Chapter 1: Limits and Continuity

Summary: This chapter explores the behavior of functions as they approach certain x-values. First, some graphical and numerical methods are used to try to ascertain the behavior of a functions output values near a particular input value. Later, the chapter introduces some algebraic techniques for finding exact answers to this question. This idea is made more precise using ε and δ to help represent the tolerances in the outputs and inputs respectively while identifying how a function behaves.

Generally, if a function approaches a consistent output value as the input values become closer to one value (such as $x = a$), then a limit of the function values is said to exist (i.e., $\lim_{x \to a} f(x) = L$). Naturally, such limits can be discussed while input values approach $x = a$ from the positive side, the negative side, or both. This directional idea is described accordingly as one-sided or two-sided limits.

Limits become an even more useful tool when they are used to describe the notion of a function being continuous. A function that is continuous can be said to have the property of $\lim_{x \to a} f(x) = f(a)$ which implies a limiting value, a function definition, and that the two values are equal. This formalizes the idea from the last chapter of polynomials having unbroken curves. Now polynomials can be described as being continuous everywhere. As it turns out, the trigonometric functions $\sin x$ and $\cos x$ are continuous everywhere, which greatly aids in studying the behavior of all six trigonometric ratios.

OBJECTIVES: After reading and working through this chapter you should be able to do the following:

1. Describe the behavior of a function using limits (§1.1, 1.4).
2. Evaluate the limiting behavior of a function graphically, numerically or algebraically (§1.1, 1.2).
3. Discuss the relationship between vertical asymptotes and infinite limits (§1.1).
4. Discuss the relationship between horizontal asymptotes and the end behavior of a function (§1.3).
5. Describe the connection between limits and continuity (§1.5).

6. Use the continuity of a function to evaluate the limit of a function at a point (§1.2, 1.5).
7. Determine on what intervals a function is continuous (§1.5).
8. Describe the continuity of an inverse function (§1.6).
9. Use the Squeezing Theorem to evaluate limits (§1.6).
10. Use the Intermediate-Value Theorem to approximate roots of a function (§1.5).

1.1 Limits (An Intuitive Approach)

PURPOSE: To introduce the notion of a limit.

This section introduces the idea of a limit by discussing how to find the slope of a tangent line to a curve. This is shown using the slopes of secant lines to a function (lines through two points on the function) as the points become closer together. The emphasis here is on what value a function approaches as the inputs become closer and closer to a prescribed value (say $x = a$). The actual function value at the prescribed value, $x = a$, is irrelevant and may not even be defined.

CAUTION: A function, $f(x)$, does not have to be defined at $x = a$ for $\lim_{x\to a} f(x)$ to exist.

informal definition of a limit

The **informal definition of a limit** then is to determine what value the output of a function can be made "arbitrarily close" to a value, L, provided that the inputs are "sufficiently close" to the prescribed input ($x = a$). This is written in the following mathematical expression.

$$\lim_{x\to a} f(x) = L$$

one-sided limits
two-sided limits

The input values can be controlled to only approach $x = a$ from the positive side or the negative side giving rise the terminology of **one-sided limits**. **Two-sided limits** occur when the inputs are allowed to approach from both sides.

IDEA: Both one-sided limits (as $x \to a^-$ and $x \to a^+$) must exist for the two-sided limit to exist.

Limits do not have to exist. If a particular value is not approached by the function or if different values are approached from either side of the point $x = a$ then the limit is said to not exist. An example is the function

$$f(x) = \begin{cases} 2x, & x < 0 \\ x+3 & x \geq 0 \end{cases}$$

In this case, $\lim_{x\to 0^-} f(x) = 0$ and $\lim_{x\to 0^+} f(x) = 3$. Since different values are approached at $x = 0$ from the left and right, the limit at $x = 0$ does not exist (i.e., the two-sided limit does not exist).

If a limit does not exist but instead appears to approach a value of $\lim_{x\to a} f(x) = \frac{\#}{0}$ then this is an indication that the function has a **vertical asymptote** at this point. If this same behavior is seen at either $x \to a^+$ or $x \to a^-$ then this is also an indication of a vertical asymptote at $x = a$. An example of this is the function $f(x) = 1/x$ at $x = 0$.

vertical asymptote

CAUTION: Saying that $\lim_{x\to a^-} f(x) = \infty$ does not mean that $f(a) = \infty$.

It should also be pointed out that if $\lim_{x\to a^-} f(x) = \infty$ then this does not mean that $f(a) = \infty$. Instead ∞ here is an indication of the **limit <u>not</u> existing**. Instead, the ∞ says that the function does not approach a limit but that it does have the consistent behavior of getting arbitrarily large as x approaches a from the negative side (or the left).

limit does not exist

<u>Checklist of Key Ideas:</u>

- ☐ tangent line problem
- ☐ secant and tangent lines
- ☐ slope of a secant line
- ☐ concept of a limit
- ☐ limit of function value
- ☐ informal definition of limit
- ☐ sampling pitfalls
- ☐ one-sided vs. two-sided limits
- ☐ limits that do not exist
- ☐ infinite limits and vertical asymptotes

1.2 Computing Limits

PURPOSE: To compute the values of limits of algebraically.

