Exam 1. a)

$$A = \begin{bmatrix} 1 & a & 0 \\ b & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow[\text{row } 2,3]{\text{① swap}} \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 1 \\ b & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 1 \\ 0 & 1-ab & 0 \end{bmatrix}$$

② Eliminate (3,1)

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -b & 0 & 1 \end{bmatrix}$$

③ E_{32}

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & ab-1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 1 \\ 0 & 0 & ab-1 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b & 1-ab & 1 \end{bmatrix}$$

b) $b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ solve $Ax = b$

$Lc = b$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b & 1-ab & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$c_1 = c_2 = 1$

$b\cancel{c_1} + (1-ab)\cancel{c_2} + c_3 = 1$

$$C_3 = 1 - b - (1 - ab)$$
$$= ab - b$$

$Ux = c$

$$\begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 1 \\ 0 & 0 & ab-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ ab-b \end{bmatrix}$$

$$\therefore \quad x_3 = \frac{ab-b}{ab-1}$$

$$x_2 = 1 - \frac{ab-b}{ab-1}$$

2. xy^T

$$xy^T = \begin{bmatrix} 3 & 4 & 5 \\ 6 & 8 & 10 \\ 3 & 4 & 5 \end{bmatrix}$$

(b) $C(A) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ should have only one vector.

3. $Ax = 0$

$$\begin{bmatrix} 1 & 0 & a & 0 & b & 0 \\ 0 & 1 & c & 0 & 0 & d \\ 0 & 0 & 0 & 1 & e & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

pivots → x_1, x_2, x_4

free vars → x_3, x_5, x_6

$x_3 = 1, \quad x_5 = x_6 = 0$

$$\left.\begin{aligned} x_1 &= -a \\ x_2 &= -c \\ x_4 &= 0 \end{aligned}\right] \rightarrow \begin{bmatrix} -a \\ -c \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_3$$

A is $m \times n$

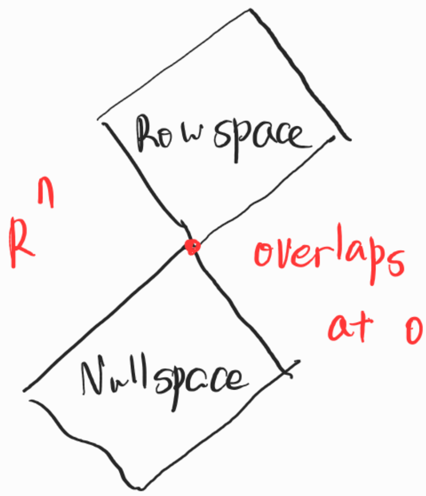
$C(A)$ – columns → $\mathbb{R}^m$ dim = # of pivots

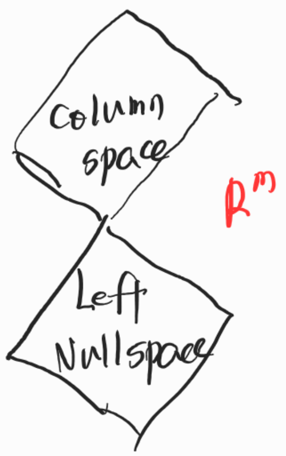
$C(A^T)$ – Row space → $\mathbb{R}^n$ dim = # of pivots

\# of basis vectors

$N(A)$ – Null space → $\mathbb{R}^n$ dim = $n-r$ (# of independent vectors)

$N(A^T)$ – Nullspace → $\mathbb{R}^m$





$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix}$$

2×3

dim(A) = # of pivots = 2

r is rank

Null vectors = n − r

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} R$$

$C(A) \neq C(R)$

$C(A^T) = C(R^T)$

$$C(A) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

$$C(A^T) = \left\{ \begin{matrix} [1\ 2\ 3\ 1] \\ [1\ 1\ 2\ 1] \end{matrix} \right\}$$

$$\left\{ \begin{matrix} [1\ 0\ 1\ 1] \\ [0\ 1\ 1\ 0] \end{matrix} \right\}$$

Same x's that make $Ax = 0$

also solve $RX = 0$

$$\boxed{N(A) = N(R)}$$