Discrete Exterior Calculus

Peter Schröder

with Mathieu Desbrun and the rest of the DEC crew

Big Picture

Deriving a whole *Discrete Calculus*

- first: **discrete domain**
  - notion of chains and discrete representation of geometry
- second: **discrete "differential" operators**
  - applied to discrete geometric set-up
  - defined as cochains (discrete forms)

Discrete Setup

Starting with a **discrete domain**

- think of as “approximation”
  - cell decomp. of smooth manifold

Nice simplicial mesh

- vertex, edge, triangle, ...

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- 2D domain example

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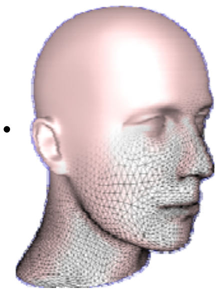
Nice simplicial mesh

- vertex, edge, triangle, ...
- curved 2D domain



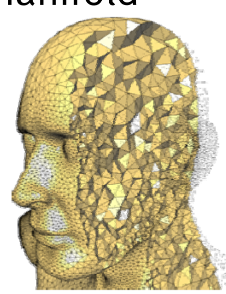
Discrete Setup

Starting with a **discrete domain**

- think of as "approximation"
  - cell decomp. of smooth manifold

Nice simplicial mesh

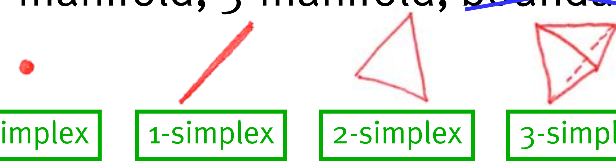
- vertex, edge, triangle, ...
- 3D domain
- and beyond



Enter DEC…

Start with simplicial complex

- 2-manifold, 3-manifold, ~~boundary~~



- DOFs on simplices: co-chains

Coeff i

differential k-form

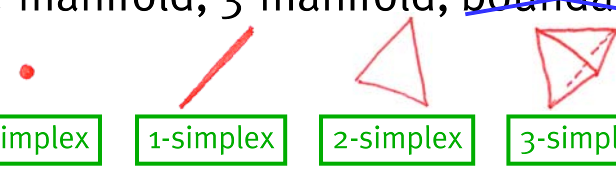
$$c_i = \int_{\sigma_i^k} \omega^k$$

k-simplex

Enter DEC…

Start with simplicial complex

- 2-manifold, 3-manifold, ~~boundary~~



- DOFs on simplices: co-chains

$$c_i = \int_{\sigma_i^k} \omega^k$$

function at point

normal over area

vectorfield along curve

"stuff" over volume

Enter DEC…

Start with simplicial complex

- 2-manifold, 3-manifold, ~~boundary~~

0-simplex | 1-simplex | 2-simplex | 3-simplex

- DOFs on simplices: co-chains

$$c_i = \int_{\sigma_i^k} \omega^k$$

Measurable physical quantities: circulation, flux, total mass, etc.

Chains & Co-Chains

Pieces of the manifold

- formal linear combinations

$$\sigma = \sum_{\sigma_i \in K} c_i \sigma_i \qquad \sigma \in C(K) \qquad \sigma^p \in C^p(K)$$

Integration over pieces

- functionals

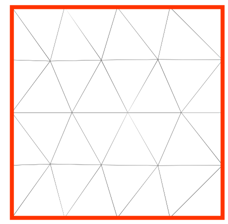
$$\alpha^p : C^p(K) \to \mathbb{R} \qquad \alpha^p(\sigma^p) = \int_{\sigma^p} \alpha^p$$

- exterior p-forms $\alpha^p \in \Omega^p(K)$

Discrete Subdomains

How to define geometric subsets?

