Numerical Methods Project
Lorenz Attractor

1A)
$$\begin{cases} \dot{x} = P(y-x) \\ \dot{y} = Rx - y - xz \\ \dot{z} = -\beta z + xy \end{cases}$$

fixed point $\Rightarrow \dot{x} = \dot{y} = \dot{z} = 0$

$\dot{x} = P(y-x) = 0 \Rightarrow \boxed{y = x}$

$\dot{y} = Rx - y - xz = 0 \Rightarrow Rx - x - xz = 0 \Rightarrow x(R-1-z) = 0$

$\boxed{x = 0}$ or $\boxed{z = R-1}$

$\dot{z} = -\beta z + xy = 0 \Rightarrow -\beta z + x^2 = 0 \Rightarrow x^2 = \beta z = \beta(R-1) \Rightarrow \boxed{x = \pm\sqrt{\beta(R-1)}}$

Fixed points:

$p = (0,0,0)$

$q_+ = \left(\sqrt{\beta(R-1)}, \sqrt{\beta(R-1)}, R-1\right)$

$q_- = \left(-\sqrt{\beta(R-1)}, -\sqrt{\beta(R-1)}, R-1\right)$

1B) Stability:

$$\left.\begin{array}{l} \partial_x \dot{x} = -P \; ; \; \partial_y \dot{x} = P \; ; \; \partial_z \dot{x} = 0 \\ \partial_x \dot{y} = R-z \; ; \; \partial_y \dot{y} = -1 \; ; \; \partial_z \dot{y} = -x \\ \partial_x \dot{z} = y \; ; \; \partial_y \dot{z} = x \; ; \; \partial_z \dot{z} = -\beta \end{array}\right\} J = \begin{pmatrix} -P & P & 0 \\ R-z & -1 & -x \\ y & x & -\beta \end{pmatrix}$$

$$\det(J - \lambda I) = \begin{vmatrix} -P-\lambda & P & 0 \\ R-z & -1-\lambda & -x \\ y & x & -\beta-\lambda \end{vmatrix} = 0$$

* for $p = (0,0,0)$:

$$= \begin{vmatrix} -P-\lambda & P & 0 \\ R & -1-\lambda & 0 \\ 0 & 0 & -\beta-\lambda \end{vmatrix} = (-\beta-\lambda) \begin{vmatrix} -P-\lambda & P \\ R & -1-\lambda \end{vmatrix}$$

$$= -(\beta+\lambda)\left[(-P-\lambda)(-1-\lambda) - RP\right] = 0$$

$$\lambda = -\beta \quad \text{or} \quad (-P-\lambda)(-1-\lambda) - RP = 0$$

$$(P+\lambda)(1+\lambda) - RP = 0$$

$$P + P\lambda + \lambda + \lambda^2 - RP = 0$$

$$\lambda^2 + \lambda(P+1) + P(1-R) = 0$$

$$\Delta = (P+1)^2 - 4.1.\,P(1-R)$$

$$\lambda = \frac{-(P+1) \pm \sqrt{\Delta}}{2}$$

To be stable: real part <0.

If λ is purely real, is going to be stable, all real parts are negative.

If λ is complex ($\lambda = \lambda_r + i\lambda_i$) it can have three roots, and if $\lambda_r < 0$ is stable and if $\lambda_r > 0$ is unstable.

For $0<R<1$, all solutions are stable.

For $R>1$, we have one $\lambda_r > 0$, and therefore unstable behaviour.

* for $x = y = \pm[\beta(R-1)]^{1/2}$; $z = R-1$

$$\begin{vmatrix} -P-\lambda & P & 0 \\ R-(R-1) & -1-\lambda & \mp[\beta(R-1)]^{1/2} \\ \pm[\beta(R-1)]^{1/2} & \pm[\beta(R-1)]^{1/2} & -\beta-\lambda \end{vmatrix} = 0$$

$$= (-P-\lambda)\begin{vmatrix} -1-\lambda & \mp[\beta(R-1)]^{1/2} \\ \pm[\beta(R-1)]^{1/2} & -\beta-\lambda \end{vmatrix} - P\begin{vmatrix} R-R+1 & \mp[\beta(R-1)]^{1/2} \\ \pm[\beta(R-1)]^{1/2} & -\beta-\lambda \end{vmatrix}$$

$$= (-P-\lambda)\left[(-1-\lambda)(-\beta-\lambda) + [\beta(R-1)]^{2/2}\right] - P\left[(-\beta-\lambda) + [\beta(R-1)]^{2/2}\right]$$

$$= -(P+\lambda)\left[(1+\lambda)(\beta+\lambda) + \beta R - \beta\right] - P\left[-(\beta+\lambda) + \beta R - \beta\right]$$

$$= -(P+\lambda)(1+\lambda)(\beta+\lambda) - (P+\lambda)(\beta R) - (P+\lambda)(-\beta) + P(\beta+\lambda) - P\beta R + P\beta$$

$$= -(P + P\lambda + \lambda + \lambda^2)(\beta+\lambda) - (P\beta R + \lambda\beta R) + P\beta + \lambda\beta + P\beta + P\lambda - P\beta R + P\beta$$

$$= -(P\beta + P\lambda\beta + \lambda\beta + \lambda^2\beta + P\lambda + P\lambda^2 + \lambda^2 + \lambda^3) - P\beta R - \lambda\beta R + P\beta + \lambda\beta + P\beta + P\lambda -$$

$$- P\beta R + P\beta$$

$$= -\cancel{P\beta} - P\lambda\beta - \cancel{\lambda\beta} - \lambda^2\beta - \cancel{P\lambda} - P\lambda^2 - \lambda^2 - \lambda^3 - \underbrace{P\beta R} - \lambda\beta R + \underbrace{P\beta} + \cancel{\lambda\beta} + \underbrace{P\beta} + \cancel{P\lambda} -$$

