Topic 4: Fundamentals of Differentiation

Topic Outline for Week 1 (3 lectures)

The chapter references are to the recommended textbook: *Calculus: Early Transcendentals* by Anton, Bivens and Davis, 9th or 10th edition. For the 8th edition, see Chapters 3 - 4 and 5.7. We will be going fairly quickly through the first week's material as it should be revision from high school.

- Rates of change, tangent lines and derivatives (Chapter 2.1 - 2.2)
- Techniques of differentiation (Chapter 2.3 - 2.5)
- The Chain Rule (Chapter 2.6)

Topic Outline for Week 2 (4 lectures)

The material this week is mostly new and important for understanding and applying the techniques of calculus, so we will be moving more slowly with this material.

- Implicit Differentiation (Chapter 3.1)
- Derivatives of Exponential Functions (Chapter 3.3)
- Derivatives of Inverse Functions (Chapter 3.3)
- The Mean Value Theorem and Rolle's Theorem (Chapter 4.8)

There were four major problems confronting mathematicians in the 17th century:

Calculating the "instantaneous" velocity of a moving body.

Area under a curve

All solved by CALCULUS!

Maximum/minimum problems

Finding the slope of the line that is tangent to a given point. (Needed for construction of lenses.) Also, what exactly is a tangent line?

For interest only: A brief historical background of the development of calculus

Some of the crucial ideas behind differential calculus can be traced back to the ancient Greeks. (Greek geometers are credited with a significant use of infinitesimals.) But the Greeks were too focused on geometry to develop these ideas very far.

The Indian mathematician, Bhaskara (12th century), developed a proto-derivative for describing infinitesimal change, as well as an early form of Rolle's theorem. Another Indian mathematician, Madhava, (14th century), described special cases of Taylor series which give useful approximations to functions such as trigonometric functions.

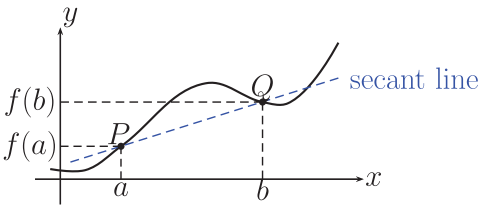
Isaac Newton (1642 - 1727) from England and Gottfried Leibniz (1646 - 1716) from Germany are usually credited with the invention of modern calculus in the late 1600s. Their most important contributions were the development of the fundamental theorem of calculus. Leibniz did a great deal of work with developing consistent and useful notation and concepts. Newton was the first to organize the field into one consistent subject, and also to provide some of the first and most important applications, especially of integral calculus.

Historically, there was much debate over whether it was Newton or Leibniz who first "invented" calculus. This argument was at the heart of a rift in the mathematical community. Both Newton and Leibniz claimed that the other plagiarized their respective works. It is now thought that they developed their ideas independently. If interested, see the Wikipedia article on the Leibniz-Newton calculus controversy!

1 Rates of Change, Tangent Lines, and Derivatives (AC 2.1-2.2)

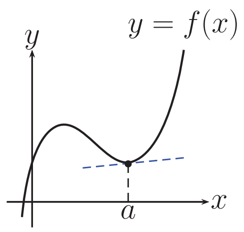
Calculus has an enormous number of applications. In particular, differential calculus provides a precise mathematical description of the concept of rate of change. This concept is seen in terms such as “growth rate”, “velocity”, “acceleration”, “rates of reaction”, “interest rates”, etc.

First let’s consider the average rate of change of a function.

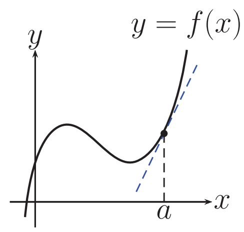


The average rate of change of a function between $x = a$ and $x = b$ is the slope of the secant line through the points $(a, f(a))$ and $(b, f(b))$.

Next let’s consider the instantaneous rate of change of a function at a point $x = a$:



changing slowly near $x = a$



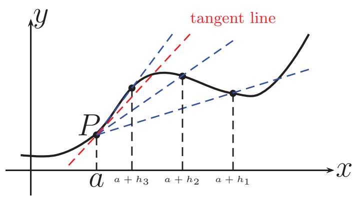
changing rapidly near $x = a$

We define the *instantaneous rate of change* of a curve at a point $x = a$ to be the slope of the tangent line at $x = a$.

Why is it called a *tangent* line?

