



离散数学

Discrete Mathematics

for Computer Science

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Discrete Mathematics, 5 Set Theory



第5讲 集合论 Set Theory

No one will drive us from the paradise which Cantor created for us.
——David Hilbert

Discrete Mathematics, 5 Set Theory

Outline

- 关于集合论
- 集合
- 关系
- 函数
- 基数、序数、公理化



Discrete Mathematics, 5 Set Theory

关于无穷与无穷集合的数学理论

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三次“数学危机”

对“无穷”的认识过程？

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Hilbert's paradox of the Grand Hotel

假设有一个拥有可数无限多个房间的旅馆，且所有的房间均已客满。此时这一旅馆将可否再接纳新的客人？

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巴拿赫-塔斯基悖论



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Galileo's paradox

Consider the set of positive natural numbers,

$N = \{1, 2, 3, 4, 5, 6, 7, 8, 9, \dots\}$.

Applying the "square" function to each number in N gives the set $S = \{1, 4, 9, 16, 25, 36, 49, 64, 81, \dots\}$, the set of all squares of positive integers.

"Therefore if I assert that all numbers, including both squares and non-squares, are more than the squares alone, I shall speak the truth, shall I not?"

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贝克莱悖论

"无穷小量究竟是否为0"的问题:就无穷小量在当时实际应用(求导数)而言,它必须既是0,又不是0。但从形式逻辑而言,这无疑是一个矛盾。

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柯西创立"极限"理论

魏尔斯特拉斯创立 " $\varepsilon - \delta$ " 语言

数学分析中的所有基本概念都可以通过实数和它们的基本运算和关系精确地表述出来。

建立数学分析(或者说微积分)基础的"逻辑顺序"是:

实数理论—极限理论—微积分

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微积分中的"极限"?

实数理论、极限理论、变量与函数

康托尔 (Cantor) :

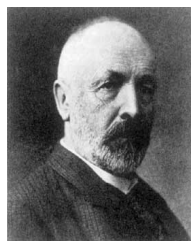
任意函数的三角级数的表达式是否唯一?

正整数集和实数集合之间能否建立一一对应?

(实数全体不可数性)

——>无穷集合的一般理论研究:集合论基础

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Georg Cantor, 1845-1918

数学中的无穷无尽·其诱人之处在于它的最棘手的悖论能够盛开出美丽的理论之花。

——E.Kasner and J.Newman

无穷大!任何一个其它问题都不能如此深刻地影响人类的精神;任何一个其它观点都不曾如此有效地激励人类的智力;但是,没有任何概念比无穷大更需要澄清.....

——Hilbert,David

没有人能把我们从康托为我们创造的乐园中赶走。

——Hilbert,David

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十九世纪数学最伟大成就之一

集合论体系

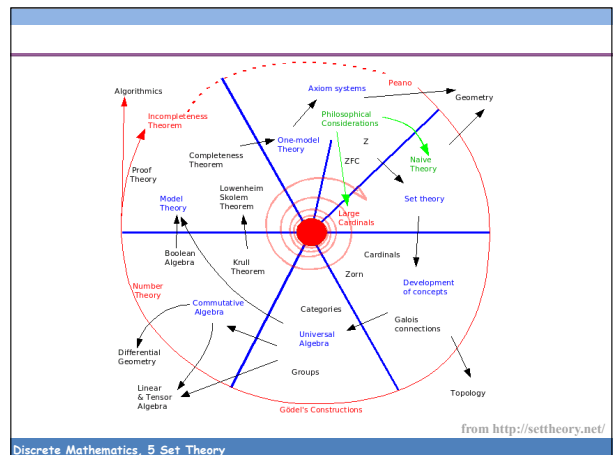
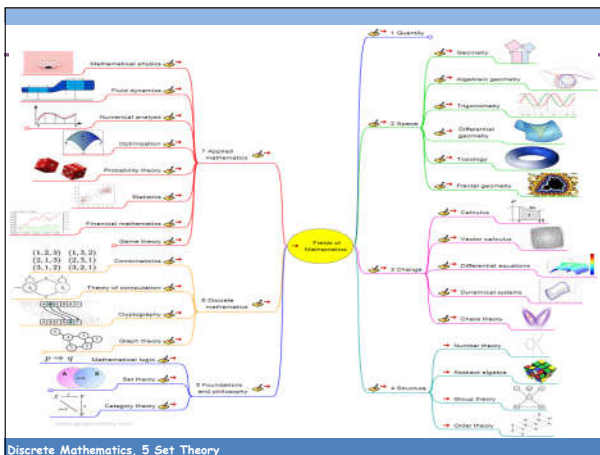
- 朴素(naive)集合论
- 公理(axiomatic)集合论

