离散数学

Discrete Mathematics

for Computer Science

计算机学院计科系

薛思清 xuesiqing@cug.edu.cn

第8讲 集合基数 Cardinality

"The infinite! No other question has ever moved so profoundly the spirit of man."

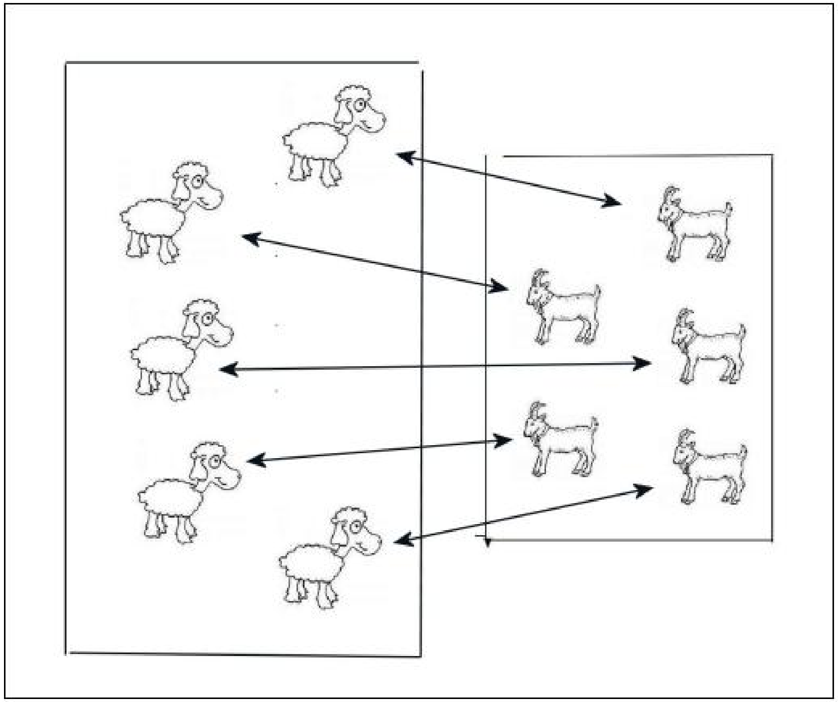
——David Hilbert

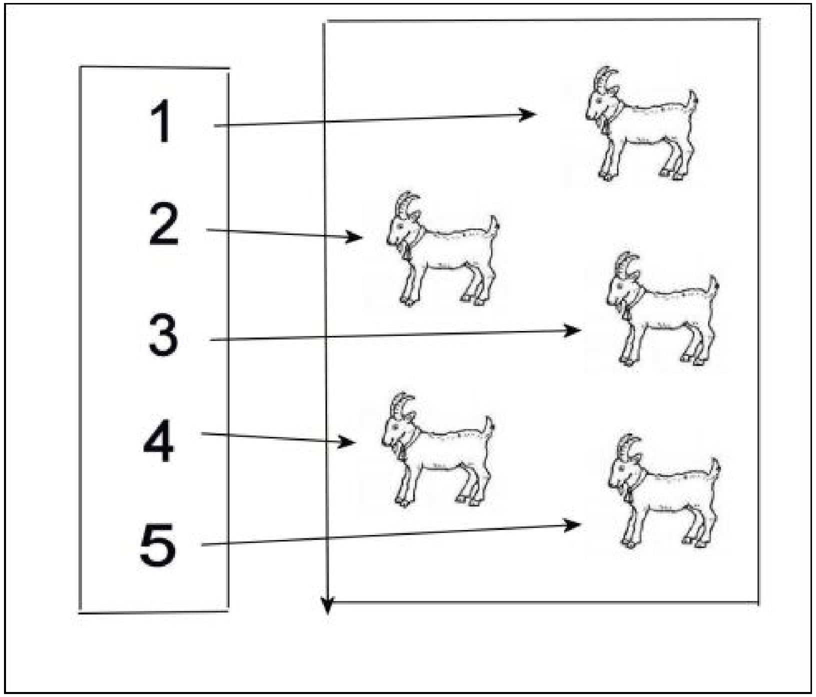
Outline

- 集合基数（Cardinality）
- 序数（Ordinals）

Reference: http://www.cs.utexas.edu/~isil/cs311h/

- Earlier we talked about sets and cardinality of sets
- Recall: Cardinality of a set is number of elements in that set
- This definition makes sense for sets with finitely many element, but more involved for infinite sets





Why these are called countable?