This section goes beyond the graphical and numerical determination or approximation of limits and starts to develop some basic limit rules and laws which can be used to obtain the values of more complicated limits. The most important laws are summarized in Theorem 1.2.2 which essentially says that limits can be easily added, subtracted, multiplied, divided (if the numerator doesn't approach a limit of zero) or raised to certain powers. "Easily" here means that some basic arithmetic is all that is required to evaluate the limits. The biggest difficulty arises when division occurs as follows.

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

Here if $\lim_{x \to a} g(x) = 0$ then these "easy" rules cannot be directly applied. If this is the situation, then there are few things that may happen especially if f and g are polynomials.

IDEA: Finding limits of polynomials and rational functions will depend upon whether division by zero occurs at $x = a$:

1. if no division by zero? $\to$ evaluate at $x = a$
2. division by zero:
   (a) if $\#/0 \to$ no limit; vertical asymptote
   (b) if $0/0 \to$ try canceling a factor

Essentially, rational functions can be evaluated at $x = a$ just like polynomials to find the limit as $x \to a$. If division by zero occurs, however, then you may either obtain $\#/0$ or $0/0$. In the first case, the limit does not exist and there is a vertical asymptote at that point. In the second case, the numerator and denominator share a factor which may be canceled. Then the new rational function should be tested in the limit again. This process should end in either obtaining $\#/0$ (**a vertical asymptote**) or a number (which would be the limit). Obtaining a number means that there is a **hole** (or **removable discontinuity**) in the graph at that point. Sometimes, through algebraic manipulation (such as rationalizing a radical), zeros in the denominator can also be removed with the same results as canceling a factor.

vertical asymptote

hole or removable discontinuity

piecewise functions

To determine the limits of **piecewise functions** at a breakpoint, the one-sided limits of the function need to be evaluated on either side of the breakpoint. If they have the same value then that is the value of the two-sided limit. Otherwise, no limiting value exists for the two-sided limit.

Checklist of Key Ideas:

- ☐ algebraic techniques for finding limits
- ☐ sums, differences, products, quotients and n-th roots of limits
- ☐ limits of polynomials and rational functions
- ☐ method of test points
- ☐ canceling all common factors of $x - a$ in rational functions
- ☐ indeterminate form of type $0/0$
- ☐ one-sided and two-sided limits and piecewise functions

1.3 Limits at Infinity; End Behavior of a Function

PURPOSE: To find limits of functions as $x \to \pm\infty$.

Up until now, limits have been restricted towards looking at $x = a$, which is some finite value. Here, limits are considered where $x \to \infty$ or $x \to -\infty$. This is called the **end behavior of the function**.

end behavior of a function

Limits at infinity obey the same limit laws as limits as $x \to a$ (see previous section). Limits at infinity can fail to exist if the function is oscillating continuously (i.e., trigonometric functions) or if the function increases (or decreases) without bound (for example, $y = x^2$). Any limit at infinity that does exist means that there is a **horizontal asymptote** of the function.

horizontal asymptote

IDEA: If $\lim_{x\to\infty} f(x) = L$ for some finite value L then the line $y = L$ is a horizontal asymptote of the function $y = f(x)$.

The limits at infinity for rational functions are controlled by their **dominant terms**. Depending upon the ratio of the highest powers of x on the top and bottom of the expression then the limit will either be $\pm\infty$ or L.

dominant terms

Checklist of Key Ideas:

- ☐ end behavior
- ☐ limits at infinity
- ☐ horizontal asymptotes
- ☐ limit laws at infinity
- ☐ how limits at infinity can fail to exist
- ☐ infinite limits at infinity
- ☐ end behavior of polynomials and rational functions
- ☐ dominant terms
- ☐ end behavior of trigonometric, exponential and logarithmic functions

1.4 Limits (Discussed More Rigorously)

PURPOSE: To give a more rigorous (or thorough) definition of limit.

The goal of this section is to explore limits more rigorously. With this in mind, the ideas of "arbitrarily close" and "sufficiently close" from Section 1.1 are replaced by the phrases $|f(x) - L| < \varepsilon$ and $0 < |x - a| < \delta$ respectively. The aim here is to first describe how close a function should be to value L (within ε) and then try to show that if x is close enough to a (within δ) that this will hold.

IDEA: The goal is to find δ so that $f(x)$ is within ε of a some value L (ε is some known value).

1. Pick ε: how close to L should $f(x)$ be?
2. Find δ so that this works if $|x-a| < \delta$.

Most problems in this section have the following format: it must be shown that given any positive number ε, a positive number δ can be found such that

$$|f(x)-L| < \varepsilon \text{ if } 0 < |x-a| < \delta.$$

Then the process to be followed is to start with $|f(x)-L|$ and try to transform this into an inequality involving x of the form $|x-a| < \delta$. When starting with

$$-\varepsilon < f(x) - L < \varepsilon$$

it very often occurs that an inequality involving x turns out to be

$$-\delta_1 < x - a < \delta_2$$

Then to satisfy the problem δ is chosen as the smaller of δ_1 or δ_2.