Georg Cantor, 1845-1918

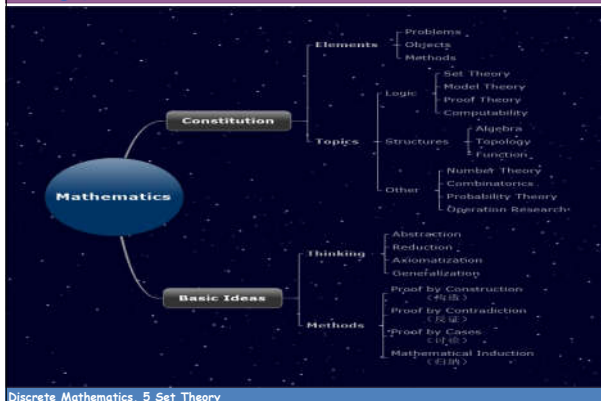
集合论-数学之基石

About—— The point is that without a precise scientific language it becomes virtually impossible, or at least enormously burdensome and awkward, to think scientifically. This is particularly true in mathematics. One of the crowning scientific achievements of the 20th century was the development of set theory as a precise language for all of mathematics, thanks to the efforts of Cantor, Dedekind, Frege, Peano, Russell and Whitehead, Zermelo, Fraenkel, Skolem, Hilbert, von Neumann, Godel, Bernays, Cohen, and others. This achievement has been so important and definitive that it led David Hilbert to say, already in 1925, that “no one will drive us from the paradise which Cantor created for us”.

About—— Set theory is not really the only rigorous mathematical language. The languages of set theory and of mathematical logic were developed together, so that, as a mathematical discipline, set theory is a branch of mathematical logic. Technically, as we shall see shortly, we can view the language of set theory as a special sublanguage of first-order logic. Furthermore, other theories such as category theory and intuitionistic type theory have been proposed as alternatives to set theory to express all of mathematics.



a big picture of mathematics



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集合论与一阶逻辑

一阶逻辑(FOL)

关注无限结构，但FOL的有穷性原则(只能刻画局部或有界的属性，而计算机科学中非常有用的各种有限但无上界的构造与过程，如递归构造、迭代过程，FOL无法规范)

→ FOL²，以及公理化

$$\forall F(F(0) \wedge \forall x(F(x) \rightarrow F(x+1)) \rightarrow \forall x F(x))$$

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集合论与计算机科学

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Motivation

Why learn Set Theory? Set Theory is an important language and tool for reasoning. It's a basis for Mathematics—pretty much all Mathematics can be formalised in Set Theory.

Why is Set Theory important for Computer Science? It's a useful tool for formalising and reasoning about computation and the objects of computation. Set Theory is indivisible from Logic where Computer Science has its roots. It has been and is likely to continue to be a source of fundamental ideas in Computer Science from theory to practice; Computer Science, being a science of the artificial, has had many of its constructs and ideas inspired by Set Theory. The strong tradition, universality and neutrality of Set Theory make it firm common ground on which to provide unification between seemingly disparate areas and notations of Computer Science. Set Theory is likely to be around long after most present-day programming languages have faded from memory. A knowledge of Set Theory should facilitate your ability to think abstractly. It will provide you with a foundation on which to build a firm understanding and analysis of the new ideas in Computer Science that you will meet.

Set Theory for Computer Science, Glynn Winskel

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集合论与计算机科学

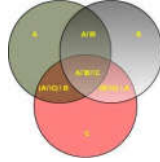
理论基础：如，关系演算的相关思想与方法对于数据库、模型检测、计算理论；为递归等计算机科学核心的构造性方法提供严格的集合论依据。构造性数学之重要基础。

应用基础：关系数据库技术、计数、数据结构、程序设计语言、编译原理等

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Outline

- 关于集合论
- **集合**
- 关系
- 函数
- 基数、序数、公理化



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1 集合(Set, Collection)?

“吾人直观或思维之对象，如为**相异**而**确定**之物，其总括之全体即谓之集合(Set)；其组成此集合之物谓之集合元素(Element)。”

“所谓**相异**者，取二物于此，其为同一，其为相异，可得而决定。而集合所含之元素乃有彼此不同之意味。”

“所谓**确定**者，此物是否属于此集合，一望而知，至少在概念上可以断定其是否为该集合之元素。盖合于某条件之集合，须其界限分明，不容有模糊不清之弊。”

“确定”？

- 模糊集合(Fuzzy Set)
- 粗糙集合(Rough Set)

“A set is a Many that allows itself to be thought of as a One.”
——Georg Cantor

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1 集合(Set, Collection)?

