Numbers and Functions

1 Numbers

Much of calculus is based on properties of the real numbers. The real numbers can be represented geometrically as points on a number line, called the *real line*. The real numbers (or equivalently, the real line) is denoted by $\mathbb{R}$.

Other important sets of numbers we deal with are:

- The set of *natural numbers* is $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$.
- The set of *integers* is $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.
- The *rational numbers*, namely those numbers which can be written as the ratio $\dfrac{m}{n}$, $(n \neq 0)$ of two integers. Some examples are

$$\frac{1}{3}, \quad -\frac{2}{7} = \frac{2}{-7}, \quad 30 = \frac{30}{1}$$

The decimal expansion for rational numbers either:

(a) terminates (ends in all zeros), such as $\frac{3}{4} = 0.75000\ldots$, or

(b) eventually repeats in blocks, for example, $\frac{23}{11} = 2.09090909\ldots$

Note that the set of rational numbers has "gaps" which would cause problems if we tried to do calculus just with rational numbers. For example there is no rational number whose square is 2; there is a gap in the set of rationals where $\sqrt{2}$ should be.

Set notation is very useful for specifying particular subsets of the real numbers.

A *set* is a collection of objects or points, which are called the elements of the set. In calculus, these elements are usually numbers. If A is a set, the notation $a \in A$ means that a is an element of the set A and $a \notin A$ means that a is not an element of A.

If A and B are two sets, then $A \bigcup B$ is the set of all elements which belong to *either* A or B (or both) and is called the *union* of A and B. Similarly $A \bigcap B$ is the set of all elements which belong to *both* A and B and is called the *intersection* of A and B.

It is sometimes useful to define the *empty set* $\emptyset$ which is the set which contains no elements. So for example the intersection of the set of positive integers, $A = \{1, 2, 3, \ldots\}$ and the set of negative integers $B = \{-1, -2, -3, \ldots\}$ is the empty set and we can write $A \bigcap B = \emptyset$. If every element of S is an element of T, we say that S is a *subset* of T and write $S \subset T$. An example would be the case where $S = \{1, 2, 3\}$ and $T = \{1, 2, 3, 4\}$.

Some sets can be described by listing their elements. For example, the set S consisting of all natural numbers (or positive integers) less than 6 can be written as

$$S = \{1, 2, 3, 4, 5\}$$

Another way to describe a set is by a *rule* that describes the elements of the set. For example, the set S above can also be written as

$$S = \{x : x \text{ is an integer and } 0 < x < 6\}$$

Order properties

The order properties of the real numbers give rise to the following rules.

1. $a < b \Longrightarrow a + c < b + c$

2. $a < b$ and $c > 0 \Longrightarrow ac < bc$

3. $a < b$ and $c < 0 \Longrightarrow ac > bc$

4. $a > 0 \Longrightarrow \dfrac{1}{a} > 0$

5. If a and b are both positive or both negative, then $a < b \Longrightarrow \dfrac{1}{b} < \dfrac{1}{a}$.

Note carefully the rules for multiplying by a number. Multiplying by a positive number preserves the inequality; multiplying by a negative number reverses the inequality.

Intervals and interval notation

Certain sets of numbers, called *intervals*, occur frequently in calculus and correspond geometrically to line segments. The main types are:

- *Open Intervals:* If $a < b$ then the open interval from a to b consists of all numbers *between* a and b and is denoted by the symbol (a, b).

  In set notation we write
  $$(a, b) = \{x : a < x < b\}$$

- *Closed Intervals:* If $a < b$ then the closed interval from a to b consists of all numbers *between* and *including* a and b and is denoted by the symbol $[a, b]$.

  In set notation we write
  $$[a, b] = \{x : a \le x \le b\}$$

We can define *half open* and *half closed* intervals in a similar way.

Exercise 1 ***Set descriptions of intervals***

Write down a set description for the following intervals.

(a) $(a, b]$

(b) $[a, b)$

(c) (a, ∞)

(d) $(-\infty, b)$

(e) $(-\infty, b]$

(f) $(-\infty, \infty)$

Absolute values — solving inequalities

The *absolute value* of a number x is defined as

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases} .$$

So, for example, $|2| = 2, \quad |-2| = 2, \quad |0| = 0.$

Note that $|x|$ is always positive. Geometrically, $|x|$ is the distance from x to 0 on the real line. More generally, $|x - y|$ is the distance between the two point x and y. Can you see why? As a hint, take any two points x and y on the real line and slide them along (so that the distance between them doesn't change) until y becomes 0.

Also since the symbol $\sqrt{a}$ always means the positive square root, an alternative definition of $|x|$ is

$$|x| = \sqrt{x^2}$$

It is important to realise that $\sqrt{x^2} = |x|$. Don't write $\sqrt{x^2} = x$ unless you know that $x \geq 0$!

When dealing with equations or inequalities, the following statements are useful to know.

Suppose $a \geq 0$. Then

1. $|x| = a$ is the same as $x = \pm a$
2. $|x| < a$ is the same as $-a < x < a$
3. $|x| > a$ is the same as $x > a$ or $x < -a$
4. $|x| \leq a$ is the same as $-a \leq x \leq a$
5. $|x| \geq a$ is the same as $x \geq a$ or $x \leq -a$

Exercise 2 *Solving inequalities*

Solve the following inequalities and show the solution sets on the real line.

(a) $-\dfrac{x}{3} < 2x + 1$

(b) $\dfrac{6}{x-1} \geq 5$

Exercise 3 ***Solving Absolute Value inequalities***

(a) Solve the inequality $|-2x-3| > 5$. Write your answer using interval notation.

(b) Solve the inequality $\left|5 - \dfrac{2}{x}\right| < 1$.

2 Introduction to Functions

The fundamental objects that we deal with in calculus are functions. This section discusses the basic ideas about functions, their graphs, and ways of transforming and combining them.

Functions arise whenever one quantity depends on another. For example, the equation

$$R = \frac{100x}{b+x}, \quad x \geq 0$$

arises in biology and describes the response of the frog's muscle to acetylcholine (b is a positive constant which depends on the particular frog). Here, x is the *independent* variable, and since R varies with x, R is the *dependent* variable.

We can think of this as an input/output situation:

$$\text{an input } x \quad \longrightarrow \quad \boxed{\text{RULE}} \quad \longrightarrow \quad \text{an output } R$$

Exercise 4

Write down an expression describing the function $R = \dfrac{100x}{b+x}$ but where

(a) x is the independent variable and y is the dependent variable,

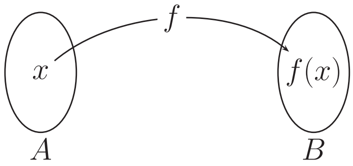
(b) z is the independent variable and h is the dependent variable,

(c) y is the independent variable and x is the dependent variable.

Moral: Don't get hung up on the notation! In this exercise, it is the SAME rule each time; we are just describing it with different variables.

More formally,

> A *function* f from a set A to a set B is a rule that assigns to each element x in A, exactly *one* element, called $f(x)$, in B.