Similar ideas hold for working with infinite limits.

Checklist of Key Ideas:

- ☐ formal limit definition
- ☐ "epsilon", ε, and "delta", δ
- ☐ definition of limits as $x \to \pm\infty$
- ☐ definition of infinite limits

1.5 Continuity

PURPOSE: To establish what is meant by a "continuous function."

Continuity is what has allowed the easy computation of limits for polynomial and rational functions (when there is not division by zero). That is to say, the limit of a polynomial is equal to its function value.

IDEA: Find limits of polynomials by evaluating the polynomial.

This idea of continuity can be summarized by the following.

$$\lim_{x \to a} f(x) = f(a)$$

This statement implies that the function is defined at $x = a$, the limit exists at $x = a$, and that the value of the limit and the function value are equal. In other words, the limiting value that the function approaches is the value that it is defined to have. Visually this amounts to having an unbroken graph with no holes or vertical asymptotes.

IDEA: The limits of continuous functions may be found by evaluating the function if the point is within the domain of the function.

Continuous functions have limits that are easy to evaluate within their domains. Rational functions for example are continuous whenever their denominator is not zero and then their limit is equivalent to their function values. If the denominator was zero, this would be a hole or a vertical asymptote which would not be in their domain.

The **Intermediate-Value Theorem** capitalizes on this idea. If a function is continuous on an interval then the function must attain all function values between the function values at the endpoints. In particular, if the function crosses from negative to positive, or from positive to negative, then it must have at least one root in this interval. By smaller and smaller intervals where this occurs, better and better approximations of roots can be found in this way.

Intermediate-Value Theorem

Checklist of Key Ideas:

- ☐ definition of continuity
- ☐ three conditions of continuity
- ☐ continuous on an interval (a,b)
- ☐ continuous everywhere
- ☐ continuity from the left or right at a point
- ☐ continuous on a closed interval
- ☐ continuity of $f \pm g$, fg and f/g
- ☐ continuity of polynomials and rational functions
- ☐ continuity of compositions
- ☐ the Intermediate-Value Theorem
- ☐ approximating roots

1.6 Continuity of Trigonometric, Exponential, and Inverse Functions

PURPOSE: To discuss the continuity of trigonometric, exponential, and inverse functions.

Simply stated, trigonometric functions and exponential functions are continuous in their domains. Also, if a continuous function has an inverse (which is not necessarily a guarantee), then the inverse will be continuous. But the interval where the inverse is continuous will be equal to the range of values attained by the original function. That is to say that the **domain of f^{-1} is equal to the range of f.**

domain of f^{-1} = range of f

Squeeze Theorem

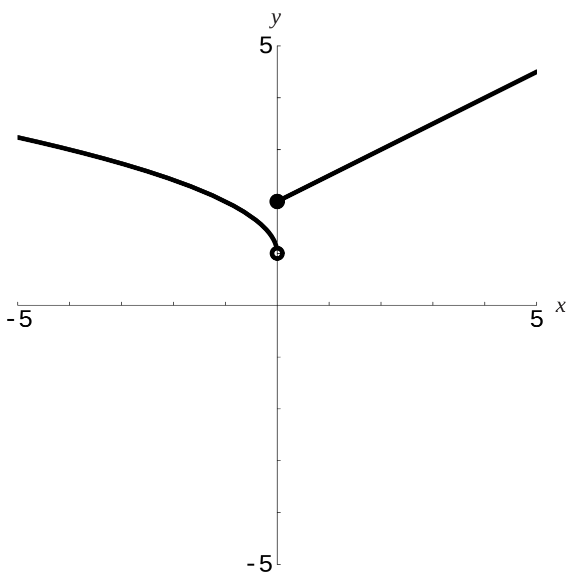
The **Squeezing Theorem** offers a powerful tool for finding other limits such as $\lim_{x\to 0} \frac{\sin x}{x}$. This is accomplished by finding two functions that have the same limit at a point and which also bound the function in question from above and below. Then the function is squeezed in between the two functions and must have the same limit. This theorem is not always easy to apply because the two bounding functions must be obtained before the theorem can be applied.

Checklist of Key Ideas:

- ☐ continuity of $\sin x$ and $\cos x$
- ☐ continuity of general trigonometric functions
- ☐ continuity of exponential and logarithmic functions
- ☐ continuity of inverse functions
- ☐ The Squeezing Theorem

