Mesh Parameterizations

Xiao-Ming Fu

Outline

- Definition
- Tutte’s barycentric mapping
- Least squares conformal maps(LSCM, ASAP)
- Angle-Based Flattening (ABF)
    - ABF++, LABF
- As-rigid-as-possible (ARAP)
    - Simplex Assembly

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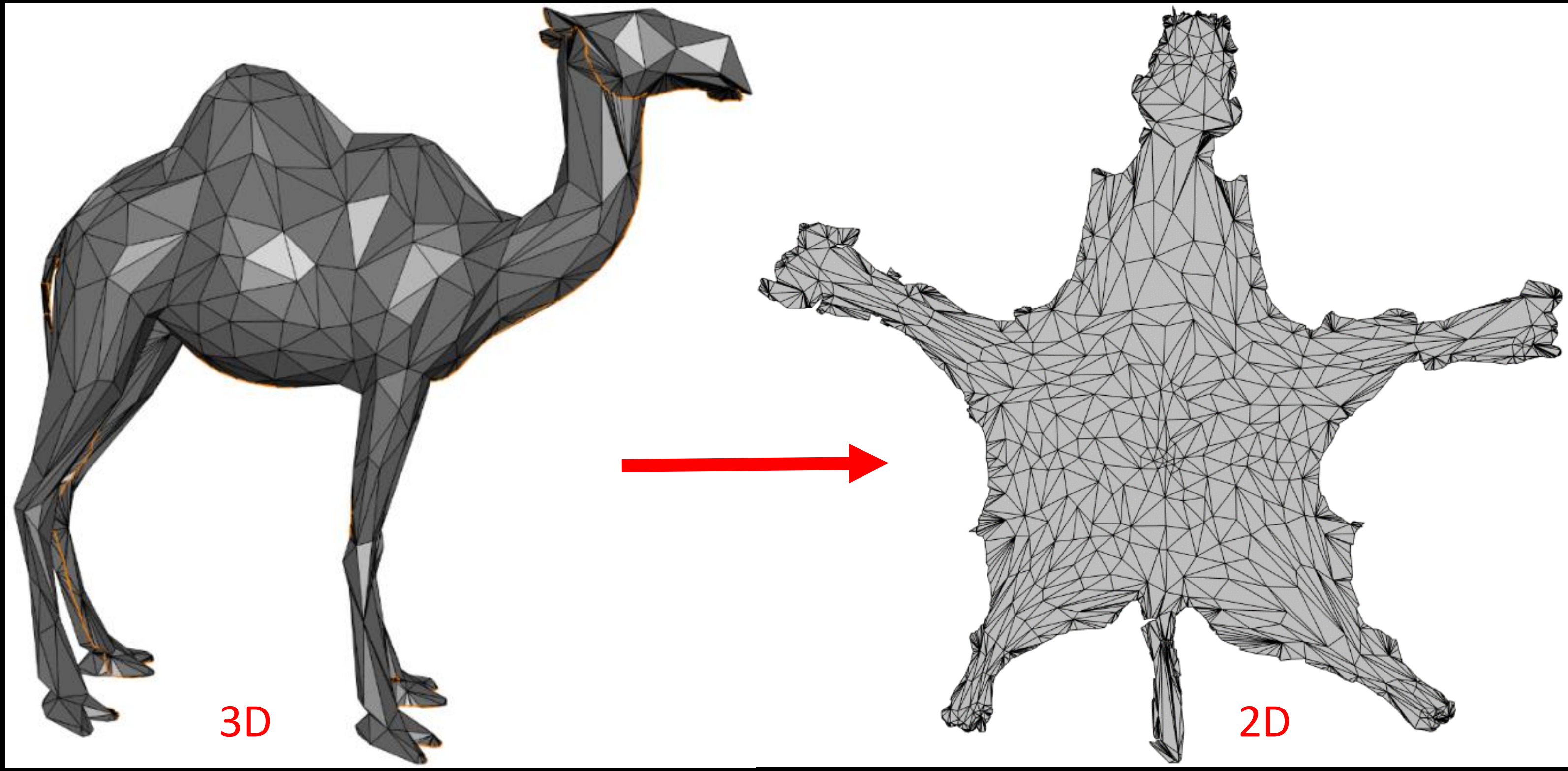
Definition

- A function that puts input surface in one-to-one correspondence with a 2D domain.
- Parameterization of a Triangulated Surface
  - all (u_i, v_i) coordinates associated with each vertex $\boldsymbol{v}_i = (x_i, y_i, z_i\ )^T$