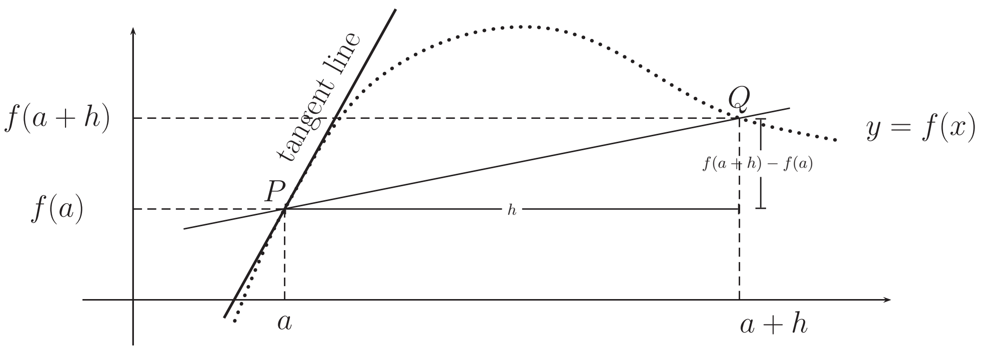
How do we obtain the slope of the tangent line?

Consider what happens when we take the average rate of change over smaller and smaller intervals:



The secant line converges to the tangent line as we take smaller and smaller intervals.

The instantaneous rate of change at the point $P = (a,\, f(a))$ is the limiting value of the slope of the secant line through $P(a, f(a))$ and $Q(a+h,\, f(a+h))$ as $h \to 0$.



The slope of the secant line is

$$m_{sec} \;=\; \frac{f(a+h) - f(a)}{h}.$$

As $h \to 0$, so that Q approaches P along the graph of the curve, we obtain the formula

$$m_{\tan} \;=\; \lim_{h \to 0} m_{\sec} \;=\; \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \qquad (*)$$

This represents the *instantaneous rate of change of f with respect to x at P.*

We can generalise formula $(*)$ to find the slope of the tangent line at any point x along the curve $y = f(x)$ by replacing a with x:

$$m_{\tan} \; = \; \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \, .$$

This limit is so important we give it a special name and notation.

Definition 1 If the limit

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists, it is called the ***derivative*** of the function $f(x)$ at x and is denoted by $f'(x)$ or $\dfrac{df}{dx}$.

Exercise 2 Derivatives by first principles

Use this definition to find the following:

(a) The slope of the curve $f(x) = x^2 - 4x$ at the point $P(1, -3)$.

(b) The derivative of $f(x) = x^2 - 4x$.

Exercise 3 Derivatives by first principles

Use the definition of the derivative to show that the function $f(x) = |x|$ is not differentiable at $x = 0$.

Differentiability: the existence of derivatives

There are three common ways that a derivative at a point fails to exist.

- **At a point of discontinuity:** Consider the function $f(x) = \begin{cases} x+1 & \text{if } x \geq 1 \\ x & \text{if } x < 1 \end{cases}$.

  This function is not continuous at $x = 1$ (the limit does not even exist), so there is no reasonable way to draw a unique tangent line at this point.

- **At a corner:** For example, $f(x) = |x|$ is not differentiable at the point $x = 0$.

  The graph of a continuous function can have sharp corners, but the graph of a differentiable function cannot. It is smooth.

- **At a vertical tangent:** For example, $f(x) = \sqrt[3]{x}$ is not differentiable at $x = 0$.
  The tangent line to the graph at this point is vertical (sketch it!), so the slope at $x = 0$ is undefined.

It seems reasonable that a function has to be continuous at a point in order to be differentiable at that point, but continuity itself is not enough to guarantee differentiability. The function must be both continuous *and* smooth to be differentiable. But differentiability guarantees continuity:

$$\boxed{\text{If } f(x) \text{ is differentiable at a point } a\text{, then it is also continuous at } a.}$$

2 Techniques of differentiation (AC 2.3-2.5)

Differentiation is a major tool of mathematics. In practice, rather than using "first principles" to find derivatives, we have the basic derivatives at our fingertips and some rules from which we work out others. Below are the basic rules which you will have used at school. We will not work through the proofs for these rules, but if you are interested you can look these up in the recommended textbook.