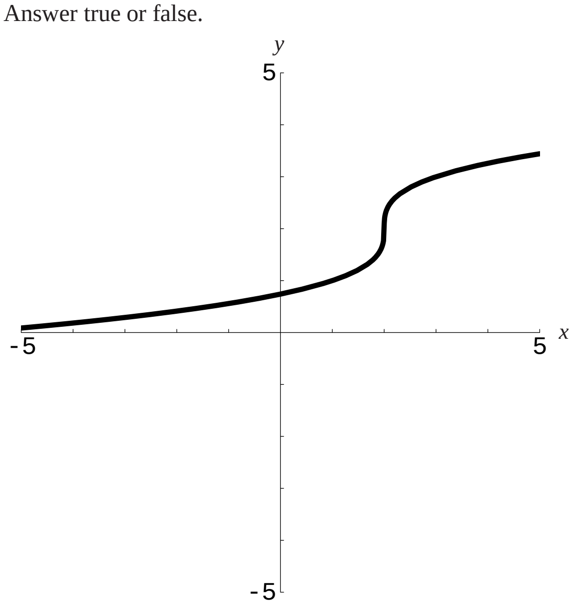
Chapter 1 Sample Tests

Section 1.1



1. The function $f(x)$ is graphed above. $\lim_{x\to 0^-} f(x) =$
   (a) 1
   (b) 2
   (c) $\frac{3}{2}$
   (d) undefined

2. Answer true or false.



For the function graphed $\lim_{x\to 2} f(x)$ is undefined.

3. Approximate $\lim_{x\to 2^-} \frac{x^2-4}{x-2}$ by evaluating $f(x) = \frac{x^2-4}{x-2}$ at $x = 3$, 2.5, 2.1, 2.01, 2.001, 1, 1.5, 1.9, 1.99, and 1.999.
   (a) 2
   (b) -2
   (c) 0
   (d) 4

4. Answer true or false. If $\lim_{x\to 0^+} f(x) = 4$ and $\lim_{x\to 0^-} f(x) = 4$, then $\lim_{x\to 0} f(x) = 4$.

5. Approximate $\lim_{x\to 2^-} \frac{x}{x-2}$ by evaluating $f(x) = \frac{x}{x-2}$ at appropriate values of x.
   (a) 1
   (b) 0
   (c) ∞
   (d) $-\infty$

6. If $f(x) = \frac{\sin x}{5x}$ then approximate $\lim_{x\to 0^-} f(x)$ by evaluating $f(x)$ at appropriate values of x.
   (a) 1
   (b) 5
   (c) $\frac{1}{5}$
   (d) ∞

7. If $f(x) = \frac{x}{\sin x}$ then approximate $\lim_{x\to 0} f(x)$ by evaluating $f(x)$ at appropriate values of x.
   (a) 1
   (b) -1
   (c) 0
   (d) ∞

8. If $f(x) = \frac{\sqrt{x+4}-2}{x}$ then approximate $\lim_{x\to 0^-} f(x)$ by evaluating $f(x)$ at appropriate values of x.
   (a) $\frac{1}{4}$
   (b) 0
   (c) ∞
   (d) $-\infty$

9. Use a graphing utility to approximate the y-coordinates of any horizontal asymptotes of $y = \frac{4x-9}{x+3}$.
   (a) $y = 4$

(b) $y=1$

(c) None exist

(d) $y=-3$

10. Use a graphing utility to approximate the y-coordinates of any horizontal asymptotes of $y=\frac{\cos x}{x}$.

(a) $y=0$

(b) only $y=1$

(c) both $y=-1$ and $y=1$

(d) only $y=-1$

11. Use a graphing utility to approximate the y-coordinates of any horizontal asymptotes of $y=\frac{x^2+4}{x-2}$.

(a) $y=0$

(b) None exist

(c) $y=1$

(d) Both $y=-1$ and $y=1$

12. Answer true or false. A graphing utility can be used to show $f(x)=\left(1+\dfrac{2}{x}\right)^x$ has a horizontal asymptote.

13. Answer true or false. A graphing utility can be used to show that $f(x)=\left(6+\dfrac{1}{x}\right)^x$ has a horizontal asymptote.

14. Answer true or false. $\lim\limits_{x\to\infty}\dfrac{2+x}{1-x}$ is equivalent to $\lim\limits_{x\to 0^+}\dfrac{\frac{2}{x}+1}{\frac{1}{x}-1}$.

15. Answer true or false. $\lim\limits_{x\to\infty}\dfrac{\sin(2\pi x)}{x-2}$ is equivalent to $\lim\limits_{x\to 0^+}\sin(2\pi x)$.

16. Answer true or false. The function $f(x)=\dfrac{x}{x^2-4}$ has no horizontal asymptotes.

Section 1.2

1. Given that $\lim\limits_{x\to a} f(x)=3$ and $\lim\limits_{x\to a} g(x)=5$, then find $\lim\limits_{x\to a}[f(x)-g(x)]^2$ if it exists.

(a) -4

(b) 4

(c) 22

(d) It does not exist.

2. $\lim\limits_{x\to 3} 4=$

(a) 4

(b) 3

(c) 7

(d) 12

3. Answer true or false. $\lim\limits_{x\to 2} 4x=8$.

4. $\lim\limits_{x\to -3}\dfrac{x^2-9}{x+3}=.$

(a) $-\infty$

(b) -6

(c) 6

(d) 1

5. $\lim\limits_{x\to 5}\dfrac{4}{x-5}=.$

(a) ∞

(b) $-\infty$

(c) 0

(d) It does not exist.

6. $\lim\limits_{x\to 1}\dfrac{4x}{x^2-6x+5}=$

(a) ∞

(b) $-\infty$

(c) 0

(d) It does not exist.

