1
Sensors

Sensor fusion is commonly used in mobile robotics to combine (fuse) measurements from different sensors. The resulting estimate has a lower variance than the individual measurements. An example is the fusing of measurements from an inertial measurement unit[1] (IMU) with a global positioning system (GPS). To understand how a better estimate is obtained, it is necessary to consider some probability and statistics, in particular, estimation theory.

[1] These contain accelerometers, gyroscopes, and magnetometers.

1.1 A little problem

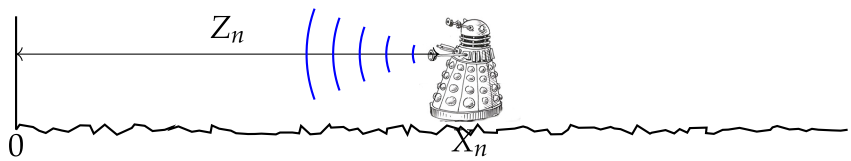

Figure 1.1: Robot position estimation problem.

Consider a robot that has been instructed to move[2] at a speed of 3 m/s for 3 s. When it has stopped, an ultrasonic range sensor says that it moved 8 m, however, a laser range finder says that it moved 11 m. So how do we decide how far the robot moved?

[2] With open-loop control.

1. Do we ignore the sensors and say that the robot moved 9 m since that is what we told it to do?
2. Do we choose the laser measurement of 11 m since laser range sensors are usually more accurate than ultrasonic range sensors?
3. Do we choose the ultrasonic measurement of 8 m since it is closest to what we expect?
4. Do we average the two measurements? If so, do we give each measurement equal weighting?

To find the best solution requires a little estimation theory, but first let's consider some sensor models.

1.2 *Sensor models*

Sensors produce a measurement, z, of an unknown state, x. If the sensor is *linear* and noiseless,

$$z = cx + d, \tag{1.1}$$

where c is a constant scale factor and d is a constant offset.

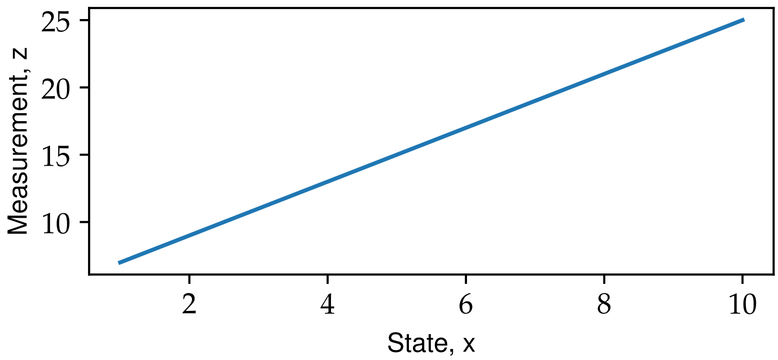

Figure 1.2: Example relationship of a linear sensor: $z = 2x + 5$.

Some sensors have a non-linear relationship between the state and measurement. For example, an IR triangulating range sensor has an approximate model

$$z = \frac{k_1}{k_2 + x}, \tag{1.2}$$

where k_1 and k_2 are constant parameters.

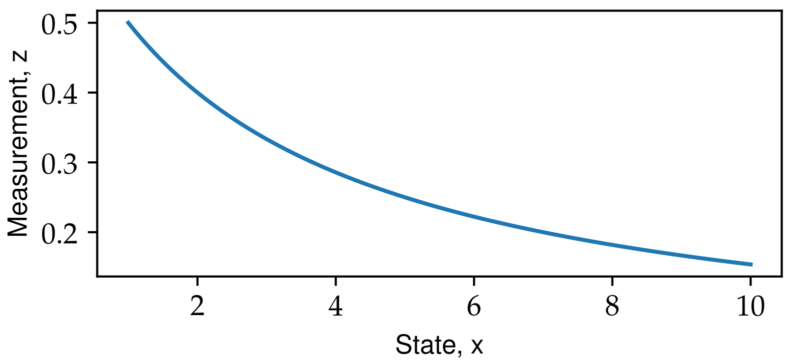

Figure 1.3: Example relationship of a non-linear sensor: $z = 2/(3 + x)$.