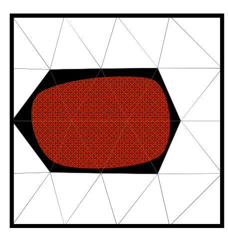
- we only have the mesh
  - simple example:



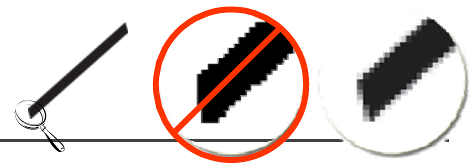
Discrete Subdomains

How to define geometric subsets?

- we only have the mesh
  - simple example: define this region?
  - “voxelized” is ok, but not great



- heard about antialiasing?

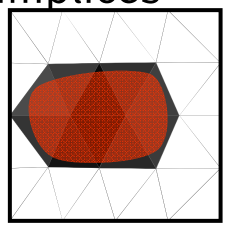


Notion of Chains

Allow lin. combination of simplices

- assign coefficients to simplices
  - not just 0 or 1
  - it's called a chain

$$c = \sum_{\sigma_i \in K} \alpha_i \, \sigma_i$$



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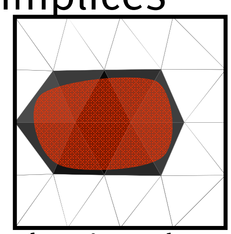


Definition:

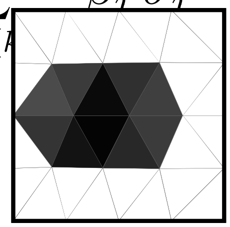
- k-chain = one value per k-simplex
  - think "column vector" (storage: array)

Chains and Boundaries

Boundary of a chain is a chain:

$$\partial c = \sum_{\sigma_i \in K^k} \alpha_i \, \partial \sigma_i = \sum_{\sigma_j \in K^{(k-1)}} \beta_j \, \sigma_j$$

- chains allow "sub-simplex" accuracy



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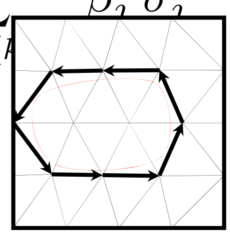

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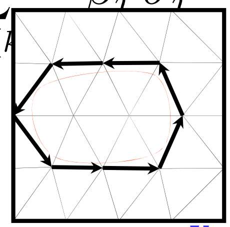
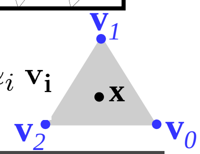


- chains allow "sub-simplex" accuracy
  - anti-aliased version...
- chains extend bary-centric coords $\mathbf{x} = \sum_{i=0}^{d} \alpha_i \, \mathbf{v_i}$



Chains & Co-Chains

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$$\sigma = \sum_{\sigma_i \in K} c_i \sigma_i \qquad \sigma \in C(K) \qquad \sigma^p \in C^p(K)$$

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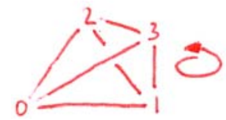
Simplicies

Primal

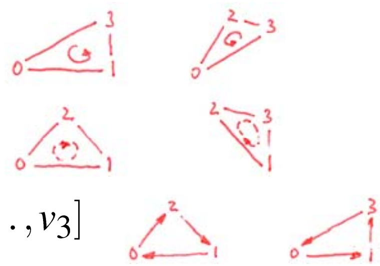
- orientation
  - from volume form
  - ... and on down



$$\mathrm{vol}^3(\sigma^3) = \pm 1$$

$$\sigma^3 = [v_0, v_1, v_2, v_3]$$

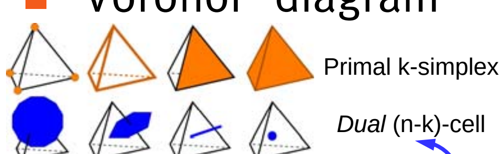
$$\partial\sigma^3 = \sum_{i=0}^{3} (-1)^i [v_0, \ldots, \hat{v}_i, \ldots, v_3]$$