7. $\lim\limits_{x\to 1}\dfrac{x-4}{\sqrt{x}-4}=$

(a) ∞

(b) $-\infty$

(c) 1

(d) It does not exist.

8. Let $f(x)=\begin{cases} x+4, & x\le 2 \\ x^2, & x>2 \end{cases}$. Then $\lim\limits_{x\to 2} f(x)=.$

(a) 6

(b) 4

(c) 3

(d) It does not exist.

9. Let $g(x)=\begin{cases} x^2+4, & x\le 1 \\ x^3, & x>1 \end{cases}$. Then $\lim\limits_{x\to 1} g(x)=.$

(a) 5

(b) 1

(c) 3

(d) It does not exist.

10. Answer true or false. $\lim\limits_{x\to 0}\dfrac{\sqrt{x^2+25}-5}{x}=\dfrac{1}{10}$.

Section 1.3

1. $\lim_{x\to\infty} \frac{10x}{x^2-5x+3} =$
   (a) 0
   (b) 2
   (c) 5
   (d) It does not exist.

2. $\lim_{x\to-\infty} \frac{2x^2-x}{x^2} =$
   (a) 2
   (b) ∞
   (c) $-\infty$
   (d) It does not exist.

3. $\lim_{x\to-\infty} \sqrt[4]{\frac{32x^8-6x^5+2}{2x^8-3x^3+1}} =$
   (a) ∞
   (b) $-\infty$
   (c) 2
   (d) It does not exist.

4. $\lim_{x\to\infty} (x^4-500x^3) =$
   (a) ∞
   (b) $-\infty$
   (c) -500
   (d) It does not exist.

5. Answer true or false. $\lim_{x\to\infty} \frac{\sqrt{x^2+9}-3}{x}$ does not exist.

Section 1.4

1. If $\lim_{x\to5} 4x = 20$ and $\varepsilon = 0.1$ then find a least number δ such that $|f(x)-L| < \varepsilon$ when $0 < |x-a| < \delta$.
   (a) 0.1
   (b) 0.25
   (c) 0.5
   (d) 0.025

2. If $\lim_{x\to2} 3x-4 = 2$ and $\varepsilon = 0.1$ then find a least number δ such that $|f(x)-L| < \varepsilon$ when $0 < |x-a| < \delta$.
   (a) 0.033
   (b) 0.33
   (c) 3.0
   (d) 0.3

3. Answer true or false. If $f(x) = x^3$, $L = 27$, $a = 3$ and $\varepsilon = 0.05$ then a least number δ such that $|f(x)-L| < \varepsilon$ if $0 < |x-a| < \delta$ is $\delta = \sqrt[3]{27.05}-3$.

4. If $\lim_{x\to5} \frac{x^2-25}{x-5} = 10$ and $\varepsilon = 0.001$ then find a least number δ such that $|f(x)-L| < \varepsilon$ when $0 < |x-a| < \delta$.
   (a) 0.001
   (b) 0.000001
   (c) 0.005
   (d) 0.025

5. If $\lim_{x\to\infty} \frac{12}{x^3} = 0$ and $\varepsilon = 0.1$ then find the least positive integer N such that $|f(x)-L| < \varepsilon$ when $x > N$.
   (a) $N = 100$
   (b) $N = 1,000$
   (c) $N = 4$
   (d) $N = 5$

6. If $\lim_{x\to-\infty} \frac{1}{x^5} = 0$ and $\varepsilon = 0.1$ then find the greatest negative integer N such that $|f(x)-L| < \varepsilon$ when $x < N$.
   (a) $N = -100,000$
   (b) $N = -10,000$
   (c) $N = -1$
   (d) $N = -2$

7. Answer true or false.
   It is reasonable to prove that $\lim_{x\to\infty} \frac{1}{x^2+1} = 0$.

8. Answer true or false.
   It is reasonable to prove that $\lim_{x\to-\infty} \frac{1}{x+6} = 0$.

9. Answer true or false.
   It is reasonable to prove that $\lim_{x\to\infty} \frac{1}{x-5} = 0$.

10. Answer true or false.
    It is reasonable to prove that $\lim_{x\to5} \frac{1}{x-5} = \infty$.

11. To prove that $\lim_{x\to5} (x+2) = 7$, a reasonable relationship between δ and ε would be
    (a) $\delta = 5\varepsilon$
    (b) $\delta = \varepsilon$
    (c) $\delta = \sqrt{\varepsilon}$
    (d) $\delta = \frac{1}{\varepsilon}$

12. Answer true or false. To use a δ-ε approach to show that $\lim_{x\to 0^+} \frac{1}{x^3} = \infty$, a reasonable first step would be to change the limit to $\lim_{x\to\infty} x^3 = 0$.