In calculus we usually consider functions for which the sets A and B are sets of real numbers. The set A is the called the *domain* of the function. The set of all possible values of $f(x)$ as x varies in A is called the *range* of the function. It is a subset of B.

In the frog example, the domain is all real numbers greater than or equal to zero, that is, $x \geq 0$. This restriction on the domain also restricts the range to all reals greater than or equal to zero, that is we could take $B = [0, \infty)$.

When we define a function $y = f(x)$ by a formula and the domain is not stated specifically, the domain is assumed to be the largest set of real numbers for which the formula gives real values. This is the *natural domain* of the function. If we want to restrict the domain further, we must say so. The domain of $f(r) = \pi r^2$ is the set of all real numbers. In practice we might want to restrict the domain to positive values of r. If we want to be specific we would write something like $f(r) = \pi r^2,\ r > 0$.

Exercise 5 *Domain & range*

Find the natural domain and the range of each of the following functions.

(a) $f(x) = x^2$

(b) $f(x) = \dfrac{1}{x}$

(c) $f(x) = \sqrt{4 - x}$

(d) $f(x) = \sqrt{1 - x^2}$

Function notation

In our frog model, we say "x is mapped to $\dfrac{100x}{b+x}$", and write:

$$R: \ x \ \to \ \frac{100x}{b+x}, \quad x \geq 0$$

or more commonly,

$$R(x) \ = \ \frac{100x}{b+x}, \quad x \geq 0$$

Sometimes we will want to denote the output of a function by a single letter, and write:

$$y \ = \ R(x)$$

Exercise 6 ***Evaluating functions***

If $f(x) = 3x^2 - 2x - 4$, find

(a) $f(1)$

(b) $f(0)$

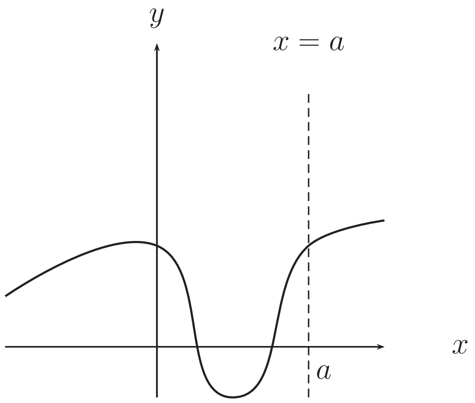
(c) $f(h)$

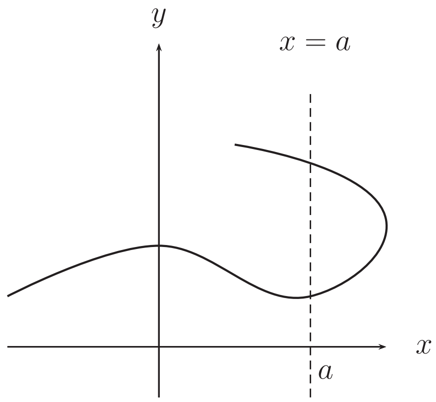
(d) $f(h^2)$

(e) $f(x+h)$

The Graph of a Function

The most common method for visualizing a function is its graph. If f is a function with domain A, then its *graph* is the set of ordered pairs $(x,\ f(x))$ for all x in A. (Notice that these are input/output pairs.) A curve in the xy-plane is the graph of a function if and only if a vertical line intersects the curve no more than once.





Exercise 7 *Visualising functions*

(a) Sketch the graph of the parabola $x = y^2 - 2$ and explain why this is not the graph of a function.

(b) Write down the equations of the upper and lower halves of this parabola and explain why they are functions.

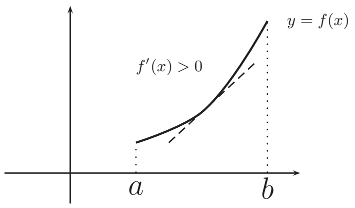
Increasing and Decreasing Functions

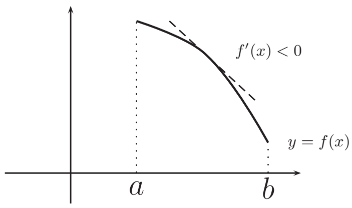
Let f be defined on an interval, and let x_1 and x_2 denote points in that interval.

- f is ***increasing*** on the interval if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$.
- f is ***decreasing*** on the interval if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.
- f is ***constant*** on the interval if $f(x_1) = f(x_2)$ for all points x_1 and x_2.

From school, you might remember that if $f'(x) > 0$, then f is increasing (the tangent lines to the graph of f have a positive slope).

If $f'(x) < 0$, then f is decreasing (tangent lines to the graph of f have a negative slope).





Exercise 8 *Increasing and decreasing functions*

(a) Give an example of a function that is increasing on its entire domain.

(b) Give an example of a function that is decreasing on its entire domain.

Even and odd functions

The function f is called *even* if $f(-x) = f(x)$ for every number x in the domain of f. Geometrically, its graph is symmetric about the y-axis.

The function f is called *odd* if $f(-x) = -f(x)$ for every number x in the domain of f. Geometrically, its graph is symmetric about the origin, that is, rotating the graph through 180° maps the graph back onto itself.

Exercise 9 ***Symmetries of functions***

Determine whether each of the following functions is even, odd or neither.

(a) $f(x) = x^3 + x$

(b) $g(x) = 1 - x^2$

(c) $h(x) = 2x - x^2$

Standard types of functions

- A function p is called a *polynomial* if it is an algebraic expression of the form

$$p(x) \;=\; a_n x^n \;+\; a_{n-1} x^{n-1} \;+\; \cdots \;+\; a_1 x \;+\; a_0$$

 where n is a non-negative integer and the *coefficients* a_n, $a_{n-1} \cdots a_1$, a_0 are constants.
 If the leading co-efficient $a_n \neq 0$ then the *degree* of the polynomial is n.

- A function of the form

$$f(x) \;=\; x^a$$

 where a is a constant, is called a *power function*.
 If $a = n$ where n is a positive integer then we get polynomials with only one term.
 If $a = \frac{1}{n}$ where n is a positive integer then we get a *root function*, $f(x) \;=\; \sqrt[n]{x}$.

- A function is a *rational function* if it is the ratio of two polynomials P and Q:

$$f(x) \;=\; \frac{P(x)}{Q(x)}\,, \qquad Q(x) \;\neq\; 0\,,$$

 A simple example is

$$f(x) \;=\; \frac{1}{x}$$

 whose domain is $\{x \;:\; x \neq 0\}$. (Also a power function with $a = -1$ if you like.)

- A function f is called an *algebraic function* if it can be constructed using algebraic operations, starting with polynomials. For example,

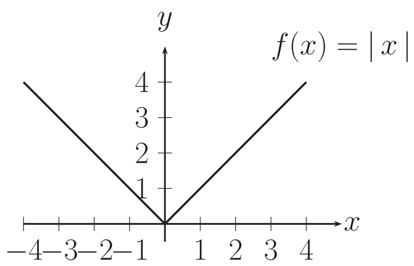
$$f(x) \;=\; \frac{x^3 - 2x}{x + \sqrt{x}} \;+\; (x-2)\,\sqrt[3]{x}$$

- A function f is called a *transcendental function* if it is not algebraic. Such functions include the exponential and logarithmic functions, trigonometric functions, and hyperbolic functions. These important functions will be covered later in this topic.