Differentiation Rules

Linearity

- $\dfrac{d}{dx}(f \pm g) = \dfrac{df}{dx} \pm \dfrac{dg}{dx}$

The Power Rule

- $\dfrac{d}{dx}(x^n) = n\,x^{n-1}$

The Product Rule

- $\dfrac{d}{dx}(f\,g) = \dfrac{df}{dx}\,g + f\,\dfrac{dg}{dx}$

The Quotient Rule

- $\dfrac{d}{dx}\left(\dfrac{f}{g}\right) = \dfrac{\dfrac{df}{dx}\,g - f\,\dfrac{dg}{dx}}{g^2}$

Table of Basic Derivatives

- $\dfrac{d}{dx}(\sin x) = \cos x$
- $\dfrac{d}{dx}(\cos x) = -\sin x$
- $\dfrac{d}{dx}(\tan x) = \sec^2 x$
- $\dfrac{d}{dx}(e^x) = e^x$
- $\dfrac{d}{dx}(\csc x) = -\csc x \cot x$
- $\dfrac{d}{dx}(\sec x) = \sec x \tan x$
- $\dfrac{d}{dx}(\cot x) = -\csc^2 x$
- $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$

Comment

- You should always first write square roots and quotients such as $\dfrac{1}{x^2}$ in terms of fractional and negative exponents before differentiating, as it makes it easier to apply the power rule.
- We will verify some of these rules from first principles. For the derivation of other rules such as those for exponential and log functions, interested students can check out the relevant section in the recommended textbook.

What is the derivative of a constant function, $f(x) = c$?

What is the derivative of a linear function, $f(x) = mx + b$?

Exercise 4 Finding Derivatives

Find the derivatives of the following functions

(a) $f(x) \;=\; \sqrt{x}$

(b) $h(y) = \dfrac{y^4 - 36}{y^2}$

(c) $g(u) \;=\; \dfrac{1 - u^2}{1 + u^2}$

(d) $h(x) \;=\; e^x \, \sin x$

(e) Find the point or points on the function $f(x) = x(\ln x - 1)$ where the tangent line is horizontal.

Exercise 5 Tangent Lines

Find the equation of the line tangent to the graph of $y = \dfrac{1}{x}$ at the point $(a, 1/a)$.

Derivatives of Trigonometric Functions

In this section, we will verify the derivative of the trigonometric function $f(x) = \sin x$. The same technique can be used to find the derivative of $f(x) = \cos x$ (see tutorial exercises). Note that when differentiating and integrating with trigonometric functions, we require angles to be taken in radians.

The trig functions $\tan x$, $\sec x$, $\csc x$, and $\cot x$ are all defined in terms of $\sin x$ and $\cos x$. Therefore once we have verified the derivatives of these two functions, the other derivatives follow via the quotient rule.

To verify the derivative of $f(x) = \sin x$, we will need the following limits, which were obtained in the previous topic.

$$\lim_{h\to 0} \frac{\sin h}{h} = 1 \quad \text{and} \quad \lim_{h\to 0} \frac{1-\cos h}{h} = 0$$

We will also need the addition properties of the sin and cos functions.

$$\begin{aligned} \sin(a+b) &= \sin a\ \cos b + \cos a\ \sin b \\ \cos(a+b) &= \cos a\ \cos b - \sin a\ \sin b \end{aligned}$$

Exercise 6 Derivative of $\sin x$

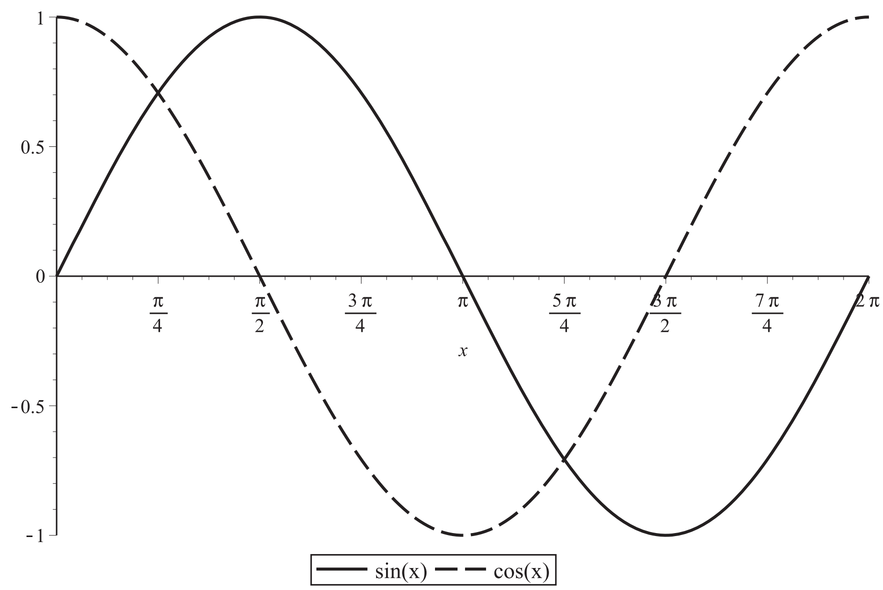
Verify that $\dfrac{d}{dx}(\sin x) = \cos x$.

