is it possible to know if the tank was full and when?
The answer is No. How do we explain?

Answer: Let be $x(t)$ the liquid height from the tank at the moment t. We denote by A the area of the cylinder base, and by a the area of the hole. According to Toricelli's law we have:

$$\dot{x} = -\frac{a}{A}\sqrt{2g}\cdot\sqrt{x}$$

If I is the tank height, then $x = I$ will correspond to the situation when the tank is full and $x = 0$ will correspond to the situation when the tank is empty. If at $t = 0$ the tank is full, then $x(0) = I$ and we have:

$$2\sqrt{u}|_I^x = -\frac{a}{A}\sqrt{2g}\cdot t$$

from which, we obtain:

$$x(t) = \left(\sqrt{I} - \frac{a}{2A}\sqrt{2g}\cdot t\right)^2$$

The period of time during which the tank empties is $t_* = \dfrac{2A}{a}\sqrt{\dfrac{I}{2g}}$ and hence:

$$x(t) = \begin{cases} \left(\dfrac{a}{2A}\sqrt{2g}\cdot t_* - \dfrac{a}{2A}\cdot\sqrt{2g}\cdot t\right)^2 & \text{for} \quad 0 \le t \le t_* \\ 0 & \text{for} \quad t \ge t_* \end{cases}$$

represents the "emptying law" of the tank, supposing that it was full at the moment $t = 0$.

There is an infinity of solutions $x(t)$ of the equation

$$\dot{x} = -\frac{a}{A}\sqrt{2g}\cdot\sqrt{x}$$

which are equal to zero for $t = t_*$. These are given by:

$$x_\tau(t) = \begin{cases} \dfrac{2g\cdot a^2}{4A^2}(t_* - t - \tau)^2 & \text{for} \quad -\tau \le t \le t_* - \tau \\ 0 & \text{for} \quad t \ge t_* - \tau \end{cases}$$

The solution $x_\tau(t)$ represents the emptying law of the tank which was full at the moment $-\tau$. Moreover, the Cauchy problem

$$\begin{cases} \dot{x} = -\dfrac{a\sqrt{2g}}{A} \cdot \sqrt{x} \\ \\ x(t_*) = 0 \end{cases}$$

has as solution the null function $x(t) = 0$. This solution corresponds to the situation when the tank has never been full. In this way, we have obtained that the solutions of the Cauchy problem describe all the possible situations, meaning the non-uniqueness of the solution of the Cauchy problem.

Conclusions:

1. If the state variable of the physical or chemical process $x(t)$ is a solution of the initial value problem $\dot{x} = f(t,x)$, $x(t_0) = x_0$, then this state variable must be searched between the solutions of this Cauchy problem.

2. If the initial value problem has a unique solution, then, finding this solution implies that it will describe the evolution in time of the state variable.

3. If the initial value problem has many solutions, then by finding one of them, we cannot assert that this solution describes the evolution in time of the searched state variable. More precisely, we need more information in order to allow us to identify the searched state variable.

4. In the non-uniqueness case, the physical problem is not solved by finding one solution of the initial value problem.

4.2 Theorems of existence and uniqueness of first order systems of nonlinear differential equations

Definition 4.2.1. *A first order system of nonlinear differential differential equations is of the form:*

$$\begin{cases} \dot{x}_1 = f_1(t, x_1, x_2, ..., x_n) \\ \dot{x}_2 = f_2(t, x_1, x_2, ..., x_n) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \dot{x}_n = f_n(t, x_1, x_2, ..., x_n) \end{cases} \tag{4.2}$$

in which $x_1,\ x_2, ...,\ x_n$ are n unknown functions and $\dot{x}_1,\ \dot{x}_2, ...,\ \dot{x}_n$ are their derivatives.

In the system (4.2), $f_1,\ f_2, ...,\ f_n$ are n known real functions defined on $I \times D$; $I \subset \mathbb{R}^1$, where I is an open interval and $D \subset \mathbb{R}^n$ is a domain.

Definition 4.2.2. *The ordered system of n real C^1-functions $x_1,\ x_2, ...,\ x_n$ defined on the interval $J \subset I$ is a solution of the system (4.2) if $(t,\ x_1(t), ...,\ x_n(t)) \in I \times \Omega$, $(\forall)t \in J$*

and:

$$\begin{cases} \dfrac{dx_1}{dt} = f_1(t, x_1(t), x_2(t), ..., x_n(t)) \\ \dfrac{dx_2}{dt} = f_2(t, x_1(t), x_2(t), ..., x_n(t)) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \dfrac{dx_n}{dt} = f_n(t, x_1(t), x_2(t), ..., x_n(t)) \end{cases} \qquad (\forall)t \in J$$

Definition 4.2.3. *Being given $t_0 \in I$ and $(x_1^0, ..., x_n^0) \in D$, the problem of determining the solution $x_1(t), ..., x_n(t)$ of the system (4.2) which verifies $x_i(t_0) = x_i^0$ for $i = \overline{1, n}$, is called initial value problem or Cauchy problem.*

For the matrix representation of the system (4.2) we denote by $F : I \times D \to \mathbb{R}^n$ the matrix function

$$F(t, x_1, x_2, ..., x_n) = (f_1(t, x_1, ..., x_n), f_2(t, x_1, ..., x_n), ..., f_n(t, x_1, ..., x_n))^T$$

and by X the column matrix $(x_1, x_2, ..., x_n)^T$. With these notations the system (4.2) can be written in the matrix form:

$$\dot{X} = F(t, X) \tag{4.3}$$

In this problem, the derivation of the matrix function $X(t)$ means the derivation of the matrix elements.

Observation 4.2.1. The initial value problem (Cauchy problem) is written in the matrix form:

$$\begin{cases} \dot{X} = F(t, X) \\ X(t_0) = X^0 \end{cases} \tag{4.4}$$

and consists of determining the matrix function $X(t)$ which verifies (4.3) and initial condition $X(t_0) = X^0$.

Observation 4.2.2. A function $X : J \subset I \to \mathbb{R}^n$ of class C^1 is a solution of the problem (4.4) if $\dot{X} = F(t, X(t))$, $(\forall)t \in J$ and $X(t_0) = X^0$ (it is supposed that $t_0 \in J$).

Let us consider $a > 0$, $b > 0$ such that the cylinder Δ:

$$\Delta = \left\{(t, X) : |t - t_0| \leq a \text{ and } \| X - X^0 \| \leq b\right\}$$

is in the domain $I \times D$.

Theorem 4.2.1 (The Cauchy-Lipschitz theorem of existence of a local solution). *If the function F is continuous on Δ and it is a Lipschitz function with respect to X on Δ, then the initial value problem (4.4) has a local solution defined on $I_h = [t_0 - h, t_0 + h]$ where $h = \min\left\{a, \dfrac{b}{M}, \dfrac{1}{K+1}\right\}$, $M = \max\limits_{(t,x)\in\Delta} \|F(t, X)\|$ and K is a Lipschitz constant:*

$$\|F(t, X') - F(t, X'')\| \leq K \|X' - X''\|, \quad (\forall)\ (t, X'), (t, X'') \in \Delta$$

Proof. We construct the following sequence of functions:

$$\begin{aligned}
X^0(t) &= X^0 \\
X^1(t) &= X^0 + \int_{t_0}^{t} F(\tau, X^0(\tau))d\tau \\
X^2(t) &= X^0 + \int_{t_0}^{t} F(\tau, X^1(\tau))d\tau \\
&\ldots\ldots\ldots\ldots\ldots \\
X^{k+1}(t) &= X^0 + \int_{t_0}^{t} F(\tau, X^k(\tau))d\tau \\
&\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

The functions of this sequence are well defined because for any $t \in I_h$ and $k \in I\!N$, $(t, X^k(t)) \in \Delta$ takes place (proof by induction). Following the same steps as in the previous section we evaluate the difference $\max\limits_{t\in I_h} \|X^{k+1}(t) - X^k(t)\|$ and we find:

$$\max_{t\in I_h} \|X^{k+1}(t) - X^k(t)\| \leq \frac{K}{K+1} \max_{t\in I_h} \|X^k(t) - X^{k-1}(t)\|$$

from which we deduce the inequality

$$\max_{t\in I_h} \|X^{k+1}(t) - X^k(t)\| \leq \left(\frac{K}{K+1}\right)^k \cdot b$$

Writing the function $X^k(t)$ in the form:

$$X^k(t) = X_0(t) + \sum_{i=0}^{k-1} \left(X^{i+1}(t) - X^i(t)\right)$$

we remark that, the sequence $\left\{X^k(t)\right\}_{k\in I\!N}$ is the sequence of partial sums of the function series

$$X^0(t) + \sum_{i=0}^{\infty} \left[X^{i+1}(t) - X^i(t)\right]$$

As

$$\|X^{k+1}(t) - X^k(t)\| \leq \left(\frac{K}{K+1}\right)^k \cdot b \qquad (\forall)t \in I_h$$

and as the series $b \cdot \sum\limits_{k=0}^{\infty} \left(\dfrac{K}{K+1}\right)^k$ is convergent, using the Weierstrass theorem, we have that the series

$$X^0(t) + \sum_{k=0}^{\infty} \left[X^{k+1}(t) - X^k(t)\right]$$

is absolutely and uniformly convergent on I_h, to a function $X(t)$. Thus, the sequence of partial sums $X^k(t)$ is uniformly convergent to the function $X(t)$.

Inequality:

$$\left\| \int_{t_0}^{t} [F(\tau, X^k(\tau)) - F(\tau, X(\tau))]d\tau \right\| \leq K \cdot h \cdot \max_{\tau\in I_h} \left\|X^k(\tau) - X(\tau)\right\|$$

for any $t \in I_h$ and $k \in \mathbb{N}$ allows us to obtain the equality:

$$\lim_{k\to\infty} \int_{t_0}^{t} F(\tau, X^k(\tau))d\tau = \int_{t_0}^{t} F(\tau, X(\tau))d\tau$$

Passing to the limit in equality:

$$X^{k+1}(t) = X^0 + \int_{t_0}^{t} F(\tau, X^k(\tau))d\tau$$

we obtain:

$$X(t) = X^0 + \int_{t_0}^{t} F(\tau, X(\tau))d\tau$$

This shows that the function $X(t)$ is of class C^1 and verifies

$$\begin{cases} \dot{X}(t) = F(t, X(t)) \\ X(t_0) = X^0 \end{cases}$$

Consequently, the initial value problem (4.4) has a solution defined on the interval I_h. The problem (4.4) might have a solution defined on the interval J larger than I_h. This is the reason for which the solution constructed in this theorem is called *local solution.* □

Theorem 4.2.2 (The Cauchy- Lipschitz theorem of uniqueness of the local solution). *If the hypothesis from the Cauchy-Lipschitz theorem of the existence of a local solution for (t_0, X^0) are satisfied, then the problem (4.4) cannot have two different solutions on the interval $J \subset I_h$, $t_0 \in J$.*

Proof. We suppose the contrary, that there are two functions

$$X', X'' : J \subset I_h \to \mathbb{R}^n \ (t_0 \in J)$$

which are local solutions of the initial value problem (4.4). These solutions verify:

$$X'(t) = X^0 + \int_{t_0}^{t} F(\tau, X'(\tau))d\tau, \quad X''(t) = X^0 + \int_{t_0}^{t} F(\tau, X''(\tau))d\tau$$

From here, we have that $X'(t)$, $X''(t)$ satisfy the inequality:

$$\|X'(t) - X''(t)\| < \varepsilon + K \left| \int_{t_0}^{t} \|X'(\tau) - X''(\tau)\| d\tau \right| \quad (\forall)\ t \in J,\ (\forall)\ \varepsilon > 0$$

and hence:

$$\|X'(t) - X''(t)\| < \varepsilon e^{K|t-t_0|} \quad (\forall) t \in J, \quad (\forall)\varepsilon > 0$$

Passing to the limit for $\varepsilon \to 0$ we obtain $X'(t) = X''(t)$, $(\forall) t \in J$. □

4.3 Qualitative properties of the solutions

Let us consider $I \subset \mathbb{R}^1$ an open interval, $D \subset \mathbb{R}^n$ a domain $F : I \times D \to \mathbb{R}^n$ a vector function and the first order system of differential equation written in matrix form:

$$\dot{X} = F(t, X) \tag{4.5}$$

Let be $X^1 : J_1 \subset I \to D$ and $X^2 : J_2 \subset I \to D$ two local solutions of the system (4.5).

Definition 4.3.1. *The local solution X^2 is said to be a prolongation of the local solution X^1 and we denote by $X^1 \leq X^2$, if $J_1 \subset J_2$ and $X^1(t) = X^2(t)$ for any $t \in J_1$.*

The binary relation $X^1 \leq X^2$ introduced in the set of local solutions of the system (4.5) is a relation of partial order.

Definition 4.3.2. *Any local solution of the system (4.5) which is a maximal element of the set of local solutions of (4.5), i.e. cannot be prolonged, is called saturated solution.*

Theorem 4.3.1. *If the function $F : I \times D \to \mathbb{R}^n$ is of class $\mathcal{C}^1$ on the domain $\Omega = I \times D$ and $(t_0, X^0) \in I \times D$, then the initial value problem:*

$$\begin{cases} \dot{X} = F(t, X) \\ \\ X(t_0) = X^0 \end{cases} \tag{4.6}$$

has a unique saturated solution $X(t; t_0, X^0)$.

Proof. We prove this assertion for the case $n = 1$. For $n \geq 2$, the proof is similar.

We consider the family of functions $\{x_\alpha\}_{\alpha \in \Lambda}$, $x_\alpha : I_\alpha \to \mathbb{R}^1$, $\quad t_0 \in I_\alpha$, of all local solutions of the problem:

$$\begin{cases} \dot{x} = f(t, x) \\ \\ x(t_0) = x_0 \end{cases} \tag{4.7}$$

The Cauchy-Lipschitz theorem of existence and uniqueness of the local solution of the initial value problem in the case of a first order differential equation, assures that this family of functions has at least one element.

We consider the open interval $I = \bigcup\limits_{\alpha \in \Lambda} I_\alpha$, and the function $x = x(t)$ defined on I.

For $t \in I$ we consider $\alpha \in \Lambda$, such that $t \in I_\alpha$ and we define $x(t) = x_\alpha(t)$.

This definition is correct if, for any $t \in I_{\alpha'} \cap I_{\alpha''}$, we have $x_{\alpha'}(t) = x_{\alpha''}(t)$. We analyze this implication for $t > t_0$ (the case for $t < t_0$ is similar).

We suppose the contrary, that there exists $t_1 > t_0, \quad t_1 \in I_{\alpha'} \cap I_{\alpha''}$, such that $x_{\alpha'}(t_1) \neq x_{\alpha''}(t_1)$. Considering

$$t_* = \inf\{t_1 : t_1 > t_0, t_1 \in I_{\alpha'} \cap I_{\alpha''}, x_{\alpha'}(t_1) \neq x_{\alpha''}(t_1)\}$$

we see that

$$x_{\alpha'}(t_*) = x_{\alpha''}(t_*) = x_*$$

The point (t_*, x_*) belongs to Ω, $(t_*, x_*) \in \Omega$, and we can consider the positive constants a_*, b_*, K_* such that:

- the rectangle $\Delta_* = \{(t,x) : \ |t - t_*| \le a_*, |x - x_*| \le b_*\}$ is in Ω ($\Delta_* \subset \Omega$);
- the interval $[t_*, t_* + a_*]$ is in the intersection $I_{\alpha'} \cap I_{\alpha''}$;
- for any $t \in [t_*, t_* + a_*]$ we have:

$$|x_{\alpha'}(t) - x_*| \le b_* \ \text{ and } \quad |x_{\alpha''}(t) - x_*| \le b_*;$$

- for any $(t,x), (t,y) \in \Delta_*$:

$$|f(t,x) - f(t,y)| \le K_* |x - y|$$

Let be $t_1 \in [t_*, t_* + a_*]$ such that $x_{\alpha'}(t_1) \neq x_{\alpha''}(t_1)$. From the definition of t_* it results that t_1 can be chosen close to t_*.

For a fixed t_1, let be $\varepsilon > 0$ such that $\varepsilon < |x_{\alpha'}(t_1) - x_{\alpha''}(t_1)| e^{-K_* \cdot a_*}$.

On the other hand, for any $t \in [t_*, t_* + a_*]$ we have:

$$x_{\alpha'}(t) = x_* + \int_{t_*}^{t} f(s, x_{\alpha'}(s)) ds$$

$$x_{\alpha''}(t) = x_* + \int_{t_*}^{t} f(s, x_{\alpha''}(s)) ds$$

and hence, the following inequalities hold:

$$|x_{\alpha'}(t) - x_{\alpha''}(t)| \le \int_{t_*}^{t} |f(s, x_{\alpha'}(s)) - f(s, x_{\alpha''}(s))| ds \le$$

$$\le K_* \int_{t_*}^{t} |x_{\alpha'}(s) - x_{\alpha''}(s)| ds < \varepsilon + K_* \int_{t_*}^{t} |x_{\alpha'}(s) - x_{\alpha''}(s))| \ ds$$

Thus, we obtain:

$$|x_{\alpha'}(t) - x_{\alpha''}(t)| < \varepsilon e^{K_* a_*}, \quad (\forall) \ t \in [t_*, t_* + a_*]$$

and from the way in which ε has been chosen, we have:

$$|x_{\alpha'}(t) - x_{\alpha''}(t)| < |x_{\alpha'}(t_1) - x_{\alpha''}(t_1)|, \quad (\forall) \ t \in [t_*, t_* + a_*]$$

which is a contradiction.

Thus, we have obtained that the function $x = x(t)$ is well defined on the interval I.

In the followings, we show that the function $x = x(t)$ is a solution of the initial value problem (4.7).

Because $x_\alpha(t_0) = x_0$ for any $\alpha \in \Lambda$ we have $x(t_0) = x_0$.

Let be $t_1 \in I$ and $\alpha_1 \in \Lambda$ such that $t_1 \in I_{\alpha_1}$. For any $t \in I_{\alpha_1}$, we have $x(t) = x_{\alpha_1}(t)$ and it follows that

$$\dot{x}(t) = \dot{x}_{\alpha_1}(t) = f(t, x_{\alpha_1}(t)) = f(t, x(t)), \quad (\forall) t \in I_{\alpha_1}$$

In particular, for $t = t_1$ we have

$$\dot{x}(t_1) = f(t, x(t_1))$$

We have obtained in this way that the function $x = x(t)$ is a solution of the initial value problem (4.7).

We still have to show that the function $x = x(t)$ defined on I is a saturated solution. Let be $y : J \to \mathbb{R}^1$ (J an open interval, $t_0 \in J$) a local solution of the initial value problem (4.7).

It's easy to see that the function y is in the family $\{x_\alpha\}_{\alpha\in\Lambda}$ and it follows that

$$J \subset I = \bigcup_{\alpha\in\Lambda} I_\alpha \text{ and } x(t) = y(t)$$

Hence, we have proved the existence and the uniqueness of the saturated solution of the initial value problem (4.7). This saturated solution will be denoted by $x = x(t; t_0, x_0)$ and the open interval on which it is defined will be denoted by I_0. $\square$

Assuming that the hypothesis of the previous theorem hold, we consider the saturated solution $X(t; t_0, X^0)$ of the Cauchy problem (4.6) defined on the open interval $I_0 = (\alpha_0, \beta_0) \subset I$.

Theorem 4.3.2. *For any $t_1 \in I_0$ the saturated solution $X(t;t_1,X(t_1;t_0,X^0))$ of the initial value problem*

$$\dot{X} = F(t, X), \quad X(t_1) = X(t_1; t_0, X^0) = X^1 \tag{4.8}$$

coincides with the saturated solution $X(t; t_0, X^0)$.

Proof. We prove this assertion for $n = 1$. In a similar way, the case $n \geq 2$ can be proved.

We denote by $I_1 = (\alpha_1, \beta_1)$ the interval of definition of the saturated solution of the initial value problem:

$$\dot{x} = f(t, x), \quad x(t_1) = x(t_1; t_0, x_0) = x_1 \tag{4.9}$$

Because $x(t_1; t_0, x_0) = x_1$, the function $x = x(t; t_0, x_0)$ is a local solution of the initial value problem (4.9).

We have that $I_0 \subset I_1$ and $x(t; t_0, x_0) = x(t; t_1, x_1)$, $(\forall)\ t \in I_0$.

On the other hand, because $x(t; t_0, x_0)$ is a saturated solution, we obtain that $I_0 \supset I_1$ and $x(t; t_0, x_0) = x(t; t_1, x_1)$. $\square$

Theorem 4.3.3. *If the following conditions are satisfied:*

(i) $\beta_0 < +\infty$ *($\alpha_0 > -\infty$, respectively)*

(ii) the sequence $\{t_n\}_n$ *from* I_0 *converges to* β_0 *(*α_0*, respectively)*

(iii) the sequence $\{X(t_n; t_0, X^0)\}_n$ *is convergent to a vector* Λ

then the point (β_0, Λ) *(*(α_0, Λ)*, respectively) is on the border of the domain* $\Omega = I \times D$*;* $(\beta_0, \Lambda) \in \partial\Omega$ *(*$(\alpha_0, \Lambda) \in \partial\Omega$*, respectively).*

Proof. We prove the above assertion for $n = 1$ (the proof in the case $n \geq 2$ is similar).

For any $t \in I_0$ the point $(t, x(t; t_0, x_0))$ is in the domain Ω and hence, the point (β_0, λ) is in the closure of the domain Ω, $(\beta_0, \lambda) \in \overline{\Omega}$. We show that (β_0, λ) is on the border $\partial\Omega$, showing that (β_0, λ) is not in Ω.

We suppose the contrary, namely that $(\beta_0, \lambda) \in \Omega$.

We consider $a > 0, b > 0, K > 0, M > 0$ such that the rectangle

$$\Delta = \{(t, x) : |t - \beta_0| \leq a, |x - \lambda| \leq b\}$$

is included in the domain Ω $(\Delta \subset \Omega)$, the function f verifies $|f(t, x)| \leq M$ for any $(t, x) \in \Delta$, and $|f(t, x) - f(t, y)| \leq K|x - y|$ for any $(t, x), (t, y) \in \Delta$.

Let be $\varepsilon > 0$ such that $\varepsilon < \min\left\{a - 2\varepsilon, \dfrac{b - 2\varepsilon}{M}, \dfrac{1}{K+1}\right\}$ and $t_1 < \beta_0$ such that $|t_1 - \beta_0| < \varepsilon$ and $|x(t_1; t_0, x_0) - \lambda| < \varepsilon$.

The saturated solution $x = x(t; t_1, x(t_1; t_0, x_0))$ of the initial value problem

$$\dot{x} = f(t, x), \quad x(t_1) = x(t_1; t_0, x_0)$$

is defined at least on the interval

$$I_\delta = [t_1 - \delta, t_1 + \delta] \quad \text{with} \quad \delta = \min\left\{a - 2\varepsilon, \frac{b - 2\varepsilon}{M}, \frac{1}{K+1}\right\}$$

and verifies

$$x(t; t_1, x(t_1; t_0, x_0)) = x(t; t_0, x_0), \quad (\forall)\ t \in I_\delta \cap I_0$$

As $t_1 + \delta > t_1 + \varepsilon > \beta_0$, we obtain that the saturated solution $x(t; t_0, x_0)$ admits a prolongation, which is absurd. $\square$

Theorem 4.3.4. *If* $I = \mathbb{R}^1$*,* $D = \mathbb{R}^n$ *and* $\beta_0 < +\infty$ *(*$\alpha_0 > -\infty$*, respectively), then the saturated solution* $X(t; t_0, X^0)$ *is unbounded on* $[t_0, \beta_0)$ *(*$(\alpha_0, t_0]$*, respectively).*

Proof. For the case $n = 1$, we suppose the contrary, that the saturated solution $x = x(t; t_0, x_0)$ is bounded on $[t_0, \beta_0)$. We consider $m > 0$ such that $|x(t; t_0, x_0)| \leq m$, $(\forall)\ t \in [t_0, \beta_0)$ and $t_n \in [t_0, \beta_0)$ such that $\lim\limits_{n\to\infty} t_n = \beta_0$. The sequence $\{x(t_n; t_0, x_0)\}_n$ is bounded and has a convergent subsequence with the limit λ. Because $\Omega = \mathbb{R}^n$ the point (β_0, λ) belongs to Ω, which is contrary to statement of the previous theorem.

The cases $n \geq 2$ are similar. $\square$

Theorem 4.3.5. *If $I = \mathbb{R}^1$, $D = \mathbb{R}^n$ and F is a Lipschitz function on any band of the form $\Delta = J \times \mathbb{R}^n$, where $J \subset \mathbb{R}^1$ is an arbitrary compact interval, then any saturated solution is defined on $\mathbb{R}^1$.*

Proof. For the case $n = 1$, we suppose the contrary, that the interval of definition $I_0 = (\alpha_0, \beta_0)$ of the saturated solution $x = x(t; t_0, x_0)$ is righthand-side bounded: $\beta_0 < +\infty$.

For $t \in [t_0, \beta_0)$ we write the inequalities:

$$|x(t;t_0,x_0) - x_0| \le \int_{t_0}^{t} |f(s, x(s : t_0, x_0)) - f(s, x_0)| ds + \int_{t_0}^{t} |f(s, x_0)| ds \le$$
$$\le K_{\beta_0} \int_{t_0}^{t} |x(s; t_0, x_0) - x_0| ds + (\beta_0 - t_0) \sup_{s \in [t_0, \beta_0]} |f(s, x_0)|.$$

From here, we have that for any $t \in [t_0, \beta_0]$ the following holds:

$$|x(t;t_0,x_0) - x_0| \le (\beta_0 - t_0) \cdot \sup_{s \in [t_0, \beta_0]} |f(s, x_0)| \cdot e^{K_{\beta_0}(\beta_0 - t_0)}$$

This inequality shows that the function $x(t; t_0, x_0)$ is bounded on the interval $[t_0, \beta_0)$, which is contrary to the conclusion of the previous theorem.

The case $n \geq 2$ can be proved similarly. $\square$

Consequence 4.3.1. *If $I = (a, b)$, $D = \mathbb{R}^n$ and $\beta_0 < b$ ($\alpha_0 > a$, respectively), then the saturated solution is unbounded on the interval $[t_0, \beta_0)$ ($(\alpha_0, t_0]$, respectively).*

Consequence 4.3.2. *If $I = (a, b)$, $D = \mathbb{R}^n$ and F is a Lipschitz function with respect to X on any band on the form $J \times I$, where $J \subset \mathbb{R}^1$ is a compact interval included in I, then any saturated solution is defined on I.*

Theorem 4.3.6. *If the C^1-function $F : \mathbb{R}^1 \times \mathbb{R}^n \to \mathbb{R}^n$ does not depend on t, then for any $X^0 \in \mathbb{R}^n$ and $t_1, t_2 \in \mathbb{R}^1$, we have:*

$$X(t_1 + t_2; 0, X^0) = X(t_2; 0, X(t_1; 0, X^0)) = X(t_1; 0, X(t_2; 0, X^0))$$

Proof. We prove the statement for $n = 1$ (the proof for $n \geq 2$ is similar).

Let us observe that for $t_2 = 0$ the following holds:

$$x(t_1 + t_2; 0, x_0) = x(t_2; 0, x(t_1; 0, x_0)) = x(t_1; 0, x(t_2; 0, x_0))$$

From the equality

$$\frac{d}{dt} x(t + t_1; 0, x_0) = f(x(t + t_1; 0, x_0))$$

we deduce that $x(t + t_1; 0, x_0)$ is a saturated solution of the initial value problem

$$\dot{x} = f(x), \quad x(0) = x(t_1; 0, x_0)$$

Hence, we obtain the equality:

$$x(t + t_1; 0, x_0) = x(t; 0, x(t_1 : 0, x_0) \qquad (\forall) t \in \mathbb{R}^1$$

In particular, for $t = t_2$, we obtain the first equality of the theorem. $\square$

Let be $I \subset \mathbb{R}^1$ a real open interval, Ω an open domain in $\mathbb{R}^n$ ($\Omega \subset \mathbb{R}^n$) and $F : I \times \Omega \to \mathbb{R}^n$, $F = F(t, X)$ a C^1-function. We consider in the followings the initial condition $(t_0,\ X^0) \in I \times \Omega$ and the maximal solution $X = X(t; t_0, X^0)$ of the Cauchy problem

$$\dot{X} = F(t, X), \qquad X(t_0) = X^0$$

We denote by I_0 the interval of definition of the maximal solution $X = X(t; t_0, X^0)$.

Theorem 4.3.7 (Continuous dependence on the initial state X^0). *For any compact $I_* = [T_1, T_2]$ included in $I_0 (I_* \subset I_0)$, which contains the point t_0 ($t_0 \in \overset{\circ}{I}_*$) and for any $\varepsilon > 0$, there is $\delta = \delta(\varepsilon, I_*)$ such that for any X^1 with $\|X^1 - X^0\| < \delta$, the saturated solution $X = X(t; t_0, X^1)$ of the Cauchy problem*

$$\begin{cases} \dot{X} = F(t, X) \\ \\ X(t_0) = X^1 \end{cases}$$

is defined at least on the interval I_* and verifies the inequality:

$$\|X(t; t_0, X^1) - X(t; t_0, X^0)\| < \varepsilon, \quad (\forall)\ t \in I_*$$

Proof. For $t \in I_*$, let be $a_t > 0$ and $b_t > 0$ such that the cylinder $\Delta_t = \{(\tau, X) : |\tau - t| \leq a_t$ and $\|X - X(t; t_0, X^0)\| \leq b_t\}$ is included in the set $I \times \Omega$ ($\Delta_t \subset I \times \Omega$).

The set Γ defined by $\Gamma = \{(t, X(t; t_0, X^0)) : t \in I_*\}$ is compact and it is included in the set $\bigcup\limits_{t \in I_*} \overset{\circ}{\Delta}_t; \Gamma \subset \bigcup\limits_{t \in I_*} \overset{\circ}{\Delta}_t$. It follows that there exists a finite number of points $t_1, t_2, \ldots, t_q$ from I_*, such that $\Gamma \subset \bigcup\limits_{j=1}^{q} \overset{\circ}{\Delta}_{tj}$.

We consider the function $d(Y, Z) = \|Y - Z\|$ defined for $Y \in \Gamma$ and Z on the border of the set $\bigcup\limits_{j=1}^{q} \Delta_{t_j}$, $Z \in \partial(\bigcup\limits_{j=1}^{q} \Delta_{t_j})$.

It exists $r > 0$, such that $d(Y, Z) > r$ for any $Y \in \Gamma$ and $Z \in \partial(\bigcup\limits_{j=1}^{q} \Delta_{t_j})$, the security tube Δ, defined as:

$$\Delta = \{(t, X) : t \in I_* \text{ and } \|X - X(t; t_0, X^0)\| \leq r\}$$

verifies the following inclusions:

$$\Delta \subset \bigcup_{j=1}^{q} \overset{\circ}{\Delta}_{t_j} \subset I \times \Omega$$

and there exists $K > 0$, such that for any $(t, X^1), (t, X^2) \in \Delta$ we have:

$$\|F(t, X^1) - F(t, X^2)\| \leq K \cdot \|X^1 - X^2\|$$

(a local Lipschitz function is a global Lipschitz function on the compact sets).

We denote by $h = \max\{T_2 - t_0, t_0 - T_1\}$ and we consider ε, $0 < \varepsilon < r$. Let be $\delta = \delta(\varepsilon, I_*) = \varepsilon \cdot 2^{-1} \cdot e^{-Kh}$ and X^1, such that $\|X^1 - X^0\| < \delta$. We denote by I_1 the interval of definition of the saturated solution $X = X(t; t_0, X^1)$ of the Cauchy problem:

$$\begin{cases} \dot{X} = F(t, X) \\ \\ X(t_0) = X^1 \ . \end{cases}$$

We show now, that for any $t \in I_* \cap I_1$ the following inequality holds:

$$\|X(t; t_0, X^1) - X(t; t_0, X^0)\| < \frac{\varepsilon}{2}$$

We suppose the contrary, that there exists $t_1 \in I_* \cap I_1$ such that $\|X(t_1; t_0, X^1) - X(t_1; t_0, X^0)\| \geq \frac{\varepsilon}{2}$. From here, we have that for at least one number α_1 or α_2, defined by

$$\alpha_1 = \inf \left\{ t \in I_* \cap I_1 : \|X(\tau; t_0, X^1) - X(\tau; t_0, X^0)\| < \frac{\varepsilon}{2},\ (\forall)\ \tau \in [t, t_0] \right\},$$

$$\alpha_2 = \sup \left\{ t \in I_* \cap I_1 : \|X(\tau; t_0, X^1) - X(\tau; t_0, X^0)\| < \frac{\varepsilon}{2},\ (\forall)\ \tau \in [t_0, t] \right\},$$

the following equality takes place:

$$\|X(\alpha_i; t_0, X^1) - X(\alpha_i; t_0, X^0)\| = \frac{\varepsilon}{2}, \quad i = 1, 2$$

For example, we suppose that $\|X(\alpha_2; t_0, X^1) - X(\alpha_2; t_0, X^0)\| = \frac{\varepsilon}{2}$.

On the other hand, for any $t \in [t_0, \alpha_2]$ we have:

$$\|X(t; t_0, X^1) - X(t; t_0, X^0)\| \leq \|X^1 - X^0\| +$$

$$+K \cdot \int_{t_0}^{t} \|X(\tau; t_0, X^1) - X(\tau; t_0, X^0)\| d\tau \leq \|X^1 - X^0\| \cdot e^{K \cdot h} < \frac{\varepsilon}{2}$$

which is a contradiction. It follows that:

$$\|X(t; t_0, X^1) - X(t; t_0, X^0)\| < \frac{\varepsilon}{2} \qquad (\forall) t \in I_* \cap I_1$$

We will show in the followings that $\alpha = \inf I_1 \leq T_1$ and $\beta = \sup I_1 \geq T_2$. We suppose contrary that $\beta < T_2$. For any $t \in [t_0, \beta)$ we have the inequalities:

$$\|X(t; t_0, X^1) - X(t; t_0, X^0)\| \leq \|X^1 - X^0\| +$$

$$+K \cdot \int_{t_0}^{t} \|X(\tau; t_0, X^1) - X(\tau; t_0, X^0)\| d\tau \leq \|X^1 - X^0\| \cdot e^{K \cdot h} < \frac{\varepsilon}{2}$$

Moreover, for any t', t'' with $t_0 < t' < t'' < \beta$ we have the inequality:

$$\|X(t'; t_0, X^1) - X(t''; t_0, X^1)\| \leq M \cdot |t'' - t'|$$

where $M = \sup\limits_{(t,X)\in\Delta} \|F(t,X)\|$.

This proves that there exists the limit:

$$\lambda = \lim_{t\to\beta} X(t;t_0,X^1)$$

and $(\beta,\lambda) \in \Delta$, which is a contradiction.

The established inequalities are available on I_* and hence, the theorem is proved. $\square$

Theorem 4.3.8 (Continuous dependence on the initial condition (t_0, X^0)**).** *For any compact interval $I_* = [T_1, T_2]$ included in the interval I_0 ($I_* \subset I_0$), which contains in the interior the point t_0 ($t_0 \in \overset{\circ}{I}_*$) and for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, I_*)$, such that for any initial condition (t_1, X^1) with $|t_1 - t_0| < \delta$ and $\|X^1 - X^0\| < \delta$, the saturated solution $X = X(t;t_1,X^1)$ of the Cauchy problem*

$$\begin{cases} \dot{X} = F(t,X) \\ X(t_1) = X^1 \end{cases}$$

is defined at least one the interval I_* and verifies the inequality:

$$\|X(t;t_1,X^1) - X(t;t_0,X^0)\| < \varepsilon,\ (\forall)\ t \in I_*$$

Proof. Let $r > 0$ and $K > 0$ be such that the tube Δ, defined by

$$\Delta = \{(t,X) : t \in I_*,\quad \|X - X(t;t_0,X^0)\| \le r\}$$

is included in the set $I \times \Omega$ ($\Delta \subset I \times \Omega$) and for any $(t,X^1), (t,X^2) \in \Delta$ we have

$$\|F(t,X^1) - F(t,X^2)\| \le K \cdot \|X^1 - X^2\|$$

We consider the numbers:

$$h_1 = \min\{T_2 - t_0, t_0 - T_1\};\quad h_2 = \max\{T_2 - t_0, t_0 - T_1\}$$

$$M = \sup_{(t,X)\in\Delta} \|F(t,X)\|$$

a number ε, $0 < \varepsilon < r$ and the number $\delta = \delta(\varepsilon, I_*)$, defined as:

$$\delta = 2^{-1} \cdot \min\{h_1, \varepsilon \cdot (M+1)^{-1} \cdot e^{-K(h_1+h_2)}\}$$

For (t_1, X^1) with $|t_1 - t_0| < \delta$ and $\|X^1 - X^0\| < \delta$ we have the inequalities:

$$T_1 < t_1 < T_2$$

$$\begin{aligned} \|X^1 - X(t_1;t_0,X^0)\| &\le \|X^1 - X^0\| + \|X^0 - X(t_1;t_0,X^0)\| < \\ &< \delta \cdot (M+1) < \frac{\varepsilon}{2} < r \end{aligned}$$

and hence $(t_1, X^1) \in \Delta \subset I \times \Omega$.

Let be $X = X(t; t_1, X^1)$ the maximal solution of the Cauchy problem:

$$\begin{cases} \dot{X} = F(t, X) \\ X(t_1) = X^1 \end{cases}$$

defined on I_1. We will show that $\|X(t; t_1, X^1) - X(t; t_0, X^0)\| < \dfrac{\varepsilon}{2}$ for any $t \in I_* \cap I_1$.

We suppose the contrary, that there exists $t_2 \in I_* \cap I_1$ such that $\|X(t_2; t_1, X^1) - X(t_2, t_0, X^0)\| \geq \dfrac{\varepsilon}{2}$. Form here, we obtain that for at least one number α_1, α_2 defined by

$$\alpha_1 = \inf \left\{ t \in I_* \cap I_1 : \|X(\tau; t_0, X^1) - X(\tau; t_0, X^0)\| < \frac{\varepsilon}{2},\ (\forall)\ \tau \in [t, t_1] \right\}$$

$$\alpha_2 = \sup \left\{ t \in I_* \cap I_1 : \|X(\tau; t_0, X^1) - X(\tau; t_0, X^0)\| < \frac{\varepsilon}{2},\ (\forall)\ \tau \in [t_1, t] \right\}$$

the following equality holds:

$$\|X(\alpha_i; t_1, X^1) - X(\alpha_i; t_0, X^0)\| = \frac{\varepsilon}{2}, \quad i = 1, 2$$

For example, we suppose that:

$$\|X(\alpha_2; t_1, X^1) - X(\alpha_2; t_0, X^0)\| = \frac{\varepsilon}{2}$$

On the other hand, for any $t \in [t_1, \alpha_2]$ we have the inequalities:

$$\|X(t; t_1, X^1) - X(t; t_0, X^0)\| \leq$$

$$\leq \|X^1 - X(t_1; t_0, X^0)\| + K \int_{t_1}^{t} \|X(\tau; t_1, X^1) - X(\tau; t_0, X^0)\| d\tau \leq$$

$$\leq \|X^1 - X(t_1; t_0, X^0)\| \cdot e^{K(h_1 + h_2)} < (M + 1) \cdot \delta \cdot e^{K(h_1 + h_2)} < \frac{\varepsilon}{2}$$

which is a contradiction. It follows that $\|X(t; t_1, X^1) - X(t; t_0, X^0)\| < \dfrac{\varepsilon}{2}$, $(\forall)\ t \in I_* \cap I_1$.

In the followings, we show that $\alpha = \inf\ I_1 \leq T_1$ and $\beta = \sup\ I_1 \geq T_2$.

We suppose the contrary, that $\beta < T_2$. For any $t \in [t_1, \beta)$ we have the inequality:

$$\|X(t; t_1, X^1) - X(t; t_0, X^0)\| < \frac{\varepsilon}{2}$$

and for any $t', t'' \in [t_1, \beta)$:

$$\|X(t'; t_1, X^1) - X(t''; t_1, X^1)\| \leq M \cdot |t' - t''|$$

This shows that exists $\lambda = \lim\limits_{t \to \beta} X(t; t_1, X^1)$ and $(\beta, \lambda) \in I \times \Omega$. Contradiction.

The established inequalities are available on I_* and hence the theorem is proved. $\square$

Let be $I \subset \mathbb{R}^1$ an open interval, $D \subset \mathbb{R}^n$ a domain, $\Omega \subset \mathbb{R}^m$ a domain, $F : I \times D \times \Omega \to \mathbb{R}^n$ a vector function and the first order, parameter dependent, system of differential equation written in the matrix form:

$$\dot{X} = F(t, X, \mu) \qquad t \in I,\ X \in D,\ \mu \in \Omega. \tag{4.10}$$

We consider a point $(t_0, X^0, \mu^0) \in I \times D \times \Omega$ and the Cauchy problem:

$$\begin{cases} \dot{X} = F(t, X, \mu^0) \\ X(t_0) = X^0 \end{cases} \tag{4.11}$$

We suppose that the function F is of C^1 class with respect to (t, X) and that it is continuous with respect to the parameter μ and we consider the saturated solution $X(t; t_0, X^0, \mu^0)$ of the Cauchy problem (4.11) defined ion the interval I_0.

Theorem 4.3.9 (continuous dependence on the parameter). *For any compact interval $I_* = [T_1, T_2] \subset I_0$, which contains the interior point t_0 ($t_0 \in \overset{\circ}{I}_*$) and for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, I_*) > 0$, such that if $\|\mu - \mu^0\| < \delta$, then the saturated solution $X = X(t; t_0, X^0, \mu)$ of the Cauchy problem*

$$\begin{cases} \dot{X} = F(t, X, \mu) \\ X(t_0) = X^0 \end{cases}$$

is defined on the interval I_* and verifies the inequality:

$$\|X(t; t_0, X^0, \mu) - X(t; t_0, X^0, \mu^0)\| < \varepsilon \qquad (\forall) t \in I_*$$

Proof. Let be $r_1 > 0$ and $r_2 > 0$ such that the sets Δ and $\overline{S}$ defined by:

$\Delta = \{(t, X) : t \in I_*, ||X - X(t; t_0, X^0, \mu^0)|| \le r_1\}$

$\overline{S} = \overline{S}(\mu^0, r_2) = \{\mu : ||\mu - \mu^0|| \le r_2\}$

verify $\Delta \subset I \times \Omega$ and $\overline{S} \subset \Omega_1$, respectively.

