Curvatures, Invariants and How to Get Them Without (M)Any Derivatives

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Classical Notions

Curves

- arclength parameterization

  $c: I \to \mathbb{R}^3 \qquad |\ddot{c}(t)| = \kappa(t)$

- center of osculating ("kissing") circle (also defines osc. plane)
  - tilt of plane is torsion
- Euclidian motion invariant
  - uniquely characterizes curve!

Surfaces

First fundamental form

- parameterized surface

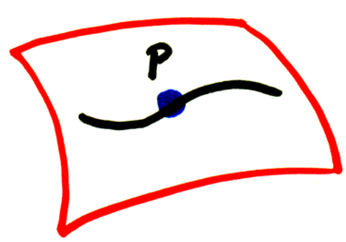
$$S : \mathbb{R}^2 \supset \Omega \to \mathbb{R}^3$$
$$S(u,v) = (x(u,v), y(u,v), z(u,v))$$

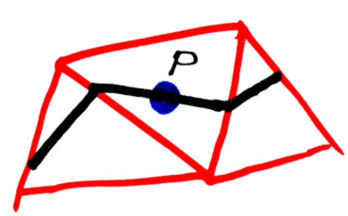
- tangent vectors

$$c : I \to S \quad c(0) = p \quad \dot{c}(0) = \alpha$$

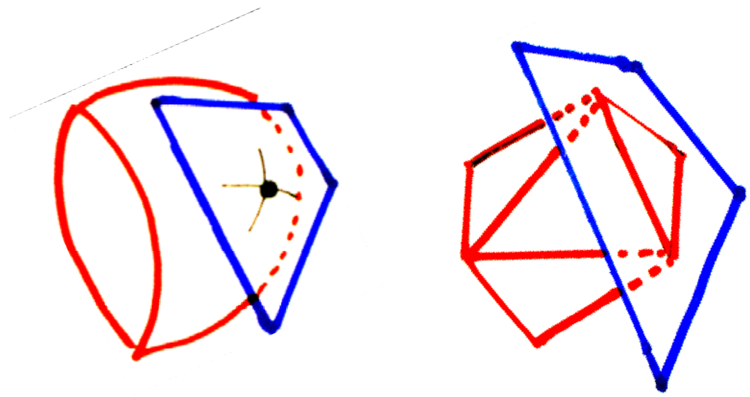
- tangent space $\quad T_pS$

Tangent Vector

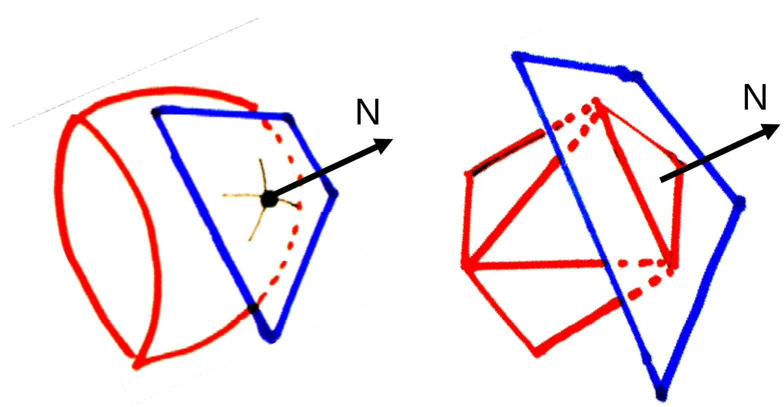




Tangent Plane



Normal Vector

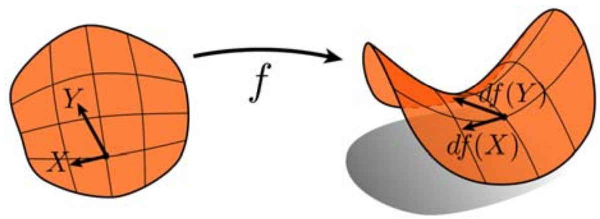


Metric

Measure stuff

- angle, length, area
  - symmetric, bilinear form $I_p : (T_pM)^2 \to \mathbb{R}$

$$I_p(X,Y) = g(X,Y) = df(X) \cdot df(Y)$$



Metric

Measuring area

- areas in tangent space

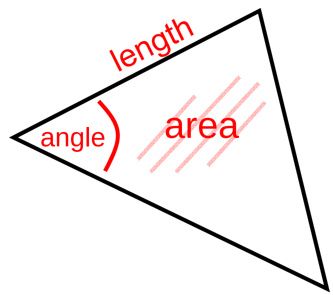
$$\int\int_{\Omega} |df(X) \times df(Y)| \, dx\, dy = A(S) = \int_S 1 \, dA$$

  - no dependence on parameterization
- discrete setting... easy
  - sum areas of triangles

Metric

Within each triangle

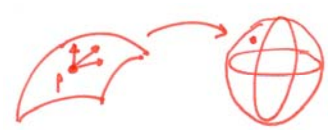
- the metric is obvious



Geometry of the Normal

Gauss map

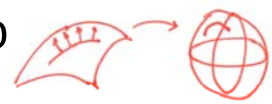
- normal at point



$$N(p) = \frac{df(X) \times df(Y)}{|df(X) \times df(Y)|}(p) \quad N : M \to \mathbb{S}^2 \subset \mathbb{R}^3$$

- consider curve in surface again
  - study its curvature at p
  - “tilting” of normal along the curve

Shape Operator

Derivative of Gauss map 

- tangent space to itself

$$dN_p : T_pM \to T_pM$$ linear map

- express in tangent space

$$dN(X) = df(SX)$$

- second fundamental form

$$II_p(X,Y) = g(SX,Y)$$ self-adjoint

Curvature of Surfaces

Normal curvature

$$\kappa_n(X) = \frac{df(X) \cdot dN(X)}{df(X) \cdot df(X)}$$

$$= \frac{g(SX,X)}{g(X,X)}$$