The derivative of $f(x) = \cos x$ can be verified similarly (see tutorial exercise). The other trigonometric functions can now be differentiated using the results for the sine and cosine functions and techniques of differentiation such as the quotient rule.

Exercise 7 Derivatives of trig functions

Find $\dfrac{d}{dx}(\csc x)$.

Geometric viewpoint for $\dfrac{d}{dx}(\sin x) = \cos x$: consider the graphs of $f(x) = \sin x$ and $g(x) = \cos x$ below. Try sketching the graph of the derivative of $g(x) = \cos x$ and verify that this is the graph of $-\sin(x)$.



Trigonometric functions are widely used in modelling situations such as waves, elastic motion and other quantities which vary in a periodic manner. Next we consider an example from simple harmonic motion.

Higher Derivatives

If $y = f(x)$ is a differentiable function, then its derivative $f'(x)$ is also a function. If f' is also differentiable, then we can differentiate f' to get a new function of x which we write as f''. This function f'' is called the *second derivative* of f.

If instead we use the alternative notation $\dfrac{dy}{dx}$ for $f'(x)$, then for the second derivative we write

$$f''(x) = \frac{d^2y}{dx^2}.$$

We can repeat this process to get higher derivatives,

$$f^{(n)}(x) = \frac{d^ny}{dx^n}.$$

Exercise 8 Higher derivatives

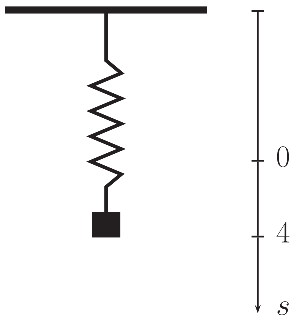
Find the first four derivatives of $y = x^3 - 3x^2 + 2$.

Exercise 9 Derivatives of trig functions

An object at the end of a vertical spring is stretched 4 cm beyond its equilibrium position and released at time $t = 0$. Its position at time t is

$$s = f(t) = 4 \cos t$$

Find the velocity and acceleration at time t and use them to analyze the motion.



3 The Chain Rule (AC 2.6)

The chain rule is one of the most important skills in all of calculus. Before we look at this rule, let's review composite functions.

$f(g(x))$ or $(f \circ g)(x)$ describes a composite function where the output of the "inner" function g becomes the input of the "outer" function f.

If a mountain climber's height at time t is given by the function $h(t)$, and the oxygen concentration at altitude x is given by the function $c(x)$, then what does $c(h(t))$ describe?

Suppose that each person in Christchurch contributes 0.02 ppm (parts per million) to the carbon monoxide level in the air, and suppose that the city's population is growing at a rate of 10,000 people per year. What is the rate of change of the CO levels with respect to time? Express this as a mathematical function.

Exercise 10 The chain rule

An environmental study of a suburban community suggests that the daily level of carbon monoxide in the air is described by the formula

$$L(p) = 2\sqrt{p} + 17$$

parts per million when the population is p thousand. It is estimated that t years from now, the population of the community (in thousands of people) will be

$$p(t) = 3.1 + 0.1t^2 ,$$

At what rate will the carbon monoxide level be changing with respect to time 3 years from now?

The chain rule

If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x, then the composition $y = f(g(x))$ is a differentiable function of x and

$$\frac{dy}{dx} = \frac{dy}{du}\,\frac{du}{dx}\,.$$

This is sometimes written in the form: $(f \circ g)'(x) = [f(g(x))]' = f'(g(x))\; g'(x)\,.$

Exercise 11 The chain rule

Differentiate the following functions using the chain rule.

(a) $f(x) = \sqrt{x^2 + 1}$

(b) $f(x) = \sin^2 x$

(c) $f(x) = \dfrac{1}{x^3 + 2x + 1}$

(d) $f(x) = \ln(\cos x)$

(e) $g(t) = e^{-2t}\ (A\cos(5t) + B\sin(5t))$

(f) $y(t) = A\sin\left(\sqrt{\dfrac{k}{m}}\ t\right)$

(g) Challenge! $\left[1+\sin^3(x^2)\right]^{12}$

Exercise 12 Related Rates

A spherical balloon is being inflated at a constant rate of F litres per second. Using that the volume of a sphere of radius r is $4\pi r^3/3$, determine the rate of change of the radius of the balloon with respect to time.