示例 数的集合

N：自然数(natural numbers)集合

$$N = \{0, 1, 2, 3, \dots\}$$

Z：整数(integers)集合(或者I表示)

$$Z = \{0, \pm 1, \pm 2, \dots\} = \{\dots, -2, -1, 0, 1, 2, \dots\}$$

Q：有理数(rational numbers)集合

R：实数(real numbers)集合

C：复数(complex numbers)集合

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1 集合(Set, Collection)?

- ▶ 集合与集合元素： $a \in A, a \notin A$, $\text{sort}(\text{esti})$, Peano
- ▶ 集合元素之无序性、不重复性 · n元集, $|A|$:基数(cardinality)
- ▶ 集合的表示
 - **外延**：列表法、枚举法、图示法 · 元素有限或无限但可数
 - **内涵**：用性质界定(概括法) · 谓词表达式： $P(a), \neg P(a)$

$$S = \{x | P(x)\}, A = \{a : P(a)\}$$

静态法

动态法

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Ideas and mehtods

计算机科学的典型方法：

抽象方法，
内涵与外延，
公理化方法，
构造性方法，
演化方法，
形式化方法，
原型方法

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2 集合间关系

- ▶ 子集(Subset)? 真子集(Proper Subset)?
- ▶ 包含关系(Inclusion Relation, $A \subseteq B, A \supseteq B$)? 真包含关系(Proper \sim , $A \subset B$)?
 - 集合相等(Equal)?
- ▶ 空集(Empty Set), 全集(Universal Set), 幂集(Power Set), 族(Set Family)

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练习

- 1 设A, B和C为任意三个集合, 则有
 - 1) $\Phi \subseteq A$; 2) $A \subseteq A$;
 - 3) 若 $A \subseteq B$ 且 $B \subseteq C$, 则 $A \subseteq C$; 4) 若 $A \supseteq B$ 且 $B \supseteq C$, 则 $A \supseteq C$.
- 2 空集是唯一的吗?
- 3 列出集合 $A = \{1, \{2\}\}$ 的全部子集。

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4 设 $A = \{a, b, \{c\}, \{a\}, \{a, b\}\}$, 试指出下列论断是否正确?

- | | |
|-----------------------------|--------------------------------|
| (1) $a \in A$ | (8) $\{b\} \subseteq A$ |
| (2) $\{a\} \in A$ | (9) $\{a, b\} \in A$ |
| (3) $\{a\} \subseteq A$ | (10) $\{a, b\} \subseteq A$ |
| (4) $\emptyset \in A$ | (11) $c \in A$ |
| (5) $\emptyset \subseteq A$ | (12) $\{c\} \in A$ |
| (6) $b \in A$ | (13) $\{c\} \subseteq A$ |
| (7) $\{b\} \in A$ | (14) $\{a, b, c\} \subseteq A$ |

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5 设有集合A, B, C和D, 下述论断是否正确? 说明理由。

- (1) 若 $A \in B, B \subseteq C$, 则 $A \in C$
- (2) 若 $A \in B, B \subseteq C$, 则 $A \subseteq C$
- (3) 若 $A \subseteq B, B \in C$, 则 $A \in C$
- (4) 若 $A \subseteq B, B \in C$, 则 $A \subseteq C$

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6 求下列集合的幂集。

- (1) $A = \emptyset$;
- (2) $B = \{\emptyset\}$;
- (3) $C = \{\emptyset, \{\emptyset\}\}$;
- (4) $D = \{a, b, c\}$ 。

解 (1) $P(A) = \{\emptyset\}$;
(2) $P(B) = \{\emptyset, \{\emptyset\}\}$;
(3) $P(C) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$;
(4) $P(D) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$ 。

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7 若A为有限集, 则 $|P(A)| = 2^{|A|}$ 。

8 设A, B为任意两个集合, 则有

- (1) $\Phi \in P(A)$;
- (2) $A \in P(A)$;
- (3) 若 $A \subseteq B$, 则 $P(A) \subseteq P(B)$ 。