There exists $K > 0$ such that we have:

$$||F(t, X^1, \mu) - F(t, X^2, \mu)|| \le K \cdot ||X^1 - X^2||,\ (\forall)(t, X^1, \mu), (t, X^2, \mu) \in \Delta \times \overline{S}$$

We denote by $h = \max\{T_2 - t_0, t_0 - T_1\}$ and we consider a number ε, having the property $0 < \varepsilon < r$. Let be $\delta = \delta(\varepsilon, I_*)$ such that for $0 < \delta < r_2$ and $||\mu - \mu^0|| < \delta$ we have:

$$||F(t, X, \mu) - F(t, X, \mu_0)|| < \varepsilon \cdot 2^{-1} \cdot e^{-Kh}\ ,\ \ (\forall)\ (t, X) \in \Delta$$

For μ satisfying $||\mu - \mu^0|| < \delta$ we consider the saturated solution $X = X(t; t_0, X^0, \mu)$ of the Cauchy problem

$$\begin{cases} \dot{X} = F(t, X, \mu) \\ X(t_0) = X^0 \end{cases}$$

defined on I_μ. We will show that:

$$||X(t;t_0,X^0,\mu)-X(t;t_0,X^0,\mu^0|| < \frac{\varepsilon}{2} \qquad (\forall)t\in I_*\cap I_\mu$$

We suppose the contrary, that there exists $t_1 \in I_*\cap I_\mu$ such that

$$||X(t_1;t_0,X^0,\mu)-X(t_1;t_0,X^0,\mu^0)|| \geq \frac{\varepsilon}{2}$$

From here we obtain that, at least for one of the numbers α_1, α_2 defined by:

$$\alpha_1 = \inf\left\{t\in I_*\cap I_\mu : ||X(\tau;t_0,X^0,\mu)-X(\tau;t_0,X^0,\mu^0)||<\frac{\varepsilon}{2},\ (\forall)\tau\in[t,t_0]\right\}$$

$$\alpha_2 = \sup\left\{t\in I_*\cap I_\mu : ||X(\tau;t_0,X^0,\mu)-X(\tau;t_0,X^0,\mu^0)||<\frac{\varepsilon}{2},\ (\forall)\tau\in[t,t_0]\right\}$$

the following equality holds:

$$||X(\alpha_i;t_0,X^0,\mu)-X(\alpha_i;t_0,X^0,\mu^0)|| = \frac{\varepsilon}{2}\ ,\quad i=1,2$$

Let us admit, for example, that:

$$||X(\alpha_2;t_0,X^0,\mu)-X(\alpha_2;t_0,X^0,\mu^0)|| = \frac{\varepsilon}{2}$$

On the other hand, for any $t\in[t_0,\alpha_2]$ the following inequalities hold

$||X(t;t_0,X^0,\mu)-X(t;t_0,X^0,\mu^0)|| \leq$

$$\begin{aligned}
&\leq \int_{t_0}^{t} ||F(\tau,X(\tau;t_0,X^0,\mu),\mu)-F(\tau,X(\tau;t_0,X^0,\mu^0),\mu^0)||d\tau \leq\\
&\leq \int_{t_0}^{t} ||F(\tau,X(\tau;t_0,X^0,\mu),\mu)-F(\tau,X(\tau;t_0,X^0,\mu^0),\mu)||d\tau \leq\\
&\leq \int_{t_0}^{t} ||F(\tau,X(\tau;t_0,X^0,\mu^0),\mu)-F(\tau,X(\tau;t_0,X^0,\mu^0),\mu^0)||d\tau \leq\\
&\leq K\cdot\int_{t_0}^{t} ||X(\tau;t_0X^0,\mu)-X(\tau;t_0,X^0,\mu^0)||d\tau+\varepsilon\cdot 2^{-1}\cdot e^{-K\cdot h} \leq\\
&\leq \varepsilon\cdot 2^{-1}\cdot e^{-K\cdot h}\cdot e^{K(t-t_0)} < \frac{\varepsilon}{2}.
\end{aligned}$$

Contradiction. It follows:

$$||X(t;t_0,X^0,\mu)-X(t;t_0,X^0,\mu^0)|| < \frac{\varepsilon}{2},\ (\forall)t\in I_*\cap I_\mu$$

We show in the followings that $\alpha = \inf I_\mu \leq T_1$ and $\beta = \sup I_\mu \geq T_2$. We suppose the contrary, that $\beta < T_2$.

For any $t \in [t_0, \beta)$ the following inequality holds:

$$||X(t; t_0, X^0, \mu) - X(t; t_0, X^0, \mu^0)|| < \frac{\varepsilon}{2}$$

and for $t', t'' \in [t_0, \beta]$ we have:

$$||X(t'; t_0, X^0, \mu) - X(t''; t_0, X^0, \mu)|| < M \cdot |t' - t''|$$

with

$$M = \sup_{\Delta \times \overline{S}} ||F(t, X, \mu)||$$

We obtain form here that the limit $\lambda = \lim_{t \to \beta} X(t; t_0, X^0, \mu)$ exists and $(\beta, \lambda) \in I \times \Omega$. Contradiction.

The computations are available on I_* and hence the theorem is proved. □

Consequence 4.3.3. *If the function $F = F(t, X, \mu)$ is linear with respect to $X \in \mathbb{R}^n$, then for any interval I_* ($I_* \subset I$) which contains in the interior the point t_0 ($\mathring{I}_* \ni t_0$) and for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, I_*) > 0$ such that if $||\mu - \mu^0|| < \delta$, we have:*

$$||X(t; t_0, X^0, \mu) - X(t; t_0, X^0, \mu^0)|| < \varepsilon \qquad (\forall) t \in I_*$$

Proof. Using the Banach-Steinhaus theorem we obtain that the function $||F(t, \cdot, \mu)||$ is bounded on compacts and

$$\lim_{\mu \to \mu_0} F(t, X, \mu) = F(t, X, \mu^0)$$

□

Theorem 4.3.10 (Differentiability with respect to the initial conditions). *In the hypothesis of Theorem 4.3.7, the function $(t, t_1, X^1) \mapsto X(t; t_1, X^1)$ is differentiable with respect to t_1, X^1 and the following equalities hold:*

$$\begin{aligned}
\frac{d}{dt}\left(\partial_{X^1} X(t; t_0, X^0)\right) &= \partial_X F(t, X(t; t_0, X^0)) \cdot \partial_{X^1} X(t; t_0, X^0) \\
\partial_{X^1} X(t_0; t_0, X^0) &= I \\
\frac{d}{dt}\left(\partial_{t_1} X(t; t_0, X^0)\right) &= \partial_X F(t, X(t; t_0, X^0)) \cdot \partial_{t_1} X(t; t_0, X^0) \\
\partial_{t_1} X(t_0; t_0, X^0) &= -F(t_0, X^0) \\
\partial_{t_1} X(t; t_0, X^0) &= -\partial_{X^1} X(t; t_0, X^0) \cdot F(t_0, X^0)
\end{aligned}$$

Proof. For $t \in I_\delta$, $X^1 \in S(X^0, \delta/2)$, $h \in S(0, \delta/2)$ and $Y \in \mathbb{R}^n$ we consider the function:

$$H(t, t_0, X^1, h, Y) =$$
$$= \left\{ \int_0^1 \partial_X F(t, X(t; t_0, X^1) + s \cdot [X(t; t_0, X^1 + h) - X(t; t_0, X^1)]) ds \right\} \cdot Y$$

The function H defined in this way is continuous with respect to t and linear with respect to Y. Moreover, the function H is continuous with respect to (t, h).

Let be $e^1, e^2, \dots, e^n$ the canonical basis from $\mathbf{R}^n$ and $\xi \in \mathbf{R}$ such that $|\xi| < \delta/2$.

The Cauchy problems:

$$\begin{cases} \dot{Y}_\xi^k = H(t, t_0, X^1, \xi \cdot e^k, Y_\xi^k) \\ Y_\xi^k(t_0) = e^k \end{cases}$$

$$\begin{cases} \dot{Y}_\xi^k = H(t, t_0, X^1, 0, Y^k) \\ Y^k(t_0) = e^k \end{cases}$$

have solutions defined on I_δ for $k = 1, 2, \dots, n$. Moreover, for any compact interval $I_* \subset I_\delta$ we have $\lim\limits_{\xi \to 0} Y_\xi^k(t) = Y^k(t)$ uniformly with respect to $t \in I_*$.

On the other hand, for $\xi \neq 0$ we have:

$$Y_\xi^k(t) = \frac{1}{\xi}[X(t; t_0, X^1 + \xi \cdot e^k) - X(t; t_0, X^1)] \qquad (\forall) t \in I_\delta$$

It follows that the following limit exists

$$\lim_{\xi \to 0} \frac{1}{\xi}[X(t; t_0, X^1 + \xi \cdot e^k) - X(t; t_0, X^1)]$$

and it is equal to $Y^k(t)$ for $t \in I_\delta$.

This proves that the function $X(t; t_0, X^1)$ has partial derivatives with respect to X^1 at the points (t, t_0, X^1) and we have:

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial X}{\partial x_k^1}(t, t_0, X^1)\right) &= H\left(t, t_0, X^1, 0, \frac{\partial X}{\partial x_k^1}(t, t_0, X^1)\right) \\ \frac{\partial X}{\partial x_k^1}(t_0, t_0, X^1) &= e^k \end{aligned}$$

The function $H(t, t_0, X^1, 0, Y)$ is continuous with respect to (t, X^1) and linear with respect to Y, so we obtain that the solution of the previous Cauchy problem $\dfrac{\partial X}{\partial x_k^1}(t, t_0, X^1)$ converges uniformly to the solution of the Cauchy problem:

$$\begin{cases} \dfrac{dY^k}{dt} = H(t, t_0, X^0, 0, Y^k) \\ Y^k(t_0) = e^k \end{cases}$$

for $X^1 \to X^0$, on the compacts $I_1 \subset I_\delta$.

This implies that the partial derivatives of the function $X(t; t_0, X^1)$ with respect to X^1 are continuous functions with respect to X^1 at the points (t, t_0, X^0). Hence, the function

$X(t;t_1,X^1)$ is differentiable with respect to X^1 at the points (t,t_0,X^0) and satisfies:

$$\begin{aligned}\frac{d}{dt}\left(\partial_{X^1}X(t;t_0,X^0)\right) &= \partial_X F\left(t,X(t;t_0,X^0)\right)\cdot\partial_{X^1}X(t;t_0,X^0)\\ \partial_{X^1}X(t_0;t_0,X^0) &= I.\end{aligned}$$

Let be $\tau\in\mathbb{R}^1$ such that $0<|\tau|<\delta$ and the function

$$X_\tau(t)=\frac{1}{\tau}\cdot\left[X(t;t_0+\tau,X^0)-X(t;t_0,X^0)\right]\ \text{ for } t\in I_\delta$$

We have the equalities:

$$\begin{aligned}\tau\cdot X_\tau(t) &= X(t;t_0+\tau,X^0)-X(t;t_0,X^0)=\\ &= X(t;t_0,X(t_0;t_0+\tau,X^0))-X(t;t_0,X^0)=\\ &= \partial_{X^1}X(t;t_0,X^0)\cdot[X(t_0;t_0+\tau,X^0)-X^0]+\\ &\quad +O(||X(t_0;t_0+\tau,X^0)-X^0||)=\\ &= \partial_{X^1}X(t;t_0,X^0)\cdot[X(t_0;t_0+\tau,X^0)-X(t_0+\tau;t_0+\tau,X^0)]+\\ &\quad +O(||X(t_0;t_0+\tau,X^0)-X^0||)=\\ &= -\tau\partial_{X^1}X(t;t_0,X^0)\sum_{k=1}^{n}F_k(t_0+\theta_k\tau,X(t_0+\theta_k\tau;t_0+\tau,X^0)\cdot e^k+\\ &\quad +O(||X(t_0;t_0+\tau,X^0)-X^0||)\end{aligned}$$

with $0<\theta_k<1$ for $k=\overline{1,n}$.

Thus,

$$X_\tau(t)=-\partial_{X^1}X(t;t_0,X^0)\Big(\sum_{k=1}^{n}F_k(t_0+\theta_k\cdot\tau,X(t_0+\theta_k\tau;t_0+\tau,X^0))e^k\Big)+$$

$$+\frac{O(||X(t_0;t_0+\tau,X^0)-X^0||)}{||X(t_0;t_0+\tau,X^0)-X^0||}\cdot\frac{||X(t_0;t_0+\tau,X^0)-X(t_0+\tau;t_0+\tau,X^0)||}{\tau}$$

Because the ratio $\frac{1}{\tau}\cdot||X(t_0;t_0+\tau,X^0)-X(t_0+\tau,t_0+\tau,X^0)||$ is bounded for $\tau\to 0$ and $||X(t_0;t_0+\tau,X^0)-X^0||\to 0$ uniformly for $\tau\to 0$ on any compact interval $I_*\subset I_\delta(I_*\ni t_0)$ we obtain that

$$\lim_{\tau\to 0}X_\tau(t)=-\partial_{X^1}X(t;t_0,X^0)\cdot F(t_0,X^0)$$

and hence, the function $X(t;t_1,X^1)$ is differentiable with respect to t_1 at $(t;t_0,X^0)$. Moreover,

$$\partial_{t_1}X(t;t_0,X^0)=-\partial_{X^1}X(t;t_0,X^0)\cdot F(t_0,X^0)$$

The differentiability with respect to t of the function $\partial_{t_1}X(t;t_0,X^0)$ is a consequence of this equality.

Moreover, we have the equalities:

$$
\begin{aligned}
&\frac{d}{dt}\left(\partial_{t_1} X(t;t_0,X^0)\right) = \\
= & -\frac{d}{dt}\left(\partial_{X^1} X(t;t_0,X^0)\right) F(t_0,X^0) = \\
= & -\partial_X F(t,X(t;t_0,X^0)) \cdot (\partial_{X^1} X(t;t_0,X^0)) \cdot F(t_0,X^0) = \\
= & \ \partial_X F\left(t,X(t;t_0,X^0)\right) \cdot \partial_{t_1} X(t;t_0,X^0)
\end{aligned}
$$

$$\partial_{t_1} X(t_0;t_0,X^0) = -F(t_0,X^0).$$

In this way the theorem concerning the differentiability with respect to the initial conditions is proved. □

Theorem 4.3.11 (Differentiability with respect to the parameter). *If the function $F = F(t,X,\mu)$ satisfies the hypothesis from Theorem 4.3.9 and is of class C^1 with respect to μ, then the function*

$$(t;t_0,X^0,\mu) \to X(t;t_0,X^0,\mu)$$

is differentiable with respect to μ and the following equalities hold:

$$
\begin{aligned}
\frac{d}{dt}(\partial_\mu X(t;t_0,X^0,\mu^0)) &= \partial_X F(t,X(t;t_0,X^0,\mu^0),\mu^0)\cdot\partial_\mu X(t;t_0,X^0,\mu^0)+ \\
&+ \partial_\mu F(t;X(t;t_0,X^0,\mu^0),\mu^0)
\end{aligned}
$$

$$\partial_\mu X(t_0;t_0,X^0,\mu^0) = 0.$$

Proof. Let be $e^1,e^2,...,e^m$ the canonical basis in the $\mathbb{R}^m$ space.

For $t \in I_\delta$, $\mu^1 \in S(\mu^0,\frac{\delta}{2})$, $h \in \mathbb{R}^1$, $|h| < \frac{\delta}{2}$, $Y \in \mathbb{R}^n$ and $k = \overline{1,m}$ we define the function $H_k = H_k(t,t_0,X^0,\mu^1,h,Y)$ as:

$$
\begin{aligned}
&H_k(t,t_0,X^0,\mu^1,h,Y) = \\
= &\left\{\int_0^1 \partial_X F(t,X(t;t_0,X^0,\mu^1)+s\left[X(t;t_0,X^0,\mu^1+h\cdot e^k)-X(t;t_0,X^0,\mu^1)\right],\right. \\
&\left.\mu^1+h\cdot e^k)ds\right\}Y + \left[\int_0^1 \partial_\mu F(t,X(t;t_0,X^0,\mu^1),\mu^1+s\cdot h\cdot e^k)ds\right]\cdot e^k.
\end{aligned}
$$

The function H_k is continuous with respect to $t \in I_\delta$, and a Lipschitz function with respect to Y on the compacts $I_* \subset I$. Moreover, the function $H_k(t,t_0,X^0,\mu^1,h,Y)$ tends to $H_k(t,t_0,X^0,\mu^1,0,Y)$ for $h \to 0$ uniformly on compacts with respect to (t,Y).

The Cauchy problems:

$$
\begin{cases}
\dfrac{dY_\xi^k}{dt} = H_k(t,t_0,X^0,\mu^1,Y_h^k) \\[2ex]
Y_h^k(t_0) = 0
\end{cases}
$$

$$\begin{cases} \dfrac{dY^k}{dt} = H_k(t, t_0, X^0, \mu^1, Y^k) \\ \\ Y^k(t_0) = 0 \end{cases}$$

have solutions defined on I_δ and $\lim\limits_{h \to 0} Y_h^k(t) = Y^k(t)$ uniformly with respect to t for any interval $J \subset I$.

On the other hand, it easy to verify that for $h \neq 0$ we have:

$$Y_h^k(t) = \frac{1}{h}[X(t; t_0, X^0, \mu^1 + h \cdot e^k) - X(t; t_0, X^0, \mu^1)]$$

and we deduce that the function $X(t; t_0, X^0, \mu)$ has partial derivatives with respect to μ_k 'in (t, t_0, X^0, μ^1) and

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial X}{\partial \mu_k}(t, t_0, X^0, \mu^1)\right) &= H_k\left(t, t_0, X^0, \mu^1, 0, \frac{\partial X}{\partial \mu_k}(t, t_0, X^0, \mu^1)\right) \\ \frac{\partial X}{\partial \mu_k}(t_0, t_0, X^0, \mu^1) &= 0 \end{aligned}$$

On the other hand:

$$\lim_{\mu^1 \to \mu^0} H^k(t, t_0, X^0, \mu^1, 0, Y) = H^k(t, t_0, X^0, \mu^0, 0, Y)$$

uniformly with respect to (t, Y) on compact sets.

Hence, $\dfrac{\partial X}{\partial \mu_k}(t, t_0, X^0, \mu^1)$ tends uniformly to $\dfrac{\partial X}{\partial \mu_k}\left(t, t_0, X^0, \mu^0\right)$ for $\mu \to \mu^0$ on any compact interval $I_* \subset I$.

This proves that the partial derivatives with respect to μ_k of the function $X(t; t_0, X^0, \mu)$ are continuous with respect to μ at $(t; t_0, X^0, \mu^0)$.

Moreover, we have:

$$\frac{d}{dt}\left(\frac{\partial X}{\partial \mu_k}(t, t_0, X^0, \mu^0)\right) =$$

$$= \partial_X F\left(t, X(t; t_0, X^0, \mu^0), \mu^0\right) \cdot \frac{\partial X}{\partial \mu_k}\left(t; t_0, X^0, \mu^0\right) + \frac{\partial F}{\partial \mu_K}\left(t, X(t; t_0, X^0, \mu^0), \mu^0\right)$$

$$\frac{\partial X}{\partial \mu_k}\left(t_0; t_0, X^0, \mu^0\right) = 0$$

It follows that the function $X = X(t; t_0, X^0, \mu)$ is differentiable with respect to μ at the point (t, t_0, X^0, μ^0) and for any $t \in I_\delta$ we have:

$$\frac{d}{dt}\left(\partial_\mu X(t; t_0, X^0, \mu^0)\right) =$$

$$\partial_X F(t, X(t; t_0, X^0, \mu^0), \mu^0) \cdot \partial_\mu X(t; t_0, X^0, \mu^0) + \partial_\mu F(t, X(t; t_0, X^0, \mu^0), \mu^0)$$

$$\partial_\mu X(t_0; t_0, X^0, \mu^0) = 0. \qquad \square$$

4.4 Numerical methods

4.4.1 The Euler method of polygonal lines for numerical determination of the local solution in the case of first order systems of differential equations

Let I be an open real and non-empty interval $I \subset \mathbb{R}^1$, D a non-empty domain in $\mathbb{R}^n$, $D \subset \mathbb{R}^n$ and F a $\mathcal{C}^\infty$-function, $F : I \times D \to \mathbb{R}^n$.

For $(t_0, X^0) \in I \times D$ we consider the saturated solution $X = X(t; t_0, X^0)$ of the initial value problem

$$\dot{X} = F(t, X); \qquad X(t_0) = X^0 \tag{4.12}$$

and we denote by $I_0 = (\alpha_0, \beta_0) \subset I$ the interval on which the solution is defined.

The simplest of all numerical methods for determining the local and saturated solution $X = X(t; t_0, X^0)$ is the Euler method of a polygonal lines; however, it is computationally inefficient. Nevertheless, the study of this method is very instructive and serves as a basis for analyzing more accurate methods. In the followings, we will present this method.

We consider two positive constants a and b, $a > 0, b > 0$ such that the cylinder

$$\Delta = \{(t, X) | \; |t - t_0| \leq a \quad \text{and} \quad \|X - X^0\| \leq b\}$$

is included in the domain $\Omega = I \times D$; $\Delta \subset I \times D$.

We denote by M the maximum of the function F on the compact cylinder Δ :

$$M = \max_{(t,X)\in\Delta} \|F(t, X)\|$$

with α the positive number $\alpha = \min\left\{a, \dfrac{b}{M}\right\}$ and by I_α the interval $I_\alpha = [t_0 - \alpha,\; t_0 + \alpha]$.

We consider a real and positive number $\varepsilon > 0$ and we chose $\delta = \delta(\varepsilon)$ such that for any $(t', X'), (t'', X'') \in \Delta$ which satisfy the inequalities:

$$|t' - t''| < \delta(\varepsilon) \quad \text{and} \quad \|X' - X''\| < \delta(\varepsilon)$$

we have $\|F(t', X') - F(t'', X'')\| < \varepsilon$.

The function $F = F(t, X)$ is uniformly continuous on the compact cylinder Δ and hence, for any $\varepsilon > 0$, it is possible to chose $\delta(\varepsilon) > 0$ with the above property.

We consider a positive number $h > 0$, called *step*, chosen such that it satisfies the inequalities $0 < h < \dfrac{\delta}{M}$. Let be q the natural number with the properties:

$$t_q = t_0 + q \cdot h \leq t_0 + \alpha \quad \text{and} \quad t_{q+1} = t_0 + (q + 1) \cdot h > t_0 + \alpha$$

For $i = \overline{1, q}$ we consider the numbers $t_i = t_0 + ih$ and $t_{-i} = t_0 - ih$. These numbers define a division of the segment I_α.

$$t_0 - \alpha \leq t_{-q} < t_{-q+1} < \cdots < t_{-1} < t_0 < t_1 < \cdots < t_{q-1} < t_q \leq t_0 + \alpha$$

On the interval $[t_0, t_1]$ we define the function $X^1 = X^1(t)$ as

$$X^1(t) = X^0 + (t - t_0) \cdot F(t_0, X^0)$$

and on the interval $[t_{-1}, t_0]$ the function $X^{-1} = X^{-1}(t)$ as

$$X^{-1}(t) = X^0 + (t - t_0) \cdot F(t_0, X^0)$$

These functions satisfy the following inequalities:

$$\begin{array}{lll} \|X^1(t) - X^0\| \le b & \text{and} \quad \|\dot{X}^1(t) - F(t, X^1(t))\| < \varepsilon, & (\forall)\ t \in [t_0, t_1] \\ \|X^{-1}(t) - X^0\| \le b & \text{and} \quad \|\dot{X}^{-1}(t) - F(t, X^{-1}(t))\| < \varepsilon, & (\forall)\ t \in [t_1, t_0] \end{array}$$

In the followings, on the interval $[t_1, t_2]$ we define the function $X^2 = X^2(t)$ as

$$X^2(t) = X^1(t_1) + (t - t_1) \cdot F(t_1, X^1(t_1))$$

and on the interval $[t_{-2}, t_{-1}]$ the function $X^{-2} = X^{-2}(t)$ as

$$X^{-2}(t) = X^{-1}(t_{-1}) + (t - t_1) \cdot F(t_{-1}, X^{-1}(t_{-1}))$$

The functions $X^2(t)$ and $X^{-2}(t)$ defined in this way verify the following inequalities:

$$\begin{array}{lll} \|X^2(t) - X^0\| \le b & \text{and } \|\dot{X}^2(t) - F(t, X^2(t))\| < \varepsilon, & (\forall)\ t \in [t_1, t_2] \\ \|X^{-2}(t) - X^0\| \le b & \text{and } \|\dot{X}^{-2}(t) - F(t, X^{-2}(t))\| < \varepsilon, & (\forall)\ t \in [t_{-2}, t_{-1}] \end{array}$$

We suppose that, in this way, we can construct the functions $X^j = X^j(t)$ and $X^{-j} = X^{-j}(t)$, respectively, defined on the intervals $[t_{j-1}, t_j]$ and $[t_{-j}, t_{-j+1}]$, respectively, for $j = \overline{1, j_0}$, $(j_0 \le q - 1)$ and verifying the inequalities:

$$\begin{array}{lll} \|X^j(t) - X^0\| \le b & \text{and } \|\dot{X}^j(t) - F(t, X^j(t))\| < \varepsilon, & (\forall)\ t \in [t_{j-1}, t_j] \\ \|X^{-j}(t) - X^0\| \le b & \text{and } \|\dot{X}^{-j}(t) - F(t, X^{-j}(t))\| < \varepsilon, & (\forall)\ t \in [t_{-j}, t_{-j+1}] \end{array}$$

Defining in the followings the functions $X^{j_0+1}(t)$ and $X^{-j_0-1}(t)$, respectively, on the intervals $[t_{j_0}, t_{j_0+1}]$ and $[t_{-j_0-1}, t_{-j_0}]$, respectively, as:

$$\begin{array}{rcll} X^{j_0+1}(t) & = & X^{j_0}(t_{j_0}) & + (t - t_{j_0}) \cdot F(t_{j_0}, X^{j_0}(t_{j_0})) \\ X^{-j_0-1}(t) & = & X^{-j_0}(t_{-j_0}) & + (t - t_{-j_0}) \cdot F(t_{-j_0}, X^{-j_0}(t_{-j_0})) \end{array}$$

we can obtain easily that they verify the inequalities:

$$\begin{array}{lll} \|X^{j_0+1}(t) - X^0\| \le b & \text{and} \quad \|\dot{X}^{j_0+1}(t) - F(t, X^{j_0+1}(t)\| < \varepsilon & (\forall)\ t \in [t_{j_0}, t_{j_0+1}] \\ \|X^{-j_0-1}(t) - X^0\| \le b & \text{and} \quad \|\dot{X}^{-j_0-1}(t) - F(t, X^{-j_0-1}(t)\| < \varepsilon & (\forall)\ t \in [t_{-j_0-1}, t_{-j_0}] \end{array}$$

We obtain in this way that for any $i = \overline{1, q}$, the following formulas:

$$X^i(t) = X^{i-1}(t_{i-1}) + (t - t_{i-1}) \cdot F(t_{i-1}, X^{i-1}(t_{i-1})) \qquad (\forall) t \in [t_{i-1}, t_i]$$

$$X^{-i}(t) = X^{-i+1}(t_{-i+1}) + (t - t_{-i+1}) \cdot F(t_{-i+1}, X^{-i+1}(t_{-i+1})) \qquad (\forall) t \in [t_{-i}, t_{-i+1}]$$

define some functions which satisfy the inequalities:

$$\begin{array}{lll} \|X^i(t) - X^0\| \le b & \text{and } \|\dot{X}^i(t) - F(t, X^i(t))\| < \varepsilon, & (\forall)\ t \in [t_{i-1}, t_i] \\ \|X^{-i}(t) - X^0\| \le b & \text{and } \|\dot{X}^{-i}(t) - F(t, X^{-i}(t))\| < \varepsilon, & (\forall)\ t \in [t_{-i}, t_{-i+1}] \end{array}$$

We consider the function $X^\varepsilon(t)$ defined for $t \in I_\alpha$ as:

$X^\varepsilon(t) = X^i(t) \quad \text{if} \quad t \in [t_{i-1}, t_i]$

$X^\varepsilon(t) = X^{-i}(t) \quad \text{if} \quad t \in [t_{-i}, t_{-i+1}]$

$X^\varepsilon(t) = X^q(t_q) + (t - t_q) \cdot F(t_q, X^q(t_q)) \quad \text{if} \quad t \in [t_q, t_0 + \alpha]$

$X^\varepsilon(t) = X^{-q}(t_{-q}) + (t - t_{-q}) \cdot F(t_{-q}, X^{-q}(t_{-q})) \quad \text{if} \quad t \in [t_0 - \alpha, t_{-q}]$

The function $X^\varepsilon(t)$ defined in this way is continuous on the interval I_α,, derivable on this interval, except eventually at the points $\{t_i\}_{i=\overline{1,q}}$ and $\{t_{-i}\}_{i=\overline{1,q}}$ and verifies:

$$\|X^\varepsilon(t) - X^0\| \leq b \ \text{ and } \ \|\dot{X}^\varepsilon(t) - F(t, X^\varepsilon(t))\| < \varepsilon, \quad t \in I_\alpha$$

If we define the function $\theta^\varepsilon(t)$ as:

$$\theta^\varepsilon(t) = \begin{cases} X^\varepsilon(t) - F(t, X^\varepsilon(t)) & \text{for} \quad t \neq t_i, t_{-i}, \quad i = \overline{1,q} \\ 0, & \text{for} \quad t = t_i \ \text{ or } \ t_{-i}, \quad i = \overline{1,q} \end{cases}$$

then we have:

$$X^\varepsilon(t) = X^0 + \int_{t_0}^{t} F(\tau, X^\varepsilon_\tau))d\tau + \int_{t_0}^{t} \theta^\varepsilon(\tau)d\tau$$

for any $t \in I_\alpha$ with $\|\theta^\varepsilon(t)\| < \varepsilon$ for any $t \in I_\alpha$.

The inequality $\|X^\varepsilon(t) - X^0\| \leq b$, true for any $t \in I_\alpha$, shows that $\|X^\varepsilon(t)\| \leq b + \|X^0\|$, $(\forall)$ $t \in I_\alpha$. From here we have that the family of functions $\{X^\varepsilon(t)\}_{\varepsilon>0}$ is equally bounded on $I_\alpha = [t_0 - \alpha, t_0 + \alpha]$.

The equality:

$$X^\varepsilon(t) = X^0 + \int_{t_0}^{t} F(\tau, X^\varepsilon(\tau))d\tau + \int_{t_0}^{t} \theta^\varepsilon(\tau)d\tau, \qquad (\forall)\ t \in I_\alpha$$

together with the inequality:

$$\|\theta^\varepsilon(\tau)\| < \varepsilon. \ (\forall) \ \ \tau \in I_\alpha$$

imply:

$$\begin{aligned} \|X^\varepsilon(t_1) - X^\varepsilon(t_2)\| &\leq \int_{t_1}^{t_2} \|F(\tau, X^\varepsilon(\tau))\|d\tau + \int_{t_1}^{t_2} \|\theta^\varepsilon(\tau)\|d\tau \leq \\ &\leq M|t_2 - t_1| + \varepsilon|t_2 - t_1| \leq (M + \varepsilon)|t_2 - t_1| \end{aligned}$$

hence, it is proved that the functions $X^\varepsilon(t)$ are equicontinuous on I_α.

Using the Arzela-Ascoli theorem we obtain that there is a sequence $\varepsilon_n \to 0$, such that the sequence $\{X^{\varepsilon_n}\}_{\varepsilon_n}$ is uniformly convergent on the interval I_α to a continuous function X defined on the interval I_α which satisfies $\|X(t) - X^0\| \leq b$, for any $t \in I_\alpha$.

The uniform continuity of the function F on the cylinder Δ and uniform convergence of the functions sequence X^{ε_n} to the function X assure the uniform convergence of the sequence of functions $F(\tau, X^{\varepsilon_n}(\tau))$ to $F(\tau, X(\tau))$ on I_α.

Passing to the limit in the equality:

$$X^{\varepsilon_n}(t) = X^0 + \int_{t_0}^{t} F(\tau, X^{\varepsilon_n}(\tau))d\tau + \int_{t_0}^{t} \theta^{\varepsilon_n}(\tau)d\tau$$

we obtain that the function $X(t)$ verifies

$$X(t) = X^0 + \int_{t_0}^{t} F(\tau, X(\tau))d\tau, \qquad (\forall)\ t \in I_\alpha$$

This proves that the limit $X = X(t)$ is a solution of the initial value problem

$$\dot{X} = F(t, X), \qquad X(t_0) = X^0$$

From the uniqueness theorem we have that the function $X(t)$ coincides with the saturated solution $X(t; t_0, X_0)$ on the interval I_α:

$$X(t) = X(t; t_0, X^0), \qquad (\forall)\ \ t \in I_\alpha$$

We obtain in this way that the function $X^\varepsilon(t)$ approximates the non-prolonged solution $X(t; t_0, X^0)$ on the interval I_α.

The values of the function $X^\varepsilon(t)$ at the points t_i are obtained by recurrence:

$$X^\varepsilon(t_i) = X^\varepsilon(t_{i-1}) + h \cdot F(t_{i-1}, X^\varepsilon(t_{i-1})), \quad i = \overline{1, q}$$

and at the points t_{-i} by the recurrence formula:

$$X^\varepsilon(t_{-i}) = X^\varepsilon(t_{-i+1}) - h \cdot F(t_{-i+1}, X^\varepsilon(t_{-i+1})), \quad i = \overline{1, q}$$

These procedures of passing from $(t_{i-1}, X^\varepsilon_{i-1})$ to (t_i, X^ε_i) or from $(t_{-i+1}, X^\varepsilon_{-i+1})$ to $(t_{-i}, X^\varepsilon_{-i})$ are easy to programme.

For exemplification, using the iteration procedure of Euler:

$$t_{i+1} = t_i + h$$

$$X^{i+1} = X^i + h \cdot m_E \qquad \text{where} \qquad m_E = F(t_i, X_i)$$

we determine numerically the solutions of the first order differential equation and of the first order system of differential equations, respectively. First, we consider the linear differential equation:

$$\dot{x} = -x + 2e^t, \tag{4.13}$$

Its solution has been determined by symbolic calculations in Chapter 1 for the initial condition $x(0) = 2$:

$$x(t) = e^t + e^{-t}$$

Programming in *Maple* using the Euler iteration procedure we obtain:

```
> h:=0.1:  n:=10:
> f:=(t,x)->-x(t)+2*exp(t):
> t:=(n,h)->n*h:
> x:=proc(n,h);
> if n=0 then x(0) else
x(n-1,h)+h*f(t(n-1,h),x(n-1,h)) end if;
> end proc:
> x(0):=2:
> x(t):=[seq(x(i,h),i=0..n)];
```

$$x(t) := [2, 2.0, 2.021034184, 2.063211317, 2.126861947, 2.212540692, 2.321030877, 2.453351549, 2.610766936, 2.794798428, 3.007239207]$$

```
> t:=[seq(t(i,h),i=0..n)];
```

$$t := [0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0]$$

To compare the numerical values obtained by the Euler iterations with those obtained by symbolic calculations, we compute the solution (obtained in Chapter 1) in different points:

```
> sol_x(t):=exp(t)+exp(-t):
> eval(sol_x(t),t=0);
eval(sol_x(t),t=0.1); eval(sol_x(t),t=0.2);
eval(sol_x(t),t=0.3);eval(sol_x(t),t=0.9);
```

$$2$$
$$2.010008336$$
$$2.040133511$$
$$2.090677029$$
$$2.866172771$$

We can observe that the results obtained numerically by the Euler iteration procedure are close to those obtained by symbolic calculations only for values of t close to the initial condition (zero) because the domain of convergence is small. So, the Euler's method gives us a good approximation of the solution only on small intervals. In the followings, we give another example in which we solve numerically (using the Euler method) a first order system of differential equation:

$$\begin{cases} \dot{x_1} = -x_1 + 8x_2 \\ \dot{x_2} = x_1 + x_2 \end{cases} \tag{4.14}$$

its solution having been determined by symbolic calculation in Chapter 3 for the initial conditions $x_1(0) = 0$, $x_12(0) = 1$:

$$x_1(t) := \frac{5}{3} \cdot e^{3t} - \frac{2}{3} \cdot e^{-3t}$$

$$x_2(t) := \frac{5}{6} \cdot e^{3t} + \frac{1}{6} \cdot e^{-3t}$$

Programming in *Maple* we obtain:

```
> h:=0.1:  n:=10:
> f1:=(t,x1,x2)->-x1(t)+8*x2(t):  f2:=(t,x1,x2)->x1(t)+x2(t):
> t:=(n,h)->n*h:
> x1:=proc(n,h);
> if n=0 then x1(0) else
x1(n-1,h)+h*f1(t(n-1,h),x1(n-1,h),x2(n-1,h))end if;
> end proc:
> x2:=proc(n,h)
> if n=0 then x2(0)else
x2(n-1,h)+h*f2(t(n-1,h),x1(n-1,h),x2(n-1,h))end if;
> end proc:
> x1(0):=1:  x2(0):=1:
> x1(t):=[seq(x1(i,h),i=0..n)];
```

$$x1\ (t) := [1, 1.7, 2.49, 3.433, 4.6001, 6.07617, 7.966249,$$
$$10.4031833, 13.55708001, 17.64726322, 22.95758363]$$

```
> x2(t):=[seq(x2(i,h),i=0..n)];
```

$$x2\ (t) := [1, 1.2, 1.49, 1.888, 2.4201, 3.12212, 4.041949,$$
$$5.2427688, 6.80736401, 8.843808412, 11.49291558]$$

```
> t:=[seq(t(i,h),i=0..n)];
```

$$t := [0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0]$$

To compare the numerical values of the solution to those obtained by symbolic calculations, we compute its values in some points:

```
> sol_x1:=5/3*exp(3*t)-2/3*exp(-3*t):
> sol_x2:=5/6*exp(3*t)+1/6*exp(-3*t):
> eval(sol_x1(t),t=0);
eval(sol_x1(t),t=0.1);eval(sol_x1(t),t=0.2);
eval(sol_x1(t),t=0.3);eval(sol_x1(t),t=0.9);
```

$$1$$
$$1.755885866$$
$$2.670990243$$
$$3.828292078$$
$$24.75474919$$

```
> eval(sol_x2(t),t=0);
eval(sol_x2(t),t=0.1); eval(sol_x2(t),t=0.2);
```

```
eval(sol_x2(t),t=0.3); eval(sol_x2(t),t=0.9);
```

1
1.248352044
1.609900939
2.117430869
12.41097735

In this case we observe that the obtained results using Euler's method are close to those obtained by symbolic calculus only for small intervals, as well. So, in the case of systems the Euler's method give us a good approximation of the solution only the small intervals.

4.4.2 The Runge-Kutta method for numerical determination of the non-prolonged solution in the case of first order systems of differential equations

For better accuracy in obtaining the numerical solution, we must use Taylor series of higher order (the Euler method uses Taylor series of the first order); however, in order to use higher orders, we need to evaluate higher order derivatives.

The Runge-Kutta method does not require the evaluation of the derivatives (these are approximated by the forward difference) and at the same time, keeps the desirable property of higher-order local truncation error. These facts imply a better convergence to the saturated solution. In practice, there are some particular forms of this method: the Runge-Kutta method of second order $rk2$, of third order $rk3$, of fourth order (standard) $rk4$ and of fifth order called the Fehlberg-Runge-Kutta method $rkf45$.

In the followings, we will present the iteration procedure for the (standard) Runge-Kutta method $rk4$, without details about the convergence:

$$\begin{aligned} t_{i+1} &= t_i + h \\ X^{i+1} &= X^i + h \cdot m_{R-K} \end{aligned}$$

where

$$\begin{aligned} m_{R-K} &= \frac{1}{6}(m_1 + 2m_2 + 2m_3 + m_4) \\ m_1 &= F(t_i, X^i) \\ m_2 &= F(t_i + h/2, X^i + h/2 \cdot m_1) \\ m_3 &= F(t_i + h/2, X^i + h/2 \cdot m_2) \\ m_4 &= F(t_i + h, X^i + h \cdot m_3). \end{aligned}$$

This procedure for passing from (t_i, X^i) to (t_{i+1}, X^{i+1}) is easy to programme.

For exemplification, to compare the results obtained by the Euler method with those obtained by the Runge-Kutta method of fourth order, we will consider the programming of the same exercises presented in the case of the Euler method. Programming in *Maple* the iteration procedure of the standard Runge-Kutta method $rk4$ for the equation (4.13) we obtain:

```
> h:=0.1:  n:=10:

> f:=(t,x)->-x(t)+2*exp(t):
```

```
> t:=(n,h)->n*h:
> x:=proc(n,h) local k1,k2,k3,k4;
> if n=0 then x(0) else
k1:=f(t(n-1,h),x(n-1,h));
k2:=f(t(n-1,h)+h/2,x(n-1,h)+h*k1/2);
k3:=f(t(n-1,h)+h/2,x(n-1,h)+h*k2/2);
k4:=f(t(n-1,h)+h/2,x(n-1,h)+h*k3);
x(n-1,h)+h/6*(k1+2*k2+2*k3+k4)
> end if;
> end proc:
> x(0):=2:
> x(t):=[seq(x(i,h),i=0..n)];
```

$$x(t) := [2, 2.008211921, 2.036522685, 2.085215637, 2.154778115, 2.245906324, 2.359512308, 2.496733074, 2.658941974, 2.847762451, 3.065084284]$$

```
> t:=[seq(t(i,h),i=0..n)];
```

$$t := [0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0]$$

```
> sol_x(t):=exp(t)+exp(-t):
> eval(sol_x(t),t=0);
eval(sol_x(t),t=0.1); eval(sol_x(t),t=0.2);
eval(sol_x(t),t=0.3); eval(sol_x(t),t=0.9);
```

$$2$$
$$2.010008336$$
$$2.040133511$$
$$2.090677029$$
$$2.866172771$$

Comparing these results with those obtained by the Euler method and then with those obtained by symbolic calculations, we observe that, for the standard Runge-Kutta method, the domain of convergence is the interval that we have considered. The solution obtained by $rk4$ is very close to the one obtained by symbolic calculations.

Programming in *Maple* the standard Runge-Kutta iteration procedure $rk4$ for the system of differential equations (4.14) we obtain:

```
> h:=0.1:  n:=10:
> f1:=(t,x1,x2)->-x1(t)+8*x2(t):f2:=(t,x1,x2)->x1(t)+x2(t):
> t:=(n,h)->n*h:
> x1:=proc(n,h) local k1,k2,k3,k4;
> if n=0 then x1(0) else
```

```
    k1:=f1(t(n-1,h),x1(n-1,h),x2(n-1,h));
k2:=f1(t(n-1,h)+h/2,x1(n-1,h)+h*k1/2,x2(n-1,h)+h*k1/2);
k3:=f1(t(n-1,h)+h/2,x1(n-1,h)+h*k2/2,x2(n-1,h)+h*k2/2);
k4:=f1(t(n-1,h)+h/2,x1(n-1,h)+h*k3,x2(n-1,h)+h*k3);
x1(n-1,h)+h/6*(k1+2*k2+2*k3+k4)
> end if;
> end proc:
> x2:=proc(n,h) local m1,m2,m3,m4;
> if n=0 then x2(0) else
    m1:=f2(t(n-1,h),x1(n-1,h),x2(n-1,h));
m2:=f2(t(n-1,h)+h/2,x1(n-1,h)+h*m1/2,x2(n-1,h)+h*m1/2);
m3:=f2(t(n-1,h)+h/2,x1(n-1,h)+h*m2/2,x2(n-1,h)+h*m2/2);
m4:=f2(t(n-1,h)+h/2,x1(n-1,h)+h*m3,x2(n-1,h)+h*m3);
x2(n-1,h)+h/6*(m1+2*m2+2*m3+m4)
> end if;
> end proc:
> x1(0):=1:  x2(0):=1:
> x1(t):=[seq(x1(i,h),i=0..n)];
```

$$x1\,(t) := [1., 1.75588586516103406, 2.67099024240189342, 3.82829207712650410, 5.33273206041661840, 7.32072833903442266, 9.97254650756094564, 13.5286455567960680, 18.3114819804437801, 24.7547491716790910, 33.4427034511831920]$$

```
> x2(t):=[seq(x2(i,h),i=0..n)];
```

$$x2\,(t) := [1., 1.24835204296884550, 1.60990093931829237, 2.11743086851170714, 2.81696313624160988, 3.77192924966291354, 5.06892269795481632, 6.82555099257945131, 9.20109996691309817, 12.4109773422481773, 16.7462452598074308]$$

```
> t:=[seq(t(i,h),i=0..n)];
```

$$t := [0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0]$$

```
> sol_x1:=5/3*exp(3*t)-2/3*exp(-3*t):
> sol_x2:=5/6*exp(3*t)+1/6*exp(-3*t):
> eval(sol_x1(t),t=0);
```

```
eval(sol_x1(t),t=0.1);eval(sol_x1(t),t=0.2);

eval(sol_x1(t),t=0.3);eval(sol_x1(t),t=0.9);
```

$$1$$
$$1.755885866$$
$$2.670990243$$
$$3.828292078$$
$$24.75474919$$

```
> eval(sol_x2(t),t=0);

eval(sol_x2(t),t=0.1); eval(sol_x2(t),t=0.2);

eval(sol_x2(t),t=0.3); eval(sol_x2(t),t=0.9);
```

$$1$$
$$1.248352044$$
$$1.609900939$$
$$2.117430869$$
$$12.41097735$$

In this case, we observe a good convergence for the standard Runge-Kutta method $rk4$.

4.4.3 Numerical computation of the solutions of the differential equations and of the systems of differential equations

In this section, using the numerical methods offered by *Maple*, in which the Euler or Runge-Kutta procedures of iteration are included, we determine the solutions of differential equations and systems of differential equations, respectively; after that, we represent graphically these solutions.