Exercise 13 Deriving Trigonometric Identities

(a) Consider the trigonometric identity $\cos 2t = \cos^2 t - \sin^2 t$. Use differentiation to obtain a trigonometric identity for $\sin 2t$.

(b) Verify that you obtain the same identity for $\sin 2t$ if you differentiate the identity $\cos 2t = 2\cos^2 t - 1$.

Exercise 14 Absolute Value Differentiation

Using the identity $|x| = \sqrt{x^2}$ and the chain rule, show that

$$\frac{d}{dx}|x| = \frac{x}{|x|}, \qquad x \neq 0$$

Decide whether the following statements are true or false.

(a) The function $f(x) = x \sin x$ can be differentiated without using the chain rule.

(b) The function $f(t) = (t^2 + 1)^{-2}$ must be differentiated using the chain rule.

(c) The derivative of a product is *not* the product of the derivatives, but the derivative of a composition is a product of derivatives.

(d) $\dfrac{d}{dx} P(Q(x)) = P'(x)\, Q'(x)$

4 Implicit Differentiation (AC 3.1)

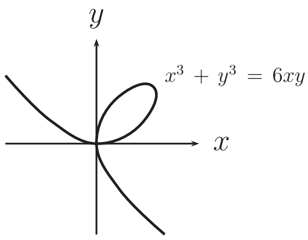
So far we have been differentiating functions of the form $y = f(x)$ where the dependent variable is expressed explicitly in terms of the independent variable x. But equations are not always given in this form. For example, in the equations

$$x^2 + y^2 = 1, \quad x\,y = 1, \quad \text{and} \quad x^3 + y^3 = 6xy$$

y, the dependent variable, is defined *implicitly* in terms of x, the independent variable.

- An equation of the form $y = f(x)$ and defines y **explicitly** as a function of x.
- **Implicit** equations such as $x^2 + y^2 = 4$ express y **implicitly** as a function of x.

Finding derivative of an implicit equation by first expressing y explicitly in terms of x can be difficult and sometimes impossible. For example, we cannot write $x^3 + y^3 = 6xy$ as a function of y. This equation is a curve called the **folium of Descartes**, as shown below.



In this equation the value of y is dependent on the value of x. We can see from the graph above that this curve is not a function (why?), but it can be described as three separate functions. We say that the equation implicitly defines y as three functions of x.

Compare this with the implicit equation for a circle, e.g. $x^2 + y^2 = 1$. This equation implicitly defines two functions:

$$y = \sqrt{1 - x^2} \qquad \text{and } y = -\sqrt{1 - x^2}.$$

Implicit Differentiation

Fortunately we don't need to solve such an equation for y in terms of x to find the derivative of y. We use a simple technique, based on the chain rule, called ***implicit differentiation***.

For example, to differentiate y^3 with respect to x we remember that y is a function of x, that is, $y = f(x)$. So we are actually trying to differentiate $(f(x))^3$ without having an explicit expression for $f(x)$. Therefore we need to use the chain rule for a composite function:

$$\frac{d}{dx}\left([f(x)]^3\right) = 3[f(x)]^2\, f'(x)$$

where

$3[f(x)]^2$ is the "outside" function differentiated wrt f, and

$f'(x)$ is the "inside" function differentiated wrt x.

In practice we use y rather than $f(x)$ but we remember that y is a function of x. That is,

$$\frac{d}{dx}\left(y^3\right) = 3y^2\,\frac{dy}{dx}$$

Exercise 15 Implicit differentiation

Use implicit differentiation to find $\dfrac{dy}{dx}$ in terms of x and y.

(a) $x^2 + y^2 = 9$

(b) $e^x + e^y = 1$

(c) $\cos(2y) = x$

(d) $\ln(x) + \ln(y^2) = 3$

Exercise 16 Information from implicit differentiation

(a) Find $\dfrac{dy}{dx}$ given that $x^2 + y^2 - 4x + 7y = 15$.

(b) For what values of x and/or y is the tangent line to this curve horizontal? Vertical?

Implicit Differentiation and the Product Rule

To differentiate an expression like $6xy$ we must use the product rule, since we are differentiating the product of $6x$ and y. Remember that y is a function of x, that is, y is dependent on x.

$$\frac{d}{dx}(6xy) = 6 \times y + 6x \times \frac{d}{dx}(y) = 6y + 6x\frac{dy}{dx}$$

When using implicit differentiation, it is important to keep in mind which is the independent variable (often this is x or t, but not always!) and which is the dependent variable.

