7.2 The Königsberg Bridges; Traversability

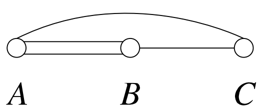
Modeling Road Systems

Suppose we want to describe the roads joining various towns to plan an itinerary. If our only desire is to list the towns that will be visited in order, we do not need to know the physical properties of the region, such as hills, whether different roads cross, or whether there are overpasses. The important information is whether or not there is a road joining a given pair of towns. In these cases we can use a complete road map, with the exact shapes of the roads and various other details shown, but it will be less confusing to make a diagram as shown in Figure 7.6(a) that indicates roads joining B to C, A to each of B and C, and C to D, with no direct roads joining A to D or B to D. The diagram is obviously a graph whose vertices are the towns and whose edges are the road links.

In some cases there is a choice of routes between two towns. This information may be important: you might prefer the freeway, which is faster, or you might prefer the more scenic coast road. In this case the graph does not contain enough information. For example, if there had been two different ways to travel from B to C, this information can be represented as in the diagram of Figure 7.6(b). This new diagram is a *multigraph*; that is, there is a *multiple edge* (of *multiplicity* 2) joining B to C.

Sample Problem 7.4. *Suppose the road system connecting towns A, B, and C consists of two roads from A to B, one road from B to C, and a bypass road directly from A to C. Represent this road system graphically.*

Solution.



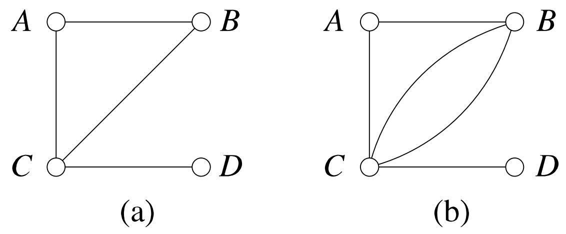


Fig. 7.6. Representing a road system

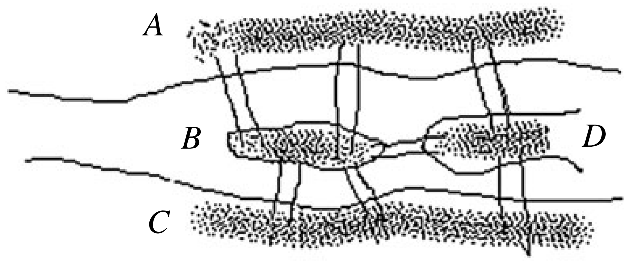


Fig. 7.7. The Königsberg bridges

Practice Exercise. Represent graphically the road system connecting towns A, B, C, D, with one road from A to B, one from A to C, two from A to D, and one from B to C.

The Königsberg Bridges

The first mathematical paper on graph theory was published by the great Swiss mathematician Leonhard Euler in 1736, and had been delivered by him to the St. Petersburg Academy one year earlier.

Euler's paper grew out of a famous old problem. The town of Königsberg in Prussia is built on the river Pregel. The river divides the town into four parts, including an island called The Kneiphof, and in the eighteenth century the town had seven bridges; the layout is shown in Figure 7.7. The question under discussion was whether it was possible from any point on Königsberg to take a walk in such a way as to cross each bridge exactly once.

Euler set for himself the more general problem as follows: given any configuration of river, islands, and bridges, find a general rule for deciding whether there is a walk that covers each bridge precisely once.

We first show that it is impossible to walk over the bridges of Königsberg. Suppose there was such a walk. There are three bridges leading to the area C: you can traverse two of these, one leading into C and the other leading out, at one time in your tour. There is only one bridge left: if you cross it going into C, then you cannot leave C again, unless you use one of the bridges twice, so C must be the finish of the walk; if you cross it in the other direction, C must be the start of the walk. In either event, C is either the place where you start or the place where you finish.

