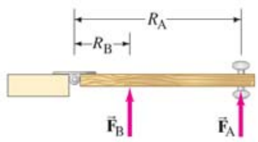
Lecture 7: Torque and Moment of Inertia

SJW^2 9.4-9.8

The majority of material in this Lecture is review material and has previously been covered in NCEA (Standard 90521) and/or equivalents.

Torque

- Forces can cause a change in translational motion described by Newton's Second Law.
- Forces can also cause a change in rotational motion. The effectiveness of this change depends on the force and the moment arm (perpendicular distance from the rotation axis to the line of action of the force). The change in rotational motion depends on the torque.



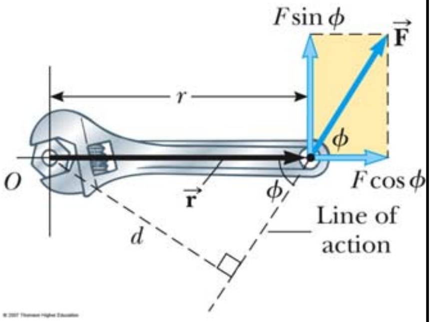
Top view of a door. Applying the same force with different lever arms, R_A and R_B. If $R_A = 3R_B$, then to create the same effect (angular acceleration), $3F_A = F_B$.

Torque

- Torque, τ, is the tendency of a force to rotate an object about some axis.
- Torque is a vector, but we will deal with its magnitude initially:

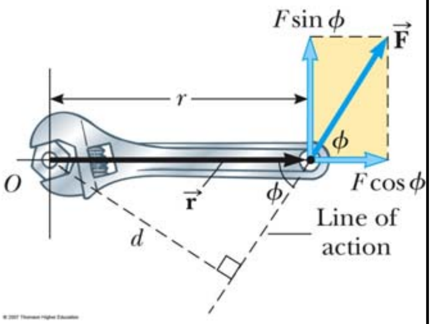
  $\tau = r\,F \sin\phi = F\,d$

  where F is the force, ϕ is the angle the force makes with the horizontal and d is the *moment arm* (or lever arm) of the force.



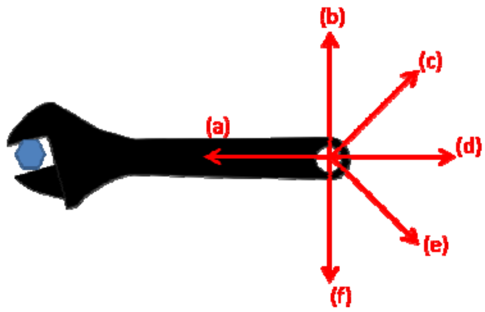
Torque

- The horizontal component of the force (F cos ϕ) has no tendency to produce a rotation. While the vertical component of the force (F sin ϕ) produces rotation.
- Torque will have direction.
  - If the turning tendency of the force is counterclockwise, the torque will be positive.
  - If the turning tendency is clockwise, the torque will be negative.



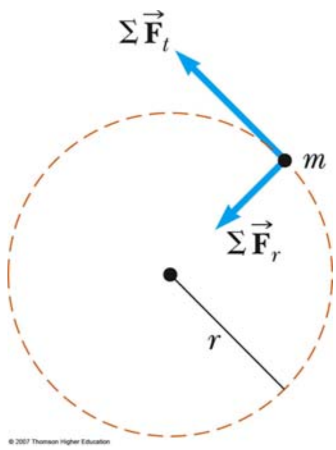
Check Understanding

Identify the largest torque for a force applied in directions (a) to (f). You may assume that the pivot point is at the centre of the bolt.



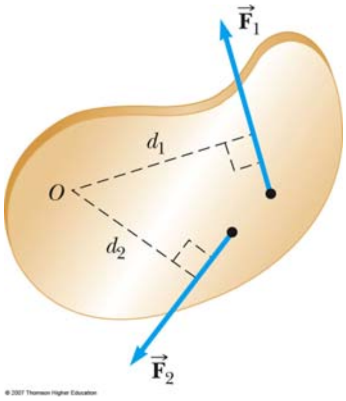
Torque and Angular Acceleration

- Consider a particle of mass *m* rotating in a circle of radius *r* under the influence of tangential force $\vec{\mathbf{F}}_t$.
- The tangential force provides a tangential acceleration:
  - $F_t = ma_t$
- The radial force, $\vec{\mathbf{F}}_r$ causes the particle to move in a circular path.



Net Torque

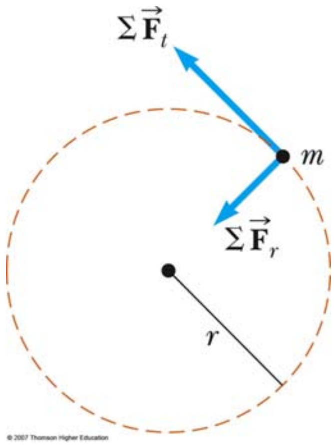
- The force $\vec{F}_1$ will tend to cause a counterclockwise rotation about the rotation axis (*O*).
- The force $\vec{F}_2$ will tend to cause a clockwise rotation about *O*.
- $\Sigma\tau = \tau_1 + \tau_2 = F_1 d_1 - F_2 d_2$



PLAY
ACTIVE FIGURE 9.17

Torque and Angular Acceleration

- The magnitude of the torque produced by $\sum \vec{F}_t$ around the center of the circle:
  - $\Sigma\tau = \Sigma F_t\, r = (ma_t)\, r$
- The tangential acceleration is related to the angular acceleration by:
  - $\Sigma\tau = (ma_t)\, r = (mr\alpha)\, r = (mr^2)\, \alpha$
- We now introduce a new variable, the moment of inertia, $I = mr^2$, of the particle:
  - $\Sigma\tau = I\alpha$
- The torque is directly proportional to the angular acceleration and the constant of proportionality is the moment of inertia.