The first example consists of a nonlinear differential equation of third order, with variable coefficients:

$$t^2\dddot{x} + 5t\ddot{x} + 4\dot{x} = \ln x, \quad x > 0 \tag{4.15}$$

```
> eq3:=t^2*diff(x(t),t,t,t)+5*t*diff(x(t),t,t)+4*diff(x(t),t)
  =ln(x(t));
```

$$eq3 := t^2 \frac{d^3}{dt^3}x(t) + 5\,t\frac{d^2}{dt^2}x(t) + 4\,\frac{d}{dt}x(t) = \ln(x(t))$$

```
> dsolve({eq3,x(2)=2,D(x)(2)=1/2,(D@@2)(x)(2)=3});
```

As *Maple* gives no output, it means that it cannot give a solution obtained by symbolic calculus; more precisely, it cannot give us the solution expressed using elementary functions. In this case, we solve numerically this equation using the syntax *dsolve* which allows us to solve the equation by one of the classical numerical method: Euler, Runge-Kutta of order 2, 3 or 4, etc. The new syntax *dsolve/numeric/classical* (numerical solution of ordinary differential equations), which is specific for numerical computations, has one of the following forms:

dsolve(odesys, numeric, method=classical);

dsolve(odesys, numeric, method=classical[choice], vars, options);

in which:

odesys	- equation or list of equations and initial conditions
numeric	- indicates to *dsolve* to solve by numerical methods
method = classical	- optional, which indicates the name of the numerical method: *impoly* for Euler; *rk2, rk3, rk4* for Runge-Kutta of order 2, 3, 4, etc.
vars	- list of dependent variables (optional)
options	- different options: output, number of points, etc.

If we don't use the option which specifies the numerical method we want to use, then the computer will chose the Fehlberg-Runge-Kutta method of fifth order (*method=rkf45*), which is the method with the best convergence.

The equation (4.15) solved by *rk*4 is:

```
> a:=dsolve({eq3,x(2)=2,D(x)(2)=1/2,(D@@2)(x)(2)=3},numeric,
  method=classical[rk4],output=listprocedure):
> sol_x := subs(a,x(t)):
> sol_x(0.2);sol_x(0.4);sol_x(1);sol_x(2);
```

```
sol_x(5);sol_x(8);sol_x(10);sol_x(30);
```

178.442355332346864
52.2875370423634607
6.30572074481224831
2.0
6.08914601990852234
9.31175162767340581
11.0536870708637434
25.1681506399649386

By the instruction *dsolve/numeric/classical*, *Maple* computed the numerical values of the solution of the differential equation in the considered points of the definition domain. For displaying $x(t)$ in $t = 0.2, 0.4, 1, 2, 5, 8, 10, 30$ we used the function *subs*.

For representing graphically the solution of the differential equation solved numerically, we chose a special function for plotting which is specific for numerical methods. This function has the following syntax:

odeplot(dsn, vars, range, options);

in which:

dsn	- the name of the output of the equation solved numerically
vars	- independent variable and the function we want to plot (optional)
range	- optional
options	- number of points, different method for displaying the solution.

Using this instruction for plotting we obtain the representation from (Figure 4.1) for the solution of the equation (4.15):

```
> with(plots):odeplot(sol_x,t=0.2..30,numpoints=100);
```

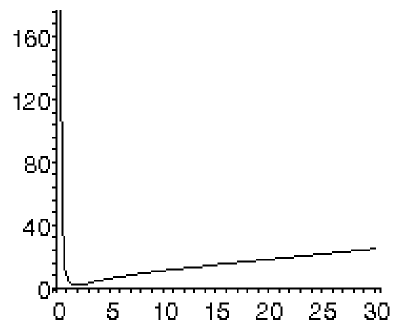


Figure 4.1:

Another example of differential equation solved numerically is the linear equation of second order:

$$L \cdot \frac{d^2 i}{dt^2} + R \cdot \frac{di}{dt} + \frac{1}{C} i = -E_0 \cdot \omega \cdot \sin \omega t \tag{4.16}$$

Its solution $i(t)$ expresses the current intensity through a R-L-C circuit. We consider some numerical values for R, L, C, E_0 and reducing the equation to a first order system of two differential equations we obtain:

```
> R:=2:  C:=0.1:  L:=1:  E=10:
                          ω:=Pi/4:
> Eq:=L*diff(i(t),t,t)+R*diff(i(t),t)+(1/C)*i(t)=-E*
> sys_Eq1:=diff(x1(t),t)=x2(t):                  ω *sin(t*ω):
> sys_Eq2:=diff(x2(t),t)=-10*x1(t)-2*x2(t)-10*Pi/4*sin(t*Pi/4):
```

For the numerical computation of the solution of this system and for the visualization of the solutions for different initial conditions (Figure 4.2 and Figure 4.3) we will use the functions:

$$with(DEtools) : DEplot$$
$$with(DEtools) : phaseportrait$$
$$with(DEtools) : DEplot3d$$

```
> with(DEtools):DEplot({sys_Eq1,sys_Eq2},{x1(t),x2(t)},
  t=0..15,[[x1(0)=0,x2(0)=0],[x1(0)=1,x2(0)=1],
[x1(0)=3,x2(0)=3]],scene=[t,x1(t)],method=classical[rk4]);
```

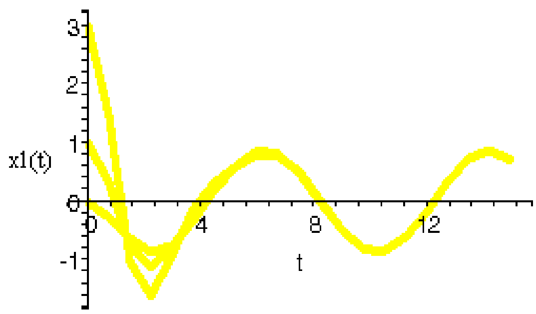


Figure 4.2:

```
> with(DEtools):DEplot({sys_Eq1,sys_Eq2},{x1(t),x2(t)},
    t=0..15,[[x1(0)=0,x2(0)=0],[x1(0)=1,x2(0)=1],
[x1(0)=3,x2(0)=3]],scene=[t,x2(t)],method=classical[rk4]);
```

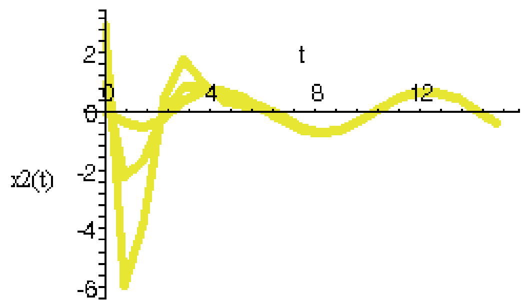


Figure 4.3:

In these figures, the solutions $(x_1(t), x_2(t))$ are represented for three initial conditions. We see that, for any initial condition, after a certain time, the solution is stabilized around the periodic solution (stabilization of the current intensity and its variation in the circuit). This fact results from the phase portrait, as well:

```
> with(DEtools):phaseportrait([sys_Eq1,sys_Eq2],
[x1(t),x2(t)],t=0..15,[[x1(0)=0,x2(0)=0]],
scene=[x1(t),x2(t)],method=classical[rk4]);
```

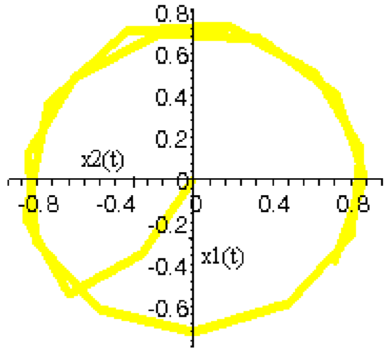


Figure 4.4:

```
> with(DEtools):phaseportrait([sys_Eq1,sys_Eq2],
  [x1(t),x2(t)],t=0..15,[[x1(0)=1,x2(0)=1]],
scene=[x1(t),x2(t)],method=classical[rk4]);
```

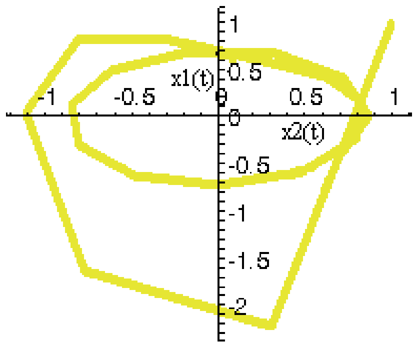


Figure 4.5:

The phase portraits (Figure 4.4 and Figure 4.5) show that the solutions of the system (current intensity and its variation) are stabilized after a certain moment of time, tending to the limit cycle. More precisely, using initial conditions from the inside or the outside of the limit cycle, solutions are stabilized around the periodic solutions. The same stabilization phenomenon can be observed in the three-dimensional figure (Figure 4.6):

```
> with(DEtools):DEplot3d({sys_Eq1,sys_Eq2},
```

```
{x1(t),x2(t)},t=0..15,[[x1(0)=0,x2(0)=0]],
scene=[t,x1(t),x2(t)],method=classical[rk4]);
```

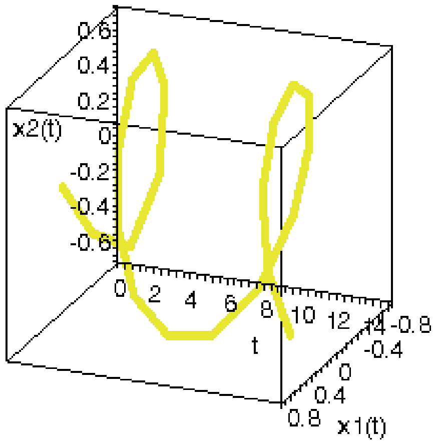


Figure 4.6:

Another example is the Lotka-Volterra system of two nonlinear differential equations, which is used in biology and describes the predator population growth $x = x(t)$ and the evolution of the pray population $y = y(t)$ (for example, foxes and rabbits):

$$\left\{ \begin{array}{lcl} \dot{x} & = & x(1-y) \\ \dot{y} & = & 0.3 \cdot y(x-1). \end{array} \right. \tag{4.17}$$

Using $with(DEtools)$: $DEplot$ (Figure 4.7 and Figure 4.8) and $with(DEtools)$: $DEplot3d$ (Figure 4.9), respectively, we obtain the evolution in time of the two species:

```
> with(DEtools):DEplot({diff(x(t),t)=x(t)*(1-y(t)),
diff(y(t),t)=.3*y(t)*(x(t)-1)},{x(t),y(t)},
t=0..50,[[x(0)=0.8,y(0)=0.5]],scene=[t,x(t)],
linecolor=t/2,method=rkf45);
> with(DEtools):DEplot({diff(x(t),t)=x(t)*(1-y(t)),
diff(y(t),t)=.3*y(t)*(x(t)-1)},{x(t),y(t)},
t=0..50,[[x(0)=0.8,y(0)=0.5]],scene=[t,y(t)],
linecolor=t/2,method=rkf45);
> with(DEtools):DEplot3d({diff(x(t),t)=x(t)*(1-y(t)),
diff(y(t),t)=.3*y(t)*(x(t)-1)},{x(t),y(t)},t=0..50,
[[x(0)=0.8,y(0)=0.5]],scene=[t,x(t),y(t)],
stepsize=.2,linecolor=t/2,method=rkf45);
```

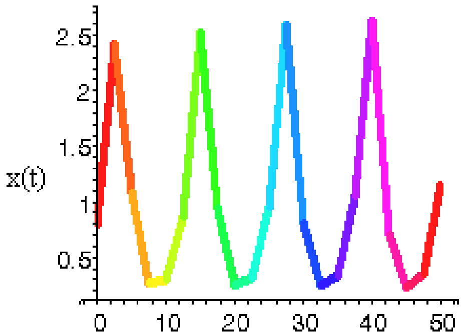


Figure 4.7:

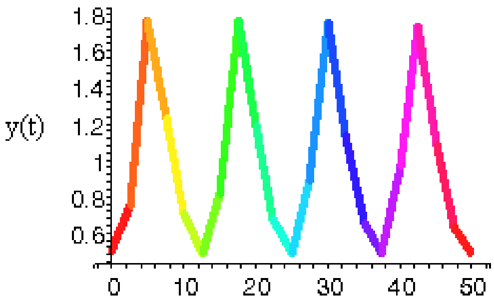


Figure 4.8:

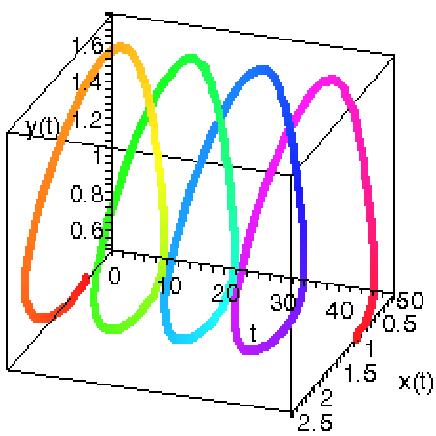


Figure 4.9:

The phase portraits of the two species are represented in Figure 4.10:

```
> with(DEtools):DEplot([diff(x(t),t)=x(t)*(1-y(t)),
diff(y(t),t)=.3*y(t)*(x(t)-1)],[x(t),y(t)],t=0..13,
[[x(0)=1.2,y(0)=1.2],[x(0)=1,y(0)=.7],[x(0)=.8,y(0)=.5]],
stepsize=.2,title=‘Lotka-Volterra’s model‘,
color=[.3*y(t)*(x(t)-1),x(t)*(1-y(t)),.1],
linecolor=t/2,arrows=MEDIUM,method=rkf45);
```

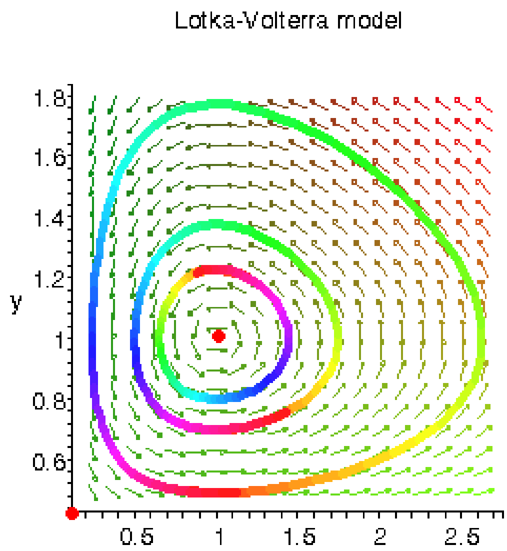


Figure 4.10:

This system has two stationary solutions $(0,0)$, $(1,1)$ and periodic solutions (Figure 4.10). The interpretation of these is the following:

i) the stationary solution $(0,0)$ represents that both species disappear;

ii) the stationary solution $(1,1)$ represents an equilibrium situation (the number of foxes is equal to the number of rabbits);

iii) the periodic solutions which are around the point $(1,1)$ represent variations of the number of foxes and rabbits between two limits. These variations (increasing or decreasing) describe if the food quantities are big or small.

Another interesting phase portrait is the following:

$$\left\{\begin{array}{lcl}\dot{x} & = & y - z\\ \dot{y} & = & z - x\\ \dot{z} & = & x - 2y,\end{array}\right. \tag{4.18}$$

which shows that solutions starting from any point will evolve to zero (Figure 4.11):

```
> with(DEtools):phaseportrait([D(x)(t)=y(t)-z(t),
D(y)(t)=z(t)-x(t),D(z)(t)=x(t)-y(t)*2],[x(t),y(t),z(t)],
t=-10..50,[[x(0)=3,y(0)=3,z(0)=3]],stepsize=.05,
scene=[z(t),y(t)],linecolour=sin(t*Pi/2),
method=classical[rk4]);
```

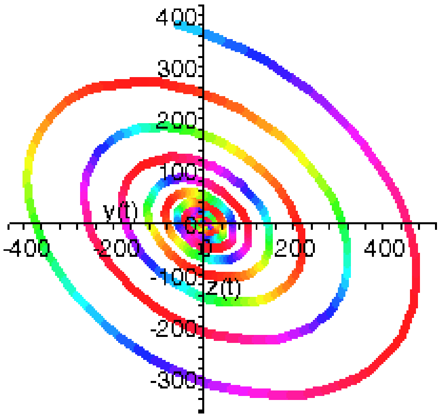


Figure 4.11:

4.5 Prime integrals

We consider the first order system of nonlinear differential equations given in the explicit form:

$$\begin{cases} \dot{x_1} = f_1(t, x_1, x_2, ..., x_n) \\ \dot{x_2} = f_2(t, x_1, x_2, ..., x_n) \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \dot{x_n} = f_n(t, x_1, x_2, ..., x_n) \end{cases} \tag{4.19}$$

in which $f_1, f_2, ..., f_n$ are real C^1-functions defined on $I \times D$; $I \subset \mathbb{R}^1$ is an open interval and $D \subset \mathbb{R}^n$ is a domain.

Definition 4.5.1. *We call prime integral of the system (4.19) a C^1-function $U : I \times D \to \mathbb{R}^1$ which is not constant but is constant along the solutions of the system (4.19).*

Theorem 4.5.1. *The necessary and sufficient condition for a nonconstant function $U : I \times D \to \mathbb{R}^1$ of class C^1 to be a prime integral is that U verifies:*

$$\frac{\partial U}{\partial t} + \sum_{i=1}^{n} \frac{\partial U}{\partial x_i} \cdot f_i = 0 \quad (\forall)\ (t, x_1, ..., x_n) \in I \times D$$

Proof. Let be $(t_0, x_1^0, ..., x_n^0) \in I \times D$ and $x_1(t), ..., x_n(t)$ the solution of the system (4.19) which verifies $x_i(t_0) = x_i^0$, $i = \overline{1, n}$ and $U(t, x_1(t), ..., x_n(t))$ a prime integral. Because $U(t, x_1(t), ..., x_n(t))$ is constant we obtain that

$$\frac{dU}{dt}(t, x_1(t), ..., x_n(t)) \equiv 0$$

from here we have

$$\frac{\partial U}{\partial t}(t, x_1(t), ..., x_n(t)) + \sum_{i=1}^{n} \frac{\partial U}{\partial x_i}(t, x_1(t), ..., x_n(t)) \cdot f_i(t, x_1(t), ..., x_n(t)) = 0 \qquad (\forall)\ t \in I$$

Hence, for $t = 0$ we have the equality:

$$\frac{\partial U}{\partial t}(t_0, x_1^0, ..., x_n^0) + \sum_{i=1}^{n} \frac{\partial U}{\partial x_i}(t_0, x_1^0, ..., x_n^0) \cdot f_i(t_0, x_1^0, ..., x_n^0) = 0$$

Because $(t_0, x_1^0, ..., x_n^0)$ is an arbitrary point from the set $I \times D$ we obtain

$$\frac{\partial U}{\partial t} + \sum_{i=1}^{n} \frac{\partial U}{\partial t} \cdot f_i = 0,\ (\forall)(t, x_1, ..., x_n) \in I \times D$$

Let us show now the reciprocal implication i.e. if $U : I \times D \to \mathbb{R}^1$ is a function of class C^1, which is not constant and verifies

$$\frac{\partial U}{\partial t} + \sum_{i=1}^{n} \frac{\partial U}{\partial t} \cdot f_i = 0$$

then U is constant along the solutions of the system (4.19).

For this, we consider an arbitrary solution $x_1(t), ..., x_n(t)$ of the system (4.19) and the function $\varphi(t) = U(t, x_1(t), ..., x_n(t))$. To show that the function $\varphi(t)$ is constant we compute its derivative:

$$\begin{aligned} \frac{d\varphi}{dt} &= \frac{\partial U}{\partial t}(t, x_1(t), ..., x_n(t)) + \\ &+ \sum_{i=1}^{n} \frac{\partial U}{\partial x_i}(t, x_1(t), ..., x_n(t)) \cdot f_i(t, x_1(t), ..., x_n(t)) = 0, \quad (\forall)\ t \in I. \end{aligned}$$

Thus, we have that the function $\varphi(t)$ is constant. □

Observation 4.5.1. A nonconstant function $U : D \to \mathbb{R}^1$ of $\mathcal{C}^\infty$ independent on t is a prime integral, if and only if U verifies $\displaystyle\sum_{i=1}^{n} \frac{\partial U}{\partial x_i} f_i = 0$.

Definition 4.5.2. *The system (4.19) is called autonomous if the functions f_i are independent on t; $f_i : D \to \mathbb{R}^n$, $f_i = \dot{f}_i(x_1, x_2, ..., x_n)$, $i = \overline{1,n}$.*

Observation 4.5.2. In order to find the movement of a point of mass m in a force field having the potential V, we solve than canonical Hamilton's system:

$$\begin{cases} \dfrac{dx_i}{dt} = \dfrac{\partial H}{\partial p_i} \\ \\ \dfrac{dp_i}{dt} = -\dfrac{\partial H}{\partial x_i} \end{cases} \qquad i = 1, 2, 3 \tag{4.20}$$

where

$$H(x_1, x_2, x_3, p_1, p_2, p_3) = \frac{1}{2m}(p_1^2 + p_2^2 + p_3^2) + V(x_1, x_2, x_3)$$

represents the Hamilton function associated to the material point.

The system (4.20) is autonomous, and the condition that a function $U(x_1, x_2, x_3, p_1, p_2, p_3)$ is a prime integral for (4.20) is that U verifies

$$\sum_{i=1}^{3} \left(\frac{\partial U}{\partial x_i} \cdot \frac{\partial H}{\partial p_i} - \frac{\partial U}{\partial p_i} \cdot \frac{\partial H}{\partial x_i} \right) = 0$$

In particular, we have that the Hamilton function is a prime integral for the canonical system.

We return to the general case of the autonomous systems :

$$\begin{cases} \dot{x_1} = f_1(x_1, ..., x_n) \\ \dot{x_2} = f_2(x_1, ..., x_n) \\ \dots\dots\dots\dots\dots\dots \\ \dot{x_n} = f_n(x_1, ..., x_n) \end{cases} \tag{4.21}$$

$f_i : D \subset \mathbb{R}^n \to \mathbb{R}^1,\ i = \overline{1,n}$ for which we consider the prime integrals U_j, $j = \overline{1,m}$ which do not depend on the independent variable t; $U_j = U_j(x_1, ..., x_n)$.

Definition 4.5.3. *A system of $m \leq n$ prime integrals of the system (4.19) is called independent if the rank of matrix:*

$$\begin{pmatrix} \dfrac{\partial U_1}{\partial x_1} & \dfrac{\partial U_1}{\partial x_2} & \cdots & \dfrac{\partial U_1}{\partial x_n} \\ \dfrac{\partial U_2}{\partial x_1} & \dfrac{\partial U_2}{\partial x_2} & \cdots & \dfrac{\partial U_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial U_m}{\partial x_1} & \dfrac{\partial U_m}{\partial x_2} & \cdots & \dfrac{\partial U_m}{\partial x_n} \end{pmatrix}$$

is equal to m.

In the followings, we will show the importance of prime integrals.

Theorem 4.5.2. *If $m < n$ prime integrals of the system (4.19) are known, then the problem of determining the solutions of the system (4.19) is reduced to finding the solutions of a system of $n - m$ first order differential equations.*

If n prime integrals are known, then the determination of the solutions of the system (4.19) is reduced to solving an algebraic system.

Proof. We will consider $m < n$ independent prime integrals $U_1, ..., U_m$ of the system (4.19). The independence assures that we can express from the system of equations:

$$\begin{cases} U_1(t, x_1, ..., x_n) - c_1 = 0 \\ U_2(t, x_1, ..., x_n) - c_2 = 0 \\ \cdots\cdots\cdots \\ U_m(t, x_1, ..., x_n) - c_m = 0 \end{cases} \tag{4.22}$$

m of the unknowns $x_1, ..., x_n$ in function of the other unknowns and in function of t and the constants $c_1, ..., c_m$.

Thus, we suppose that $x_1, ..., x_m$ are expressed in function of $t, x_{m+1}, ..., x_n$:

$$\begin{cases} x_1 = \varphi_1(t, x_{m+1}, ..., x_n, c_1, ..., c_m) \\ x_2 = \varphi_2(t, x_{m+1}, ..., x_n, c_1, ..., c_m) \\ \cdots\cdots\cdots \\ x_m = \varphi_m(t, x_{m+1}, ..., x_n, c_1, ..., c_m) \end{cases}$$

If $x_1(t), ..., x_n(t)$ is a solution of the system (4.19), then we have:

$$\begin{cases} x_1(t) = \varphi_1(t, x_{m+1}(t), ..., x_n(t), c_1, ..., c_m) \\ x_2(t) = \varphi_2(t, x_{m+1}(t), ..., x_n(t), c_1, ..., c_m) \\ \cdots\cdots\cdots \\ x_m(t) = \varphi_m(t, x_{m+1}(t), ..., x_n(t), c_1, ..., c_m) \end{cases}$$

In this way, we obtain that m compounds of the solution $x_1(t), ..., x_m(t)$ are constructed with the remaining compounds $x_{m+1}(t), ..., x_n(t)$ and the constants $c_1, ..., c_m$ with the help of the functions $\varphi_1, \varphi_2, ..., \varphi_m$.

Replacing in the system (4.19), we obtain that $x_{m+1}(t), ..., x_n(t)$ verify the following system of differential equations

$$\begin{cases} \dfrac{dx_{m+1}}{dt} = f_{m+1}(t, \varphi_1(t, x_{m+1},..,x_n, c_1,..,c_m),..,\varphi_m(t, x_{m+1},..,x_n, c_1,..,c_m), x_{m+1},..,x_n) \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \dfrac{dx_n}{dt} = f_n(t, \varphi_1(t, x_{m+1},..,x_n, c_1,..,c_m),..,\varphi_m(t, x_{m+1},..,x_n, c_1,..,c_m), x_{m+1},..,x_n) \end{cases}$$

Thus, we obtain that the functions $x_{m+1}, ..., x_n$ verify a system of $n - m$ differential equations of first order.

If $m = n$ then the algebraic system (4.22) is

$$U_k(t, x_1, x_2, ..., x_n) - c_k = 0, \ \ k = \overline{1, n}$$

from which we obtain:

$$x_k = \varphi(t, c_1, ..., c_m) \quad k = \overline{1, n}$$

$\square$

In the following we will reconsider some examples from the case of one equation $n = 1$ where we search for a prime integral.

Examples:

1. Show that, if $f : (a, b) \to \mathbb{R}^1$ is continuous, then the function
$$U(t, x) = x - \int_{t_*}^{t} f(\tau)d\tau$$
is a prime integral for the differential equation $\dot{x} = f(t)$. Deduce in this way that the solutions of this differential equation are the solutions of the following algebraic equation:
$$x - \int_{t_*}^{t} f(\tau)d\tau - c = 0$$

2. Show that, if $g : (c, d) \to \mathbb{R}^1$ is a continuous function which doesn't take null values, then the function
$$U(t, x) = t - \int_{x_*}^{x} \frac{du}{g(u)}$$
is a prime integral for the equation $\dot{x} = g(x)$. Deduce in this way that the solutions of this differential equation are solutions of the algebraic equation
$$t - \int_{x_*}^{x} \frac{du}{g(u)} - c = 0$$

3. Show that, if $f : (a, b) \to \mathbb{R}^1$ and $g : (c, d) \to \mathbb{R}^1$ are continuous functions and the function g doesn't take null values, then the function
$$U(t, x) = \int_{t_*}^{t} f(\tau)d\tau - \int_{x_*}^{x} \frac{du}{g(u)}$$
is a prime integral for the differential equation $\dot{x} = f(t) \cdot g(x)$. Deduce in this way that the solutions of this differential equation are solutions of an algebraic equation.

We reconsider the case $n > 1$, for which the following problem can be posed: *What is the maximal number of independent prime integrals for the system (4.19)?*

The answer to this question is given by the following theorem.

Theorem 4.5.3. *For any point $(t_0, x_1^0, ..., x_n^0) \in I \times D$ there exists an open neighborhood $V \subset I \times D$ such that the system (4.19) has n independent prime integrals in V and any prime integral defined on V can be expressed with the help of n independent prime integrals.*

Proof. Let be $X(t; t_0, X^0)$ the saturated solution of the system (4.19) which satisfies $X(t_0) = X^0$ and I_0 its interval of definition. We consider the compact interval $[T_1, T_2]$ included in I_0 which contains in the interior the point t_0.

According to the theorem of continuity with respect to the initial conditions, there is a neighborhood U_0 of $x^0 = (x_1^0, ..., x_n^0)$ such that for any $X' = (x_1', ..., x_n') \in U_0$ the solution of the system (4.19) which coincides with X' at $t = t_0$ is defined on $[T_1, T_2]$; we denote by $X(t; t_0, X')$ this saturated solution.

The function $(t, X') \xrightarrow{\varphi} X(t; t_0, X')$ is of class C^1, based on the theorem of differentiability with respect to the initial conditions. From the same theorem we have that $\dfrac{\partial}{\partial X'}[X(t; t_0, X')]$ is non-singular and it follows that the application $X' \xrightarrow{\varphi} X(t; t_0, X')$ is invertible. There are two open neighborhoods W_1, W_2 of the point (t_0, X^0) and a function ψ of class C^1, $\psi : W_2 \to W_1$ with the properties: $\psi(t, X(t; t_0, X')) \equiv (t, X')$ and $X(t; t_0, \psi(t, X'')) \equiv X''$ $(\forall)$ $(t, X') \in W_1$ and $(t, X'') \in W_2$.

The scalar compounds ψ_k of the function ψ remain constant when $x_1, ..., x_n$ is replaced with a solution of the system defined on the considered neighborhood so, ψ_k are prime integrals. Moreover, $\psi(t_0, X') = (t_0, X')$ so we have $rang \left(\dfrac{\partial \psi_k}{\partial x_r'}\right) = n$, i.e. the prime integrals $\psi_1, ..., \psi_n$ are independent.

Let be U an arbitrary integral of the system (4.1). According to the definition, $U(t, X(t; t_0, X'))$ doesn't depend on t and we can write

$$U(t, X(t; t_0, X')) = h(X')$$

Replacing X' with $\psi(t, X'')$ and taking into account the equality

$$X(t; t_0, \psi(t, X'')) \equiv X''$$

we obtain:

$$U(t, X'') = h(\psi(t, X''))$$

This last equality shows that the prime integral U is a function h of n independent prime integrals $\psi_1, ..., \psi_n$.

A method of determining prime integrals is given by the equation:

$$\frac{\partial U}{\partial t} + \sum_{i=1}^{n} \frac{\partial U}{\partial x_i} \cdot f_i = 0$$

In this equation U is an unknown function and appears in the equation with its partial derivatives of first order. Any solution of this equation is a prime integral.

We observe that, if functions $\mu_0, \mu_1, ..., \mu_n$ are such that

$$\mu_0 + \sum_1^n \mu_i \cdot f_i = 0$$

and there exists a function U having the properties $\dfrac{\partial U}{\partial t} = \mu_0$ and $\dfrac{\partial U}{\partial x_i} = \mu_i$, then U is a prime integral for system (4.19). □

Exercises:

1. Determine a prime integral for the system of equations:

$$\begin{cases} \dot{x_1} = x_2 \\ \dot{x_2} = -x_1 \end{cases}$$

A: $\begin{cases} \mu_0 = 0,\ \mu_1 = x_1,\ \mu_2 = x_2 \\ U(x_1, x_2) = \dfrac{1}{2}(x_1^2 + x_2^2) \end{cases}$

2. Determine two prime integrals for the following system which describes the movement of a rigid body:

$$\begin{cases} A \cdot \dot{p} &= (B - C)g \cdot r \\ B \cdot \dot{q} &= (C - A)r \cdot p \\ C \cdot \dot{r} &= (A - B)p \cdot q \end{cases}$$

A: $U_1(p, q, r) = Ap^2 + Bq^2 + cr^2$ and $U_2(p, q, r) = A^2p^2 + B^2q^2 + C^2r^2$.

3. Determine two prime integrals for the systems:

$a)$ $\dfrac{dx}{x} = -\dfrac{dy}{2y} = \dfrac{dz}{-z}$

A:$U_1(x, y, z) = x\sqrt{y}$ and $U_2(x, y, z) = xz$.

$b)$ $\dfrac{dx}{z - y} = \dfrac{dy}{x - z} = \dfrac{dz}{y - x}$

A: $U_1(x, y, z) = x + y + z$ and $U_2(x, y, z) = x^2 + y^2 + z^2$.

$c)$ $\dfrac{dx}{x^2(y + z)} = \dfrac{dy}{-y^2(z + x)} = \dfrac{dz}{z^2(y - x)}$

A: $U_1(x, y, z) = xyz$ and $U_2(x, y, z) = \dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z}$.

$d)$ $\dfrac{dx}{xy^2} = \dfrac{dy}{x^2y} = \dfrac{dz}{z(x^2 + y^2)}$

A: $U_1(x, y, z) = x^2 - y^2$ and $U_2(x, y, z) = \frac{xy}{z}$.

Chapter 5

First order partial differential equations

5.1 Linear first order partial differential equations

Definition 5.1.1. *A linear first order partial differential equation is a functional dependence of the form*

$$\sum_{i=0}^{n} \frac{\partial u}{\partial x_i} \cdot f_i(x_0, x_1, \ldots, x_n) = 0 \tag{5.1}$$

between the first order partial derivatives of the unknown function $u = u(x_0, x_1, \ldots, x_n)$.

The functions $f_0, f_1, \ldots, f_n$ *from (5.1)* $f_i : \Omega \subset \mathbb{R}^{n+1} \to \mathbb{R}^1$ *are known and are of class* $\mathcal{C}^1$ *on* Ω.

Definition 5.1.2. *A solution of the equation (5.1) is a function*

$$u : \Omega' \subset \Omega \to \mathbb{R}^1$$

of class $\mathcal{C}^1$ *such that for any* $(x_0, x_1, \ldots, x_n) \in \Omega$ *the identity*

$$\sum_{i=0}^{n} \frac{\partial u}{\partial x_i}(x_0, x_1, \ldots, x_n) \cdot f_i(x_0, x_1, \ldots, x_n) \equiv 0$$

is satisfied.

Let be $\xi = (\xi_0, \xi_1, \ldots, \xi_n) \in \Omega$ such that

$$\sum_{i=0}^{n} f_i^2(\xi_0, \xi_1, \ldots, \xi_n) \neq 0.$$

We can admit the existence of a point $\xi \in \Omega$ having this property because, if the contrary holds, all functions f_i will be identically zero on Ω and hence, the equation (5.1) will be verified for any function u of class $\mathcal{C}^1$ on Ω. Moreover, we can admit that $f_0(\xi_0, \xi_1, \ldots, \xi_n) \neq 0$. That is because from $\sum_{i=0}^{n} f_i(\xi_0, \xi_1, \ldots, \xi_n) \neq 0$ we obtain the existence of $i_0 \in \{0, 1, \ldots, n\}$ such that $f_{i_0}(\xi) \neq 0$ and we change the notation.

If $f_0(\xi) = 0$, then the reasoning will be made for f_{i_0}.

Let us consider $f_0(\xi) \neq 0$. Because f_0 is continuous, there exists a neighborhood V_ξ of ξ such that $f_0(x) \neq 0$ for any $x \in V_\xi$. Let us consider now the system of differential equations:

$$\dot{x}_i = g_i(t, x_1, \ldots, x_n), \quad i = \overline{1, n} \tag{5.2}$$

where

$$g_i(t, x_1, \ldots, x_n) = \frac{f_i(t, x_1, \ldots, x_n)}{f_0(t, x_1, \ldots, x_n)}$$

t being the x_0 component of the vector $x = (x_0, x_1, \ldots, x_n) \in V_\xi$, $t = x_0$. The system (5.2) is defined for $(t, x_1, \ldots, x_n) \in V_\xi$ and the functions g_i are of class $\mathcal{C}^1$ on V_ξ.

Theorem 5.1.1. *The solutions of the equation (5.1) defined on $\widetilde{V} \subset V_\xi$ are prime integrals of the system (5.2) and vice-versa.*

Proof. Let be $u : \widetilde{V} \to \mathbb{R}^1$ a solution of the equation (5.1). Using the introduced notations we have:

$$\frac{\partial u}{\partial t}(t, x_1, \ldots, x_n) \cdot f_0(t, x_1, \ldots, x_n) + \sum_{k=1}^{n} \frac{\partial u}{\partial x_k}(t, x_1, \ldots, x_n) \cdot f_k(t, x_1, \ldots, x_n) \equiv 0$$

hence,

$$\frac{\partial u}{\partial t}(t, x_1, \ldots, x_n) + \sum_{k=1}^{n} \frac{\partial u}{\partial x_k}(t, x_1, \ldots, x_n) \cdot \frac{f_k(t, x_1, \ldots, x_n)}{f_0(t, x_1, \ldots, x_n)} \equiv 0$$

and it follows that:

$$\frac{\partial u}{\partial t}(t, x_1, \ldots, x_n) + \sum_{k=1}^{n} \frac{\partial u}{\partial x_k}(t, x_1, \ldots, x_n) \cdot g_k(t, x_1, \ldots, x_n) \equiv 0$$

which shows that u is a prime integral of the system (5.2).

In the same manner, it is shown that, if u is a prime integral for the system (5.2) then it is a solution of the equation (5.1). $\square$

It follows that the problem of determining the solutions of the equation (5.1) is reduced to finding the prime integrals for the system (5.2).

Taking into account the proved properties for prime integrals we obtain the following properties for the solutions of the linear first order partial differential equation (5.1):

i) for any $(x_0^0, x_1^0, \ldots, x_n^0) \in \Omega$, if $f_0(x_0^0, x_1^0, \ldots, x_n^0) \neq 0$ then there exists an open neighborhood $V \subset \Omega$ of the point $(x_0^0, x_1^0, \ldots, x_n^0)$ and n functions $u_1, \ldots, u_n : V \to \mathbb{R}^1$ of class $\mathcal{C}^1$ such that:

    a) $u_1, u_2, \ldots, u_n$ are solutions of the equation (5.1);

b) $u_k(x_0^0, x_1, \dots, x_n) = x_k$;

c) $\det\left(\dfrac{\partial u_k}{\partial x_l}\right) \neq 0, \quad k, l = 1, 2, \dots, n.$

ii) if u is an arbitrary solution of the equation (5.1), defined on $\widetilde{V} \subset V$, then there exists an open neighborhood D of $(x_1^0, \dots, x_n^0)$ and a function $\gamma : D \subset \mathbb{R}^n \to \mathbb{R}^1$ such that
$u(x_0, x_1, \dots, x_n) = \gamma(u_1(x_0, x_1, \dots, x_n), u_2(x_0, x_1, \dots, x_n), \dots, u_n(x_0, x_1, \dots, x_n))$.

Definition 5.1.3 (The Cauchy Problem for the equation (5.1)). *We call the Cauchy Problem for the equation (5.1), the problem of determination of the solution u of (5.1) such that*

$$u(x_0^0, x_1, \dots, x_n) = h(x_1, \dots, x_n), \quad (\forall)(x_1, \dots, x_n) \in D' \subset D,$$

where x_0^0 and the function $h : D \subset \mathbb{R}^n \to \mathbb{R}^1$ of class $\mathcal{C}^1$ are given, D being an open neighborhood of $(x_1^0, \dots, x_n^0)$.

Theorem 5.1.2. *If at the point $(x_0^0, x_1^0, \dots, x_n^0) \in \Omega$ we have*

$$f_0(x_0^0, x_1^0, \dots, x_n^0) \neq 0$$

then for h of class $\mathcal{C}^1$ defined on an open neighborhood D of $(x_1^0, \dots, x_n^0)$ the Cauchy Problem has a unique solution.

Proof. We consider $u_1, u_2, \dots, u_n$ solutions of the equation (5.1) defined on the open neighborhood D of $(x_1^0, \dots, x_n^0)$ having the property

$$u_k(x_0^0, x_1, \dots, x_n) = x_k, \quad k = 1, \dots, n.$$

There is a neighborhood $\widetilde{V} \subset D$ of $(x_1^0, \dots, x_n^0)$ such that for any $(x_0, x_1, \dots, x_n) \in \widetilde{V}$ we have

$$(u_1(x_0, x_1, \dots, x_n), \dots, u_n(x_0, x_1, \dots, x_n)) \in D.$$

Let be

$$u(x_0, x_1, \dots, x_n) = h(u_1(x_0, x_1, \dots, x_n), \dots, u_n(x_0, x_1, \dots, x_n)).$$

The function $u(x_0, x_1, \dots, x_n)$ defined in this way is of class $\mathcal{C}^1$ and verifies the equation (5.1):

$$\begin{aligned} \frac{\partial u}{\partial x_0} \cdot f_0^0 + \sum_{k=1}^{n} \frac{\partial u}{\partial x_k} \cdot f_k &= \sum_{l=1}^{n} \frac{\partial h}{\partial y_l} \cdot \frac{\partial u_l}{\partial x_0} \cdot f_0 + \sum_{k=1}^{n} \sum_{l=1}^{n} \frac{\partial h}{\partial y_l} \cdot \frac{\partial u_l}{\partial x_k} \cdot f_k = \\ &= \sum_{l=1}^{n} \frac{\partial h}{\partial y_l} \left(\frac{\partial u_l}{\partial x_0} \cdot f_0 + \sum_{k=1}^{n} \frac{\partial u_l}{\partial x_k} \cdot f_k \right) = 0. \end{aligned}$$

Moreover,

$$u(x_0^0, x_1, \dots, x_n) = h(u_1(x_0^0, x_1, \dots, x_n), \dots, u_n(x_0^0, x_1, \dots, x_n)) = h(x_1, \dots, x_n)$$

and hence, u is a solution of the Cauchy Problem.
For the proof of uniqueness, we suppose that $\widetilde{u}$ is another solution of the Cauchy Problem, defined on $\widetilde{V}$. We obtain that there is γ such that

$$\widetilde{u}(x_0, x_1, \ldots, x_n) = \gamma(u_1(x_0, x_1, \ldots, x_n), \ldots, u_n(x_0, x_1, \ldots, x_n)).$$

From here we have

$$h(x_1, \ldots, x_n) = \widetilde{u}(x_0^0, x_1, \ldots, x_n) = \gamma(x_1, \ldots, x_n);$$

and hence $\widetilde{u}$ coincides with u. □

If at the point $(x_0^0, x_1^0, \ldots, x_n^0)$ we have

$$f_k(x_0^0, x_1^0, \ldots, x_n^0) \neq 0,$$

then a similar result can be formulated for a function h defined on the open neighborhood of $(x_0^0, x_1^0, \ldots, x_{k-1}^0, x_{k+1}^0, \ldots, x_n^0)$.

Problem 5.1.1. A function $u = u(x_1, x_2, \ldots, x_n)$ is called homogeneous of degree zero in the Euler sense if for any $\lambda \in \mathbb{R}_+^1$ we have

$$u(\lambda \cdot x_1, \lambda \cdot x_2, \ldots, \lambda \cdot x_n) = u(x_1, x_2, \ldots, x_n).$$

Show that the function $u = u(x_1, x_2, \ldots, x_n)$ of class $\mathcal{C}^1$ is homogeneous of degree zero in the Euler sense if and only if there exists $\gamma = \gamma(\xi_1, \xi_2, \ldots, \xi_{n-1})$ such that:

$$u(x_1, x_2, \ldots, x_n) = \gamma\left(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \ldots, \frac{x_{n-1}}{x_n}\right).$$

Answer: From

$$u(\lambda \cdot x_1, \lambda \cdot x_2, \ldots, \lambda \cdot x_n) = u(x_1, x_2, \ldots, x_n)$$

we have $\dfrac{d}{d\lambda}[u(\lambda \cdot x_1, \lambda \cdot x_2, \ldots, \lambda \cdot x_n)] = 0$ i.e.

$$x_1 \frac{\partial u}{\partial x_1}(\lambda \cdot x_1, \lambda \cdot x_2, \ldots, \lambda \cdot x_n) + \ldots + x_n \frac{\partial u}{\partial x_n}(\lambda \cdot x_1, \lambda \cdot x_2, \ldots, \lambda \cdot x_n) = 0, \ (\forall)\lambda > 0.$$

For $\lambda = 1$ we obtain:

$$x_1 \frac{\partial u}{\partial x_1}(x_1, x_2, \ldots, x_n) + \ldots + x_n \frac{\partial u}{\partial x_n}(x_1, x_2, \ldots, x_n) = 0, \ (\forall)(x_1, x_2, \ldots, x_n),$$

and hence:

$$u(x_1, x_2, \ldots, x_n) = \gamma\left(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \ldots, \frac{x_{n-1}}{x_n}\right).$$

Exercises:

1. Solve the following linear first order partial differential equations:

$a)\quad y \cdot \dfrac{\partial u}{\partial x} - x \cdot \dfrac{\partial u}{\partial y} = 0$

$$\textbf{A:}\quad u(x,y) = \gamma(x^2+y^2)$$

$b)\quad x \cdot \dfrac{\partial u}{\partial x} + y \cdot \dfrac{\partial u}{\partial y} = 0$

$$\textbf{A:}\quad u(x,y) = \gamma\left(\frac{y}{x}\right)$$

$c)\quad x \cdot \dfrac{\partial u}{\partial x} - 2y \cdot \dfrac{\partial u}{\partial y} - z \cdot \dfrac{\partial u}{\partial z} = 0$

$$\textbf{A:}\quad u(x,y,z) = \gamma\left(x\sqrt{y}, xz\right)$$

$d)\quad xy \cdot \dfrac{\partial u}{\partial x} - \sqrt{1-y^2}\left(y \cdot \dfrac{\partial u}{\partial y} - z \cdot \dfrac{\partial u}{\partial z}\right) = xy \cdot \dfrac{\partial u}{\partial z}$

$$\textbf{A:}\quad u(x,y,z) = \gamma\left(2yz + x \cdot \left(y + \sqrt{1-y^2}\right), x \cdot e^{\arcsin y}\right)$$

2. Solve the following Cauchy Problems:

$a)\quad x \cdot \dfrac{\partial u}{\partial x} + y \cdot \dfrac{\partial u}{\partial y} = 0; \quad u(x,1) = x$

$$\textbf{A:}\quad u(x,y) = \frac{x}{y}$$

$b)\quad \sqrt{x} \cdot \dfrac{\partial u}{\partial x} + \sqrt{y} \cdot \dfrac{\partial u}{\partial y} + \sqrt{z} \cdot \dfrac{\partial u}{\partial z} = 0; \quad u(1,y,z) = y - z$

$$\textbf{A:}\quad u(x,y,z) = y - z + 2 \cdot (1-\sqrt{x}) \cdot (\sqrt{y} - \sqrt{z})$$

$c)\quad (1+x^2) \cdot \dfrac{\partial u}{\partial x} + xy \cdot \dfrac{\partial u}{\partial y} = 0; \quad u(0,y) = y^2$

$$\textbf{A:}\quad u(x,y) = \frac{y^2}{1+x^2}$$

5.2 Quasi-linear first order partial differential equations

Definition 5.2.1. *A quasi-linear first order partial differential equation is a functional dependence of the form*

$$\sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \cdot f_i(x_1, \ldots, x_n, u) - g(x_1, \ldots, x_n, u) = 0 \tag{5.3}$$

between the unknown function $u = u(x_1, \ldots, x_n)$ *and its partial derivatives* $\dfrac{\partial u}{\partial x_i}$, $i = \overline{1, n}$.