Exercise 17 Implicit differentiation

Use implicit differentiation to find $\frac{dy}{dx}$ in terms of x and y.

(a) $x\,y^2 = 1$

(b) $\sin(xy) = x^2$

Exercise 18 Implicit differentiation

For the folium of Descartes, $x^3 + y^3 = 6xy$, find $\dfrac{dy}{dx}$ in terms of x and y.

Exercise 19 Review: Two methods compared

Consider the equation $xy = 1$. We can now differentiate this function in two different ways.

(a) First write as an explicit function $y = f(x)$ and differentiate.

(b) Use implicit differentiation and compare your answer to part (a).

Exercise 20 Finding tangent lines

Find the equation of the tangent line at the point (4, 0) on the curve

$$7y^4 + x^3y + x = 4\,.$$

Exercise 21 Solving problems involving tangent lines

Find the points where the two lines through the origin are tangent to the circle

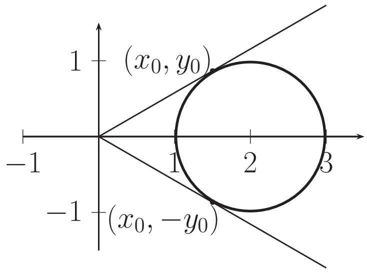
$$x^2 - 4x + y^2 + 3 = 0\,.$$

Note: This equation can be written in the standard form

$$(x-2)^2 + y^2 = 1$$

which is the equation of a circle of radius 1 and center (2, 0).

First draw a diagram of the situation and observe any geometric features that may help, for example, symmetry.



5 Derivatives of Exponential Functions (AC 3.3)

We know that the derivative of $y = e^x$ is $\dfrac{dy}{dx} = e^x$, that is, the slope of the function e^x at a given point is equal to the value of the function at that point. However this is only true for base e.

Can we find the derivative of a general function $y = a^x$ where $a > 0$?

We may attempt to do this using the formal definition of the derivative:

$$\frac{d}{dx}\, a^x = \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} = \lim_{h\to 0} \frac{a^x(a^h - 1)}{h} = \left(\lim_{h\to 0} \frac{a^h - 1}{h} \right) a^x.$$

This does not represent any real progress since we still have a difficult limit to evaluate.

Instead we can use the properties of the natural logarithm function and implicit differentiation to find the derivative.

Exercise 22 Differentiation of a^x
Find the derivative of $f(x) = a^x$ where $a > 0$.

Using logarithms and implicit differentiation also gives us a method for differentiating functions such as $f(x) = x^{\sin x}$. This method is sometimes called logarithmic differentiation. The idea is that when x appears in the exponent, we take logs before differentiating.

Exercise 23 Logarithmic Differentiation

Find the derivative of $f(x) = x^{\sin x}$.

6 Derivatives of Inverse Functions (AC (3.3)

Using the Chain Rule to Differentiate Inverse Functions

Recall the chain rule for a composite function:

$$[f(g(x))]' \;=\; f'(g(x))\,g'(x).$$

Using the chain rule, write down the formula for the following derivative:

$$[g(f(x))]' \;=$$

We can apply the chain rule in the special case when the functions f and g are inverse functions to allow us to find the derivatives of inverse functions.

If f and g are inverse functions, then $f(g(y)) = y$ and $g(f(x)) = x$.

Consider $g(f(x)) = x$ and differentiate both sides with respect to x.

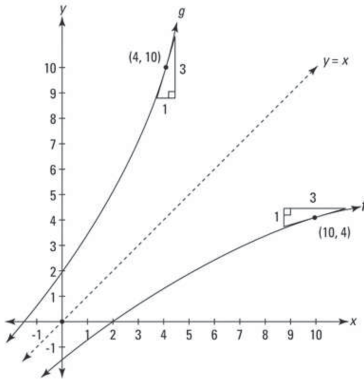
$$\begin{aligned}
\Rightarrow \quad \frac{d}{dx}\,(g(f(x))) &= \frac{d}{dx}\,(x) \\
\Rightarrow \quad g'(f(x))\,f'(x) &= 1 \\
\Rightarrow \quad g'(y)\,f'(x) &= 1 \\
\Rightarrow \quad \frac{dx}{dy}\,\frac{dy}{dx} &= 1 \\
\Rightarrow \quad \frac{dy}{dx} &= \frac{1}{\frac{dx}{dy}}
\end{aligned}$$

So if we have a function $y = f(x)$, we can take its inverse function $x = f(y)$ and find $\dfrac{dx}{dy}$,

then we obtain the derivative of the original function y by using the formula $\dfrac{dy}{dx} \;=\; \dfrac{1}{\frac{dx}{dy}}$.