9 证明: 对任意的集合S, 有
 $\{\emptyset, \{\emptyset\}\} \in PPP(S)$ 。

证明:
 $\emptyset \subseteq S$
 $\therefore \emptyset \in P(S)$

 $\text{又 } \{\emptyset\} \subseteq P(S)$
 $\therefore \{\emptyset\} \in PP(S)$

 $\text{又 } \emptyset \subseteq PP(S)$
 $\therefore \emptyset \in PPP(S)$

 $\therefore \{\emptyset, \{\emptyset\}\} \subseteq PPP(S)$
 $\therefore \{\emptyset, \{\emptyset\}\} \in PPP(PPP(S))$

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3 从逻辑角度看集合

空集

对任意集合A, $\emptyset \subseteq A$

证明: $\emptyset \subseteq A \Leftrightarrow \forall x(x \in \emptyset \rightarrow x \in A)$

$$\Leftrightarrow \forall x(0 \rightarrow x \in A) \Leftrightarrow 1.$$

3 从逻辑角度看集合

子集、包含关系

B为A的子集($B \subseteq A$, 或 $A \supseteq B$):

$$B \subseteq A \Leftrightarrow \forall x(x \in B \rightarrow x \in A)$$

B不是A的子集($B \not\subseteq A$):

$$B \not\subseteq A \Leftrightarrow \exists x(x \in B \wedge x \notin A)$$

$$\neg \forall x(x \in B \rightarrow x \in A) \Leftrightarrow \exists x \neg (x \in B \rightarrow x \in A)$$

$$\Leftrightarrow \exists x(x \in B \wedge \neg x \in A) \Leftrightarrow \exists x(x \in B \wedge x \notin A)$$

3 从逻辑角度看集合

► 真子集: B真包含A: $A \subset B \Leftrightarrow A \subseteq B \wedge A \neq B$

练习 如何用逻辑语言表示 $A \subset B$ 关系?

$A \subset B$

$$\Leftrightarrow \neg(A \subseteq B \wedge A \neq B) \quad (\text{c定义})$$

$$\Leftrightarrow \neg(A \subseteq B) \vee (A = B) \quad (\text{德·摩根律})$$

$$\Leftrightarrow \exists x(x \in A \wedge x \notin B) \vee (A = B) \quad (\text{c定义})$$

3 从逻辑角度看集合

集合相等

$A = B$

$$\Leftrightarrow A \subseteq B \wedge B \subseteq A$$

$$\Leftrightarrow \forall x(x \in A \rightarrow x \in B) \wedge \forall x(x \in B \rightarrow x \in A)$$

$$\Leftrightarrow \forall x((x \in A \rightarrow x \in B) \wedge (x \in B \rightarrow x \in A))$$

$$\Leftrightarrow \forall x(x \in A \leftrightarrow x \in B)$$

3 从逻辑角度看集合

一些性质

$A \subseteq A$:

$$A \subseteq A \Leftrightarrow \forall x(x \in A \rightarrow x \in A) \Leftrightarrow 1$$

若 $A \subseteq B$, 且 $A \neq B$, 则 $B \not\subseteq A$

证明: 1) $A \neq B$

$$2) \neg(A = B)$$

$$3) \neg(A \subseteq B \wedge B \subseteq A)$$

$$4) \neg(A \subseteq B) \vee \neg(B \subseteq A)$$

$$5) A \subseteq B$$

$$6) \neg(B \subseteq A) \quad (\text{即 } B \not\subseteq A)$$

3 从逻辑角度看集合

► 若 $A \subseteq B$, 且 $B \subseteq C$, 则 $A \subseteq C$

证明: $A \subseteq B \Leftrightarrow \forall x(x \in A \rightarrow x \in B)$

$\forall x, x \in A$

$\Rightarrow x \in B \quad (A \subseteq B)$

$\Rightarrow x \in C \quad (B \subseteq C)$

$\therefore \forall x(x \in A \rightarrow x \in C)$, 即 $A \subseteq C$.

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3 从逻辑角度看集合

► $A \not\subseteq A$:

$A \subset A \Leftrightarrow A \subseteq A \wedge A \neq A \Leftrightarrow 1 \wedge 0 \Leftrightarrow 0$.

► 若 $A \subset B$, 则 $B \not\subset A$

证明: (反证) 设 $B \subset A$, 则

$A \subset B \Leftrightarrow A \subseteq B \wedge A \neq B \Rightarrow A \subseteq B$ (化简)

$B \subset A \Leftrightarrow B \subseteq A \wedge B \neq A \Rightarrow B \subseteq A$

所以 $A \subseteq B \wedge B \subseteq A \Leftrightarrow A = B$ (定义)

但是 $A \subset B \Leftrightarrow A \subseteq B \wedge A \neq B \Rightarrow A \neq B$ (化简) 矛盾!