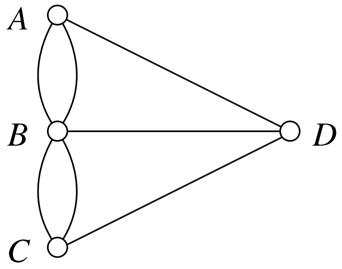


Fig. 7.8. The multigraph representing the Königsberg bridges

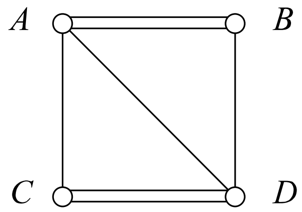
A similar analysis can be applied to A, B, and D since each has an odd number of bridges—five for B and three for the others. But any walk starts at one place and finishes at one place; either there can be two endpoints, or the walk can start and finish at the same place. Therefore it is impossible for A, B, C, and D all to be either the start or the finish.

Euler started by finding a multigraph that models the Königsberg bridge problem (considered as a road network, with the islands and river banks as separate "towns"). Vertices A, B, C, and D represent the parts A, B, C, and D of the town, and each bridge is represented by an edge. The multigraph is shown in Figure 7.8. In terms of this model, the original problem becomes can a simple walk be found that contains every edge of the multigraph?

These ideas can be applied to more general configurations of bridges and islands, and to other problems. A simple walk that contains every edge of a given multigraph will be called an *Euler walk* in the multigraph. Finding whether a given multigraph (or a given road network) has an Euler walk is called the *traversability problem*.

Sample Problem 7.5. *The islands A, B, C, D are joined by seven bridges—two joining A to B, two from C to D, and one each joining the pairs AC, AD, and BD. Represent the system graphically. Can one walk through this system, crossing each bridge exactly once?*

Solution. Islands B and D each have an odd of bridges, so one must be the start of the walk and the other the finish. With a little experimentation you will find a solution—one example is $BACDABDC$ and there are others.



Practice Exercise. The islands X, Y, Z, T are joined by seven bridges—three joining X to Y, and one each joining the pairs XZ, XT, and YZ. Represent the system graphically. Can one walk through this system, crossing each bridge exactly once?

In proving that a solution to the Königsberg bridge problem is impossible, we used the fact that certain vertices had an odd number of edges incident with them.

(The precise number was not important; oddness was the significant feature.) Let us call a vertex *even* if its degree is even, and *odd* otherwise. It was observed that an odd vertex must be either the first or the last point in the walk. In fact, if a multigraph is traversable, then either the multigraph has two odd vertices, the start and finish of the Euler walk, or the multigraph has no odd vertices, and the Euler walk starts and finishes at the same point. Another obvious necessary condition is that the multigraph must be connected. These two conditions are together sufficient.

Theorem 44. *If a connected multigraph has no odd vertices, then it has a Euler walk starting from any given point and finishing at that point. If a connected multigraph has two odd vertices, then it has a Euler walk whose start and finish are the odd vertices.*

(A closed Euler walk—one that starts and ends at the same point—is called a *Euler circuit*.)

Proof. Consider any simple walk in a multigraph that starts and finishes at the same vertex. If one erases every edge in that walk, one deletes two edges touching any vertex that was crossed once in the walk, four edges touching any vertex that was crossed twice, and so on. (For this purpose, count "start" and "finish" combined as one crossing.) In every case an *even* number of edges is deleted.

First, consider a multigraph with no odd vertex. Select any vertex x, and select any edge incident with x. Go along this edge to its other endpoint, say y. Then choose any other edge incident with y. In general, on arriving at a vertex, select any edge incident with it that has not yet been used, and go along the edge to its other endpoint. At the moment when this walk has led into the vertex z, where z is not x, an odd number of edges touching z has been used up (the last edge to be followed, and an even number previously). Since z is even, there is at least one edge incident with z that is still available. Therefore, if the walk is continued until a further edge is impossible, the last vertex must be x—that is, the walk is closed. It will necessarily be a simple walk and it must contain every edge incident with x.