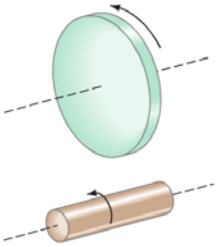
MYTHBUSTERS : Hellboy Punch

Moment of Inertia

- The definition of moment of inertia is

$$I = \sum_i r_i^2 m_i$$

- Note that the moment of inertia of a body depends not only on its mass but also on how the mass is distributed.
- **Moment of inertia is the quantity which expresses a body's tendency to resist angular acceleration**
- The dimensions of moment of inertia are ML^2 and its SI units are kg·m^2.
- We can calculate the moment of inertia of an object more easily by assuming it is divided into many small volume elements, each of mass Δm_i.



A large diameter cylinder has a greater rotational inertia than one of equal mass but smaller diameter.

Moment of Inertia

- We can rewrite the expression for I in terms of Δm_i

$$I = \lim_{\Delta m_i \to 0} \sum_i r_i^2 \Delta m_i = \int r^2 dm$$

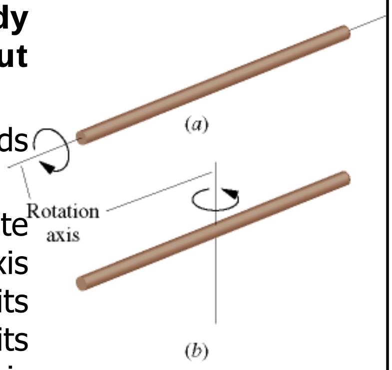
- With the small volume segment assumption,

$$I = \int \rho r^2 dV$$

- If ρ is constant, the integral can be evaluated with known geometry, otherwise its variation with position must be known.

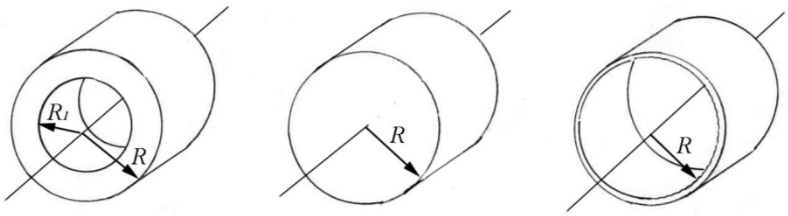
Moment of Inertia

- **The moment of inertia of a body depends not only on its mass but how the mass is distributed.**
- The moment of inertia also depends on the axis of rotation.
- A long rod is much easier to rotate about its central (longitudinal) axis (a) than about an axis through its centre (b) and perpendicular to its length because the mass is distributed closer to the rotation axis in (a) than in (b).



Check Understanding

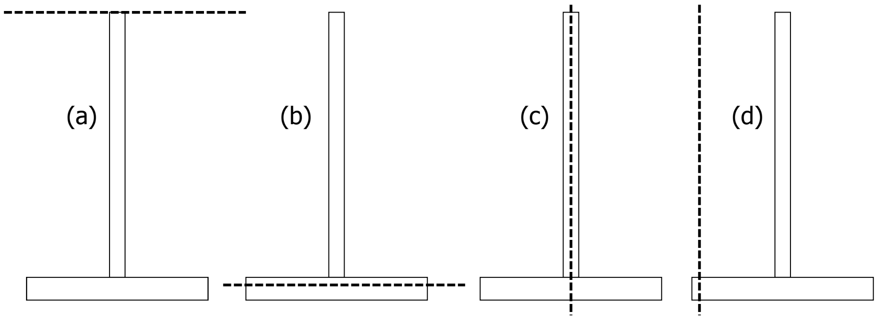
Which of the objects below has the largest moment of inertia? You may assume that they all have the same mass.



a) b) c)

Check Understanding

Four Ts are made from two identical rods of equal mass and length. Identify the one with the largest moment of inertia about the dashed line.

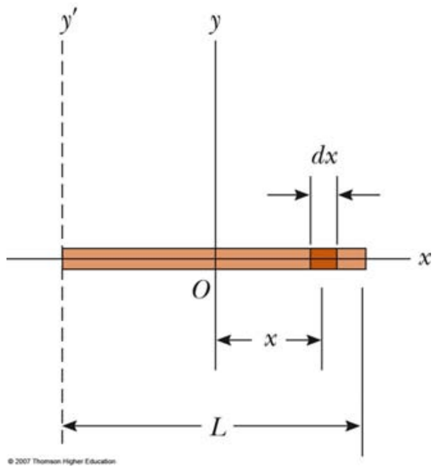


Moment of Inertia of a Uniform Rigid Rod

- The shaded area has a mass
  - $dm = \lambda\ dx$
- Then the moment of inertia is

$$I_y = \int r^2 dm = \int_{-L/2}^{L/2} x^2 \frac{M}{L} dx$$

$$I = \frac{1}{12} ML^2$$

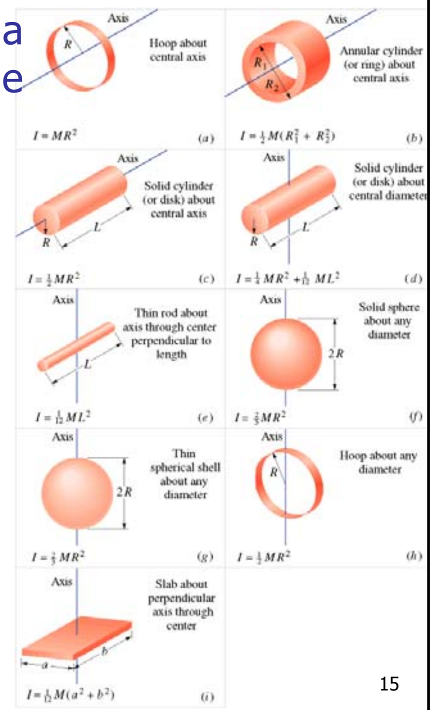


Proof

Moments of inertia for some simple geometrical bodies

- The moment of inertia for bodies of different shapes and sizes can be computed and results in a simple formula in terms of the total mass *M* and dimensions.

- The formulae for some simple bodies of uniform density are shown in the figure on the right.