可以写出逻辑形式的证明过程吗?

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3 从逻辑角度看集合

若 $A \subset B$, 且 $B \subset C$, 则 $A \subset C$

证明:

1) $A \subset B$

2) $A \subseteq B \wedge A \neq B$

3) $A \subseteq B$

4) $B \subset C$

5) $B \subseteq C \wedge B \neq C$

6) $B \subseteq C$

7) $A \subseteq C$

8) $A = C$

9) $B \subseteq A$

10) $A = B$

11) $A = B \wedge A \subset B$, 矛盾

12) $A \neq C$

13) $A \subset C$

证明:

$A \subset B \Leftrightarrow A \subseteq B \wedge A \neq B \Rightarrow$

$A \subseteq B$ (化简),

同理 $B \subset C \Rightarrow B \subseteq C$,

所以 $A \subseteq C$.

假设 $A = C$, 则 $B \subseteq C \Leftrightarrow B \subseteq A$,
又 $A \subseteq B$, 故 $A = B$, 此与 $A \subset B$
矛盾,
所以 $A \neq C$.

于是, $A \subset C$.

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示例

There is a barber who just shaves anybody which don't shave themselves.

Russell's barber paradox

$\{x \mid B(x) \wedge \forall y(\neg s(y, y) \leftrightarrow s(x, y))\}$

$\exists x(B(x) \wedge \forall y(\neg s(y, y) \leftrightarrow s(x, y)))$?

罗素悖论

设 $A = \{S \mid S \text{ 是集合, 且 } S \notin S\}$, 则 A 不是集合。

Let $R = \{x \mid x \notin x\}$, then $R \in R \Leftrightarrow R \notin R$?

Discrete Mathematics, 5 Set Theory

4 集合运算



设 A, B 为任意两个集合。令

$A \cup B = \{x \mid x \in A \vee x \in B\}$

$A \cap B = \{x \mid x \in A \wedge x \in B\}$

$A - B = \{x \mid x \in A \wedge x \notin B\}$

$A \oplus B = \{x \mid (x \in A \vee x \in B) \wedge x \notin A \cap B\}$ (Complement Set), 用 A' 来表示。

$= (A \cup B) - (A \cap B)$

并 (Union)

交 (Intersection), 如果 $A \cap B = \emptyset$, 称 A 和 B 不相交。

差 (Subtraction)

称 $U - A$ 为 A 对于某 U (Universal) 的

绝对补/相对补

对称差 (Symmetric Difference)

初级并、初级交; 广义并、广义交

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4 集合运算

示例

1 若取 $U = \{0, 1, 2, 3, 4, 5\}$, $A = \{1, 2, 5\}$,
 $B = \{2, 4\}$ 时, 则有
 $A \cup B = \{1, 2, 4, 5\}$ $A \cap B = \{2\}$
 $A - B = \{1, 5\}$ $A \oplus B = \{1, 4, 5\}$
 $A' = \{0, 3, 4\}$ $B' = \{0, 1, 3, 5\}$

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4 集合运算

示例

2 设 A , B 和 C 为任意三个集合, 则有
 (1) $A \subseteq A \cup B$ 且 $B \subseteq A \cup B$; (2) $A \cap B \subseteq A$ 且 $A \cap B \subseteq B$;
 (3) $A - B \subseteq A$; (4) $A - B = A \cap B'$;
 (5) 若 $A \subseteq B$, 则 $B' \subseteq A'$; (6) 若 $A \subseteq C$ 且 $B \subseteq C$, 则
 $A \cup B \subseteq C$;
 (7) 若 $A \subseteq B$ 且 $A \subseteq C$, 则 $A \subseteq B \cap C$.
 3 设 A , B 为任意两个集合, 则以下条件互相等价:
 (1) $A \subseteq B$; (2) $A \cup B = B$; (3) $A \cap B = A$.

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4 集合运算

(1) 初级并

$$A_1 \cup A_2 \cup \dots \cup A_n = \{x \mid \exists i (1 \leq i \leq n \wedge x \in A_i)\}$$

$$\bigcup_{i=1}^n A_i = A_1 \cup A_2 \cup \dots \cup A_n$$

$$\bigcup_{i=1}^{\infty} A_i = A_1 \cup A_2 \cup \dots$$

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4 集合运算

示例

1 $A_n = \{x \in \mathbb{R} \mid n-1 \leq x \leq n\}, n=1, 2, \dots, 10$, 求 $\bigcup_{i=1}^{10} A_i$