Goal

- Map attributes
  - Color
  - Normal
  - ……

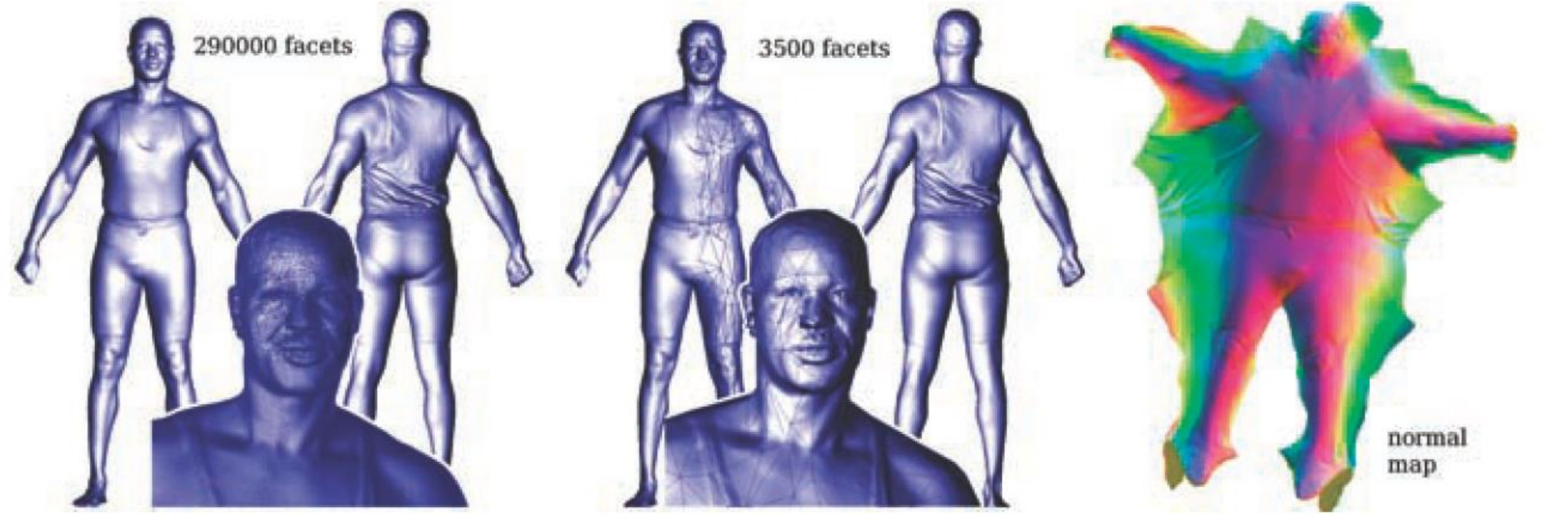
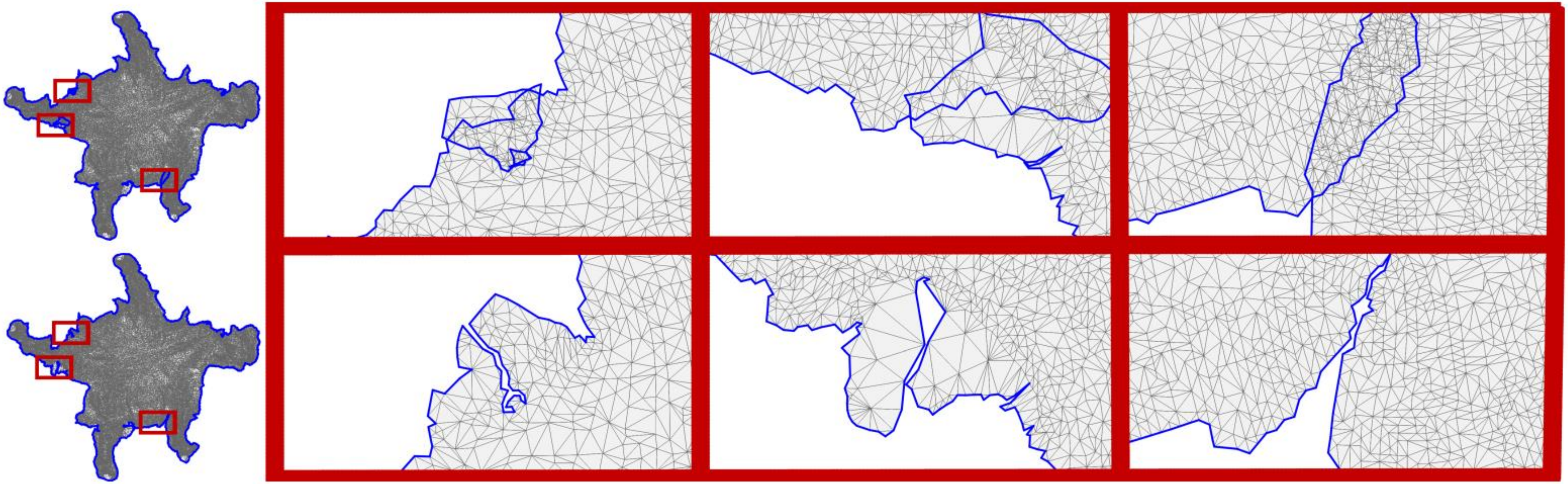


Figure 5.2. Appearance-preserving simplification as another application of parameterization: The initial object (left) is decimated to 1.5% of the original size (center). High-resolution geometric details are encoded in a normal map (right) and mapped to the simplified model, thereby preserving the original appearance. (Model courtesy of Cyberware. Image taken from [Hormann et al. 07]. ©2007 ACM, Inc. Included here by permission.)

