离散数学

Discrete Mathematics

for Computer Science

计算机学院计科系

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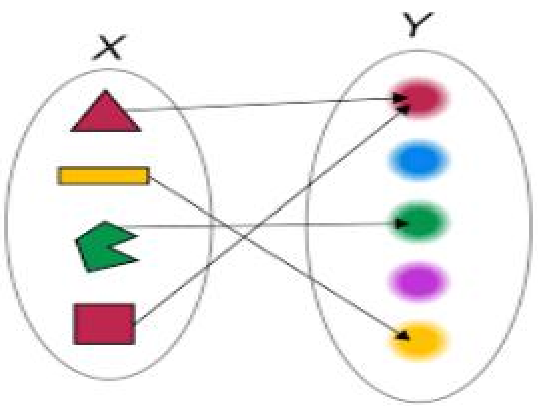
第7讲 函数 Function

"One of the most important concepts in all of mathematics is that of function."

——T.P. Dick and C.M. Patton

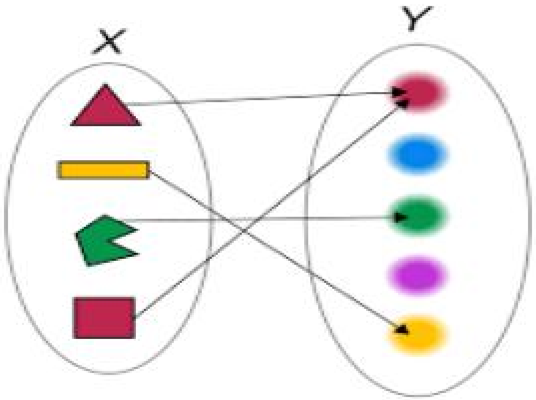
Outline

- 从关系到函数
- 函数与映射
- 函数类型
- 函数运算



Reference: http://www.cs.utexas.edu/~isil/cs311h/

Function



A **Function** or Mapping assigns to each element of a set, exactly one element of a related set.

（total and partial functions）

Functions find their application in various fields like representation of the computational complexity of algorithms, counting objects, study of sequences and strings, to name a few.

- A function f from a set A to a set B assigns each element of A to exactly one element of B.

- A is called domain of f, and B is called codomain of f.

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- If f maps element $a \in A$ to element $b \in B$, we write $f(a) = b$

- If $f(a) = b$, b is called image of a; a is in preimage of b.

  像 原像

- Range of f is the set of all images of elements in A.

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Function

Is this mapping a function?