The elements of the set can be enumerated and listed.

Cardinality

Cardinality of Infinite Sets

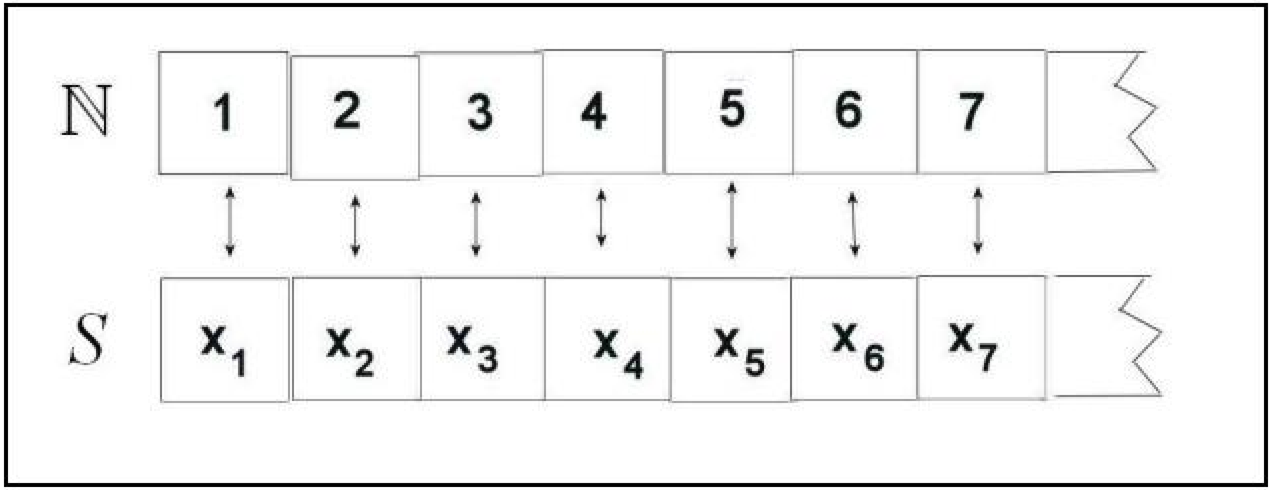
- Sets with infinite cardinality are classified into two classes:
  1. Countably infinite sets (e.g., natural numbers)
  2. Uncountably infinite sets (e.g., real numbers)
- A set A is called countably infinite if there is a bijection between A and the set of positive integers.
- A set A is called countable if it is either finite or countably infinite
- Otherwise, the set is called uncountable or uncountably infinite

Note: Roughly speaking an uncountable set has so many points it cannot be put in a sequence.

Margin Note: The "lazy eight" symbol "∞" does not represent the number infinity; it is simply a symbol used to denote that a set of real numbers is unbounded, such as $(a,\infty),(-\infty,b),(-\infty,\infty)$ and so on.

the natural numbers is an infinite set whose cardinality is called **aleph null**[1], and denoted by $\aleph_0$.

Sets of cardinality $\aleph_0$ are those sets which can be "counted" or arranged in a sequence $S = (x_1, x_2, \ldots)$.



Theorem 1 ($\aleph_0$ is the Smallest Infinity)

Example

Prove: The set of odd positive integers is countably infinite.

- Need to find a function f from $\mathbb{Z}^+$ to the set of odd positive integers, and prove that f is bijective
- Consider $f(n) = 2n - 1$ from $\mathbb{Z}^+$ to odd positive integers
- We need to show f is bijective (i.e., one-to-one and onto)

Another Example

Prove that the set of all integers is countable

Another Example

Prove that the set of all integers is countable

- We can list all integers in a sequence, alternating positive and negative integers:

$$a_n = 0, 1, -1, 2, -2, 3, -3, \ldots$$

- Observe that this sequence defines the bijective function:

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ even} \\ -(n-1)/2 & \text{if } n \text{ odd} \end{cases}$$

