4
Prior information and dead-reckoning

With estimation problems we often have some prior information about the state being estimated. For example, if a robot starting at the origin moves at 3 m/s for 3 s, we would expect that it would move 9 m. Of course, something may go wrong. The batteries may go flat, the wheels may slip, it may crash into something, or it may be kidnapped. So there is always some uncertainty.

Prior information improves the estimate. In some situations, it simply can be fused with the measurements using a weighted average. The trick is to represent the unknown parameter, x, being estimated as a random variable X[1].

In probabilistic robotics, it is common to refer to X as the belief of the robot's state. Using a random variable allows the prior belief to be represented as a PDF.

[1] At the expense of some notational confusion!

4.1 Beliefs

Most of the time, beliefs are assumed to have a Gaussian distribution since this only requires two parameters: mean and variance. However, there are many other possible distributions.

4.1.1 Uniform distribution

If the state is equally likely between x_{min} and x_{max} (and zero outside) then the belief can be represented using a uniform random variable, with a PDF,

$$f_X(x) = \begin{cases} 0 & x < x_{\text{min}}, \\ \frac{1}{x_{\text{max}} - x_{\text{min}}} & x_{\text{min}} \leq x \leq x_{\text{max}}, \\ 0 & x > x_{\text{max}}. \end{cases} \tag{4.1}$$

This is shown in Figure 4.1.

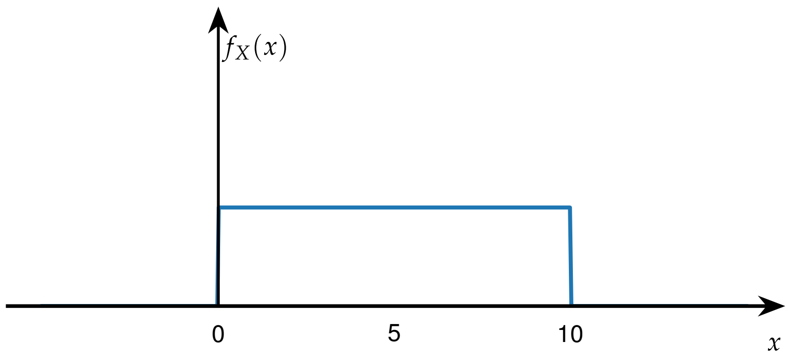

Figure 4.1: Uniform distribution, $\mathcal{U}(0, 10)$.

The shorthand notation that a random variable, X, is uniformly distributed between $x_{\min}$ and $x_{\max}$ is

$$X \sim \mathcal{U}\left(x_{\min}, x_{\max}\right). \tag{4.2}$$

The uniform distribution can be roughly approximated by a Gaussian PDF by only considering the first two moments: the mean and variance,

$$\mu_X = \frac{x_{\min} + x_{\max}}{2}, \tag{4.3}$$

$$\sigma_X^2 = \frac{\left(x_{\max} - x_{\min}\right)^2}{12}. \tag{4.4}$$

4.1.2 *Multiple hypotheses*

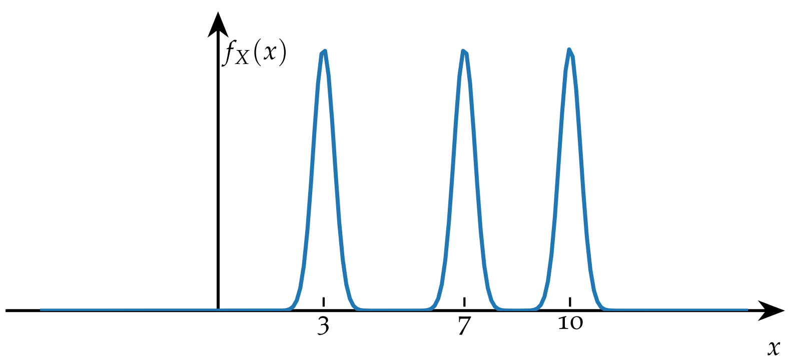

Figure 4.2: Multimodal distribution.