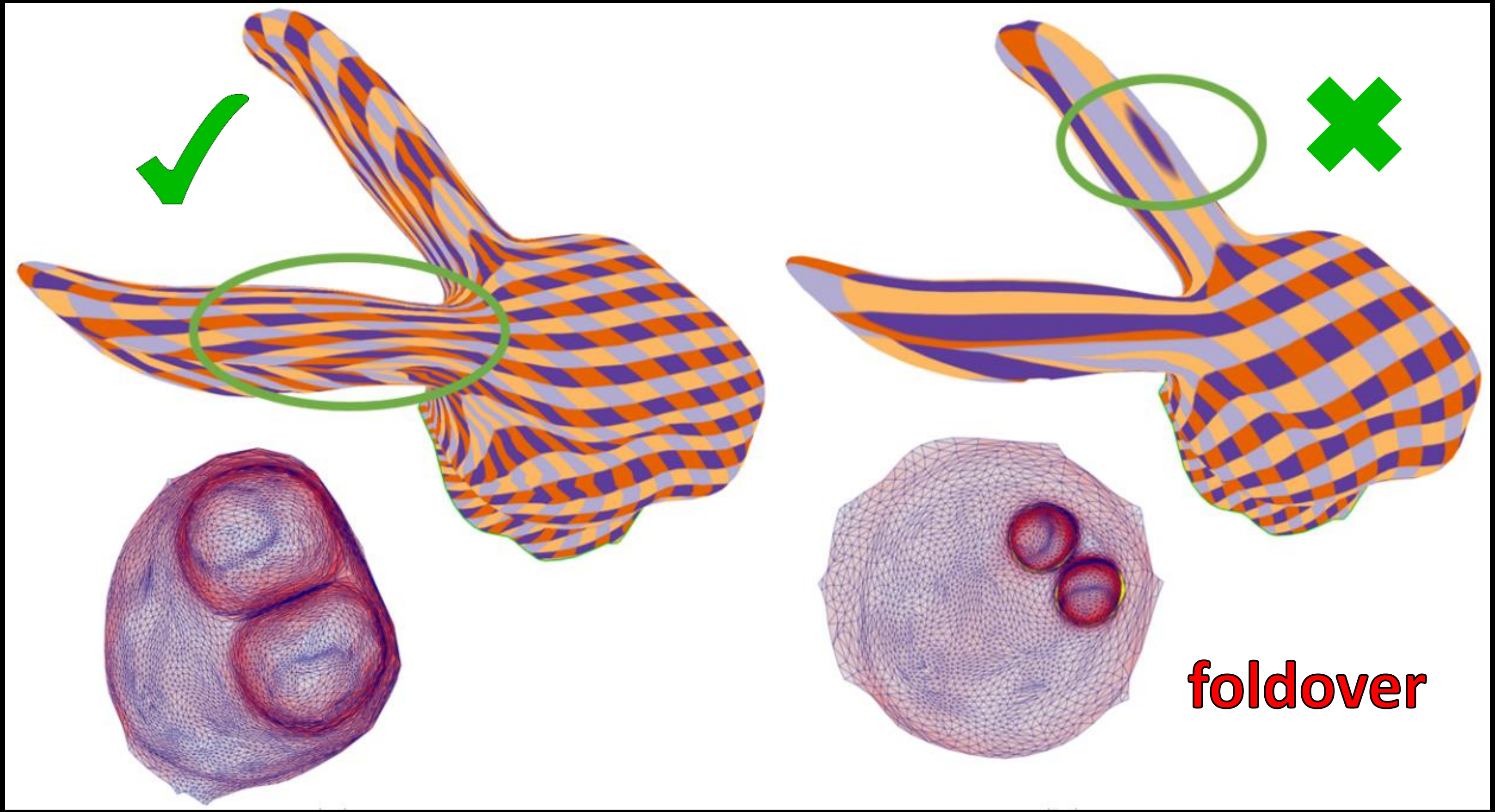
Constraints

- Bijective
  - The image of the surface in parameter space does not self-intersect.
  - The intersection of any two triangles in parameter space is either a common edge, a common vertex, or empty.



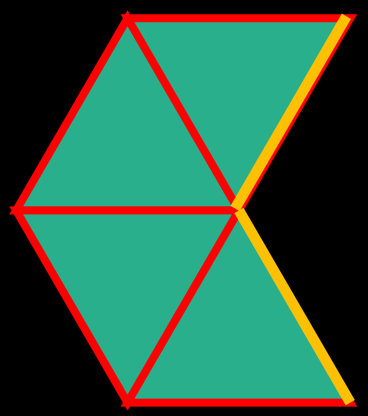
Constraints

- Inversion-free
  - The orientation of each triangle is positive.

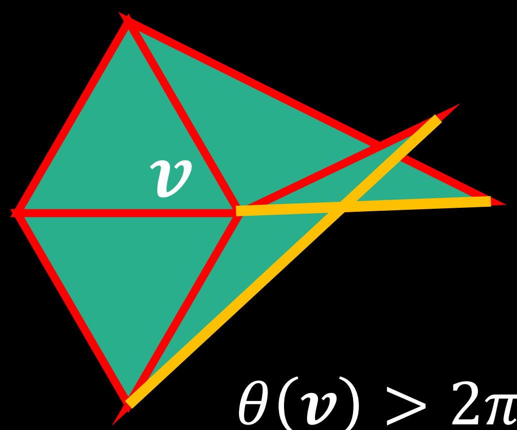


Constraints

- Locally injective
  - The orientation of each triangle is positive → $\det J > 0$.
  - For boundary vertex, the mapping is locally bijective → $\theta(\boldsymbol{v}) < 2\pi$.