13. Answer true or false.
It is possible to show that $\lim_{x\to 4^-} \frac{1}{x-4} = -\infty$.

14. To prove that $\lim_{x\to 3} f(x) = 6$ where

$$f(x) = \begin{cases} 2x, & x < 3 \\ x+3, & x \geq 3 \end{cases}$$

a reasonable relationship between δ and ε would be

(a) $\delta = \varepsilon/2$

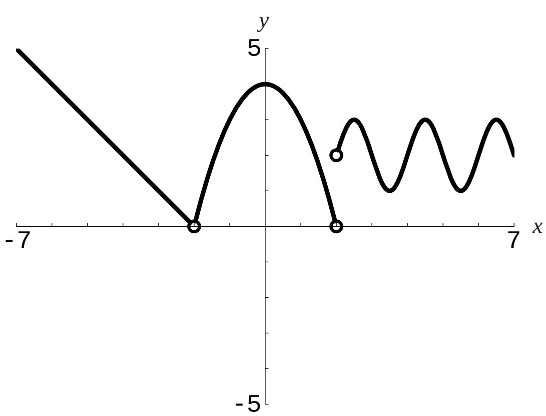
(b) $\delta = \varepsilon$

(c) $\delta = \varepsilon + 3$

(d) $\delta = \varepsilon/2 + 3$

15. Answer true or false. It is possible to show that $\lim_{x\to\infty} \frac{x}{5} = 5$.

Section 1.5



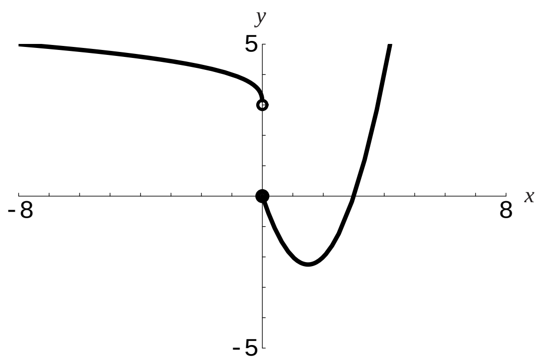
1. On the interval $[-7,7]$, where is f not continuous?

(a) $-2, 2$

(b) 2

(c) -2

(d) nowhere



2. On the interval $[-8,8]$, where is f not continuous?

(a) 3

(b) $0, 3$

(c) 0

(d) nowhere

3. Answer true or false. $f(x) = x^6 - 4x^4 + 2x^2 - 8$ has no point of discontinuity.

4. Answer true or false. $f(x) = |x+4|$ has a point of discontinuity at $x = -4$.

5. Find the x-coordinates for all points of discontinuity for the function $f(x) = \frac{x+6}{x^2+8x+12}$.

(a) $-6, -2$

(b) -2

(c) $2, 6$

(d) 2

6. Find the x-coordinates for all points of discontinuity for the function $f(x) = \frac{|x+5|}{x^2+5x}$.

(a) 0

(b) -5

(c) $-5, 0$

(d) $-5, 0, 5$

7. Find the x-coordinates for all points of discontinuity for the function $f(x) = \begin{cases} x^2-5, & x \leq 1 \\ -4, & x > 1 \end{cases}$.

(a) 1

(b) $-\sqrt{5}, \sqrt{5}$

(c) $-4, -\sqrt{5}, \sqrt{5}$

(d) None exists.

8. If $f(x) = \begin{cases} x-2, & x \leq 1 \\ kx^3, & x > 1 \end{cases}$, then find the value k, if possible, that will make $f(x)$ continuous everywhere.

(a) 1
(b) -1
(c) 2
(d) None exists.

9. Answer true or false. The function $f(x) = \dfrac{x^3-1}{x-1}$ has a removable discontinuity at $x = 1$.

10. Answer true or false. The function

$$f(x) = \begin{cases} x^2, & x \le 2 \\ x-4, & x > 2 \end{cases}$$

is continuous everywhere.

11. Answer true or false. If f and g are continuous at $x = c$, then $f+g$ may be discontinuous at $x = c$.

12. Answer true or false. The Intermediate-Value Theorem can be used to approximate the locations of all discontinuities for $f(x) = \dfrac{x}{x^3-5x+13}$. [Hint: It may be applied to the denominator only.]

13. Answer true or false. $f(x) = x^5 - 6x^2 + 2 = 0$ has at least one solution on the interval $[-1,0]$.

14. Answer true or false. $f(x) = x^3 + 2x + 9 = 0$ has at least one solution on the interval $[0,1]$.

15. Use the fact that $\sqrt{8}$ is a solution of $x^2 - 8 = 0$ to approximate $\sqrt{8}$ with an error of at most 0.005.