Parallel-Axis Theorem

- In the previous examples, the axis of rotation coincided with the axis of symmetry of the object
- For an arbitrary axis, the parallel-axis theorem can be used. The theorem states:

$$I = I_{CM} + MD^2$$

  - I is the moment of inertia about any axis parallel to the axis through the center of mass of the object
  - I_{CM} is the moment of inertia about the axis through the center of mass
  - D is the distance from the center of mass axis to the arbitrary axis

Moment of Inertia for a Rod Rotating Around One End

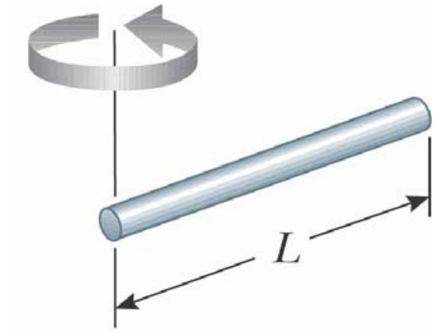
- The moment of inertia of the rod about its center is :

$$I_{CM} = \frac{1}{12}ML^2$$

- D is ½ L
- Therefore:

$$I = I_{CM} + MD^2$$

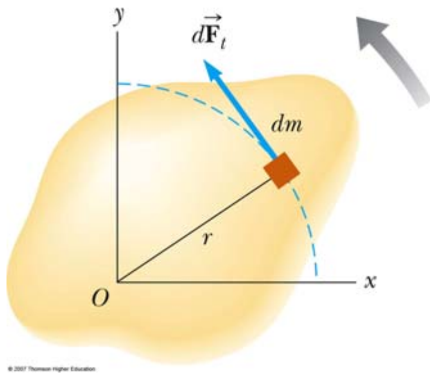
$$I = \frac{1}{12}ML^2 + M\left(\frac{L}{2}\right)^2 = \frac{1}{3}ML^2$$



Proof

Torque and Angular Acceleration

- For an extended object, consider the object to consist of an infinite number of mass elements *dm* of infinitesimal size.
- Each mass element rotates in a circle about the origin, *O*.
- Each mass element has a tangential acceleration.
- From Newton's Second Law
  $dF_t = (dm)\, a_t$



Torque and Angular Acceleration

- The torque associated with the force and using the angular acceleration gives
  - $d\tau = r\,dF_t = a_t r\,dm = \alpha r^2\,dm$
- Finding the net torque
  - $\sum \tau = \int \alpha r^2 dm = \alpha \int r^2 dm$ which becomes $\Sigma\tau = I\alpha$
- This is the same relationship that applied to a particle.
- This is the mathematic representation of the analysis model of a rigid body under a net torque.

Torque

- The torque $\vec{\tau}$ produced by a force $\vec{F}$ about some axis of rotation is actually better defined by the vector equation:

$$\vec{\tau} = \vec{r} \times \vec{F}$$

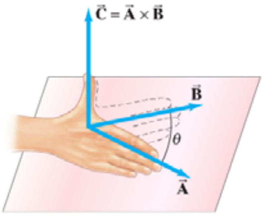
- where $\vec{r}$ is the position vector for the point through which the force acts.
- The torque vector lies in a direction perpendicular to the plane formed by the position vector and the force vector.
- The torque is the vector (or cross) product of the position vector and the force vector.

The Vector Product

- There are instances where the product of two vectors is another vector.
- The vector product of two vectors is also called the cross product.
- Given two vectors, $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$
- The vector (cross) product of $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ is defined as a *third vector*:

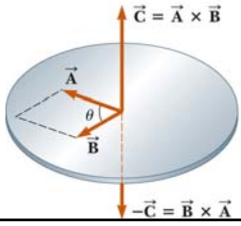
$$\vec{\mathbf{C}} = \vec{\mathbf{A}} \times \vec{\mathbf{B}}$$

The vector $\vec{\mathbf{C}} = \vec{\mathbf{A}} \times \vec{\mathbf{B}}$ is perpendicular to the plane containing $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$; its direction is given by the right hand rule.



The Vector Product

- The vector (cross) product of $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ is defined as a *third vector*, $\vec{\mathbf{C}} = \vec{\mathbf{A}} \times \vec{\mathbf{B}}$
  - **C** is read as "**A** cross **B**"
- The magnitude of vector C is $AB\sin\theta$ *where* θ is the angle between $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$
- The direction of $\vec{\mathbf{C}}$ is perpendicular to the plane formed by $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ and can be found using the Right hand rule.



Right-hand rule

Vector Product

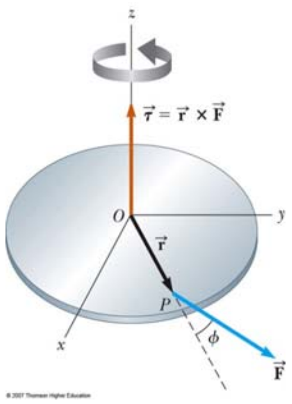
- Given:
$$\hat{\mathbf{i}}\times\hat{\mathbf{i}}=\hat{\mathbf{j}}\times\hat{\mathbf{j}}=\hat{\mathbf{k}}\times\hat{\mathbf{k}}=0$$
$$\hat{\mathbf{i}}\times\hat{\mathbf{j}}=-\hat{\mathbf{j}}\times\hat{\mathbf{i}}=\hat{\mathbf{k}}$$
$$\hat{\mathbf{j}}\times\hat{\mathbf{k}}=-\hat{\mathbf{k}}\times\hat{\mathbf{j}}=\hat{\mathbf{i}}$$
$$\hat{\mathbf{k}}\times\hat{\mathbf{i}}=-\hat{\mathbf{i}}\times\hat{\mathbf{k}}=\hat{\mathbf{j}}$$
- The cross product can also be expressed as:

$$\vec{\mathbf{A}}\times\vec{\mathbf{B}}=\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}=\begin{vmatrix}A_y & A_z\\ B_y & B_z\end{vmatrix}\hat{\mathbf{i}}-\begin{vmatrix}A_x & A_z\\ B_x & B_z\end{vmatrix}\hat{\mathbf{j}}+\begin{vmatrix}A_x & A_y\\ B_x & B_y\end{vmatrix}\hat{\mathbf{k}}$$