Another Way to Prove Countableness

- One way to show a set A is countably infinite is to give bijection between $\mathbb{Z}^+$ and A

Another Way to Prove Countableness

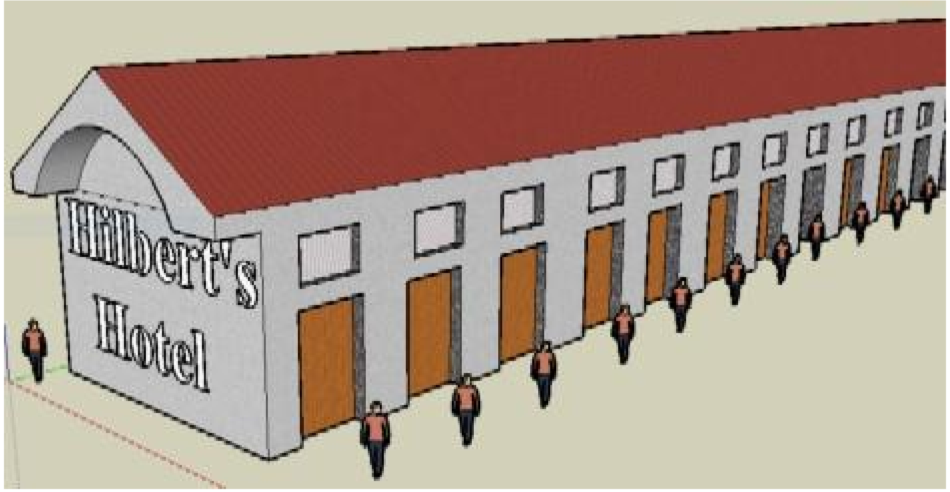
- One way to show a set A is countably infinite is to give bijection between $\mathbb{Z}^+$ and A
- Another way is by showing members of A can be written as a sequence $(a_1, a_2, a_3, \ldots)$
- Since such a sequence is a bijective function from $\mathbb{Z}^+$ to A, writing A as a sequence $a_1, a_2, a_3, \ldots$ establishes one-to-one correspondence