In general, a noiseless non-linear sensor be modelled as

$$z = h(x). \tag{1.3}$$

1.3 Uncertainty

All sensors have uncertainty due to electronic noise. This is modelled by considering the measurement as a random variable[3]. Usually, the noise is additive and thus a linear sensor can be modelled by

$$Z = cx + d + V, \tag{1.4}$$

where V is a random variable describing the additive noise.

[3] A random variable, Z, can be considered a placeholder for the outcome of a measurement. The actual measured value is denoted z.

1.3.1 Gaussian noise

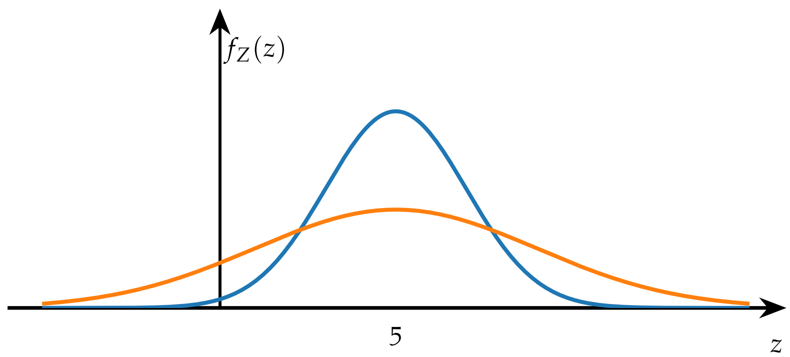

Figure 1.4: Gaussian distribution with mean, $\mu_V = \mathrm{E}[V] = 5$, and variance, $\sigma_V^2 = \mathrm{Var}[V] = 2$ and 4.

Random variables have an associated probability density function (PDF). This is denoted[4],

$$V \sim f_V(v). \tag{1.5}$$

The PDF of electronic noise is a Gaussian (normal) distribution[5], denoted

$$f_V(v) = \mathcal{N}\left(\mu_V, \sigma_V^2\right). \tag{1.6}$$

This has two parameters:

Mean $\mu_V = \mathrm{E}[V]$,

Variance $\sigma_V^2 = \mathrm{Var}[V]$.

An example of the Gaussian distribution is shown in Figure 1.4. In general, it is described by

$$f_V(x) = \frac{1}{\sqrt{2\pi\sigma_V^2}} \exp\left(-\frac{1}{2}\frac{(x-\mu_V)^2}{\sigma_V^2}\right). \tag{1.7}$$

All PDFs integrate to unity and so a Gaussian distribution with a smaller variance has a higher peak.

[4] The $\sim$ symbol is read "is distributed as" or "has a probability density".

[5] The Gaussian distribution has some interesting properties: the product of two Gaussians is a Gaussian and so is the convolution of two Gaussians.

1.3.2 *Linear sensor mean and variance*

The noise is usually zero mean, i.e.,

$$\mathrm{E}\left[V\right] = 0, \tag{1.8}$$

and so the expected measurement for a linear sensor is

$$\mathrm{E}\left[Z\right] = cx + d. \tag{1.9}$$

The noise has a variance

$$\mathrm{Var}\left[V\right] = \sigma_V^2, \tag{1.10}$$

and so the measurement variance for a linear sensor is

$$\mathrm{Var}\left[Z\right] = \mathrm{Var}\left[V\right] = \sigma_V^2. \tag{1.11}$$

1.4 *Properties of random variables*

1.4.1 *Linear transformation*

Consider a random variable X that is scaled by a factor a and offset by b,

$$Y = aX + b. \tag{1.12}$$

The result Y is another random variable. It has an expected value

$$\mathrm{E}\left[Y\right] = \mathrm{E}\left[cX + d\right] = c\,\mathrm{E}\left[X\right] + d, \tag{1.13}$$

and a variance

$$\mathrm{Var}\left[Y\right] = \mathrm{Var}\left[cX + d\right] = c^2\,\mathrm{Var}\left[X\right]. \tag{1.14}$$

Note, if X has a Gaussian PDF then Y also has a Gaussian PDF.