(a) 1.25
(b) 2.81
(c) 2.83
(d) 2.84

Section 1.6

1. Answer true or false. $f(x) = \cos(x^2+5)$ has no point of discontinuity.

2. A point of discontinuity of $f(x) = \dfrac{1}{|-1+2\cos x|}$ is at

(a) $\dfrac{\pi}{2}$
(b) $\dfrac{\pi}{3}$
(c) $\dfrac{\pi}{4}$
(d) $\dfrac{\pi}{6}$

3. Find the limit. $\lim\limits_{x\to\infty} \cos\left(\dfrac{4}{x}\right) =$

(a) 0
(b) 1
(c) -1
(d) ∞

4. Find the limit. $\lim\limits_{x\to 0^-} \dfrac{\sin x}{x^3} =$

(a) ∞
(b) 0
(c) 1
(d) $-\infty$

5. Find the limit. $\lim\limits_{x\to 0} \dfrac{\sin(3x)}{\sin(8x)} =$

(a) ∞
(b) 0
(c) $\dfrac{3}{8}$
(d) 1

6. Find the limit. $\lim\limits_{x\to 0} \dfrac{4}{1-\sin x} =$

(a) 1
(b) -1
(c) 4
(d) 0

7. Find the limit. $\lim\limits_{x\to 0} \dfrac{1+\cos x}{1-\cos x} =$

(a) 0
(b) ∞
(c) $-\infty$
(d) 2

8. Find the limit. $\lim\limits_{x\to 0} \dfrac{\tan x}{\cos x} =$

(a) 0
(b) 1
(c) ∞
(d) $-\infty$

9. Find the limit. $\lim\limits_{x\to 0^-} \cos\left(\dfrac{1}{x}\right) =$

(a) 1
(b) -1
(c) $-\infty$
(d) does not exist

10. Find the limit. $\lim\limits_{x\to 0^+} \dfrac{-1.3x+\cos x}{x} =$

(a) 0

(b) 1

(c) -1

(d) ∞

11. Answer true or false. If $f(x) = \begin{cases} \dfrac{\cos x - 1}{x}, & x \leq 0 \\ \cos x + k, & x > 0 \end{cases}$, then f is continuous if $k = 0$.

12. Answer true or false. The fact that $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x} = 0$ and that $\lim\limits_{x\to 0} x = 0$ guarantees that $\lim\limits_{x\to 0} \dfrac{(1-\cos x)^2}{x} = 0$ by the Squeeze Theorem.

13. Answer true or false. The Squeezing Theorem can be used to show $\lim\limits_{x\to 0} \dfrac{\sin(4x)}{6x} = 1$ by utilizing $\lim\limits_{x\to 0} \dfrac{\sin(4x)}{4x} = 1$ and $\lim\limits_{x\to 0} \dfrac{\sin(6x)}{6x} = 1$.

14. Answer true or false. The Intermediate-Value Theorem can be used to show that the equation $y^3 = \sin^2 x$ has at least one solution on the interval $[-5\pi/6, 5\pi/6]$.

15. Find the limit. $\lim\limits_{x\to 0} \left(\dfrac{\sin x}{2x} + \dfrac{x}{2\sin x} \right) =$

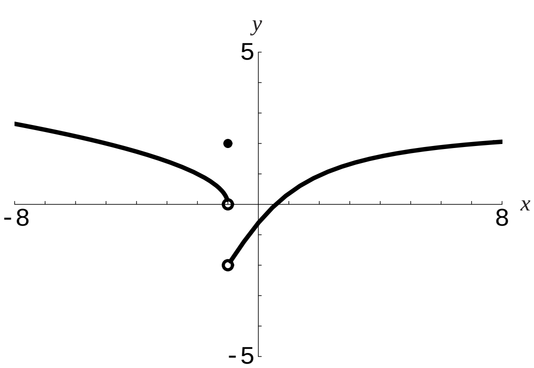
(a) 1

(b) 2

(c) $\dfrac{1}{2}$

(d) 0

Chapter 1 Test



1. The function f is graphed. $\lim\limits_{x\to -1} f(x) =$

(a) 2

(b) -2

(c) 0

(d) undefined

2. Approximate $\lim\limits_{x\to -5} \dfrac{x^2-25}{x+5}$ by evaluating $f(x) = \frac{x^2-25}{x+5}$ at $x = -4,\ -4.5,\ -4.9,\ -4.99,\ -4.999,\ -6,\ -5.5,\ -5.1,\ -5.01$, and -5.001.

(a) 5

(b) -5

(c) 10

(d) -10

3. Use a graphing utility to approximate the y-coordinate of the horizontal asymptote of $y = f(x) = \dfrac{9x+3}{4x+2}$.

(a) $\dfrac{9}{4}$

(b) $\dfrac{4}{9}$

(c) $\dfrac{3}{2}$

(d) $\dfrac{2}{3}$

4. Answer true or false. A graphing utility can be used to show that $f(x) = \left(8 + \dfrac{2}{2x}\right)^{2x}$ has a horizontal asymptote.