The functions $f_1, \ldots, f_n$ and g from (5.3) are given: $f_i, g : \Omega \subset \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^1$ and are supposed to be of class $\mathcal{C}^1$.

The equation (5.3) is called quasi-linear because it is linear in the partial derivatives $\dfrac{\partial u}{\partial x_k}$, but generally it is not linear in u.

Definition 5.2.2. *A solution of the equation (5.3) is a function* $u : D \subset \mathbb{R}^n \longrightarrow \mathbb{R}^1$ *of class* $\mathcal{C}^1$ *having the property that for any* $(x_1, \ldots, x_n) \in D$ *we have*

$$(x_1, \ldots, x_n, u(x_1, \ldots, x_n)) \in \Omega$$

and

$$\sum_{i=1}^{n} \frac{\partial u}{\partial x_i}(x_1, \ldots, x_n) \cdot f_i(x_1, \ldots, x_n, u(x_1, \ldots, x_n)) - g(x_1, \ldots, x_n, u(x_1, \ldots, x_n)) \equiv 0$$

In order to determine the solutions of the equation (5.3) we consider the linear first order partial differential equation:

$$\sum_{i=1}^{n} \frac{\partial v}{\partial x_i} \cdot f_i(x_1, \ldots, x_n, u) + \frac{\partial v}{\partial u} \cdot g(x_1, \ldots, x_n, u) = 0 \tag{5.4}$$

where $(x_1, \ldots, x_n, u) \in \Omega$.

Theorem 5.2.1. *If* v *is a solution of the equation (5.4) defined on* $\Omega' \subset \Omega$, *and* $(x_1^0, \ldots, x_n^0, u^0) \in \Omega'$ *such that*

$$\frac{\partial v}{\partial u}(x_1^0, \ldots, x_n^0, u^0) \neq 0$$

then the function $u = u(x_1, \ldots, x_n)$ *defined implicitly by the equation*

$$v(x_1, \ldots, x_n, u) - v(x_1^0, \ldots, x_n^0, u^0) = 0$$

is a solution of the equation (5.3).

Proof. Let be $u = u(x_1, \dots, x_n)$ defined implicitly by the equation

$$v(x_1, \dots, x_n, u) - v(x_1^0, \dots, x_n^0, u^0) = 0.$$

We have:

$$v(x_1, \dots, x_n, u(x_1, \dots, x_n)) - v(x_1^0, \dots, x_n^0, u^0) \equiv 0$$

and differentiating with respect to x_k we find:

$$\frac{\partial v}{\partial x_k}(x_1, \dots, x_n, u(x_1, \dots, x_n)) + \frac{\partial v}{\partial u}(x_1, \dots, x_n, u(x_1, \dots, x_n)) \cdot \frac{\partial u}{\partial x_k}(x_1, \dots, x_n) \equiv 0$$

for $k = 1, 2, \dots, n$.

Multiplying these equalities with $f_k(x_1, \dots, x_n, u(x_1, \dots, x_n))$ and summing them, we obtain:

$$\sum_{k=1}^{n} \frac{\partial v}{\partial x_k}(X, u(X)) \cdot f_k(X, u(X)) + \frac{\partial v}{\partial u}(X, u(X)) \cdot \sum_{k=1}^{n} \frac{\partial u}{\partial x_k}(X, u(X)) \cdot f_k(X, u(X)) \equiv 0$$

where $X = (x_1, \dots, x_n)$.

But as v is a solution of the equation (5.4)we have

$$\sum_{k=1}^{n} \frac{\partial v}{\partial x_k}(x_1, \dots, x_n, u(x_1, \dots, x_n)) \cdot f_k(x_1, \dots, x_n, u(x_1, \dots, x_n)) +$$

$$\frac{\partial v}{\partial u}(x_1, \dots, x_n, u(x_1, \dots, x_n)) \cdot g(x_1, \dots, x_n, u(x_1, \dots, x_n)) \equiv 0$$

and hence

$$\frac{\partial v}{\partial u}(x_1, \dots, x_n, u(x_1, \dots, x_n)) \cdot g(x_1, \dots, x_n, u(x_1, \dots, x_n))$$

$$= \frac{\partial v}{\partial u}(x_1, ..., x_n, u(x_1, ..., x_n)) \cdot \sum_{k=1}^{n} \frac{\partial u}{\partial x_k}(x_1, ..., x_n) \cdot f_k(x_1, ..., x_n, u(x_1, ..., x_n)).$$

Taking into account that $\frac{\partial v}{\partial u}(x_1, ..., x_n, u(x_1, ..., x_n)) \neq 0$ we obtain:

$$\sum_{k=1}^{n} \frac{\partial u}{\partial x_k}(x_1, ..., x_n) \cdot f_i(x_1, ..., x_n, u(x_1, ..., x_n)) = g(x_1, ..., x_n, u(x_1, ..., x_n)).$$

We found in this way that the function $u = u(x_1, \dots, x_n)$ is a solution of the equation (5.3). $\square$

This theorem shows that, the problem of solving a quasi-linear first order partial differential equation is reduced to solving a linear first order partial differential equation.

Definition 5.2.3. *For $(x_1^0, \dots, x_n^0) \in \Omega$ the problem of determining a solution $u(x_1, \dots, x_n)$ of (5.3) such that*

$$u(x_1^0, x_2, \dots, x_n) = \xi(x_2, \dots, x_n),$$

ξ being a given function of class $\mathcal{C}^1$, is called Cauchy Problem for the equation (5.3).

Theorem 5.2.2. *If $f_1(x_1^0,\dots,x_n^0,\xi(x_2^0,\dots,x_n^0))\neq 0$, then there exists an open neighborhood V of $(x_1^0,\dots,x_n^0)$ and a solution $u{=}u(x_1,...,x_n)$ of (5.3) defined on V such that*

$$u(x_1^0,x_2,\dots,x_n)=\xi(x_2,\dots,x_n).$$

Proof. There are n solutions $v_1,v_2,\dots,v_n$ of (5.4) defined on the neighborhood of a point $(x_1^0,\dots,x_n^0,\xi(x_2^0,\dots,x_n^0))$ such that

$$v_k(x_1^0,x_2,\dots,x_n,u)=x_{k+1},\quad k=1,2,\dots,n-1$$

and

$$v_n(x_1^0,x_2,\dots,x_n,u)=u.$$

The function v defined by

$$v(x_1,\dots,x_n,u)=v_n(x_1,\dots,x_n,u)-\xi(v_1(x_1,\dots,x_n,u),\dots,v_{n-1}(x_1,\dots,x_n,u))$$

is a solution of the equation (5.4) (obtained in function of the n independent prime integrals) and

$$v(x_1^0,x_2,\dots,x_n,u)=u-\xi(x_2,\dots,x_n).$$

From the implicit function theorem we have that there exists a neighborhood V of $(x_1^0,\dots,x_n^0)$ and a function $u=u(x_1,\dots,x_n)$ of class $\mathcal{C}^1$ defined on this neighborhood such that

$$v(x_1,\dots,x_n,u(x_1,\dots,x_n))\equiv 0.$$

Because

$$\frac{\partial v}{\partial u}(x_1,\dots,x_n,u(x_1,\dots,x_n))\neq 0$$

from the previous theorem we have that $u=u(x_1,\dots,x_n)$ is a solution of the equation (5.3). Hence:

$$v(x_1,\dots,x_n,u(x_1,\dots,x_n)){-}\xi(v_1(x_1,...,x_n,u(x_1,...,x_n)),\dots,v_{n-1}(x_1,...,x_n,u(x_1,...,x_n))\equiv 0$$

and, at x_i, we have:

$$u(x_1^0,x_2,\dots,x_n)-\xi(x_2,\dots,x_n)\equiv 0.$$

□

Observation 5.2.1. A first order partial differential equation of the form:

$$\sum_{i=1}^{n}\frac{\partial u}{\partial x_i}\cdot f_i=g \tag{5.5}$$

is called non-homogeneous linear first order partial differential equation. This fact is due to that, for $g=0$, the equation (5.5) is a homogeneous linear first order partial differential equation. In the equation (5.5), functions $f_1,\dots,f_n$ and g are of class $\mathcal{C}^1$ and depend on the variables $(x_1,\dots,x_n)\in\Omega$. The equation (5.5) is solved as the quasi-linear first order partial differential equations.

Exercises:

1. Solve the following quasi-linear first order partial differential equation:

$$a)\quad \frac{\partial u}{\partial x} \cdot (1 + \sqrt{u - x - y}) + \frac{\partial u}{\partial y} = 2$$

A: $\gamma(u - 2y,\, y + 2\sqrt{u - x - y}) = 0$

$$b)\quad \sum_{i=1}^{n} x_i \cdot \frac{\partial u}{\partial x_i} = m \cdot u$$

A: $\gamma\left(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \ldots, \frac{x_{n-1}}{x_n}, \frac{u}{x_n^m}\right) = 0$

$$c)\quad xy \cdot \frac{\partial u}{\partial x} - y^2 \cdot \frac{\partial u}{\partial y} + x(1 + x^2) = 0$$

A: $\gamma\left(xy\, u + \frac{x^2}{2} + \frac{x^2}{4},\, xy\right) = 0$

$$d)\quad 2y^4 \cdot \frac{\partial u}{\partial x} - xy \cdot \frac{\partial u}{\partial y} = x\sqrt{u^2 + 1}$$

A: $\gamma\left(x^2 + y^4,\, y\left(u + \sqrt{u^2 + 1}\right)\right) = 0$

2. Solve the following Cauchy Problems:

$$a)\quad \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = \frac{y - x}{u}, \quad u(1, y) = y^2$$

A: $u^2(x, y, z) = (x + y - 1)^4 + 2xy - 2(x + y - 1)$

$$b)\quad xy \cdot \frac{\partial u}{\partial x} - y^2 \cdot \frac{\partial u}{\partial y} = x, \quad u(1, y) = \frac{3}{2y}$$

A: $u(x, y) = \frac{x^2 + 2}{2xy}$

$$c)\quad u \cdot \frac{\partial u}{\partial x} + (u^2 - x^2) \cdot \frac{\partial u}{\partial y} + x = 0, \quad u(x, x^2) = 2x$$

A: $y^2 + u^2(x, y) = 5\,(x \cdot u(x, y) - y)$

5.3 Symbolic calculus of the solutions of first order partial differential equations

For the symbolic calculus of the solutions of first order partial differential equations, *Maple* uses the function *pdsolve* (finds solutions for partial differential equations (PDEs) and systems of PDEs) of one of the following syntaxes :

pdsolve(PDE, f, INTEGRATE, build);

pdsolve(PDE system, funcs, other options);

in which:

PDE	- partial differential equation which we want to solve
f	- name of the unknown function (imply the existence of the partial derivatives
$INTEGRATE$	-(optional) indicate the automat integration of PDE which appear when PDE is solved by the separate variables method
build	- (optional) indicate the display of the explicit form of the solution (if it is possible)

For exemplification, we consider the first order partial differential equation:

$$y \cdot \frac{\partial u}{\partial x} - x \cdot \frac{\partial u}{\partial y} = 0 \tag{5.6}$$

Using *pdsolve* we obtain the general solution (all solutions) of the above equation:

```
> PDE1 := y*diff(u(x,y),x)-x*diff(u(x,y),y) = 0;
```

$$PDE1 := y\frac{\partial}{\partial x}u(x,y) - x\frac{\partial}{\partial y}u(x,y) = 0$$

```
> pdsolve(PDE1);
```

$$u(x,y) = \_F1\left(x^2+y^2\right)$$

It can be seen that the general solution is displayed with the help of the function $F1$ which can be any function of class $\mathcal{C}^1$. To determine one solution of the equation, we need an initial condition but, in *pdsolve* there doesn't exist any parameter which would permit us to specify the initial condition. All these are due to the fact that, from mathematical point of view, there doesn't exist a theorem which proves the uniqueness of the solution of the IVP for quasi-linear first order partial differential equations.

In the following we will determine the general solution of a first order PDE in which the unknown function has three variables x, y, z:

$$\sqrt{x} \cdot \frac{\partial u}{\partial x} + \sqrt{y} \cdot \frac{\partial u}{\partial y} + \sqrt{z} \cdot \frac{\partial u}{\partial z} = 0 \tag{5.7}$$

```
> PDE3 :=sqrt(x)*diff(u(x,y,z),x)+sqrt(y)*diff(u(x,y,z),y)+

     sqrt(z)*diff(u(x,y,z),z) = 0;
```

$$PDE3 := \sqrt{x}\frac{\partial}{\partial x}u(x,y,z) + \sqrt{y}\frac{\partial}{\partial y}u(x,y,z) + sqrtz\frac{\partial}{\partial z}u(x,y,z) = 0$$

```
> pdsolve(PDE3);
```

$$u(x,y,z) = \_F1\left(\sqrt{x}-\sqrt{y}, \sqrt{z}-\sqrt{y}\right)$$

Chapter 6

Linear second order partial differential equations

6.1 Classification of linear second order partial differential equations

Definition 6.1.1. *A linear second order partial differential equation is a functional dependence of the form*

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\cdot\frac{\partial^2 u}{\partial x_i \partial x_j}+\sum_{i=1}^{n} b_i\cdot\frac{\partial u}{\partial x_i}+c\cdot u+f=0 \tag{6.1}$$

between the unknown function $u = u(x_1, \ldots, x_n)$ and its first and second order partial derivatives.

The functions $a_{ij}, b_i, c, f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^1$ form the equation (6.1) are known and at least are continuous on Ω.

Definition 6.1.2. *A classical solution of the equation (6.1) is a function $u : D \subset \Omega \to \mathbb{R}^1$ of class $\mathcal{C}^2$ having the property that for any $(x_1, \ldots, x_n) \in D$ we have:*

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x_1,\ldots,x_n)\cdot\frac{\partial^2 u}{\partial x_i \partial x_j}(x_1,\ldots,x_n)+\sum_{i=1}^{n} b_i(x_1,\ldots,x_n)\cdot\frac{\partial u}{\partial x_i}(x_1,\ldots,x_n)+$$

$$+c(x_1,\ldots,x_n)\cdot u(x_1,\ldots,x_n)+f(x_1,\ldots,x_n)=0.$$

Let be $T : \Omega \to \Omega'$ a diffeomorphism of class $\mathcal{C}^2$ having the components $\xi_k = \xi_k(x_1, \ldots, x_n), k = \overline{1,n}$ and $u = u(x_1, \ldots, x_n)$ a solution of the equation (6.1). Let us consider $v = u \circ T^{-1}$. Differentiating the composed functions we obtain:

$$\frac{\partial u}{\partial x_i}=\sum_{k=1}^{n}\frac{\partial v}{\partial \xi_k}\cdot\frac{\partial \xi_k}{\partial x_i}$$

$$\frac{\partial^2 u}{\partial x_i \partial x_j}=\sum_{k=1}^{n}\sum_{l=1}^{n}\frac{\partial^2 v}{\partial \xi_k \partial \xi_l}\cdot\frac{\partial \xi_k}{\partial x_i}\cdot\frac{\partial \xi_l}{\partial x_j}+\sum_{k=1}^{n}\frac{\partial v}{\partial \xi_k}\cdot\frac{\partial^2 \xi_k}{\partial x_i \partial x_j}.$$

Replacing these derivatives in the equation (6.1) we obtain

$$\sum_{k=1}^{n}\sum_{l=1}^{n}\overline{a}_{kl}\cdot\frac{\partial^2 v}{\partial\xi_k\partial\xi_l}+\sum_{k=1}^{n}\overline{b}_k\cdot\frac{\partial v}{\partial\xi_k}+c\cdot v+f=0 \tag{6.2}$$

where

$$\overline{a}_{kl}=\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}\cdot\frac{\partial\xi_k}{\partial x_i}\cdot\frac{\partial\xi_l}{\partial x_j}$$

$$\overline{b}_k=\sum_{i=1}^{n}b_i\cdot\frac{\partial\xi_k}{\partial x_i}+\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}\cdot\frac{\partial^2\xi_k}{\partial x_i\partial x_j}.$$

The equation (6.2) is equivalent to the equation (6.1), the solutions u of (6.1) being obtained from the solutions v of the equation (6.2) with the formula $u=v\circ T$.

On the other hand, at any fixed arbitrary point $(x_1^0,\ldots,x_n^0)\in\Omega$ we can consider the quadratic form

$$\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}^0\cdot y_i\cdot y_j$$

where $a_{ij}^0=a_{ij}(x_1^0,\ldots,x_n^0)$. Making the changement of variables

$$y_i=\sum_{k=1}^{n}\frac{\partial\xi_k}{\partial x_i}\cdot\eta_k$$

the considered quadratic form will be transformed into

$$\sum_{k=1}^{n}\sum_{l=1}^{n}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}^0\cdot\frac{\partial\xi_k}{\partial x_i}\cdot\frac{\partial\xi_l}{\partial x_j}\right)\eta_k\eta_l=\sum_{k=1}^{n}\sum_{l=1}^{n}\overline{a}_{kl}\cdot\eta_k\eta_l.$$

It follows that the coefficients of the second order partial derivatives are changed in the same way as the coefficients of the quadratic form, by the action of the linear transformation:

$$y_i=\sum_{k=1}^{n}\frac{\partial\xi_k}{\partial x_i}\cdot\eta_k.$$

It is known that, by choosing the adequate linear transformation, the quadratic form can be reduced to the canonic form (i.e. the matrix a_{ij}^0 can be reduced to the diagonal form: $|\overline{a}_{ii}^0|=1$ or 0 and $\overline{a}_{ij}^0=0$ if $i\neq j$). This means that, choosing an adequate linear transformation $\xi_k=\xi_k(x_1,...,x_n)$, $k=\overline{1,n}$, the coefficients of the partial derivatives $\dfrac{\partial^2 v}{\partial\xi_k^2}$ at the point $\xi_k=\xi_k(x_1^0,...,x_n^0)$, $k=\overline{1,n}$ will be equal to $+1,-1$ or 0, and the coefficients of the partial derivatives $\dfrac{\partial^2 v}{\partial\xi_k\partial\xi_l}$ will be null.

According to the law of inertia, the number of the positive, negative or null coefficients $\overline{a}_{ii}^0$ doesn't depend on the linear transformation which reduces the quadratic form to its canonic form. This allows us to give the following definition:

Definition 6.1.3. *i. The equation (6.1) is called* elliptic *at the point* $(x_1^0, \ldots, x_n^0)$ *if all* n *coefficients* $\overline{a}_{ii}^0$ *have the same sign and* $\overline{a}_{ij}^0 = 0$, $i \neq j$.

ii. The equation (6.1) is called hyperbolic *at the point* $(x_1^0, \ldots, x_n^0)$ *if* $(n-1)$ *coefficients* $\overline{a}_{ii}^0$ *have the same sign, one coefficient has opposite sign, and* $\overline{a}_{ij}^0 = 0$, $i \neq j$.

iii. The equation (6.1) is called ultra-hyperbolic *at the point* $(x_1^0, \ldots, x_n^0)$ *if there are* m *coefficients* $\overline{a}_{ii}^0$ *of one sign,* $(n-m)$ *coefficients of the opposite sign, and* $\overline{a}_{ij}^0 = 0$, $i \neq j$.

iv. The equation (6.1) is called parabolic *if at least one coefficient* $\overline{a}_{ii}^0$ *is null, and* $\overline{a}_{ij}^0 = 0$, $i \neq j$.

Observation 6.1.1. According to the previous definition the equation (6.1) has one of the following standard form:

i) The elliptic equation at the point $(x_1^0, \ldots, x_n^0)$:

$$\frac{\partial^2 v}{\partial \xi_1^2} + \frac{\partial^2 v}{\partial \xi_2^2} + \ldots + \frac{\partial^2 v}{\partial \xi_n^2} + \Phi = 0; \tag{6.3}$$

ii) The hyperbolic equation at the point $(x_1^0, \ldots, x_n^0)$:

$$\frac{\partial^2 v}{\partial \xi_1^2} = \sum_{i=2}^{n} \frac{\partial^2 v}{\partial \xi_i^2} + \Phi; \tag{6.4}$$

iii) The ultra-hyperbolic equation at the point $(x_1^0, \ldots, x_n^0)$:

$$\sum_{i=1}^{m} \frac{\partial^2 v}{\partial \xi_i^2} = \sum_{i=m+1}^{n} \frac{\partial^2 v}{\partial \xi_i^2} + \Phi; \tag{6.5}$$

iv) The parabolic equation at the point $(x_1^0, \ldots, x_n^0)$:

$$\sum_{i=1}^{n-m} \left(\pm \frac{\partial^2 v}{\partial \xi_i^2} \right) + \Phi = 0, \ (m > 0). \tag{6.6}$$

The standard form of the equation (6.1) at $(x_1^0, \ldots, x_n^0)$ can be obtained by reducing the quadratic form:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^0 \, y_i \, y_j$$

to the canonic form. For this, the transformation $\xi_k = \xi_k(x_1, \ldots, \xi_n), k = \overline{1, n}$ is chosen such that the linear transformation:

$$y_i = \sum_{k=1}^{n} \frac{\partial \xi_k}{\partial x_i}(x_1^0, \ldots, x_n^i) \cdot \eta_k$$

transforms the quadratic form into the canonic form.

Let us study the possibility to transform the equation (6.1) into the (standard) canonic form in a neighborhood of a point $(x_1, \ldots, x_n)$.

To transform the equation (6.1) into the canonic form on a domain, first of all, we must impose that the functions $\xi_k(x_1 \ldots, x_n), k = \overline{1,n}$ are differentiable $\overline{a}_{kl} = 0$ for $k \neq l$. The number of these conditions is $\dfrac{n(n-1)}{2}$ and is less than n (n represents the number of the functions $\xi_k, k = \overline{1,n}$) if $n < 3$. Due to this reason, for $n > 3$ the equation (6.1) cannot be written in the canonic form in the neighborhood of a point $(x_1, \ldots, x_n)$.

For $n = 3$ generally, the non-diagonal elements can be null but, the diagonal elements would be different of $\pm$1,0. Hence, the equation (6.1) cannot be reduced to the canonic form in the neighborhood of a point.

For the $n = 2$ case only, we can write the partial differential equation in the canonic form in a neighborhood of a point.

Exercises

1. Write the canonic form of the equation:

$$a_{11} \cdot \frac{\partial^2 u}{\partial x^2} + 2a_{12} \cdot \frac{\partial^2 u}{\partial x \partial y} + a_{22} \cdot \frac{\partial^2 u}{\partial y^2} + b_1 \cdot \frac{\partial u}{\partial x} + b_2 \cdot \frac{\partial u}{\partial y} + c \cdot u + f(x,y) = 0$$

where a_{ij}, b_i, c are constants.

A: Writing the associated characteristic equation:

$$a_{11} \left(\frac{dy}{dx}\right)^2 - 2a_{12} \left(\frac{dy}{dx}\right) + a_{22} = 0$$

we obtain the discriminant: $\Delta = (a_{12})^2 - a_{11} \cdot a_{22}$.

- if $\Delta > 0$ *(hyperbolic case)*, then the characteristic equation has two real roots:

$$\begin{cases} \dfrac{dy}{dx} = \dfrac{a_{12} + \sqrt{(a_{12})^2 - a_{11} \cdot a_{22}}}{a_{11}} \\ \dfrac{dy}{dx} = \dfrac{a_{12} - \sqrt{(a_{12})^2 - a_{11} \cdot a_{22}}}{a_{11}} \end{cases}$$

  Solving these ordinary differential equations we obtain $f(x,y) = c_1$, and $g(x,y) = c_2$, respectively.

  The change of variables which leads to the first canonic form will be:

$$\begin{cases} \xi(x,y) = f(x,y) \\ \eta(x,y) = g(x,y). \end{cases}$$

  After that, the second canonic form corresponding to the hyperbolic equation can be obtained by

$$\begin{cases} s(x,y) = \xi(x,y) + \eta(x,y) \\ t(x,y) = \xi(x,y) - \eta(x,y). \end{cases}$$

or can be obtained directly by the change:

$$\begin{cases} s(x,y) = f(x,y) + g(x,y) \\ t(x,y) = f(x,y) - g(x,y). \end{cases}$$

- if $\Delta = 0$ *(parabolic case)*, then the characteristic equation has two real and equal roots:

$$\frac{dy}{dx} = \frac{a_{12}}{a_{11}}.$$

Solving this ordinary differential equation we obtain $f(x,y) = c_1$ which leads us to the changing $\xi(x,y) = f(x,y)$. For $\eta(x,y)$ we adequately choose a function $g(x,y)$ such that the properties of the change of variables are satisfied, for example:

$$\begin{cases} \xi(x,y) = f(x,y) \\ \eta(x,y) = x \end{cases} \qquad \begin{cases} \xi(x,y) = f(x,y) \\ \eta(x,y) = y. \end{cases}$$

- if $\Delta < 0$ *(elliptic case)*, then the equation has two complex roots:

$$\begin{cases} \dfrac{dy}{dx} = \dfrac{a_{12}}{a_{11}} + i\sqrt{a_{11} \cdot a_{22} - (a_{12})^2} \\ \\ \dfrac{dy}{dx} = \dfrac{a_{12}}{a_{11}} - i\sqrt{a_{11} \cdot a_{22} - (a_{12})^2} \end{cases}$$

Solving these ordinary differential equations we obtain $\alpha(x,y) \pm i\beta(x,y) = \tau$ which leads us to the changes:

$$\begin{cases} \xi(x,y) = \alpha(x,y) \\ \eta(x,y) = \beta(x,y). \end{cases}$$

2. Find the elliptical, hyperbolical and parabolical domains for the equation:

$$y \cdot \frac{\partial^2 u}{\partial x^2} + x \cdot \frac{\partial^2 u}{\partial y^2} = 0$$

A: Because $\Delta = b^2 - ac = -xy$ we have:

- the equation is hyperbolic for ($\Delta > 0$) in cadres II and IV;

- the equation is elliptic for ($\Delta < 0$) in cadres I and III.

3. Write the canonic forms for the following partial differential equations:

a) $\dfrac{\partial^2 u}{\partial x^2} + 2 \cdot \dfrac{\partial^2 u}{\partial x \partial y} + \dfrac{\partial^2 u}{\partial y^2} = 0$

A: The equation is parabolic $(\Delta = 0)$; the canonic form is: $\dfrac{\partial^2 v}{\partial \eta^2} = 0$;

b) $x^2 \cdot \dfrac{\partial^2 u}{\partial x^2} - 4 \cdot y^2 \cdot \dfrac{\partial^2 u}{\partial y^2} = 0, \; x, y > 0$

A: The equation is hyperbolic $(\Delta > 0)$; the first canonic form is:

$$\frac{\partial^2 v}{\partial \xi \partial \eta} - \frac{3}{8} \cdot \frac{\partial v}{\partial \xi} - \frac{1}{8} \cdot \frac{\partial v}{\partial \eta} = 0;$$

c) $x^2 \dfrac{\partial^2 u}{\partial x^2} + 4 \cdot y^2 \cdot \dfrac{\partial^2 u}{\partial y^2} = 0, \; x, y > 0$

A: The equation is ellpitic$(\Delta < 0)$; the canonic form is:

$$\frac{\partial^2 v}{\partial \xi^2} + \frac{\partial^2 v}{\partial \eta^2} - \frac{\partial v}{\partial \xi} + \frac{1}{2} \cdot \frac{\partial v}{\partial \eta} = 0;$$

6.2 Green formulas and the representation formulas in two dimensions

In this section, we present some results of differential and integral calculus for functions of two variables which appear frequently in PDEs.

Theorem 6.2.1 (relation between the double integral on a bounded domain and the curve integral on its border). *If Ω is a bounded domain in $\mathbb{R}^2$ having smooth (or partially smooth) boundary $\partial\Omega$, and the functions $P, Q : \overline{\Omega} \to \mathbb{R}^1$ are continuous in $\overline{\Omega}$ and of class $\mathcal{C}^1$ in Ω, then the following equality holds:*

$$\iint_{\Omega} \left(\frac{\partial P}{\partial x_1} + \frac{\partial Q}{\partial x_2} \right) dx_1 dx_2 = \int_{\partial\Omega} (P \cdot \cos\alpha_1 + Q \cdot \cos\alpha_2) ds \tag{6.7}$$

where: α_i is the angle between the unit vector $\overline{n}$ of the external normal vector at $\partial\Omega$ and the $\overline{e}_i$ axis, ds is the measure of the arc element $\partial\Omega$.

Proof. We present the idea of the proof of this theorem.

The equality (6.7) is obtained by summing the equalities:

$$\begin{aligned} \iint_{\Omega} \frac{\partial P}{\partial x_1} dx_1 dx_2 &= \int_{\partial\Omega} P \cdot \cos\alpha_1 \cdot ds \\ \iint_{\Omega} \frac{\partial Q}{\partial x_2} dx_1 dx_2 &= \int_{\partial\Omega} Q \cdot \cos\alpha_2 \cdot ds \end{aligned} \tag{6.8}$$

and it follows that the proof of (6.7) is reduced to the proof of the equalities (6.8).

The first equality from (6.8) is obtained by computing the double integral as an iterated integral and afterwards, computing the result as a curve integral. Thus, we have:

$$\iint_\Omega \frac{\partial P}{\partial x_1} dx_1 dx_2 = \int_c^d \left(\int_{x_1^-(x_2)}^{x_1^+(x_2)} \frac{\partial P}{\partial x_1} dx_1 \right) dx_2 = \int_c^d \left[P(x_1^+(x_2), x_2) - P(x_1^-(x_2), x_2) \right] dx_2$$

in which the following curves are used:

$\Gamma^+ : \begin{cases} x_1 = x_1^+(x_2) \\ x_2 = x_2 \end{cases}$ $\Gamma^- : \begin{cases} x_1 = x_1^-(x_2) \\ x_2 = x_2 \end{cases}$ In this figure, we see that

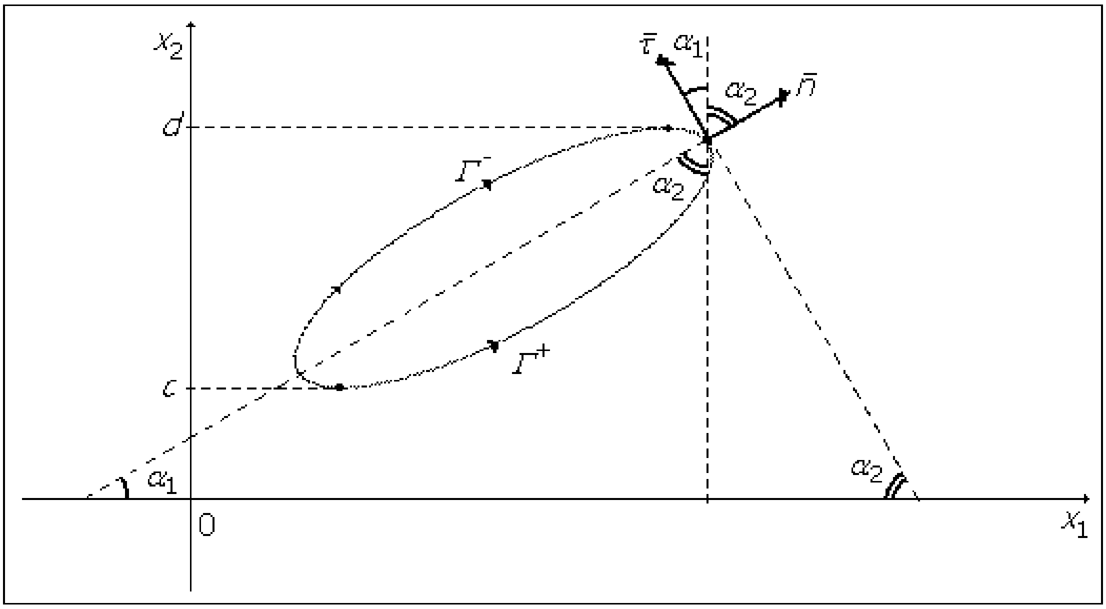


Figure 6.1:

$$\cos \alpha_1 = \cos \sphericalangle(\tau, Ox_2) = \frac{1}{\sqrt{1 + \left(\frac{dx_1^+}{dx_2} \right)^2}}$$

and it follows that:

$$\int_c^d \left[P(x_1^+(x_2), x_2) - P(x_1^-(x_2), x_2) \right] dx_2 =$$

$$= \int_c^d P(x_1^+(x_2), x_2) \cdot \cos \alpha_1 \cdot \sqrt{1 + \left(\frac{dx_1^+}{dx_2} \right)^2} dx_2 +$$

$$+ \int_c^d P(x_1^-(x_2), x_2) \cdot \cos \alpha_1 \cdot \sqrt{1 + \left(\frac{dx_1^-}{dx_2} \right)^2} dx_2 = \int_{\partial\Omega} P \cos \alpha_1 ds$$

The second equality from (6.8) can be obtained in the same way. □

Theorem 6.2.2 (Integration by parts). *If Ω is a bounded domain in $\mathbb{R}^2$ with smooth (partially smooth) boundary $\partial\Omega$, and $P, Q : \overline{\Omega} \to \mathbb{R}^1$ are continuous functions in $\overline{\Omega}$ and of class $\mathcal{C}^1$ in Ω then the following equalities hold:*

$$\iint_\Omega \frac{\partial P}{\partial x_i} \cdot Q dx_1 dx_2 = \int_{\partial\Omega} P \cdot Q \cdot \cos\alpha_i ds - \iint_\Omega P \frac{\partial Q}{\partial x_i} dx_1 dx_2,\ i = \overline{1,2} \tag{6.9}$$

Proof. Because:

$$\iint_\Omega \frac{\partial}{\partial x_i}(P \cdot Q) dx_1 dx_2 = \iint_\Omega \frac{\partial P}{\partial x_i} \cdot Q dx_1 dx_2 + \iint_\Omega \frac{\partial Q}{\partial x_i} \cdot P dx_1 dx_2,$$

and on the other hand:

$$\iint_\Omega \frac{\partial}{\partial x_i}(P \cdot Q) dx_1 dx_2 = \int_{\partial\Omega} P \cdot Q \cdot \cos\alpha_i \cdot ds$$

by equating we obtain:

$$\iint_\Omega \frac{\partial P}{\partial x_i} \cdot Q dx_1 dx_2 + \iint_\Omega \frac{\partial Q}{\partial x_i} \cdot P dx_1 dx_2 = \int_{\partial\Omega} P \cdot Q \cos\alpha_i \cdot ds$$

and in this way (6.9) is obtained. □

Theorem 6.2.3 (The first Green formula). *If Ω is a bounded domain in $\mathbb{R}^2$ with smooth (partially smooth) boundary $\partial\Omega$, and the functions $P, Q : \overline{\Omega} \to \mathbb{R}^1$ are of class $\mathcal{C}^1$ in $\overline{\Omega}$ and of class $\mathcal{C}^2$ in Ω then the following equality holds:*

$$\iint_\Omega P \Delta Q dx_1 dx_2 = \int_{\partial\Omega} P \cdot \frac{\partial Q}{\partial \overline{n}} ds - \iint_\Omega \nabla P \cdot \nabla Q dx_1 dx_2 \tag{6.10}$$

Proof. In formula (6.10), ΔQ represents the Laplacian of the function Q i.e.

$$\Delta Q = \frac{\partial^2 Q}{\partial x_1^2} + \frac{\partial^2 Q}{\partial x_2^2}$$

and $\frac{\partial Q}{\partial \overline{n}}$ is the normal derivative defined on $\partial\Omega$ by:

$$\frac{\partial Q}{\partial \overline{n}} = \frac{\partial Q}{\partial x_1} \cdot n_1 + \frac{\partial Q}{\partial x_2} \cdot n_2 == \frac{\partial Q}{\partial x_1} \cdot \cos\alpha_1 + \frac{\partial Q}{\partial x_2} \cdot \cos\alpha_2$$

where $\cos\alpha_i =< \overline{n}, \overline{e_i} >= n_i$, $i = 1, 2$; ∇P and ∇Q are vector functions (the gradients of the functions P and Q) defined by

$$\nabla P = \frac{\partial P}{\partial x_1} \cdot \overline{e_1} + \frac{\partial P}{\partial x_2} \cdot \overline{e_2}$$

$$\nabla Q = \frac{\partial Q}{\partial x_1} \cdot \overline{e_1} + \frac{\partial Q}{\partial x_2} \cdot \overline{e_2}.$$

We compute the left term of the equality (6.10), taking into account the formula of integration by parts (6.9):

$$\begin{aligned}
&\iint_\Omega P\Delta Q dx_1dx_2 = \\
&= \iint_\Omega P\left[\frac{\partial}{\partial x_1}\left(\frac{\partial Q}{\partial x_1}\right)+\frac{\partial}{\partial x_2}\left(\frac{\partial Q}{\partial x_2}\right)\right]dx_1dx_2 = \\
&= \int_{\partial\Omega} P\cos\alpha_1\frac{\partial Q}{\partial x_1}ds - \iint_\Omega \frac{\partial P}{\partial x_1}\frac{\partial Q}{\partial x_1}dx_1dx_2 + \int_{\partial\Omega} P\cos\alpha_2\frac{\partial Q}{\partial x_2}ds - \iint_\Omega \frac{\partial P}{\partial x_2}\frac{\partial Q}{\partial x_2}dx_1dx_2 = \\
&= \int_{\partial\Omega} P\left[\frac{\partial Q}{\partial x_1}\cos\alpha_1 + \frac{\partial Q}{\partial x_2}\cos\alpha_2\right]ds - \iint_\Omega\left(\frac{\partial P}{\partial x_1}\frac{\partial Q}{\partial x_1}+\frac{\partial P}{\partial x_2}\frac{\partial Q}{\partial x_2}\right)dx_1dx_2 = \\
&= \int_{\partial\Omega} P\left[\frac{\partial Q}{\partial x_1}\cos\alpha_1 + \frac{\partial Q}{\partial x_2}\cos\alpha_2\right]ds - \iint_\Omega \nabla P\nabla Q dx_1dx_2 = \\
&= \int_{\partial\Omega} P\frac{\partial Q}{\partial \overline{n}}ds - \iint_\Omega \nabla P\nabla Q dx_1dx_2
\end{aligned}$$

□

Theorem 6.2.4 (The second Green formula). *If Ω is a bounded domain in $\mathbb{R}^2$ with smooth (partially smooth) boundary $\partial\Omega$, and the functions $P, Q : \overline{\Omega} \to \mathbb{R}^1$ are of class $\mathcal{C}^1$ in $\overline{\Omega}$ and of class $\mathcal{C}^2$ in Ω, then the following equality holds:*

$$\iint_\Omega (P\Delta Q - Q\Delta P)dx_1dx_2 = \int_{\partial\Omega}\left(P\cdot\frac{\partial Q}{\partial\overline{n}} - Q\cdot\frac{\partial P}{\partial\overline{n}}\right)ds. \tag{6.11}$$

Proof. The equality (6.11) is obtained by writing the equality (6.10) for the functions $P\cdot\nabla Q$ and $Q\cdot\nabla P$, and subtracting them. □

Theorem 6.2.5 (Representation theorem for functions of two variables). *If Ω is a bounded domain in $\mathbb{R}^2$ having smooth (partially smooth) boundary $\partial\Omega$, and the function $u : \overline{\Omega} \to \mathbb{R}^1$ is of class $\mathcal{C}^1$ in $\overline{\Omega}$ and of class $\mathcal{C}^2$ in Ω, then for any $X = (x_1, x_2) \in \Omega$ the following equality takes place:*

$$\begin{aligned}
u(X) &= -\frac{1}{2\pi}\iint_\Omega \ln\frac{1}{\|X-Y\|}\cdot\Delta u(Y)dy_1dy_2 + \\
&+ \frac{1}{2\pi}\int_{\partial\Omega}\ln\frac{1}{\|X-Y\|}\cdot\frac{\partial u}{\partial\overline{n}_Y}(Y)ds_Y - \\
&- \frac{1}{2\pi}\int_{\partial\Omega}u(Y)\cdot\frac{\partial}{\partial\overline{n}_Y}\left(\ln\frac{1}{\|X-Y\|}\right)ds_Y.
\end{aligned} \tag{6.12}$$

Proof. The function

$$E(X) = -\frac{1}{2\pi}\ln\frac{1}{\|X\|}$$

defined for any $X \in \mathbb{R}^2$, $X \neq 0$ is of class $\mathcal{C}^2$ in $\mathbb{R}^2\setminus\{0\}$ and satisfies $\Delta E = 0$. For a fixed $X \in \mathbb{R}^2$, we consider the function:

$$E(X-Y) = -\frac{1}{2\pi}\ln\frac{1}{\|X-Y\|}$$

defined for any $Y \in \mathbb{R}^2 \backslash \{X\}$. This function of Y is of class $\mathcal{C}^2$ and satisfies $\Delta_Y E(X-Y) = 0$. Moreover, for any $Y \in \mathbb{R}^2$, Y-fixed, the function $E(X-Y) = -\frac{1}{2\pi}\ln\frac{1}{\|X-Y\|}$ defined for any $X \in \mathbb{R}^2, X \neq Y$ is of class $\mathcal{C}^2$ and verifies $\Delta_X E(X-Y) = 0$.

