离散数学

Discrete Mathematics

for Computer Science

计算机学院计科系

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第6讲 关系 Relation（3）

Good order is the foundation of all things.

—Edmund Burke (1729–1797)

Outline

几个特殊的二元关系

- 等价关系
- 偏序关系
- 函数

In mathematics, an **equivalence relation** is a binary relation that is at the same time a reflexive relation, a symmetric relation and a transitive relation. As a consequence of these properties an equivalence relation provides a partition of a set into equivalence classes.

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关于等价关系——Why these three properties

In math you may want to ask why these three properties have been selected to represent equivalence between sets? If you are indeed curious, try to think of other properties common to all equivalence relations you may think of. I can only offer a very lame excuse: over many-many years mathematicians agreed that the above three are the simplest and the commonest.

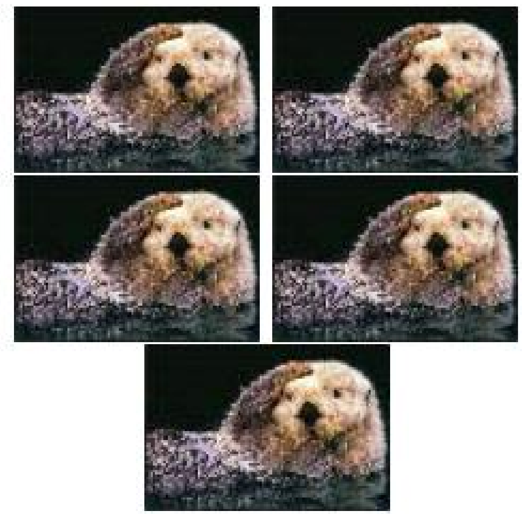
关于等价关系——Why these three properties

Somehow, by the age of five, we know that

is

equivalent

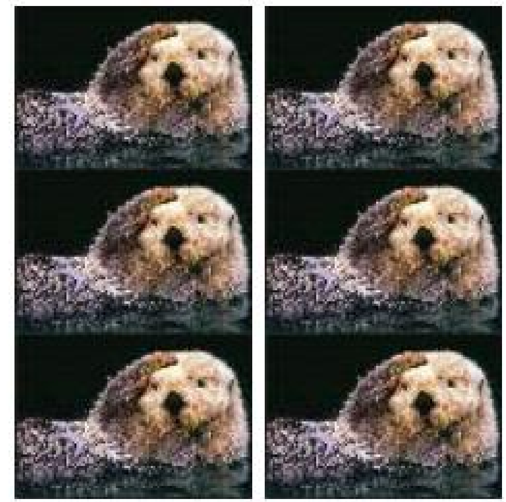
to



关于等价关系——Why these three properties

while

is not
equivalent
to



关于等价关系——Why these three properties

But then again

is

equivalent

to

Fuzzy equivalence relations

Fuzzy equivalence relations were introduced by Zadeh [37] as a generalization of the concept of an equivalence relation. They have been since widely studied as a way to measure the degree of indistinguishability or similarity between the objects of a given universe of discourse, and they have shown to be useful in different contexts such as fuzzy control, approximate reasoning, fuzzy cluster analysis, etc. Depending on the authors and the context in which they appeared, they have received other names such as similarity relations (original Zadeh's name [37]), indistinguishability operators ([36], [23], [24], [6], [25], [14], [15]), $\mathscr{T}$-equivalences ([10], [11]), many-valued equivalence relations ([12], [13]), etc.

The first definition of a fuzzy partition was given by Ruspini [33], and it has played a significant role in many studies in fuzzy cluster analysis. Butnariu [1] proposed another definition which was originally based on the Lukasiewicz t-norm, and later defined for an arbitrary t-norm. But, none of these definitions of a fuzzy partition gives a bijective