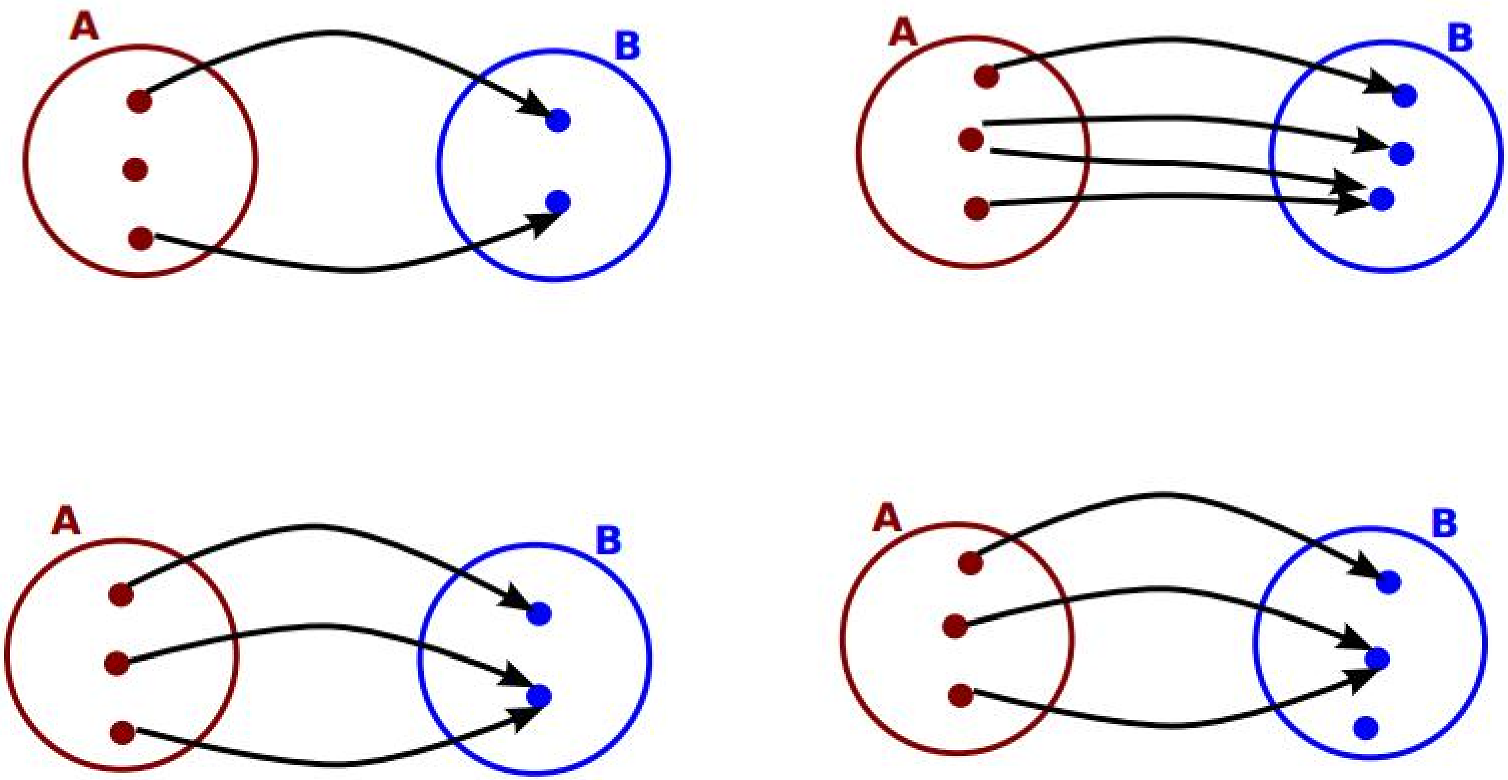


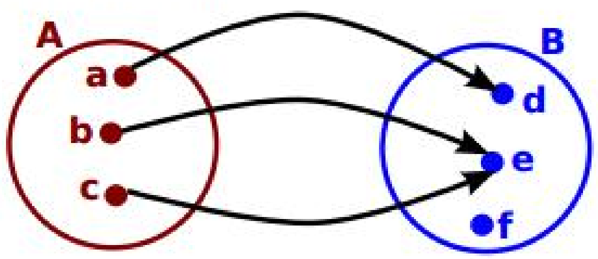
Image of a Set

Image of a Set

- We can extend the definition of image to a set
- Suppose f is a function from A to B and S is a subset of A
- The image of S under f includes exactly those elements of B that are images of elements of S:

$$f(S) = \{t \mid \exists s \in S.\ t = f(s)\}$$

- What is the image of $\{b, c\}$?



One-to-One Functions (一对一映射/单射)

One-to-One Functions (一对一映射/单射)

- A function f is called one-to-one if and only if $f(x) = f(y)$ implies $x = y$ for every x, y in the domain of f:

$$\forall x, y.\ (f(x) = f(y) \rightarrow x = y)$$

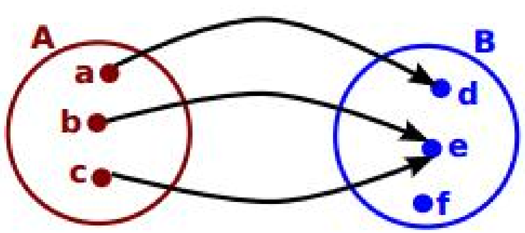
- One-to-one functions never assign different elements in the domain to the same element in the codomain:

$$\forall x, y.\ (x \neq y \rightarrow f(x) \neq f(y))$$

- A one-to-one function also called injection or injective function

单射

- Is this function one-to-one?



