CHAPTER 3

Chinese postman problem

Learning objectives

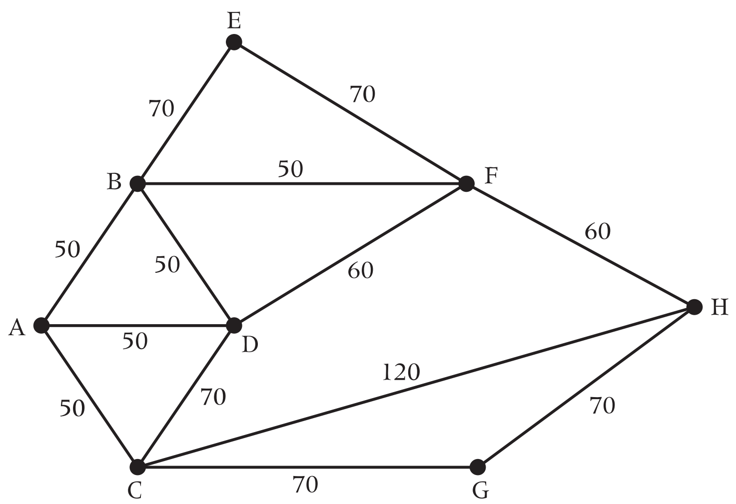
After studying this chapter, you should be able to:

- understand the Chinese postman problem
- apply an algorithm to solve the problem
- understand the importance of the order of vertices of graphs.

3.1 Introduction

In 1962, a Chinese mathematician called Kuan Mei-Ko was interested in a postman delivering mail to a number of streets such that the total distance walked by the postman was as short as possible. How could the postman ensure that the distance walked was a minimum?

In the following example a postman has to start at A, walk along all 13 streets and return to A. The numbers on each edge represent the length, in metres, of each street. The problem is to find a trail that uses all the edges of a graph with minimum length.

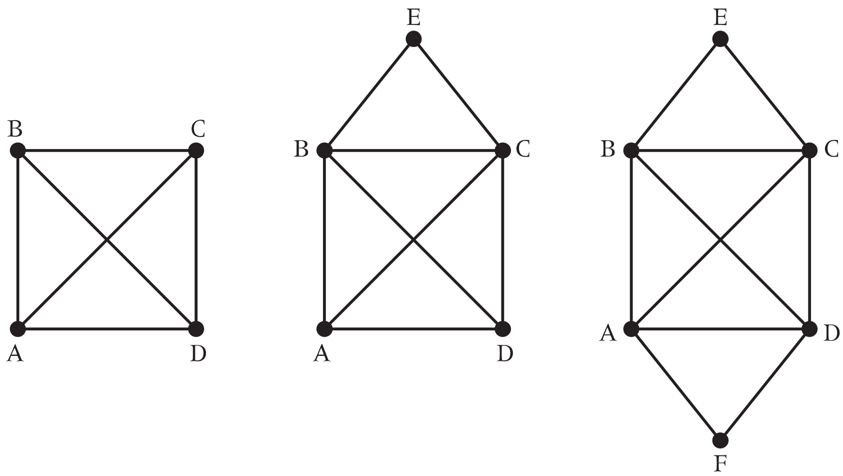


We will return to solving this actual problem later, but initially we will look at drawing various graphs.

3.2 Traversable graphs

A **traversable** graph is one that can be drawn without taking a pen from the paper and without retracing the same edge. In such a case the graph is said to have an Eulerian trail.

Eulerian trails are dealt with in detail in Chapter 5.



Graph 1 Graph 2 Graph 3

If we try drawing the three graphs shown above we find:

- it is impossible to draw Graph 1 without either taking the pen off the paper or re-tracing an edge
- we can draw Graph 2, but only by starting at either A or D – in each case the path will end at the other vertex of D or A
- Graph 3 can be drawn regardless of the starting position and you will always return to the start vertex.

What is the difference between the three graphs?
In order to establish the differences, we must consider the order of the vertices for each graph. We obtain the following:

Graph 1

Vertex	Order
A	3
B	3
C	3
D	3

Graph 2

Vertex	Order
A	3
B	4
C	4
D	3
E	2

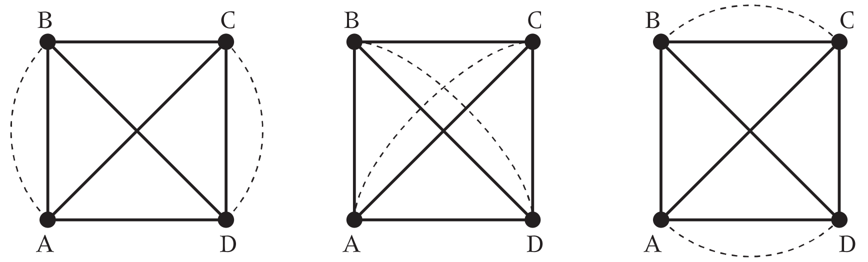
Graph 3

Vertex	Order
A	4
B	4
C	4
D	4
E	2
F	2

When the order of all the vertices is even, the graph is traversable and we can draw it. When there are two odd vertices we can draw the graph but the start and end vertices are different. When there are four odd vertices the graph can't be drawn without repeating an edge.