BIG
DATA
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PETABYTES
SOFTWARE
MANAGEMENT
SETS
MILLION
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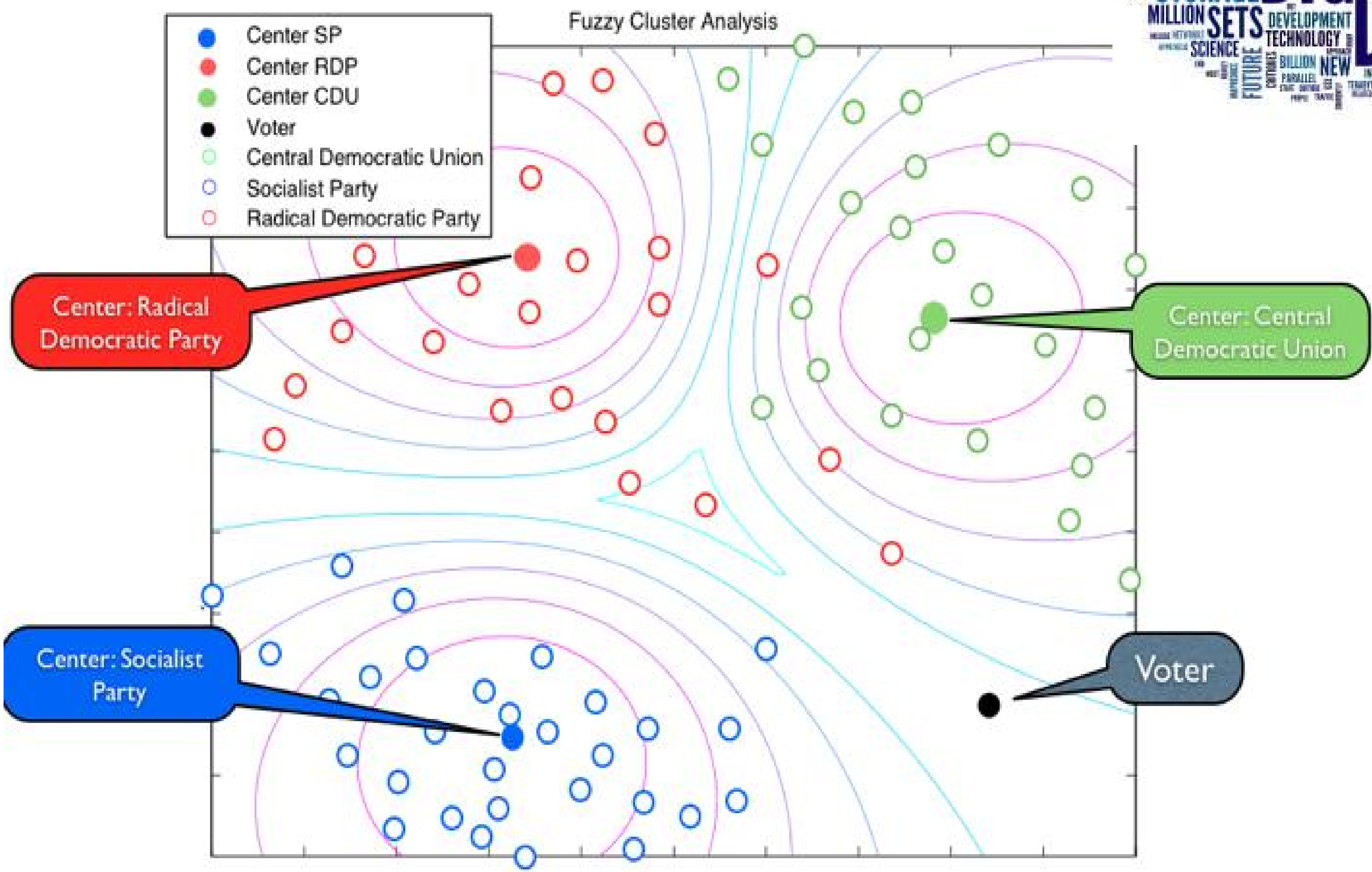
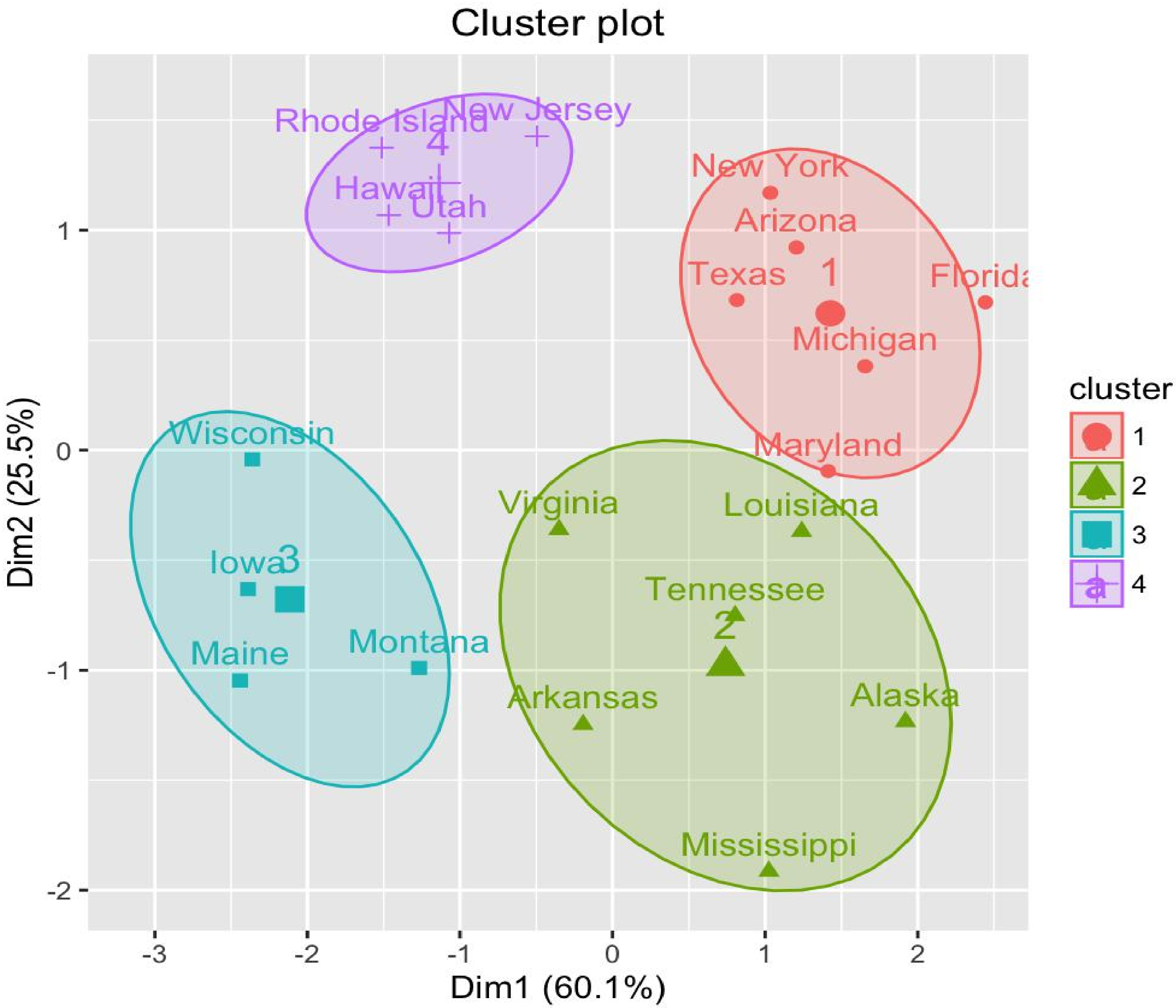
PROCESS
TECHNOLOGY
DEVELOPMENT
SCIENCE
NEW
INTERNET
TECHNOLOGIES
GOVERNMENT
FUTURE