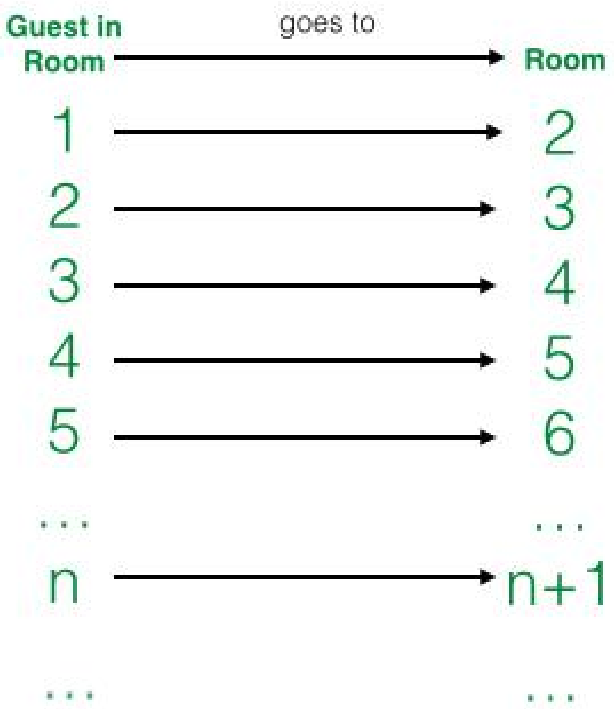
Rational Numbers are Countable

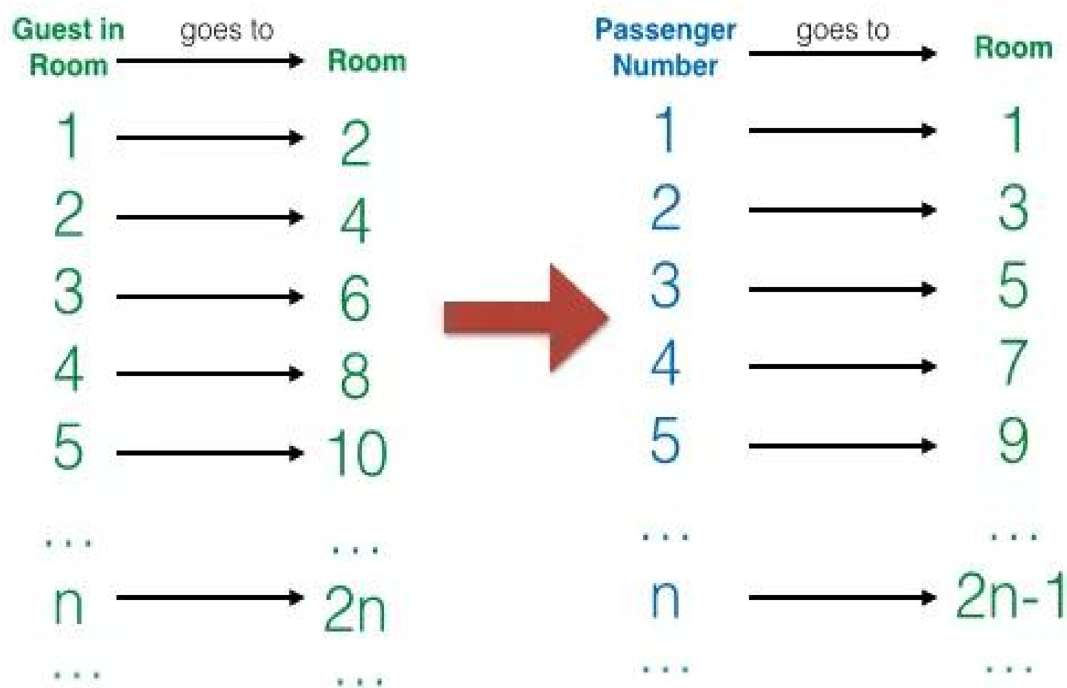
- Not too surprising $\mathbb{Z}$ and odd $\mathbb{Z}^+$ are countably infinite
- More surprising: Set of rationals is also countably infinite!

Example

Understanding Hilbert's Grand Hotel Paradox




Guest in Room
goes to
Room
1 2
2 3
3 4
4 5
5 6
... ...
n n+1
... ...


Guest in Room
goes to
Room
1 2
2 4
3 6
4 8
5 10
... ...
n 2n
... ...
Passenger Number
goes to
Room
1 1
2 3
3 5
4 7
5 9
... ...
n 2n-1
... ...

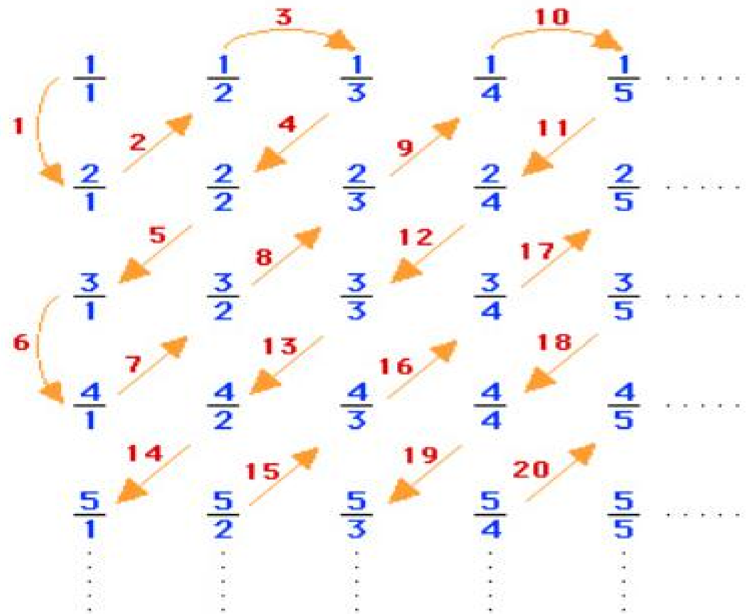
Rational Numbers are Countable

- Not too surprising $\mathbb{Z}$ and odd $\mathbb{Z}^+$ are countably infinite
- More surprising: Set of rationals is also countably infinite!
- We'll prove that the set of positive rational numbers is countable by showing how to enumerate them in a sequence
- Recall: Every positive rational number can be written as the quotient p/q of two positive integers p, q