We consider the fixed $X \in \Omega$ and $\varepsilon > 0$ such that for any Y with $\|X-Y\| \leq \varepsilon$, we have $Y \in \Omega$. Let us denote by $\overline{B}(X,\varepsilon)$ the disc centered in X of radius ε:

$$\overline{B}(X,\varepsilon) = \{Y : \|X-Y\| \leq \varepsilon\}$$

and $\Omega_\varepsilon = \Omega - \overline{B}(X,\varepsilon)$. We consider the functions

$$Y \longmapsto u(Y) \text{ and } Y \longmapsto -\frac{1}{2\pi}\ln\frac{1}{\|X-Y\|} = E(X-Y)$$

which are of class $\mathcal{C}^2$ in Ω_ε and of class $\mathcal{C}^1$ in $\overline{\Omega}_\varepsilon$. Writing the second Green formula for these functions we have:

$$\begin{aligned}
-\frac{1}{2\pi}\iint_{\Omega_\varepsilon} \ln\frac{1}{\|X-Y\|}\cdot(\Delta u)(Y)dy_1dy_2 &= -\frac{1}{2\pi}\int_{\partial\Omega_\varepsilon} \ln\frac{1}{\|X-Y\|}\cdot\frac{\partial u}{\partial\overline{n}_Y}(Y)ds_Y + \\
&+ \frac{1}{2\pi}\int_{\partial\Omega_\varepsilon} u(Y)\cdot\frac{\partial}{\partial\overline{n}_Y}\left(\ln\frac{1}{\|X-Y\|}\right)\cdot ds_Y
\end{aligned}$$

The boundary $\partial\Omega_\varepsilon$ of the set Ω_ε has two parts: $\partial\Omega_\varepsilon = \partial\Omega \cup S_\varepsilon$ where $S_\varepsilon = \{Y : \|Y-X\| = \varepsilon\}$ such that the integrals from the right hand side can be written as follows:

$$\begin{aligned}
&-\frac{1}{2\pi}\int_{\partial\Omega_\varepsilon} \ln\frac{1}{\|X-Y\|}\ \frac{\partial u}{\partial\overline{n}_Y}(Y)ds_Y = \\
&\qquad = -\frac{1}{2\pi}\int_{\partial\Omega} \ln\frac{1}{\|X-Y\|}\cdot\frac{\partial u}{\partial\overline{n}_Y}ds_Y - \frac{1}{2\pi}\int_{S_\varepsilon} \ln\frac{1}{\|X-Y\|}\cdot\frac{\partial u}{\partial\overline{n}_Y}ds_Y \\
&\qquad = -\frac{1}{2\pi}\int_{\partial\Omega} \ln\frac{1}{\|X-Y\|}\cdot\frac{\partial u}{\partial\overline{n}_Y}(Y)ds_Y + \frac{1}{2\pi}\ln\varepsilon\cdot 2\pi\varepsilon\cdot\frac{\partial n}{\partial\overline{n}_Y}(Y^*)
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{2\pi}\int_{\partial\Omega_\varepsilon} u(Y)\frac{\partial}{\partial\overline{n}_Y}\left(\ln\frac{1}{\|X-Y\|}\right)ds_Y = \\
&= \frac{1}{2\pi}\int_{\partial\Omega} u(Y)\frac{\partial}{\partial\overline{n}_Y}\left(\ln\frac{1}{\|X-Y\|}\right)ds_Y + \frac{1}{2\pi}\int_{S_\varepsilon} u(Y)\frac{\partial}{\partial\overline{n}_Y}\left(\ln\frac{1}{\|X-Y\|}\right)ds_Y = \\
&= \frac{1}{2\pi}\int_{\partial\Omega} u(Y)\frac{\partial}{\partial\overline{n}_Y}\left(\ln\frac{1}{\|X-Y\|}\right)ds_Y + \frac{1}{2\pi}\int_{S_\varepsilon} u(Y)\left[\frac{x_1-y_1}{\|X-Y\|^2}\cos\alpha_1 + \frac{x_2-y_2}{\|X-Y\|^2}\cos\alpha_2\right]ds_Y = \\
&= \frac{1}{2\pi}\int_{\partial\Omega} u(Y)\frac{\partial}{\partial\overline{n}_Y}\left(\ln\frac{1}{\|X-Y\|}\right)ds_Y + \frac{1}{2\pi}\int_{S_\varepsilon} u(Y)\left[\frac{(x_1-y_1)^2+(x_2-y_2)^2}{\|X-Y\|^3}\right]ds_Y = \\
&= \frac{1}{2\pi}\int_{\partial\Omega} u(Y)\frac{\partial}{\partial\overline{n}}\left(\ln\frac{1}{\|X-Y\|}\right)ds_Y + \frac{1}{2\pi}\int_{S_\varepsilon} u(Y)\frac{1}{\|X-Y\|}ds_Y = \\
&= \frac{1}{2\pi}\int_{\partial\Omega} u(Y)\frac{\partial}{\partial\overline{n}_Y}\left(\ln\frac{1}{\|X-Y\|}\right)ds_Y + \frac{1}{2\pi\varepsilon}\int_{S_\varepsilon} u(Y)ds_Y
\end{aligned}$$

For $\varepsilon \longmapsto 0$ we have:

$$\lim_{\varepsilon\to 0} -\frac{1}{2\pi}\int_{\partial\Omega_\varepsilon} \ln\frac{1}{\|X-Y\|}\cdot\frac{\partial u}{\partial\overline{n}_Y}u(Y)ds_Y = -\frac{1}{2\pi}\int_{\partial\Omega} \ln\frac{1}{\|X-Y\|}\cdot\frac{\partial u}{\partial\overline{n}_Y}u(Y)ds_Y$$

$$\lim_{\varepsilon\to 0}\frac{1}{2\pi}\int_{\partial\Omega_\varepsilon}u(Y)\cdot\frac{\partial}{\partial\overline{n}_Y}\left(\ln\frac{1}{\|X-Y\|}\right)ds_Y=\frac{1}{2\pi}\int_{\partial\Omega}u(Y)\cdot\frac{\partial}{\partial\overline{n}_Y}\left(\ln\frac{1}{\|X-Y\|}\right)ds_Y+u(X)$$

From here, we obtain:

$$\begin{aligned}-\frac{1}{2\pi}\iint_\Omega\ln\frac{1}{\|X-Y\|}\cdot(\Delta u)(Y)dy_1dy_2 &= -\frac{1}{2\pi}\int_{\partial\Omega}\ln\frac{1}{\|X-Y\|}\cdot\frac{\partial u}{\partial\overline{n}_Y}(Y)ds_Y+\\ &+\frac{1}{2\pi}\int_{\partial\Omega}u(Y)\cdot\frac{\partial}{\partial\overline{n}_Y}\left(\ln\frac{1}{\|X-Y\|}\right)ds_Y+u(X)\end{aligned}$$

or

$$\begin{aligned}u(X) &= -\frac{1}{2\pi}\iint_\Omega\ln\frac{1}{\|X-Y\|}\cdot(\Delta u)(Y)dy_1dy_2+\frac{1}{2\pi}\int_{\partial\Omega}\ln\frac{1}{\|X-Y\|}\cdot\frac{\partial u}{\partial\overline{n}_Y}(Y)ds_Y-\\ &-\frac{1}{2\pi}\int_{\partial\Omega}u(Y)\cdot\frac{\partial}{\partial\overline{n}_Y}\left(\ln\frac{1}{\|X-Y\|}\right)ds_Y\end{aligned}$$

□

Comments: This representation formula permits us to determine the function u in all points of Ω knowing the following elements: the Laplacian of the function u on Ω, the values of the normal derivative $\dfrac{\partial u}{\partial\overline{n}}$ on $\partial\Omega$ and the values of the function u on $\partial\Omega$.

In the followings, we will show the importance of this representation formula for the case of harmonic functions.

Definition 6.2.1. *A function $u:\Omega\to\mathbb{R}^1$ is called harmonic in Ω if the function u is of classs $\mathcal{C}^2$ and $\Delta u=0$, $(\forall)(x_1,x_2)\in\Omega$ where Δu represents the Laplacian of the function u:*

$$\Delta u=\frac{\partial^2u}{\partial x_1^2}+\frac{\partial^2u}{\partial x_2^2}.$$

Observation 6.2.1. In the conditions of the representation theorem (6.2.5), if the function u is harmonic in Ω and of class $\mathcal{C}^1$ in $\bar{\Omega}$ then we have:

$$u(X)=\frac{1}{2\pi}\int_{\partial\Omega}\ln\frac{1}{\|X-Y\|}\cdot\frac{\partial u}{\partial\overline{n}_Y}(Y)ds_Y-\frac{1}{2\pi}\int_{\partial\Omega}u(Y)\cdot\frac{\partial}{\partial\overline{n}_Y}\left(\ln\frac{1}{\|X-Y\|}\right)ds_Y$$

Theorem 6.2.6 (The representation of harmonic functions). *If $u:\overline{\Omega}\subset\mathbb{R}^2\to\mathbb{R}^1$ is a harmonic function in the domain Ω then $(\forall)X\in\Omega$ and $(\forall)r>0$ for which the disc:*

$$\overline{B}(X,r)=\{Y:\|Y-X\|\le r\}$$

is included in Ω, the following equality holds:

$$u(X)=\frac{1}{2\pi r}\int_{\partial\overline{B}(X,r)}u(Y)ds_Y \tag{6.13}$$

Proof. Writing the representation formula (6.12) for the harmonic function u on $\overline{B}(X,r)$ we obtain the equality:

$$u(X)=\frac{1}{2\pi}\int_{\partial\overline{B}(X,r)}\ln\frac{1}{r}\cdot\frac{\partial u}{\partial\overline{n}_Y}(Y)ds_Y+\frac{1}{2\pi}\int_{\partial\overline{B}(X,r)}u(Y)\cdot\frac{1}{r}ds_Y$$

or

$$u(X) = \frac{1}{2\pi r}\int_{\partial\overline{B}(X,r)} u(Y)ds_Y + \frac{1}{2\pi}\cdot\ln\frac{1}{r}\int_{\partial\overline{B}(X,r)}\frac{\partial u}{\partial\overline{n}_Y}(Y)ds_Y.$$

We will show, in the followings, that

$$\int_{\partial\overline{B}(X,r)}\frac{\partial u}{\partial\overline{n}_Y}(Y)ds_Y = 0.$$

For this, let us consider $\overline{B}(X,r)$, and the functions $P\equiv 1$ and $Q = u$. Applying the first Green formula we obtain:

$$\iint_{\overline{B}(X,r)} P\cdot\Delta Q dy_1dy_2 = \int_{\partial\overline{B}(X,r)} P\cdot\frac{\partial Q}{\partial\overline{n}_Y}ds_Y - \int_{B(X,r)}\nabla P\cdot\nabla Q dy_1dy_2$$

so we have:

$$0 = \int_{\partial\overline{B}(X,r)} P\cdot\frac{\partial Q}{\partial\overline{n}_Y}ds_Y = \int_{\partial\overline{B}(X,r)}\frac{\partial u}{\partial\overline{n}_Y}ds_Y.$$

□

Observation 6.2.2. From this proof, we have that the integral of the normal derivative of a harmonic function on the circle is zero:

$$\int_{\partial\overline{B}(X,r)}\frac{\partial u}{\partial\overline{n}_Y}(Y)ds_Y = 0.$$

This result can be generalized.

Consequence 6.2.1. *If u is a harmonic function of class $\mathcal{C}^2$ in Ω and of class $\mathcal{C}^1$ in $\overline{\Omega}$, then the following equality holds:*

$$\int_{\partial\Omega}\frac{\partial u}{\partial\overline{n}_Y}(Y)ds_Y = 0.$$

Proof. To prove this equality, we apply the first Green formula for the functions $P\equiv 1$ and $Q = u$:

$$\iint_{\Omega} P\cdot\Delta Q dx_1dx_2 = \int_{\partial\Omega} P\cdot\frac{\partial Q}{\partial\overline{n}_Y}ds_Y - \iint_{\Omega}\nabla P\cdot\nabla Q dx_1dx_2$$

i.e.,

$$0=\iint_{\Omega} 1\cdot\Delta u dx_1dx_2=\int_{\partial\Omega} 1\cdot\frac{\partial u}{\partial\overline{n}_Y}ds_Y-\iint_{\Omega}\nabla 1\cdot\nabla u dx_1dx_2=\int_{\partial\Omega}\frac{\partial u}{\partial\overline{n}_Y}ds_Y.$$

□

Another important property of the harmonic functions is the localization of the points in which these functions have extreme values:

Theorem 6.2.7 (The extreme value theorem for harmonic functions). *If $\Omega\in\mathbb{R}^2$ is a bounded domain having smooth boundary $\partial\Omega$ and $u:\overline{\Omega}\to\mathbb{R}^1$ is a harmonic function of class $\mathcal{C}^1$ in $\overline{\Omega}$, then u is constant on $\overline{\Omega}$ or reaches its extremes on the boundary $\partial\Omega$.*

Proof. We suppose that the function u is not constant on $\overline{\Omega}$ and we want to show that u reaches its extremes on the boundary $\partial\Omega$. We suppose the contrary, i.e. we suppose that there exists $X^0 \in \Omega$ such that for any $X \in \overline{\Omega}$ we have $u(X) \leq u(X^0)$.

We consider $r_0 > 0$ such that

$$\overline{B}(X^0, r_0) = \{Y : \|Y - X^0\| \leq r_0\} \subset \Omega$$

and we show that u is constantly equal to $u(X^0)$ on $\overline{B}(X^0, r_0)$.

If u is not constantly equal to $u(X^0)$ on $\overline{B}(X^0, r_0)$ then there exists $X^1 \in \overline{B}(X^0, r_0)$ such that $u(X^1) < u(X^0)$. For X^1, there exists $r_1 > 0$ such that for any $X \in \overline{B}(X^1, r_1)$ we have:

$$u(X) < \frac{1}{2}\left[u(X^0) + u(X^1)\right]$$

We can admit that $r_1 < \min\{r_0 - \|X^1 - X^0\|, \|X^1 - X^0\|\}$ and we consider the number $\rho = \|X^1 - X^0\| > 0$. Applying the representation formula for $u(X^0)$ on $\partial\overline{B}(X^0, \rho)$ we find:

$$u(X^0) = \frac{1}{2\pi\rho}\int_{\partial\overline{B}(X^0,\rho)} u(Y)ds_Y$$

where the boundary $\partial\overline{B}(X^0, \rho)$ is decomposed as follows:

$$\partial\overline{B}(X^0, \rho) = \partial\overline{B}(X^0, \rho) \cap \overline{B}(X^1, r_1) \cup \overline{B}(X^0, \rho) \cap (\Omega \setminus \overline{B}(X^1, r_1)) = \sigma_1 \cup \sigma_2$$

Using this decomposition, the representation formula becomes:

$$\begin{aligned} u(X^0) &= \frac{1}{2\pi\rho} ds \int_{\sigma_1} u(Y)ds_Y + \frac{1}{2\pi\rho}\int_{\sigma_2} u(Y)ds_Y < \\ &< \frac{1}{2\pi\rho} \cdot \frac{1}{2}\left[u(X^0) + u(X^1)\right] \cdot \int_{\sigma_1} ds_Y + \frac{1}{2\pi\rho} \cdot u(X^0) \cdot \int_{\sigma_2} ds_Y < \\ &< \frac{1}{2\pi\rho} \cdot u(X^0) \cdot 2\pi\rho = u(X^0) \text{ absurd.} \end{aligned}$$

We have obtained in this way that the function u is constantly equal to $u(X^0)$ on $B(X^0, r)$.

In order to show that for any $X \in \Omega$ we have $u(X) = u(X^0)$, let us consider an arbitrary $X^* \in \Omega$, X^* fixed and P a polygonal line included in Ω which binds the points X^0 and X^*. Let us consider a direction on the polygonal line P from X^0 to X^*. The sets P and $\partial\Omega$ are compact and don't have a common point. It follows that there exists $r > 0$ such that for any $X \in P$ the closed disc: $\overline{B}(X, r) = \{Y : \|Y - X\| \leq r_0\}$ is included in Ω; $\overline{B}(X, r) \subset \Omega$. Let be X^2 the intersection point of the polygonal line P and the boundary of $\overline{B}(X^0, r_0)$, being the first point on the direction from X^0 to X^*. In X^2 we have $u(X^2) = u(X^0)$ and hence for any $X \in \overline{B}(X^2, r)$ we have $u(X) = u(X^0)$. Thus, we obtain the equality $u(X) = u(X^0)$ for any $X \in \overline{B}(X^0, r_0) \cup \overline{B}(X^2, r)$.

In the followings, we consider the intersection point X^3 of P and $\partial\overline{B}(X^2, r)$, being the first point on the direction from X^2 to X^*. In X^3 we have $u(X^3) = u(X^0)$ and hence $u(X) = u(X^0)$ for any $X \in \overline{B}(X^3, r)$. In this way, after a finite number of steps, we reach the point X^* and obtain the equality $u(X^*) = u(X^0)$. This means that the function u is constant on Ω which is impossible.

It has been proved in this way that, if the harmonic function u reaches the maximum value at a point $X^0 \in \Omega$, then u is constant on Ω.

It follows that if a harmonic function is not constant on Ω, then it reaches its extremes on $\partial\Omega$. $\square$

Consequence 6.2.2. *If the function u is harmonic on Ω and $u/_{\partial\Omega} = 0$ then $u \equiv 0$.*

Consequence 6.2.3. *There is at most one function u of class $\mathcal{C}^2$ in Ω and of class $\mathcal{C}^1$ in $\overline{\Omega}$ which verifies:*

$$\begin{cases} \Delta u = F \\ u/_{\partial\Omega} = f. \end{cases}$$

where F and f are given functions: $F : \overline{\Omega} \to \mathbb{R}^1$ continuous on $\overline{\Omega}$, and $f : \partial\Omega \to \mathbb{R}^1$ continuous on $\partial\Omega$.

Proof. By supposing the contrary. $\square$

6.3 Green formulas and the representation formulas in dimension $n \geq 3$

In this section we will present some results of differential and integral calculus for functions of n variables ($n \geq 3$), which appear when solving partial differential equations in dimension n.

Theorem 6.3.1 (Relation between the integral on a bounded domain and the integral on its border). *If $\Omega \subset \mathbb{R}^n$ is a bounded domain having smooth (partially smooth) boundary $\partial\Omega$, $f_1, f_2, ..., f_n : \bar{\Omega} \to \mathbb{R}^1$ are n continuous functions in $\bar{\Omega}$ and of class $\mathcal{C}^1$ in Ω then the following equality holds:*

$$\int_\Omega \left(\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} \right) dx_1 \cdots dx_n = \int_{\partial\Omega} \sum_{i=1}^{n} f_i \cdot \cos(\bar{n}, \bar{e}_i) dS. \tag{6.14}$$

Proof. The significance of the symbols from (6.14) is the same as in the case $n = 2$, and the proof of theorem can be made in the same way. $\square$

Theorem 6.3.2 (Integration by parts). *If $\Omega \subset \mathbb{R}^n$ is a bounded domain having smooth (partially smooth) boundary $\partial\Omega$, $f, g : \bar{\Omega} \to \mathbb{R}^1$ are two continuous functions in $\bar{\Omega}$ and of class $\mathcal{C}^1$ in Ω, then the following equality holds:*

$$\int_\Omega \frac{\partial f}{\partial x_i} \cdot g \, dx_1 \cdots dx_n = \int_{\partial\Omega} f \cdot g \cdot \cos(\bar{n}, \bar{e}_i) dS - \int_\Omega f \cdot \frac{\partial g}{\partial x_i} \, dx_1 \cdots dx_n \tag{6.15}$$

Proof. Similar to the case $n = 2$. $\square$

Theorem 6.3.3 (The first Green formula). *If $\Omega \subset \mathbb{R}^n$ is a bounded domain having smooth (partially smooth) boundary $\partial\Omega$, $f, g : \bar{\Omega} \to \mathbb{R}^1$ are two functions of class $\mathcal{C}^1$ in $\bar{\Omega}$ and of class $\mathcal{C}^2$ in Ω, then the following equality holds:*

$$\int_\Omega f \cdot \Delta g \, dx_1 \cdots dx_n = \int_{\partial\Omega} f \cdot \frac{\partial g}{\partial \bar{n}} \, dS - \int_\Omega \nabla f \cdot \nabla g \, dx_1 \cdots dx_n \tag{6.16}$$

Proof. The symbols in (6.16) have the following meaning:

$$\Delta g = \sum_{i=1}^{n} \frac{\partial^2 g}{\partial x_i^2}, \qquad \frac{\partial g}{\partial \bar{n}} = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i} \cos(\bar{n}, \bar{e}_i), \qquad \nabla f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \bar{e}_i$$

The theorem is proved in the same way as in the case $n = 2$. □

Theorem 6.3.4 (The second Green formula). *If $\Omega \subset \mathbb{R}^n$ is a bounded domain having smooth (partially smooth) boundary $\partial\Omega$, $f, g : \bar{\Omega} \to \mathbb{R}^1$ are two functions of class $\mathcal{C}^1$ in $\bar{\Omega}$ and of class $\mathcal{C}^2$ in Ω, then the following equality holds:*

$$\int_{\Omega} (f \cdot \Delta g - g \cdot \Delta f)\, dx_1 \cdots dx_n = \int_{\partial\Omega} \left(f \cdot \frac{\partial g}{\partial \bar{n}} - g \cdot \frac{\partial f}{\partial \bar{n}} \right) dS \tag{6.17}$$

Proof. The theorem is proved in the same way as in the case $n = 2$. □

Theorem 6.3.5 (Representation theorem for functions of n variables). *If $\Omega \subset \mathbb{R}^n$ is a bounded domain having smooth (partially smooth) boundary $\partial\Omega$, $u : \bar{\Omega} \to \mathbb{R}^1$ is a function of class $\mathcal{C}^1$ in $\bar{\Omega}$ and of class $\mathcal{C}^2$ in Ω, then the following formula of representation takes place:*

$$\begin{aligned} u(X) &= -\frac{1}{(n-2)\sigma_n} \int_{\Omega} \frac{1}{\|X - Y\|^{n-2}} \Delta u(Y)\, dy_1 \cdots dy_n + \\ &+ \frac{1}{(n-2)\sigma_n} \int_{\partial\Omega} \frac{1}{\|X - Y\|^{n-2}} \frac{\partial u}{\partial \bar{n}_Y}(Y)\, dS_Y - \\ &- \frac{1}{(n-2)\sigma_n} \int_{\partial\Omega} u(Y) \frac{\partial}{\partial \bar{n}_Y} \left(\frac{1}{\|X - Y\|^{n-2}} \right) dS_Y \end{aligned} \tag{6.18}$$

where σ_n represents the area of the surface of

$$\bar{B}(0,1) = \{Y = (y_1, \ldots, y_n) \in \mathbb{R}^n \mid \|Y\| \leq 1\}$$

and the subscript Y from $\dfrac{\partial}{\partial \bar{n}_Y}$ shows that the normal derivative of the function $Y \mapsto \dfrac{1}{\|X - Y\|^{n-2}}$ is computed (the same for dS_Y).

Proof. Similar to the case $n = 2$. □

Comments The formula (6.18) allows to construct the function u knowing the values of Laplacian of the function Δu in Ω, the values of the normal derivative $\dfrac{\partial u}{\partial \bar{n}}$ of u in $\partial\Omega$ and the values of u in $\partial\Omega$.

In the followings, we will show what this formula of representation becomes in the case of harmonic functions.

Definition 6.3.1. *The function $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^1$ is called harmonic in Ω if the function is of class $\mathcal{C}^2$ in Ω and if*

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} = 0, \quad (\forall)(x_1, \ldots, x_n) \in \Omega$$

Observation 6.3.1. If conditions of the Theorem 6.3.5 (of representation) are fulfilled and if the function u is harmonic in Ω, then we have:

$$u(X) = \frac{1}{(n-2)\sigma_n}\left[\int_{\partial\Omega}\frac{1}{\|X-Y\|^{n-2}}\frac{\partial u}{\partial \bar{n}_Y}(Y)\,dS_Y - \int_{\partial\Omega} u(Y)\,\frac{\partial}{\partial \bar{n}_Y}\left(\frac{1}{\|X-Y\|^{n-2}}\right)dS_Y\right]$$

Theorem 6.3.6 (The representation of harmonic functions). *If $\Omega \subset \mathbb{R}^n$ is a domain and $u : \bar{\Omega} \to \mathbb{R}^1$ is a harmonic function in Ω ($\Delta u = 0$), then for any $x \in \Omega$ and any $r > 0$ such that the closed ball $\bar{B}(0, r) = \{Y \in \mathbb{R}^n | \, \|Y\| \le r\}$ is included in Ω, the following equality holds:*

$$u(X) = \frac{1}{\sigma_n\, r^{n-2}}\int_{\partial\bar{B}(X,r)} u(Y)\,dS_Y \tag{6.19}$$

Proof. Similar to the case $n = 2$. □

Observation 6.3.2. If $\Omega \subset \mathbb{R}^n$ is a bounded domain having smooth (partially smooth) boundary $\partial\Omega$, $u : \bar{\Omega} \to \mathbb{R}^1$ is a function of class $\mathcal{C}^1$ in $\bar{\Omega}$ and harmonic in Ω ($\Delta u = 0$ in Ω), then:

$$\int_{\partial\Omega}\frac{\partial u}{\partial \bar{n}}\,dS = 0.$$

Theorem 6.3.7 (The extreme values of harmonic functions). *If $\Omega \subset \mathbb{R}^n$ is a bounded domain having smooth (partially smooth) boundary $\partial\Omega$, $u : \bar{\Omega} \to \mathbb{R}^1$ is a continuous function in $\bar{\Omega}$ and harmonic in Ω ($\Delta u = 0$ in Ω), then the function u is constant or reaches its extreme values on the boundary $\partial\Omega$.*

Proof. Similar to the case $n = 2$. □

Consequence 6.3.1. *If $\Delta u = 0$ and $u|_{\partial\Omega} = 0$ then $u = 0$.*

Consequence 6.3.2. *There is at most one function u of class $\mathcal{C}^2$ in Ω and of class $\mathcal{C}^1$ on $\bar{\Omega}$ which verifies*

$$\begin{cases} \Delta u = F \\ u|_{\partial\Omega} = f \end{cases}$$

where F and f are two given functions, $F : \bar{\Omega} \to \mathbb{R}^1$ is continuous in $\bar{\Omega}$ and $f : \partial\Omega \to \mathbb{R}^1$ is continuous in $\partial\Omega$.

6.4 The limit problem for the Poisson equation

Let be $\Omega \subset \mathbb{R}^n$ a bounded domain having smooth (partially smooth) boundary $\partial\Omega$ and f a function $f : \bar{\Omega} \to \mathbb{R}^1$ of class $\mathcal{C}^1$ on Ω.

Definition 6.4.1. *The Poisson equation is a functional dependence of the form*

$$-\Delta u = f(X), \quad (\forall) X = (x_1, \ldots, x_n) \in \Omega \tag{6.20}$$

between an unknown function $u : \Omega \to \mathbb{R}^1$ and the given function f of class $\mathcal{C}^1$ in $\bar{\Omega}$.

In equation (6.20) Δu means $\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$ (i.e. the Laplacian of the function u) and the function $u : \bar{\Omega} \to \mathbb{R}^1$ is a classical solution if it is continuous in $\bar{\Omega}$, of class $\mathcal{C}^2$ in Ω and satisfies the equality:

$$\sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}(x_1, ..., x_n) = f(x_1, ..., x_n), \quad (\forall)(x_1, ..., x_n) \in \Omega.$$

Definition 6.4.2. *The Dirichlet Problem for the Poisson equation is the problem of determining those solutions of the equation (6.20) which satisfy the boundary condition*

$$u|_{\partial\Omega} = h \tag{6.21}$$

where $h : \partial\Omega \to \mathbb{R}^1$ is a known continuous function in $\partial\Omega$.

The Dirichlet Problem for the Poisson equation is denoted by:

$$\begin{cases} -\Delta u = f, & (\forall)(x_1, ..., x_n) \in \Omega \\ u|_{\partial\Omega} = h \end{cases} . \tag{6.22}$$

Definition 6.4.3. *The Neumann Problem for the Poisson equation is the determination of those solutions of the equation (6.20) which verify the boundary condition*

$$\left.\frac{\partial u}{\partial \bar{n}}\right|_{\partial\Omega} = g \tag{6.23}$$

where $g : \partial\Omega \to \mathbb{R}^1$ is a known continuous function in $\partial\Omega$ and $\bar{n}$ is the unit vector of the external normal.

The Neumann Problem for the Poisson equation is denoted by:

$$\begin{cases} -\Delta u = f, & (\forall)(x_1, ..., x_n) \in \Omega \\ \left.\frac{\partial u}{\partial \bar{n}}\right|_{\partial\Omega} = g \end{cases} . \tag{6.24}$$

Theorem 6.4.1 (Uniqueness of the solution of the Dirichlet Problem). *If the Dirichlet Problem (6.22) has a solution then this is unique.*

Proof. Let be u_1 and u_2 two solutions of the Dirichlet Problem (6.22). We will consider $u = u_1 - u_2$ for which we have:

$$\Delta u = 0 \quad \text{and} \quad u|_{\partial\Omega} = 0.$$

From here, on the basis of the maximum principle of harmonic functions, we have $u \equiv 0$. It follows that $u_1 = u_2$. $\square$

Theorem 6.4.2 (Non-uniqueness of the solution of the Neumann Problem). *If u is a solution of the Neumann Problem (6.24), then $u + C$ is also a solution of the Neumann Problem, where C is a constant. If u, v are solutions of Neumann Problem (6.24), then $u - v = const$.*

Proof. Let be u a solution of the problem (6.24) and $v = u + C$. Because $\Delta v = \Delta u$ and $\left.\dfrac{\partial v}{\partial \bar{n}}\right|_{\partial\Omega} = \left.\dfrac{\partial u}{\partial \bar{n}}\right|_{\partial\Omega} = g$, we obtain that v is solution of the problem (6.24). If u, v are solutions of the problem (6.24) then $w = u - v$ verifies

$$\Delta w = 0 \quad \text{and} \quad \left.\frac{\partial w}{\partial \bar{n}}\right|_{\partial\Omega} = 0.$$

On the basis of the representation formulas we obtain $w = const.$ □

Theorem 6.4.3. *The necessary condition for the existence of a solution of the Neumann Problem (6.24) is that the functions f and g verify the equality:*

$$\int_{\partial\Omega} g(Y)\, dS_Y + \int_{\Omega} f(Y)\, dY = 0. \tag{6.25}$$

Proof. Let us suppose that the Neumann Problem (6.24) has a solution u. Considering the functions u, 1, and applying the second Green formula we have:

$$\int_{\Omega} (\Delta u \cdot 1 - u\,\Delta 1)\, dY = \int_{\partial\Omega} \left(\frac{\partial u}{\partial \bar{n}_Y} - u\,\frac{\partial 1}{\partial \bar{n}_Y} \right) dS_Y.$$

Taking into account the equalities $-\Delta u = f$, $\left.\dfrac{\partial u}{\partial \bar{n}_Y}\right|_{\partial\Omega} = g$, $\Delta 1 = 0$ and $\dfrac{\partial 1}{\partial \bar{n}_Y} = 0$ we obtain (6.25). □

From the above considerations, it doesn't result that the Dirichlet Problem (6.22) or the Neumann Problem (6.24) have solutions. In the following sections we will present the method of Green functions for showing that in some conditions these problems have solutions and we will represent these solutions.

6.5 Green function for the Dirichlet Problem

Let us consider a bounded domain $\Omega \subset \mathbb{R}^n$ having smooth (partially smooth) boundary, the function $f : \bar{\Omega} \to \mathbb{R}^1$ of class $\mathcal{C}^1$ in $\bar{\Omega}$ and the continuous function $h : \partial\Omega \to \mathbb{R}^1$ in $\partial\Omega$. These elements define the Dirichlet Problem:

$$\begin{cases} -\Delta u = f, & (\forall)(x_1, ..., x_n) \in \Omega \\ u|_{\partial\Omega} = h \end{cases}. \tag{6.26}$$

Definition 6.5.1. *We call Green function for the Dirichlet Problem, a function G of the form:*

$$G(X, Y) = E(X, Y) - v(X, Y) \tag{6.27}$$

where

$$E(X,Y) = \begin{cases} \dfrac{1}{2\pi} \ln \dfrac{1}{\|X - Y\|}, & (\forall) X, Y \in \Omega, X \neq Y, n = 2 \\ \dfrac{1}{(n-2)\sigma_n} \cdot \dfrac{1}{\|X - Y\|^{n-2}}, & (\forall) X, Y \in \Omega, X \neq Y, n > 2 \end{cases} \tag{6.28}$$

and the function $v(X, Y)$ has the following properties:

a) $Y \mapsto v(X,Y)$ *is harmonic in* Ω *and continuous in* $\bar{\Omega}$ *for any fixed* $X = (x_1, \ldots, x_n) \in \Omega$.

b) for a fixed $X \in \Omega$, *we have* $G(X,Y) = 0$, $(\forall)Y = (y_1, \ldots, y_n) \in \partial\Omega$.

Proposition 6.5.1. *If there exists a Green function* $G(X,Y)$ *for the Dirichlet problem then it is unique.*

Proof. If G_1, G_2 are two Green functions for the Dirichlet problem, then for any fixed $X \in \Omega$, the function

$$v(X,Y) = v_1(X,Y) - v_2(X,Y)$$

is harmonic on Ω and identically zero on $\partial\Omega$. Thus, $v(X,Y) = 0$, for $(\forall)Y \in \Omega$ and $(\forall)X \in \Omega$. This shows that $v(X,Y) \equiv 0$, i.e. $v_1(X,Y) = v_2(X,Y)$. $\square$

Proposition 6.5.2. *If for any* $X \in \bar{\Omega}$, *the function* $Y \mapsto v(X,Y)$ *is of class* $\mathcal{C}^1$ *in* $\bar{\Omega}$, *then the Green function is symmetric i.e.,*

$$G(X_1, X_2) = G(X_2, X_1),\ (\forall)X_1, X_2 \in \Omega \ \ \textit{and} \ \ X_1 \neq X_2. \tag{6.29}$$

Proof. Let us consider the functions $P(Y) = (X_1, Y)$ and $Q(Y) = G(X_2, Y)$ for which we apply the second Green formula in the domain $\Omega_\varepsilon = \Omega \setminus (B(X_1,\varepsilon) \cup B(X_2,\varepsilon))$, where $B(X_i,\varepsilon) = \{X : \|X - X_i\| < \varepsilon\}$, $i = 1, 2$. We obtain the equality:

$$\int_{\partial B(X_1,\varepsilon)} \left(P\,\frac{\partial Q}{\partial \bar{n}_Y} - Q\,\frac{\partial P}{\partial \bar{n}_Y} \right) dS_Y + \int_{\partial B(X_2,\varepsilon)} \left(P\,\frac{\partial Q}{\partial \bar{n}_Y} - Q\,\frac{\partial P}{\partial \bar{n}_Y} \right) dS_Y = 0.$$

Passing to the limit in this equality for $\varepsilon \to 0$ we obtain the symmetry. $\square$

Theorem 6.5.1. *If* G *is the Green function for the Dirichlet Problem (6.26) and* u *is the solution of this problem, then the following equality holds:*

$$u(X) = -\int_\Omega G(X,Y) \cdot f(Y)\, dY - \int_{\partial\Omega} h(Y) \cdot \frac{\partial G(X,Y)}{\partial \bar{n}_Y} dS_Y. \tag{6.30}$$

Proof. Writing the second Green formula for the functions $u(Y)$ and $Y \mapsto v(X,Y)$ and the general formula of representation of the function $u(X)$ (formula (6.18)), by summation we obtain (6.30). $\square$

Observation 6.5.1. This theorem shows that if there exists a Green function G for the Dirichlet Problem (6.26) having the solution u, then u is represented in the form (6.30) using the function G. The reciprocal assertion is also true: if G is a Green function for the Dirichlet Problem, then u defined by (6.30) is the solution of the Dirichlet Problem (6.26). This method of determining the solution of the Dirichlet Problem will be called the Green function method.

Example 6.5.1. Let be $\Omega = \{X \in \mathbb{R}^3 \,|\, \|X\| < r\}$ and $\partial\Omega = \{X \in \mathbb{R}^3 \,|\, \|X\| = r\}$. The solution of the Dirichlet problem

$$\begin{cases} -\Delta u = 0, & (\forall)X \in \Omega \\ u|_{\partial\Omega} = h \end{cases}$$

where h is a continuous function in $\partial\Omega$, is given by the Poisson formula:

$$u(X) = \frac{1}{4\pi r}\int_{\partial\Omega}\frac{r^2-\|X\|^2}{\|X-Y\|^3}h(Y)\,dS_Y \tag{6.31}$$

because the function $G(X,Y)$ defined by

$$G(X,Y) = \frac{1}{4\pi\|X-Y\|} - \frac{r}{4\pi\|X^*-Y\|} \tag{6.32}$$

where X^* represents the conjugate of X with respect to the sphere $\partial\Omega$ (i.e. X, X^* are collinear and $\|X\|\cdot\|X^*\| = r$), is the Green function for this Dirichlet Problem.

Observation 6.5.2. The determination of the Green function for the Dirichlet Problem will be reduced to the determination of $v = v(X,Y)$ which, according to the definition of the Green function, will be reduced to solving the Dirichlet Problem:

$$\begin{cases} -\Delta_Y v(X,Y) = 0 \\ v(X,Y)|_{Y\in\partial\Omega} = E(X,Y)|_{Y\in\partial\Omega} \end{cases} \tag{6.33}$$

This problem is apparently more simple than the Dirichlet Problem (6.26) but, in reality, it is solved only in particular domains Ω, by geometrical methods. From this reason, the proof of the existence of the solution of the Dirichlet Problem (6.26) using the Green function method, can be made only in particular domains Ω for which it is known that there exists the Green function.

Exercises

1. Determine the solution of the Dirichlet Problem:

$$\begin{cases} \dfrac{\partial^2 u}{\partial x_1^2} + \dfrac{\partial^2 u}{\partial x_2^2} + \dfrac{\partial^2 u}{\partial x_3^2} = 0, \quad \text{in } \Omega = \{(x_1,x_2,x_3)\,|\,x_1^2+x_2^2+x_3^2 < r^2\} \\ \\ u(x_1,x_2,x_3) = (x_1+x_2+x_3)^2, \quad \text{pentru } (x_1,x_2,x_3)\in\partial\Omega. \end{cases}$$

2. Determine the Green function of the Dirichlet Problem in the circle and solve it.

6.6 Green function for the Neumann problem

We consider the Neumann problem

$$\begin{cases} -\Delta u = f,\ (\forall)(x_1,\ldots,x_n)\in\Omega \\ \\ \left.\dfrac{\partial u}{\partial \bar{n}}\right|_{\partial\Omega} = g \end{cases} \tag{6.34}$$

where $f:\Omega\to\mathbb{R}^1$ is a function of class $\mathcal{C}^1$ in $\bar{\Omega}$ and $g:\partial\Omega\to\mathbb{R}^1$ is a continuous function. We suppose that the functions f and g verify:

$$\int_\Omega f(Y)\,dY + \int_{\partial\Omega} g(Y)\,dS_Y = 0. \tag{6.35}$$

Definition 6.6.1. *We call Green function for the Neumann Problem (6.34) any function G of the form*

$$G(X,Y) = E(X,Y) - v(X,Y) \tag{6.36}$$

which verifies

$$\frac{\partial G}{\partial \bar{n}_Y}(X,Y) = 0,\ (\forall)Y = (y_1,\ldots,y_n) \in \partial\Omega \tag{6.37}$$

where $v : \Omega \times \Omega \to \mathbb{R}^1$ and for any fixed $Y \in \Omega$, the function $Y \mapsto v(X,Y)$ is harmonic in Ω, and $E(X,Y)$ is defined for any $X, Y \in \Omega$, $X \neq Y$, being given by:

$$E(X,Y)=\begin{cases} \dfrac{1}{2\pi}\ln\dfrac{1}{\|X-Y\|}, & (\forall)X,Y \in \Omega, X\neq Y, n=2 \\ \dfrac{1}{(n-2)\sigma_n}\cdot\dfrac{1}{\|X-Y\|^{n-2}}, & (\forall)X,Y \in \Omega, X\neq Y, n>2 \end{cases} \tag{6.38}$$

Observation 6.6.1. If G_1 and G_2 are Green functions for the Neumann Problem, then $G_1 - G_2 = const.$

Proposition 6.6.1. *If for any $X \in \bar{\Omega}$, the function $Y \mapsto v(X,Y)$ is of class $\mathcal{C}^1$ on $\bar{\Omega}$, then the Green function for the Neumann Problem is symmetric:*

$$G(X_1,X_2) = G(X_2,X_1),\ (\forall)X_1,X_2 \in \Omega \ \ \textit{and} \ \ X_1 \neq X_2. \tag{6.39}$$

Proof. Similar to the results presented in the case of the Green function for the Dirichlet Problem. □

Theorem 6.6.1. *If G is a Green function for the Neumann Problem (6.34) and u is a solution of this problem, then there exists a positive constant C such that the following equality holds:*

$$u(X) = -\int_\Omega G(X,Y)\cdot f(Y)\,dY + \int_{\partial\Omega} E(X,Y)\cdot g(Y)\,dS_Y + C. \tag{6.40}$$

Proof. Similar to the case of the Dirichlet Problem. □

Observation 6.6.2. The determination of the Green function for the Neumann Problem will be reduced to the determination of the function $v(X,Y)$ which means, according to the definition of the Green function for the Neumann Problem, solving the Neumann Problem

$$\begin{cases} -\Delta_Y v(X,Y) = f,\ (\forall)(x_1,\ldots,x_n) \in \Omega \\ \left.\dfrac{\partial v}{\partial \bar{n}_Y}\right|_{\partial\Omega} = \left.\dfrac{\partial E}{\partial \bar{n}_Y}\right|_{\partial\Omega} \end{cases} \tag{6.41}$$

This problem is apparently more simple than the Neumann Problem (6.34) but, in reality, it is more complex. From this reason, the proof of the existence of the solution of the Neumann Problem (6.34) using the Green function can be made only for the cases for which it is known that there exists the Green function.

6.7 Dirichlet and Neumann problems for the Laplace equation on the disc. Separation of variables.

Let us consider the set $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 | \; x_1^2 + x_2^2 < R^2\}$ which will be called the disc centered in the origin of radius R.

The Poisson equation on Ω is the equation:

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} = -f(x_1, x_2) \tag{6.42}$$

where f is a function of class $\mathcal{C}^1$ in $\bar{\Omega}$.

If the function f is identically null, then the Poisson equation becomes:

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} = 0 \tag{6.43}$$

and it is called the Laplace equation on the disc Ω of radius R.

The Dirichlet Problem for the Laplace equation on the disc of radius R means the determination of those solutions of the equation (6.43) which, on $\partial\Omega$ coincide with a given continuous function h, i.e.

$$\begin{cases} \dfrac{\partial^2 u}{\partial x_1^2} + \dfrac{\partial^2 u}{\partial x_2^2} = 0, \; (\forall)(x_1, x_2) \in \Omega \\ u \mid_{\partial\Omega} = h \end{cases} \tag{6.44}$$

The solution of this Dirichlet Problem can be determined using the Green function for the Dirichlet Problem.

The Neumann Problem for the Laplace equation on the disc of radius R means the determination of those solutions of the equation (6.43) for which the derivative with respect to unit vector of the external normal at $\partial\Omega$ coincides with a given continuous function g on $\partial\Omega$, i.e.

$$\begin{cases} \dfrac{\partial^2 u}{\partial x_1^2} + \dfrac{\partial^2 u}{\partial x_2^2} = 0, \; \forall(x_1, x_2) \in \Omega \\ \left.\dfrac{\partial u}{\partial n}\right|_{\partial\Omega} = g \end{cases} \tag{6.45}$$

If the problem (6.45) has a solution, then

$$\int_{\partial\Omega} g(Y) ds_Y = 0 \tag{6.46}$$

and if this condition is satisfied then, using the Green function for the Neumann Problem, we can determine the solutions of the problem (6.45), (which differ by an additive constant).

The aim of this section is to present another method, called "separation of variables", which can be used for determining the solutions of the problems (6.44) and (6.45). Because the domain Ω is considered as a circular domain, we make a change of variables, more exactly we write the problems (6.44) and (6.45) in polar coordinates:

$$\begin{cases} x_1 = r\,\cos\varphi \\ x_2 = r\,\sin\varphi \end{cases}, \; r > 0,\; \varphi \in [0, 2\pi). \tag{6.47}$$

We denote by T the transformation $(r,\varphi) \to (x_1, x_2)$ defined by (6.47). If u is a function which satisfies the equation (6.43) then we will denote by $\widetilde{u}$ the function $\widetilde{u} = u \circ T$.

Using the rules of differentiating the composed functions we deduce that $\widetilde{u}$ satisfies the equation:

$$\frac{\partial^2 \widetilde{u}}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 \widetilde{u}}{\partial \varphi^2} + \frac{1}{r}\frac{\partial \widetilde{u}}{\partial r} = 0 \tag{6.48}$$

which is called the Laplace equation in polar coordinates.