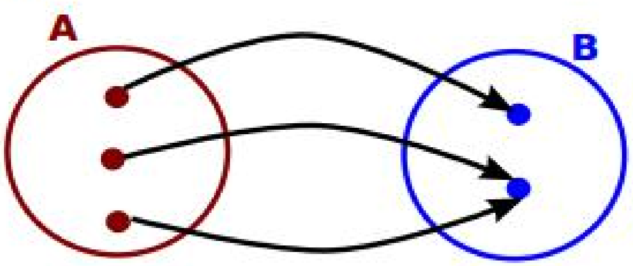
Onto Functions (满射)

Onto Functions (满射)

- A function f from A to B is called onto iff for every element $y \in B$, there is an element $x \in A$ such that $f(x) = y$:

$$\forall y \in B. \exists x \in A.\ f(x) = y$$

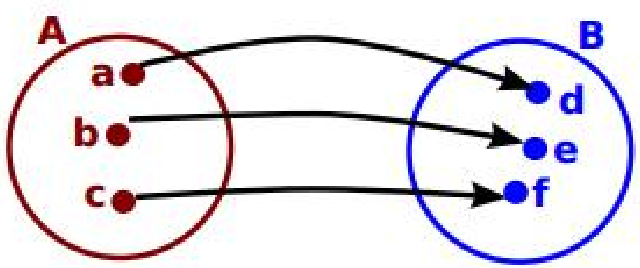
- Onto functions also called surjective functions or surjections
满射
- For onto functions, range and codomain are the same
- Is this function onto?



Bijective Functions(双射)

Bijective Functions(双射)

- Function that is both onto and one-to-one called bijection
- Bijection also called one-to-one correspondence or invertible function
  一一对应 可逆函数
- Example of bijection:



Bijection Example

(恒等函数)

- The identity function I on a set A is the function that assigns every element of A to itself, i.e., $\forall x \in A.\ I(x) = x$
- Prove that the identity function is a bijection.

Bijection Example

- The identity function I on a set A is the function that assigns every element of A to itself, i.e., $\forall x \in A.\ I(x) = x$
- Prove that the identity function is a bijection.
- Need to prove I is both one-to-one and onto.
- One-to-one: We need to show $\forall x, y.\ (x \neq y \rightarrow I(x) \neq I(y))$
- Suppose $x \neq y$.
- Since $I(x) = x$ and $I(y) = y$, and $x \neq y$, $I(x) \neq I(y)$

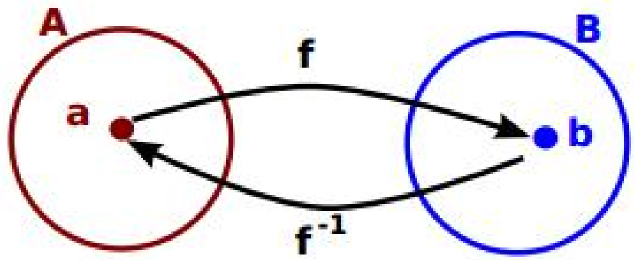
Bijection Example, cont.

- Now, prove I is onto, i.e., for every b, there exists some a such that $f(a) = b$
- For contradiction, suppose there is some b such that $\forall a \in A.\ I(a) \neq b$
- Since $I(a) = a$, this means $\forall a \in A.\ a \neq b$
- But since b is itself in A, this would imply $b \neq b$, yielding a contradiction.
- Since I is both onto and one-to-one, it is a bijection. $\square$

Inverse Functions (逆映射)

Inverse Functions (逆映射)

- Every bijection from set A to set B also has an inverse function
- The inverse of bijection f, written f^{-1}, is the function that assigns to $b \in B$ a unique element $a \in A$ such that $f(a) = b$



- Observe: Inverse functions are only defined for bijections, not arbitrary functions!
- This is why bijections are also called invertible functions

Function Composition (函数复合)