- Which on expanding the determinants gives:

$$\vec{\mathbf{A}}\times\vec{\mathbf{B}}=\left(A_yB_z-A_zB_y\right)\hat{\mathbf{i}}-\left(A_xB_z-A_zB_x\right)\hat{\mathbf{j}}+\left(A_xB_y-A_yB_x\right)\hat{\mathbf{k}}$$

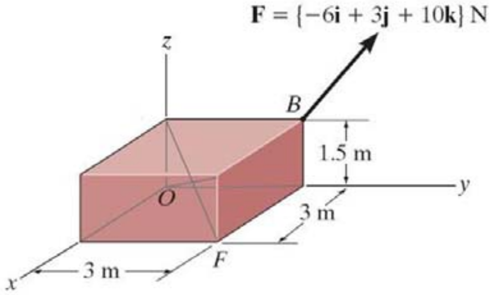
Torque



- A force, lying in the *xy* plane, acts on a particle at point P.
- The right hand rule is used to determine the direction of the cross product result.
- This force produces a torque τ on the particle with respect to the origin O. The torque vector points in the positive direction of *z*. Its magnitude is given by
$$\tau = rF\sin\theta$$
where θ is the angle between $\vec{r}$ and $\vec{F}$.

Example

- Determine the Torque associated with the Force, F, around point O (see Figure below)?

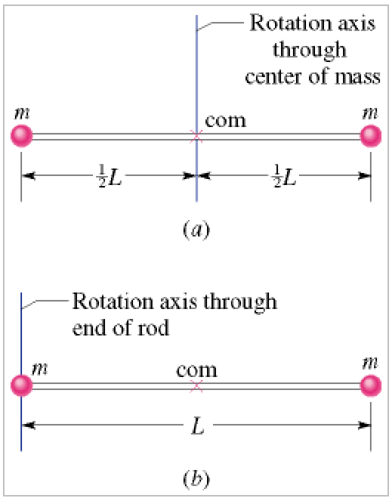


Practice Example

The figure shows a rigid body consisting of two particles of equal mass *m* connected by a rod of length *L* and negligible mass.

(a) What is the rotational inertia I_{com} of this body about an axis through its centre of mass, perpendicular to the rod as shown?

(b) What is the rotational inertia I of the body about an axis through the left end of the rod and parallel to the first axis?



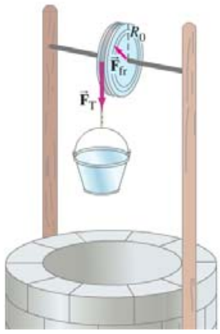
We will complete as many Practice examples as time allows. Students are encouraged to complete these examples in their own time if not completed in class. Model answers will be provided on the LEARN website.

Practice Example

A bucket of mass, 1.5kg, is suspended from a weightless cord wrapped around a pulley with a mass of 4kg and a radius R_o of 33 cm.

Calculate the angular acceleration of the pulley and the linear acceleration of the bucket?

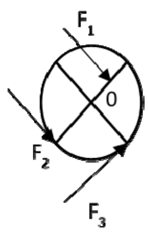
[You may assume that there is a frictional torque of 1.1Nm at the axle.]



We will complete as many Practice examples as time allows. Students are encouraged to complete these examples in their own time if not completed in class. Model answers will be provided on the LEARN website.

Practice Example

Three people reach a rotating door at the same moment and apply three forces as shown in the diagram below. The magnitudes of the forces are F_1= 50 N, F_2= 20 N and F_3= 50 N and are applied at distances of r_1=0.25 m, r_2=0.5 m and r_3=0.5 m from the rotation axis (O). Determine the net torque on the rotating door. Assuming that the moment of inertia of the door is 100 kgm^2, also identify the angular speed of the door after 2 seconds assuming that the rotating door was initially at rest.



We will complete as many Practice examples as time allows. Students are encouraged to complete these examples in their own time if not completed in class. Model answers will be provided on the LEARN website.

Practice Example

- A space station is composed of a 50 m long central tunnel which connects two rockets at either of its end, these rockets have masses of 50,000 kg and 75,000 kg respectively and their engines are perpendicular to the long axis of the tunnel. The rockets both fire their engines simultaneously, each generating a thrust of 100,000 N in opposite directions (acting to produce an anticlockwise rotation around the centre of mass of the space station).
  - Determine the moment of inertia of the space station perpendicular to the long axis of the connecting tunnel around its centre of mass assuming that the tunnel has negligible mass.
  - Determine the corresponding angular velocity of the space station after 20 seconds assuming that the space station was initially at rest and that the thrust from the rockets is constant.

We will complete as many Practice examples as time allows. Students are encouraged to complete these examples in their own time if not completed in class. Model answers will be provided on the LEARN website.

Lecture 8: Rotational Kinetic Energy, Work in Rotational Motion and Angular Momentum SJW^2 10.1,10.3-10.4

Some of the material in this Lecture is review material and has previously been covered in NCEA (Standard 90521) and/or equivalents.

Rotational Kinetic Energy

- A rotating body can be considered to be comprised of many particles, each of which describes circular motion at angular velocity, ω.
- An object rotating about some axis with an angular speed, ω, has rotational kinetic energy even though it may not have any translational kinetic energy.
- Each particle in the rigid body has a kinetic energy equal to $K_i = ½\, m_i v_i^2$
- Since the tangential velocity depends on the distance, r, from the axis of rotation, we can substitute $v_i = \omega_i\, r$.

Rotational Kinetic Energy

- The total rotational kinetic energy of the rigid object is the sum of the energies of all its particles:

$$K = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \frac{1}{2}m_3v_3^2 + \ldots$$

$$= \sum \frac{1}{2}m_i v_i^2,$$

$$K_R = \sum_i K_i = \sum_i \frac{1}{2}m_i r_i^2 \omega^2$$

$$K_R = \frac{1}{2}\left(\sum_i m_i r_i^2\right)\omega^2 = \frac{1}{2}I\omega^2$$

where I is called the moment of inertia.

Rotational Kinetic Energy

- Defining the moment of inertia I in this way we can write:

$$K = ½I\omega^2$$

which is the kinetic energy of a rotating body.