- The *absolute value function* is defined as

$$f(x) \;=\; |\,x\,| \;=\; \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

 This is an example of a function defined in a piecewise manner.



Exercise 10 *Classifying functions*

Classify the following functions as one of the types of functions we have discussed.

(a) $f(x) = 5^x$

(b) $g(x) = x^5$

(c) $h(x) = \dfrac{1+x}{1-\sqrt{x}}$.

(d) $u(t) = 1 - t + 5t^2$

Exercise 11 *Sketching functions*

Sketch the graph of $g(x) = -|x - 2| + 2$.

3 Operations on Functions

Basic operations on functions

Given the two functions, f and g, we define:

- the *sum/difference* of f and g as
$$(f \pm g)(x) \;=\; f(x) \;\pm\; g(x);$$
- the *product* of f and g as
$$(fg)(x) \;=\; f(x)\, g(x);$$
- the *quotient* of f and g as, where $g(x) \;\neq\; 0$,
$$\left(\frac{f}{g}\right)(x) \;=\; \frac{f(x)}{g(x)}.$$

Composition of Functions

A lot of situations in life can be modelled by a *composite* function where one function is *followed* by another to create a new function. For example, suppose that the contamination, C, in a lake is modelled by the function
$$C(p) \;=\; \sqrt{p} \qquad (*)$$
parts per million when the surrounding community population is p people.

Suppose also that the population of the community can be modelled by the function
$$p(t) \;=\; 1000(0.4t^2 + 2.5)$$
where this gives the number of people t years after 1980.

Then the level of contamination, C, is a function of the population, p, which is itself a function of time, t. So C can be thought of as a function of t.

$$t \xrightarrow{\;p\;} p(t) \xrightarrow{\;C\;} C(p(t))$$

Substituting the expression for p in terms of t into $(*)$ gives the composite function that models the contamination in the lake t years after 1980,
$$C(p(t)) \;=\; \sqrt{(1000(0.4t^2 + 2.5))}$$

In general, given two function f and g, we start with a number x in the domain of g and find its image, $g(x)$. If this image is in the domain of f then we can calculate the value $f(g(x))$.

Given two functions f and g, the composite function $f \circ g$ is defined by $$(f \circ g)(x) \;=\; f(g(x)).$$ Note that in general, $f(g(x)) \neq g(f(x))$.

Exercise 12 *Compositions of functions*

If $f(x) = x^3$ and $g(x) = x - 2$, find

(a) $(f + g)(x)$

(b) $(f - g)(x)$

(c) $(fg)(x)$

(d) $\left(\frac{f}{g}\right)(x)$

(e) $(f \circ g)(2)$

(f) $(g \circ f)(2)$

(g) $(f \circ g)(x)$

(h) $(g \circ f)(x)$

4 Functions and their inverses

Intuitively, an inverse function "reverses" the effect of a given function.

$$\begin{tikzcd} x = g(y) \arrow[rr, bend left, "f"] & & y = f(x) \arrow[ll, bend left, "g"] \end{tikzcd}$$

If the function f maps x to y, then the inverse function g takes y back to x.

This means that the *domain* of a function becomes the *range* of its inverse, and the *range* of a function becomes the *domain* of its inverse.

Starting at x in the above diagram and moving (clockwise) around the diagram gives

$$g(f(x)) \;=\; x \quad \text{for all } x \text{ in the domain of } f\,.$$

Similarly, starting at y, gives

$$f(g(y)) \;=\; y \quad \text{for all } y \text{ in the domain of } g\,.$$

Example One: The function

$$y \;=\; f(x) \;=\; \frac{5}{9}(x \,-\, 32)$$

converts degrees Fahrenheit into Celsius.

For example, f maps $x = 32$ to $y = f(32) = 0$. The inverse takes 0 back to 32.

$$\begin{tikzcd} x = 32 \arrow[rr, bend left, "f"] & & y = 0 \arrow[ll, bend left, "g"] \end{tikzcd}$$

The inverse function, for converting degrees Celsius into degrees Fahrenheit, is found by unraveling the given equation to get x in terms of y.

$$x \;=\; g(y) \;=\; \frac{9}{5}y \,+\, 32\,.$$

where we are now thinking of x as a function of y.

The convention (and the textbook by Anton nearly always follows this) is to write the inverse function as a function of x, so here we would write:

$$f^{-1}(x) \;=\; \frac{9}{5}x \,+\, 32\,.$$

This is no different from what you did at school when you swapped x and y in the original equation and then solved for y in terms of x.

Example Two: Consider the function $y = f(x) = x^3$.

The inverse function is $x = y^{\frac{1}{3}}$, or writing this as a function of x,

$$f^{-1}(x) \;=\; x^{\frac{1}{3}}.$$

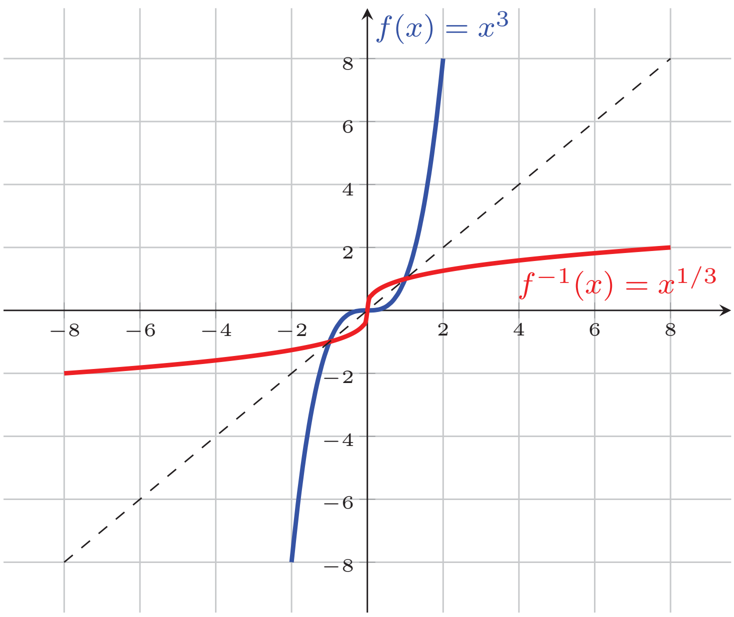
We can sketch the graph of f and its inverse f^{-1} (both as functions of x) by swapping the x and y values of key points on f to get points on f^{-1}. For example, (2, 8) on the graph of f becomes (8, 2) on the graph of f^{-1}.

Note also that $f(x) = x^3$ is an *increasing* function for all $x \neq 0$, since

$$f'(x) \;=\; 3x^2 \;>\; 0 \quad \text{for all } x \neq 0.$$

(The function is actually stationary at $x = 0$.)

So $f(x) = x^3$ maps *different* values of x to *different* y values, and is hence *one-to-one*.