Now assume that a connected multigraph with every vertex even is given, and a simple closed walk has been found in it by the method just described. Delete all the edges in the walk, forming a new multigraph. From the first paragraph of the proof it follows that every vertex of the new multigraph is even. It may be that we have erased every edge in the original multigraph; in that case we have already found a Euler walk. If there are edges still left, there must be at least one vertex, say c, that was in the original walk and that is still on an edge in the new multigraph—if there were no such vertex, then there can be no connection between the edges of the walk and the edges left in the new multigraph, and the original multigraph must have been disconnected. Select such a vertex c, and find a closed simple walk starting from c. Then unite the two walks as follows: at one place where the original walk contained c, insert the new walk. For example, if the two walks are

$$x, y, \ldots, z, c, u, \ldots, x,$$

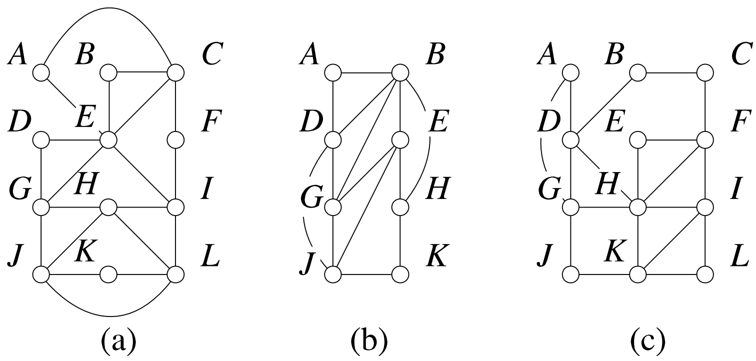


Fig. 7.9. Find Euler walks

and

$$c, s, \ldots, t, c,$$

then the resulting walk will be

$$x, y, \ldots, z, c, s, \ldots, t, c, u, \ldots, x.$$

(There may be more than one possible answer if c occurred more than once in the first walk. Any of the possibilities may be chosen.) The new walk is a closed simple walk in the original multigraph. Repeat the process of deletion, this time deleting the newly formed walk. Continue in this way. Each walk contains more edges than the preceding one, so the process cannot go on indefinitely. It must stop. But this will only happen when one of the walks contains all edges of the original multigraph, and that walk is a Euler walk.

Finally, consider the case where there are two odd vertices p and q and every other vertex is even. Form a new multigraph by adding an edge pq to the original. This new multigraph has every vertex even. Find a closed Euler walk in it, choosing p as the first vertex and the new edge pq as the first edge. Then delete this first edge; the result is a Euler walk from q to p. □

It is clear that loops make no difference as to whether or not a graph has a Euler walk. If there is a loop at vertex x, it can be added to a walk at some traversing of x.

A good application of Euler walks is planning the route of a highway inspector or mail contractor who must travel over all the roads in a highway system. Suppose the system is represented as a multigraph G, as was done earlier. Then the most efficient route will correspond to a Euler walk in G.

Sample Problem 7.6. *Find a Euler walk in the road network represented by Figure 7.9(a,b).*

Solution. In the first example, starting from A, we find the walk $ACBEA$. When these edges are deleted (see Figure 7.10(a)) there are no edges remaining through A. We choose C, a vertex from the first walk that still has edges

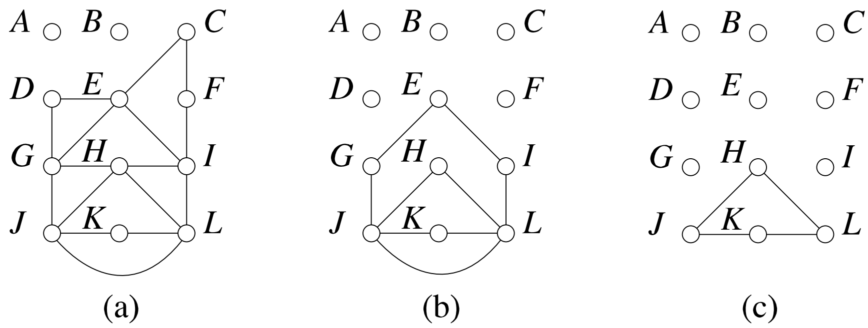


Fig. 7.10. Constructing a Euler walk

adjacent to it, and trace the walk $CFIHGDEC$, after which there are no edges available at C (see Figure 7.10(a)). E is available, yielding walk $EGJLIE$. As is clear from Figure 7.10(c), the remaining edges form a walk $HJKLH$.