Fuzzy Cluster Analysis
Center SP
Center RDP
Center CDU
Voter
Central Democratic Union
Socialist Party
Radical Democratic Party
Center: Radical Democratic Party
Center: Central Democratic Union
Center: Socialist Party
Voter

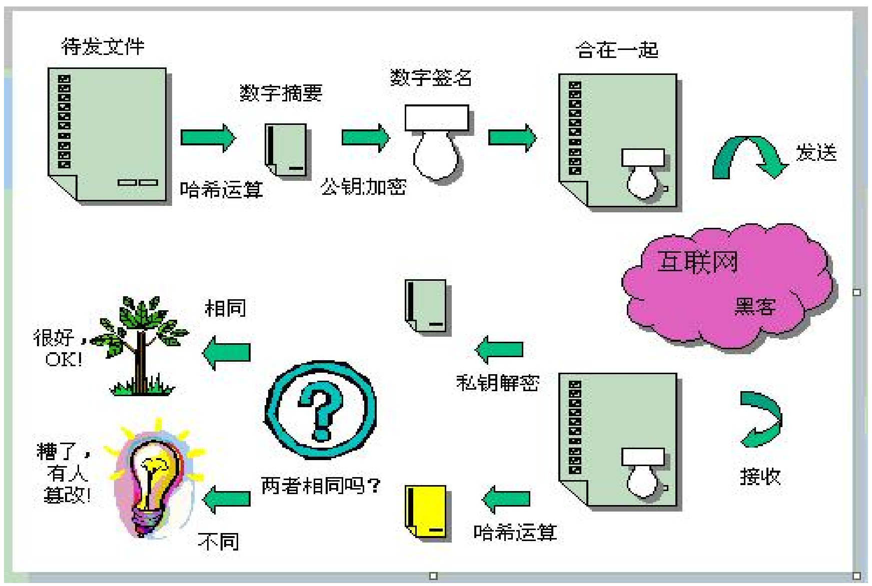

Cluster plot
Rhode Island
New Jersey
4
Hawaii
Utah
New York
Arizona
Texas
1
Florida
Michigan
Maryland
Wisconsin
Iowa
3
Maine
Montana
Virginia
Louisiana
Tennessee
2
Arkansas
Alaska
Mississippi
cluster
1
2
3
4
Dim1 (60.1%)
Dim2 (25.5%)

A set of stamps partitioned in bundles: No stamp is in two bundles, and no bundle is empty.

软件等价类测试

序号	【A, B, C】	覆盖等价类	输 出
1	【3, 4, 5】	(1), (2), (3), (4), (5), (6)	一般三角形
2	【0, 1, 2】	(7)	不能构成三角形
3	【1, 0, 2】	(8)	
4	【1, 2, 0】	(9)	
5	【1, 2, 3】	(10)	
6	【1, 3, 2】	(11)	
7	【3, 1, 2】	(12)	
8	【3, 3, 4】	(1), (2), (3), (4), (5), (6), (13)	等腰三角形
9	【3, 4, 4】	(1), (2), (3), (4), (5), (6), (14)	
10	【3, 4, 3】	(1), (2), (3), (4), (5), (6), (15)	
11	【3, 4, 5】	(1), (2), (3), (4), (5), (6), (16)	非等腰三角形
12	【3, 3, 3】	(1), (2), (3), (4), (5), (6), (17)	是等边三角形
13	【3, 4, 4】	(1), (2), (3), (4), (5), (6), (14), (18)	非等边三角形
14	【3, 4, 3】	(1), (2), (3), (4), (5), (6), (15), (19)	
15	【3, 3, 4】	(1), (2), (3), (4), (5), (6), (13), (20)	

RSA Cryptography



Modulo equivalence

Equivalent code (compilers)

Compiler optimisation of source code by determination and utilization of the <u>equivalence of algebraic expressions</u> in the source code

等价关系?

等价类

设R是$A\neq\varnothing$上等价关系,$\forall x\in A$,令 $[x]_R=\{ y \mid y\in A \wedge xRy \}$,称$[x]_R$为x关于R的等价类, 简称x的等价类,可简记为[x].

示例 设$A=\{a,b,c,d,e\}$，A上的关系

$\rho=\{(a,a),(a,b),(b,a),(b,b),(c,c),(d,d),(d,e),(e,d),(e,e)\}$,

确定由集合A中的元素产生的等价类。

示例 设 A={1,2,3,4,5,8}, 求

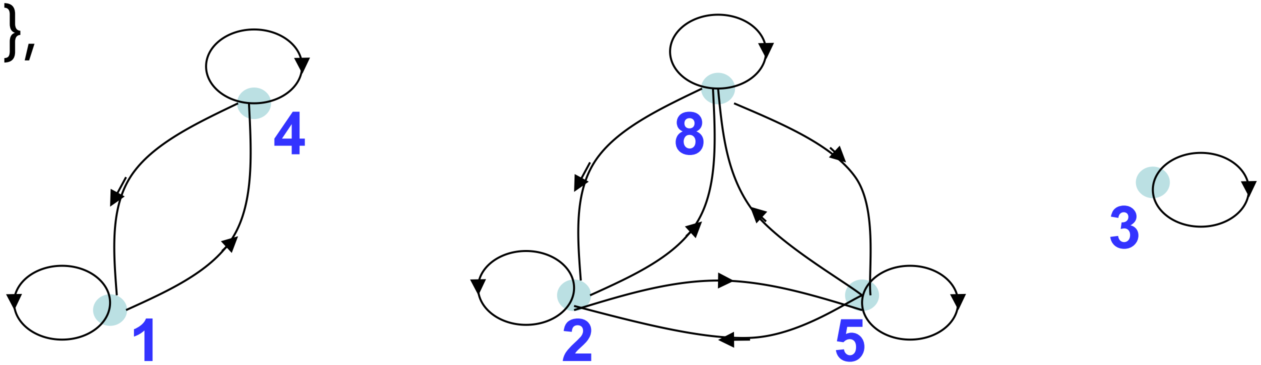
R_3 = { <x,y> | $x,y \in A \wedge x \equiv y \pmod 3$ }的等价类, 画出R_3的关系图.