The graphs of f and f^{-1} are reflections of each other in the line $y = x$.

Example Three: Consider the function $y = f(x) = x^2$.

This function maps both -2 and 2 to the value 4.

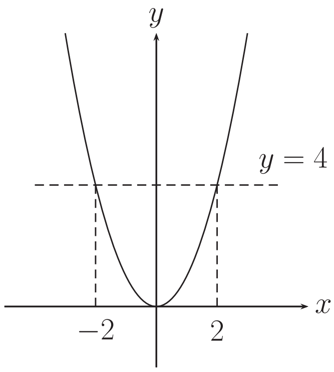
If this function had an inverse, say g, then we would require that g would map 4 to *both* -2 and 2. But since functions can only have *one* output value, g would not be a function.

$$\begin{tikzcd} x = 2, -2 \arrow[r, bend left, "f = x^2"] & y = 4 \arrow[l, bend left, "g = ??"] \end{tikzcd}$$

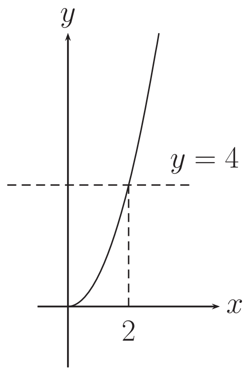
A way to deal with this is as follows. Solving $y = x^2$ for x in terms of y gives $x = \pm\sqrt{y}$.

This is not a function, but $x = +\sqrt{y}$ is. We need to restrict the domain of $f(x)$ to $x \geq 0$.

Geometrically, since every y value ($\neq 0$) has two corresponding x values, a horizontal line cuts the graph of $f(x) = x^2$ at two points. $f(x) = x^2$ is a function but it is *not a one-to-one* function. Restricting the domain to $x \geq 0$ means that $f(x) = x^2$ is now increasing and will be one-to-one *on this interval*!



$f(x) = x^2$



$f(x) = x^2,\ x \geq 0$

That is,

$$f(x) = x^2, \text{ where } x \geq 0, \quad \text{and} \quad g(y) = \sqrt{y}$$

will now be inverse functions.

(Note that we could have also chosen to restrict the domain to $x \leq 0$.)

Summarising the Important Points

- The domain of a function is the range of its inverse.
  The range of a function is the domain of its inverse.

- For a function f to have an inverse, f must map *different* values x_1 and x_2 to *different* values. That is,
  $$\text{if} \quad x_1 \neq x_2 \quad \text{then} \quad f(x_1) \neq f(x_2)\,.$$
  Such functions are called one-to-one. Geometrically this means that any horizontal line cuts the graph of $y = f(x)$ in at most one point.

- The domain of a function is very important. There are many functions which are not invertible *but which become so when the domain is suitably restricted*.

- Since an increasing or decreasing function is cut at most once by any horizontal line, such a function is one-to-one, and so has an inverse.

Exercise 13 ***Inverse Functions***

Given $g(x) = \sqrt{x-3}$, find $g^{-1}(x)$. Find $\left(g^{-1} \circ g\right)(x)$. Is this what you expect?
Sketch both functions and verify that g and g^{-1} are reflections of each other in the line $y = x$.

5 Special Functions

The exponential and logarithmic functions

An exponential function is a function of the form

$$f(x) = b^x, \qquad \text{where } b \text{ is a positive constant.}$$

Note that the function $f(x) = b^x$ is called an *exponential* function because the variable x is in the exponent. This should not be confused with the power function $g(x) = x^a$ where the variable is the base.

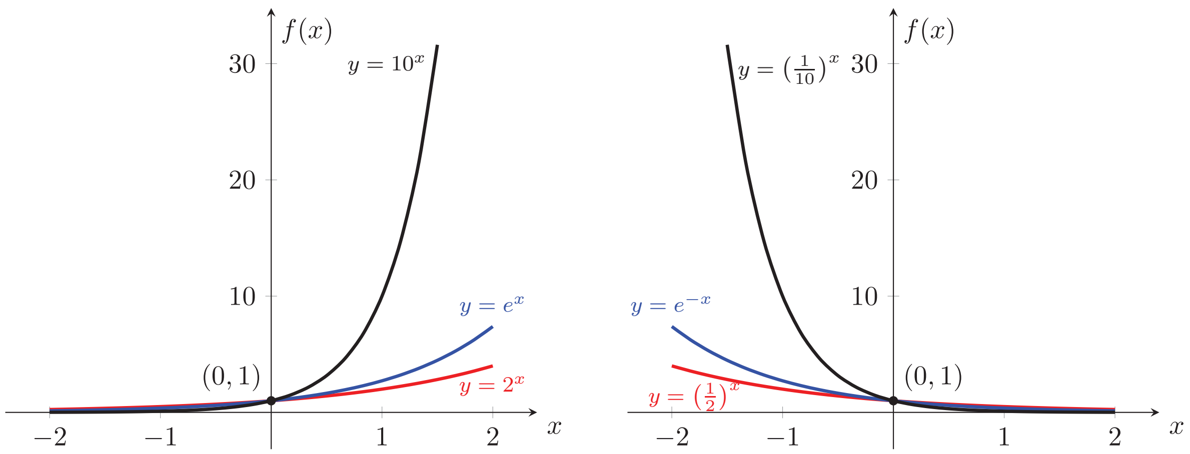
Some examples of exponential functions are

$$f(x) = 2^x, \qquad f(x) = \left(\frac{1}{10}\right)^x, \qquad f(x) = e^x.$$

The exponential function occurs in mathematical models in biology, social sciences, physics, chemistry, engineering, and economics. One key property of this function is that the growth rate of the function (that is, its derivative) is directly proportional to the value of the function. When we cover differentiation, we will see that the constant of proportionality of this relationship is the natural logarithm of the base of the function b.

If $x = 0$, then $f(0) = b^0 = 1$ and so the graphs of the family of exponential functions of the form $f(x) = b^x$ all go through the point $(0, 1)$.

- If $b > 1$, then the function increases as x increases (exponential growth).
- If $0 < b < 1$, then the function decreases as x increases (exponential decay).
- If $b = 1$, then the value of b^x is a constant, and the function is constant.



(a) $b > 1$ (b) $0 < b < 1$

Exercise 14 *Domain and Range*

Give the domain and range for the family of exponential functions $y = b^x$ for $b > 0, \quad b \neq 1$. Write your answer in interval notation.

The Natural Exponential Function

Among all the possible bases b for exponential functions, one particular base is very important in calculus.

This base is denoted by the letter e and is an irrational number whose value to 11 decimal places is

$$e \approx 2.71828182846...$$

This famous mathematical constant was identified in the 17th century. The first direct record we have of e is given by Jacob Bernoulli when he was investigating the limit

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n .$$

Of all the possible exponential functions $y = b^x$, the function $f(x) = e^x$ is the only one for which the slope of the tangent line to the curve $y = e^x$ at any point (x, y) is equal to the y-coordinate at that point (that is, the derivative of e^x is the exact same function, e^x).