The method of separation of variables consists of searching for solutions $\widetilde{u}(r,\varphi)$ of the form:

$$\widetilde{u}(r,\varphi) = P(r)\cdot Q(\varphi) \tag{6.49}$$

i.e. solutions which are products of one-variable functions, namely of variable r, and of variable φ, respectively. Imposing that (6.49) verifies (6.48) we obtain:

$$r^2\frac{P''}{P} + r\frac{P'}{P} = -\frac{Q''}{Q}. \tag{6.50}$$

The left term of this equality depends only on r and the right term depends on φ. Because r and φ are independent variables we have that every term is constant. If we denote by λ this constant, then we deduce from (6.50) the equalities:

$$r^2P'' + rP' - \lambda P = 0 \tag{6.51}$$

$$Q'' + \lambda Q = 0 \tag{6.52}$$

Because the function $\widetilde{u}$ is of class $\mathcal{C}^2$, the solution of the equation (6.52) must verify $Q(\varphi) = Q(\varphi + 2\pi)$. From here and from the homogeneous condition on the boundary, we have that $\lambda_n = n^2$, $n \in I\!N$.
For $\lambda_n = n^2$ we have:

$$Q_n(\varphi) = A_n \cos n\varphi + B_n \sin n\varphi \tag{6.53}$$

in which A_n and B_n are arbitrary constants.

The equation (6.51) is solved making the change of variables $r = e^t$. For $\lambda = n^2$ the general solution is:

$$\begin{aligned} P_n(r) &= C_n r^n + D_n r^{-n}, \text{ if } n = 1, 2, ... \qquad (6.54)\\ P_0(r) &= A_0 + B_0 \ln r, \quad \text{ if } n = 0. \qquad (6.55)\end{aligned}$$

We obtain in this way that (6.48) admits the following family of solutions:

$$\widetilde{u}_n(r,\varphi) = \begin{cases} A_0 + B_0 \ln r, & n = 0 \\ r^n(A_n \cos n\varphi + B_n \sin n\varphi), & n = 1, 2, ... \\ r^{-n}(A_{-n} \cos n\varphi + B_{-n} \sin n\varphi), & n = 1, 2, ... \end{cases} \tag{6.56}$$

in which $A_0, B_0, A_n, B_n, A_{-n}, B_{-n}$ are arbitrary constants.

Observation 6.7.1. Any finite sum of solutions of the form (6.48) is a solution for equation (6.48).

Definition 6.7.1. *A formal solution of the Laplace equation in polar coordinates is a "function" $\widetilde{u}(r, \varphi)$ of the form:*

$$\widetilde{u}(r, \varphi) = A_0 + B_0 \ln r + \sum_{n=1}^{\infty} [(A_n r^n + A_{-n} r^{-n}) \cos n\varphi + (B_n r^n + B_{-n} r^{-n}) \sin n\varphi] \quad (6.57)$$

The name of formal solution is due to the fact that we don't have information concerning convergence of the series (6.57). It is easy to see that the series (6.57) is a solution. If the series has a finite number of non-null terms, the sum of the series is a function which is solution of the Laplace equation.

In order to determine the solution of the Dirichlet Problem (6.44) we first write the equation in polar coordinates:

$$\begin{cases} \dfrac{\partial^2 \widetilde{u}}{\partial r^2} + \dfrac{1}{r^2} \dfrac{\partial^2 \widetilde{u}}{\partial \varphi^2} + \dfrac{1}{r} \dfrac{\partial \widetilde{u}}{\partial r} = 0, \quad r < R, \varphi \in [0, 2\pi) \\ \widetilde{u}(r, \varphi) = u(r, \varphi + 2\pi) \\ \lim\limits_{r \to 0} \mid \widetilde{u}(r, \varphi) \mid < +\infty \\ \widetilde{u}(R, \varphi) = \widetilde{h}(\varphi) \end{cases} \quad (6.58)$$

where $\widetilde{h}(\varphi) = h(R \cos \varphi, R \sin \varphi)$.

We consider the function $\widetilde{h}$ written as Fourier series

$$\widetilde{h}(\varphi) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\varphi + b_n \sin n\varphi) \quad (6.59)$$

and we impose that the formal solution $\widetilde{u}(r, \varphi)$ given by (6.57) verifies the conditions (6.58). We deduce that:

$$A_0 = a_0, A_n = \frac{a_n}{R^n} \text{ and } B_n = \frac{b_n}{R^n}$$

which leads us to the formal solution:

$$\widetilde{u}(r, \varphi) = a_0 + \sum_{n=1}^{\infty} \frac{r^n}{R^n} (a_n \cos n\varphi + b_n \sin n\varphi). \quad (6.60)$$

To prove that the formal solution is a solution of the problem we suppose that the function $\widetilde{h}$ is of class $\mathcal{C}^2$ and $\widetilde{h}(0) = \widetilde{h}(2\pi)$. In these conditions, for the Fourier coefficients a_n, b_n of the function $\widetilde{h}$, we can write:

$$a_n = -\frac{1}{n} b'_n = -\frac{1}{n^2} a''_n \text{ and } b_n = \frac{1}{n} a'_n = -\frac{1}{n^2} b''_n. \quad (6.61)$$

where a'_n, b'_n and a''_n, b''_n are the Fourier coefficients of the functions $\widetilde{h}'$ and $\widetilde{h}''$, respectively.

As $\widetilde{h}'' \in L^2[0, 2\pi)$ we have $\sum_{n=1}^{\infty}(|a''_n|^2 + |b''_n|^2) < +\infty$. So, there exists $M > 0$ such that $|a''_n| \leq M$ and $|b''_n| \leq M$ for any $n \in I\!N$. From here and from (6.61) we obtain the inequality:

$$\sum_{n=1}^{\infty}(|a_n|^2 + |b_n|^2) \leq 2M \sum_{n=1}^{\infty} \frac{1}{n^2} < +\infty$$

According to the Weierstrass criteria we have that the series (6.60) is absolutely and uniformly convergent on $[0, R] \times [0, 2\pi]$ thus, the function $\widetilde{u}(r, \varphi)$ defined by (6.60) is well defined and continuous on $[0, R] \times [0, 2\pi]$.

The derivative of $\widetilde{u}(r, \varphi)$ is obtained from the derivatives of the terms of the series (6.60) and due to the fact that the series obtained by differentiating term by term are absolutely and uniformly convergent being bounded by series of the form

$$\sum_{n=1}^{\infty} n^p \left(\frac{r}{R}\right)^n [|\ a_n \ | + |\ b_n \ |], \ p = 1, 2, ...$$

Computing the sum of the series, we obtain the formula:

$$\widetilde{u}(r, \varphi) = \frac{1}{2\pi} \int_0^{2\pi} \widetilde{h}(\theta) \frac{R^2 - r^2}{r^2 - 2Rr\cos(\varphi - \theta) + R^2} d\theta \tag{6.62}$$

called the Poisson formula.

In order to determine the solution of the Neumann Problem we proceed in the same way.

Exercises

1. Find the solution of Dirichlet Problem:

$$\begin{cases} \dfrac{\partial^2 u}{\partial x_1^2} + \dfrac{\partial^2 u}{\partial x_2^2} = 0, & x_1^2 + x_2^2 < R^2 \\ u(x_1, x_2) = (x_1 + x_2)^2, & x_1^2 + x_2^2 = R^2 \end{cases}$$

A: $u(x_1, x_2) = R^2 + 2x_1x_2$

2. Find the solution of the Dirichlet Problem:

$$\begin{cases} \dfrac{\partial^2 u}{\partial x_1^2} + \dfrac{\partial^2 u}{\partial x_2^2} = 0, & x_1^2 + x_2^2 < 1 \\ u(x_1, x_2) = x_1 \cdot x_2, & x_1^2 + x_2^2 = 1 \end{cases}$$

A: $u(x_1, x_2) = x_1x_2$

3. Find the solution of the Neumann Problem:

$$\begin{cases} \dfrac{\partial^2 u}{\partial x_1^2} + \dfrac{\partial^2 u}{\partial x_2^2} = 0, & x_1^2 + x_2^2 < R^2 \\ \dfrac{\partial u}{\partial \widetilde{n}} = x_1 + x_1^2 - x_2^2, & x_1^2 + x_2^2 = R^2 \end{cases}$$

A: $u(x_1, x_2) = A_0 + Rx_1 + \dfrac{R}{2}(x_1^2 - x_2^2)$

6.8 Symbolic calculus for the solution of the Dirichlet problem for Laplace equation on the disc

We consider the Dirichlet Problem for the Laplace equation on the disc of radius R:

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0,\ (\forall)(x,y) \in \Omega \\ \\ u \mid_{\partial\Omega} = h \end{cases} \tag{6.63}$$

The function *pdsolve* cannot find directly, by symbolic computation, the solution corresponding to the above problem. Thus, on the basis of the theoretical results presented in the previous section (the formula for the formal solution), we will present a program in *Maple* which displays the analytic solution of the Dirichlet Problem (6.63).

Writing the problem in polar coordinates:

$$\begin{cases} \dfrac{\partial^2 \widetilde{u}}{\partial r^2} + \dfrac{1}{r^2}\dfrac{\partial^2 \widetilde{u}}{\partial \varphi^2} + \dfrac{1}{r}\dfrac{\partial \widetilde{u}}{\partial r} = 0, \quad r < R, \varphi \in [0, 2\pi) \\ \\ \widetilde{u}(r,\varphi) = u(r, \varphi + 2\pi) \\ \\ \lim\limits_{r \to 0} \mid \widetilde{u}(r,\varphi) \mid < +\infty \\ \\ \widetilde{u}(R,\varphi) = \widetilde{h}(\varphi) \end{cases} \tag{6.64}$$

and writing the Fourier series of $\widetilde{h}$ we obtain the formal solution:

$$\widetilde{u}(r,\varphi) = a_0 + \sum_{n=1}^{\infty} \frac{r^n}{R^n}(a_n \cos n\varphi + b_n \sin n\varphi). \tag{6.65}$$

in which a_0, a_n, b_n are the Fourier coefficients:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \widetilde{h}(\varphi) d\varphi \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \widetilde{h}(\varphi) \cdot \cos n\varphi d\varphi \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \widetilde{h}(\varphi) \cdot \sin n\varphi d\varphi$$

This formal solution can be obtained in *Maple* with the help of the *DirichletInt* procedure:

```
> restart;
> DirichletInt:=proc(f,R)

local a0,a,b;
> a0:=(1/(2*Pi))*Int(f,phi=-Pi..Pi);
> a:=n->1/Pi*Int(f*cos(n*phi),phi=-Pi..Pi);
> b:=n->1/Pi*Int(f*sin(n*phi),phi=-Pi..Pi);
> a0+add(r^n/R^n*(a(n)*cos(n*phi)+b(n)*sin(n*phi)),n=1..Order);
> RETURN(map(simplify,value(%)));
> end:
```

This procedure is used with the instruction $DirichletInt(f, R)$ in which f is the function $\widetilde{h}$ from (6.64), as it can be seen in the following examples:

Example 1:

$$\begin{cases} \dfrac{\partial^2 u}{\partial x_1^2} + \dfrac{\partial^2 u}{\partial x_2^2} = 0, & x_1^2 + x_2^2 < R^2 \\ u(x_1, x_2) = (x_1 + x_2)^2, & x_1^2 + x_2^2 = R^2 \end{cases}$$

```
> f:=(x1+x2)^2:  R:=R:
> f:=subs(x1=r*cos(phi),x2=r*sin(phi),r=R,f);
```

$$f := (R\cos(\phi) + R\sin(\phi))^2$$

```
> sol1:=DirichletInt(f,R);
```

$$sol1 := R^2 + r^2\sin(2\phi)$$

Example 2:

$$\begin{cases} \dfrac{\partial^2 u}{\partial x_1^2} + \dfrac{\partial^2 u}{\partial x_2^2} = 0, & x_1^2 + x_2^2 < 1 \\ u(x_1, x_2) = x_1 \cdot x_2, & x_1^2 + x_2^2 = 1 \end{cases}$$

```
> f:=x1*x2:  R:=1:
> f:=subs(x1=r*cos(phi),x2=r*sin(phi),r=R,f);
```

$$f := \cos(\phi)\sin(\phi)$$

```
> sol2:=DirichletInt(f,R);
```

$$sol2 := 1/2\, r^2\sin(2\phi)$$

Example 3:

$$\begin{cases} \dfrac{\partial^2 \widetilde{u}}{\partial r^2} + \dfrac{1}{r^2}\dfrac{\partial^2 \widetilde{u}}{\partial \varphi^2} + \dfrac{1}{r}\dfrac{\partial \widetilde{u}}{\partial r} = 0, \quad r < 1, \varphi \in [0, 2\pi) \\ \widetilde{u}(r, \varphi) = u(r, \varphi + 2\pi) \\ \lim\limits_{r \to 0} \mid \widetilde{u}(r, \varphi) \mid < +\infty \\ \widetilde{u}(1, \varphi) = \sin^3 \varphi \end{cases}$$

```
> f:=sin(phi)^3:  R:=1:
> sol3:=DirichletInt(f,R);
```

$$sol3 := 3/4\, r\sin(\phi) - 1/4\, r^3\sin(3\phi)$$

Example 4:

$$\begin{cases} \dfrac{\partial^2 \widetilde{u}}{\partial r^2} + \dfrac{1}{r^2}\dfrac{\partial^2 \widetilde{u}}{\partial \varphi^2} + \dfrac{1}{r}\dfrac{\partial \widetilde{u}}{\partial r} = 0, \quad r < 1, \varphi \in [0, 2\pi) \\ \widetilde{u}(r,\varphi) = u(r, \varphi + 2\pi) \\ \lim\limits_{r \to 0} \mid \widetilde{u}(r,\varphi) \mid < +\infty \\ \widetilde{u}(1,\varphi) = \sin^6 \varphi + \cos^6 \varphi \end{cases}$$

```
> f:=sin(phi)^6+cos(phi)^6:  R:=1:
> sol4:=DirichletInt(f,R);
```

$$sol4 := 5/8 + 3/8\, r^4 \cos(4\,\phi)$$

Chapter 7

Generalized solutions. Variational methods

In the previous chapter, we concentrated our attention on solving the Dirichlet and Neumann Problems in the case of the Poisson equation. The particularity of this equation consists on the fact that the elliptic equation is defined by the Laplacian $\triangle$.

In the followings, we consider more general elliptic equations, called divergence type elliptic equations, for which we formulate the Dirichlet Problem. Besides the concept of classical solution, we introduce the concept of generalized solution and we give conditions for the existence and the uniqueness of the generalized solution.

After that, we define the Cauchy-Dirichlet Problem for parabolic and hyperbolic equations, and we give conditions for the existence of the generalized solution.

7.1 The divergence type elliptic equation and the Dirichlet Problem

Let us consider the bounded domain $\Omega \subset \mathbb{R}^n$ having smooth (partially smooth) boundary $\partial\Omega$, the real valued functions a_{ij}, c and F defined on $\overline{\Omega}$ $(i, j = \overline{1, n})$ with the following properties:

1) a_{ij} are of class $\mathcal{C}^1$ on $\overline{\Omega}$ and there exists $\mu_0 > 0$ such that for any $X = (x_1, ..., x_n) \in \overline{\Omega}$ and any $(\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n$ we have

$$\sum_{i,j=1}^{n} a_{ij}(X) \cdot \xi_i \cdot \xi_j \geq \mu_0 \cdot \sum_{i=1}^{n} \xi_i^2 \quad \text{and} \quad a_{ij}(X) = a_{ji}(X)$$

2) the function c is continuous on $\overline{\Omega}$ and $c(X) \geq 0, (\forall)X \in \overline{\Omega}$.

3) the function F is continuous on $\overline{\Omega}$.

Definition 7.1.1. *We call divergence type elliptic partial differential equation a functional dependence of the form*

$$-\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n} a_{ij}(X)\cdot\frac{\partial u}{\partial x_j}\right) + c(X)\cdot u = F(X), (\forall)\ X\in\Omega \tag{7.1}$$

between the unknown function u and its first and second order partial derivatives.

In equation (7.1), the functions a_{ij}, c and F are known.

Definition 7.1.2. *A real valued function u of class $\mathcal{C}^2$ on Ω which verifies*

$$-\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n} a_{ij}(X)\cdot\frac{\partial u}{\partial x_j}(X)\right) + c(X)\cdot u(X) = F(X),\ \ (\forall)\ X\in\Omega$$

is called classical solution of the equation (7.1).

Definition 7.1.3. *The problem of determining those solutions u of equation (7.1) which are continuous on $\overline{\Omega}$ and verify the boundary condition*

$$u|_{\partial\Omega} = h \tag{7.2}$$

is called Dirichlet Problem for the equation (7.1).

In equality (7.2), h is a known real valued continuous function on $\partial\Omega$.

The Dirichlet Problem for the divergence type elliptic partial differential equation is denoted by

$$\begin{cases} -\displaystyle\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n} a_{ij}(X)\cdot\frac{\partial u}{\partial x_j}\right) + c(X)\cdot u = F(X), (\forall)\ X\in\Omega \\ u|_{\partial\Omega} = h. \end{cases} \tag{7.3}$$

Observation 7.1.1. If there exists a function $H:\Omega'\supset\overline{\Omega}\to\mathbb{R}'$ of class $\mathcal{C}^2$ such that $H|_{\partial\Omega} = h$, then u is a solution of the Dirichlet Problem (7.3) if and only if the function $v = u - H$ is a solution of the Dirichlet Problem:

$$\begin{cases} -\displaystyle\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n} a_{ij}(X)\cdot\frac{\partial v}{\partial x_j}\right) + c(X)\cdot v = G(X), (\forall)\ X\in\Omega \\ v|_{\partial\Omega} = 0 \end{cases} \tag{7.4}$$

where $G(X) = F(X) - c(X)\cdot H(X) + \displaystyle\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n} a_{ij}(X)\cdot\frac{\partial H}{\partial x_j}\right)$.

Thus, by a changement of functions, the Dirichlet Problem with non-homogeneous boundary conditions is reduced to the Dirichlet Problem with homogeneous boundary condition. Due to this fact, in the followings, we will consider the Dirichlet Problems in which the known function h is identically null on the boundary.

Observation 7.1.2. If we consider the vector space of real valued continuous functions u on $\overline{\Omega}$ which are of class $\mathcal{C}^2$ on Ω and null on the boundary:

$$D = \{u \,|\, u : \overline{\Omega} \to \mathbb{R}^1,\, u \in \mathcal{C}(\overline{\Omega}) \cap \mathcal{C}^2(\Omega) \textit{ and } u|_{\partial\Omega} = 0\},$$

and the differential operator A defined on this vector space D by:

$$(Au)(X) = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n} a_{ij}(X) \cdot \frac{\partial u}{\partial x_j}\right) + c(X) \cdot u \tag{7.5}$$

then the Dirichlet Problem:

$$\begin{cases} -\displaystyle\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n} a_{ij}(X) \cdot \frac{\partial u}{\partial x_j}\right) + c(X) \cdot u = F(X), (\forall)\ X \in \Omega \\ \\ u|_{\partial\Omega} = 0 \end{cases} \tag{7.6}$$

is equivalent to the following problem :

Determine the function $u \in D$ such that we have:

$$Au = F. \tag{7.7}$$

In this formulation of the Dirichlet Problem, the limit condition doesn't appear separately, being included in the definition of the vector space D (i.e. the domain of definition of A).

The problem of uniqueness of the classical solution of the Dirichlet Problem is equivalent with the injectivity of the operator $A : D \to \mathcal{C}(\overline{\Omega})$, and the problem of existence of the classical solution is equivalent with the surjectivity of the operator A.

In the followings, we give some properties of the operator A for answering the existence and uniqueness problems of the solution of the Dirichlet Problem (7.6).

Theorem 7.1.1. *The operator A defined on the function space D by (7.5) has the following properties:*

i) the domain of definition D of the operator A is a dense subspace in the Hilbert space $L^2(\Omega)$;

ii) the operator A is linear;

iii) for any $u,\ v \in D$ the following equality holds: $\displaystyle\int_\Omega Au \cdot v dX = \int_\Omega Av \cdot u dX$

Proof. i) We know the fact that the vector space $\mathcal{D}(\Omega)$ of real functions u of class $\mathcal{C}^\infty$ on Ω and with compact support included in Ω:

$$\mathcal{D}(\Omega) = \{u \,|\, u : \Omega \to \mathbb{R}^1,\, u \in \mathcal{C}^\infty(\Omega),\ \mathrm{supp}\ u \subset \Omega\}$$

is dense in $L^2(\Omega)$. Because the function space D contains the vector space $\mathcal{D}(\Omega)$ we have that the space D is dense in $L^2(\Omega)$.

ii) For proving that A is linear, we will show that

$$\begin{cases} A(u+v) = Au + Av \\ A(\alpha \cdot u) = \alpha \cdot Au \end{cases} \qquad (\forall)\, u,\, v \in D,\ (\forall)\, \alpha \in \mathbb{R}^1.$$

Indeed,

$$\begin{aligned}
A(u+v) &= -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n} a_{ij}(X)\frac{\partial(u+v)}{\partial x_j}\right) + c(X)(u+v) = \\
&= -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n} a_{ij}(X)\frac{\partial u}{\partial x_j}\right) + c(X)u - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n} a_{ij}(X)\frac{\partial v}{\partial x_j}\right) + c(X)v = \\
&= Au + Av
\end{aligned}$$

$$A(\alpha \cdot u) = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n} a_{ij}(X)\cdot\frac{\partial(\alpha \cdot u)}{\partial x_j}\right) + c(X)\cdot(\alpha\cdot u) = \alpha \cdot Au$$

which prove the linearity of A.

iii) For $u, v \in D$ we compute $\int_\Omega Au \cdot v dX$ and we find

$$\begin{aligned}
&\int_\Omega Au \cdot v\, dX = \\
&= -\int_\Omega \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n} a_{ij}(X)\frac{\partial u}{\partial x_j}\right) v\, dX + \int_\Omega c(X)uv\, dX \\
&= -\int_{\partial\Omega} \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(X)\frac{\partial u}{\partial x_j}\cos(\overline{u}, \overline{e}_i)v\, dS + \int_\Omega \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(X)\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i}\, dX + \int_\Omega c(X)uv\, dX \\
&= \int_\Omega \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(X)\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i}dX + \int_\Omega c(X)uv\, dX
\end{aligned}$$

Computing $\int_\Omega Av \cdot u\, dX$ we find:

$\int_\Omega Av \cdot u\, dX = \int_\Omega \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(X)\cdot\frac{\partial v}{\partial x_j}\cdot\frac{\partial u}{\partial x_i}dX + \int_\Omega c(X)\cdot u \cdot v\, dX.$

Because $a_{ij}(X) = a_{ji}(X)$ we obtain $\int_\Omega Au \cdot v dX = \int_\Omega Av \cdot u dX.$ □

Theorem 7.1.2. *There exists $\gamma > 0$ such that for any $u \in D$ we have*

$$\int_\Omega Au \cdot u dX \geq \gamma \int_\Omega u^2 dX \tag{7.8}$$

Proof. Computing $\int_\Omega Au \cdot udX$ we obatin :

$$\begin{aligned}\int_\Omega Au \cdot udX &= \int_\Omega \sum_{i=1}^n \sum_{j=1}^n a_{ij}(X) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \, dX + \int_\Omega c(X)u^2(X) \, dX \geq \\ &\geq \int_\Omega \mu_0 \sum_{i=1}^n \left| \frac{\partial u}{\partial x_i} \right|^2 \, dX + \int_\Omega c(X)u^2(X) \, dX\end{aligned}$$

It remains to evaluate $\int_\Omega \sum_{i=1}^n \left| \frac{\partial u}{\partial x_i} \right|^2 dX$ for $u \in D$. This evaluation is the content of Friedrichs theorem (which will be proved). According to this theorem (inequality) there is a constant $k > 0$, such that for any $u \in C^1(\bar{\Omega})$ with $u|_{\partial\Omega} = 0$ we have:

$$\int_\Omega |u(X)|^2 dX \leq k \int_\Omega \sum_{i=1}^n \left| \frac{\partial u}{\partial x_i} \right|^2 dX$$

We suppose for the moment that this evaluation is true, and continue the evaluation:

$$\int_\Omega Au \cdot udX \geq \mu_0 \int_\Omega \sum_{i=1}^n \left| \frac{\partial u}{\partial x_i} \right|^2 dX + \int_\Omega c(X)u^2(X)dX,$$

obtaining

$$\int_\Omega Au \cdot udX \geq \frac{\mu_0}{k} \cdot \int_\Omega |u(X)|^2 dX + \int_\Omega c(X)u^2(X)dX \geq \frac{\mu_0}{k} \int_\Omega |u(X)|^2 dX$$

Thus, the inequality (7.8) is proved. □

It remains to prove the Friedrichs inequality used in the previous proof.

Theorem 7.1.3 (Friedrichs inequality). *There is a positive constant $k > 0$, such that for any $u \in C^1(\bar{\Omega})$ with $u|_{\partial\Omega} = 0$ we have:*

$$\int_\Omega |u(X)|^2 dX \leq k \int_\Omega \sum_{i=1}^n \left| \frac{\partial u}{\partial x_i} \right|^2 dX. \tag{7.9}$$

Proof. The domain Ω being bounded, there is a translation $T : \mathbb{R}^n \to \mathbb{R}^n$, ($TX = X + X^0$) such that for any $X' \in \Omega' = T\Omega$ we have $x_i' \geq 0$. Since

$$\int_{\Omega'} |u'(X')|^2 \, dX' = \int_\Omega |u(X)|^2 \, dX \quad \text{and} \quad \int_{\Omega'} \left| \frac{\partial u'}{\partial x_i'} \right|^2 \, dX' = \int_\Omega \left| \frac{\partial u}{\partial x_i} \right|^2 \, dX$$

where $u'(X') = u(TX)$ and $X' = TX$, we obtain that it is sufficient to prove the inequality (7.9) on Ω'. Moreover, Ω' being bounded, there exists $a > 0$ such that $\Omega' \subset \Gamma_a = \{X' \,|\, 0 \leq x_i' \leq a\}$ and prolonging by 0 the function u' on $\Gamma_a \setminus \Omega'$ we obtain:

$$\int_{\Omega'} |u'(X')|^2 \, dX' = \int_{\Gamma_a} |u'(X')|^2 \, dX' \quad \text{and} \quad \int_{\Omega'} \left| \frac{\partial u'}{\partial x_i'} \right|^2 \, dX' = \int_{\Gamma_a} \left| \frac{\partial u'}{\partial x_i'} \right|^2 \, dX'$$

Thus, it is sufficient to prove the inequality (7.9) only for $\Omega = \Gamma_a$. For this purpose we use the Leibnitz-Newton formula

$$u(x_1, x_2, ...x_n) = \int_0^{x_i} \frac{\partial u}{\partial x_i}(x_1, ..., x_{i-1}, \xi, x_{i+1}, ...x_n)d\xi$$

from which, using the Cauchy-Buniakovski inequality we have:

$$\begin{aligned} |u(X)| &\leq \int_0^{x_i} \frac{\partial u}{\partial x_i}(x_1, ..., x_{i-1}, \xi, x_{i+1}, ..., x_n)d\xi \leq \\ &\leq \left(\int_0^{x_i} d\xi\right)^{1/2} \left(\int_0^{x_i} \left|\frac{\partial u}{\partial x_i}(x_1, ..., x_{i-1}, \xi, x_{i+1}, ..., x_n)\right|^2 d\xi\right)^{1/2} \leq \\ &\leq a^{1/2} \left(\int_0^a \left|\frac{\partial u}{\partial x_i}|(x_1, ..., x_{i-1}, \xi, x_{i+1}, ...x_n)\right|^2 d\xi\right)^{1/2}. \end{aligned}$$

From here we obtain:

$$\int_{\Gamma_a} |u(X)|^2 dX \leq a^2 \int_{\Gamma_a} \left|\frac{\partial u}{\partial x_i}\right|^2 dX$$

$$\int_{\Gamma_a} |u(X)|^2 dX \leq \frac{a^2}{n} \int_{\Gamma_a} \sum_{i=1}^{n} \left|\frac{\partial u}{\partial x_i}\right|^2 dX$$

In this way, we have shown that for $k = \dfrac{a^2}{n}$ the inequality (7.9) holds. □

Theorem 7.1.4 (Variational characterization of the solution of the equation $Au = F$). *The function $u_0 \in D$ is a solution of the equation $Au = F$ ($F \in C(\overline{\Omega})$) if and only if u_0 is a minim point for the functional:*

$$\Phi_F : D \to \mathbb{R}^1, \qquad \Phi_F(u) = \int_\Omega Au \cdot udX - 2\int_\Omega F \cdot udX. \tag{7.10}$$

Proof. If $u_0 \in D$ is a solution of the equation $Au = F$ then $Au_0 = F$ and for any $u \in D$ we have:

$$\begin{aligned} \Phi_F(u) - \Phi_F(u_0) &= \int_\Omega Au \cdot udX - 2\int_\Omega F \cdot udX - \int_\Omega Au_0 \cdot u_0 dX + 2\int_\Omega F \cdot u_0 dX = \\ &= \int_\Omega Au \cdot udX - 2\int_\Omega Au_0 \cdot u - \int_\Omega Au_0 \cdot u_0 dX + 2\int_\Omega Au_0 \cdot u_0 dX = \\ &= \int_\Omega Au \cdot udX - \int_\Omega Au_0 \cdot u - \int_\Omega u_0 \cdot AudX + \int_\Omega Au_0 \cdot u_0 dX = \\ &= \int_\Omega A(u - u_0) \cdot (u - u_0)dX \geq \gamma \int_\Omega |u - u_0|^2 dX \geq 0 \end{aligned}$$

We obtain in this way that u_0 is a global minimum of the functional Φ_F.

For proving the reciprocal assertion, we suppose that $u_0 \in D$ is a global minimum point of the functional Φ_F. Then, for any $u \in D$ we have

$$\Phi_F(u) \geq \Phi_F(u_0)$$

For a fixed $u \in D$ and $t \in \mathbb{R}^1$ we have

$$\Phi_F(u_0 + tu) \geq \Phi_F(u_0)$$

for any $t \in \mathbb{R}^1$. Hence the function

$$\varphi(t) = \Phi_F(u_0 + tu)$$

has a minimum at $t = 0$. Therefore $\varphi'(0) = 0$ and we have:

$$\int_\Omega (Au_0 - F)udX = 0 \quad (\forall)u \in D$$

Since the vector space D is dense in the Hilbert space $L^2(\Omega)$ finally we obtain $Au_0 = F$. □

Observation 7.1.3. This theorem is not an existence theorem because we didn't establish that the functional Φ_F has an absolute minimum point.

This theorem reduces the problems of existence and uniqueness of the classical solution of equation $Au = F$ to the problem of existence and uniqueness of the minimum point of the functional Φ_F.

Theorem 7.1.5 (Uniqueness theorem). *For $F \in C(\bar{\Omega})$ there exists at most one $u_0 \in D$ such that $Au_0 = F$.*

Proof. We suppose that for $F \in C(\bar{\Omega})$ there exists $u_1, u_2 \in D$ such that $Au_1 = F$ and $Au_2 = F$. Hence $\Phi_F(u_2) \geq \Phi_F(u_1)$ and $\Phi_F(u_1) \geq \Phi_F(u_2)$, which show that $\Phi_F(u_2) = \Phi_F(u_1)$. Taking into account the inequality:

$$\Phi_F(u_1) - \Phi_F(u_2) \geq \gamma \int_\Omega |u_1 - u_2|^2 dX$$

we obtain that $\int_\Omega |u_1 - u_2|^2 dX = 0$, thus, we have $u_1 = u_2$. □

Observation 7.1.4. i) The uniqueness theorem proves that the Dirichlet Problem has at most one classical solution.
ii) The existence of the global minimum in D of the functional $\Phi_F(u)$ is not true. There are examples which prove that, generally, the Dirichlet Problem doesn't have a classical solution.

In order to assure the existence of the global minimum of the functional Φ_F we enlarge the vector space D such that it becomes a Hilbert space. For this, we define on $D \times D$ the following function:

$$D \times D \ni (u, v) \to < u, v >_A = \int_\Omega Au \cdot vdX \tag{7.11}$$

Lemma 7.1.1. *The function defined by (7.11) is a scalar product on the vector space D.*

Proof. Direct verification. □

Observation 7.1.5. For $(u, v) \in D$ the following equality holds:

$$< u, v >_A = \int_\Omega \sum_{i=1}^n \sum_{j=1}^n a_{ij}(X) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dX + \int_\Omega c(X)u(X)v(X)dX$$

Definition 7.1.4. *The completed vector space D in the norm $\| \cdot \|_A$, generated by the scalar product*

$$< u, v >_A = \int_\Omega Au \cdot vdx$$

is called the energetic space of the operator A and it is denoted by X_A.

Observation 7.1.6. The elements of the energetic space X_A are elements of the vector space D, as limits in the $\| \cdot \|_A$ norm of fundamental sequences u_n form D. This means that, if $u \in X_A$ then $u \in D$ or there exists $(u_n)_n$ with $u_n \in D$ such that $\|u_n - u\|_A \to 0$. An element $u \in X_A$ has the property $u \in L^2(\Omega)$, $\dfrac{\partial u}{\partial x_i} \in L^2(\Omega)$ and $u|_{\partial\Omega} = 0$.

Observation 7.1.7. The energetic space X_A together with the scalar product $< u, v >_A$ is a Hilbert space.

Lemma 7.1.2. *The functional $\Phi_F : D \to \mathbb{R}^1$ defined by the formula*

$$\Phi_F(u) = \int_\Omega Au \cdot udX - 2 \int_\Omega F \cdot udX \tag{7.12}$$

where $F \in L^2(\Omega)$, is prolonged by continuity to a continuous functional $\widetilde{\Phi}_F$ defined on the energetic space X_A.

Proof. First, we will show that the functional $\Phi_F(u)$ defined by (7.12) is continuous with respect to the energetic space topology. For this, we consider $u \in D$, a sequence $(u_n)_n$, $u_n \in D$ having the property $\|u_n - u\|_A \underset{n}{\to} 0$ and then the sequence

$$\Phi_F(u_n) = \int_\Omega Au_n \cdot u_n dX - 2 \int_\Omega F \cdot u_n dX$$

We prove that this sequence is convergent and its limit is

$$\Phi_F(u) = \int_\Omega Au \cdot udX - 2 \int_\Omega F \cdot udX$$

Indeed, we have:

$$\Phi_F(u_n) = \|u_n\|_A^2 - 2 \int_\Omega F \cdot u_n dX$$

$$\Phi_F(u) = \|u\|_A^2 - 2 \int_\Omega F \cdot udX$$

and hence,

$$|\Phi_F(u_n) - \Phi_F(u)| \leq | \, \|u_n\|_A^2 - \|u\|_A^2 \, | + 2 \int_\Omega |F| \cdot |u_n - u| dX \leq$$

$$\leq | \, \|u_n\|_A^2 - \|u\|_A^2 \, | + 2\|F\|_{L^2(\Omega)} \cdot \|u_n - u\|_{L^2(\Omega)}.$$

Because the convergence $\|u_n - u\|_A \to 0$ implies $|\ \|u_n\|_A^2 - \|u\|_A^2|\ \to 0$ and $\|u_n - u\|_{L^2(\Omega)} \to 0$, we obtain the convergence

$$|\Phi_F(u_n) - \Phi_F(u)| \to 0$$

If $u \not\in D$ and $u \in X_A$ then we define $\Phi_F(u)$ by

$$\Phi_F(u) = \int_\Omega \sum_{i=1}^n \sum_{j=1}^n a_{ij}(X) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dX - 2 \int_\Omega F \cdot u dX$$

and for $u_n \in D$, we have $\|u_n - u\|_A \to 0$ by the same reasoning. □

Theorem 7.1.6 (Existence and uniqueness of the global minimum point). *For any $F \in L^2(\Omega)$, the functional $\widetilde{\Phi}_F$ obtained prolonging by continuity the functional Φ_F to the energetic space X_A has a unique minimum point.*

Proof. For the existence of the minimum point we will consider the linear and continuous functional

$$u \longmapsto \int_\Omega F \cdot u dX$$

defined on the energetic space X_A. According to the theorem of F. Riesz, there exists a function $u_F \in X_A$ such that we have:

$$< u_F, u >_A = \int_\Omega F \cdot u dX$$

for any $u \in X_A$. It remains to show that u_F is the global minimum point of the functional $\widetilde{\Phi}_F(u)$. For this, we compute $\widetilde{\Phi}_F(u_F)$ and $\widetilde{\Phi}_F(u)$ for $u \in X_A$. We have:

$$\begin{aligned}
\widetilde{\Phi}_F(u_F) &= < u_F, u_F >_A -2 < F, u_F >_{L^2(\Omega)} = \|u_F\|_A^2 - 2\|u_F\|_A^2 = -\|u_F\|_A^2 \\
\widetilde{\Phi}_F(u) &= < u, u >_A -2 < F, u >_{L^2(\Omega)} = \\
&= < u - u_F, u - u_F >_A +2 < u_F, u >_A - < u_F, u_F >_A -2 < F, u >_{L^2(\Omega)} = \\
&= \|u - u_F\|_A^2 - \|u_F\|_A^2 = \|u - u_F\|_A^2 + \widetilde{\Phi}_F(u_F)
\end{aligned}$$

The last equality

$$\widetilde{\Phi}_F(u) = \|u - u_F\|_A^2 + \widetilde{\Phi}_F(u_F)$$

is true for any $u \in X_A$ and shows that

$$\widetilde{\Phi}_F(u) \geq \widetilde{\Phi}_F(u_F)$$

Moreover, the equality shows that

$$\widetilde{\Phi}_F(u) > \widetilde{\Phi}_F(u_F)$$

for any $u \neq u_F$, which proves the uniqueness. □

Observation 7.1.8. If the global minimum point u_F of the prolonged functional $\tilde{\Phi}_F$ belongs to $D \subset X_A$ then $Au_F = F$, and it follows that u_F is a classical solution of the Dirichlet Problem.

Observation 7.1.9. If the global minimum point u_F of the prolonged functional $\tilde{\Phi}_F$ doesn't belong to the vector space $D \subset X_A$ then we cannot apply the operator A to see if it is a solution of the equation $Au = F$. In this case the following question is natural: *What does u_F represent in the context of solving the equation $Au = F$?*

We will give an answer to this question showing that the operator A has a prolongation $\tilde{A} : D(\tilde{A}) \to L^2(\Omega)$, called Friedrichs prolongation ($D \subset D(\tilde{A}) \subset X_A$) and showing that u_F verifies $\tilde{A}u_F = F$. This means precisely that the function u_F is not necessary a classical solution:

$$u \in D \quad \text{and} \quad -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n} a_{ij}(X) \cdot \frac{\partial u}{\partial x_j}\right) + c(X) \cdot u = F$$

but it verifies $u_F \in X_A$ and

$$\int_\Omega \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(X) \cdot \frac{\partial u_F}{\partial x_j} \cdot \frac{\partial v}{\partial x_i} dx + \int_\Omega c(X) \cdot u_F \cdot v \, dX = \int_\Omega F \cdot v \, dX, (\forall)\, v \in X_A$$

Theorem 7.1.7 (Friedrichs prolongation). *The operator $G : L^2(\Omega) \to X_A$ defined by $G(F) = u_F \in X_A$ where u_F verifies*

$$< u_F, u >_A = < F, u >_{L^2(\Omega)}, \quad (\forall)\, u \in X_A$$

has the following properties:

i) G is linear and bounded

ii) G is injective

iii) G is self-adjoint

iv) G is compact

v) $G^{-1} : Im\, G \subset X_A \to L^2(\Omega)$ is a prolongation of A.

Proof. First, we remark that the operator G is well defined due to the existence and the uniqueness of the function $u_F = G(F)$.

i) For $\alpha, \beta \in \mathbb{R}^1$, $F_1, F_2 \in L^2(\Omega)$ and $u \in X_A$ we have:

$$\begin{aligned} < G(\alpha F_1 + \beta F_2), u >_A &= < \alpha F_1 + \beta F_2, u >_{L^2(\Omega)} \\ &= < \alpha F_1, u >_{L^2(\Omega)} + < \beta F_2, u >_{L^2(\Omega)} \\ &= \alpha < F_1, u >_{L^2(\Omega)} + \beta < F_2, u >_{L^2(\Omega)} \\ &= < \alpha G(F_1) + \beta G(F_2), u >_{L^2(\Omega)} \end{aligned}$$

so, we obtain the equality:

$$G(\alpha F_1 + \beta F_2) = \alpha G(F_1) + \beta G(F_2)$$

which shows that the operator G is linear.

For proving that the operator G is bounded, we consider $F \in L^2(\Omega)$ and we evaluate $\|G(F)\|_A^2$. We find:

$$\frac{\mu_0^2}{k^2} \cdot \|G(F)\|_{L^2(\Omega)}^2 \leq \|G(F)\|_A^2 =< F, G(F) >_{L^2(\Omega)} \leq \|F\|_{L^2(\Omega)} \cdot \|G(F)\|_{L^2(\Omega)}$$

where $k > 0$ is the constant from the Friedrichs inequality and $\mu_0 > 0$ is the constant from the ellipticity condition.

Simplifying by $\|G(F)\|_{L^2(\Omega)}$ we obtain:

$$\|G(F)\|_{L^2(\Omega)} \leq \frac{k^2}{\mu_0^2} \cdot \|F\|_{L^2(\Omega)}$$

which shows that the operator $G : L^2(\Omega) \to L^2(\Omega)$ is bounded.

ii) $G(F) = 0 \Rightarrow < u_F, u >_A = 0, (\forall) u \in X_A \Rightarrow < F, u >_{L^2(\Omega)} = 0, (\forall) u \in X_A \Rightarrow F = 0$

iii) The operator $G : L^2(\Omega) \to Im\, G \subset X_A$ being injective, we consider the operator $G^{-1} : Im\, G \to L^2(\Omega)$ and we show that it is self-adjoint i.e.

$$< G^{-1}u, v >_{L^2(\Omega)} = < u, G^{-1}v >_{L^2(\Omega)}, \quad (\forall) u, v \in Im\, G$$

Indeed, we have

$$< G^{-1}u, v >_{L^2(\Omega)} = < u, v >_A = < v, u >_A = < G^{-1}v, u >_{L^2(\Omega)} = < u, G^{-1}v >_{L^2(\Omega)}$$

therefore

$$< G(F), H >_{L^2(\Omega)} = < G(F), G^{-1}(G(H)) >_{L^2(\Omega} = < G^{-1}(G(F)), G(H) >_{L^2(\Omega)} = < F, G(H) >_{L^2(\Omega)}$$

iv) The operator G is compact if it transforms the closed sphere of radius 1 from $L^2(\Omega)$ in a compact set from $L^2(\Omega)$. In order to show the compactness of G, we consider $F \in L^2(\Omega)$ with $\|F\|_{L^2(\Omega)} \leq 1$. Evaluating $\|G(F)\|_A^2$, we find:

$$\|G(F)\|_A^2 =< F, G(F) >_{L^2(\Omega)} \leq \|F\|_{L^2(\Omega)}^2 \cdot \|G(F)\|_{L^2(\Omega)}^2 \leq \|F\|_{L^2(\Omega)} \cdot \frac{k}{\mu_0} \cdot \|G(F)\|_A$$

Simplifying by $\|G(F)\|_A$ we obtain:

$$\|G(F)\|_A \leq \frac{k}{\mu_0} \cdot \|F\|_{L^2(\Omega)} \leq \frac{k}{\mu_0}$$

which shows that the image of the closed sphere of radius 1 from $L^2(\Omega)$ (through the operator G) is a bounded set in the energetic space X_A. Since a bounded sub-set of X_A is pre-compact in $L^2(\Omega)$ we obtain that the operator G is compact.

v) We show now that the operator $G^{-1} : Im\, G \subset X_A \to L^2(\Omega)$ is a prolongation of the operator A.

We know that $G^{-1} : Im\, G \subset X_A \to L^2(\Omega)$ is an injective operator. We consider $u \in D$ and therefore $Au \in L^2(\Omega)$. The function GAu belongs to $Im\ G$ and we have:

$$< GAu, v >_A = < Au, v >_{L^2(\Omega)} = < u, v >_A \quad (\forall) v \in X_A$$

Therefore we obtain:

$$GAu = u, \quad (\forall)\, u \in D$$

which proves that $u \in ImG$ and $G^{-1}u = Au$. We have shown in this way that the operator $G^{-1} : Im\, G \subset X_A \to L^2(\Omega)$ is a prolongation of the operator A. □

Definition 7.1.5. *The operator G^{-1} is called the Friedrichs prolongation of the operator A and it is denoted by $\tilde{A}$.*

Theorem 7.1.8 (Characterization of the global minimum of the functional Φ_F). *The function $u_F \in X_A$ is the global minimum of the functional*

$$\tilde{\Phi}_F : X_A \to \mathbb{R}^1, \ \tilde{\Phi}_F(u) =< u, u >_A -2 < F, u >_{L^2(\Omega)}$$

if and only if u_F is a solution of the equation $\tilde{A}u = F$, where $\tilde{A}$ is the Friedrichs prolongation of the operator A.

