Euler Circuits and Euler Paths

This handout is meant to serve two purposes: it supplies a real generalized-induction proof of a real theorem, and it introduces a little *graph theory* which is a natural topic to touch upon in this course.

1 Graphs

The rough idea is this: a graph is a nonempty (usually finite) set of *vertices*,[1] with *edges* running between some[2] of the vertices. Stated like this, though, the idea turns out to be too vague; it could mean one of *eight* different things, depending on your answers to these three questions:

1. Is it permitted to have multiple edges running between one pair of vertices? If the answer is *yes*, the graphs are called *multigraphs*.

2. Is it permitted to have edges that begin and end at the same vertex? Such an edge is called a *loop*, so if the answer is *yes*, one says that *loops are permitted*.

3. Are the edges "one-way streets" (so that each edge comes with one permissible and one impermissible direction)? If the answer is *yes*, the corresponding graphs are called *directed graphs* (or *digraphs* for short); if the answer in *no* the graphs are called *undirected*.

Some more terms:

- If the answers to the three questions are all *no*—*no* multiple edges, *no* loops, and *no* directions on the edges—the graph is called a *simple* graph.

- If there is at least one edge between vertices v and w, v and w are said to be *adjacent*.

- If edge e runs between vertices v and w, one says that e is *incident* to v and to w.

For this writeup, let's take our graphs to be the *yes/yes/no* graphs—undirected multigraphs with loops permitted—with only finitely many vertices and only finitely many edges.

[1] Singular is *vertex*.
[2] Zero or more

2 The Handshaking Lemma.

To state and prove this, I need to define the *degree* ("$d(v)$") of a vertex v.

Definition 2.1 *For a vertex v in a graph G, the* ***degree of* v** *is given by*

$$d(v) := \left(\begin{array}{c}\textit{the number of non-loop edges incident to} \\ v\end{array}\right) \;+\; 2\cdot\left(\begin{array}{c}\textit{the number of loops incident} \\ \textit{to } v\end{array}\right)$$

The idea is that $d(v)$ should be the number of edge-*ends* incident to v; that's why loops are counted twice here.

Observe that for a graph G with E edges,

$$2E = \sum_{v\in G} d(v), \tag{1}$$

because each edge (*including each loop!*) is being counted twice, once at each end.

Now, the sum on the right side of (1) can clearly be decomposed into two sums, one over odd-degree vertices and the other over even-degree vertices. This gives an alternate way to write (1):

$$2E = \underbrace{\sum_{\substack{v\in G: \\ d(v)\text{is even}}} d(v)}_{(\mathbf{A})} \;+\; \underbrace{\sum_{\substack{v\in G: \\ d(v)\text{is odd}}} d(v)}_{(\mathbf{B})}. \tag{2}$$

In (2), observe that $2E$ (obviously) is even, and also that sum **(A)** (being a sum of even numbers) is also even. Therefore, sum **(B)** must be even as well. But sum **(B)** is a sum of *odd* numbers. It follows that there must be an *even* number of terms **(B)**. We have proved what is known as the *Handshaking Lemma*; the precise statement is:

Theorem 2.1 (Handshaking Lemma) *In any graph G, the number of odd-degree vertices is even.* ∎

3 Euler Paths and Circuits.

For brevity, I am going to assume that the idea of a *path* is clear and not put in a careful definition. A *circuit* is a path that ends where it begins.

Definition 3.1 *An* ***euler*** $\left\{\begin{array}{c}\textbf{\textit{path}} \\ \\ \textbf{\textit{circuit}}\end{array}\right\}$ *in a graph is a* $\left\{\begin{array}{c}\textit{path} \\ \\ \textit{circuit}\end{array}\right\}$ *that uses each edge exactly once.*

In solving the Königsberg Bridge Problem, Euler observed:

- If G has an euler circuit $\mathcal{C}$,[3] then $\mathcal{C}$ must leave each vertex exactly as often as it enters it; therefore, every vertex of G must have even degree.[4]
- If G has an euler path $\mathcal{P}$ that starts at vertex v and ends at vertex w, where $w \neq v$, then $\mathcal{P}$
  - enters v *one less time* than it leaves it;
  - enters w *one more time* than it leaves it; and
  - enters every other vertex *exactly as many times* as it leaves it.

