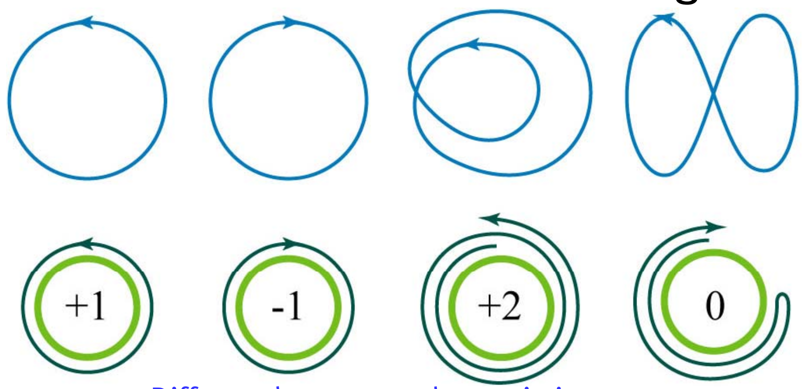
Turning Number

Number of orbits in Gauß image

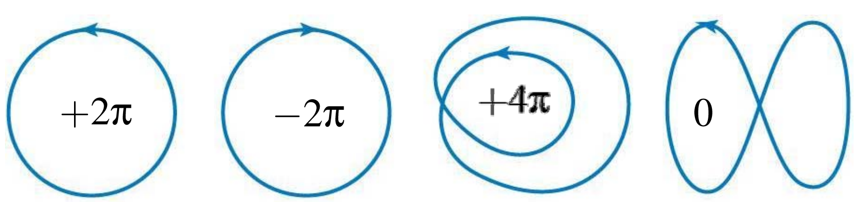


Different homotopy classes in image

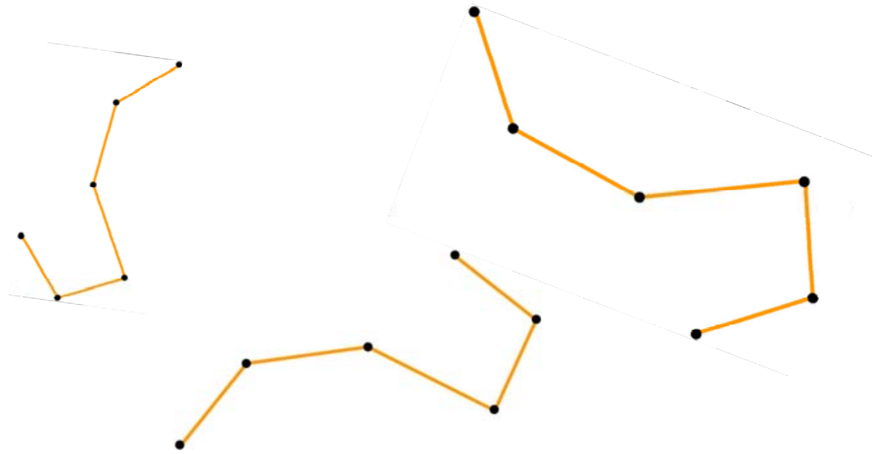
Turning Number Thm.

