离散数学

Discrete Mathematics

for Computer Science

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第6讲 关系 Relation（2）

Good order is the foundation of all things.

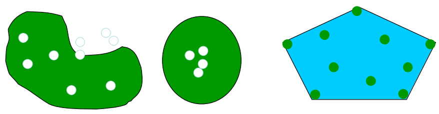
—Edmund Burke (1729–1797)

Outline

- 关系闭包

闭包(closure):
包含一些给定对象, 具有指定性质的最小集合 “最小”： 任何包含同样对象, 具有同样性质的集合, 都包含这个闭包集合。

示例



Computer science

- Closure (computer programming), an abstraction binding a function to its scope
- Clojure, a dialect of the Lisp programming language
- Kleene closure
- Syntactic closure
- Google Closure Tools, a set of JavaScript tools created by Google
- Relational database model: Set-theoretic formulation and Armstrong's axioms for its use in database theory

In graph theory
In logic and computational complexity
In database query languages
Algorithms

自反闭包: 包含给定关系R的最小自反关系, 称为R的自反闭包:

(1) $R \subseteq R'$;
(2) R '是自反的;
(3) $\forall S((R \subseteq S \wedge S$自反$) \rightarrow R' \subseteq S)$.

R'记作：r(R)

对称闭包 s(R)
传递闭包 t(R)

设$R \subseteq A \times A$且$A \neq \varnothing$,则

(1) R自反 $\Leftrightarrow r(R) = R$;

(2) R对称 $\Leftrightarrow s(R) = R$;

(3) R传递 $\Leftrightarrow t(R) = R$.

(1) $r(R)$是R的自反闭包,

$R \subseteq R \wedge R$自反 $\Rightarrow r(R) \subseteq R$, 且 $R \subseteq r(R)$,

所以，$r(R) = R$.

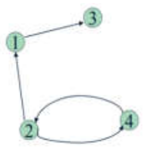
如何求闭包?

(1) $r(R) = R \cup ?$

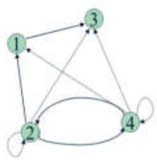
(2) $s(R) = R \cup ?$

(3) $t(R) = R \cup ?$

- Computes the transitive closure of a relation
- (Alternatively: all paths in a directed graph)
- Example of transitive closure:



$$\begin{matrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{matrix}$$



$$\begin{matrix} 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{matrix}$$

设 $R \subseteq A \times A$ 且 $A \neq \varnothing$, 则

(1) $r(R) = R \cup I_A$;

(2) $s(R) = R \cup R^{-1}$;

(3) $t(R) = R \cup R^2 \cup R^3 \cup \ldots$ (记为R^+, R的自反传递闭包记为R^*)

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证明: (1)

$R \subseteq R \cup I_A \wedge R \cup I_A$自反 $\Rightarrow r(R) \subseteq R \cup I_A$;

$R \subseteq r(R) \wedge r(R)$自反 $\Rightarrow R \subseteq r(R) \wedge I_A \subseteq r(R) \Rightarrow R \cup I_A \subseteq r(R)$

于是， $r(R) = R \cup I_A$

(3) $t(R) = R \cup R^2 \cup R^3 \cup \ldots$

1) $R \subseteq R \cup R^2 \cup R^3 \cup \ldots$;

2) $(R \cup R^2 \cup R^3 \cup \ldots)^2 = R^2 \cup R^3 \cup \ldots \subseteq R \cup R^2 \cup R^3 \cup \ldots$

$\Leftrightarrow R \cup R^2 \cup R^3 \cup \ldots$传递

$\boxed{R\text{传递} \Leftrightarrow R^2 \subseteq R}$

3) 若有R', $R \subseteq R' \wedge R'$ 传递

$\Rightarrow R \subseteq R' \wedge R^2 \subseteq R' \wedge R^3 \subseteq R' \wedge \ldots$

$\Rightarrow R \cup R^2 \cup R^3 \cup \ldots \subseteq R'$

$\therefore t(R) = R \cup R^2 \cup R^3 \cup \ldots$

$\boxed{R \subseteq G \wedge G\text{传递} \Rightarrow R^n \subseteq G}$

$$|A| = n,\ t(R) = \bigcup_{i=1}^{n} R^i$$

示例 $A=\{a, b, c\}$, $R=\{\langle a, b\rangle,\langle b, c\rangle,\langle c, a\rangle\}$,求$r(R)$,$S(R)$,$t(R)$.

解：$r(R)=R \cup I_A=\{\langle a, b\rangle, \langle b, c\rangle, \langle c, a\rangle, \langle a, a\rangle, \langle b, b\rangle, \langle c, c\rangle\}$

$s(R)= R \cup R^{-1} =\{\langle a, b\rangle,\langle b, a\rangle,\langle b, c\rangle,\langle c, b\rangle,\langle c, a\rangle,\langle a, c\rangle\}$

为求$t(R)$先求R^2，R^3，R^4

即$R^2 = \{\langle a, c\rangle,\langle b, a\rangle,\langle c, b\rangle\}$

$R^3=\{\langle a, a\rangle,\langle b, b\rangle,\langle c, c\rangle\}$

$R^4=\{\langle a, b\rangle,\langle b, c\rangle,\langle c, a\rangle\}$

可见$R=R^4=R^{3n+1}$，$R^2= R^5= R^{3n+2}$，$R^3= R^6= R^{3n+3}$

故$t(R) = R \cup R^2 \cup R^3$

$= \{\langle a, a\rangle, \langle b, b\rangle, \langle c, c\rangle, \langle a, b\rangle, \langle b, c\rangle, \langle c, a\rangle, \langle a\ c\rangle, \langle b, a\rangle, \langle c, b\rangle\}$

Warshall's Algorithm (pseudocode and analysis)

```
ALGORITHM  Warshall(A[1..n, 1..n])
    //Implements Warshall's algorithm for computing the transitive closure
    //Input: The adjacency matrix A of a digraph with n vertices
    //Output: The transitive closure of the digraph
    R^(0) ← A
    for k ← 1 to n do
        for i ← 1 to n do
            for j ← 1 to n do
                R^(k)[i, j] ← R^(k−1)[i, j] or (R^(k−1)[i, k] and R^(k−1)[k, j])
    return R^(n)
```

Time efficiency: $\Theta(n^3)$

Space efficiency: Matrices can be written over their predecessors