Proof. The result follows from the construction of the Friedrichs prolongation of the operator A. □

Observation 7.1.10. Form the results presented above, we have that for any $F \in L^2(\Omega)$ the equation $\tilde{A}u = F$ has a unique solution in the Hilbert space X_A.

Definition 7.1.6. *The solution u_F of the equation $\tilde{A}u = F$ is called the generalized solution of the equation $Au = F$.*

Observation 7.1.11. If $u_F \in D \subset X_A$ then u_F is a classical solution of the Dirichlet Problem. If $u_F \in X_A$ doesn't belong to D ($u_F \not\in D$) then it only verifies:

$$\int_\Omega \sum_{i=1}^n \sum_{j=1}^n a_{ij}(x) \cdot \frac{\partial u_F}{\partial x_j} \cdot \frac{\partial v}{\partial x_i} dx + \int_\Omega c(x) \cdot u_F(x) \cdot v(x) dx = \int_\Omega F(x) \cdot v(x) dx$$

for any $v \in X_A$.

In the followings, we will describe a method for determining the generalized solution u_F of the equation $Au = F$ (we know that this solution exists and it is unique). The method is based on the determination of the eigenvalues and eigenvectors of the Friedrichs prolongation $\tilde{A}$.

Definition 7.1.7. *A number λ is an eigenvalue for the operator $\tilde{A}$ if there exists a function u (called eigenfunction or eigenvector) in $D(\tilde{A})$ (the domain of definition of the operator $\tilde{A}$), $u \neq 0$, such that we have:*

$$\tilde{A}u = \lambda \cdot u$$

Theorem 7.1.9. *For the operator $\tilde{A}$, there exists an infinite sequence of eigenvalues*

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le ... \le \lambda_m \le ...$$

and an infinite sequence of eigenfunctions (eigenvectors) which correspond to the above eigenvalues

$$u_1, u_2, u_3, ..., u_n, ...$$

having the following properties:

$$\begin{aligned} \lim_{n\to\infty} \lambda_n &= +\infty \\ < u_i, u_j >_{L^2(\Omega)} &= \delta_{ij} \end{aligned}$$

Proof. The operator $G : L^2(\Omega) \to L^2(\Omega)$ is linear, self-adjoint and completely continuous. On the basis of the theorem concerning this class of linear operators we obtain that G admits a sequence of eigenvalues and a sequence of eigenfunctions. We will denote the eigenvalues of G by

$$\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, ..., \frac{1}{\lambda_m}, ...$$

and the eigenfunctions by

$$u_1, u_2, ..., u_m, ...$$

We assume that this sequence of eigenvalues is arranged such that:

$$\left|\frac{1}{\lambda_1}\right| \geq \left|\frac{1}{\lambda_2}\right| \geq ... \geq \left|\frac{1}{\lambda_m}\right| \geq ...$$

and the eigenfunctions are chosen such that

$$< u_i, u_j >_{L^2(\Omega)} = \delta_{ij}$$

Taking into account the equality $\tilde{A} = G^{-1}$ we have that $\tilde{A}$ admits the sequence of eigenvalues $|\lambda_1| \leq |\lambda_2| \leq ...$ and the sequence of eigenfunctions $u_1, u_2, ..., u_m, ...$

Now, we prove that $\lambda_m > 0$ for any m. Indeed, from $G(u_m) = \frac{1}{\lambda_m} \cdot u_m$ we have

$$< G(u_m), u_m >_{L^2(\Omega)} = \frac{1}{\lambda_m} \|u_m\|^2_{L^2(\Omega)} = \frac{1}{\lambda_m}$$

On the other hand,

$$< G(u_m), u_m >_{L^2(\Omega)} = < u_m, u_m >_A \geq \gamma \|u_m\|^2_{L^2(\Omega)}$$

and hence,

$$\frac{1}{\lambda_m} \geq \gamma \|u_m\|^2_{L^2\Omega} > 0$$

Now, we show that the sequence of the eigenvalues of $\widetilde{A}$ is infinite. First, we will show that the domain of definition ImG of $\widetilde{A}$ (formed by the elements of $G(F)$ with $F \in L^2(\Omega)$) is an infinite dimensional vector space. For this, let be u a function of class $\mathcal{C}^\infty$ having compact support in Ω. We consider the function $F = Au$ and we observe that u is a classical solution of the Dirichlet Problem

$$Au = F$$

It results that u is a generalized solution of the problem, and $G(F) = u$. We have shown in this way that any function of class $\mathcal{C}^\infty$ with compact support included in Ω belongs to the set $Im(G)$ and it follows that the vector space $Im(G)$ is infinite dimensional.

Let us suppose the contrary, that the operator G has a finite number of eigenvalues: $\lambda_1, \lambda_2, \ldots, \lambda_m$. Taking into account the fact that G is self-adjoint and compact we have:

$$G(F) = \sum_{k=1}^{m} < G(F), u_k > u_k, \ (\forall) F \in L^2(\Omega)$$

This equality shows that the sequence $u_1, u_2, \ldots, u_m$ is a basis in $Im(G)$ and hence, $Im(G)$ is a finite dimensional vector space. Thus, we obtain a contradiction and hence, G has an infinite sequence of eigenvalues. Moreover, we observe that $\lim_{m\to\infty} \lambda_m = +\infty$. $\square$

Theorem 7.1.10. *The sequence of eigenfunctions $\{u_m\}_m$ of the operator $\widetilde{A}$ is orthogonal, $\|u_m\| = 1$ and it is complete in the Hilbert space $L^2(\Omega)$. The sequence of eigenfunctions $\left\{\dfrac{u_m}{\sqrt{\lambda_m}}\right\}_m$ is orthogonal, $\|u_m\| = 1$ and it is complete in the energetic space X_A.*

Proof. The proof is technique and it well stepped. □

Now, we can formulate a theorem concerning the generalized solution of the equation $Au = F$, $F \in L^2(\Omega)$.

Theorem 7.1.11. *For any $F \in L^2(\Omega)$, the generalized solution u_F of the equation $Au = F$ is given by*

$$u_F = \sum_{m=1}^{+\infty} \frac{1}{\lambda_m} < F, u_m >_{L^2(\Omega)} \cdot u_m$$

where $\{u_m\}_m$ is the sequence of the eigenfunctions of the Friedrichs prolongation $\widetilde{A}$. This sequence is orthogonal , $\|u_m\| = 1$ and it is complete in the Hilbert space $L^2(\Omega)$.

Proof. Since

$$u_F = \sum_{m=1}^{+\infty} < u_F, u_m >_{L^2} \cdot u_m \quad \text{or} \quad u_F = \sum_{m=1}^{+\infty} < u_F, \frac{u_m}{\sqrt{\lambda_m}} >_A \cdot \frac{u_m}{\sqrt{\lambda_m}}$$

using the equality

$$< u_F, v >_A = < F, v >_{L^2}, \ (\forall) v \in X_A$$

we have

$$u_F = \sum_{m=1}^{+\infty} < F, \frac{u_m}{\sqrt{\lambda_m}} >_{L^2} \cdot \frac{u_m}{\sqrt{\lambda_m}} = \sum_{m=1}^{+\infty} \frac{1}{\lambda_m} < F, u_m >_{L^2} \cdot u_m$$

□

Exercises:

Let be $\Omega = (0, l_1) \times (0, l_2)$ and the operator A defined by

$$Au = -\left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}\right)$$

for $u \in D = \{u \in C^2(\Omega)$ and $u|_{\partial\Omega} = 0\}$.

Determine the generalized solution of the Dirichlet Problem $Au = F$ where:

a) $F(x_1, x_2) = x_1 \cdot x_2$;
b) $F(x_1, x_2) = x_1^2 + x_2^2$;
c) $F(x_1, x_2) = x_1 - x_2$;

A: From $Au = \lambda u$ we obtain the eigenvalues and eigenvectors:

$$\lambda_{m,n} = \left(\frac{n\pi}{l_1}\right)^2 + \left(\frac{m\pi}{l_2}\right)^2$$

and

$$u_{m,n} = \sin \frac{n\pi}{l_1} x_1 \cdot \sin \frac{m\pi}{l_2} x_2,$$

respectively. Computing $\|u_{m,n}\|_{L^2} = \frac{1}{2}\sqrt{l_1 \cdot l_2}$ we obtain the basis of orthogonal vectors of norm 1

$$\left\{ \frac{2}{\sqrt{l_1 l_2}} \cdot \sin \frac{n\pi}{l_1} x_1 \cdot \sin \frac{m\pi}{l_2} x_2 \right\}_{m,n}$$

The generalized solution is:

$$u_F(x_1, x_2) =$$
$$= \sum_{m=1}^{+\infty} \sum_{n=1}^{+\infty} \frac{1}{\left(\frac{n\pi}{l_1}\right)^2 + \left(\frac{m\pi}{l_2}\right)^2} \int_0^{l_1}\int_0^{l_2} \left(F(x_1, x_2) \frac{2}{\sqrt{l_1 l_2}} \sin \frac{n\pi}{l_1} x_1 \sin \frac{m\pi}{l_2} x_2 \right) dx_1 dx_2 \frac{2}{\sqrt{l_1 l_2}} \sin \frac{n\pi}{l_1} x_1 \sin \frac{m\pi}{l_2} x_2$$

where $F(x_1, x_2)$ represents the function given at a), b), c).

7.2 Cauchy-Dirichlet Problem for parabolic equations

Let be $\Omega \subset \mathbb{R}^n$ a bounded domain having smooth (partially smooth) boundary $\partial\Omega$ and the real valued functions:

$$a_{ij}, c, u_0 : \overline{\Omega} \to \mathbb{R}^1, \ f : [0, +\infty) \times \overline{\Omega} \to \mathbb{R}^1, \ g : [0, +\infty) \times \partial\Omega \to \mathbb{R}^1$$

satisfying the following properties:

i) a_{ij} are functions of class $\mathcal{C}^1$ on $\overline{\Omega}$ and $a_{ij} = a_{ji} \quad, i, j = \overline{1, n}$
c is a continuous function on $\overline{\Omega}$
u_0 is a continuous function on $\overline{\Omega}$ and of class $\mathcal{C}^2$ on Ω

ii) there exists $\mu_0 > 0$ such that for any $(\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ the following inequality holds

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(X) \cdot \xi_i \cdot \xi_j \geq \mu_0 \sum_{i=1}^{n} \xi_i^2, \quad (\forall) \ X \in \overline{\Omega}$$

iii) $c(X) \geq 0, \quad (\forall) \ X \in \overline{\Omega}$

iv) f is a continuous function on $[0, \infty) \times \overline{\Omega}$ and g is a continuous function on $[0, +\infty) \times \partial\Omega$

Definition 7.2.1. *The problem which consists of determining the real valued functions* $u : [0, +\infty) \times \overline{\Omega} \to \mathbb{R}^1$ *having the following properties:*

1. *u is continuous on $[0, +\infty) \times \overline{\Omega}$, it is of class $\mathcal{C}^1$ on $(0, +\infty) \times \Omega$ and for any fixed $t \in (0, +\infty)$, u is of class $\mathcal{C}^2$ on Ω*

2. $\dfrac{\partial u}{\partial t} - \sum_{i=1}^{n} \dfrac{\partial}{\partial x_i}\left(\sum_{j=1}^{n} a_{ij}(X) \cdot \dfrac{\partial u}{\partial x_j}\right) + c(X) \cdot u(t,X) = f(t,X), (\forall)\ t>0$ *and* $(\forall) X \in \Omega$

3. $u(t,X) = g(t,X), \quad (\forall)\ (t,X) \in [0,+\infty) \times \partial\Omega$

4. $u(0,X) = u_0(X), \quad (\forall) x \in \overline{\Omega}$

is called the Cauchy-Dirichlet Problem for the parabolic equation.

Definition 7.2.2. *A function u which satisfies the conditions from the previous definition is called classical solution of the Cauchy-Dirichlet Problem for the parabolic equation.*

Proposition 7.2.1. *If there exists a domain $\Omega' \subset \mathbb{R}^n$ which contains the set $\overline{\Omega}$ and a function $G : [0,+\infty) \times \Omega' \to \mathbb{R}^1$ of class $\mathcal{C}^1$ on $[0,+\infty) \times \Omega'$ such that*

$$G(t,X) = g(t,X)\ \forall (t,X) \in [0,+\infty) \times \partial\Omega,$$

then the non-homogeneous Cauchy-Dirichlet Problem for the parabolic equation, by the changes $v(t,X) = u(t,X) - G(t,X)$ is reduced to a homogeneous Cauchy-Dirichlet Problem for the parabolic equation:

$$\frac{\partial v}{\partial t} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n} a_{ij}(X) \frac{\partial v}{\partial x_j}\right) + c(X) v(t,X) = \tag{7.13}$$

$$= f(t,X) - \frac{\partial G}{\partial t} + \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n} a_{ij}(X) \frac{\partial G}{\partial x_j}\right) - c(X) G(t,X) \quad (\forall)\ t > 0,\ (\forall) X \in \Omega$$

$$v(t,X) = 0 \qquad (\forall)\ (t,X) \in [0,+\infty) \times \partial\Omega \tag{7.14}$$

$$v(0,X) = u_0(X) - G(0,X) \qquad (\forall)\ X \in \overline{\Omega} \tag{7.15}$$

Proof. By verification. □

Observation 7.2.1. The Cauchy-Dirichlet Problem for the parabolic equation having non-homogeneous conditions on the boundaries (presented in Def. 7.2.1), can be reduced to a Cauchy-Dirichlet Problem for the parabolic equation with homogeneous conditions on the boundaries. Due to this fact, in the followings, we will consider only the Cauchy-Dirichlet Problem for the parabolic equation of the type:

$$\frac{\partial u}{\partial t} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n} a_{ij}(X) \cdot \frac{\partial v}{\partial x_j}\right) + c(X) \cdot u(t,X) = f(t,X) \tag{7.16}$$

$$u(t,X) = 0 \qquad (\forall)\ (t,X) \in [0,+\infty) \times \partial\Omega \tag{7.17}$$

$$u(0,X) = u_0(X) \qquad (\forall)\ X \in \overline{\Omega} \tag{7.18}$$

where the functions a_{ij}, c, f have the properties presented above: (i), (ii), (iii), (iv), the function u_0 is continuous on $\overline{\Omega}$ and of class $\mathcal{C}^2$ in Ω.

Definition 7.2.3. *A classical solution of the Cauchy-Dirichlet Problem (7.16)-(7.18) is a function $u : [0,+\infty) \times \overline{\Omega} \to \mathbb{R}^1$ which has the following properties: u is continuous on*

$[0, +\infty) \times \overline{\Omega}$, *it is of class* $\mathcal{C}^1$ *on* $(0, +\infty) \times \Omega$ *and it is of class* $\mathcal{C}^2$ *on* Ω *for* $(\forall) t \in (0, +\infty)$, t *- fixed and verifies:*

$$\frac{\partial u}{\partial t} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n} a_{ij}(X)\frac{\partial u}{\partial x_j}\right) + c(X)u(t, X) = f(t, X) \quad (\forall)\ t > 0, (\forall)\ X \in \Omega \tag{7.19}$$

$$u(t, X) = 0 \qquad (\forall)\ (t, X) \in [0, \infty) \times \partial\Omega \tag{7.20}$$

$$u(0, X) = u_0(X) \qquad (\forall)\ x \in \overline{\Omega} \tag{7.21}$$

If we consider the differential operator A defined on the function space:

$$D = \{w | w : \overline{\Omega} \to \mathbb{R}^1, w \in C(\overline{\Omega}) \cap C^2(\Omega) \ \text{ and } \ w|_{\partial\Omega} = 0\}$$

with the formula:

$$Aw = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n} a_{ij}(X) \cdot \frac{\partial w}{\partial x_j}\right) + c(X) \cdot w$$

then the Cauchy-Dirichlet Problem (7.16)-(7.18) is:

$$\begin{cases} \dfrac{\partial u}{\partial t} + Au = f \\ \\ u(0, X) = u_0 \end{cases} \tag{7.22}$$

In the followings, we will formulate a more general problem for which we prove an existence and uniqueness theorem.

Theorem 7.2.1. *If the function* $u : [0, +\infty) \times \overline{\Omega} \to \mathbb{R}^1$ *is a classical solution of the Cauchy-Dirichlet Problem (7.16)-(7.18), then the function* V *defined as*

$$V(t)(X) = u(t, X)$$

has the following properties:

a) $V : [0, +\infty) \to L^2(\Omega)$ *is continuous*

b) $V : (0, +\infty) \to L^2(\Omega)$ *is of class* $\mathcal{C}^1$

c) $V : (0, +\infty) \to X_A$ *is continuous, where* X_A *represents the energetic space of the operator* A

d) $V(0) = u_0$

e) $\dfrac{dV}{dt} + AV = F(t)$, $(\forall)\ t > 0$ *where* $F(t)(X) = f(t, X)$

Proof. a) Let be $t_0 \in [0, +\infty)$ and $\eta > 0$. The function $u(t, X)$ is uniformly continuous on the compact set $[t_0 - \eta, t_0 + \eta] \times \overline{\Omega}$ (if $t_0 = 0$, then $[0, \eta] \times \overline{\Omega}$).

It follows, $(\forall)\varepsilon > 0, (\exists)\delta(\varepsilon) > 0$ such that $(\forall)(t', X'), (t'', X'') \in [t_0 - \eta, t_0 + \eta] \times \overline{\Omega}$ with $|t' - t''| < \delta(\varepsilon)$ and $||X' - X''|| < \delta(\varepsilon)$ we have:

$$|u(t', X') - u(t'', X'')| < \varepsilon/\sqrt{|\Omega|}$$

where $|\Omega|$ represents the measure of Ω.

We obtain from here that, the following inequality holds:

$$\int_{\Omega} |u(t,X) - u(t_0,X)|^2 dX < \varepsilon^2, \quad (\forall)\ t \ \text{ s.t. } \ |t - t_0| < \delta(\varepsilon).$$

We deduce that, if $|t - t_0| < \delta(\varepsilon)$ then

$$||V(t) - V(t_0)||_{L^2(\Omega)} < \varepsilon$$

This fact proves the continuity of the function $V : [0, +\infty) \to L^2(\Omega)$ at an arbitrary point t_0.

b) In order to prove that the function $V : (0, +\infty) \to L^2(\Omega)$ is of class $\mathcal{C}^1$, we will consider a point $t_0 \in (0, +\infty)$ and, with a similar reasoning, we show that:

$$\lim_{t \to t_0} \int_{\Omega} \left| \frac{u(t,X) - u(t_0,X)}{t - t_0} - \frac{\partial u}{\partial t}(t_0, X) \right|^2 dX = 0$$

which proves that the function $V : (0, +\infty) \to \mathbb{L}^2(\Omega)$ is of class $\mathcal{C}^1$ and $\dfrac{dV}{dt} + AV = F(t)$.

c) For proving that the function $V : (0, +\infty) \to X_A$ is continuous, a point $t_0 \in (0, +\infty)$ is considered for which it can be shown that $\lim\limits_{t \to t_0} ||V(t) - V(t_0)||_{X_A} = 0$.

d) $V(0)(X) = u(0, X) = u_0(X)$

e) This statement has been proved together with (b). □

Let us consider the Friedrichs prolongation $\widetilde{A}$ (of the operator A) defined on $D(\widetilde{A})$, a continuous function $F : [0, +\infty) \to L^2(\Omega)$, $V_0 \in L^2(\Omega)$ and the Cauchy Problem:

$$\begin{cases} \dfrac{dV}{dt} + \widetilde{A}V = F(t) \\ V(0) = V_0 \end{cases} \tag{7.23}$$

Definition 7.2.4. *A solution of this problem is a function $V : [0, +\infty) \to L^2(\Omega)$ having the following properties:*

a) $V \in C^1((0, +\infty); L^2) \cap C([0, +\infty), L^2)$

b) $V(t) \in D(\widetilde{A}), \quad (\forall)\ t \in (0, +\infty)$ *and* $\dfrac{dV}{dt} + \widetilde{A}V = F(t)$

c) $V(0) = V_0$

Definition 7.2.5. *The Cauchy Problem (7.23) will be called the abstract Cauchy Problem for the parabolic equation, and its solution will be called "strong" solution of the Cauchy-Dirichlet Problem for the parabolic equation.*

Observation 7.2.2. If $u(t, X)$ is a classical solution of the Cauchy-Dirichlet Problem for the parabolic equation, then the function V defined by $V(t)(x) = u(t, X)$ is a solution of the abstract Cauchy Problem for the parabolic equation.

Theorem 7.2.2. *The abstract Cauchy Problem (7.23) for the parabolic equation has at most one solution.*

Proof. Let be V_1, V_2 two solutions of the problem (7.23) and $V = V_1 - V_2$. The function V verifies

$$\frac{dV}{dt} + \widetilde{A}V = 0 \qquad \text{and} \qquad V(0) = 0$$

From here we obtain the equality

$$\frac{1}{2}\frac{d}{dt}||V||^2_{L^2(\Omega)} + ||V||^2_A = 0$$

from which it follows the inequality

$$\frac{d}{dt}||V||^2_{L^2(\Omega)} \leq 0$$

The function $||V||^2_{L^2(\Omega)}$ is positive, null at $t = 0$ and according to the previous inequality, it is decreasing. Thus, $||V||^2_{L^2(\Omega)} = 0$, $(\forall)\ t \geq 0$, and hence $V_1 = V_2$. □

Theorem 7.2.3. *If the function $F : [0, +\infty) \to L^2(\Omega)$ is of class $\mathcal{C}^1$ on $[0, +\infty)$ then the abstract Cauchy problem for the parabolic equation has a unique solution.*

Proof. We only prove the existence of the solution of the problem (7.23), because we already know the uniqueness form the previous theorem.

We suppose that we have a solution of the problem (7.23) and, for $t \in [0, +\infty)$, we develop this solution using the system of eigenfunctions $(u_m)_{m\in I\!N}$ of the operator $\widetilde{A}$:

$$V(t) = \sum_{m=1}^{\infty} < V(t), u_m >_{L^2(\Omega)} \cdot u_m$$

We proceed in the same way with $F(t)$ and V_0:

$$F(t) = \sum_{m=1}^{\infty} < F(t), u_m >_{L^2(\Omega)} \cdot u_m$$

$$V_0 = \sum_{m=1}^{\infty} < V_{(0)}, u_m >_{L^2(\Omega)} \cdot u_m$$

Denoting by:

$$v_m(t) =< V(t), u_m >_{L^2(\Omega)}$$

$$f_m(t) =< F(t), u_m >_{L^2(\Omega)}$$

$$v^0_m(t) =< V_0, u_m >_{L^2(\Omega)}$$

the equation

$$\frac{dV}{dt} + \widetilde{A}V = F(t)$$

and the initial condition

$$V(0) = V_0$$

give:

$$\frac{dv_m}{dt} + \lambda_m \cdot v_m = f_m \quad \text{and} \quad v_m(0) = v_m^0 \qquad m = 1, 2, 3, \dots$$

The solutions of these initial value problems are given by

$$v_m(t) = v_m^0 e^{-\lambda_m t} + \int_0^t e^{-\lambda_m (t-s)} \cdot f_m(s) ds \qquad m = 1, 2, 3, \dots$$

and it results that the solution $V(t)$ of the abstract Cauchy Problem (7.23) verifies the equality:

$$V(t) = \sum_{m=1}^{\infty} v_m^0 e^{-\lambda_m t} \cdot u_m + \sum_{m=1}^{\infty} \left(\int_0^t e^{-\lambda_m (t-s)} \cdot f_m(s) ds \right) \cdot u_m \tag{7.24}$$

Now, we will show that if $V_0 \in L^2(\Omega)$ and the function $F : [0, +\infty) \to L^2(\Omega)$ is of class $\mathcal{C}^1$ on $[0, +\infty)$, then the right hand term of (7.24) defines a function $V(t)$ which is solution for the abstract Cauchy problem i.e., V has the properties (a), (b), (c) from Definition (7.23).

At the first step, we prove the convergence of the series from the right hand term of the equality (7.24) and we analyze the smoothness of the function obtained by summing this series.

Since $\{u_m\}_m$ is an orthonormal basis in $L^2(\Omega)$, form the convergence of the series $\sum\limits_{m=1}^{\infty} |v_m^0|^2 \cdot e^{-2\lambda_m t}$ we obtain the convergence in $L^2(\Omega)$ of the function series $\sum\limits_{m=1}^{\infty} v_m^0 \cdot e^{-\lambda_m t} \cdot u_m$. The series $\sum\limits_{m=1}^{\infty} |v_m^0|^2 \cdot e^{-2\lambda_m t}$, $(\forall)\ t \geq 0$, is majored by the series $\sum\limits_{m=1}^{\infty} |v_m^0|^2$ which is convergent ($V_0 \in L^2(\Omega)$). It results in this way that the series $\sum\limits_{m=1}^{\infty} v_m^0 \cdot e^{-\lambda_m t} \cdot u_m$ is uniformly convergent for any $t \geq 0$ in the space $L^2(\Omega)$ and its sum is a continuous function which depends on t.

For showing that the series $\sum\limits_{m=1}^{\infty} \left(\int\limits_0^t e^{-\lambda_m (t-s)} \cdot f_m(s) ds \right) \cdot u_m$ converges uniformly in $L^2(\Omega)$ on an arbitrary segment $[0, T]$, we consider the series $\sum\limits_{m=1}^{\infty} | \int\limits_0^t e^{-\lambda_m (t-s)} \cdot f_m(s) ds|^2$, and we evaluate them as follows:

$$\sum_{m=1}^{\infty} \left| \int_0^t e^{-\lambda_m (t-s)} \cdot f_m(s) ds \right|^2 \leq \sum_{m=1}^{\infty} \int_0^t e^{-2\lambda_m (t-s)} ds \cdot \int_0^t f_m^2(s) ds =$$

$$= \sum_{m=1}^{\infty} e^{-2\lambda_m t} \cdot \frac{1}{2\lambda_m} e^{-2\lambda_m s} \Big|_0^t \cdot \int_0^t f_m^2 ds = \sum_{m=1}^{\infty} \left(\frac{1}{2\lambda_m} - \frac{1}{2\lambda_m} e^{-2\lambda_m t} \right) \cdot \int_0^t f_m^2(s) ds =$$

$$= \sum_{m=1}^{\infty} \frac{1}{2\lambda_m} (1 - e^{-2\lambda_m t}) \cdot \int_0^t f_m^2(s) ds \leq \sum_{m=1}^{\infty} \frac{1}{2\lambda_1} \int_0^t f_m^2(s) ds = \frac{1}{2\lambda_1} \sum_{m=1}^{\infty} \int_0^t f_m^2(s) ds$$

The series $\sum\limits_{m=1}^{\infty} f_m^2(s)$ converges for any $s \in [0,T]$, the functions are positive and the sum of the series $\sum\limits_{m=1}^{\infty} f_m^2(s) = ||F(s)||^2$ is a continuous function. According to Dini theorem we obtain that, the series $\sum\limits_{m=1}^{\infty} f_m^2(s)$ converges on $[0,T]$. Thus, we obtain the uniform convergence of the series $\sum\limits_{m=1}^{\infty} \int\limits_0^t |f_m(s)|^2\, ds$ pe $[0,T]$. We obtain in this way that the series $\sum\limits_{m=1}^{\infty} \left(\int\limits_0^t e^{-\lambda_m(t-s)} f_m(s) ds \right) u_m$ is uniformly convergent in $L^2(\Omega)$ with respect to $t \in [0,T]$ and its sum is a continuous function of t.

Thus, the function $V(t)$ defined by (7.24) is a continuous function from $[0,+\infty)$ to $L^2(\Omega)$.

In the followings, we will show that $V : (0,+\infty) \to L^2(\Omega)$ is a function of class $\mathcal{C}^1$. This fact results from the uniform convergence on $[t_0,+\infty)$ $(0 < t_0 < T$ arbitrary) of the series

$$\sum_{m=1}^{\infty} v_m^0 e^{-\lambda_m t} u_m \quad \text{and} \quad \sum_{m=1}^{\infty} \left(\int\limits_0^t e^{-\lambda_m(t-s)} f_m(s) ds \right) u_m$$

as well as the convergence of the series obtained by differentiating each term.

For the derivative of the series $\sum\limits_{m=1}^{\infty} v_m^0 e^{-\lambda_m t} u_m$, i.e. for series $-\sum\limits_{m=1}^{\infty} \lambda_m v_m^0 e^{-\lambda_m t} u_m$, the uniform convergence results form the estimation:

$$\lambda_m^2 (v_m^0)^2 e^{-2\lambda_m t} \le \lambda_m^2 (v_m^0)^2 e^{-2\lambda_m t_0} \le c \cdot (v_m^0)^2$$

where c is a constant, whih doesn't depend on m and t. For the derivative of the second series, i.e. for the series

$$\sum_{m=1}^{\infty} \left[f_m(t) - \lambda_m \int\limits_0^t e^{-\lambda_m(t-s)} f_m(s) ds \right] u_m$$

the uniform convergence is obtained using the estimation:

$$\begin{aligned}
|f_m(t) - \lambda_m \int\limits_0^t e^{-\lambda_m(t-s)} \cdot f_m(s)|^2 &= |f_m(0) e^{-\lambda_m t} + \int\limits_0^t f'_m(s) e^{-\lambda_m(t-s)} ds|^2 \le \\
&\le 2|f_m(0)|^2 e^{-2\lambda_1 t_0} + 2 \int\limits_0^t |f'_m(s)|^2 ds \cdot \int\limits_0^t e^{-2\lambda_m(t-s)} ds \le \\
&= |f_m(0)|^2 e^{-2\lambda_1 t_0} + \frac{1}{\lambda_1} \int\limits_0^T |f'_m(s)|^2 ds
\end{aligned}$$

From this estimation and form the hypothesis $F \in C^1([0,+\infty), L^2)$, we obtain the uniform convergence of the differentiated series and the continuity of sum of the series. Now, it is obvious that $V \in C^1((0,+\infty), L^2)$.

In order to prove that $V(t) \in D(\widetilde{A})$ if $t > 0$ we remark that the domain $D(\widetilde{A})$ can be characterized as follows:

$$D(\widetilde{A}) = \left\{ v = \sum_{m=1}^{\infty} c_m u_m | \sum_{m=1}^{\infty} \lambda_m^2 \cdot c_m^2 < +\infty \right\}$$

The previous reasonings show that $V(t) \in D(\widetilde{A})$, $(\forall)\ t > 0$.

It is easy to verify that the equalities $\dfrac{dV}{dt} + \widetilde{A}V = F(t)$ and $V(0) = V_0$ hold. □

Exercises:

Exercise 1. Find the abstract Cauchy problem in the case of the Cauchy-Dirichlet Problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = a^2 \dfrac{\partial^2 u}{\partial x^2} & t > 0,\ x \in (0, l) \\ u(t, 0) = u(t, l) = 0 & \\ u(0, x) = u_0(x) & \end{cases}$$

and determine the "strong" solution.

Answer: $\lambda_k = \left(\dfrac{k\pi}{l} \right)^2,\ k \in I\!N^*,\ u_k = \sin \dfrac{k\pi x}{l}$

$$u(t, x) = \sum_{k=1}^{\infty} a_k \cdot e^{-(\frac{ak\pi}{l})^2 \cdot t} \sin \frac{k\pi x}{l} \quad \text{where} \quad a_k = \frac{2}{l} \int_0^l u_0(x) \cdot \sin \frac{k\pi x}{l} dx$$

Exercise 2 Find the solutions of the following Cauchy-Dirichlet Problems:

$$a) \quad \begin{cases} \dfrac{\partial u}{\partial t} = 4 \dfrac{\partial^2 u}{\partial x^2}, & (t, x) \in (0, +\infty) \times (0, 1) \\ u(t, 0) = u(t, 1) = 0, \quad t \geq 0 & \\ u(0, x) = 3 \sin 2\pi x, \quad x \in [0, 1] & \end{cases}$$

A: $u(t, x) = 3 \cdot e^{-16\pi^2 t} \cdot \sin 2\pi x$

$$b) \quad \begin{cases} \dfrac{\partial u}{\partial t} = 4 \cdot \dfrac{\partial^2 u}{\partial x^2} + e^{-4t} \sin x & x \in (0, \pi) \times (0, \infty) \\ u(t, 0) = u(t, \pi) = 0, & t \geq 0 \\ u(0, x) = 4 \sin x \cdot \cos x, & x \in [0, \pi] \end{cases}$$

A: $u(t, x) = t \cdot e^{-4t} \cdot \sin x + 2e^{-16t} \cdot \sin 2x$

$$c) \quad \begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} + x + 1, & (t,x) \in (0,+\infty) \times (0,1) \\ u(t,0) = t+1, & u(t,1) = 2t+1, \quad t \geq 0 \\ u(0,x) = 1, & x \in [0,1] \end{cases}$$

$$\textbf{A:}\, u(t,x) = t + 1 + x \cdot t$$

7.3 Symbolic and numerical computations of the Cauchy-Dirichlet Problem for the parabolic equations

The symbolic computation of the solution of the Cauchy-Dirichlet Problem cannot be done using the function *pdsolve*. In these cases, we have to solve numerically. For example, we will consider three Cauchy-Dirichlet problems for parabolic equations and we determine their solutions using the function *pdsolve* having the following syntax for numerical computations:

pdsolve(PDE or PDE system, conds, type=numeric, other option);

where

$PDEorPDEsystem$	- PDE or system of PDE-s that we want to solve
conds	- initial conditions and limit conditions
type $=$ *numeric*	- indicates that numerical methods will be used
otheroption	- differet options (ex. numerical method, number of points

Example 1: *Heat equation*

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \dfrac{1}{10} \cdot \dfrac{\partial^2 u}{\partial x^2}(x,t) \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = 1 \end{cases}$$

```
> PDE1 :=diff(u(x,t),t)=1/10*diff(u(x,t),x,x);
```

$$PDE1 := \frac{\partial}{\partial t} u(x,t) = 1/10 \frac{\partial^2}{\partial x^2} u(x,t)$$

```
> IBC1 := {u(0,t)=0, u(1,t)=0, u(x,0)=1};
```

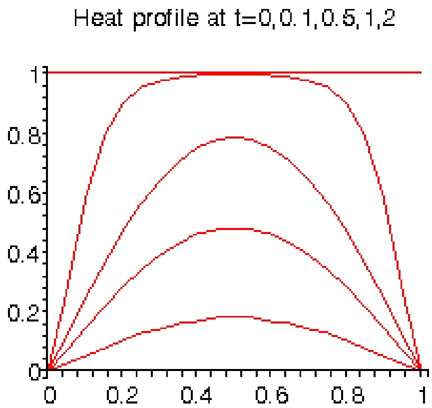
$$IBC1 := \{u(0,t) = 0, u(1,t) = 0, u(x,0) = 1\}$$

```
> pds1 := pdsolve(PDE1,IBC1,numeric);
```

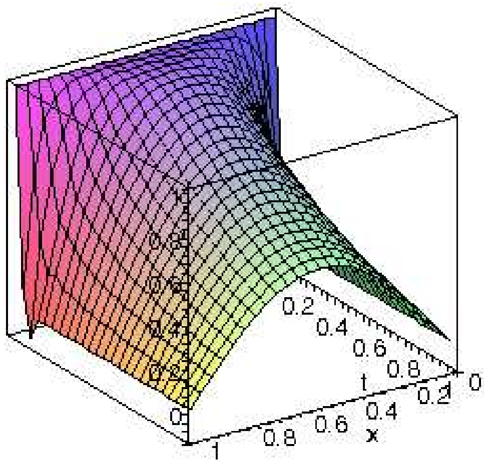
pds1 := module () local INFO; export plot, plot3d, animate, value, settings; option ‘Copyright (c) 2001 by Waterloo Maple Inc. All rights reserved.‘; end module

```
> p1 := pds1:-plot(t=0):
p2 := pds1:-plot(t=1/10):
p3 := pds1:-plot(t=1/2):
p4 := pds1:-plot(t=1):
p5 := pds1:-plot(t=2):
```

```
plots[display]({p1,p2,p3,p4,p5},
title=`Heat profile at t=0,0.1,0.5,1,2`);
```




```
> pds1:-plot3d(t=0..1,x=0..1,axes=boxed);
```



Exemple 2: *Heat equation*

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = 4\left(\dfrac{\partial^2 u}{\partial x^2}\right)(x,t) + e^{-4t}\cdot \sin x \\ u(0,t) = u(\pi,t) = 0 \\ u(x,0) = 4\cos x \sin x \end{cases}$$

```
> PDE2 :=
> diff(u(x,t),t)=4*diff(u(x,t),x,x)+(exp(-4*t))*sin(x);
```

$$PDE2 := \frac{\partial}{\partial t}u\left(x,t\right) = 4\,\frac{\partial^2}{\partial x^2}u\left(x,t\right) + e^{-4\,t}\sin\left(x\right)$$

```
> IBC2 := {u(0,t)=0,u(Pi,t)=0,u(x,0)=4*cos(x)*sin(x)};
```

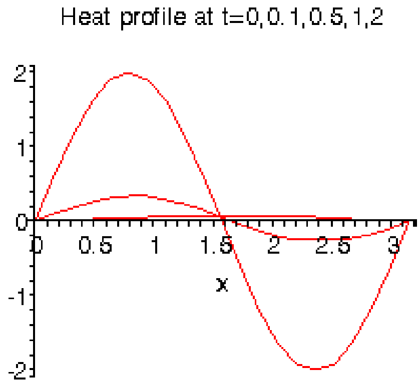
$$IBC2 := \{u\left(0,t\right) = 0, u\left(\pi,t\right) = 0, u\left(x,0\right) = 4\,\cos\left(x\right)\sin\left(x\right)\}$$

```
> pds2 := pdsolve(PDE2,IBC2,numeric);
```

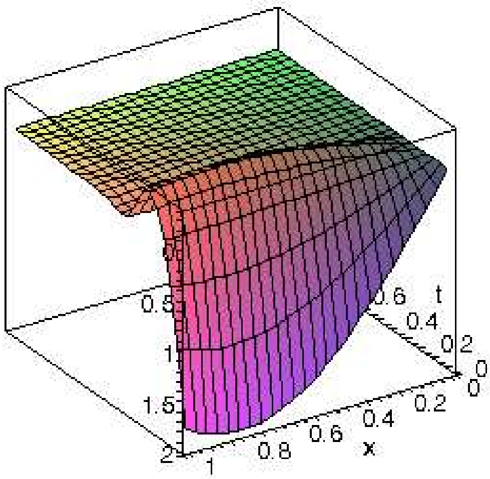
pds1 := module () local INFO; export plot, plot3d, animate, value, settings; option ‘Copyright (c) 2001 by Waterloo Maple Inc. All rights reserved.‘; end module

```
> p6 := pds2:-plot(t=0):
p7 := pds2:-plot(t=1/10):
p8 := pds2:-plot(t=1/2):
p9 := pds2:-plot(t=1):
p10 := pds2:-plot(t=2):
```

```
plots[display]({p6,p7,p8,p9,p10},
title=`Heat profile at t=0,0.1,0.5,1,2`);
```




```
> pds2:-plot3d(t=0..1,x=0..1,axes=boxed);
```



Exemple 3: *Heat equation*

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \dfrac{\partial^2 u}{\partial x^2}(x,t) + t \cdot cosx \\ \\ u(0,t) = t, u(\pi,t) = 0 \\ \\ u(x,0) = \cos 2x + \cos 3x \end{cases}$$

```
> PDE3 :=
> diff(u(x,t),t)=diff(u(x,t),x,x)+t*cos(x);
```

$$PDE3 := \frac{\partial}{\partial t}u(x,t) = \frac{\partial^2}{\partial x^2}u(x,t) + t\cos(x)$$

```
> IBC3 := {u(0,t)=t,u(Pi,t)=0,u(x,0)=cos(2*x)+cos(3*x)};
```

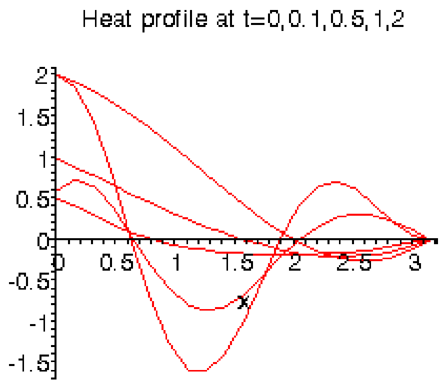
$$IBC3 := \{u(\pi,t) = 0, u(0,t) = t, u(x,0) = \cos(2x) + \cos(3x)\}$$

```
> pds3 := pdsolve(PDE3,IBC3,numeric);
```

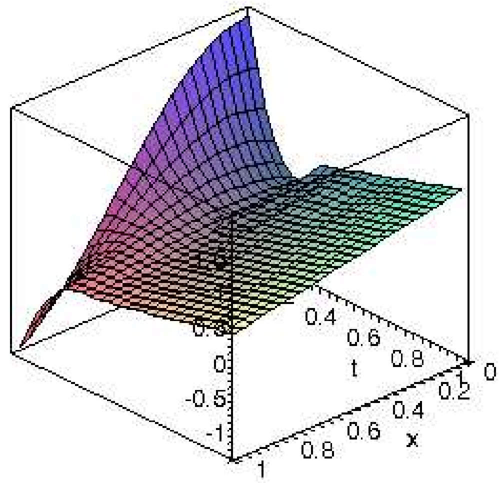
pds1 := module () local INFO; export plot, plot3d, animate, value, settings; option ‘Copyright (c) 2001 by Waterloo Maple Inc. All rights reserved.‘; end module

```
> q1 := pds3:-plot(t=0):
q2 := pds3:-plot(t=1/10):
q3 := pds3:-plot(t=1/2):
q4 := pds3:-plot(t=1):
q5 := pds3:-plot(t=2):
```

```
plots[display]({q1,pq2,q3,q4,q5},
title=‘Heat profile at t=0,0.1,0.5,1,2‘);
```




```
>  pds3:-plot3d(t=0..1,x=0..1,axes=boxed);
```



7.4 Cauchy-Dirichlet Problem for hyperbolic equations

Let be $\Omega \subset \mathbb{R}^n$ a bounded domain having smooth (partially smooth) boundary $\partial\Omega$ and the real valued functions a_{ij}, c, u_0, u_1 : $\overline{\Omega} \to \mathbb{R}^1$, $f : [0,+\infty) \times \overline{\Omega} \to \mathbb{R}^1$ and $g : [0,+\infty) \times \partial\Omega \to \mathbb{R}^1$ with the following properties:

i) a_{ij} are functions of class $\mathcal{C}^1$ on $\overline{\Omega}$ and $a_{ij} = a_{ji}$, $i, j = \overline{1,n}$
c is continuous on $\overline{\Omega}$
u_0 is of class $\mathcal{C}^1$ on $\overline{\Omega}$ and class $\mathcal{C}^2$ on Ω
u_1 is continuous on $\overline{\Omega}$ and of class $\mathcal{C}^1$ on Ω

ii) there exists $\mu_0 > 0$ such that for any $(\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n$ the following inequality holds:
$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(X)\cdot \xi_i \cdot \xi_j \geq \mu_0 \sum_{i=1}^{n} \xi_i^2, \ \ (\forall)X \in \overline{\Omega}$$

iii) $c(X) \geq 0$, $(\forall)X \in \overline{\Omega}$

iv) the functions $f : [0,+\infty) \times \overline{\Omega} \to \mathbb{R}^1$ and $g : [0,+\infty) \times \partial\Omega \to \mathbb{R}^1$ are continuous.

Definition 7.4.1. *The problem which consists of determining the real valued functions $u : [0,+\infty) \times \overline{\Omega} \to \mathbb{R}^1$, which are continuous on $[0,+\infty) \times \overline{\Omega}$, of class $\mathcal{C}^1$ on $[0,+\infty) \times \overline{\Omega}$ and of class $\mathcal{C}^2$ on $(0,+\infty) \times \Omega$, with the following properties:*

$$\frac{\partial^2 u}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n} a_{ij}(X)\frac{\partial u}{\partial x_j}\right) + c(X)u = f(t,X), \ \ (\forall)(t,X) \in (0,\infty)\times\Omega \quad (7.25)$$
$$u(t,X) = g(t,X), \quad (\forall)(t,X) \in [0,+\infty)\times\partial\Omega \quad (7.26)$$
$$u(0,X) = u_0(X), \quad (\forall)X \in \overline{\Omega} \quad (7.27)$$
$$\frac{\partial u}{\partial t}(0,X) = u_1(X), \quad (\forall)X \in \overline{\Omega} \quad (7.28)$$

is called the Cauchy-Dirichlet Problem for the hyperbolic equation.