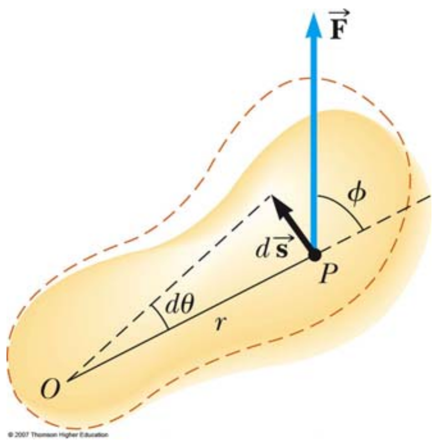
- This is the analogue of $K = ½mv^2$ for linear motion.
- Rotational kinetic energy is not a new type of energy, the form is different because it is applied to a rotating object.
- Moment of inertia in rotating dynamics is the analogue of mass in linear dynamics.

Work in Rotational Motion

- Find the work done by $\vec{\mathbf{F}}$ on the object as it rotates through an infinitesimal distance $ds = r\,d\theta$

$$dW = \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}}$$
$$= (F \sin\phi)\, r\, d\theta$$

- The radial component of the force does no work because it is perpendicular to the displacement.



Power in Rotational Motion

- The rate at which work is being done in a time interval dt is:

$$\text{Power} = \wp = \frac{dW}{dt} = \tau \frac{d\theta}{dt} = \tau\omega$$

- This is analogous to $\wp = Fv$ in a linear system.

Example

- What is the kinetic energy of the Earth's rotation about its axis?

- The radius of the Earth is 6.4 $\times 10^{6}$ m and the mass of the Earth is approximately 6.0×10^{24} kg.

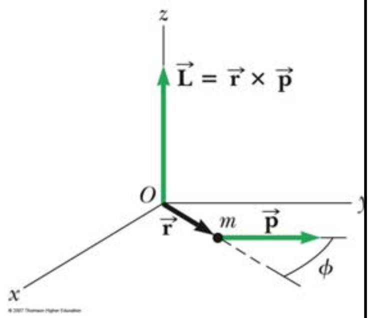
Torque and Momentum

- Consider a particle of mass *m* located at the vector position $\vec{\mathbf{r}}$ and moving with linear momentum $\vec{\mathbf{p}}$.
- The associated net torque can be written as:

$$\vec{\mathbf{r}} \times \sum \vec{\mathbf{F}} = \sum \vec{\tau} = \vec{\mathbf{r}} \times \frac{d\vec{\mathbf{p}}}{dt}$$

Add the term $\frac{d\vec{\mathbf{r}}}{dt} \times \vec{\mathbf{p}}\left(\text{since it} = 0\right)$

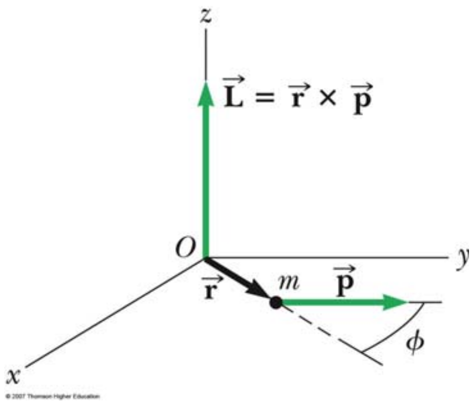
$$\sum \vec{\tau} = \frac{d(\vec{\mathbf{r}} \times \vec{\mathbf{p}})}{dt}$$



Angular Momentum

- The instantaneous angular momentum $\vec{\mathbf{L}}$ of a particle relative to the origin *O* is defined as the cross product of the particle's instantaneous position vector $\vec{\mathbf{r}}$ and its instantaneous linear momentum $\vec{\mathbf{p}}$.

$$\vec{\mathbf{L}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}}$$



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Torque and Angular Momentum

$$\sum \vec{\tau} = \frac{d(\vec{\mathbf{r}} \times \vec{\mathbf{p}})}{dt} \qquad \vec{\mathbf{L}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}}$$

- The torque is therefore related to the angular momentum.

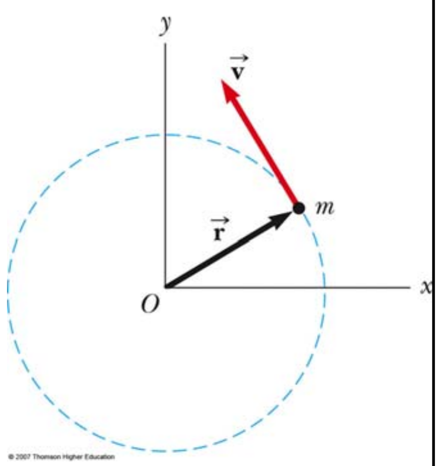
$$\sum \vec{\tau} = \frac{d\vec{\mathbf{L}}}{dt}$$

  - Similar to the way force is related to linear momentum
- The torque acting on a particle is equal to the time rate of change of the particle's angular momentum.
- This is the rotational analog of Newton's Second Law.

Angular Momentum of a Particle

- The vector $\vec{\mathbf{L}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}}$ is pointed out of the diagram.
- The magnitude of the vector is:

  $L = mvr \sin 90^{\circ} = mvr$
  - $\sin 90^{\circ}$ is used since v is perpendicular to r.
- A particle in uniform circular motion has a constant angular momentum about an axis through the center of its path.



Angular Momentum of a System of Particles

- The total angular momentum of a system of particles is defined as the vector sum of the angular momenta of the individual particles.

$$\vec{\mathbf{L}}_{tot} = \vec{\mathbf{L}}_1 + \vec{\mathbf{L}}_2 + \ldots + \vec{\mathbf{L}}_n = \sum_i \vec{\mathbf{L}}_i$$

- Differentiating with respect to time gives:

$$\frac{d\vec{\mathbf{L}}_{\text{tot}}}{dt} = \sum_i \frac{d\vec{\mathbf{L}}_i}{dt} = \sum_i \vec{\tau}_i$$

Angular Momentum of a System of Particles

- Any torques associated with the internal forces acting in a system of particles is zero.
- Therefore,

$$\sum \vec{\tau}_{\text{ext}} = \frac{d\vec{\mathbf{L}}_{\text{tot}}}{dt}$$

- The net external torque about some axis passing through the origin in an inertial frame equals the time rate of change of the total angular momentum of the system about that origin
- **Alternative statement:** The resultant torque acting on a system about an axis through the center of mass equals the time rate of change of angular momentum of the system regardless of the motion of the center of mass.