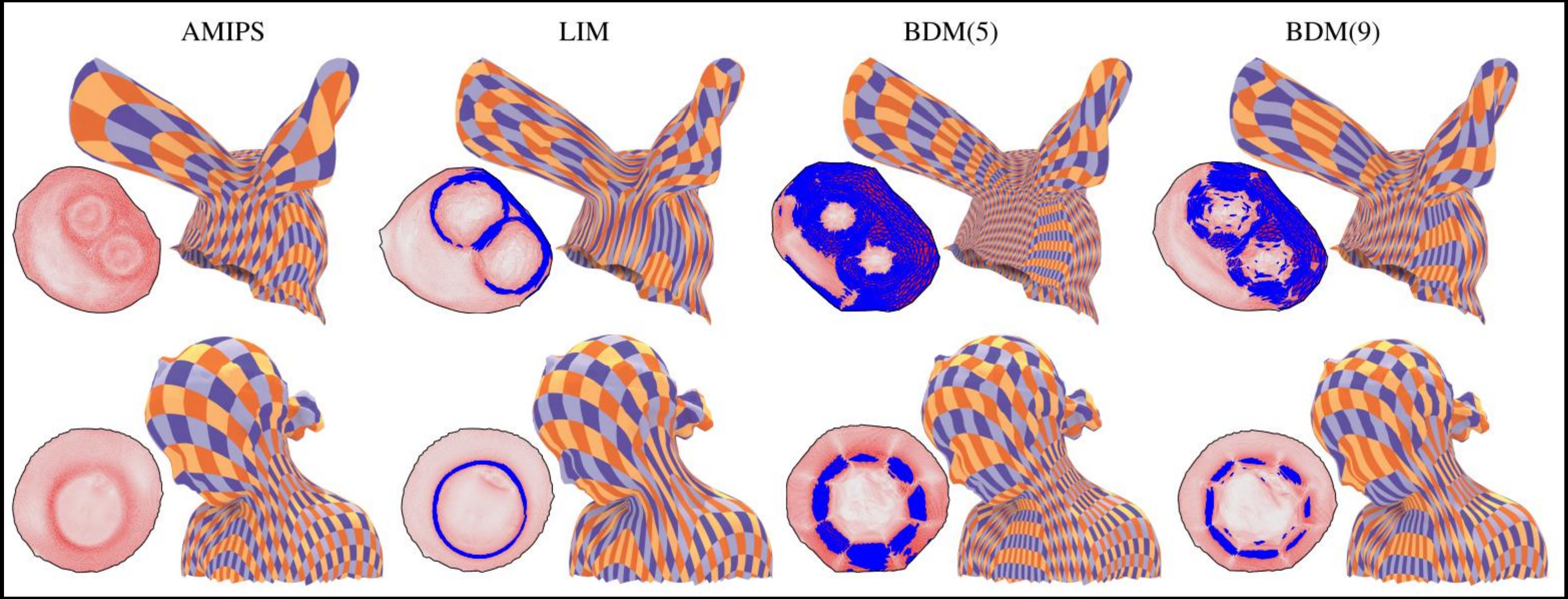
$\theta(\boldsymbol{v}) < 2\pi$



$\theta(\boldsymbol{v}) > 2\pi$

Constraints

- Low distortion



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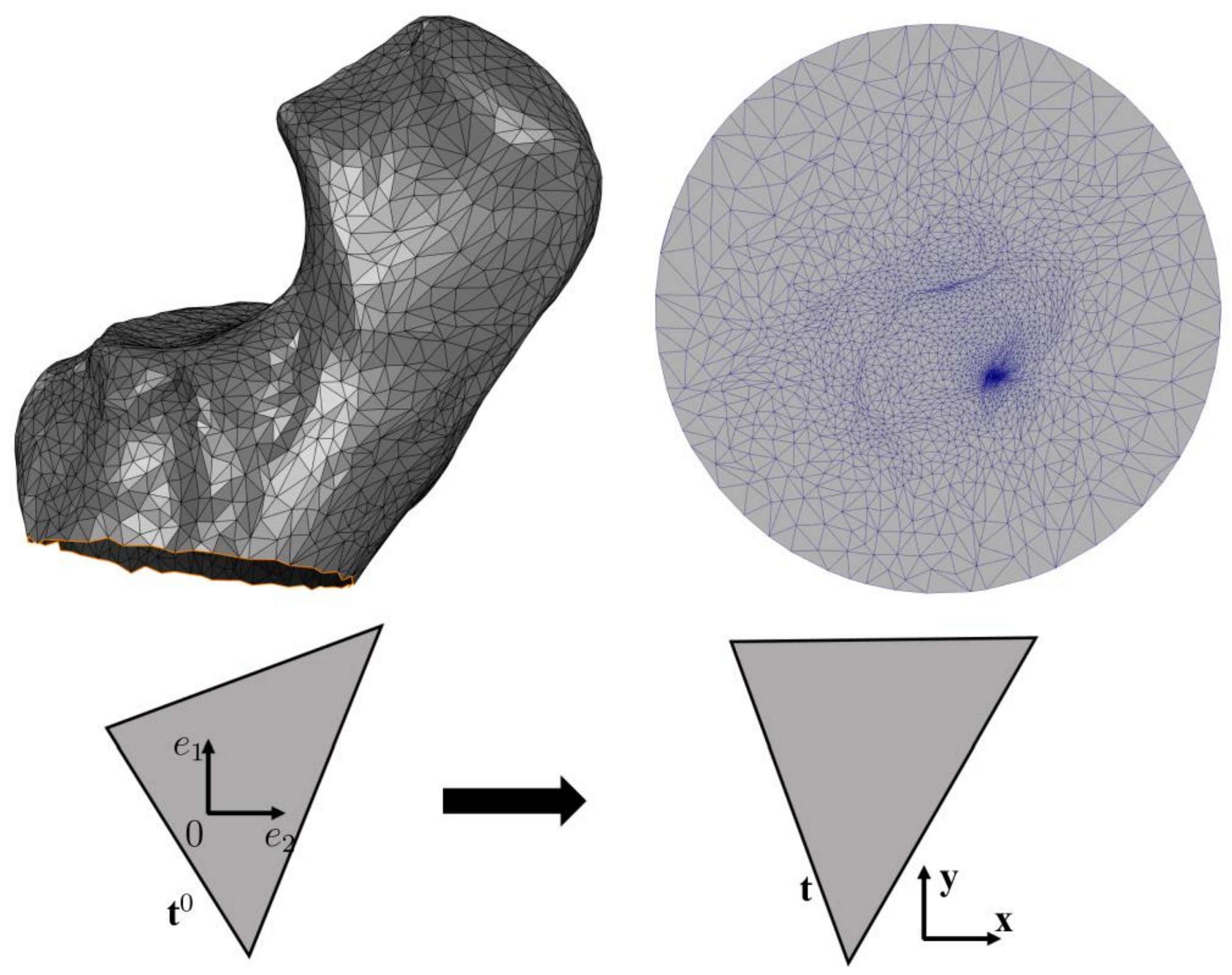
Barycentric Mapping, Tutte's embedding

- One of the most widely used methods.