2 $A_n = \{x \in \mathbb{R} \mid 0 \leq x \leq 1/n\}, n=1, 2, \dots$, 求 $\bigcup_{i=1}^{\infty} A_i$

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4 集合运算

(2) 初级交

$$A_1 \cap A_2 \cap \dots \cap A_n = \{x \mid \forall i (1 \leq i \leq n \rightarrow x \in A_i)\}$$

$$\bigcap_{i=1}^n A_i = A_1 \cap A_2 \cap \dots \cap A_n$$

$$\bigcap_{i=1}^{\infty} A_i = A_1 \cap A_2 \cap \dots$$

Discrete Mathematics, 5 Set Theory

4 集合运算

示例

1 $A_n = \{x \in \mathbb{R} \mid n-1 \leq x \leq n\}, n=1, 2, \dots, 10$, 求 $\bigcap_{i=1}^{10} A_i$

2 $A_n = \{x \in \mathbb{R} \mid 0 \leq x \leq 1/n\}, n=1, 2, \dots$, 求 $\bigcap_{i=1}^{\infty} A_i$

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4 集合运算

(3) 广义并(Union of the family of sets)

设A是族, A中所有集合的元素的全集, 称为A的广义并, 记作 $\bigcup A$.

$$\bigcup A = \{x \mid \exists Z (x \in Z \wedge Z \in A)\}$$

(4) 广义交(Intersection of the family of sets)

设A是族, A中所有集合的公共元素的全集, 称为A的广义交, 记作 $\bigcap A$.

$$\bigcap A = \{x \mid \forall Z (Z \in A \rightarrow x \in Z)\}$$

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4 集合运算

示例 设 $A1 = \{a, b, \{c, d\}\}$, $A2 = \{\{a, b\}\}$, $A3 = \{a\}$, $A4 = \{\emptyset, \{\emptyset\}\}$, $A5 = a (a \neq \emptyset)$, $A6 = \emptyset$, 则

$$\bigcup A1 = a \cup b \cup \{c, d\}, \quad \bigcap A1 = a \cap b \cap \{c, d\},$$

$$\bigcup A2 = \{a, b\}, \quad \bigcap A2 = \{a, b\},$$

$$\bigcup A3 = a, \quad \bigcap A3 = a$$

$$\bigcup A4 = \emptyset \cup \{\emptyset\} = \{\emptyset\}, \quad \bigcap A4 = \emptyset \cap \{\emptyset\} = \emptyset,$$

$$\bigcup A5 = \bigcup a, \quad \bigcap A5 = \bigcap a$$

$$\bigcup A6 = \emptyset, \quad \bigcap A6 = \emptyset$$

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5 集合运算定律

(集合代数) 设A、B、C是全集合U的任意子集, 有

(1) 等幂律

$$A \cup A = A, \quad A \cap A = A$$

(2) 结合律

$$(A \cup B) \cup C = A \cup (B \cup C),$$

$$(A \cap B) \cap C = A \cap (B \cap C)$$

(3) 交换律

$$A \cup B = B \cup A, \quad A \cap B = B \cap A$$

(4) 分配律

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C),$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

(5) 同一律

$$A \cup \emptyset = A, \quad A \cap U = A$$

(6) 零一律

$$A \cup \emptyset = A, \quad A \cap \emptyset = \emptyset$$

(7) 互补律

$$A \cup A' = U, \quad A \cap A' = \emptyset$$

(8) 吸收律

$$A \cup (A \cap B) = A, \quad A \cap (A \cup B) = A$$

(9) 德摩根律

$$(A \cup B)' = A' \cap B', \quad (A \cap B)' = A' \cup B'$$

$$U' = \emptyset, \quad \emptyset' = U$$

(10) 对合律

$$(A')' = A$$

Discrete Mathematics, 5 Set Theory

5 集合运算定律

集合等式的证明

► 逻辑演算法: 利用逻辑等值式和推理规则

► 集合演算法: 利用集合恒等式和已知结论

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5 集合运算定律——集合等式的证明

逻辑演算法

题目: $A=B$.

证明: $\forall x$,

$$x \in A$$

$$\Leftrightarrow \dots \quad (????)$$

$$\Leftrightarrow x \in B$$

$$\therefore A=B.$$

题目: $A \subseteq B$.