> An **Eulerian** trail uses all the edges of a graph. For a graph to be Eulerian all the vertices must be of even order.
>
> If a graph has two odd vertices then the graph is said to be **semi-Eulerian**. A trail can be drawn starting at one of the odd vertices and finishing at the other odd vertex.

To draw the graph with odd vertices, edges need to be repeated. To find such a trail we have to make the order of each vertex even. In graph 1 there are four vertices of odd order and we need to pair the vertices together by adding an extra edge to make the order of each vertex four. We can join AB and CD, or AC and BD, or AD and BC.

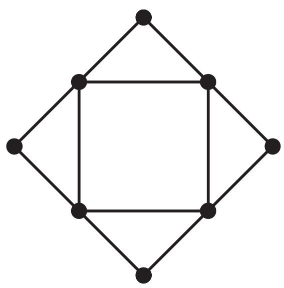


In each case the graph is now traversable.

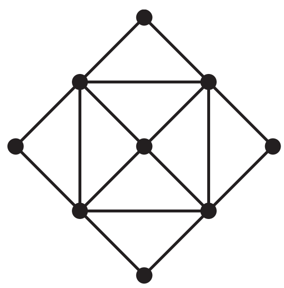
Worked example 3.1

Which of the graphs below is traversable?

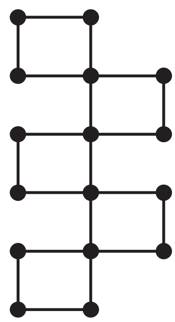
(a)



(b)



(c)



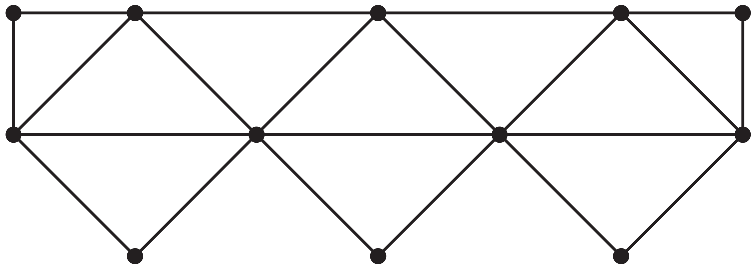
Solution

Graphs **(a)** and **(c)** are traversable as all the vertices are of even order. Graph **(b)** is not traversable as there are vertices of odd order.

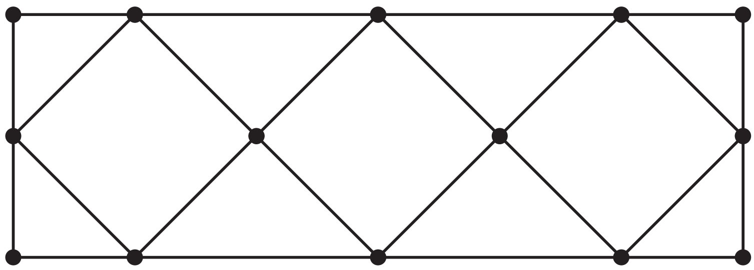
EXERCISE 3A

Which of the graphs below are traversable?

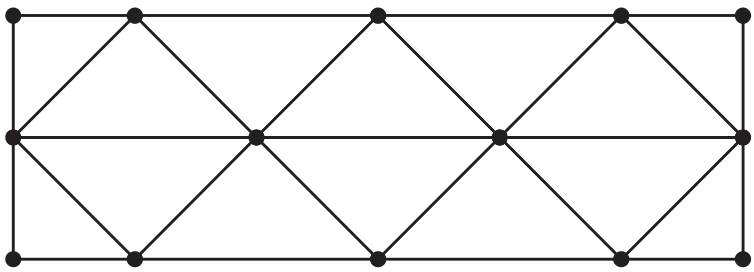
1



2



3



3.3 Pairing odd vertices

If there are two odd vertices there is only one way of pairing them together.
If there are four odd vertices there are three ways of pairing them together.
How many ways are there of pairing six or more odd vertices together?
If there are six odd vertices ABCDEF, then consider the vertex A. It can be paired with any of the other five vertices and still leave four odd vertices. We know that the four odd vertices can be paired in three ways; therefore the number of ways of pairing six odd vertices is $5 \times 3 \times 1 = 15$.