This method is useful if $\dfrac{dx}{dy}$ is an easier derivative to find, e.g. for inverse trigonometric functions.

For a geometric illustration of this result, consider the following diagram:



This figure shows a pair of inverse functions, f and g. Inverse functions are symmetrical with respect to the line $y = x$. That is, if (x, y) is a point on the graph of f, then (y, x) is a point on the graph of $g = f^{-1}$. For example, if $(10, 4)$ is a point on the graph of f, then $(4, 10)$ lies on the graph of g. Because of the symmetry of the graphs, the slopes at these points are reciprocals.

f : At the point $(10, 4)$, the slope of the tangent is $\frac{1}{3}$.

$g = f^{-1}$: At the point $(4, 10)$, the slope of the tangent is $\frac{3}{1}$.

In general,

f : At the point (x, y), the slope of the tangent is m.

$g = f^{-1}$: At the point (y, x), the slope of the tangent is $\frac{1}{m}$.

Exercise 24 Differentiation of $\sqrt{x}$

By taking the inverse function and using the result above, verify that $\dfrac{d}{dx}\sqrt{x} = \dfrac{1}{2\sqrt{x}}$.

Exercise 25 Differentiation of the Inverse Sine Function

An important exercise! Show that

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad \text{for} \quad -1 < x < 1 .$$

Exercise 26 Differentiation of an Inverse Trig Function using the Chain Rule

Use this result to find $\dfrac{d}{dx}(\sin^{-1}(3x))$.

Exercise 27 Relationship between inverse trig functions

There is a special relationship between $\sin^{-1} x$ and $\cos^{-1} x$. Show that

$$\cos^{-1} x + \sin^{-1} x = \frac{\pi}{2}$$

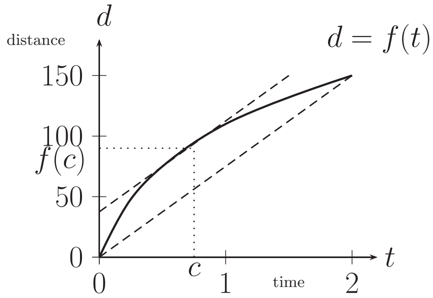
and use this identity to find the derivative of $y = \cos^{-1} x$.

7 The Mean Value Theorem and Rolle's Theorem (AC 4.8)

We now take a (slightly only!) theoretical look at one of the cornerstones of calculus: the Mean Value Theorem. This is the foundation that underlies the applications of differentiation we will meet in the next topic, because many of the proofs of the results we will use rely on this theorem.

The sort of question the MVT answers is: "If a car averages 75 km/hr on a trip is there a moment when the speedometer reads exactly 75km/hr?"

Intuitively, you would say "yes".



Here, the average speed on this trip is given by

$$\frac{\text{change in distance}}{\text{change in time}} = \frac{f(2) - f(0)}{2 - 0} = \frac{150}{2} = 75.$$

What the MVT theorem concludes is that there is at least one point in time, c, where the instantaneous velocity is also 75.

Geometrically, the average speed is the slope of the secant line through (0, 0) and (2, 150) and the instantaneous speed is the slope of the tangent line to the curve at $(c,\ f(c))$.

The Mean Value Theorem

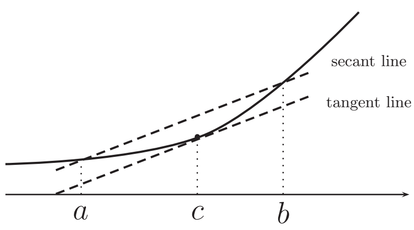
Suppose that $f(x)$ is continuous on the closed interval $[a,\ b]$ and is differentiable on the open interval $(a,\ b)$. Then there is at least one point c in $(a,\ b)$ where

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Note that the theorem doesn't tell us the value of c, just that such a point exists. In some simple cases, we can find the value of c, as shown in the following exercise.

Geometrically, this theorem says that

"there is at least one point c in the interval $(a,\ b)$ where the slope of the tangent line to the curve at $(c,\ f(c))$, is equal to the slope of the secant line through $(a,\ f(a))$ and $(b,\ f(b))$.".