For a closed curve

$$\boxed{\int_C \kappa\, ds = k 2\pi}$$



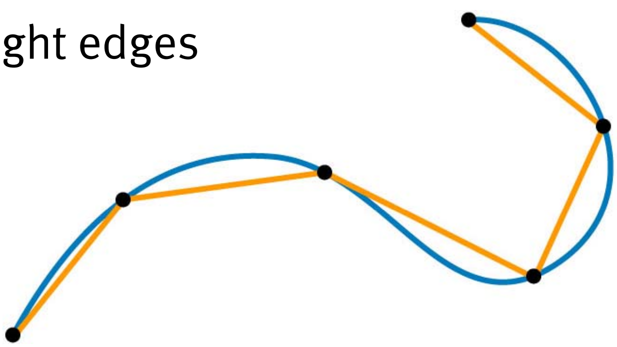
Discrete Setting



Inscribed Polygon: p

Finite number of vertices

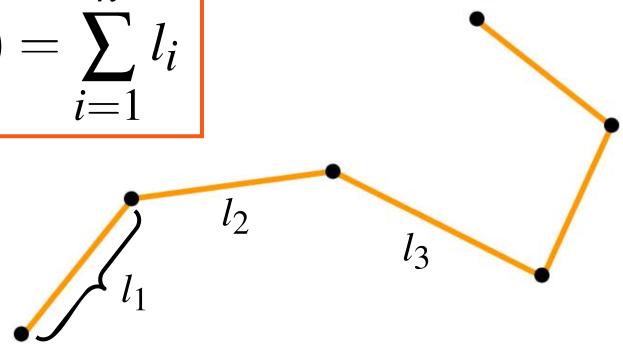
- on curve, ordered
- straight edges



Length

Sum of edge lengths

$$\boxed{l(p) = \sum_{i=1}^{n} l_i}$$

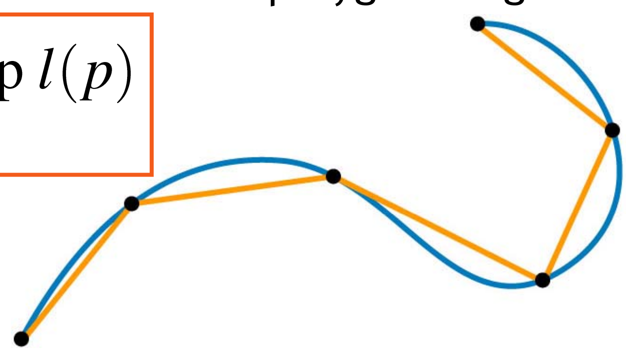


Length

Smooth curve

- limit of inscribed polygon lengths

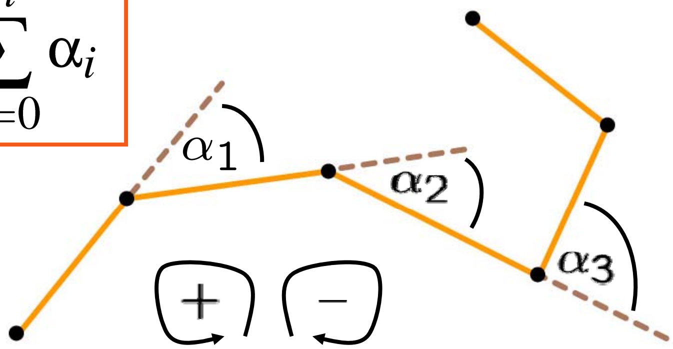
$$\boxed{\sup_p l(p)}$$



Total Signed Curvature

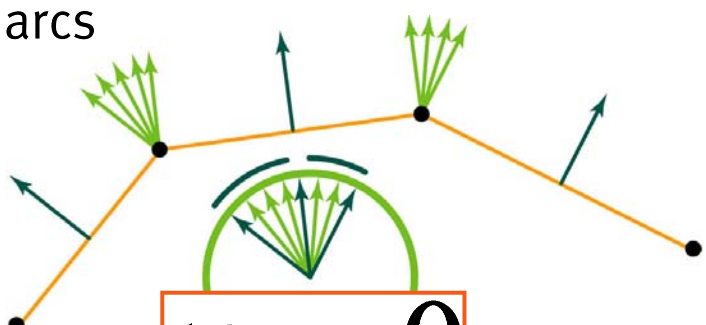
Sum of turning angles

$$\boxed{T_\kappa = \sum_{i=0}^{n} \alpha_i}$$



Discrete Gauß Map

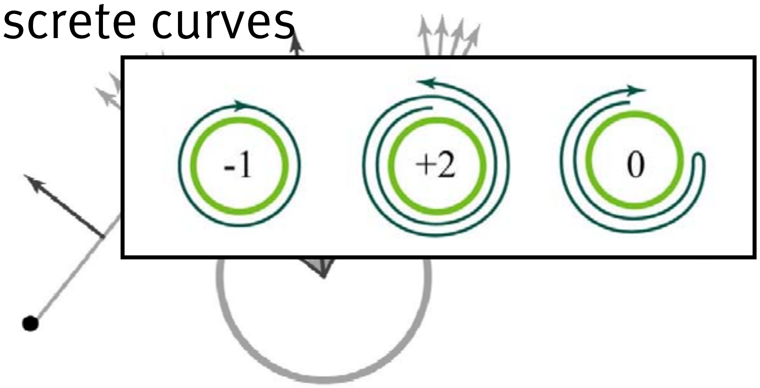
Edges map to points, vertices map to arcs