Given a triangulated surface homeomorphic to a disk, if the (u, v) coordinates at the boundary vertices lie on a convex polygon in order, and if the coordinates of the internal vertices are a convex combination of their neighbors, then the (u, v) coordinates form a valid parameterization (without self-intersections, bijective).

Barycentric Mapping

- Homeomorphic to a disk.
- A convex polygon
  - circle, square,……
- A convex combination
  - $\omega_{ij} > 0$
  - Uniform Laplacian, mean value coordinate
- Solver: linear equation.



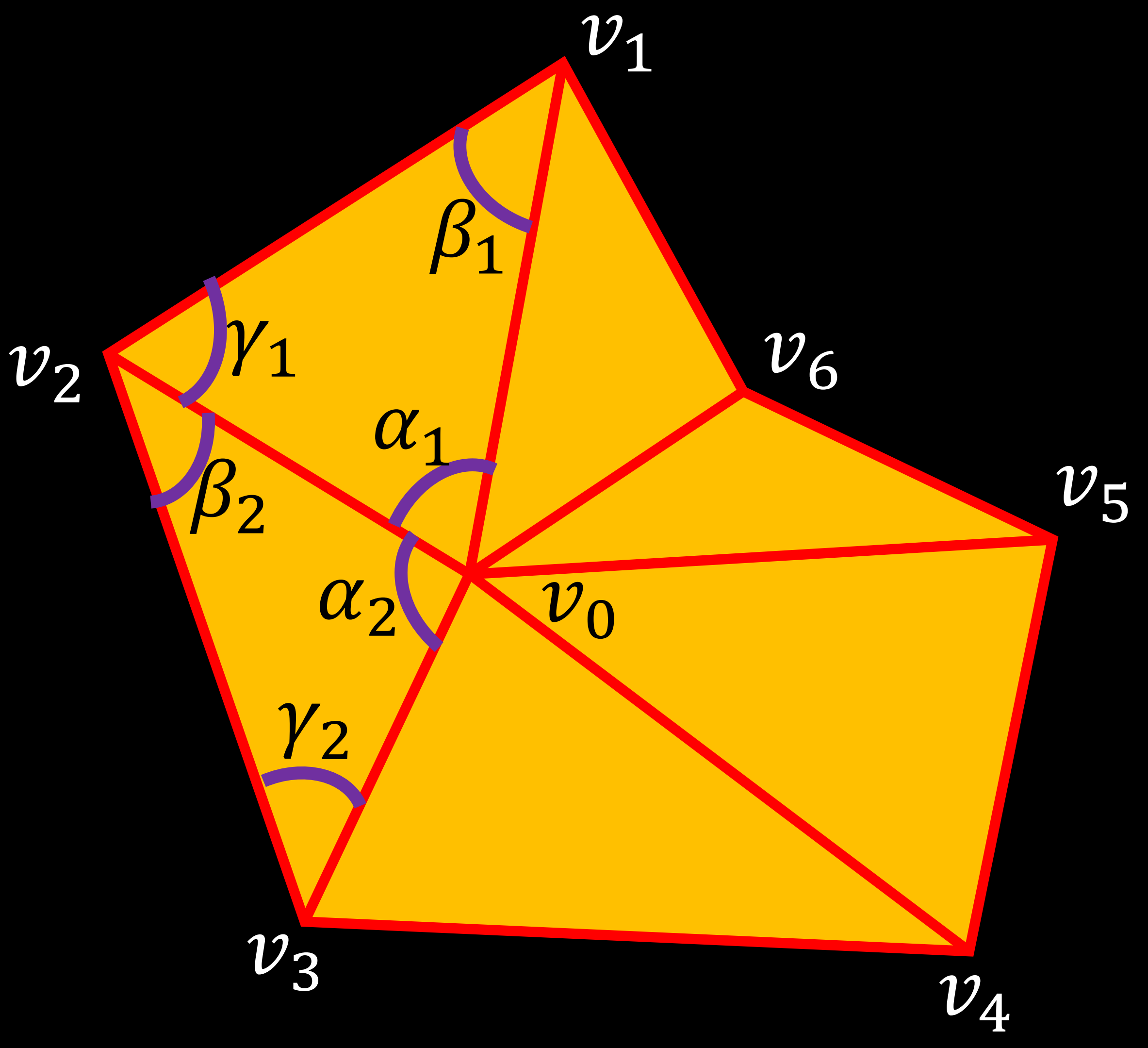
Mean value coordinates

- Our aim is to study sets of weights $\lambda_1, \ldots, \lambda_k \geq 0$ such that

$$\sum_{i=1}^{k} \lambda_i v_i = v_0$$

$$\sum_{i=1}^{k} \lambda_i = 1$$

v_i is on 2D.