证明: $\forall x$,

$$x \in A$$

$$\Rightarrow \dots \quad (????)$$

$$\Rightarrow x \in B$$

$$\therefore A \subseteq B.$$

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5 集合运算定律——集合等式的证明

示例

1 试证明分配律: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

证明: $\forall x$,

$$x \in A \cup (B \cap C)$$

$$\Leftrightarrow x \in A \vee x \in (B \cap C) \quad (\cup \text{定义})$$

$$\Leftrightarrow x \in A \vee (x \in B \wedge x \in C) \quad (\cap \text{定义})$$

$$\Leftrightarrow (x \in A \vee x \in B) \wedge (x \in A \vee x \in C) \quad (\text{命题逻辑分配律})$$

$$\Leftrightarrow (x \in A \cup B) \wedge (x \in A \cup C) \quad (\cup \text{定义})$$

$$\Leftrightarrow x \in (A \cup B) \cap (A \cup C) \quad (\cap \text{定义})$$

$$\therefore A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

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5 集合运算定律——集合等式的证明

2 零一律: $A \cap \emptyset = \emptyset$

证明:

$$\forall x,$$

$$x \in A \cap \emptyset$$

$$\Leftrightarrow x \in A \wedge x \in \emptyset \quad (\cap \text{定义})$$

$$\Leftrightarrow x \in A \wedge 0 \quad (\emptyset \text{定义})$$

$$\Leftrightarrow 0 \quad (\text{命题逻辑零一律})$$

$$\therefore A \cap \emptyset = \emptyset$$

3 排中律: $A \cup A^c = U$

证明: $\forall x$,

$$x \in A \cup A^c$$

$$\Leftrightarrow x \in A \vee x \in A^c \quad (\cup \text{定义})$$

$$\Leftrightarrow x \in A \vee x \notin A \quad (\sim \text{定义})$$

$$\Leftrightarrow x \in A \vee \neg x \in A \quad (\notin \text{定义})$$

$$\Leftrightarrow 1 \quad (\text{命题逻辑排中律})$$

$$\therefore A \cup A^c = U$$

Discrete Mathematics, 5 Set Theory

5 集合运算定律——集合等式的证明

集合演算法

题目: $A=B$.

证明: A

$$= \dots (????)$$

$$= B$$

$$\therefore A=B.$$

题目: $A \subseteq B$.

证明: A

$$\subseteq \dots (????)$$

$$\subseteq B$$

$$\therefore A \subseteq B.$$

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5 集合运算定律——集合等式的证明

集合演算法

题目: $A=B$.

证明: $(\subseteq) \dots$

$$\therefore A \subseteq B$$

$(\supseteq) \dots$

$$\therefore A \supseteq B$$

$$\therefore A = B.$$

说明: 分=成 \subseteq 与 \supseteq

题目: $A \subseteq B$.

证明: $A \cap B$ (或 $A \cup B$)

$$= \dots (????)$$

$$= A \text{ (或 } B)$$

$$\therefore A \subseteq B.$$

说明: 化 \subseteq 成=, 利用:

$$A \cap B = A \Leftrightarrow A \subseteq B$$

$$A \cup B = B \Leftrightarrow A \subseteq B$$

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5 集合运算定律——集合等式的证明

示例 试证明吸收律

1 $A \cup (A \cap B) = A$

证明: $A \cup (A \cap B)$

$$= (A \cap U) \cup (A \cap B) \quad (\text{同一律})$$

$$= A \cap (U \cup B) \quad (\text{分配律})$$

$$= A \cap U \quad (\text{零律})$$

$$= A \quad (\text{同一律})$$

$$\therefore A \cup (A \cap B) = A$$

2 $A \cap (A \cup B) = A$

证明: $A \cap (A \cup B)$

$$= (A \cap A) \cup (A \cap B) \quad (\text{分配律})$$

$$= A \cup (A \cap B) \quad (\text{等幂律})$$

$$= A \quad (\text{吸收律 (1)})$$

$$\therefore A \cap (A \cup B) = A$$

Discrete Mathematics, 5 Set Theory

5 集合运算定律——集合等式的证明

练习

1 $P(A) \cup P(B) \subseteq P(A \cup B)$

证明: $\forall x,$

$$x \in P(A) \cup P(B)$$

$$\Leftrightarrow x \in P(A) \vee x \in P(B)$$

$$\Leftrightarrow x \subseteq A \vee x \subseteq B$$

$$\Rightarrow x \subseteq A \cup B$$

$$\Leftrightarrow x \in P(A \cup B)$$

$$\therefore P(A) \cup P(B) \subseteq P(A \cup B)$$

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5 集合运算定律——集合等式的证明

2 证明 $A \cap (B - C) = (A \cap B) - (A \cap C)$

证明

$$(1) A \cap (B - C) = A \cap (B \cap C')$$

$$= A \cap B \cap C'$$

$$(2) (A \cap B) - (A \cap C) = (A \cap B) \cap (A \cap C)'$$

$$= (A \cap B) \cap (A' \cup C')$$

$$= (A \cap B \cap A') \cup (A \cap B \cap C')$$

$$= \emptyset \cup (A \cap B \cap C')$$

互补律

$$= A \cap B \cap C'$$

同一律

由 (1) (2) 知, $A \cap (B - C) = (A \cap B) - (A \cap C)$.