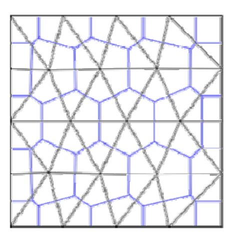
Dual Complex

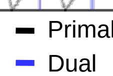
Each k-simplex gets a (n-k)-cell

- connectivity of mesh *induces* another mesh
- "Voronoi" diagram



dual of k-simplex, so same cardinality





Simplicies

Dual

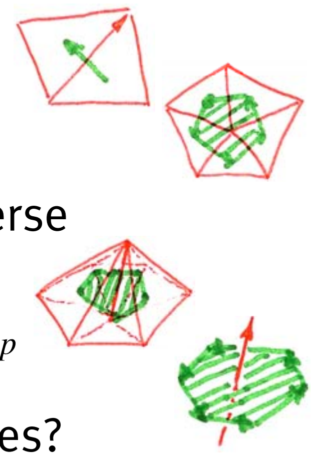
- well,… cells, actually

$$\boxed{\star\sigma^p = \sigma^{n-p}}$$

- primal & dual transverse
  - use to orient

$$\star\star\sigma^p = (-1)^{p(n-p)}\sigma^p$$

- what about 0-simplices?



Exterior Derivative

Co-boundary operator

- Stokes' by design

$$\int_{\partial\sigma^p} \alpha^{p-1} = \int_{\sigma^p} d\alpha^{p-1}$$

$$\partial\partial = 0 \implies dd = 0$$

$$\boxed{d = \partial^T}$$

$$\begin{array}{ccc} \Omega^p & & C^p(K) \\ d \uparrow & & \downarrow \partial \\ \Omega^{p-1} & & C^{p-1}(K) \end{array}$$

Example: Boundary Op.

$K = \{\{\sigma^3\},\{\sigma^2\},\{\sigma^1\},\{\sigma^0\}\}$

$\sigma^3_0 = [0,1,2,3]$	$\sigma^2_0 = [0,2,1]$	$\sigma^1_0 = [0,1]$	$\sigma^0_0 = [0]$
$\sigma^3_1 = [1,2,3,4]$	$\sigma^2_1 = [0,1,3]$	$\sigma^1_1 = [1,2]$	$\sigma^0_1 = [1]$
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$$\partial^1 : C^1(K) \to C^0(K)$$

$$\begin{pmatrix} -1 & & & & -1 & -1 & & & \\ 1 & -1 & -1 & & & & 1 & & \\ & 1 & & 1 & 1 & & & 1 & \\ & & 1 & -1 & & 1 & & & 1 \\ & & & & & & -1 & -1 & -1 \end{pmatrix}$$

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$$\partial^2 : C^2(K) \to C^1(K)$$

$$\begin{pmatrix}
-1 & 1 & & & & & \\
-1 & & & 1 & 1 & & \\
 & 1 & & -1 & & -1 & \\
 & & 1 & -1 & & & -1 \\
1 & & -1 & & & & \\
 & -1 & 1 & & & & \\
 & & & & 1 & -1 & \\
 & & & & -1 & & 1 \\
 & & & & & 1 & -1
\end{pmatrix}$$

Example: Boundary Op.

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$$\partial^3 : C^3(K) \to C^2(K)$$

$$\begin{pmatrix}
1 & \\
1 & \\
1 & \\
1 & -1 \\
 & 1 \\
 & 1 \\
 & 1
\end{pmatrix}$$

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$$\partial = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \partial^3 & 0 & 0 & 0 \\ 0 & \partial^2 & 0 & 0 \\ 0 & 0 & \partial^1 & 0 \end{pmatrix}$$