1.4.2 *Sum of two random variables*

The sum of two random variables is also a random variable,

$$Z = X + Y. \tag{1.15}$$

This has an expected value,

$$\mathrm{E}\left[Z\right] = \mathrm{E}\left[X + Y\right] = \mathrm{E}\left[X\right] + \mathrm{E}\left[Y\right], \tag{1.16}$$

and a variance

$$\mathrm{Var}\left[Z\right] = \mathrm{Var}\left[X + Y\right] = \mathrm{Var}\left[X\right] + \mathrm{Var}\left[Y\right] + \sqrt{\mathrm{Var}\left[X\right]\mathrm{Var}\left[Y\right]}\rho_{XY}. \tag{1.17}$$

The result depends on the correlation ρ_{XY} between X and Y. Usually, X and Y are independent and thus the correlation is zero. With this assumption,

$$\mathrm{Var}[Z] = \mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y]. \tag{1.18}$$

If X and Y are both Gaussian distributed, Z is also Gaussian distributed[6].

[6] In general, the PDF of Z is the convolution of the PDF of X with the PDF of Y.

1.4.3 *Average of two random variables*

The average of two random variables is

$$A = \frac{1}{2}(X+Y). \tag{1.19}$$

The expected value of the average is

$$\mathrm{E}[A] = \frac{1}{2}\mathrm{E}[X+Y] = \frac{1}{2}(\mathrm{E}[X] + \mathrm{E}[Y]), \tag{1.20}$$

and assuming X and Y are independent,

$$\mathrm{Var}[A] = \frac{1}{4}(\mathrm{Var}[X] + \mathrm{Var}[Y]). \tag{1.21}$$

Thus if X and Y have the same variance, the variance of the average is halved. This is an important result for improving sensor measurements.

1.5 *Averaging*

Consider N measurements from a linear sensor. These can be modelled by

$$Z_n = cx + d + V_n. \tag{1.22}$$

This assumes that the unknown state x is not changing. The average of the random process is

$$\overline{Z} = \frac{1}{N}\sum_{n=1}^{N} Z_n. \tag{1.23}$$

This is also a random variable.

If the parameters of the sensor noise are unchanging, the random noise process $\{V_n\}$ is identically distributed, i.e.,

$$\mathrm{E}[V_n] = \mathrm{E}[V], \tag{1.24}$$

$$\mathrm{Var}[V_n] = \sigma_V^2. \tag{1.25}$$

If each of the noise random variables are mutually independent, the random noise process is termed IID (independent and identically distributed). With this assumption, the expected value of the average is

$$\mathrm{E}\left[\overline{Z}\right] = cx + d, \tag{1.26}$$

and the variance is

$$\mathrm{Var}\left[\overline{Z}\right] = \frac{1}{N}\sigma_V^2. \tag{1.27}$$

Note, if the noise random process is Gaussian distributed, the average is also Gaussian distributed.

1.6 Other considerations

The modelling of sensors is not straightforward, since the variance of the measurement can vary with the state, the sensor can be ambiguous, the noise statistics can vary with time, and the sensor can produce erroneous results (outliers).

1.6.1 State varying variance

The variance of a range sensor increases with distance. This is because the returned signal becomes smaller[7], the farther it has to travel. This effect can be modelled by making the noise a function of the state, for example,

[7] Due to spreading and absorption effects.

$$Z = x + V(x). \tag{1.28}$$

This says that the PDF of the noise is parameterised by x, denoted $f_V(v; x)$.