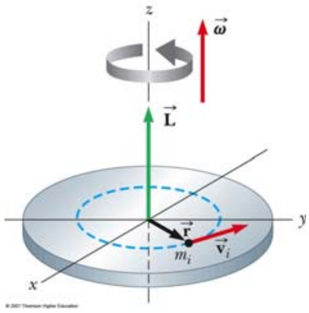
Check Understanding: Rotational Kinetic Energy

Two different spinning disks have the same angular momentum, but disk 1 has more rotational kinetic energy than disk 2. Which one has the bigger moment of inertia?

1) disk 1
2) disk 2
3) not enough information

Angular Momentum of a Rotating Rigid Object

- Consider a rigid object, in which each particle of the object rotates in the *xy* plane about the *z* axis with an angular speed of ω.
- The angular momentum of an individual particle is $L_i = m_i r_i^2 \omega$
- $\vec{L}$ and $\vec{\omega}$ are directed along the *z* axis.
- To find the angular momentum of the entire object, add the angular momenta of all the individual particles.



$$L_z = \sum_i L_i = \sum_i \left(m_i r_i^2\right)\omega = I\omega$$

Angular Momentum of a Rotating Rigid Object

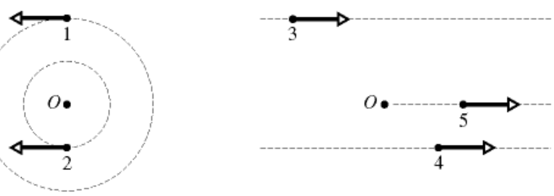
- Consider a body comprising many small particles of mass m_i.
- Then $$L = \sum L_i = \sum m_i v_i r_{i\perp} = \sum m_i(\omega r_i) r_i$$ $$= \sum m_i \omega r_i^2 = \omega \sum m_i r_i^2 = \omega I.$$
- Differentiating this also gives:

$$\tau = \frac{dL}{dt} = I\frac{d\omega}{dt} = I\alpha.$$

Check your understanding

Particles 1 and 2 move around point O in opposite directions, in circles with radii 2 m and 4 m. In the second Figure, particles 3 and 4 travel in the same direction, along straight lines at perpendicular distances of 4 m and 2 m from point O. Particle 5 moves directly away from O. All five particles have the same mass and the same constant speed.

Rank the particles according to the magnitudes of their angular momentum about point O, greatest first.

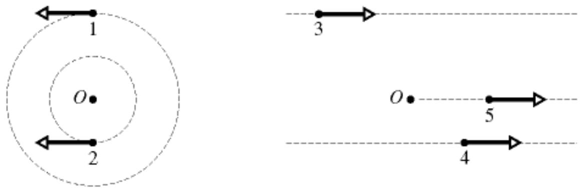


(a) (b)

Check your understanding

Particles 1 and 2 move around point O in opposite directions, in circles with radii 2 m and 4 m. In the second Figure, particles 3 and 4 travel in the same direction, along straight lines at perpendicular distances of 4 m and 2 m from point O. Particle 5 moves directly away from O. All five particles have the same mass and the same constant speed.

Which particles have negative angular momentum about point O?



(a) (b)

Practice Example

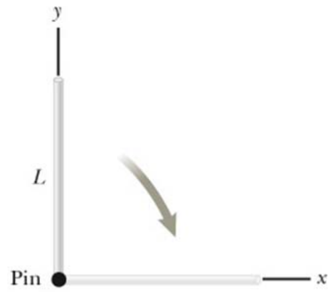
- An Attwood machine consists of two masses of 5 and 4 kg, which are connected by a cord of negligible mass that passes over a pulley, such that the two masses hang vertically either side of the pulley. The pulley has a radius of 0.5m and the moment of inertia about the rotation axis is 20 kgm^2. Determine the rate of change of the linear velocity of the masses.

[g=10ms^{-2}]

We will complete as many Practice examples as time allows. Students are encouraged to complete these examples in their own time if not completed in class. Model answers will be provided on the LEARN website.

Practice Example

- A uniform rod with mass (m) and length (L) is pivoted about a frictionless, horizontal pin through one end. The rod is disturbed from rest in a vertical position (see Figure below).
  - Derive an expression for the angular speed of the rod as the rod becomes horizontal.
  - Determine the corresponding angular acceleration when the rod is horizontal.



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Lecture 9: Angular Momentum Conservation and Rolling Motion
SJW^2 10.5 and 10.2

Some of the material in this Lecture is review material and has previously been covered in NCEA (Standard 90521) and/or equivalents.

Conservation of Angular Momentum

- **The total angular momentum of a system is constant in both magnitude and direction if the resultant external torque acting on the system is zero.**
  - Net torque = 0 -> means that the system is isolated.
- *Alternative statement:* If the net external torque acting on a system is zero, the angular momentum of the system remains constant, no matter what changes take place within the system

$$\vec{\mathbf{L}}_{tot} = \text{constant or } \vec{\mathbf{L}}_i = \vec{\mathbf{L}}_f$$

- For a system of particles,

$$\vec{\mathbf{L}}_{tot} = \sum \vec{\mathbf{L}}_n = \text{constant}$$

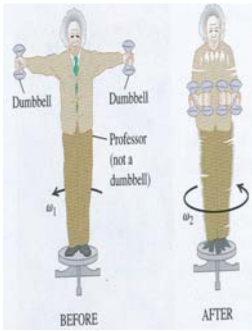
Conservation of Angular Momentum

- One consequence of the conservation of angular momentum is that if no external torque acts, then:

$$I_i\,\omega_i = I_f\,\omega_f = \text{constant}$$

where *i* represents an initial value, *f* a final value.

