Quantum Optics 2016 Spring – Homework Set #4

Date: 5/11/2016
Due: 5/18/2016

1. [Plenio and Knight, Rev. Mod. Phys. (1998)] This exercise is to implement the quantum trajectory (or called quantum jump) approach for the Lindblad master equation

$$\dot{\rho} = -\frac{i}{\hbar}[H_0, \rho] - \frac{\gamma}{2}\mathcal{L}(\rho),$$

where the Lindblad superoperator $\mathcal{L}^{-}(\rho) = c^\dagger c\rho + \rho c^\dagger c - 2c\rho c^\dagger$ corresponds to the decoherence. We call c the jumping operator that represents a "decay channel". It can be done by stochastic numerical processes that deal with a pure state $|\psi(t)\rangle$ and the following steps:
1) For a small time interval δt, the probability of having a quantum jump (a decay event) is $\delta p \equiv \gamma\delta t\langle\psi|c^\dagger c|\psi\rangle$, where γ is the decay rate. To simulate this, generate a random number r, which is uniformly distributed over $[0, 1]$.
2) If $r > \delta p$, then there is no jump event within δt. Evolve the state $|\psi(t)\rangle \to |\psi(t+\delta t)\rangle_{\text{no jump}}$ according to an effective non-Hermitian Hamiltonian $H_{eff} = H - i\hbar\frac{\gamma}{2}c^\dagger c$ and properly normalized.
3) If $r < \delta p$, then there is a jump event within δt. The evolved state becomes $|\psi(t+\delta t)\rangle_{\text{jump}} = \frac{c|\psi(t)\rangle}{\sqrt{\langle\psi(t)|c^\dagger c|\psi(t)\rangle}}$.
4) Considering an initial state $\psi(t_i = 0)$, repeat the above procedures until the final time t_f. Then you get a single "trajectory", i.e., $|\psi(t)\rangle$ from time $t_i \to t_f$, and therefore an evolution curve of the observable of interest.
5) Average the observable over many such trajectories.
Now consider the following cases:
a) The density matrix from t to $t + \delta t$

$$|\psi(t)\rangle\langle\psi(t)| \to \delta p|\psi\rangle_{\text{jump}}\langle\psi| + (1 - \delta p)|\psi\rangle_{\text{no jump}}\langle\psi|.$$

Prove that by taking $\frac{\delta\rho}{\delta t}$ and setting $\delta t \to dt$, one can recover the Lindblad master equation.
b) Consider a two-level system $|\psi(t)\rangle = a(t)|e\rangle + b(t)|g\rangle$ under no interaction, i.e. $H = 0$, with the jumping operator $c = \sigma^- = |g\rangle\langle e|$ and an initial condition $a(0) = 1$, $b(0) = 0$. Run a computer program that plots $a(t)$ for $\gamma t = 0 \to 3$ for a single trajectory, average over 10, and 1000 trajectories.
c) Consider the same system as (b), but only reset the jumping operator $c = \sigma_z$ and a different initial condition $a(0) = \sqrt{2/3}$ and $b(0) = \sqrt{1/3}$. Use a computer program to determine $\rho(t \to \infty)$.
d) Consider an equally spaced three-level cascade atom with the ground state $|g\rangle$, the excited state $|e\rangle$, and an intermediate state $|m\rangle$. Then the corresponding Hamiltonian is given by $H_0 = \hbar\omega(2|e\rangle\langle e| + |m\rangle\langle m|)$. If a single mode laser field with frequency ω and intensity χ is applied to the atom, in the interaction

picture, $V_I = \hbar\chi(|e\rangle\langle m| + |m\rangle\langle g| + h.c.) - \hbar\delta(2|e\rangle\langle e| + |m\rangle\langle m|)$, where $\delta = \omega - \omega_0$. Include the Lindblad processes:

$$-\frac{\gamma}{2}\mathcal{L}(\rho) = -\frac{\gamma}{2}\left(c^+c^-\rho + \rho c^+c^- - 2c^-\rho c^+\right),$$

where $c^- = |g\rangle\langle m| + |m\rangle\langle e|$ and $c^+ = (c^-)^\dagger$. Consider the case that $\chi = 0.5\gamma$ and all population is initially located in $|e\rangle$. Use the quantum trajectory technique to determine $\rho_{ee}(t)$ for all times, and the steady state ρ_{mm} and ρ_{mg} (for $t \to \infty$).