Exercise 28 Using the MVT

Determine whether the following functions satisfy the Mean Value Theorem on the given intervals. If so, find the points that are guaranteed to exist by this theorem.

(a) $f(x) = \sqrt{x}, \quad [1, 4]$

(b) $f(x) = \dfrac{1}{x}, \quad [-1, 1]$

(c) $f(x) = \sqrt{25 - x^2}, \quad [-5, 5]$

A special case of the MVT is called *Rolle's theorem*.

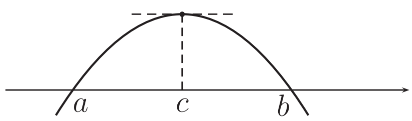
Rolle's Theorem

Suppose that $f(x)$ is continuous on the closed interval $[a,\, b]$ and is differentiable on the open interval $(a,\, b)$.

If $f(a) = f(b)$, then at some point c in $(a,\, b)$, $f'(c) \;=\; 0$.

Geometrically, this theorem says that if $f(a) = f(b)$ then

" there is at least one point c in the interval $(a,\, b)$ where the slope of the tangent line to the curve at $(c,\, f(c))$ is zero".



Learn these theorems and their geometrical interpretation. If you are interested, you can look up the proofs in Anton, but the proofs are not examinable.

Caution: You *must* make sure (and state) that the conditions of a theorem are satisfied before applying the theorem.

Exercise 29 Rolle's Theorem

Let $f(x) = 1 - x^{2/3}$. Show that $f(-1) = f(1)$ but there is no number c in $(-1, 1)$ such that $f'(c) = 0$. Why does this not contradict Rolle's Theorem?

Exercise 30 Rolle's Theorem and motion

A moving object is described by the position function $s = f(t)$, where f is a polynomial. If the object is in the same position at $t = a$ and $t = b$, that is, $f(a) = f(b)$, what can you conclude about the object's velocity?

Verify this statement for $f(t) = 60t - 5t^2$.

Two Useful Results

The first of these relies on the MVT for its proof (like many fundamental results in calculus).

> **MVT: Consequence 1**
>
> Suppose that $f(x)$ is continuous on the closed interval $[a,\, b]$ and is differentiable on the open interval $(a,\, b)$.
>
> If $f'(x) = 0$ at all points in $(a,\, b)$, then $f(x)$ is constant.

Proof:

Let x_1 and x_2 be any two distinct points in the interval $[a,\, b]$ such that $x_1 < x_2$.

Now apply the MVT to $f(x)$ on the interval $[x_1,\, x_2]$.

There is a point c in $(x_1,\, x_2)$ such that

$$f'(c) \;=\; \frac{f(x_2) - f(x_1)}{x_2 - x_1}\,.$$

Since $f'(x) = 0$ at all points in $(a,\, b)$,

$$\begin{aligned} 0 \;&=\; \frac{f(x_2) - f(x_1)}{x_2 - x_1} \\ 0 \,\times\, (x_2 \,-\, x_1) \;&=\; f(x_2) \,-\, f(x_1) \\ \text{so that} \qquad f(x_1) \;&=\; f(x_2) \end{aligned}$$

Because x_1 and x_2 were any two *general* points in $[a,\, b]$, we can conclude that $f(x)$ is constant on this interval.

The second is a consequence of this result.

> **MVT: Consequence 2**
>
> Suppose that $f(x)$ and $g(x)$ are continuous on the closed interval $[a,\, b]$ and are differentiable on the open interval $(a,\, b)$.
>
> If $f'(x) \;=\; g'(x)$ at all points in $(a,\, b)$, then $f(x)$ and $g(x)$ *differ* by a constant.

Proof:

Set

$$H(x) \;=\; f(x) \;-\; g(x)$$

The result will be proved if we can show that $H(x)$ is a constant, since $H(x)$ measures the difference between f and g.

To show that $H(x)$ is constant, we can show that the derivative of $H(x)$ is zero, that is, $H'(x) \;=\; 0$.

Differentiating $H(x)$:

$$H'(x) \;=\; f'(x) \;-\; g'(x) \;=\; 0$$

since $f'(x) = g'(x)$.

By the MVT: Consequence 1, $H'(x) = 0$ means that $H(x)$ has to be a constant.

That is, $f(x)$ and $g(x)$ differ by a constant.

Exercise 31 Applying MVT Consequence 2

Find a function f such that the graph of f contains the point $\left(\frac{\pi}{2}, -3\right)$ and the derivative of f is equal to the derivative of $g(x) = \sin(x)$ for all x.