解:

[1]=[4]={1,4},

[2]=[5]=[8]={2,5,8},

[3]={3}.



等价关系(Equivalence)?

设 $R \subseteq A \times A$； 且 $A \neq \varnothing$, 若R是自反的, 对称的, 传递的

示例: 判断是否等价关系(A是某班学生):

$R_1=\{<x,y>|x,y \in A \wedge$ x与y同年生$\}$

$R_2=\{<x,y>|x,y \in A \wedge$ x与y同姓$\}$

$R_3=\{<x,y>|x,y \in A \wedge$ x的年龄不比y小$\}$

$\varnothing$是A上等价关系吗?

$R_4=\{<x,y>|x,y \in A \wedge$ x与y选修同门课程$\}$

$R_5=\{<x,y>|x,y \in A \wedge$ x的体重比y重$\}$

示例

1）在一群人的集合上年龄相等的关系是等价关系，而朋友关系不一定是等价关系，因为它可能不是传递的。

2）命题公式间的逻辑等值关系是等价关系。

3）集合上的恒等关系I_A和普遍关系U_A都是等价关系。

4）在同一平面上直线之间的平行关系，三角形之间的相似关系都是等价关系。

练习

假设给定了正整数的序偶集合A，在$A\times A$上定义二元关系R如下：

$((x,y),(u,v))\in R$，当且仅当x*v=y*u，*为一般乘法，请证明R是一个等价关系。

示例 设$n\in\{2,3,4,\ldots\}$, $x,y\in Z$,则x与y模n同余(be congruent modulo n)$\Leftrightarrow x\equiv y(\mathrm{mod}\ n) \Leftrightarrow n|(x-y) \Leftrightarrow x-y=kn\ (k\in Z)$,

称x,y具有同余关系(Congruence)。

同余等价类

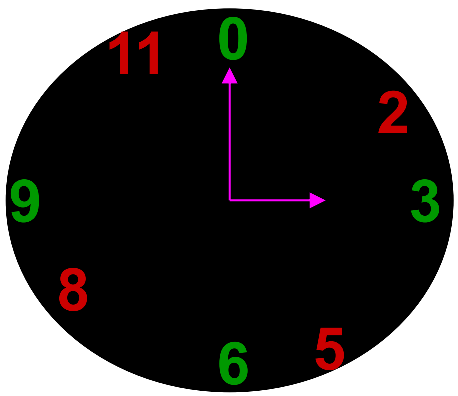
$[0]\ =\{\ kn|k\in Z\}$,

$[1]\ =\{\ 1+kn|k\in Z\}$,

$[2]\ =\{\ 2+kn|k\in Z\}$,

…,

$[n-1]=\{(n-1)+kn|k\in Z\}$.