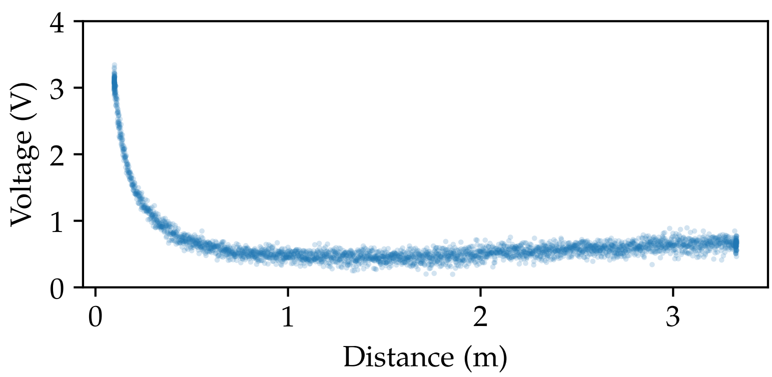

Figure 1.5: Example of range varying variance for triangulating IR sensor. This sensor gives an output voltage. Note, the non-linear relationship between distance and output voltage.

1.6.2 Ambiguities

Some sensors produce ambiguous measurements. We have all been fooled by optical illusions[8] and robots are no different. For example, consider an ultrasonic range sensor. This might produce an ambiguous measurement due to reflections from a wall. The resulting PDF is multimodal, see Figure 1.6.

[8] There are some interesting tactile and audio illusions.

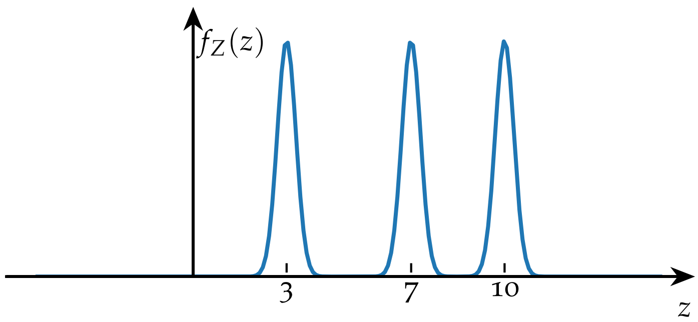

Figure 1.6: Multimodal distribution.

1.6.3 System identification

The parameters of the sensor model, including the parameters of the noise, can be found using system identification techniques. For example, $h(x)$ can be guessed and least squares employed to determine the parameters. Unfortunately, the result is biased by outliers and *robust regression*[9] techniques are required.

[9] For example, iteratively reweighted least squares (IRLS).

1.6.4 Outliers

Sensors can sometimes produce weird measurements, often due to electrical interference[10]. In this case, the simple additive Gaussian noise model is invalid. If the outliers are due to interference, then the statistics of the noise will vary with time.

[10] This might be from a robot's motors.

Outlier rejection is difficult. One approach uses Iglewicz and Hoaglin's modified Z-score:

$$M_i = 0.6745 \frac{(\epsilon_i - \text{median}(\epsilon))}{\text{median}(|\epsilon|)}, \tag{1.29}$$

where ϵ is a vector of errors. Values of ϵ_i where $|M_i| > 3.5$ are considered outliers. The effect of this is shown in Figure 1.8. The flaw with this approach is that the errors cannot be determined without a model!

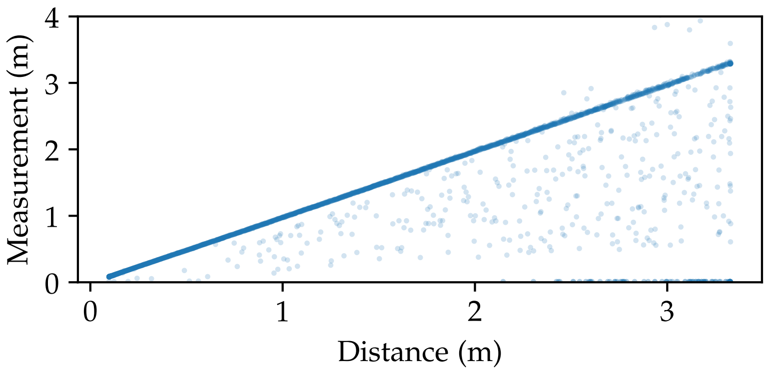

Figure 1.7: Ultrasonic range sensor measurements showing outliers. This sensor gives an output in metres.