5. Answer true or false. $\lim\limits_{x\to\infty} \dfrac{5}{x^2}$ is equivalent to $\lim\limits_{x\to 0^+} 5x^2$.

6. Find the value of $\lim\limits_{x\to a} [2f(x) + 4g(x)]$ if it is given that

$$\lim_{x\to a} f(x) = 5 \text{ and } \lim_{x\to a} g(x) = -5.$$

(a) 0

(b) -5

(c) -10

(d) 30

7. $\lim\limits_{x\to 6} 4 =$

(a) 6

(b) -6

(c) 4

(d) does not exist

8. $\lim\limits_{x\to 1} \dfrac{x^8-1}{x-1} =$

(a) 0

(b) ∞

(c) 4

(d) 8

9. $\lim\limits_{x\to 6} \dfrac{1}{x-6} =$

(a) $\frac{1}{12}$
(b) 0
(c) ∞
(d) does not exist

10. Let $f(x) = \begin{cases} x^2, & x \leq 1 \\ x, & x > 1 \end{cases}$. $\lim_{x\to 1} f(x) =$
   (a) 1
   (b) -1
   (c) 0
   (d) does not exist

11. If $\lim_{x\to 5} 4x = 20$ and $\varepsilon = 0.01$ then find a least number δ such that $|f(x) - L| < \varepsilon$ when $0 < |x-a| < \delta$.
   (a) 0.01
   (b) 0.025
   (c) 0.05
   (d) 0.0025

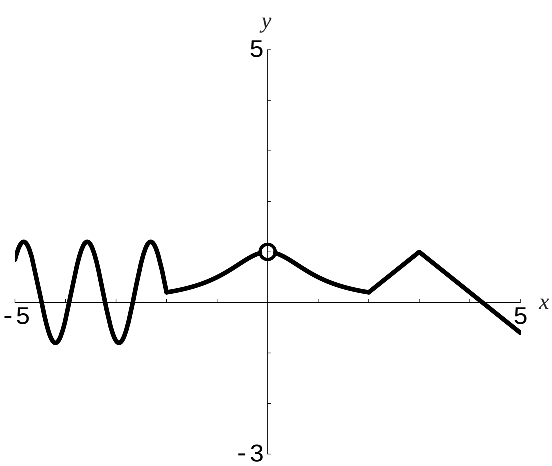
12. If $\lim_{x\to 6} \frac{x^2-36}{x+6} = -12$ and $\varepsilon = 0.001$ then find a least number δ such that $|f(x) - L| < \varepsilon$ when $0 < |x-a| < \delta$.
   (a) 0.001
   (b) 0.000001
   (c) 0.006
   (d) 0.03

13. Answer true or false. It is possible to prove that $\lim_{x\to\infty} \frac{1}{x^7} = 0$.

14. To prove $\lim_{x\to 2}(3x+1) = 7$, a reasonable relationship between δ and ε would be
   (a) $\delta = \frac{\varepsilon}{3}$
   (b) $\delta = 3\varepsilon$
   (c) $\delta = \varepsilon$
   (d) $\delta = \varepsilon - 1$

15. Answer true or false. It is possible to prove that $\lim_{x\to\infty}(x-3) = -3$.

16. The graph of the function $f(x)$ is shown below. On the interval $[-5, 5]$, where is f not continuous?



   (a) 0
   (b) -2
   (c) 2
   (d) nowhere

17. Find the x-coordinate of each point of discontinuity of $f(x) = \frac{x+2}{x^2-3x-10}$.
   (a) 5
   (b) $-2, 5$
   (c) $2, 5$
   (d) $-5, 2$

18. Find the value k, if possible, that will make the function $f(x) = \begin{cases} kx+3, & x \leq 3 \\ x^2, & x > 3 \end{cases}$ continuous everywhere.
   (a) $k = 3$
   (b) $k = 0$
   (c) $k = 2$
   (d) No such k exists.

19. Answer true or false. The function $f(x) = \frac{1}{x-6}$ has a removable discontinuity at $x = 6$.

20. Answer true or false. The equation $f(x) = x^5 + 6 = 0$ has at least one solution on the interval $[-2, -1]$.

21. Find $\lim_{x\to 0} \frac{\sin(-3x)}{\sin(2x)}$.
   (a) 0
   (b) $-\frac{3}{2}$
   (c) $\frac{3}{2}$
   (d) undefined

22. Find $\lim_{x\to 0} \frac{\sin x}{\tan x}$.
   (a) 0
   (b) -1
   (c) 1
   (d) undefined

23. Answer true or false. $\lim_{x\to 0} \frac{\sin x}{1-\cos x} = 0$.

Chapter 1: Answers to Sample Tests

Section 1.1

1. a	2. false	3. d	4. true	5. d	6. c	7. a	8. a
9. a	10. a	11. b	12. true	13. false	14. true	15. false	16. false

Section 1.2

1. b	2. a	3. true	4. b	5. d	6. d	7. c	8. d
9. d	10. false						

Section 1.3

1. a	2. a	3. c	4. a	5. false

Section 1.4

1. d	2. a	3. false	4. a	5. c	6. c	7. true	8. true
9. true	10. false	11. b	12. false	13. true	14. a	15. false	

Section 1.5

1. a	2. c	3. true	4. false	5. a	6. c	7. d	8. b
9. true	10. false	11. false	12. true	13. true	14. false	15. c	

Section 1.6

1. true	2. b	3. b	4. a	5. c	6. c	7. b	8. a
9. d	10. d	11. false	12. true	13. false	14. false	15. a	

Chapter 1 Test

1. d	2. d	3. a	4. false	5. true	6. c	7. c	8. d
9. d	10. a	11. d	12. a	13. true	14. a	15. false	16. a
17. b	18. c	19. false	20. true	21. b	22. c	23. false	