注意到，由等价关系确定的等价类具有如下性质：

$[x]_R \neq \varnothing$；

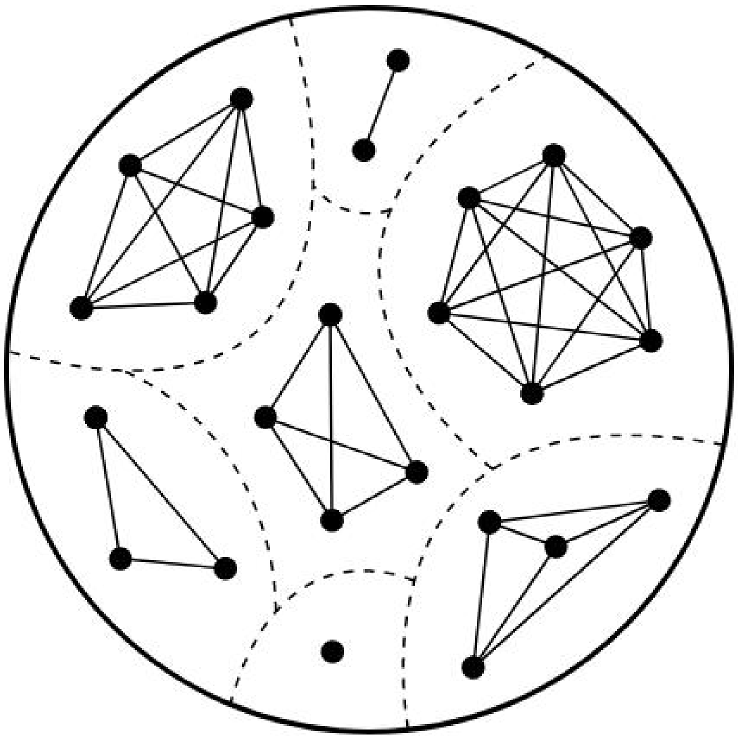
$xRy \Rightarrow [x]_R = [y]_R$；

$\neg xRy \Rightarrow [x]_R \cap [y]_R = \varnothing$；

$\cup\{ [x]_R \mid x \in A \} = A$.

$$
\begin{aligned}
A &= \cup\{ \{x\} \mid x \in A \} \\
&\subseteq \cup\{ [x]_R \mid x \in A \} \\
&\subseteq \cup\{ A \mid x \in A \} \\
&= A.
\end{aligned}
$$

$\therefore \cup\{ [x]_R \mid x \in A \} = A.$



商集: 设R是$A\neq\varnothing$上等价关系,

$$A/R = \{ [x]_R \mid x\in A \}$$

称为A关于R的商集, 简称A的商集.

显然 $\bigcup A/R = A$.

设$A\neq\varnothing$, 则

(1) R是A上等价关系 $\Rightarrow$ 商集A/R是A的划分

(2) Π是A的划分 $\Rightarrow$ R_Π是A上等价关系,其中

$$x\, R_\Pi\, y \Leftrightarrow \exists z(z\in\Pi\wedge x\in z\wedge y\in z)$$

称为由划分Π 所定义的等价关系(同块关系).

An important fact about equivalence relations is expressed by the following

Theorem

Let ~ be an equivalence relation defined between elements of a set $\mathbf{A}$. *Then the set* $\mathbf{A}$ *can be written as a union* $\cup \mathbf{A}_t$ *of pairwise disjoint subsets* $\mathbf{A}_t \subset \mathbf{A}$ *such that*

$$a \sim b \text{ iff there exists } \mathbf{A}_t \text{ such that } a \in \mathbf{A}_t \text{ and } b \in \mathbf{A}_t.$$

There are several ways to express the fact that **a** ~ **b**. For example:

- **a** is equivalent to **b**.
- **a** and **b** are equivalent.
- **a** and **b** belong to the same equivalence class.

示例 $A=\{a,b,c\}$, 求A上全体等价关系

解: A上不同划分共有5种:

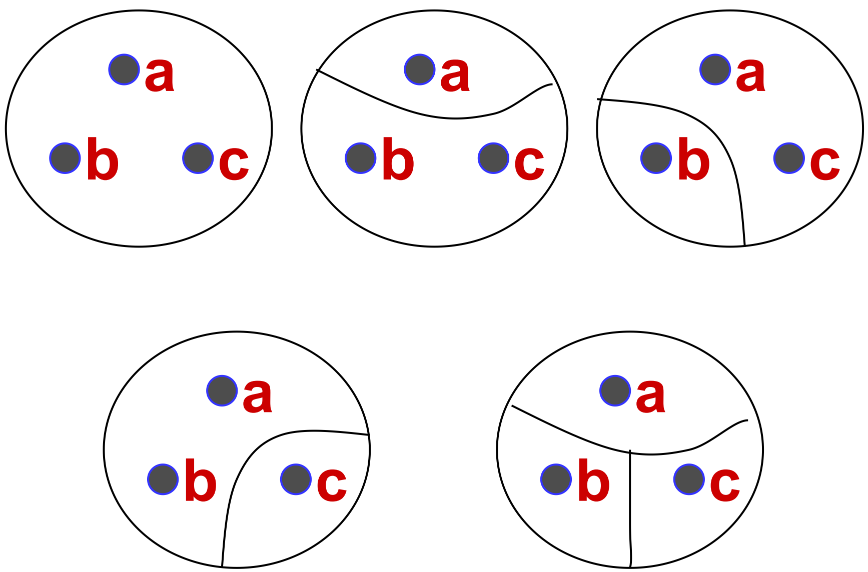
$R_1= U_A$,

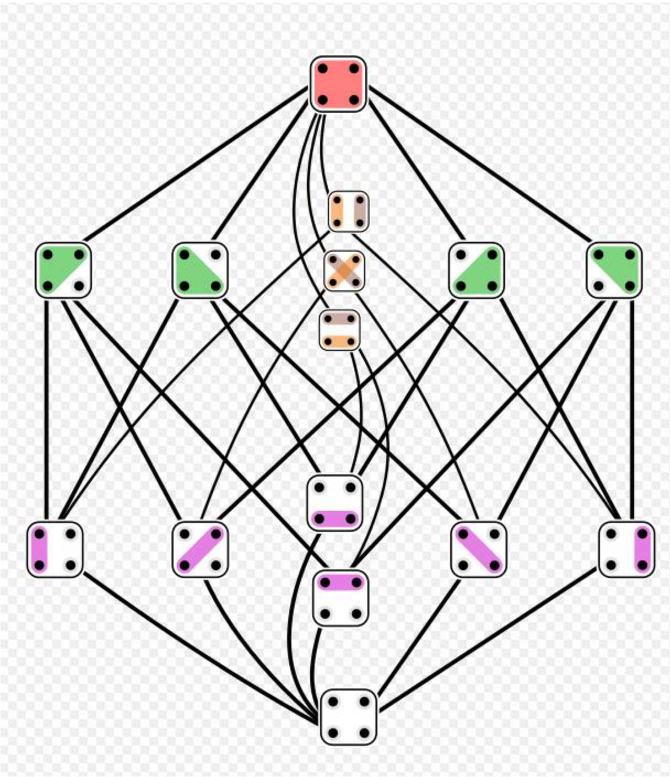
$R_2=I_A\cup\{<b,c><c,b>\}$,

$R_3=I_A\cup\{<a,c><c,a>\}$,

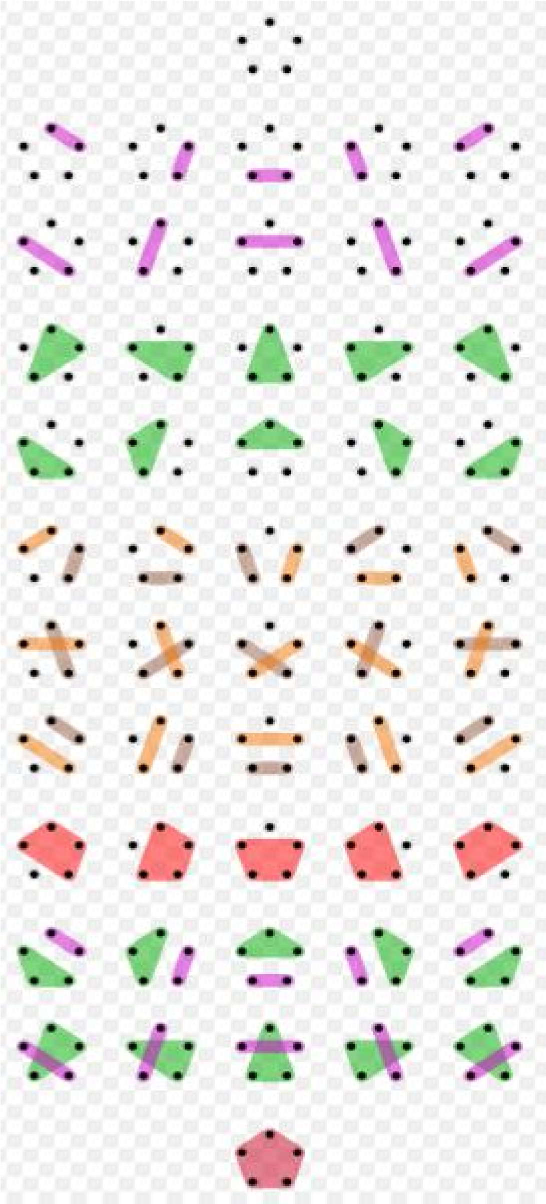
$R_4=I_A\cup\{<a,b><b,a>\}$,

$R_5=I_A$.





Set partitions 4; Hasse



The 52 partitions of a set with 5 elements

From Wikimedia

Bell triangle

1							
1	2						
2	3	5					
5	7	10	15				
15	20	27	37	52			
52	67	87	114	151	203		
203	255	322	409	523	674	877	
877	1080	1335	1657	2066	2589	3263	4140

$\vdots \qquad \vdots \qquad \vdots$