The Logarithmic Function

If $b > 0$ and $b \neq 1$, the exponential function is either always increasing, or always decreasing, and so it is a one-to-one function (check this with the horizontal line test). Therefore it has an inverse function, which is called the **logarithmic function with base b**.

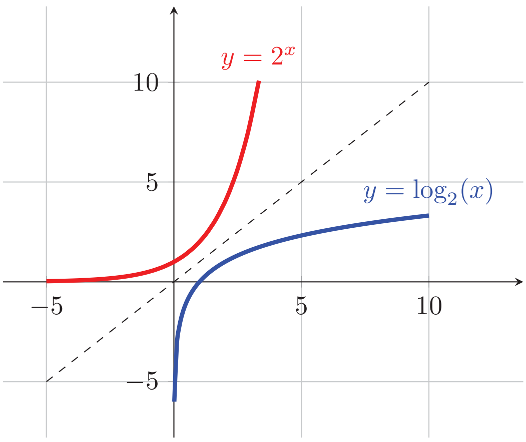
Recall that the logarithmic function is related to exponents as follows:

$$\log_b x = y \quad \Leftrightarrow \quad b^y = x.$$

Therefore we can solve the equation $x = b^y$ for y to obtain the inverse function $y = \log_b x$.
Hence

$$\boxed{\text{Let } f(x) = b^x \text{ for } b > 0 \text{ and } b \neq 1. \text{ Then } f^{-1}(x) = \log_b x.}$$

It follows from this result that the graphs of $y = b^x$ and $y = \log_b x$ are reflections of one another about the line $y = x$. Notice that the logarithmic function is an increasing function.



Example for base $b = 2$

Exercise 15 ***Domain and Range of Log Function***

Give the domain and range for the family of logarithmic functions $y = \log_b x$ for $b > 0, \;\; b \neq 1$.
Write your answer in interval notation.

Key Properties of Logarithms

If $b > 0$, $b \neq 1$, $x > 0$, $y > 0$, and n is any real number, then

(a) $\log_b(xy) = \log_b x + \log_b y$

(b) $\log_b\left(\dfrac{x}{y}\right) = \log_b x - \log_b y$

(c) $\log_b(x^n) = n \log_b x$

(d) $\log_b\left(\dfrac{1}{y}\right) = -\log_b y$

Exercise 16 ***Log Properties***

(a) Rewrite the following expression as a single logarithm.

$5\log 2 + \log 3 - \log 8$

(b) Expand the following logarithm in terms of the sums, differences, and multiples of simpler logarithms.

$\log \dfrac{xy^5}{\sqrt{z}}$

The Natural Logarithm Function

The most important logarithms in applications are those with base e. These are called ***natural logarithms*** because the function $f(x) = \log_e x$ is the inverse of the natural exponential function e^x.

We usually denote the natural logarithm function as $f(x) = \ln x$ (read "ell en" of x). The first letter of ln is a lowercase "ell" and NOT an uppercase "eye" (I).

$$y = \ln x \quad \Leftrightarrow \quad x = e^y.$$

Exercise 17 ***Natural logarithm properties***

Evaluate the following expressions.

(a) $\ln 1$

(b) $\ln e$

(c) $\ln\left(\frac{1}{e}\right)$

(d) $\ln\left(e^2\right)$

The trigonometric functions and their inverses

Exercise 18 *Graphs of the Sine and Cosine Functions*

Sketch the graphs of $f(x) = \sin x$ and $g(x) = \cos x$ for $-2\pi \leq x \leq 2\pi$. Comment on the amplitude, period, and symmetry of these functions.

Comments:

- The zeros of the sine function occur at integer multiples of π, that is,

$$\sin x = 0 \quad \text{whenever} \;\; x = n\pi, \;\; n \text{ an integer}$$

- The graph of the cosine function can be obtained by shifting the graph of the sine function by $\pi/2$ units to the left, that is,

$$\cos x \;=\; \sin\left(x + \frac{\pi}{2}\right).$$

- For both the sine and cosine functions, the domain is $(-\infty, \infty)$ and the range is the closed interval $[-1, 1]$. Therefore, for all values of x, we have

$$-1 \leq \sin x \leq 1 \qquad \text{and} \qquad -1 \leq \cos x \leq 1$$

- The reciprocal trigonometric functions are

$$\csc x = \frac{1}{\sin x} \qquad \sec x = \frac{1}{\cos x} \qquad \cot x = \frac{1}{\tan x}.$$

The tangent and cotangent functions have period π and the other four trig functions have period 2π.

The Inverse Trigonometric Functions

We will only consider the inverse sine and cosine functions here. The properties of the inverse tangent function can be developed in a similar way.

Inverse Sine

The function $y = \sin x$ is not one-to-one on its domain (the real numbers). But it is one-to-one on the interval $[-\pi/2,\, \pi/2]$, and so with this restriction we can define the inverse function.

We denote this inverse function by $x = \sin^{-1} y$ (or $\arcsin(y)$) and so

$$\boxed{x \;=\; \sin^{-1} y \quad \text{if and only if} \quad y \;=\; \sin x \quad \text{where} \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}\,.}$$

Comments:

- Because the range of $\sin x$ is $[-1\,,1]$ and its domain has been restricted to $[-\pi/2,\, \pi/2]$, the domain of $\sin^{-1} y$ is $[-1,\, 1]$ and its range is $[-\pi/2,\, \pi/2]$.
- The mathematical notation for the inverse function is $\sin^{-1} x$.
  It has nothing to do with the reciprocal function $\dfrac{1}{\sin x} \;=\; \csc x\,.$

Exercise 19 ***The graph of sine and its inverse function***

Sketch the graph of $y = \sin^{-1} x$ by reflecting $y = \sin x$ in the line $y = x$.

Exercise 20

Evaluate $\sin^{-1}(\frac{1}{2})$.

Inverse Cosine

If we restrict the domain to $[0, \pi]$ then $y = \cos x$ is 1-1 on this interval and so will have an inverse. We denote this inverse function by $x = \cos^{-1} y$ (or $\arccos(y)$). So we have

$$\boxed{x = \cos^{-1} y \quad \text{if and only if} \quad y = \cos x \quad \text{where } 0 \le x \le \pi\,.}$$

Exercise 21 ***The graph of cosine and its inverse function***

(a) What is the domain and range of the inverse cosine function?

(b) Draw the graph of the inverse cosine function (as a function of x).

Exercise 22

Simplify the expression $\tan(\cos^{-1} x)$.

6 An introduction to limits

The concept of a limit is very important in many branches of mathematics. Intuitively it is quite a straightforward concept; a value which is some kind of boundary. For example, we use the phrases: speed limits; being "pushed to the limit".

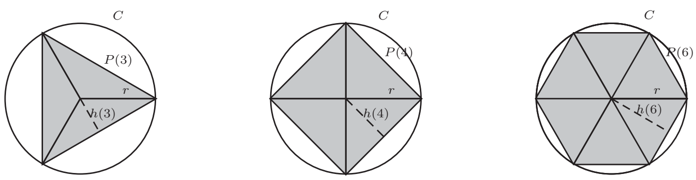
In mathematics it is similar. We talk about limiting values of processes and functions.