A belief with multiple hypotheses has as a mode for each hypothesis. For example, consider a robot passing through a door into a hallway with three doors. The position of the robot along the hallway is either one of the three doorways. One way of modelling multiple hypotheses is a weighted sum of Gaussian distributions.

4.2 *Dead reckoning*

When sensor measurements are unavailable[2], the belief of the state of a robot can be updated using *dead reckoning*. This requires a *motion model*[3] for the robot and an initial belief of the state. Unfortunately, errors accumulate with time and so the belief becomes less certain.

[2] Or when only relative measurements are available.

[3] Also known as a transition or system model.

4.2.1 *Linear motion model with additive noise*

A linear 1-D motion model can be used to predict the position of a robot constrained to a rail. Denoting the robot's unknown position at time-step n by X_n, and the speed commanded to the robot's motor by u_n, then

$$X_n = X_{n-1} + u_{n-1}\Delta t + W_n. \tag{4.5}$$

Here Δt is the duration of a time-step and W_n models the process noise, i.e., the uncertainty in the model due to wheel slippage or other disturbances.

In general, a linear 1-D motion model with additive zero-mean process noise has the form[4],

$$X_n = aX_{n-1} + bu_{n-1} + W_n, \tag{4.6}$$

where a and b are constants. This assumes a 1-D control u_n.

[4] Note, (4.5) can be seen to have the form of (4.6) with $a = 1$ and $b = \Delta t$.

4.2.2 *Propagation of uncertainty*

The motion model is a form of prediction and the effect of the process noise increases the uncertainty of the prediction. If X_{n-1} and W_n are independent, then

$$\mathrm{Var}\left[X_n\right] = a^2\,\mathrm{Var}\left[X_{n-1}\right] + \mathrm{Var}\left[W_n\right]. \tag{4.7}$$

Thus for every iteration of the model, the uncertainty increases[5]. This is depicted in Figure 4.3 to Figure 4.6.

[5] This is the disadvantage of dead-reckoning where measurements are not used.

4.2.3 *General motion models*

In general, a motion model with additive process noise has the form,

$$X_n = g(X_{n-1}, u_{n-1}) + W_n, \tag{4.8}$$

where g is a function of the previous state, X_{n-1}, and the known previous control, u_{n-1}, and W_n is the zero-mean random variable that represents the uncertainty in the motion model (the process noise). The motion model is found by fitting to a histogram obtained from many repeated measurements.

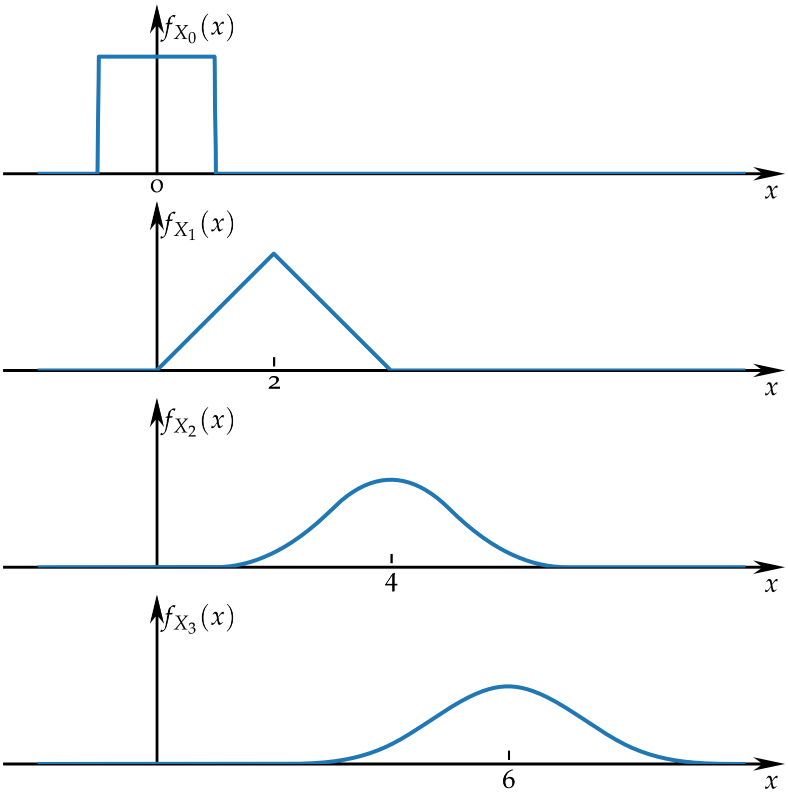

Figure 4.3: Propagation of state belief with no measurements: uniform initial belief distribution and linear motion model with additive uniform noise.