Definition 7.4.2. *A function u which verifies the conditions from the previous definition is called classical solution of the Cauchy-Dirichlet Problem for the hyperbolic equation.*

Proposition 7.4.1. *If there exists a domain $\Omega' \subset \mathbb{R}^n$ which contains the domain $\overline{\Omega}$ and a function $G : [0,+\infty) \times \Omega' \to \mathbb{R}^1$ of class $\mathcal{C}^1$ on $[0,+\infty) \times \Omega'$ such that*

$$G(t,X) = g(t,X), \ (\forall)(t,X) \in (0,+\infty)\times\partial\Omega$$

then the Cauchy-Dirichlet Problem for the hyperbolic equation, by the change of the unknown function $v(t,X) = u(t,X) - G(t,X)$, is reduced to the Cauchy-Dirichlet Problem *for the hyperbolic equation with null conditions on the boundary:*

$$\frac{\partial^2 v}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n} a_{ij}\frac{\partial v}{\partial x_j}\right) + c(X)v(t,X) = \tag{7.29}$$

$$= f(t,X) - \frac{\partial^2 G}{\partial t^2} + \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n} \frac{\partial G}{\partial x_j}\right) - c(X)G(t,X),\ (\forall)(t,X) \in (0,+\infty)\times\Omega$$

$$v(t,X) = 0, \qquad (\forall)(t,X) \in [0,+\infty)\times\Omega \tag{7.30}$$

$$u(0,X) = u_0(X) - G(0,X) = v_0(X), \qquad (\forall)X \in \overline{\Omega} \tag{7.31}$$

$$\frac{\partial v}{\partial t}(0,X) = u_1(X) - \frac{\partial G}{\partial t}(0,X) = v_1(X), \qquad (\forall)X \in \overline{\Omega}. \tag{7.32}$$

Proof. By verification. □

Observation 7.4.1. The proposition reduces the problem (7.25-7.28) with non-null boundary condition to the problem (7.29-7.32), in which the boundary condition (7.30) is zero:

$$v(t,X) = 0, \qquad (\forall)(t,X) \in [0,+\infty)\times\partial\Omega$$

Due to this fact, we will study the existence and the uniqueness of the solution of the Cauchy-Dirichlet Problem for the hyperbolic equation with null boundary condition. Thus, we will consider the Cauchy-Dirichlet Problem for the hyperbolic equation of the following form:

$$\frac{\partial^2 u}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n} a_{ij}(X)\frac{\partial u}{\partial X_j}\right) + c(X)u(t,X) = f(t,X),\ (t,X) \in (0,\infty)\times\Omega \tag{7.33}$$

$$u(t,X) = 0, \qquad (\forall)(t,X) \in [0,+\infty)\times\Omega \tag{7.34}$$

$$u(0,X) = u_0(X), \qquad (\forall)X \in \overline{\Omega} \tag{7.35}$$

$$\frac{\partial u}{\partial t}(0,X) = u_1(X), \qquad (\forall)X \in \overline{\Omega} \tag{7.36}$$

in which the functions a_{ij}, c, f have the previous properties $(i), (ii), (iii), (iv)$; the function u_0 is of class $\mathcal{C}^1$ on $\overline{\Omega}$ and of class $\mathcal{C}^2$ on Ω; the function u_1 is continuous on $\overline{\Omega}$ and of class $\mathcal{C}^1$ on Ω.
A classical solution of this problem is a function $u : [0,+\infty)\times\overline{\Omega} \to \mathbb{R}^1$ of class $\mathcal{C}^1$ on $[0,+\infty)\times\overline{\Omega}$, of class $\mathcal{C}^2$ on $(0,+\infty)\times\Omega$ which verifies (7.33)-(7.36).
Considering the differential operator A defined on the function space

$$D = \{w | w : \overline{\Omega} \to \mathbb{R}^1; w \in C(\overline{\Omega}) \cap C^2(\Omega),\ w_{|\partial\Omega} = 0\}$$

with the formula

$$Aw = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n} a_{ij}(X)\cdot\frac{\partial w}{\partial x_j}\right) + c(X)\cdot w(X)$$

the hyperbolic equation can be written as follows:

$$\frac{\partial^2 u}{\partial t^2} + A\cdot u(t,X) = f(t,X), \qquad (\forall)(t,X) \in (0,+\infty)\times\Omega$$

Theorem 7.4.1. *If the function* $u : [0, +\infty) \times \overline{\Omega} \rightarrow \mathbb{R}^1$ *is a classical solution of the problem (7.33-7.36), then the function* U *defined by:*

$$U(t)(X) = u(t, X)$$

has the following properties:

i) $U : [0, +\infty) \rightarrow L^2(\Omega)$ *is of class* $\mathcal{C}^1$ *on* $[0, +\infty)$

ii) $U : [0, +\infty) \rightarrow H_0^1$ *is continuous*

iii) $U(0) = u_0$ *and* $U'(0) = u_1$

Proof. i) First, we show that the function $U : [0, +\infty] \rightarrow L^2(\Omega)$ is differentiable at any $t_0 \in [0, +\infty)$. For this, we consider $t_0 \in [0, +\infty)$ and the quotient

$$\frac{1}{t - t_0}[U(t) - U(t_0)] \in L^2(\Omega)$$

and we prove that this quatient tends to $\dfrac{\partial u}{\partial t}(t_0, x) \in L^2(\Omega)$ in the norm $L^2(\Omega)$ when $t \rightarrow t_0$. This means that we show the equality:

$$\lim_{t \to t_0} \int_\Omega \left| \frac{1}{t - t_0} \left[U(t) - U(t_0)\right](X) - \frac{\partial u}{\partial t}(t_0, X) \right|^2 dX = 0$$

Using the Lagrange theorem we have:

$$\frac{1}{t - t_0} \left[U(t) - U(t_0)\right](X) = \frac{1}{t - t_0} \left[u(t, X) - u(t_0, X)\right] = \frac{\partial u}{\partial t}(t(X), X)$$

with $|t(X) - t_0| < |t - t_0|$. It follows that:

$$\left| \frac{1}{t - t_0} \left[U(t) - U(t_0)\right](X) - \frac{\partial u}{\partial t}(t_0, X) \right|^2 = \left| \frac{\partial u}{\partial t}(t(X), X) - \frac{\partial u}{\partial t}(t_0, X) \right|^2$$

with $|t(X) - t_0| < |t - t_0|$.

The function $(t, X) \rightarrow \dfrac{\partial u}{\partial t}(t, X)$ is continuous on $[0, +\infty) \times \overline{\Omega}$ and therefore, it is uniformly continuous on the set $[t_0 - \eta, t_0 + \eta] \times \overline{\Omega}$ $(\eta > 0)$ so we have: $(\forall)\varepsilon > 0$, $(\exists)\delta(\varepsilon)$ such that $(\forall)(t', X'), (t", X") \in [t_0 - \eta, t_0 + \eta] \times \overline{\Omega}$ with $|t' - t"| < \delta$ and $|X' - X"| < \delta$ we have

$$\left| \frac{\partial u}{\partial t}(t', X') - \frac{\partial u}{\partial t}(t", X") \right| < \sqrt{\frac{\varepsilon}{|\Omega|}}$$

where $|\Omega|$ represents the measure of the domain Ω. We obtain that, if $|t - t_0| < \delta(\varepsilon)$, then the following inequality holds:

$$\int_\Omega \left| \frac{1}{t - t_0}[U(t) - U(t_0)](X) - \frac{\partial u}{\partial t}(t_0, X) \right|^2 dX < \varepsilon$$

In this way, the equality

$$\lim_{t\to t_0}\int_\Omega \left|\frac{1}{t-t_0}[U(t)-U(t_0)](X)-\frac{\partial u}{\partial t}(t_0,X)\right|^2 dX=0$$

is proved.

In the followings, we have to show that the function $U' : [0,+\infty)\to L^2(\Omega)$ is continuous. For this purpose, we will prove the equality:

$$\lim_{t\to t_0}\|U'(t)-U'(t_0)\|_{L^2(\Omega)}=0$$

using

$$\|U'(t)-U'(t_0)\|_{L^2(\Omega)}=\int_\Omega\left|\frac{\partial u}{\partial t}(t,X)-\frac{\partial u}{\partial t}(t_0,X)\right|^2 dX$$

and the fact that the function $\dfrac{\partial u}{\partial t}$ is continuous on $[0,+\infty)\times\overline{\Omega}$. From the continuity of the function $\dfrac{\partial u}{\partial t}$ we obtain uniform continuity on the set $[t_0-\eta,t_0+\eta]\times\overline{\Omega}$, $(y>0)$ and it follows that:$(\forall)\varepsilon>0$, $(\exists)\delta(\varepsilon)$ such that $(\forall)(t',X'),(t",X")\in[t_0-\eta,t_0+\eta]\times\overline{\Omega}$, if $|t'-t"|<\delta$ and $|X'-X"|<\delta$ then

$$\left|\frac{\partial u}{\partial t}(t',X')-\frac{\partial u}{\partial t}(t",X")\right|<\sqrt{\frac{\varepsilon}{|\Omega|}}$$

From here, it results that, if $|t-t_0|<\delta(\varepsilon)$ then $\|U'(t')-U'(t_0)\|<\varepsilon$.

ii) Let us prove that the function U, having the values in the space:

$$H_0^1=\left\{u\in L^2(\Omega)\,\middle|\,(\exists)\frac{\partial u}{\partial x_i}\in L^2(\Omega)\text{ and }u_{|\partial\Omega}=0\right\}$$

is continuous. The fact that for any $t\in[0,+\infty)$ the function $U(t)$ belongs to the space H_0^1, is obtained from the properties of the solution $u(t,X)=U(t)(X)$. We only have to evaluate the norm $\|U(t)-U(t_0)\|_{H_0^1}$ and we show that this tends to zero for $t\to t_0$.

Indeed, we have:

$$\begin{aligned}\|U(t)-U(t_0)\|^2_{H_0^1}&=\|U(t)-U(t_0)\|^2_{L^2(\Omega)}+\sum_{i=1}^n\left\|\frac{\partial U(t)}{\partial x_i}-\frac{\partial U(t_0)}{\partial x_i}\right\|^2_{L^2(\Omega)}=\\
&=\int_\Omega|u(t,X)-u(t_0,X)|^2dX+\sum_{i=1}^n\int_\Omega\left|\frac{\partial u}{\partial x_i}(t,X)-\frac{\partial u}{\partial x_i}(t_0,X)\right|^2dX=\\
&=\int_\Omega\left|\frac{\partial u}{\partial t}(t(X),X)\right|^2\cdot|t-t_0|^2dX+\sum_{i=1}^n\int_\Omega\left|\frac{\partial^2 u}{\partial t\partial x_i}(t_i^*(X),X)\right|^2\cdot|t-t_0|^2dX\end{aligned}$$

where $|t(X)-t_0|\le|t-t_0|$ and $|t_i^*(X)-t_0|\le|t-t_0|$.

The functions $\dfrac{\partial u}{\partial t}$ and $\dfrac{\partial^2 u}{\partial t\partial x_i}$ are continuous and hence, they are bounded on the compact sets of the form $[t_0-\eta,t_0+\eta]\times\overline{\Omega}$. It follows that there exists a positive constant $K^2>0$ such that

$$\|U(t)-U(t_0)\|^2_{H_0^1}\le K^2\cdot|t-t_0|^2$$

form where we obtain the continuity of the function $U : [0, +\infty) \to H_0^1$.

iii) The equalities: $U(0) = u_0$ and $U'(0) = u_1$ are obvious. $\square$

We consider in the followings, the function space $\mathcal{S}$ defined by:

$$\mathcal{S} = C([0, +\infty); H_0^1) \cap C^1([0, +\infty); L^2)$$

The previous theorem shows that, if $u = u(t, X)$ is a classical solution of the problem (7.33-7.36), then the function U defined by $U(t)(X) = u(t, X)$ belongs to the function space $\mathcal{S}$ and satisfies $U(0) = u_0$, $U'(0) = u_1$.

Let be $T > 0$ and the function subspace $\mathcal{S}_T$ defined by:

$$\mathcal{S}_T = \{V \in \mathcal{S} | V(T) = 0\}$$

Theorem 7.4.2. *If the function $u = u(t, X)$ is a classical solution of the problem (7.33-7.36), then the function U defined by $U(t)(X) = u(t, X)$ belongs to the function space S, verifies $U(0) = u_0$, $U'(0) = u_1$, and for any $T > 0$ and any function $V \in S_T$ the following equality holds:*

$$-\int_0^T < U'(t), V'(t) >_{L^2(\Omega)} dt - < u_1, V(0) >_{L^2(\Omega)} + \int_0^T < U(t), V(t) >_A dt =$$

$$= \int_0^T < F(t), V(t) >_{L^2(\Omega)} dt \tag{7.37}$$

Proof. The fact that the function U belongs to the space $\mathcal{S}$ has been already proved. Moreover, it has also been proved that $U(0) = u_0$ and $U'(0) = u_1$. It remains to show that for any $T > 0$ and any $V \in \mathcal{S}_T$ the equality (7.37) holds.

For this, let be $T > 0$ and $V \in \mathcal{S}_T$. Writing the equality (7.33) in the form:

$$\frac{d^2U(t)}{dt^2}(X) + A \cdot U(t)(X) = F(t)(X)$$

(where $F(t)(X) = f(t, X)$) multiplying this equation by the function $V(t)(X)$ and integrating on Ω we obtain:

$$< \frac{d^2U}{dt^2}, V >_{L^2(\Omega)} + < U(t), V(t) >_A = < F(t), V(t) >_{L^2(\Omega)}$$

We integrate this last equality with respect to t on the interval $[0, T]$, we take into account $V(T) = 0$ and we obtain:

$$< U'(t), V(t) >_{L^2(\Omega)} \Big|_0^T - \int_0^T < U'(t), V'(t) >_{L^2(\Omega)} dt + \int_0^T < U(t), V(t) >_A dt =$$

$$= \int_0^T < F(t), V(t) >_{L^2(\Omega)} dt$$

i.e.,

$$< U'(0), V(0) >_{L^2(\Omega)} - \int_0^T < U'(t), V'(t) >_{L^2(\Omega)} dt + \int_0^T < U(t), V(t) >_A dt =$$

$$= \int_0^T < F(t), V(t) >_{L^2(\Omega)} dt$$

which is equivalent with:

$$- < u_1, V(0) >_{L^2(\Omega)} - \int_0^T < U'(t), V'(t) >_{L^2(\Omega)} dt + \int_0^T < U(t), V(t) >_A =$$

$$= \int_0^T < F(t), V(t) >_{L^2(\Omega)} dt$$

$\square$

Definition 7.4.3. *A function $U \in \mathcal{S}$ is called generalized solution of the problem (7.33-7.36) if $U(0) = u_0$, $U'(0) = u_1$ and for $(\forall)T > 0$, $(\forall)V \in \mathcal{S}_T$ it verifies:*

$$- < u_1, V(0) >_{L^2(\Omega)} - \int_0^T < U'(t), V'(t) >_{L^2(\Omega)} dt + \int_0^T < U(t), V(t) >_A dt =$$

$$= \int_0^T < F(t), V(t) >_{L^2(\Omega)} dt \tag{7.38}$$

Observation 7.4.2. A classical solution $u(t, X)$ of the problem (7.33-7.36) defines a generalized solution of the problem.

Theorem 7.4.3. *If the function $U \in \mathcal{S}$ is a generalized solution of the problem (7.33-7.36), and if the function $u : [0, +\infty) \times \overline{\Omega} \to \mathbb{R}^1$ defined by $u(t, X) = U(t, X)$ is of class $\mathcal{C}^2$ for $t > 0$ and $X \in \Omega$, then the function $u = u(t, X)$ is a classical solution of the problem (7.33-7.36).*

Proof. The equality (7.34):

$$u(t, X) = 0, \quad (\forall)t \geq 0 \text{ and } (\forall)x \in \partial\Omega$$

is obtained from the relation $U(t) \in H_0^1$. The equality (7.35): $u(0, X) = u_0(0)$, $(\forall)X \in \overline{\Omega}$ results from $U(0) = u_0$.

The equality (7.36): $\dfrac{\partial u}{\partial t}(0, X) = u_1(X)$, $(\forall)X \in \partial\Omega$ results from $U'(0) = u_1$.

It remains to show the equality (7.33) i.e.:

$$\frac{\partial^2 u}{\partial t^2} + A \cdot u(t, X) = f(t, X)$$

In order to prove this equality, we will start from the equality (7.38) written in the form:

$$-\int_0^T \left(\int_\Omega \frac{\partial u}{\partial t}(t,X)\cdot V'(t)(X)dX \right) dt - \int_\Omega u_1(X)\cdot V(0)(X)dX +$$

$$+\int_0^T \left(\int_\Omega A\cdot u(t,X)\cdot V(t)(X)dX \right) dt = \int_0^T \left(\int_\Omega f(t,X)\cdot V(t)(X)dX \right) dt$$

Inverting the order of integration and making an integration by parts with respect to t, in the first integral we obtain:

$$\int_\Omega \int_0^T \left[\frac{\partial^2 u}{\partial t^2} + A\cdot u(t,X) - f(t,X) \right] \cdot V(t)(X)dt\, dX = 0$$

for any $V \in \mathcal{S}_T$ (we used that $V(T) = 0$).

It results in this way that:

$$\frac{\partial^2 u}{\partial t^2} + A\cdot u(t,X) = f(t,X),\ (\forall)t > 0,\ (\forall)X \in \Omega$$

□

Observation 7.4.3. This theorem shows that, if a generalized solution is sufficiently smooth then it is a classical solution.

Theorem 7.4.4 (Uniqueness of the generalized solution). *The problem (7.33)-(7.36) has at most one generalized solution.*

Proof. We suppose the contrary, that the problem (7.33)-(7.36) admits two generalized solutions $U_1(t)$ and $U_2(t)$, and we consider the function $U(t) = U_1(t) - U_2(t)$. For any $T > 0$ and for any $V \in \mathcal{S}_T$, the functions U and V verify:

$$-\int_0^T < U'(t), V'(t) >_{L^2(\Omega)} dt + \int_0^T < U(t), V(t) >_A dt = 0$$

The function $\overline{V}$ defined by $\overline{V}(t) = \int_t^T U(\tau)d\tau$ belongs to the function space $\mathcal{S}$ and $\mathcal{S}_T$ has the following properties:

$$\overline{V}'(t) = -U(t) \qquad \text{and} \qquad \overline{V}''(t) = -U'(t)$$

Replacing these equalities in the above relation we have:

$$\int_0^T < \overline{V}''(t), \overline{V}'(t) >_{L^2(\Omega)} dt - \int_0^T < \overline{V}'(t), V(t) >_A dt = 0$$

Since

$$<\overline{V}''(t),\overline{V}'(t)>_{L^2(\Omega)}=\frac{1}{2}\cdot\frac{d}{dt}\|\overline{V}'(t)\|^2_{L^2(\Omega)}$$

and

$$<\overline{V}'(t),\overline{V}(t)>_A=\frac{1}{2}\cdot\frac{d}{dt}\|\overline{V}(t)\|^2_A$$

the previous equality implies:

$$\|\overline{V}'(T)\|^2_{L^2(\Omega)}-\|\overline{V}'(0)\|^2_{L^2(\Omega)}-\|\overline{V}(T)\|^2_A+\|\overline{V}(0)\|^2_A=0$$

Taking into account the equalities $\overline{V}(T)=0$, $\overline{V}'(0)=U(0)=0$ we deduce that:

$$\|\overline{V}'(T)\|^2_{L^2(\Omega)}+\|\overline{V}'(T)\|^2_A=0$$

from which we have $\overline{V}'(T)=0$ and $\overline{V}(0)=0$.

Since $T>0$ is arbitrary, we obtain $\overline{V}'(T)=-U(T)=0$. It follows that $U_1(T)=U_2(T)$, $(\forall)T\geq 0$. Thus, we have obtained that the two generalized solution coincide. $\square$

Consequence 7.4.1. *The problem (7.33)-(7.36) has at most one classical solution.*

Theorem 7.4.5 (Existence of the generalized solution). *If the function F defined by $F(T)(X)=f(t,X)$ is continuous as function with values in $L^2(\Omega)$ and if $u_0\in H^1_0$, $u_1\in L^2(\Omega)$, then the problem (7.33)-(7.36) has at least one generalized solution.*

Proof. We prove this theorem in two steps.

First, we will deduce a representation formula of the generalized solution, supposing that this solution exists.

Secondly, we will show that, the representation formula found at the first step, defines a function which is a generalized solution of the problem.

Step I. We suppose that $U=U(t)$ is a generalized solution of the problem (7.33)-(7.36) and we consider the sequence of eigenvalues

$$0<\lambda_1\leq\lambda_2\leq\lambda_3\leq\cdots\leq\lambda_k\leq\cdots\to\infty$$

of the Friedrichs prolongation $\widetilde{A}$ of the operator A, and then the corresponding ortonormal eigenfunction sequence $(\omega_k)_{k\in\mathbb{N}}$, which is complete in the space $L^2(\Omega)$.

We consider the functions

$$u_k(t)=<U(t),\omega_k>_{L^2(\Omega)}$$

and the representation

$$U(t)=\sum_{k=1}^{\infty}u_k(t)\cdot\omega_k$$

of the function $U(t)$. Because $U\in C^1([0,+\infty);L^2(\Omega))$, the functions $u_k(t)$ are derivable, with continuous derivatives and $U'(t)$ is represented as follows:

$$U'(t)=\sum_{k=1}^{\infty}u_k'(t)\cdot\omega_k$$

According to this representation formula, the equality (7.38) satisfied by the generalized solution U becomes:

$$-\int_0^T \sum_{k=1}^{+\infty} u_k'(t)\cdot < V'(t), \omega_k >_{L^2(\Omega)} dt - \sum_{k=1}^{+\infty} u_k'(0)\cdot < V(0), \omega_k >_{L^2(\Omega)} +$$

$$+\int_0^T \sum_{k=1}^{+\infty} \lambda_k \cdot u_k(t) < V(t), \omega_k >_{L^2(\Omega)} dt = \int_0^T \sum_{k=1}^{+\infty} f_k(t)\cdot < V(t), \omega_k >_{L^2(\Omega)} dt$$

Now, let be j an arbitrarily fixed natural number and the function $V_j \in \mathcal{S}_T$ defined by:

$$V_j(t) = (T-t)\cdot \omega_j$$

In the previous equality we replace V with V_j, and taking into account the equalities

$$< \omega_k, \omega_j >_{L^2(\Omega)} = \delta_{kj};\ V'(t) = -\omega_j,\ V(0) = T\cdot \omega_j$$

we obtain

$$-T\cdot < u_1, \omega_j >_{L^2(\Omega)} + \int_0^T u_j'(t)dt + \lambda_j \cdot \int_0^T (T-t)\cdot f_j(t)dt = \int_0^T (T-t) f_j(t)dt$$

for $j = 1, 2, \ldots$

Differentiating two times with respect to T we have:

$$\begin{cases} u_j''(T) + \lambda_j \cdot u_j(T) = f_j(T),\ (\forall) T > 0 \\ u_j'(0) = < u_1, \omega_j >_{L^2(\Omega)} \\ u_j(0) = < u_0, \omega_j >_{L^2(\Omega)},\ j = 1, 2, \ldots \end{cases} \tag{7.39}$$

The initial value problem (7.39) has a unique solution and this is given by:

$$\begin{aligned} u_j(t) &= < u_0, \omega_j >_{L^2(\Omega)} \cdot \cos\sqrt{\lambda_j}\cdot t + \frac{1}{\sqrt{\lambda_j}}\cdot < u_1, \omega_j >_{L^2(\Omega)} \cdot \sin\sqrt{\lambda_j}\cdot t + \\ &+ \frac{1}{\sqrt{\lambda_j}} \cdot \int_0^t f_j(\tau)\cdot \sin\sqrt{\lambda_j}\cdot (t-\tau)d\tau,\ (\forall) t \geq 0,\ j = 1, 2, \ldots \end{aligned} \tag{7.40}$$

It results in this way that the generalized solution U has the following representation:

$$\begin{aligned} U(t) &= \sum_{j=1}^{+\infty} \left[< u_0, \omega_j >_{L^2(\Omega)} \cdot \cos\sqrt{\lambda_j}\cdot t + \frac{1}{\sqrt{\lambda_j}}\cdot < u_1, \omega_j >_{L^2(\Omega)} \cdot \sin\sqrt{\lambda_j}\cdot t \right] \cdot \omega_j + \\ &+ \sum_{j=1}^{+\infty} \left(\frac{1}{\sqrt{\lambda_j}} \cdot \int_0^t f_j(\tau)\cdot \sin\sqrt{\lambda_j}\cdot (t-\tau)d\tau \right) \cdot \omega_j. \end{aligned} \tag{7.41}$$

Step II We prove that, in the conditions of the theorem, the formula (7.41) defines a function U which belongs to the space $\mathcal{S}$ and verifies (7.38) for$T > 0$.

The convergence

$$\sum_{j=1}^{+\infty} | < u_0, \omega_j >_{L^2} |^2 < +\infty \text{ and } \sum_{j=1}^{+\infty} | < u_1, \omega_j >_{L^2} |^2 < +\infty$$

implies the uniform convergence with respect to $t \in [0, +\infty)$ in the space $L^2(\Omega)$ of the function series:

$$\sum_{j=1}^{+\infty} \left[< u_0, \omega >_{L^2(\Omega)} \cdot \cos \sqrt{\lambda_j} \cdot t + \frac{1}{\sqrt{\lambda_j}} \cdot < u_1, \omega_j >_{L^2} \cdot \sin \sqrt{\lambda_j} \cdot t \right] \cdot \omega_j$$

and the fact that, the sum of the series is a continuous function of t taking values in $L^2(\Omega)$.

The inequalities:

$$\left| \frac{1}{\sqrt{\lambda_j}} \cdot \int_0^T f_j(\tau) \cdot \sin \sqrt{\lambda_j} \cdot (t - \tau) d\tau \right|^2 \le \frac{T}{\lambda_1} \int_0^T f_j^2(\tau) d\tau, \quad (\forall) t \in [0, T], j = 1, 2, \ldots$$

as well as the convergence of the continuous and positive function series $\sum_{j=1}^{+\infty} f_j^2(\tau)$ to a continuous function $\|F(\tau)\|^2_{L^2(\Omega)}$ imply the uniform convergence with respect to $t \in [0, T]$ ($(\forall)T > 0$ and $T < +\infty$) in $L^2(\Omega)$ of the function series:

$$\sum_{j=1}^{+\infty} \left(\frac{1}{\sqrt{\lambda_i}} \int_0^t f_j(\tau) \cdot \sin \sqrt{\lambda_j}(t - \tau) d\tau \right) \omega_j$$

and the fact that the sum of the series is a continuous function of t taking values in $L^2(\Omega)$. It results that, in the conditions of the theorem, the formula (7.41) defines a function $U \in C([0, +\infty); L^2(\Omega))$.

To prove that $U \in C^1([0, +\infty); L^2(\Omega))$ we will consider the series of derivatives:

$$\sum_{j=1}^{+\infty} \left[-\sqrt{\lambda_j} \cdot < u_0, \omega >_{L^2(\Omega)} \cdot \sin \sqrt{\lambda_j} \cdot t + < u_1, \omega_j >_{L^2(\Omega)} \cdot \cos \sqrt{\lambda_j} \cdot t \right] \cdot \omega_j +$$

$$\sum_{j=1}^{+\infty} \left(\int_0^T f_j(\tau) \cdot \cos(\sqrt{\lambda_j}) \cdot (t - \tau) d\tau \right) \omega_j$$

and we will show that the convergence of this series is uniform with respect to $t \in [0, T]$ ($T > 0$ and $T < +\infty$) in the space $L^2(\Omega)$.

Let us examine the convergence of the series of derivatives which is, in fact, a sum of three series.

The first series is

$$\sum_{j=1}^{+\infty} -\sqrt{\lambda_j} \cdot < u_0, \omega_j >_{L^2(\Omega)} \cdot \sin \sqrt{\lambda_j} \cdot t \cdot \omega_j$$

which is uniformly convergent with respect to $t \in [0, +\infty)$ if the series: $\sum_{j=1}^{+\infty} \lambda_j \cdot | < u_0, \omega_j >_{L^2(\Omega)} |^2$ is convergent. This last series can be written in the form:

$$\sum_{j=1}^{+\infty} \lambda_j \cdot | < u_0, \omega_j >_{L^2(\Omega)} |^2 = \sum_{j=1}^{+\infty} \frac{1}{\lambda_j} \cdot \lambda_j^2 \cdot | < u_0, \omega_j >_{L^2(\Omega)} |^2 =$$

$$= \sum_{j=1}^{+\infty} \frac{1}{\lambda_j} \cdot | < u_0, \omega_j >_A |^2 = \sum_{j=1}^{+\infty} \left| < u_0, \frac{\omega_j}{\sqrt{\lambda_j}} >_A \right|^2$$

and hence it is convergent because $u_0 \in H_0^1$.

The second series for which we examine the convergence is

$$\sum_{j=1}^{+\infty} < u_1, \omega_j >_{L^2(\Omega)} \cdot \cos \sqrt{\lambda_j} \cdot t \cdot \omega_j$$

This series is uniformly convergent with respect to $t \in [0, +\infty)$ if the series $\sum_{j=1}^{+\infty} | < u_1, \omega_j >_{L^2(\Omega)} |^2$ is convergent, which is true because $u_1 \in L^2(\Omega)$.

The third series for which we examine the convergence is:

$$\sum_{j=1}^{+\infty} \left(\int_0^T f_j(\tau) \cdot \cos \sqrt{\lambda_j} \cdot (t - \tau) d\tau \right) \omega_j$$

This series is uniform convergent with respect to $t \in (0, T]$ ($T > 0$ and $T < +\infty$) if the series:

$$\sum_{j=1}^{+\infty} \left| \int_0^T f_j(\tau) \cdot \cos \sqrt{\lambda_j} \cdot (t - \tau) d\tau \right|$$

is uniformly convergent with respect to $t \in [0, T]$. Due to the inequality:

$$\sum_{j=1}^{+\infty} \left| \int_0^T f_j(\tau) \cdot \cos \sqrt{\lambda_j} \cdot (t - \tau) d\tau \right|^2 \le \sum_{j=1}^{+\infty} T \cdot \int_0^T f_j^2(\tau) d\tau$$

the problem will be reduced to the uniform convergence on $[0, T]$ of the series $\sum_{j=1}^{+\infty} \int_0^T f_j^2(\tau) d\tau$.

The convergence of this last series is obtained form the uniform convergence of the series $\sum_{j=1}^{+\infty} f_j^2(\tau) d\tau$ on $[0, T]$, which is true on the basis of the Dini theorem, by the continuity and positivity of the functions $f_j^2(\tau)$, the continuity of the function $\|F(\tau)\|^2$ and the equality $\sum_{j=1}^{+\infty} f_j^2(\tau) = \|F(\tau)\|^2$, $(\forall) \tau \in [0, T]$.

In this way, we obtain that the formula (7.41) defines a function $U \in C^1([0,+\infty); L^2(\Omega))$.

It follows to show the inclusion $U \in C([0,+\infty); H_0^1)$.

The uniform convergence with respect to $t \in [0,+\infty)$ of the series

$$\sum_{j=1}^{+\infty} < u_0, \omega_j >_{L^2(\Omega)} \cdot \cos\sqrt{\lambda_j} \cdot t \cdot \omega_j$$

in H_0^1 can be assured by the uniform convergence with respect to $t \in [0,+\infty)$ of the series:

$$\sum_{j=1}^{+\infty} \sqrt{\lambda_j} < u_0, \omega_j >_{L^2(\Omega)} \cdot \cos\sqrt{\lambda_j} \cdot t \cdot \frac{\omega_j}{\sqrt{\lambda_j}} \quad \text{in } H_0^1$$

The convergence of this series is obtained from the convergence of the series

$$\sum_{j=1}^{+\infty} \lambda_j | < u_0, \omega_j >_{L^2(\Omega)} |^2$$

convergence which is assured by $u_0 \in H_0^1$.

The uniform convergence with respect to $t \in [0,+\infty)$ of the function series

$$\sum_{j=1}^{+\infty} \frac{1}{\sqrt{\lambda_j}} \cdot < u_1, \omega_j >_{L^2(\Omega)} \cdot \sin\sqrt{\lambda_j} \cdot t \cdot \omega_j$$

in the space H_0^1 is assured by the convergence of the series $\sum_{j=1}^{+\infty} | < u_1, \omega_j > |^2_{L^2(\Omega)}$, convergence which is obtained from $u_1 \in L^2(\Omega)$.

Finally, the uniform convergence with respect to $t \in [0,T]$ ($T > 0$ and $T < +\infty$) of the function series:

$$\sum_{j=1}^{+\infty} \left(\frac{1}{\sqrt{\lambda_j}} \int_0^T f_j(\tau) \cdot \sin\sqrt{\lambda_j} \cdot (t-\tau) d\tau \right) \omega_j$$

in the space H_0^1 is assured if the series

$$\sum_{j=1}^{+\infty} \left(\int_0^T f_j(\tau) \cdot \sin\sqrt{\lambda_j} \cdot (t-\tau) d\tau \right)$$

converge uniformly on $[0,T]$. This last convergence has already been proved.

We have obtained in this way that $U \in C([0,+\infty); H_0^1)$.

Differentiating with respect to t the function U given by (7.41), as function which takes values in $L^2(\Omega)$, and taking into account the relations (7.39), we obtain that the function U given by (7.41) is a generalized solution of the problem (7.33)-(7.36). $\square$

Exercises:

Determine the generalized solutions for the Cauchy-Dirichlet Problems of hyperbolic type:

1. $$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2}, & (t,x) \in (0,+\infty) \times (0,1) \\ u(t,0) = u(t,1) = 0, & t \geq 0 \\ u(0,x) = \sin \dfrac{4\pi}{l} x, & x \in [0,l] \\ \dfrac{\partial u}{\partial t}(0,x) = 0, & x \in [0,l] \end{cases}$$

   A: $u(t,x) = \cos \dfrac{4\pi}{l} t \cdot \sin \dfrac{4\pi}{l} x$

2. $$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2} - t \cdot \sin x, & (t,x) \in (0,+\infty) \times (0,\pi) \\ u(t,0) = u(\pi,t) = 0, & t \geq 0 \\ u(0,x) = 4 \sin x \cos x, & x \in [0,\pi] \\ \dfrac{\partial u}{\partial t}(0,x) = 2 \sin 3x, & x \in [0,\pi] \end{cases}$$

   A: $u(t,x) = (\sin t - t) \cdot \sin x + 2 \cos 2t \cdot \sin 2x + \dfrac{2}{3} \sin 3t \cdot \sin 3x$

3. $$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = 4 \dfrac{\partial^2 u}{\partial x^2}, & (t,x) \in (0,+\infty) \times (0,\pi) \\ u(t,0) = u(t,\pi) = 0, & t \geq 0 \\ u(0,x) = 2 \sin x, & x \in [0,\pi] \\ \dfrac{\partial u}{\partial t}(0,x) = \sin x + \sin 2x, & x \in [0,\pi] \end{cases}$$

   A: $u(t,x) = \left(2 \cos 2t + \dfrac{1}{2} \sin 2t\right) \cdot \sin x + \dfrac{1}{4} \sin 4t \cdot \sin 2x$

7.5 Symbolic and numerical computations for the Cauchy-Dirichlet Problem for hyperbolic equations

Since for the symbolic computations of the solution of the Cauchy-Dirichlet Problem the function *pdsolve* doesn't display anything, we have to solve it numerically, using the function *pdsolve* which has already been presented.

For example we consider the following Cauchy-Dirichlet Problems of hyperbolic type:

Example 1: *Wave equation*

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(x,t) = \dfrac{\partial^2 u}{\partial x^2}(x,t) \\ u(0,t) = u(1,t) = 0 \\ u(x,0) = x^2 - x \\ \dfrac{\partial u}{\partial t}(x,0) = 0 \end{cases}$$

```
> PDE1 :=diff(u(x,t),t,t)=diff(u(x,t),x,x);
```

$$PDE1 := \frac{\partial^2}{\partial t^2}u(x,t) = \frac{\partial^2}{\partial x^2}u(x,t)$$

```
> IBC1 := {u(0,t)=0, u(1,t)=0,u(x,0)=x^2-x, D[2](u)(x,0)=0};
```

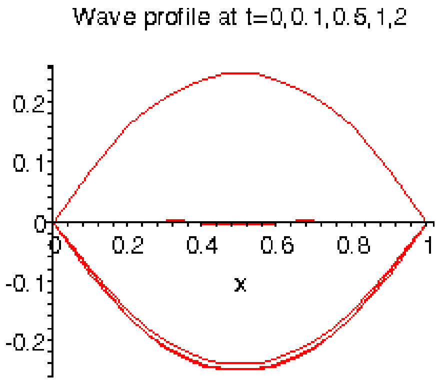
$$IBC1 := \{u(0,t) = 0, u(1,t) = 0, u(x,0) = x^2 - x, D_2(u)(x,0) = 0\}$$

```
> pds1 := pdsolve(PDE1,IBC1,numeric);
```

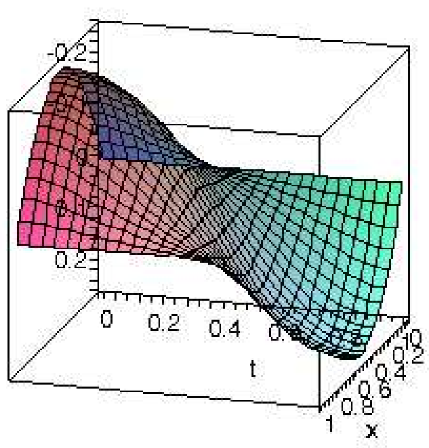
pds1 := module () local INFO; export plot, plot3d, animate, value, settings; option ‘Copyright (c) 2001 by Waterloo Maple Inc. All rights reserved.‘; end module

```
> p1 := pds1:-plot(t=0):
p2 := pds1:-plot(t=1/10):
p3 := pds1:-plot(t=1/2):
p4 := pds1:-plot(t=1):
p5 := pds1:-plot(t=2):
```

```
plots[display]({p1,p2,p3,p4,p5},
title=‘Wave profile at t=0,0.1,0.5,1,2‘);
```




```
> pds1:-plot3d(t=0..1,x=0..1,axes=boxed);
```



Example 2: *Wave equation*

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(x,t) = \dfrac{\partial^2 u}{\partial x^2}(x,t) - t \cdot sinx \\ u(0,t) = u(\pi,t) = 0 \\ u(x,0) = 4 \cdot sinx \cdot cosx \\ \dfrac{\partial u}{\partial t}(x,0) = 2 \cdot sin3x \end{cases}$$

```
> PDE2 :=diff(u(x,t),t,t)=diff(u(x,t),x,x)-t*sin(x);
```

$$PDE2 := \frac{\partial^2}{\partial t^2}u(x,t) = \frac{\partial^2}{\partial x^2}u(x,t) - t\sin(x)$$

```
> IBC2 :={u(0,t)=0,u(Pi,t)=0,u(x,0)=4*(sin(x))*(cos(x)),
      D[2](u)(x,0)=2*sin(3*x)};
```

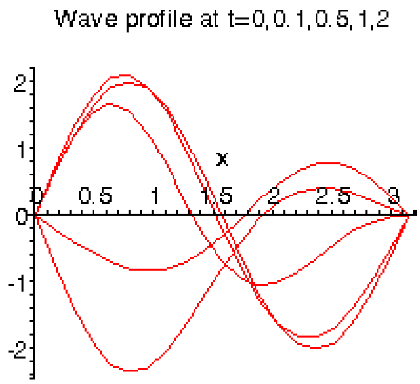
$$IBC2 := \{u(0,t) = 0, u(\pi,t) = 0, u(x,0) = 4\,\sin(x)\cos(x), D_2(u)(x,0) = 2\,\sin(3\,x)\}$$

```
> pds2 := pdsolve(PDE2,IBC2,numeric);
```

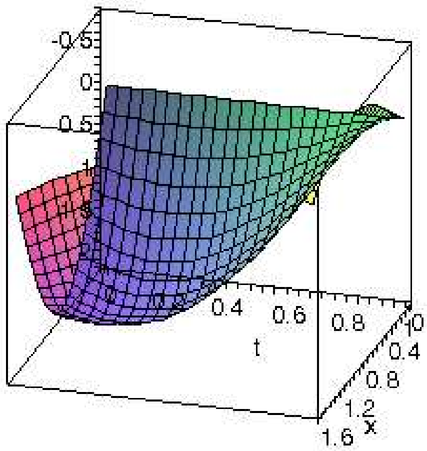
pds1 := module () local INFO; export plot, plot3d, animate, value, settings; option ‘Copyright (c) 2001 by Waterloo Maple Inc. All rights reserved.‘; end module

```
> p6 := pds2:-plot(t=0):
p7 := pds2:-plot(t=1/10):
p8 := pds2:-plot(t=1/2):
p9 := pds2:-plot(t=1):
p10 := pds2:-plot(t=2):
```

```
plots[display]({p6,p7,p8,p9,p10},
title=`Wave profile at t=0,0.1,0.5,1,2`);
```




```
> pds2:-plot3d(t=0..1,x=0..Pi/2,axes=boxed);
```



Example 3: *Wave equation*

$$\begin{cases} \dfrac{\partial^2 v}{\partial t^2}(x,t) = 4 \cdot \dfrac{\partial^2 v}{\partial x^2}(x,t) \\ v(0,t) = v(\pi,t) = 0 \\ v(x,0) = 0 \\ \dfrac{\partial v}{\partial t}(x,0) = 2 \cdot sinx \end{cases}$$

```
> PDE3 :=diff(v(x,t),t,t)=4*diff(v(x,t),x,x);
```

$$PDE2 := \frac{\partial^2}{\partial t^2} v(x,t) = 4 \frac{\partial^2}{\partial x^2} v(x,t)$$

```
> IBC3 :={v(0,t)=0,v(Pi/2,t)=0,v(x,0)=0,
        D[2](v)(x,0)=2*sin(x);
```

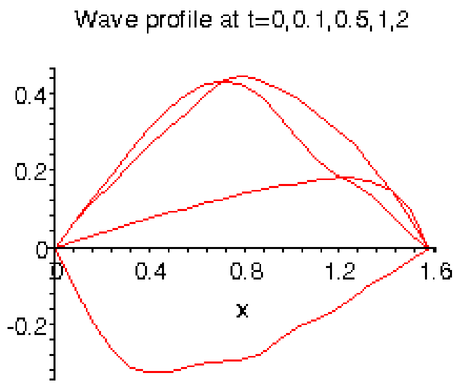
$$IBC3 := \{v(0,t) = 0, v(1/2\,\pi, t) = 0, v(x,0) = 0, D_2(v)(x,0) = 2\sin(x)\}$$

```
> pds3 := pdsolve(PDE3,IBC3,numeric);
```

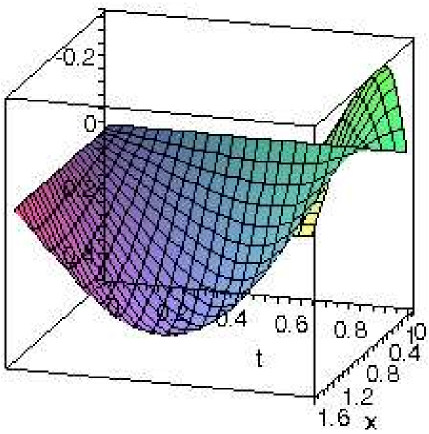
pds1 := module () local INFO; export plot, plot3d, animate, value, settings; option ‘Copyright (c) 2001 by Waterloo Maple Inc. All rights reserved.‘; end module

```
> q1 := pds3:-plot(t=0):
q2 := pds3:-plot(t=1/10):
q3 := pds3:-plot(t=1/2):
q4 := pds3:-plot(t=1):
q5 := pds3:-plot(t=2):
```

```
plots[display]({q1,pq2,q3,q4,q5},
title=`Wave profile at t=0,0.1,0.5,1,2`);
```




```
> pds3:-plot3d(t=0..1,x=0..Pi/2,axes=boxed);
```



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