$$\boxed{\kappa = \theta}$$

Discrete Gauß Map

Turning number well-defined for discrete curves

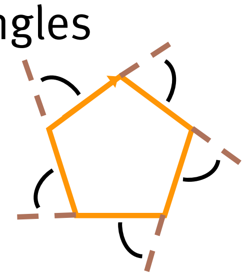


Turning Number Theorem

Closed curve

- the total signed curvature is an integer multiple of 2π.
- proof: sum of exterior angles

$$\boxed{T_\kappa = \sum_{i=1}^{n} \alpha_i = k\,2\pi}$$



Structure-Preservation

Arbitrary discrete curve

- total signed curvature obeys discrete turning number theorem

discrete analog of continuous theorem

- even on a coarse mesh
- can be crucial
  - depending on the application

Convergence

Consider refinement sequence

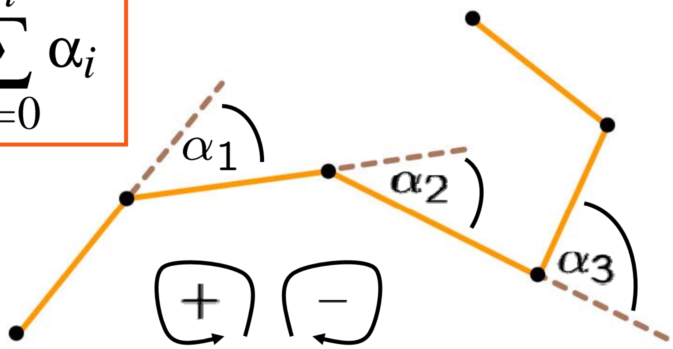
- length of inscribed polygon to length of smooth curve
- discrete measure approaches continuous analogue
- which refinement sequence?
  - depends on discrete operator
  - pathological sequences may exist

Recall:

Total Signed Curvature

Sum of turning angles

$$\boxed{T_\kappa = \sum_{i=0}^{n} \alpha_i}$$



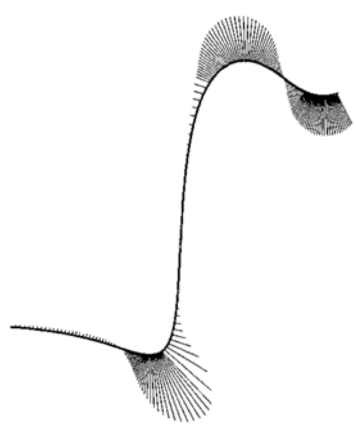
Another Definition

Curvature normal

$$\kappa\vec{n}$$

signed curvature (scalar)

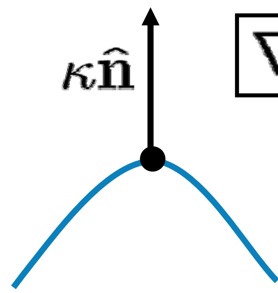
unit normal (vector)