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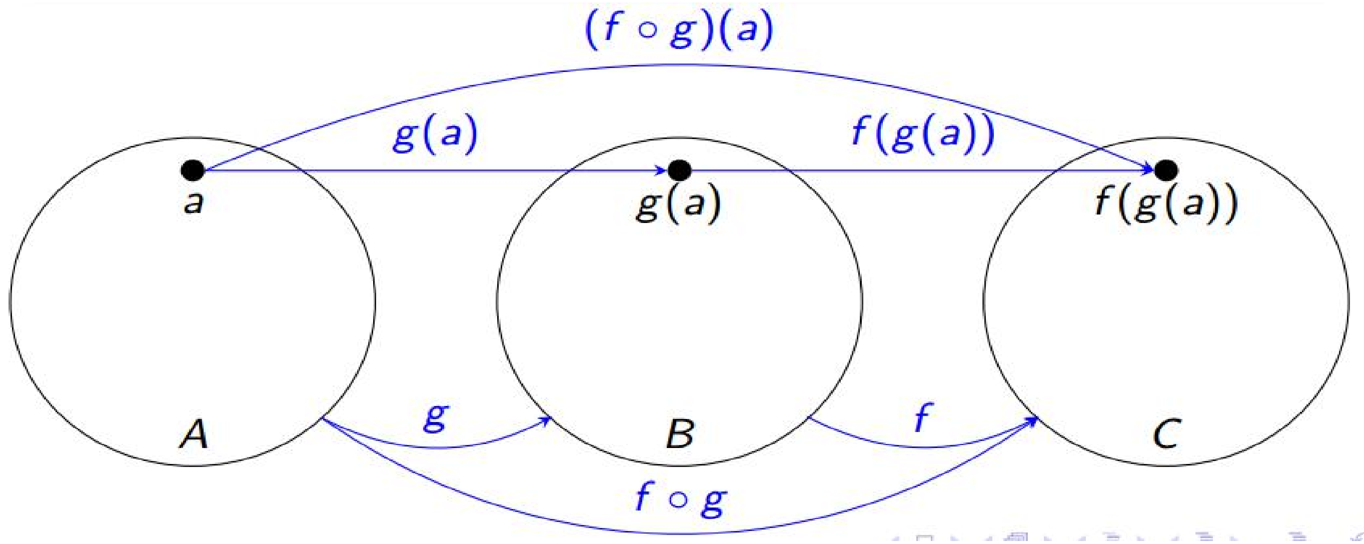
- Let g be a function from A to B, and f from B to C.
- The composition of f and g, written $f \circ g$, is defined by:

$$(f \circ g)(x) = f(g(x))$$



Composition Example

- Prove that $f^{-1} \circ f = I$ where I is the identity function.

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- Prove that $f^{-1} \circ f = I$ where I is the identity function.
- Since $I(x) = x$, need to show $(f^{-1} \circ f)(x) = x$
- First, $(f^{-1} \circ f)(x) = f^{-1}(f(x))$
- Let $f(x)$ be y
- Then, $f^{-1}(f(x)) = f^{-1}(y)$
- By definition of inverse, $f^{-1}(y) = x$ iff $f(x) = y$
- Thus, $f^{-1}(f(x)) = f^{-1}(y) = x$ $\square$

Problem

If |A|=n, |B|=m, how many injection funcitons(单射) can be defined from A to B? How about bijection(双射), and surjection(满射)?

A surjection from a set A of size n to a set B of size k may be characterized by a partition of A into k subsets, together with an permutation of the k elements of B. The partitions are counted by the Stirling numbers of the second kind S(n,k), and the permutations are counted by k!, so there are **S(n,k)k!**

The Stirling numbers of the second kind, written $S(n,k)$ or $\left\{ {n \atop k} \right\}$ or with other notations, count the number of ways to partition a set of n labelled objects into k nonempty unlabelled subsets. Equivalently, they count the number of different equivalence relations with precisely k equivalence classes that can be defined on an n element set. In fact, there is a bijection between the set of partitions and the set of equivalence relations on a given set. Obviously, The Stirling numbers of the second kind, written $S(n,k)$ or $\left\{ {n \atop k} \right\}$ or with other notations, count the number of ways to partition a set of n labelled objects into k nonempty unlabelled subsets. Equivalently, they count the number of different equivalence relations with precisely k equivalence classes that can be defined on an n element set. In fact, there is a bijection between the set of partitions and the set of equivalence relations on a given set. Obviously,

$$\left\{ {n \atop n} \right\} = 1 \text{ and for } n \geq 1, \left\{ {n \atop 1} \right\} = 1$$

The Stirling numbers can be calculated using the following formula:

$$\left\{ {n \atop k} \right\} = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n.$$

$$\left\{ {n \atop n} \right\} = 1 \text{ and for } n \geq 1, \left\{ {n \atop 1} \right\} = 1$$

Some simple identities include

$$\left\{ {n \atop n-1} \right\} = \binom{n}{2}.$$

and

$$\left\{ {n \atop 2} \right\} = 2^{n-1} - 1.$$

Another explicit expansion of the recurrence-relation gives identities in the spirit of the above example.