  It follows that vertices v and w must have odd degree, while every other vertex of G must have even degree.

We thus have a condition—a *necessary* condition—that G must satisfy before it can possess an euler circuit, and we have a similar necessary condition that G must satisfy before it can possess an euler path that is not a circuit. This much is all that is needed to solve the Königsberg problem: in the Königsberg graph, there can be no euler circuit, and there can be no euler path that is not a circuit. However, we have not yet considered the question whether or not these necessary conditions are also *sufficient*: will a graph that satisfies the vertex-degree requirements automatically have an euler circuit/path, or could something else go wrong?

The answer to *this exact* question is that there *is* something else that could go wrong: the graph G might not be connected.[5] However, Euler showed that not being connected is the *only* other thing that could go wrong: whenever G is connected and satisfies the vertex-degree requirements, G will definitely possess an Euler path/circuit. Here is the precise theorem.

Theorem 3.1 *Let G be a connected graph.*

[a]: *If G is eulerian, then G possesses euler circuits.*

[b]: *If exactly two vertices—v and w, say—of G are of odd degree, then G possesses euler paths that start at v and end at w.*

Proof of [a] by induction on the number E of edges.

Basis:

- $E = 0$: the only connected eulerian graph with no edges is a single point. In such a graph, the zero-length path (vacuously) uses every edge.
- $E = 1$: the only connected eulerian graph with one edge is one vertex and one loop. Such a graph clearly possesses an euler circuit.
- $E = 2$: if G is a connected eulerian graph with two edges, then G is either two vertices with two edges running between them or one vertex with two loops attached to it. Both of these graphs clearly possess euler circuits.

Induction: Let G be a connected, eulerian E-edge graph, and assume that for every integer $0 \leq m \leq E - 1$, every connected eulerian m-edge graph H possesses euler circuits. We can construct an euler circuit in G in three steps.

[3]Euler did not use this name, of course!

[4]Still more vocabulary: a graph G in which every vertex has even degree is said to be *eulerian*.

[5]That is, there may be vertices between which there is no path.

Step 1. Starting at any vertex (v_0, say), trace a path, being careful at each stage not to repeat an edge. Keep going until you get stuck—until you arrive at a vertex with no remaining unused edges. Observe that *you must have returned to* v_0, because v_0 is the *only* vertex which will have an even number of unused edges each time you visit it. Therefore, the path you have traced is a non-edge-repeating circuit; let's call it $\mathcal{C}_0$.

Step 2. Remove all of the edges of $\mathcal{C}_0$ from G. Doing this decreases the degree of each vertex by an even number,[6] so the graph that remains is still eulerian. The graph that remains need not be connected, of course, but each of the connected pieces[7]—call them $\{H_1, \ldots, H_t\}$—is connected and eulerian, and so, by induction, each connected piece possesses euler circuits. Trace an euler circuit in each of $\{H_1, \ldots, H_t\}$.

Step 3(splice). Put $\mathcal{C}_0$ back into the graph. Begin tracing $\mathcal{C}_0$, but every time you come to a vertex with non-$\mathcal{C}_0$ edges—that is, edges from one or more of the $\{H_1, \ldots, H_t\}$—halt the traversal of $\mathcal{C}_0$ and splice in euler circuits[8] from any of the $\{H_1, \ldots, H_t\}$ that have edges incident to this vertex. (At the same time, mark these H's as "processed.") When you finish, you will return to v having used each edge exactly once—that is, having constructed an euler circuit for G. Induction is complete. ∎(part [a])

Proof of [b]: This will be much easier: we can piggy-back on [**a**]. Let G be a connected graph with exactly two vertices v and w of odd degree. Let $\widehat{G}$ be the graph you get from G by putting in an additional edge e^* between v and w. In $\widehat{G}$, v and w have even degree, so $\widehat{G}$ is eulerian (and connected); so by part [**a**], $\widehat{G}$ has euler circuits. Trace an euler circuit $\mathcal{C}$ in $\widehat{G}$, being sure to start with edge e^*[9] going from w to v. The rest of $\mathcal{C}$—the part that follows e^*—is an euler path in G going from v to w. ∎(part [**b**]).

[6] The even number could be zero.
[7] The fancy word is *components*.
[8] Observe that a circuit can be rotated like a necklace to make it start with any of its edges.
[9] See footnote #8.