Example 23 *A limiting process*

Suppose we divide a circle with radius r, circumference C, and area A, into n equal segments. Consider the n-sided polygon inside the circle.

Find the limiting value, as n increases, of the area of the n segments $S(n)$, the perimeter of the n-sided polygon $P(n)$, and the perpendicular height of each segment $h(n)$.

Here are the diagrams for three, four and six segments.



(The shaded areas represent $S(3)$, $S(4)$ and $S(6)$ respectively.)

As the number of segments increase, the polygon increasingly "looks like" a circle.

- The total area of the segments gets close to the area of the circle.
  That is, $S(n) \to A$ as $n \to \infty$.

- The perimeter of the polygon gets close to the circumference of the circle.
  That is, $P(n) \to C$ as $n \to \infty$.

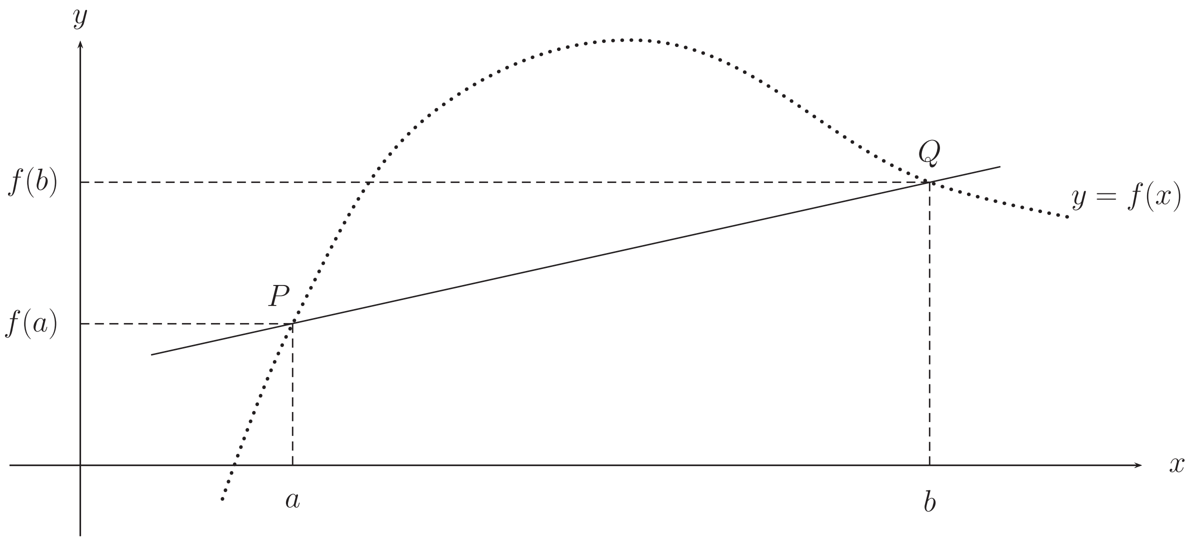
- The perpendicular height of each segment gets close to the radius of the circle.
  That is, $h(n) \to r$ as $n \to \infty$.

We use the following notation to describe these limiting processes.

$$\lim_{n\to\infty} S(n) = A, \qquad \lim_{n\to\infty} P(n) = C, \qquad \lim_{n\to\infty} h(n) = r\,.$$

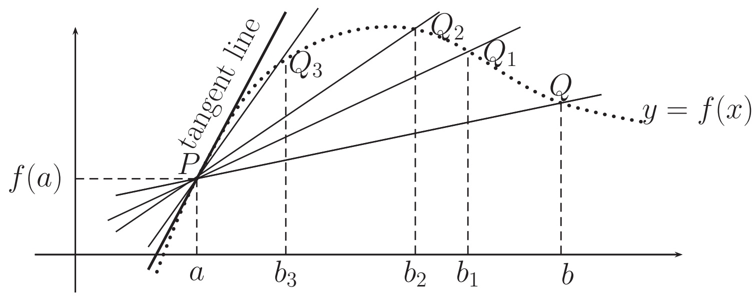
Exercise 24 ***Finding tangents***

(a) Consider a function $y = f(x)$. Find a formula for the slope of the secant line between the points $P = (a,\ f(a))$ and $Q = (b,\ f(b))$, where $b = a + h$.



(b) If we start shrinking h, that is, let $h \to 0$ so that $b \to a$, then the secant line through P and the successive Q_i's tends to the line touching the curve at point P, that is, to the *tangent* to the curve at P.

Find an expression for the slope of the tangent line at P.



Exercise 25 *Finding limits*

Consider the function $f(x) = \dfrac{1-\sqrt{x}}{1-x}$.

Explore what value $f(x)$ tends to, as x gets close to 1 from the left and the right, using the following approaches .

(a) Numerically (plug in some numbers close to $x = 1$).

(b) Geometrically (consider the graph of f).

(c) Algebraically (quite hard, but we will do more examples like this soon).

Limit notation

If the value of $f(x)$ approaches the number L_1 as x approaches a from the right, we write

$$\lim_{x \to a^+} f(x) = L_1$$

and say the limit from the right of $f(x)$ at a is L_1.

Similarly, if $f(x)$ approaches the number L_2 as x approaches a from the left, we write

$$\lim_{x \to a^-} f(x) = L_2$$

and say the limit from the left of $f(x)$ at a is L_2.

If the two one-sided limits both exist at a and are equal, then we define the double-sided limit at a to be their common value. That is, if

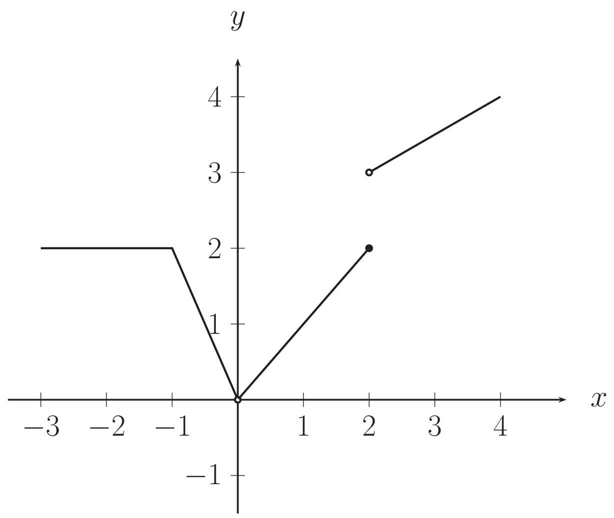
$$\lim_{x \to a^+} f(x) = L = \lim_{x \to a^-} f(x) \quad \text{then} \quad \lim_{x \to a} f(x) = L\,.$$

Note carefully that we are not saying *anything* about $f(a)$, only about the behaviour of $f(x)$ when x is *close to* a.

7 A geometrical approach to limits

Exercise 26 ***Finding limits***

Consider the following graph of a piecewise function.



For each of the following values,

(a) $a = -1$ (b) $a = 0$ (c) $a = 2$

use the graph of $f(x)$ to find these limits.