Hodge Star

Take forms to dual complex

- co-chain: numbers on simplices

$$\star : \Omega^p \longrightarrow \Omega_\star^{n-p}$$

- now the metric enters

$$\frac{1}{|\star\sigma^p|}\int_{\star\sigma^p} \star\alpha^p = \frac{1}{|\sigma^p|}\int_{\sigma^p} \alpha^p$$

average value at common point agrees

- “diagonal” hodge star
- sign: $\star\star\alpha^p = (-1)^{p(n-p)}\alpha^p$

Realizations

$$K_p \equiv_\star K^\star_{n-p} \qquad\qquad \Omega_p \xrightarrow{\star^p} \Omega^\star_{n-p}$$

$$\partial^p \qquad \partial_\star^{n-(p-1)} \qquad d^{p-1} \qquad d_\star^{n-p}$$

$$K_{p-1} \equiv_\star K^\star_{n-(p-1)} \qquad\qquad \Omega_{p-1} \xrightarrow{\star^{p-1}} \Omega^\star_{n-(p-1)}$$

$$\boxed{\partial^p = (d^{p-1})^T}$$

$$|||\qquad\qquad|||$$

$$\boxed{d_\star^{n-p} = (\partial_\star^{n-(p-1)})^T}$$

$$\star^p : \Omega^p \to \Omega_\star^{n-p}$$
$$\star^p = \mathrm{diag}\left(\frac{|*\sigma^p|}{|\sigma^p|}\right)$$
$$\star^{n-p} = (\star^p)^{-1}$$

More Operators

Co-differential

$$\langle \alpha^p, \beta^p \rangle = \int \alpha^p \wedge \star\beta^p$$

- inner product dual to differential

$$\delta^p = (-1)^{n(p+1)} (\star^{p-1})^{-1} d_\star^{n-p} \star^p$$

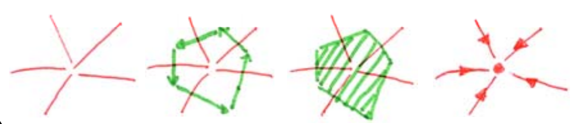
$$\delta^p : \Omega^p \to \Omega^{p-1}$$

$$\begin{array}{ccc} \Omega_p & \xrightarrow{\star^p} & \Omega^\star_{n-p} \\ \delta^p \downarrow & & \downarrow d_\star^{n-p} \\ \Omega_{p-1} & \xrightarrow{\star^{p-1}} & \Omega^\star_{n-(p-1)} \end{array}$$

Example

- “Div”

$$\delta^1 \alpha^1 (\sigma^0)$$



de Rham Complex

Putting it all together

transpose of incidence matrix

$$
\begin{tikzcd}
\text{primal} & \text{point based (scalar field)} \arrow[r, "d", "\nabla"'] \arrow[d, leftrightarrow, "\star"] & \text{edge based (vector field)} \arrow[r, "d", "\nabla\times"'] \arrow[d, leftrightarrow, "\star"] & \text{triangle based (vector field)} \arrow[r, "d", "\nabla\cdot"'] \arrow[d, leftrightarrow, "\star"] & \text{tet based (scalar field)} \arrow[d, leftrightarrow, "\star"] \\
\text{dual} & \text{cell based (scalar field)} & \text{face based (vector field)} \arrow[l, "d^T"', "\nabla\cdot"] & \text{edge based (vector field)} \arrow[l, "d^T"', "\nabla\times"] & \text{point based (scalar field)} \arrow[l, "d^T"', "\nabla"]
\end{tikzcd}
$$

diagonal matrix: ratio of primal/dual volumes

Hodge Decomposition

$$\Omega^k = d\Omega^{k-1} \oplus \delta\Omega^{k+1} \oplus \mathcal{H}^k$$

$\boxed{d\delta + \delta d}$

no boundary & compact

$\boxed{\{h|\Delta^k h = 0\}}$

$$
\begin{aligned}
h \in \mathcal{H}^k &\implies dh = 0, \delta h = 0 \\
\forall \alpha, \beta &\implies d\alpha \perp h \perp \delta\beta \\
dd = 0 &\implies d\alpha \perp \delta\beta \\
dh = 0, \delta h = 0 &\implies \Delta h = 0
\end{aligned}
$$

$$
\begin{aligned}
0 &= \langle \Delta h, h \rangle \\
&= \langle d\delta h, h \rangle + \langle \delta d h, h \rangle \\
&= \langle dh, dh \rangle + \langle \delta h, \delta h \rangle
\end{aligned}
$$