- Thus, if the mass of an isolated system undergoes redistribution, the moment of inertia changes. The conservation of angular momentum then requires a compensating change in the angular velocity.



Einstein spinning illustrates conservation of angular momentum. In BEFORE I is large and ω is small; In AFTER I is small and ω is large.

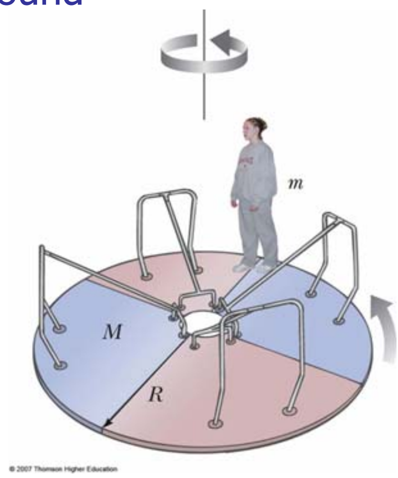
Check Understanding

A merry-go-round spins freely when Janice moves quickly to the center along a radius of the merry-go-round. It is true to say that:

a) the moment of inertia of the system decreases and the angular speed remains the same.
b) the moment of inertia of the system increases and the angular speed increases.
c) the moment of inertia of the system decreases and the angular speed decreases.
d) the moment of inertia of the system decreases and the angular speed increases.

Conservation of Angular Momentum: The Merry-Go-Round

- The moment of inertia of the system is the moment of inertia of the platform plus the moment of inertia of the person.
  - Assume the person can be treated as a particle.
- As the person moves toward the center of the rotating platform, the angular speed will increase.
  - To keep the angular momentum constant.

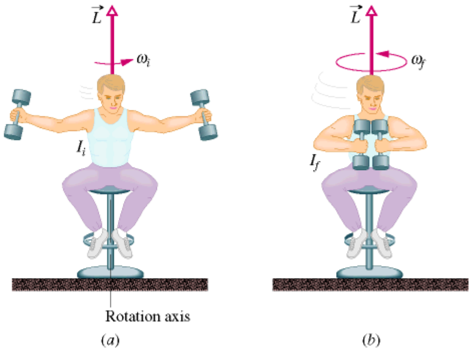


MYTHBUSTERS : Scary-go-round

The spinning volunteer

The figure shows a student seated on a stool that can rotate freely about a vertical axis. The student has been set into rotation at a modest initial angular speed ω_i, and holds two dumbbells in their outstretched hands. Their angular momentum vector $\underline{L}$ lies along the vertical rotation axis, pointing upwards.

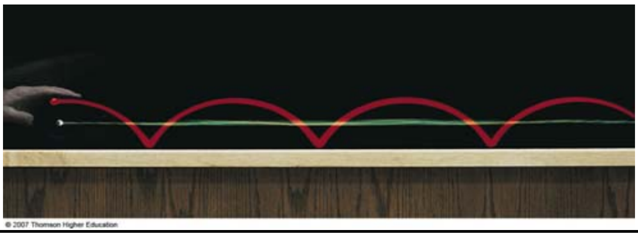
(a) The student has a relatively large rotational inertia and a relatively small angular speed.
(b) By decreasing their rotational inertia, the student automatically increases their angular speed. The angular momentum of the rotating system remains unchanged.



DEMO: Spinning Chair

Rolling Motion

- In pure rolling motion, an object rolls without slipping.
- In such a case, rolling can be considered as a combination motion in which an object rotates about an axis that is moving along a straight trajectory.
- The red curve shows the path moved by a point on the rim of the object. This path is called a ***cycloid.***
- The green line shows the path of the center of mass of the object (the straight trajectory of the rotation axis).



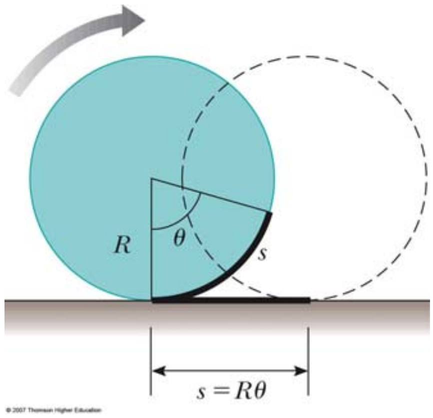
Rolling Object, Center of Mass

- As the object rotates through an angle θ, its center of mass moves a linear distance s=rθ, assuming no slipping (friction required).
- The velocity of the center of mass is:

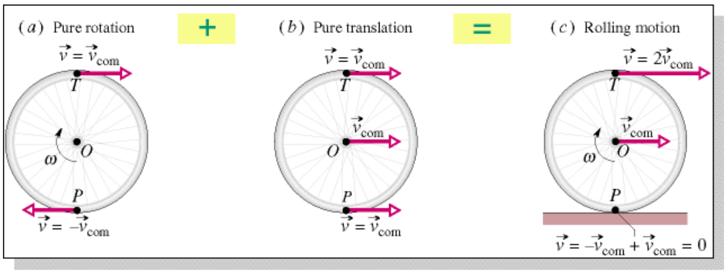
$$v_{CM} = \frac{ds}{dt} = R\frac{d\theta}{dt} = R\omega$$

- The acceleration of the center of mass is:

$$a_{CM} = \frac{dv_{CM}}{dt} = R\frac{d\omega}{dt} = R\alpha$$

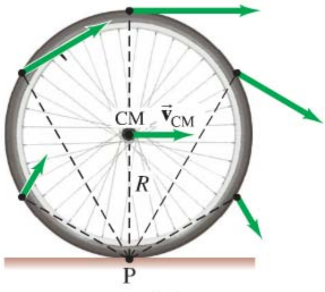


Rolling as rotation and translation combined



The rolling motion of a wheel can be considered as a combination of purely rotational motion and purely translational motion.

a) The purely rotational component: All points on the wheel move with the same angular speed ω. Points on the outside edge move with linear speed $v = v_{com}$.
b) The purely translational component: All points on the wheel move to the right with linear velocity v_{com}.
c) The rolling motion of the wheel is the combination of (a) and (b).