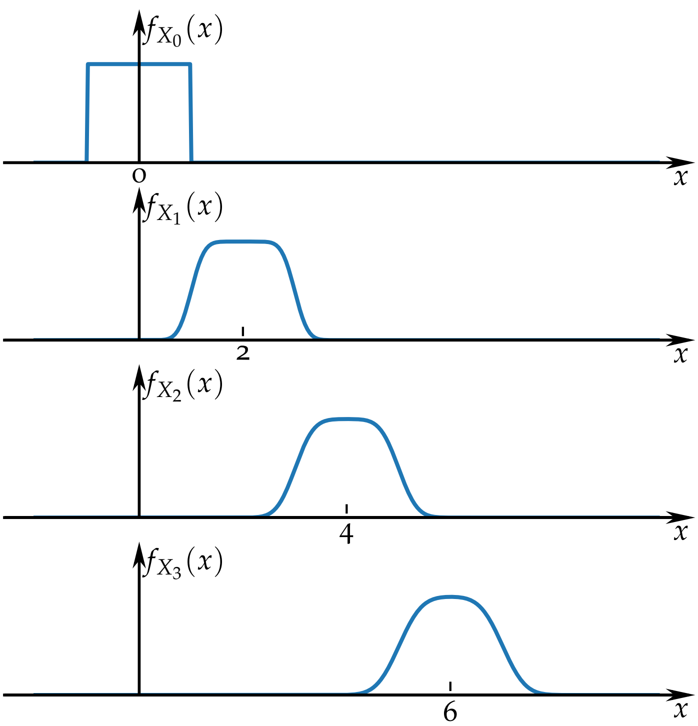

Figure 4.4: Propagation of state belief with no measurements: uniform initial belief distribution and linear motion model with additive Gaussian noise.

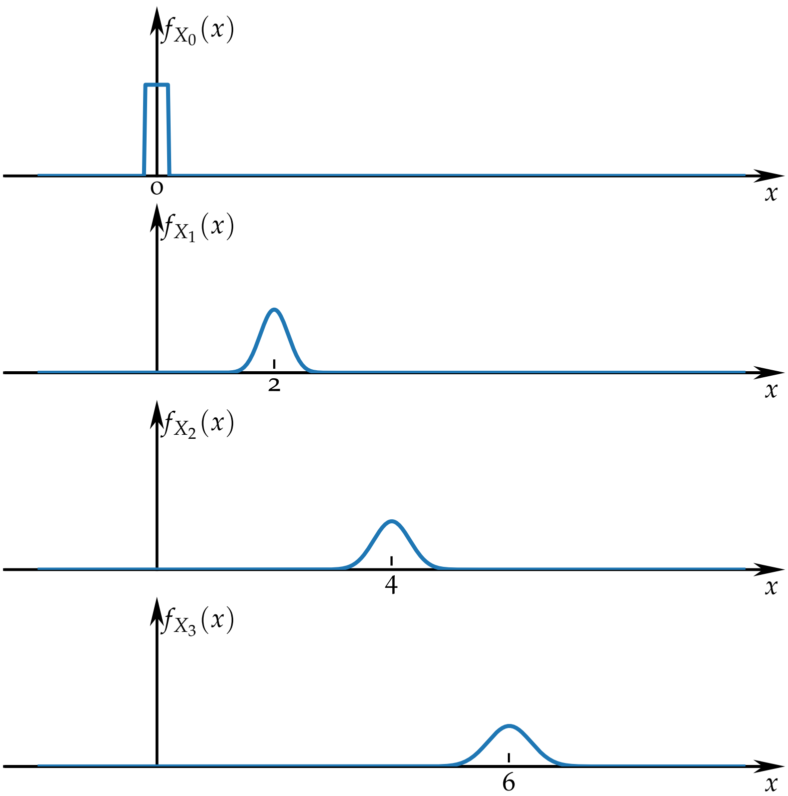

Figure 4.5: Propagation of state belief with no measurements: uniform initial belief distribution and linear motion model with additive Gaussian noise.