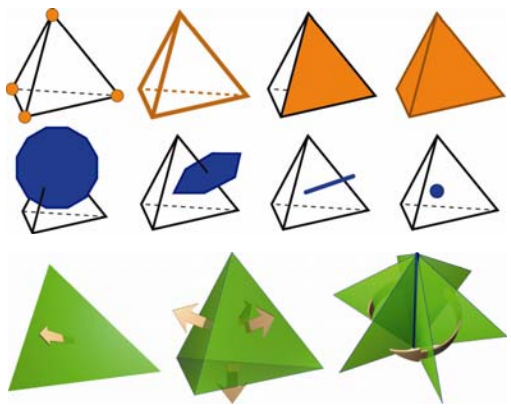
$$
\begin{aligned}
\langle d\alpha, h \rangle &= \langle \alpha, \delta h \rangle = 0 \\
\langle h, \delta\beta \rangle &= \langle dh, \beta \rangle = 0
\end{aligned}
$$

$$\langle d\alpha, \delta\beta \rangle = \langle dd\alpha, \beta \rangle = 0$$

$$\Delta h = d(\delta h) + \delta(dh) = 0$$

Forms and Simplicies

Where does what live?



Chain Map

Smooth/Discrete relationship

- discrete coefficients are generalized "samples" $c_i = \int_{\sigma_i^k} \omega^k$
- reconstruction?
  - generalized interpolation
  - relate smooth **d** and discrete *d*

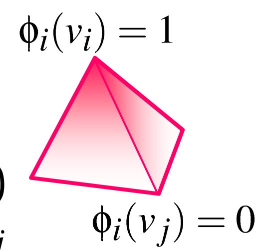
$$R(dc) = \mathbf{d}Rc$$

- how?

Whitney Elements

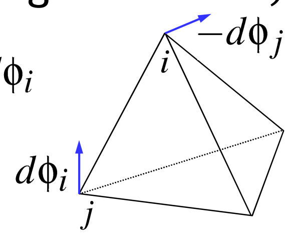
PL interpolation of forms

- 0-forms (functions)
  - "hat" functions
- 1-forms (edge elements)



$$\phi_{ij} = \phi_i d\phi_j - \phi_j d\phi_i$$

$$\phi_{ij}(e_{kl}) = \delta_{kl}^{ij}$$



$$\boxed{\begin{array}{c} d\phi_{ij} = \text{const} \\ \delta\phi_{ij} = 0 \end{array}}$$

Whitney Elements

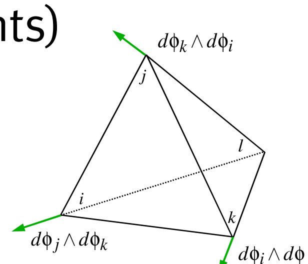
General form

$$\phi_{\sigma^p} = p! \sum_i (-1)^i \phi_i d\phi_0 \wedge \ldots \wedge \hat{d\phi_i} \wedge \ldots d\phi_p$$

- 2-forms (face elements)

$$\phi_{f_l} = 2 \sum \phi_i d\phi_j \wedge d\phi_k$$

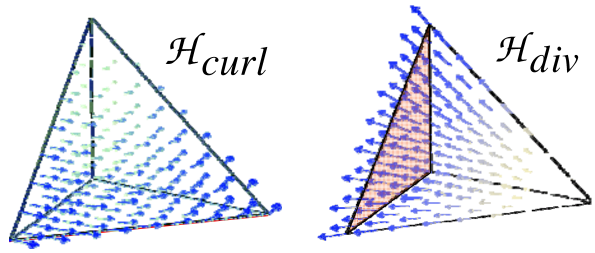
- 3-forms constant
- continuity
  - tangential (1), normal (2)



Underlying FEM

Whitney elements

- Nedelec and Raviart-Thomas type



- tangent and normal continuity