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5 集合运算定律——集合等式的证明

3 试证明 对任意集合 A, B, C, 等式 $(A - B) \cup (A - C) = A$ 成立的充要条件是 $A \cap B \cap C = \emptyset$.

证明 (1) 证必要性。

$$\text{设 } (A - B) \cup (A - C) = A,$$

$$\text{因为 } (A - B) \cup (A - C) = (A \cap B') \cup (A \cap C') = A \cap (B' \cup C') = A \cap (B \cap C)' = A - (B \cap C),$$

$$\text{所以 } A - (B \cap C) = A, \text{ 即对任意的 } x \in A, \text{ 必有 } x \in A - (B \cap C), \text{ 因而必有 } x \notin B \cap C,$$

$$\text{因此 } A \cap (B \cap C) = A \cap B \cap C = \emptyset.$$

(2) 证充分性。

$$\text{设 } A \cap B \cap C = \emptyset, \text{ 则对任意 } x \in A, \text{ 必有 } x \notin B \cap C, \text{ 即 } x \in (B \cap C)',$$

$$\text{所以 } A \subseteq (B \cap C)', \text{ 因此 } (A - B) \cup (A - C) = A \cap (B \cap C)' = A.$$

由 (1)(2) 知等式 $(A - B) \cup (A - C) = A$ 成立的充要条件是 $A \cap B \cap C = \emptyset$.

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6 集合论公理

- 第一个常用的公理系统是 E.F.F. 策梅洛和 A.A. 弗伦克尔等提出的 ZF 系统。这个系统中只有一个非逻辑二元关系符号 \in 。如果加上选择公理就构成 ZFC 系统。
- 集合论公理有：外延公理、空集公理、无序对公理、并集公理、幂集公理、无穷公理、分离公理模式、替换公理模式、正则公理、选择公理。
- 利用公理可以定义出空集、序对、关系、函数等集合，还可以给出序关系、良序关系、序数、基数，也可以给出自然数、整数、实数等概念。集合论中有关集合的性质，在公理集合论中都可以得到证明。公理系统中还可以证明公理之间的相对和谐性和独立性。

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6 集合论公理

► The Axiom of Extensionality (外延公理) : $\forall A, B (A = B \leftrightarrow (\forall C (C \in A \leftrightarrow C \in B)))$.

This is a formal way of saying that a set is described by its members: two sets are equivalent if and only if they contain the same members.

► The Empty-Set Axiom (空集公理) : $\exists A \forall B (B \subseteq A)$.

There exists a set, the empty set, which contains no members.

► The Axiom of Union (并集公理) : $\forall A (\exists B (\forall C (C \in B \leftrightarrow (\exists D (C \in D \wedge D \in A))))$

Once again, the precise formal statement in FOPL is tough going. There's an easier way to write it using the union symbol: $\forall A (\exists B (B = \cup A))$; that is, for any set A, there's a set B consisting of the unions of the members of A.

► The Axiom of Infinity (无限公理) : $\exists N (\emptyset \in N \wedge (\forall x (x \in N \rightarrow x \cup \{x\} \in N)))$

What it says is, there's a set that (a) contains the empty set as a member, and (b) for each of its members x, it also contains the singleton set {x} containing x. So, if following the formal statement, we called that set N, N contains $\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}$, etc. What the axiom of infinity does is really two basic things: it gives us our first countably infinite set; and it gives us a construction which can be turned into Peano integers. Following von Neumann, $0 = \emptyset, 1 = 0' = \{\emptyset\}, 2 = 1' = \{\emptyset, 1\}, 3 = 2' = \{\emptyset, 1, 2\}, \dots$

► The Powerset Construction Axiom (幂集公理) : $\forall A (\exists B (\forall C (C \subseteq A \leftrightarrow \forall D (D \subseteq C \rightarrow D \in A))))$

This is a nice, easy one. For any set A, the powerset - that is, the class of all subsets of A - is a set.

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小结

- ▶ 关于集合
- ▶ 集合及其表示
- ▶ 集合与数理逻辑
- ▶ 集合运算
- ▶ 集合论公理