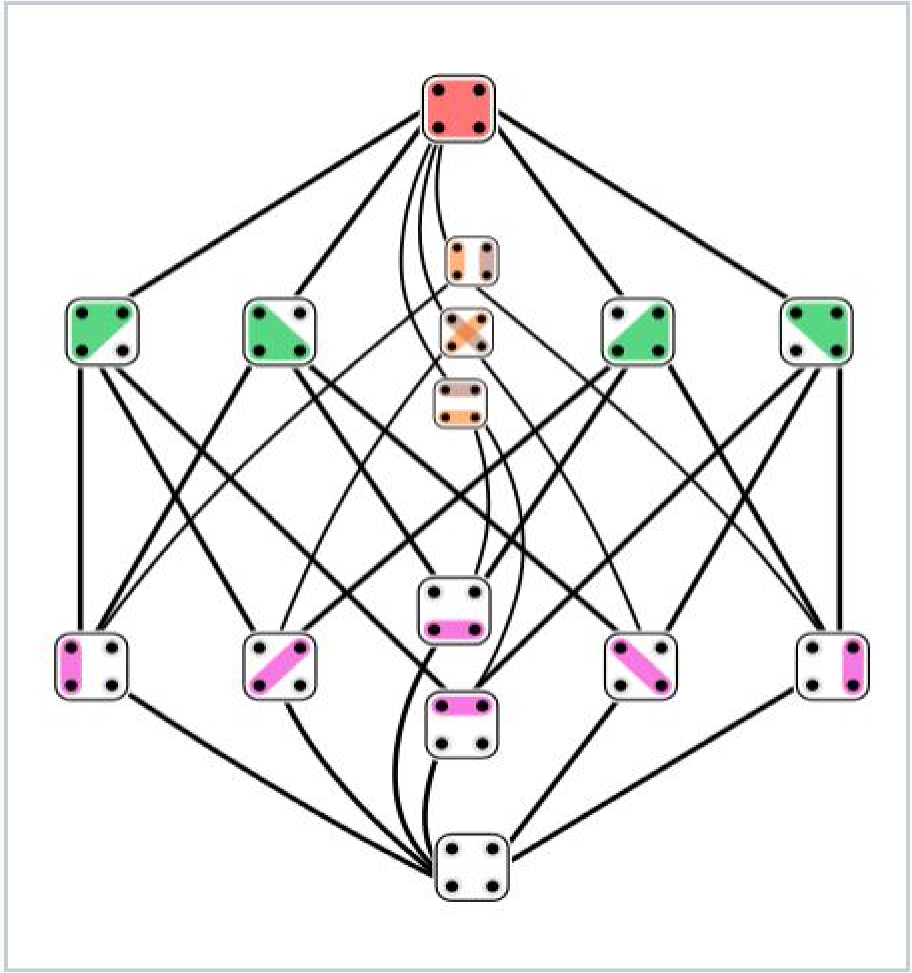
$$\left\{ {n \atop 2} \right\} = \frac{\frac{1}{1}(2^{n-1} - 1^{n-1})}{0!}$$

$$\left\{ {n \atop 3} \right\} = \frac{\frac{1}{1}(3^{n-1} - 2^{n-1}) - \frac{1}{2}(3^{n-1} - 1^{n-1})}{1!}$$

$$\left\{ {n \atop 4} \right\} = \frac{\frac{1}{1}(4^{n-1} - 3^{n-1}) - \frac{2}{2}(4^{n-1} - 2^{n-1}) + \frac{1}{3}(4^{n-1} - 1^{n-1})}{2!}$$

$$\left\{ {n \atop 5} \right\} = \frac{\frac{1}{1}(5^{n-1} - 4^{n-1}) - \frac{3}{2}(5^{n-1} - 3^{n-1}) + \frac{3}{3}(5^{n-1} - 2^{n-1}) - \frac{1}{4}(5^{n-1} - 1^{n-1})}{3!}$$

$$\vdots$$



The 15 partitions of a 4-element set ordered in a Hasse diagram

There are $S(4,1),\ldots,S(4,4) = 1,7,6,1$ partitions containing 1,2,3,4 sets.