Curvature normal

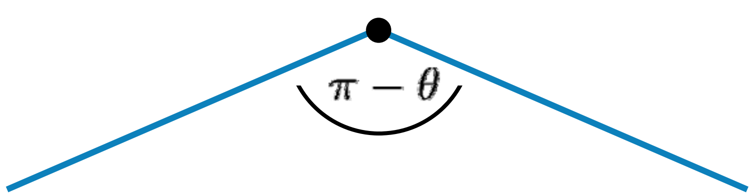
Gradient of length

- define discrete curvature

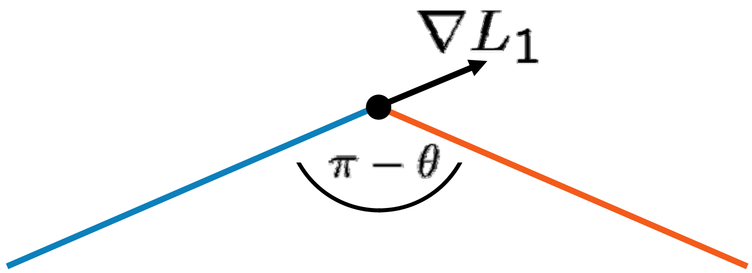


$$\boxed{\nabla L = \kappa\hat{\mathbf{n}}}$$

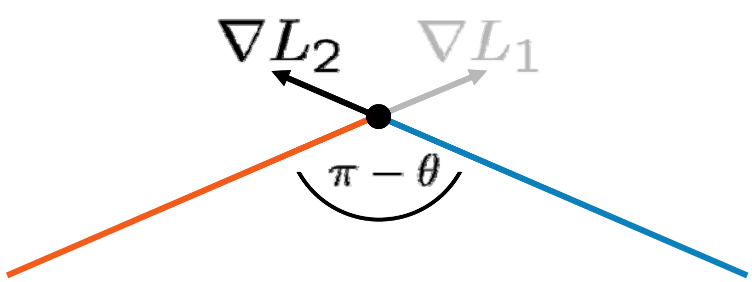
Gradient of Length



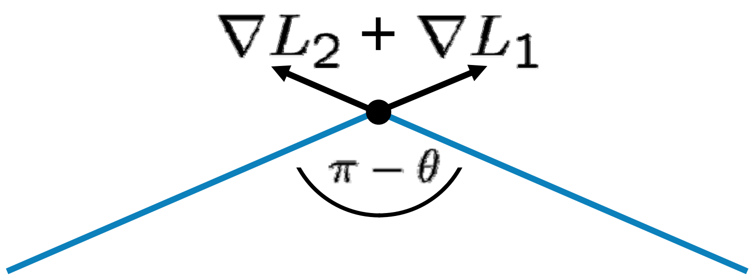
Gradient of Length



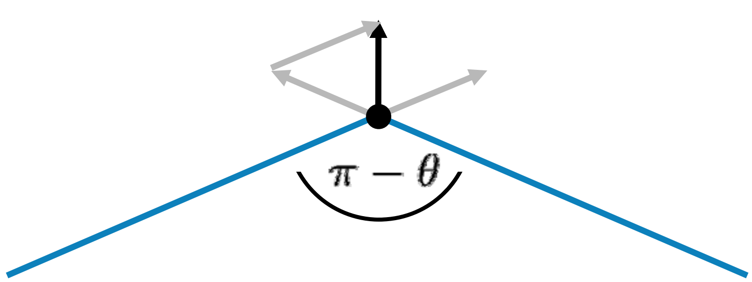
Gradient of Length



Gradient of Length

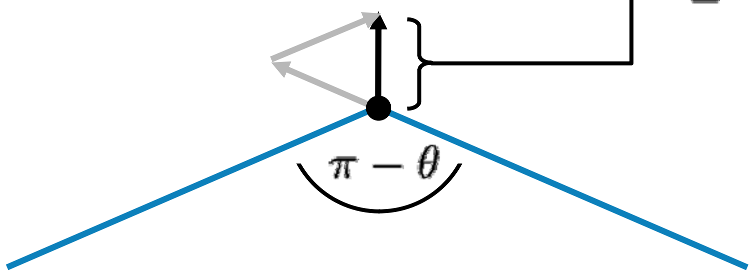


Gradient of Length

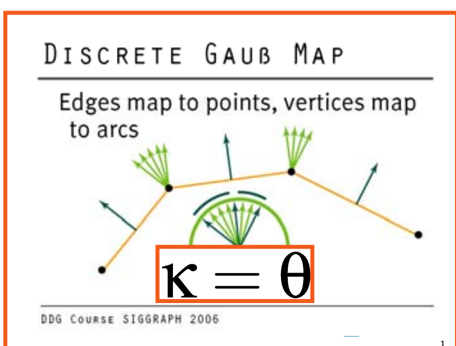


Gradient of Length

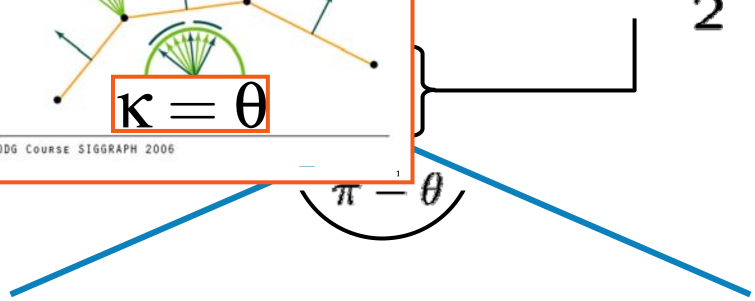
$$\nabla L = \kappa\hat{\mathbf{n}} = 2\sin\frac{\theta}{2}\hat{\mathbf{n}}$$



Gradient of Length



$$\kappa\hat{\mathbf{n}} = 2\sin\frac{\theta}{2}\hat{\mathbf{n}}$$



Moral of the Story

Structure-preservation

For an arbitrary (even coarse) discrete curve, the discrete measure of curvature obeys the discrete turning number theorem.

Convergence

In the limit of a refinement sequence, discrete measures of length and curvature agree with continuous measures.