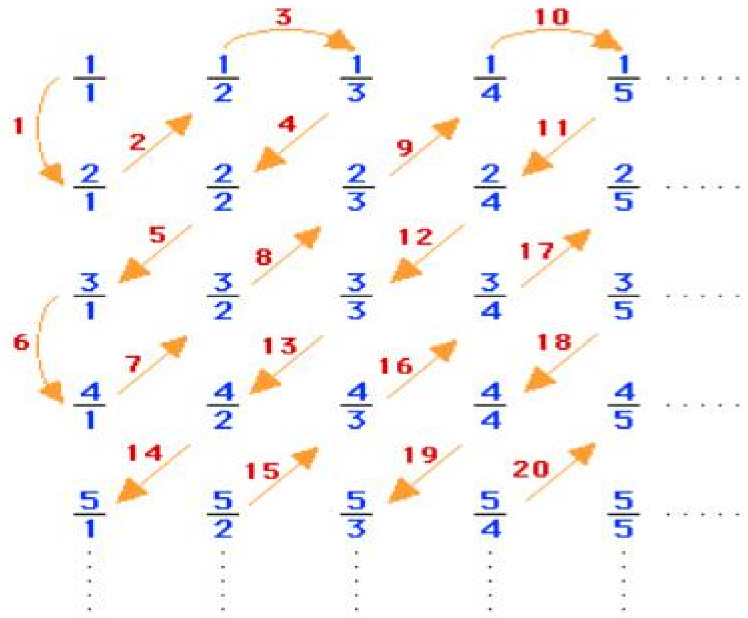
Rationals in a Table

- Now imagine placing rationals in a table such that:
  1. Rationals with $p = 1$ go in first row, $p = 2$ in second row, etc.
  2. Rationals with $q = 1$ in 1st column, $q = 2$ in 2nd column, ...
- How to enumerate entries in this table without missing any?
- Trick: First list those with $p + q = 2$, then $p + q = 3$, ...
- Traverse table diagonally from left-to-right, in the order shown by arrows

Enumerating the Rationals, cont.



Enumerating the Rationals, cont.



- This allows us to list all rationals in a sequence:

$$\frac{1}{1}, \frac{2}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{2}, \frac{3}{1}, \frac{4}{1}, \frac{3}{2}, \ldots$$

- Hence, set of rationals is countable

Uncountability of Real Numbers

- Prime example of uncountably infinite sets is real numbers
- The fact that $\mathbb{R}$ is uncountably infinite was proven by George Cantor using the famous Cantor's diagonalization argument