$$- \underbrace{P\beta R} + \cancel{P\beta}$$

$$= -\cancel{P\lambda\beta} - \cancel{\lambda^2\beta} - \cancel{P\lambda^2} - \lambda^2 - \lambda^3 - \cancel{P\beta R} - \cancel{\lambda\beta R} + \cancel{P\beta} + \cancel{P\beta} - P\beta R$$

$$= -\lambda^3 - \lambda^2(1-P-\beta) + \lambda(-P\beta - \beta R) + (-P\beta R + P\beta + P\beta - P\beta R) = 0$$

$$= -\lambda^3 - \lambda^2(1-(P+\beta)) - \lambda(\beta(R+P)) - (2P\beta R - 2P\beta) = 0$$

$$= \lambda^3 + \lambda^2(1-(P+\beta)) + \lambda(\beta(R+P)) + 2(P\beta R - P\beta) = 0$$

$$= \lambda^3 + \lambda^2 \underbrace{(1-(P+\beta))}_{A} + \lambda \underbrace{(\beta(R+P))}_{B} + \underbrace{2P\beta(R-1)}_{C} = 0$$

$$= \lambda^3 + A\lambda^2 + B\lambda + C = 0$$

$A, B, C > 0 \Rightarrow (R > 1)$

$>$ 3 roots $\sigma = \sigma_1$ and $\sigma = \alpha + i\beta$

$\alpha = \underbrace{C - AB > 0}_{\text{unstable if that happens}}$ → to study instability

$$\sigma\underbrace{(\sigma^2 + B)}_{\text{positive real}} = \underbrace{-A\sigma^2 - C}_{\text{negative real}} < 0$$

Consequently any real $\sigma < 0$ and we need to consider only the complex roots.

σ_1 : negative real root

$\sigma_{2,3} = \alpha + i\beta$

$$(\sigma - \sigma_1)(\sigma - \alpha - i\beta)(\sigma - \alpha + i\beta) = 0$$

$$A = -(\sigma_1 + 2\alpha)$$
$$B = 2\alpha\sigma_1 + \alpha^2 + \beta^2$$
$$C = -\sigma_1(\alpha^2 + \beta^2)$$

$$C - AB = 2\alpha\underbrace{[(\alpha_1 + \alpha)^2 + \beta^2]}_{\text{positive real}}$$

Since α is the real part of both complex roots we have,

$$\operatorname{sgn}(\operatorname{Re}\{\sigma_{2,3}\}) = \operatorname{sgn}(\alpha) = \operatorname{sgn}(C - AB)$$

Thus Instability occurs for $C - AB > 0$ or

(consider $\beta = 1$)

$$2P(R-1) - (P+2)(P+R) > 0$$

$$R(2P - P - 2) > 2P + P(P+2)$$

Instability occurs for

$$r > r_c = \frac{P(P+4)}{\underbrace{P-2}_{\text{for } P>2 \text{ becomes unstable.}}}$$

2A)

Start

Initial conditions (x_0, y_0, z_0)
Step h, final value x_f

Variables to determine

Define EDO $\rightarrow y' = F(x, y)$

RK4 calculation

$K_1 = h F(x_i, y_i)$

$K_2 = h F(x_i + h/2, y_i + K_1/2)$

$K_3 = h F(x_i + h/2, y_i + K_2/2)$

$K_4 = h F(x_i + h, y_i + K_3)$

Evaluate coordinates

$$y_{i+1} = y_i + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$$x_{i+1} = x_i + h$$

graph

End

2B)

$$K_1 = f(t_n, U_n) \Rightarrow K_{1x} = P(y-x)$$

$$K_{1y} = x(R-z) - y$$

$$K_{1z} = -\beta z + xy$$

$$K_2 = f(t_n + h, U_n + hK_1) \Rightarrow K_{2x} = P\Big((y_n + hK_{1y}) - (x_n + hK_{1x})\Big)$$

$$K_{2y} = (x_n + hK_{1x})\big(R - (z_n + hK_{1z})\big) - (y_n + hK_{1y})$$

$$K_{2z} = -\beta\,(z_n + hK_{1z}) + (x_n + hK_{1x})(y_n + hK_{1y})$$

$$K_3 = f\left(t_n + \frac{h}{2}, U_n + \frac{h}{4}(K_1 + K_2)\right) \Rightarrow$$

$$\Rightarrow K_{3x} = P\left[\left(y_n + \frac{h}{4}(K_{1y} + K_{2y})\right) - \left(x_n + \frac{h}{4}(K_{1x} + K_{1y})\right)\right]$$

$$K_{3y} = \left(x_n + \frac{h}{4}(K_{1y} + K_{2y})\right)\left(R - \left(z_n + \frac{h}{4}(K_{1z} + K_{2z})\right) - \left(y_n + \frac{h}{4}(K_{1y} + K_{2y})\right)\right)$$

$$K_{3z} = -\beta\left(z_n + \frac{h}{4}(K_{1z} + K_{2z})\right) + \left(x_n + \frac{h}{4}(K_{1x} + K_{2x})\right)\left(y_n + \frac{h}{4}(K_{1y} + K_{2y})\right)$$

$$U_{n+1} = U_n + \frac{h}{6}(K_1 + K_2 + 4K_3) \Rightarrow$$

$$x_{n+1} = x_n + \frac{h}{6}(K_{1x} + K_{2x} + 4K_{3x})$$

$$y_{n+1} = y_n + \frac{h}{6}(K_{1y} + K_{2y} + 4K_{3y})$$

$$z_{n+1} = z_n + \frac{h}{6}(K_{1z} + K_{2z} + 4K_{3z})$$