Proposition

- The weights

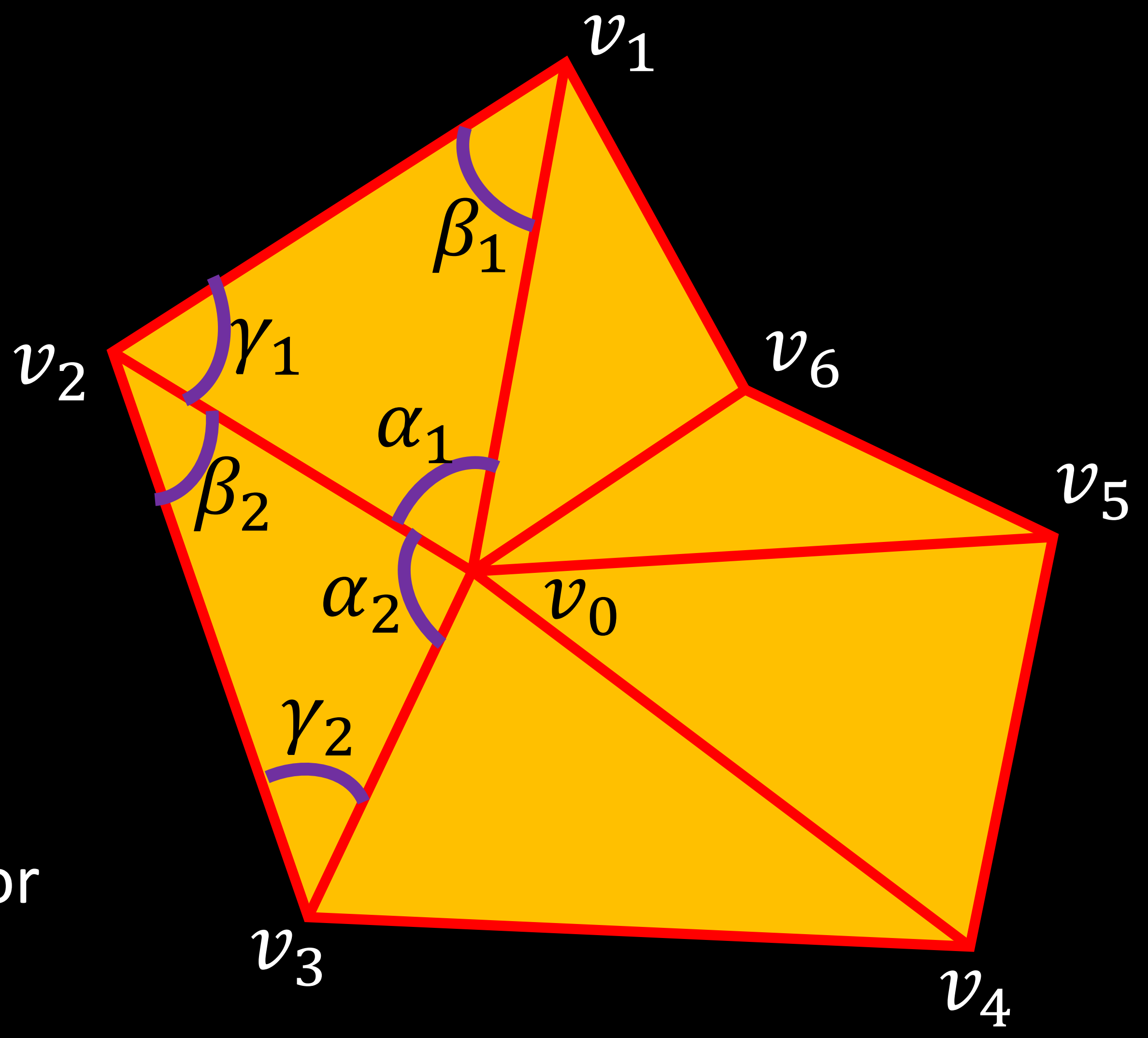
$$\lambda_i = \frac{\omega_i}{\sum_{i=1}^{k} \omega_i},$$

$$\omega_i = \frac{\tan \frac{\alpha_{i-1}}{2} + \tan \frac{\alpha_i}{2}}{\|v_i - v_0\|}$$

are the valid weights.

Proof: substitution. ???

Come from the mean value theorem for harmonic functions. ???



Mean value coordinates

- The input mesh is a spatial one.
  - $v_i \in R^3$
  - the mean value coordinates can be applied directly.
  - compute the coordinates directly form the spatial angle.

