$$S = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

$Ax = \lambda x$

$|A - \lambda I| = 0$

positive eigvals

$\lambda_i > 0$ for all i

pivots > 0

Test for λ_i

* Any positive definite is also semi definite.

1. $a > 0$
2. $|S| = a \cdot c - b \cdot b = ac - b^2$

   $|S| = ac - b^2 > 0$ (positive eigvals)

Both must be true

S is a $\boxed{\text{positive definite}}$ matrix

$$S_1 = \begin{bmatrix} -3 & 1 \\ 1 & 2 \end{bmatrix}$$ → negative definite

$a < 0$,

$|S| = -6 - 1 = -7$

$$S_2 = \begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix}$$

$a > 0$ → Not positive

$|S| = -6 - 1 = -7$

$$S_3 = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \xrightarrow{\text{Row-Reduce}} \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$$

$a > 0$

$|S| = 2 \cdot 1 - 2 \cdot 1 = 0$ → Not positive

$$S = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1.5 & .5 \\ 0 & 0.5 & 1.5 \end{bmatrix}$$

$a > 0$

$$\longrightarrow \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1.5 & .5 \\ 0 & 0 & \frac{4}{3} \end{bmatrix}$$

all pivots > 0, S is positive definite.

$$S = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$a > 0$

$\lambda_1, \lambda_2 = 0$ positive semi definite $\boxed{\lambda_i \geq 0}$

$\lambda_3 = 3$

$$X^T S X = [X_1 \; X_2] \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

$1 \times 2 \quad 2 \times 2 \quad 2 \times 1$

$$= [X_1 a + X_2 b, \; X_1 b + X_2 c] \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

$$= \boxed{a x_1^2 + 2b x_1 x_2 + c x_2^2} = Q(X)$$

$Q(X)$ = Quadratic Form

$a x_1^2 + 2b x_1 x_2 + c x_2^2$

$x_i > 0$

if a, b, c are all positive, $Q(x)$ is + def

if $a, b, c < 0$, $Q(x)$ is − def $x_i < 0$

if the signs don't match, $Q(x)$ is indefinite

$Q(x) = x_1^2 + 4x_2^2 + 2x_3^2$ +def

$Q(x) = 2x_1^2 - x_2^2 + 4x_3^2$ indefinite

$Q(x) = -4x_1^2 - 2x_2^2 - x_3^2$ - def

$Q(x) = \underset{+}{3x_1^2} + \underset{+}{x_3^2} + 0x_2^2$ $\boxed{+\text{ semi def}}$

$$\begin{bmatrix} 1 & & \\ & 4 & \\ & & 2 \end{bmatrix} \qquad \begin{bmatrix} 3 & & \\ & 0 & \\ & & 1 \end{bmatrix}$$

$\lambda = 1, 4, 2$ $\qquad$ $\lambda = 3, 0, 1$

$Q(x) = 2x_1^2 + x_2^2 + 4x_1x_2$ $\quad$ $2b$

$b = 2.$

$$\begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$$

$\boxed{\text{Indefinite}}$

$$Q\left(\begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}\right) = 2\left(\frac{3}{5}\right)^2 + 1\cdot\left(\frac{4}{5}\right)^2 + 4\cdot\left(\frac{3}{5}\right)\left(\frac{4}{5}\right) > 0$$

$$Q\left(\begin{bmatrix} -\frac{3}{5} \\ \frac{4}{5} \end{bmatrix}\right) = 2\cdot\left(-\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2 + 4\cdot\left(-\frac{3}{5}\right)\cdot\left(\frac{4}{5}\right) < 0$$

S is $+$ def, S^{-1} exist, S^{-1} is also a positive definite