We start with $ACBEA$. We replace C by $CFIHGDEC$, yielding

$$ACFIHGDECBEA,$$

then replace the first E by $EGJLIE$, with result

$$ACFIHGDEGJLIECBEA$$

(we could equally well replace the second E). Finally H is replaced by $HJKLH$, and the Euler walk is

$$ACFIHJKLHGDEGJLIECBEA.$$

In the second example, there are two odd vertices, namely B and F, so we add another edge BF and make it the first edge used. The first walk found was

$$BFHGCABFDBCEB,$$

and the second is $DEGD$, exhausting all the vertices and producing the walk

$$BFHGCABFDEGDBCEB.$$

The first edge (the new one we added) is now deleted, giving Euler walk

$$FHGCABFDEGDBCEB,$$

in the original graph.

Practice Exercise. Find a Euler walk in the road network represented by Figure 7.9(c).

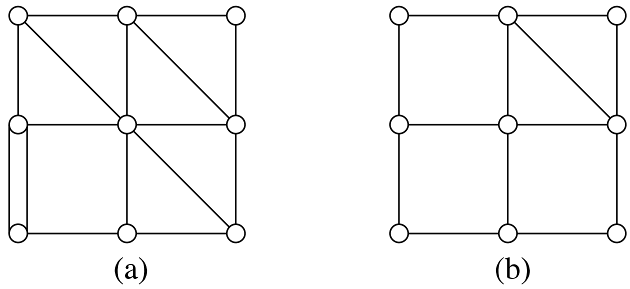


Fig. 7.11. Find Eulerizations

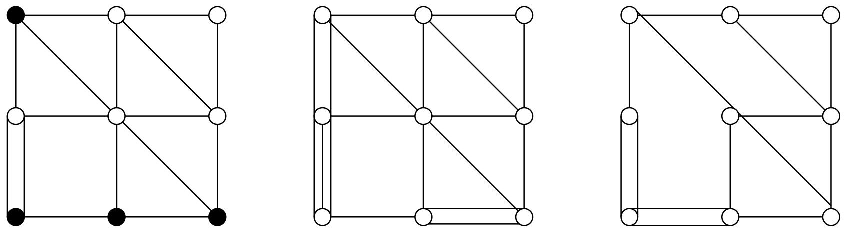


Fig. 7.12. Finding an Eulerization

Eulerization

If G contains no Euler walk, the highway inspector must repeat some edges of the graph to return to his starting point. Let us define a *Eulerization* of G to be a multigraph, with a closed Euler walk, that is formed from G by duplicating some edges. A *good* Eulerization is one that contains the minimum number of new edges, and this minimum number is the *Eulerization number* $eu(G)$ of G. For example, if two adjacent vertices have an odd degree, you can add a further edge joining them. This will mean that the inspector must travel the road between them twice. However, if the two vertices are not adjacent, one new edge will not suffice—it would be the same as requiring that a new road be built!

Sample Problem 7.7. *Consider the multigraph G shown in Figure* 7.11(a). *What is* $eu(G)$*? Find an Eulerization of the road network represented by G that uses the minimum number of edges.*

Solution. Look at the representation of the multigraph on the left of Figure 7.12. The black vertices have an odd degree, so they need additional edges. As there are four black vertices, at least two new edges are needed; but obviously no two edges will suffice. However, there are solutions with three added edges—two examples are shown in Figure 7.12—so $eu(G) = 3$.

Practice Exercise. Consider the multigraph H shown in Figure 7.11(b). What is $eu(H)$? Find an Eulerization of the road network represented by H that uses the minimum number of edges.