(i) $\lim\limits_{x \to a^-} f(x)$ (ii) $\lim\limits_{x \to a^+} f(x)$ (iii) $\lim\limits_{x \to a} f(x)$

Exercise 27 *Some special limits that do not exist*

Sketch the graph of $f(x) = \dfrac{1}{x}$ and find $\lim_{x \to 0^+} f(x)$ and $\lim_{x \to 0^-} f(x)$.

Exercise 28

Repeat this exercise with $f(x) = \left|\dfrac{1}{x}\right|$.

Definition 29 *The line $x = a$ is a* vertical asymptote *for the graph of a function $f(x)$ if*

$$f(x) \to +\infty \quad \textit{or} \quad f(x) \to -\infty$$

as x approaches a from the right or the left.

For *rational functions*, that is, functions of the form

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials, vertical asymptotes occur where the denominator is zero.

Exercise 30 ***Vertical asymptotes***

Give an example of a function with a vertical asymptote at $x = 2$.

Find $\lim_{x \to 2^-} f(x)$ and $\lim_{x \to 2^+} f(x)$ for your function.

Definition 31 *The line $y = b$ is a* horizontal asymptote *for the graph of a function $f(x)$ if*

$$\lim_{x \to +\infty} f(x) = b \quad \textit{or} \quad \lim_{x \to -\infty} f(x) = b\,.$$

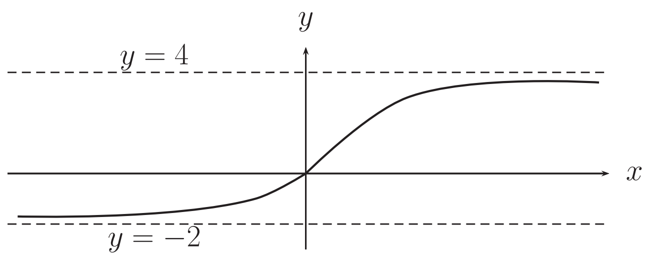
Example 32 *Horizontal Asymptotes*

The line $y = 0$ is a horizontal asymptote of the function $f(x) = e^x$ because $\lim_{x \to -\infty} e^x = 0$. Can you write a similar statement for $f(x) = e^{-x}$?

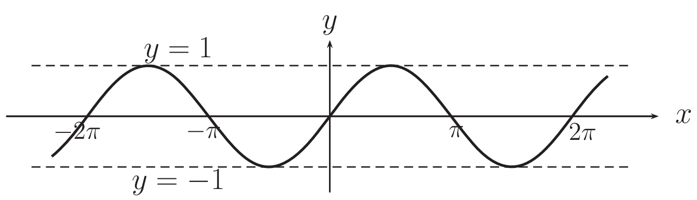
Exercise 33 ***Limits at infinity***

Find $\lim_{x \to +\infty} f(x)$ and $\lim_{x \to -\infty} f(x)$ for the following functions.

(a)



(b) $f(x) = \sin x$



8 How to work out limits

Idea 1: If f(x) is "nice", then the limit of f(x) as $x \to a$ is just $f(a)$.

Exercise 34 *Finding limits*

Find these limits using the above idea.

(a) $\lim_{x \to 2} 6$

(b) $\lim_{x \to 2} x$

(c) $\lim_{x \to 2} 5x$

(d) $\lim_{x \to 2} x^2$

(e) $\lim_{x \to 2} (x^2 + 5x + 6)$

(f) $\lim_{x \to 2} \left(\dfrac{x^2 + 5x + 6}{x} \right)$

From the above exercise, the following properties of limits should be clear.

1. $\lim_{x \to a} kf(x) \;=\; k \lim_{x \to a} f(x)$, where k is a constant.

2. $\lim_{x \to a} (f(x) \pm g(x)) \;=\; \lim_{x \to a} f(x) \;\pm\; \lim_{x \to a} g(x)$.

3. $\lim_{x \to a} (f(x)g(x)) \;=\; \lim_{x \to a} f(x) \; \lim_{x \to a} g(x)$.

4. $\lim_{x \to a} \left(\dfrac{f(x)}{g(x)} \right) \;=\; \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, as long as $\lim_{x \to a} g(x) \;\neq\; 0$.

Idea 2: Sometimes the bothersome factor in the denominator can be cancelled.

Exercise 35 ***Finding limits***

Find

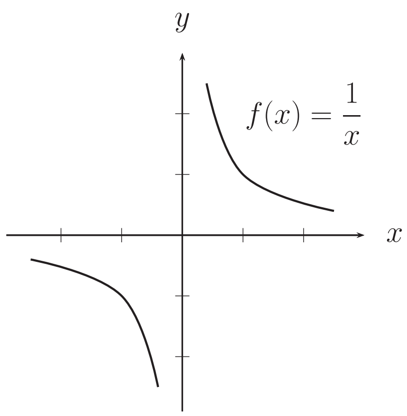
(a) $\lim\limits_{x \to 2} \dfrac{x^2 - 4}{x - 2}$

(b) $\lim\limits_{x \to -4} \dfrac{2x + 8}{x^2 + x - 12}$

Idea 3: Use the following limits:

From the graph of $f(x) = \dfrac{1}{x}$, we can see that

$$\boxed{\lim_{x \to +\infty} \frac{1}{x} = 0 \quad \text{and} \quad \lim_{x \to -\infty} \frac{1}{x} = 0\,.}$$



More generally, for $p > 0$,

$$\boxed{\lim_{x \to +\infty} \frac{1}{x^p} = 0 \quad \text{and} \quad \lim_{x \to -\infty} \frac{1}{x^p} = 0}$$

Exercise 36 *Finding limits*

Find the following limits by dividing both denominator and numerator by the highest power of x that occurs in the denominator. Write down the equations of any horizontal asymptotes.

(a) $\displaystyle\lim_{x\to\infty} \frac{3x+5}{6x-8}$

(b) $\displaystyle\lim_{x\to-\infty} \frac{2x^2+1}{3x^3+2x}$

(c) $\lim_{x \to \infty} \dfrac{4x^2 + x}{x + 1}$

Think about the relationship between the degrees of the numerator and denominator and the value of the limit you found in each of the above examples.

Exercise 37 ***Insect infestation***

The proportion y of apple trees infested with the woolly apple aphid in a particular orchid can be approximated by

$$y = 0.33 \, \frac{x}{x + 41.30}$$

where x is the aphid density measured by the average number of aphids per tree. What proportion of trees will be infested as the aphid density increases without bound?

Exercise 38 *Limits of piecewise functions*

Consider the following piecewise defined function

$$g(t) = \begin{cases} t^2 & \text{if } t \geq 2 \\ t - k & \text{if } t < 2 \end{cases}$$

where k is a constant.