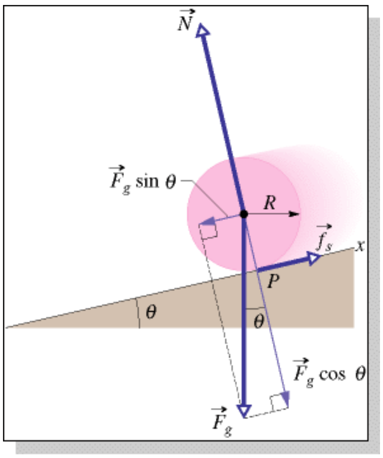
(a)


(b)

(a) Rolling can be viewed instantaneously as a pure rotation, with angular speed ω, about an axis that always extends through P (contact point with ground). The vectors show the instantaneous linear velocities of selected points on the rolling wheel. You can obtain the vectors by combining the translational and rotational motions.
(b) Photograph of a rolling wheel. The spokes are more blurred where the speed is greater.

Rolling down a ramp

The figure shows a round uniform body of mass *M* and radius *R* rolling smoothly down a ramp at angle θ, along an *x* axis. We want to find expressions for the body's acceleration $a_{com,x}$ down the ramp. We do this by using Newton's second law in both its linear version ($F_{net} = Ma$) and its angular version ($\tau_{net} = I\alpha$).



Then: $f_s - Mg\sin\theta = Ma_{\text{com,x}}$

Also the torque is $Rf_s = I_{\text{com}}\alpha$

(Using that $\alpha = -a_{com,x}/R$)

This results in $f_s = -I_{\text{com}}\dfrac{a_{com,x}}{R^2}$

PROOF

Total Kinetic Energy of a Rolling Object

- The total kinetic energy of a rolling object is the sum of the translational energy of its center of mass and the rotational kinetic energy about its center of mass.
  - $K = ½\, I_{CM}\, \omega^2 + ½\, Mv_{CM}^2$
    - The $½\, I_{CM}\omega^2$ represents the rotational kinetic energy of the cylinder about its center of mass (K_{ROT}).
    - The $½\, Mv^2$ represents the translational kinetic energy of the cylinder about its center of mass (K_{CM}).
- Alternative statement: The rolling motion of a rigid object can be described as a translation of the center of mass (with kinetic energy K_{CM}) plus a rotation about the center of mass (with kinetic energy K_{ROT}).

Check Understanding

Two cylinders of the same size and mass roll down an incline, starting from rest. Cylinder A has most of its mass concentrated at the rim, while cylinder B has most of its mass concentrated at the center. Which reaches the bottom first?

1) A

2) B

3) Both at the same time.

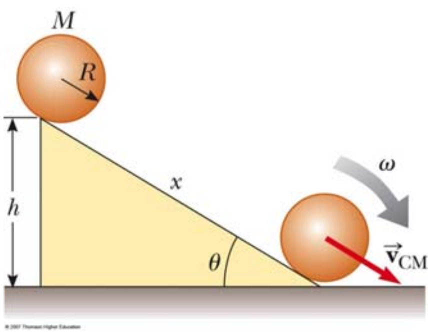
Check Understanding

Two cylinders of the same size but different masses roll down an incline, starting from rest. Cylinder A has a greater mass. Which reaches the bottom first?

1) A
2) B
3) Both at the same time.

Total Kinetic Energy

- Accelerated rolling motion is possible only if friction is present between the sphere and the incline.
  - The friction produces the net torque required for rotation.
  - No loss of mechanical energy occurs because the contact point is at rest relative to the surface at any instant.



PLAY
ACTIVE FIGURE 10.9

Practice Example

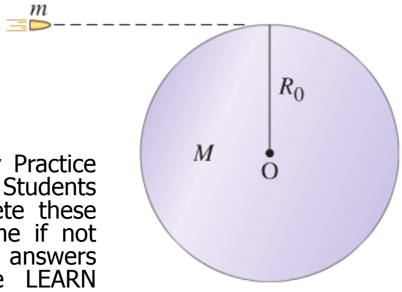
A star rotates with a period of 30 days about an axis through its center. After the star undergoes a supernova explosion, the stellar core, which had a radius of 1.0×10^4 km, collapses into a neutron star of radius 3.0 km. Determine the period of rotation of the neutron star?

Assume that the moment of inertia can be assumed to be associated with a mass distribution linked to a solid sphere.

We will complete as many Practice examples as time allows. Students are encouraged to complete these examples in their own time if not completed in class. Model answers will be provided on the LEARN website.

Practice Example

A bullet of mass m moving with velocity v strikes and becomes embedded in the edge of a cylinder of mass M and radius R_o, as shown in the figure. The cylinder, initially at rest, begins to rotate about its symmetry axis. Assuming no frictional torque, what is the angular velocity of the cylinder after this collision?



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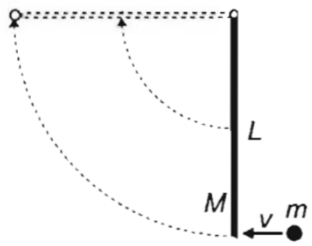
Practice Example

A rod of length L= 2m and mass M=10kg initially hangs vertically. A projectile of mass m=0.25kg is fired into the end of the rod. What is the velocity of the projectile required to move the rod to the horizontal position?



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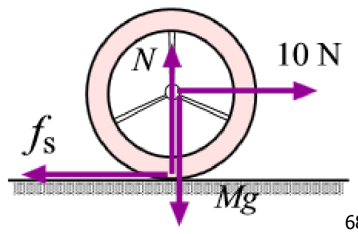
Practice Example

- A constant horizontal force of 10 N is applied to a wheel of mass 10 kg and radius 0. 30m as shown in the Figure. The wheel rolls without slipping on the horizontal surface, and the acceleration of its center of mass is 0.60 m/s^2
- What are the magnitude and direction of the frictional force on the wheel?
- What is the rotational inertia of the wheel about an axis through its center of mass and perpendicular to the plane of the wheel?



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