$$Q(x) = \sum_{i=1}^{n} a_{ii} x_i^2 + 2\sum_{i \neq j} a_{ij} x_i x_j$$

Diagonal off-Diagonal

$$S = \begin{bmatrix} 2 & -3 & 1 \\ -3 & 0 & 0 \\ 1 & 0 & 4 \end{bmatrix}$$

$$Q(x) = \underbrace{2x_1^2 + 0x_2^2 + 4x_3^2}_{\text{Diag}} + \underbrace{-6x_1x_2 + 2x_1x_3}_{\text{off-Diag}}$$

+Def : $Q(x) > 0 \quad \in x \neq 0$

unit x

$$x = \begin{bmatrix} \frac{3}{5} \\ -\frac{4}{5} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

−Def : $Q(x) < 0 \quad \in x \neq 0$

+ Semi : $Q(x) \geq 0$ …

def : $Q(x) \geq 0 \quad \in \quad x \neq 0$

- semi def : $Q(x) \leq 0 \quad \in \quad x \neq 0$

Indefinite : $Q(x) \geq 0$ for some x,

$Q(x) \leq 0$ for some x

$$\boxed{|\min \lambda| \leq Q(x) \leq \left|\max_{\lambda}\right| \quad \text{for unit } x\text{'s}}$$

λ = eigenval associated unit A

$Q(x) = 2x_1^2 + 8x_1x_2 + 2x_2^2$

① Make S.

Divide 8 by 2

$$S = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$$

② Find λ's

$|S - \lambda I| = 0$

$\begin{bmatrix} 2-\lambda & 4 \end{bmatrix}$ $(2-\lambda)(2-\lambda) - 16 = 0$

$$\begin{bmatrix} & \\ 4 & 2-\lambda \end{bmatrix}$$

$$4 - 4\lambda + \lambda^2 - 16$$

$$\lambda^2 - 4\lambda - 12 = 0$$

$$\lambda_1 = -2, \quad \lambda_2 = 6$$

$$\boxed{-2 \le Q(x) \le 6}$$

when?

③ Find min, max for x.

Unit eigenspace

$$N(S - \lambda I)$$

$\lambda_1 = -2,$

$$N\left(\begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}\right)$$

$$= N\left(\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}\right)$$

$$x = \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix} \longrightarrow \text{Min}$$

$\lambda_2 = 6,$

$$N\left(\begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix} - \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}\right) = N\left(\begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix}\right)$$

$$X = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \rightarrow \text{Max}$$

$\therefore$ $Q(X) = -2$ when $X = \pm \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$

and $Q(x) = 6$ when $X = \pm \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

If λ's have multiplicity,
we may have more than 2 X's that max/min $Q(X)$

Quiz.

$\lambda_i \le t \le h$

$Q(X) = t$ when

$X = \sqrt{\alpha}\, X_i + \sqrt{1-\alpha}\, X_{i+1}$

$$S = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

consider $Q(x)$ is $1/3$ away from λ_1 to λ_2

$\lambda_1 = 4, \ \lambda_2 = 1$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\alpha = \frac{1}{3}$

$$t = \alpha \lambda_1 + (1-\alpha) \lambda_{i+1}$$

$$= \frac{1}{3} \cdot 4 + \left(1 - \frac{1}{3}\right) \cdot 1$$

$$= 2. \longrightarrow Q(x) = 2$$

$$x = \sqrt{1/3} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \sqrt{\left(1 - \frac{1}{3}\right)} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$x = \begin{bmatrix} \sqrt{2/3} \\ \sqrt{1/3} \end{bmatrix}$$

eigvecs