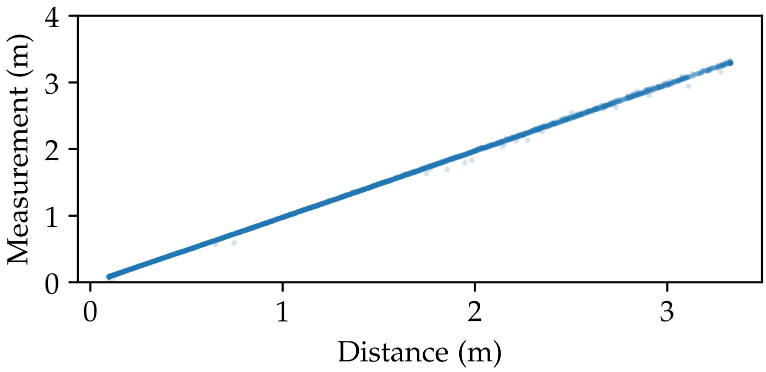

Figure 1.8: Ultrasonic range sensor measurements with outlier rejection.

1.7 Summary

1. Sensor measurements are corrupted by noise.

2. Sensor noise is modelled with random variables.

3. Continuous Random variables have probability density function (PDF).

4. Discrete Random variables have probability mass function (PMF).

5. The variance of a range sensor increases with range.

6. Averaging reduces the variance of sensor measurements.

7. Sensors can sometimes produce outliers.

8. Sensors can sometimes be ambiguous.

1.8 Exercises

1. An ultrasonic sensor with a variance of $0.01\,\mathrm{m}^2$ measures the range to a wall to be 1.2 m. A lidar with a variance $0.01\,\mathrm{m}^2$ measures the range to the wall to be 1.1 m.
   (a) If the measurements are averaged, what is the expected value?
   (b) If the measurements are averaged, what is the variance? State any assumptions.
2. An ultrasonic sensor with a variance of $0.04\,\mathrm{m}^2$ measures the range to a wall to be 1.2 m. A lidar with a variance $0.01\,\mathrm{m}^2$ measures the range to the wall to be 1.1 m.
   (a) If the measurements are averaged, what is the expected value?
   (b) If the measurements are averaged, what is the variance? State any assumptions.
   (c) Can you think of a better way of combining the measurements?
3. Consider two range sensors: one that samples at 1 Hz with a variance of $0.01\,\mathrm{m}^2$ and another that samples at 10 Hz with a variance of $0.05\,\mathrm{m}^2$. Which would you choose?
4. If X is random distance variable what are the units for its variance?
5. Emily's UFO has a altitude sensor where the output voltage can be modelled by a random variable Z, where
$$Z = 2x + 1 + V,$$
   for an unknown altitude x (in km) with additive zero-mean Gaussian random noise, represented by the random variable V with a standard deviation $\sigma_V = 2\,\mathrm{V}$.
   (a) If the expected value for the altitude is 1 km determine the expected value of Z.
   (b) Given that the altitude is 1 km determine the variance of Z.
6. Sam's UFO has an altitude sensor where the output voltage can be modelled by a random variable Z, where
$$Z = 3x^2 + V,$$

x is an unknown altitude (in km), and V is a zero-mean Gaussian random variable with a standard deviation $\sigma_V = 3\,\mathrm{V}$.

(a) If the expected value for the altitude is 1 km determine the expected value of Z.

(b) Given that the altitude is 1 km determine the variance of Z.

7. Evelyn's UFO has an ACME altitude sensor where the output voltage can be modelled by a random variable Z, where
$$Z = 2x + V(x),$$
x is an unknown altitude (in km), and V is a zero-mean Gaussian random variable with a standard deviation,
$$V(x) = 0.01x. \tag{1.30}$$

(a) If the expected value for the altitude is 1 km determine the expected value of Z.

(b) Given that the altitude is 1 km determine the variance of Z.