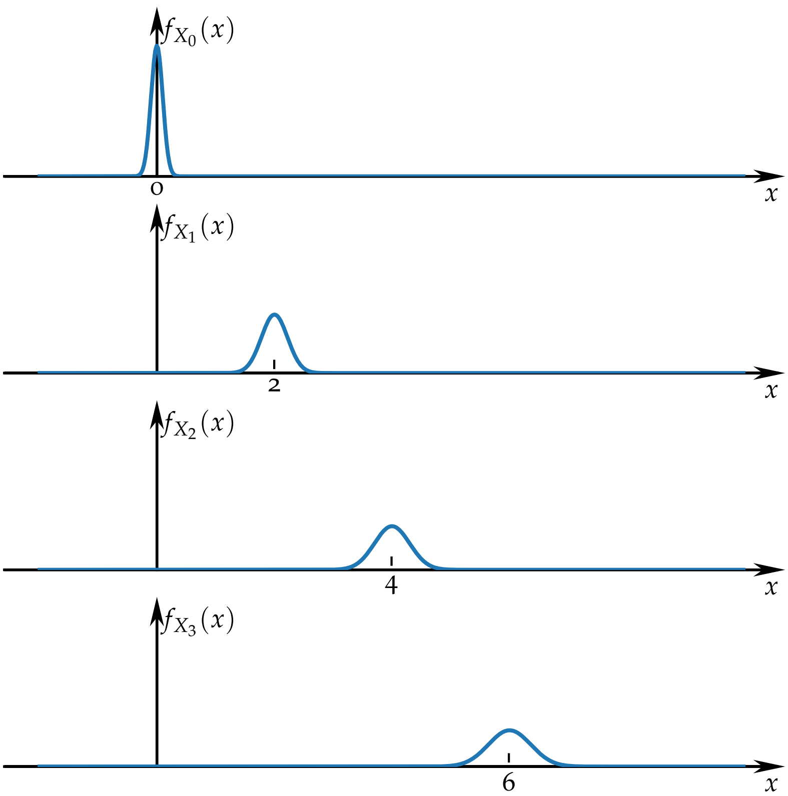

Figure 4.6: Propagation of state belief with no measurements: Gaussian initial belief distribution and linear motion model with additive Gaussian noise.

4.3 MLE and prior information

For the simple 1-D robot problem, the position of the robot's position can be inferred from the measurements. But we also have some prior information; the robot started at position 0 and was told to move 9 m. This prior information can be fused with the position estimated from the measurements using a weighted sum[6],

[6] This is the core of a Kalman filter.

$$X|Z = K\hat{X} + (1 - K)\,X, \tag{4.9}$$

where

X is the *prior belief*[7],

$\hat{X} = \hat{X}(Z)$ is the *measurement estimator*,

$X|Z$ is the *posterior belief*[8], and

K is a weighting factor.

[7] Before the measurement. From Latin *a priori*: from what is before.

[8] After the measurement. From Latin *a posteriori*: from later.

Note the weights sum to unity as required for an unbiased estimator. For a BLU estimator, the weights should be inversely proportional to the variances of the measured and prior estimators, so

$$K = \frac{\frac{1}{\mathrm{Var}[\hat{X}]}}{\frac{1}{\mathrm{Var}[\hat{X}]} + \frac{1}{\mathrm{Var}[X]}}. \tag{4.10}$$

The variance of the resulting estimate is

$$\mathrm{Var}\left[X|Z\right] = \frac{1}{\frac{1}{\mathrm{Var}[\hat{X}]} + \frac{1}{\mathrm{Var}[X]}}. \tag{4.11}$$