Similarly, if there are eight odd vertices ABCDEFGH, then consider the first odd vertex A. This could be paired with any of the remaining seven vertices and still leave six odd vertices. We know that the six odd vertices can be paired in 15 ways therefore the number of ways of pairing eight odd vertices is
$7 \times 5 \times 3 \times 1 = 105$ ways.
We can continue the process in the same way and the results are summarised in the following table.

Number of odd vertices	Number of possible pairings
2	1
4	$3 \times 1 = 3$
6	$5 \times 3 \times 1 = 15$
8	$7 \times 5 \times 3 \times 1 = 105$
10	$9 \times 7 \times 5 \times 3 \times 1 = 945$
n	$(n - 1) \times (n - 3) \times (n - 5) \ldots \times 1$

Exam questions will not be set where candidates will have to pair more than four odd vertices but students do need to be aware of the number of ways of pairing more than four odd vertices.

3.4 Chinese postman algorithm

To find a minimum Chinese postman route we must walk along each edge at least once and in addition we must also walk along the least pairings of odd vertices on one extra occasion.

An algorithm for finding an optimal Chinese postman route is:

Step 1 List all odd vertices.

Step 2 List all possible pairings of odd vertices.

Step 3 For each pairing find the edges that connect the vertices with the minimum weight.

Step 4 Find the pairings such that the sum of the weights is minimised.

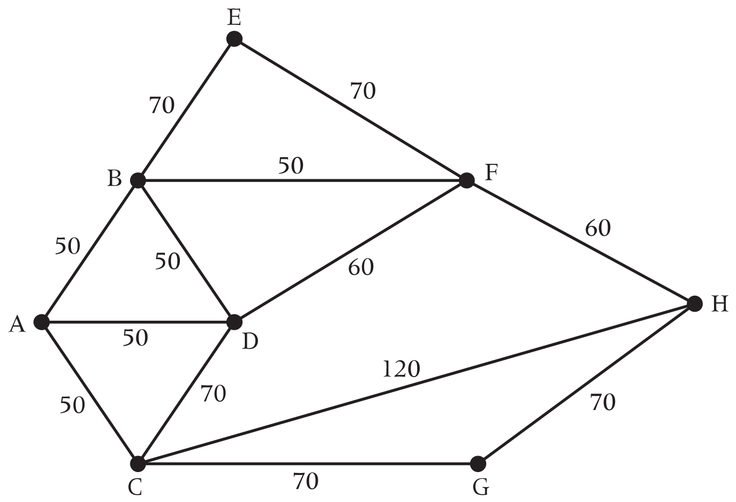
Step 5 On the original graph add the edges that have been found in Step 4.

Step 6 The length of an optimal Chinese postman route is the sum of all the edges added to the total found in Step 4.

Step 7 A route corresponding to this minimum weight can then be easily found.

Worked example 3.2

If we now apply the algorithm to the original problem:

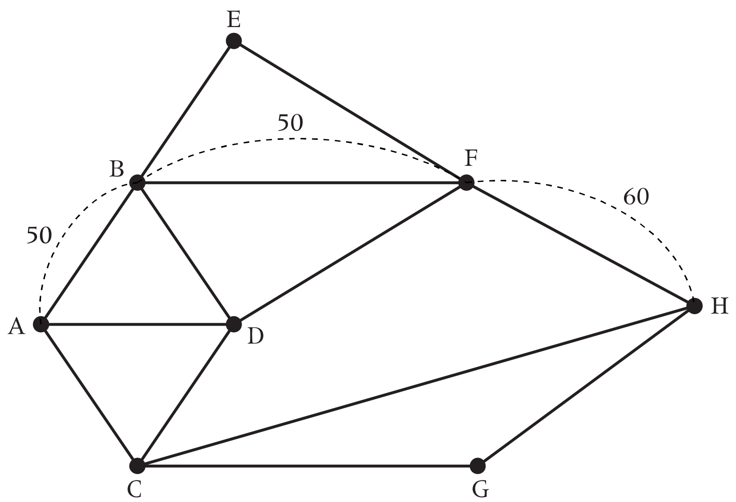


Step 1 The odd vertices are A and H.

Step 2 There is only one way of pairing these odd vertices, namely AH.

Step 3 The shortest way of joining A to H is using the path AB, BF, FH, a total length of 160.

Step 4 Draw these edges onto the original network.



Step 5 The length of the optimal Chinese postman route is the sum of all the edges in the original network, which is 840 m, plus the answer found in Step 4, which is 160 m. Hence the length of the optimal Chinese postman route is 1000 m.

Step 6 One possible route corresponding to this length is ADCGHCABDFBEFHFBA, but many other possible routes of the same minimum length can be found.