10.5 DESIGNING AN EFFICIENT COMPUTER DRUM

The position of a rotating drum is to be recognised by means of binary signals produced at a number of electrical contacts at the surface of the drum. The surface is divided into 2^n sections, each consisting of either insulating or conducting material. An insulated section gives signal 0 (no current), whereas a conducting section gives signal 1 (current). For example, the position of the drum in figure 10.9 gives a reading 0010 at the four

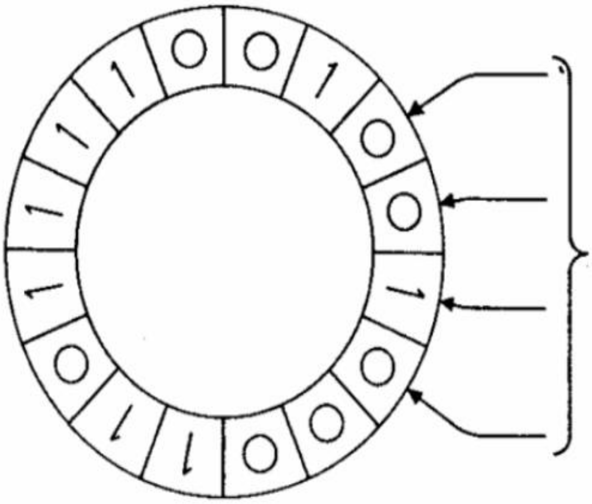


Figure 10.9. A computer drum

contacts. If the drum were rotated clockwise one section, the reading would be 1001. Thus these two positions can be distinguished, since they give different readings. However, a further rotation of two sections would result in another position with reading 0010, and therefore this latter position is indistinguishable from the initial one.

We wish to design the drum surface in such a way that the 2^n different positions of the drum can be distinguished by k contacts placed consecutively around part of the drum, and we would like this number k to be as small as possible. How can this be accomplished?

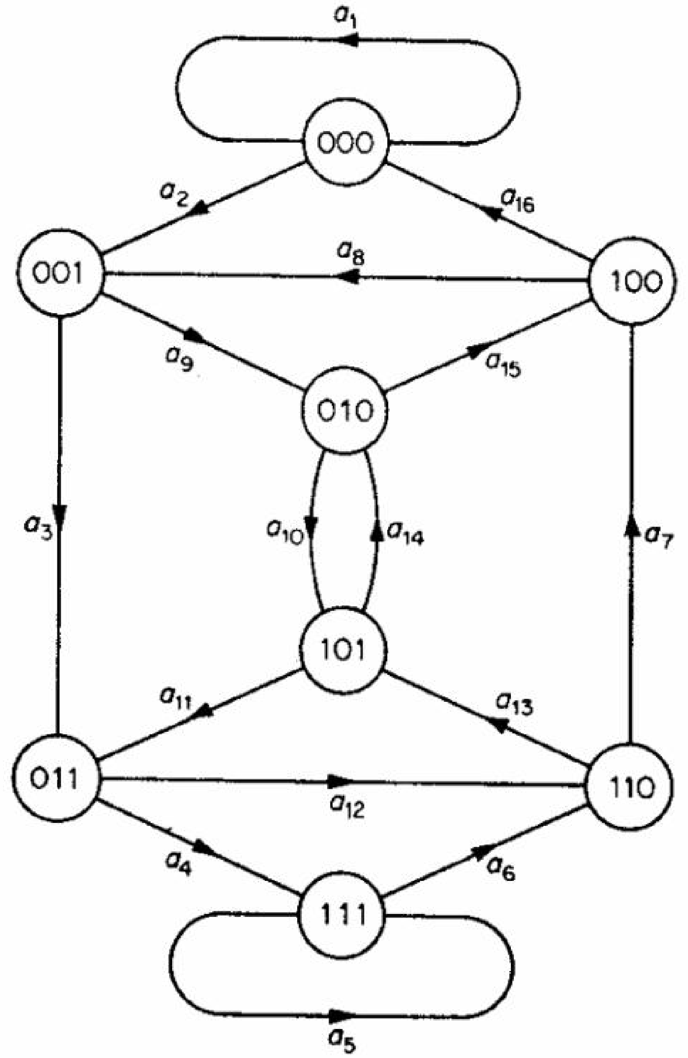
First note that k contacts yield a k-digit binary number, and there are 2^k such numbers. Therefore, if all 2^n positions are to give different readings, we must have $2^k \geq 2^n$, that is, $k \geq n$. We shall show that the surface of the drum can be designed in such a way that n contacts suffice to distinguish all 2^n positions.