For example, from fusing the measurements,

$$\hat{x} = 10.4\,\mathrm{m}, \tag{4.12}$$

$$\mathrm{Var}\left[\hat{X}\right] = 0.8\,\mathrm{m}^2. \tag{4.13}$$

We also know that the robot was told to move 9 m so,

$$\mathrm{E}\left[X\right] = 9. \tag{4.14}$$

But how sure are we that the robot moved 9 m? Perhaps the wheels slipped? To handle this uncertainty, we need the variance of the prior. Let's say that

$$\mathrm{Var}\left[X\right] = 1.2\,\mathrm{m}^2, \tag{4.15}$$

and so from (4.10)

$$K = \frac{\frac{1}{0.8}}{\frac{1}{0.8} + \frac{1}{1.2}}, \tag{4.16}$$
$$= 0.6. \tag{4.17}$$

Now using (4.9),

$$\mathrm{E}\left[X|Z\right] = 0.6 \times 1.4 + (1 - 0.6) \times 9, \tag{4.18}$$
$$= 9.94\,\mathrm{m}, \tag{4.19}$$

with a variance calculated from (4.11),

$$\mathrm{Var}\left[X|Z\right] = \frac{1}{\frac{1}{0.8} + \frac{1}{1.2}}, \tag{4.20}$$
$$= 0.47\,\mathrm{m}^2. \tag{4.21}$$

Note, this is smaller than both the variances of the measured estimate and prior estimate.

4.4 Summary

1. The belief of a state is represented by a random variable X, with a PDF $f_X(x)$.
2. The belief of a state can be predicted with a motion model from a previous belief (dead-reckoning).
3. Dead-reckoning causes the variance of the belief to increase.
4. The PDF of the sum of two independent random variables is given by a convolution of their PDFs:

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\mathrm{d}x \quad X \text{ and } Y \text{ independent.} \tag{4.22}$$

5. The sum of two independent Gaussian random variables is a Gaussian random variable[9],

$$X + Y \sim \mathcal{N}\left(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2\right). \tag{4.23}$$

6. Under certain conditions, sensor fusion can include prior information using a weighted average.

[9] A Gaussian PDF convolved with a Gaussian PDF produces a Gaussian PDF with a larger variance.

4.5 Exercises

1. Sketch the PDF of a uniform distribution with mean 5 and range 4.
2. Sketch the PDF of a uniform distribution with mean 5 and range 8.
3. Determine the variance of a uniform distribution with mean 5 and range 4.
4. Sketch the PDF of a Gaussian distribution with mean 5 and variance 1.
5. Sketch the PDF of a Gaussian distribution with mean 5 and variance 4.
6. Consider the sum of two independent random variables X and Y. Sketch the PDF of the result if X a uniform distribution with mean 5 and range 4 and Y has a uniform distribution with mean 5 and range 8.
7. Consider the average of two independent random variables X and Y. Sketch the PDF of the result if X a uniform distribution with mean 5 and range 4 and Y has a uniform distribution with mean 5 and range 8.
8. Sketch the approximate PDF of the average of ten independent random variables, each with a uniform distribution of mean 5 and range 4.
9. If X is a random variable described by a Gaussian distribution with mean 5 and variance 4, sketch the PDF of the random variable $Z = X/5$.
10. A initial belief of a robot's position at $n = 0$ can be described by a uniform distribution with a mean of 1 and a range of 2. If the robot's speed is a constant 1 m/s, sketch the belief of the robot's position for four time-steps assuming no process noise.
11. A initial belief of a robot's position at $n = 0$ can be described by a uniform distribution with a mean of 1 and a range of 2. If the robot's speed is a constant 1 m/s, roughly sketch the belief of the robot's position for four time-steps assuming zero-mean additive Gaussian process noise with a standard deviation of 0.5 m/s.
12. Consider Figure 4.3 and determine the mean speed of the robot.

13. Consider Figure 4.3 and sketch the conditional PDF for the motion model.
14. Consider Figure 4.6. If the standard deviation of the initial belief, X_0, is 0.1 and the standard deviation of the process noise is 0.2, determine the variance for X_n.