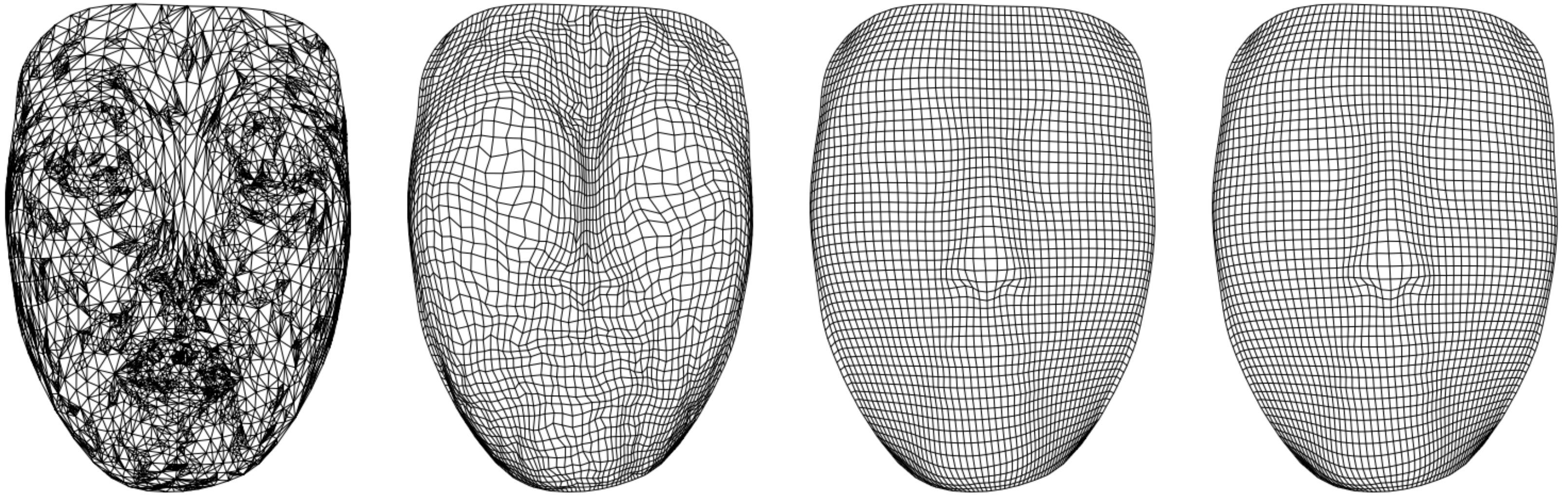


Figure 3. Comparisons from left to right:
(3a) Triangulation, (3b) Tutte, (3c) shape-preserving, (3d) mean value

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Conformal mapping

- Conformal mappings locally correspond to similarities

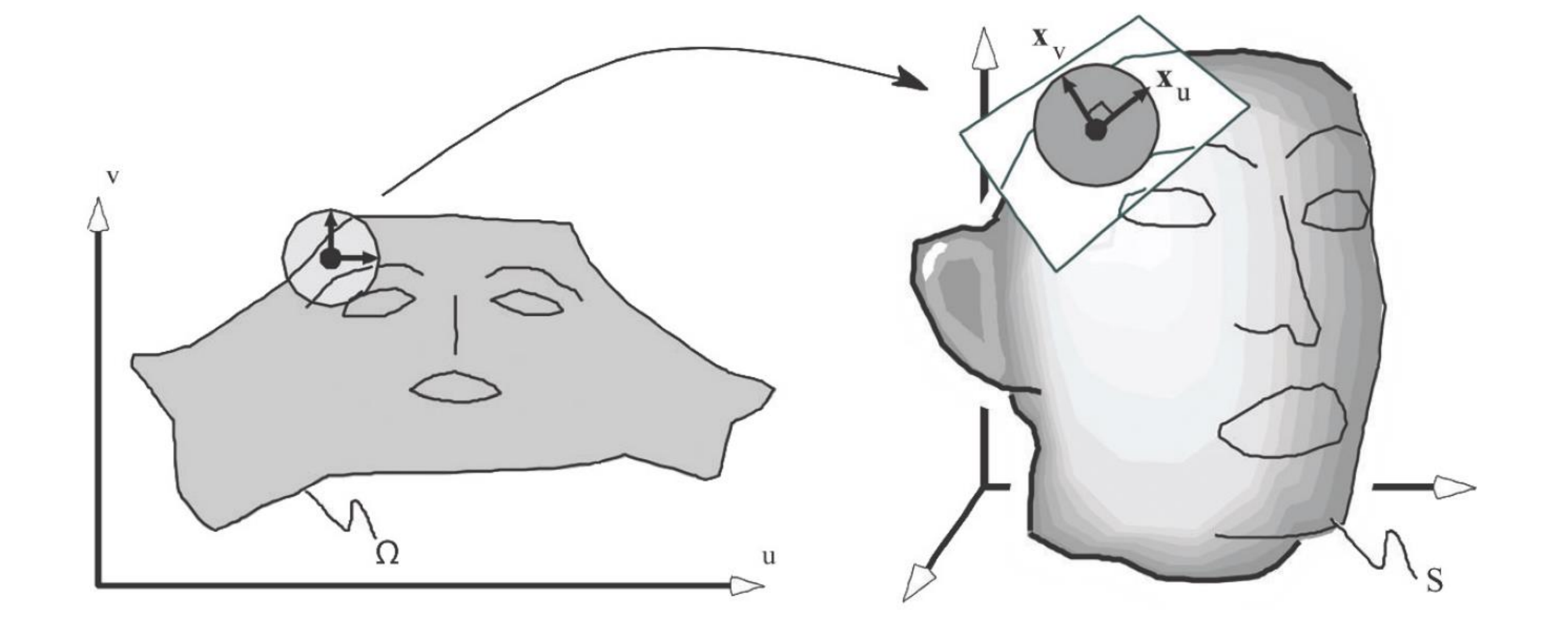
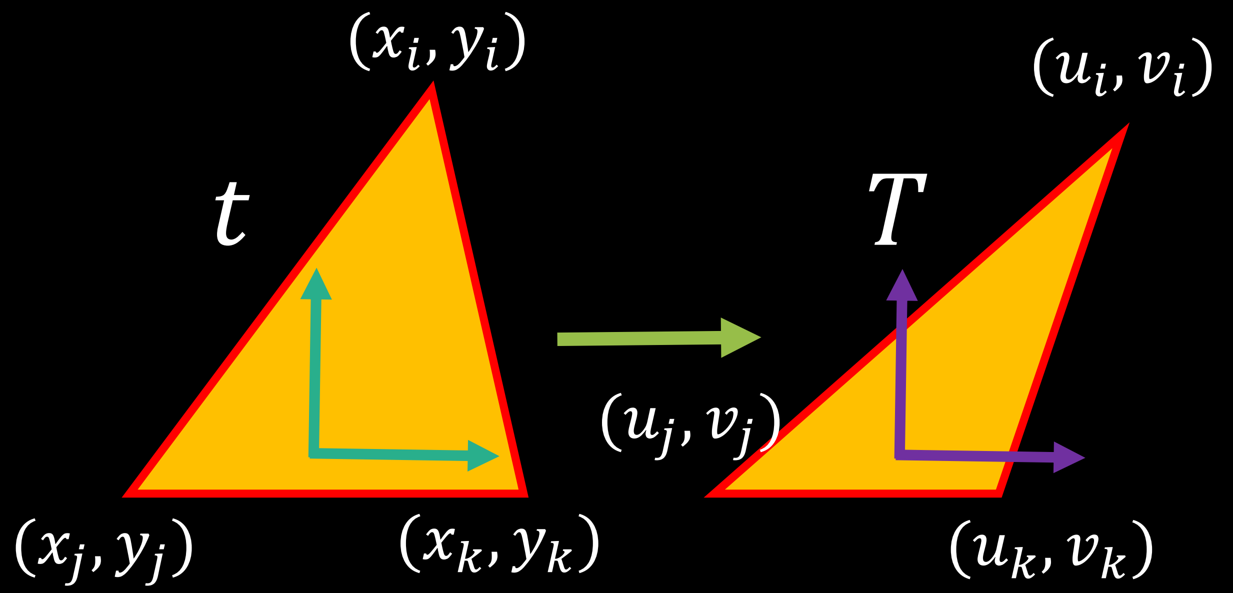