We define a digraph D_n as follows: the vertices of D_n are the $(n-1)$-digit binary numbers $p_1p_2 \ldots p_{n-1}$ with $p_i = 0$ or 1. There is an arc with tail $p_1p_2 \ldots p_{n-1}$ and head $q_1q_2 \ldots q_{n-1}$ if and only if $p_{i+1} = q_i$ for $1 \leq i \leq n-2$; in other words, all arcs are of the form $(p_1p_2 \ldots p_{n-1}, p_2p_3 \ldots p_n)$. In addition, each arc $(p_1p_2 \ldots p_{n-1}, p_2p_3 \ldots p_n)$ of D_n is assigned the label $p_1p_2 \ldots p_n$. D_4 is shown in figure 10.10.

Clearly, D_n is connected and each vertex of D_n has indegree two and outdegree two. Therefore (exercise 10.3.2) D_n has a directed Euler tour. This directed Euler tour, regarded as a sequence of arcs of D_n, yields a binary sequence of length 2^n suitable for the design of the drum surface.

For example, the digraph D_4 of figure 10.10 has a directed Euler tour $(a_1, a_2, \ldots, a_{16})$, giving the 16-digit binary sequence 0000111100101101. (Just read off the first digits of the labels of the a_i.) A drum constructed from this sequence is shown in figure 10.11.

This application of directed Euler tours is due to Good (1946).



Arc	Label
a_1	0000
a_2	0001
a_3	0011
a_4	0111
a_5	1111
a_6	1110
a_7	1100
a_8	1001
a_9	0010
a_{10}	0101
a_{11}	1011
a_{12}	0110
a_{13}	1101
a_{14}	1010
a_{15}	0100
a_{16}	1000

Figure 10.10

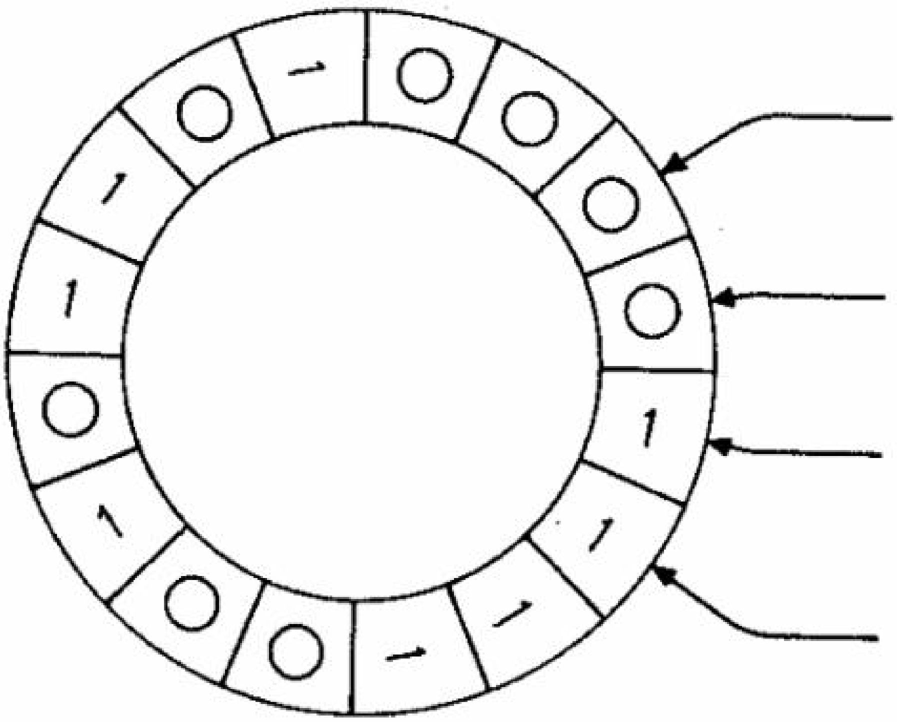


Figure 10.11