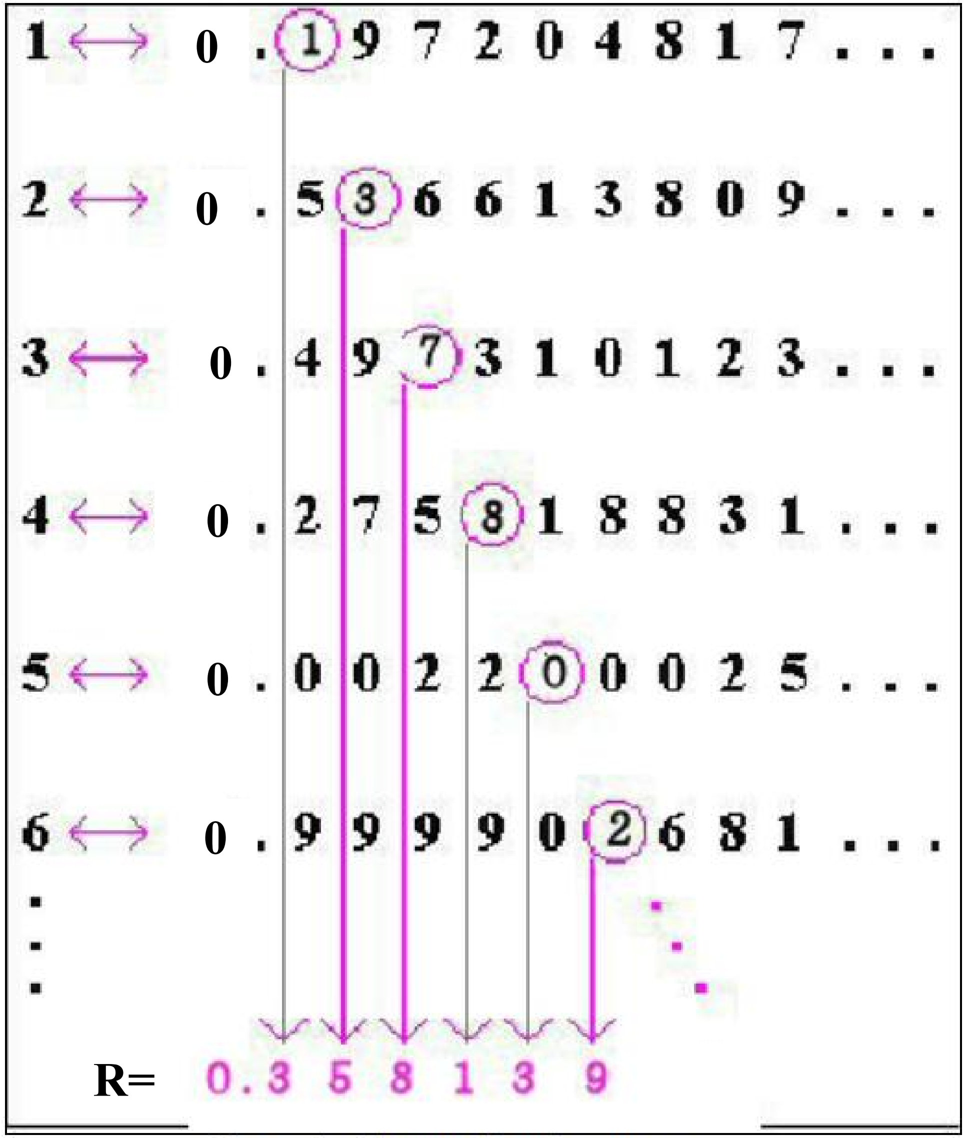
Cantor's Diagonalization Argument

- For contradiction, assume the set of reals was countable
- Since any subset of a countable set is also countable, this would imply the set of reals between 0 and 1 is also countable
- Now, if reals between 0 and 1 are countable, we can list them in the following way:

$$
\begin{array}{rclcccccc}
R_1 & = & 0. & [a_{11}] & a_{12} & a_{12} & \cdots & a_{1n} & \cdots \\
R_2 & = & 0. & a_{21} & [a_{22}] & a_{23} & \cdots & a_{2n} & \cdots \\
R_3 & = & 0. & a_{31} & a_{32} & [a_{33}] & \cdots & a_{3n} & \cdots \\
\vdots & & & & & & \ddots & \vdots & \\
R_n & = & 0. & a_{n1} & a_{n2} & a_{n3} & \cdots & [a_{nn}] & \cdots \\
\vdots & & & & & & & \vdots & \ddots
\end{array}
$$

1	$\longleftrightarrow$	0.197204817...
2	$\longleftrightarrow$	0.536613809...
3	$\longleftrightarrow$	0.497310123...
4	$\longleftrightarrow$	0.275818831...
5	$\longleftrightarrow$	0.002200025...
6	$\longleftrightarrow$	0.999902681...
⋮		⋮

Cantor's hypothesized one-to-one correspondence between the whole numbers and the real numbers.



Cantor's diagonalization process.

Diagonalization Argument, concluded

- Since R is not in the table, this is not a complete enumeration of all reals between 0 and 1
- Hence, the set of real between 0 and 1 is not countable
- Since the superset of any uncountable set is also uncountable, set of reals is uncountably infinite

$\square$

Summary:

Countable Infinite sets (cardinality $\aleph_0$): $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{N}\times\mathbb{N}$

Uncountable sets (cardinality c) $\mathbb{R}, [a,b], (a,b), \mathbb{R}^2, \mathbb{R}-\mathbb{Q}$

There are other sets the reader has seen in undergraduate mathematics. The n-dimensional Euclidean space $\mathbb{R}^n$ has cardinality c, the set of all sequences of real numbers has cardinality c, the set of continuous function defined on an interval has a *larger* cardinality.