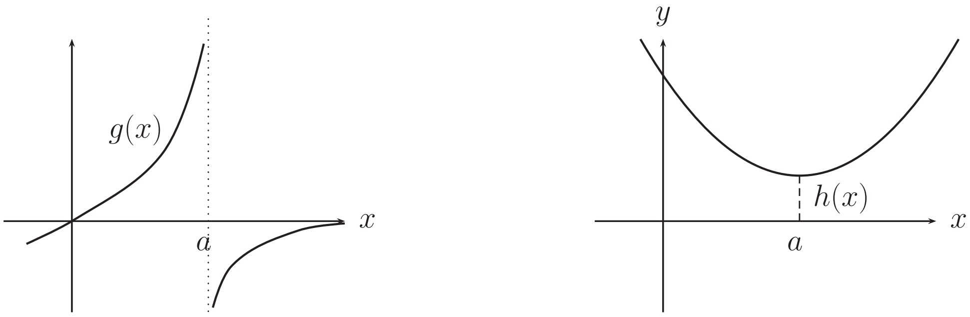
Find the value of k such that $g(t)$ has a limit at $t = 2$.

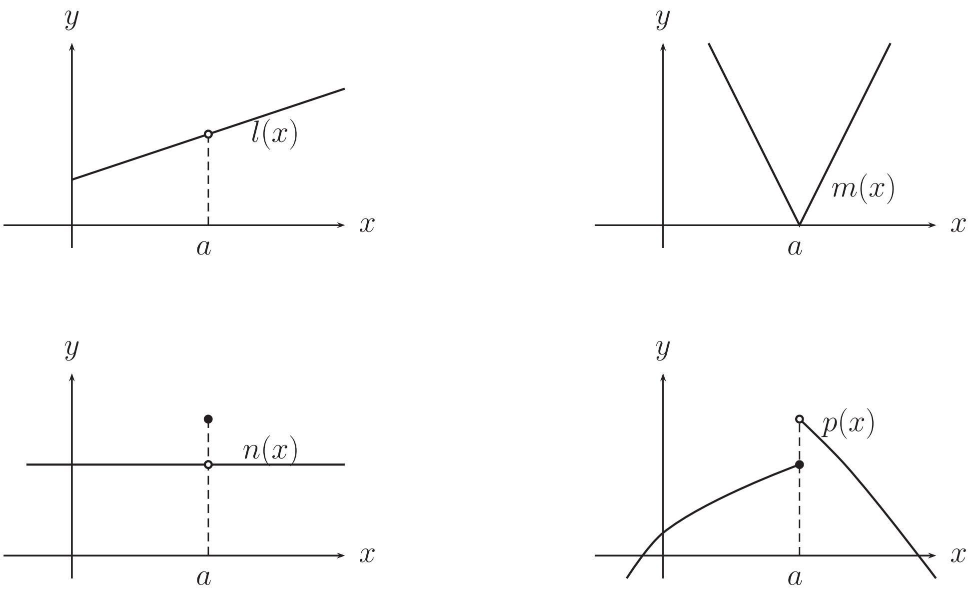
9 Continuity

Intuitively, a function is *continuous* if its graph is unbroken, free of sudden jumps or gaps.

Exercise 39 *Continuity*

Which of the following functions are continuous?





More generally, for a function $y = f(x)$ to be continuous at $x = a$:

- the function must be defined at that point;
- $\lim_{x \to a} f(x)$ must exist;
- the limit must equal the value of the function at $x = a$; that is, $\lim_{x \to a} f(x) \; = \; f(a)$.

Since if the last condition holds, the other two must also hold, we can say:

A function is continuous at $x = a$ if and only if $\lim_{x \to a} f(x) \; = \; f(a)$.

Basic properties of continuous functions

If the functions f and g are continuous at a point $x = c$ then

- $f + g$ is continuous at c
- $f - g$ is continuous at c
- fg is continuous at c
- f/g is continuous at c if $g(c) \neq 0$.

Comments:

- If $f(x)$ is continuous at all points on an open interval $(d,\, e)$, then $f(x)$ is said to be continuous on $(d,\, e)$. A function that is continuous on $(-\infty,\, +\infty)$ is said to be continuous everywhere or simply continuous.

  For example, polynomials are continuous everywhere.

- A function $f(x)$ is said to be continuous on the closed interval $[d,\, e]$, if it is continuous on $(d,\, e)$ and

$$\lim_{x \to d^+} f(x) \; = \; f(d) \quad \text{and} \quad \lim_{x \to e^-} f(x) \; = \; f(e)\,.$$

  For example, $f(x) = \sqrt{4 - x^2}$ is continuous on the closed interval $[-2,\, 2]$.

- Points where functions are not continuous are called *points of discontinuity*.

  For example, $x = a$ in the graphs of g, l, n and p is a point of discontinuity.

  Rational functions will have discontinuities at points where the denominator is zero.

Exercise 40 *Continuity*

Discuss the continuity of the following functions.

(a) $f(x) = \begin{cases} 2 & \text{if } x \geq 0 \\ x^2 & \text{if } x < 0 \end{cases}$

(b) $f(x) = \dfrac{3}{x-1}$

(c) $f(x) = \begin{cases} 5 & \text{if } x \geq 3 \\ 2x - 1 & \text{if } x < 3 \end{cases}$

(d) $f(x) = \begin{cases} 5 & \text{if } x > 3 \\ 2x - 1 & \text{if } x < 3 \\ 0 & \text{if } x = 3 \end{cases}$

Exercise 41 *Continuity*

Find the value of the constant a so that $h(x)$ is continuous everywhere.

$$h(x) = \begin{cases} 1+x & \text{if } x < 0 \\ a & \text{if } x = 0 \\ 1-x & \text{if } x > 0 \end{cases}$$

10 The "squeezing" property and applications

This is a rather obvious idea, but which is sometimes quite useful to give a quick proof of a result. Very briefly it says that if something is small and something else is even smaller, then that something else must be small.

Squeeze Theorem

Let f, g and h be functions such that

$$g(x) \leq f(x) \leq h(x)$$

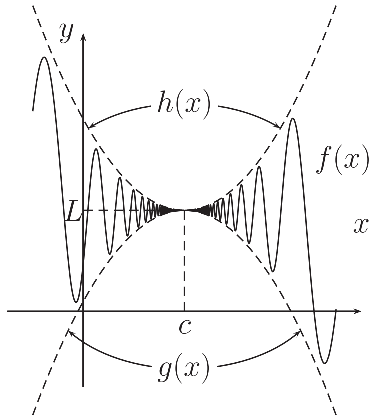
for all x in some open interval around a number c.

If g and h have the same limit L as x approaches c, that is, if

$$\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$$

then f also has this limit L, that is,

$$\lim_{x\to c} f(x) = L.$$



As an example of the squeezing property, let us derive the following important results concerning trigonometric limits.

Exercise 42 ***Some important limits***

(a) Show that $\displaystyle\lim_{x\to 0} \frac{\sin x}{x} = 1$

(b) Show that $\lim_{x \to 0} \dfrac{1 - \cos x}{x} = 0$

Exercise 43 ***Using these important limits***

(a) $\lim_{x \to 0} \dfrac{\sin 2x}{\sin 3x}$

(b) $\lim_{x \to 0} \dfrac{\tan x}{x}$

(c) $\lim_{x \to 0} \dfrac{\sin^2 x}{3x^2}$

(d) $\lim_{x \to 0} x \sin\left(\dfrac{1}{x}\right)$