Figure 5.8. A conformal parameterization transforms a small circle into a small circle, i.e., it is locally a similarity transform. (Image taken from [Hormann et al. 07]. ©2007 ACM, Inc. Included here by permission.)

Mapping

- Build a local coordinate system on input triangle t.
- The mapping is piecewise linear.
- J_t is 2×2.



$$\begin{pmatrix} u_j - u_i & u_k - u_i \\ v_j - v_i & v_k - v_i \end{pmatrix} \begin{pmatrix} x_j - x_i & x_k - x_i \\ y_j - y_i & y_k - y_i \end{pmatrix}^{-1}$$

$$f_t(\boldsymbol{x}) = J_t \boldsymbol{x} + \boldsymbol{b}_t$$

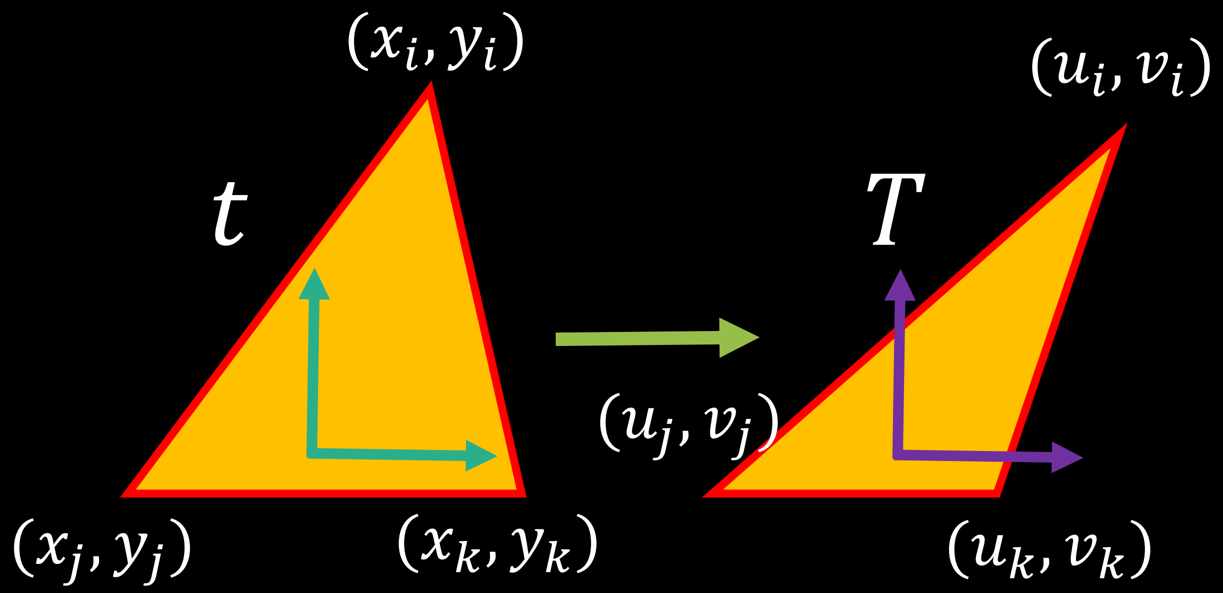
Mapping

- J_t is the Jacobian of $f_t(\boldsymbol{x})$.

$$J_t = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix} = \nabla u$$

$$= \frac{1}{2A_t} \begin{pmatrix} y_j - y_k & y_k - y_i & y_i - y_j \\ x_k - x_j & x_i - x_k & x_j - x_i \end{pmatrix} \begin{pmatrix} u_i \\ u_j \\ u_k \end{pmatrix}$$



$$f_t(\boldsymbol{x}) = J_t\boldsymbol{x} + \boldsymbol{b}_t$$

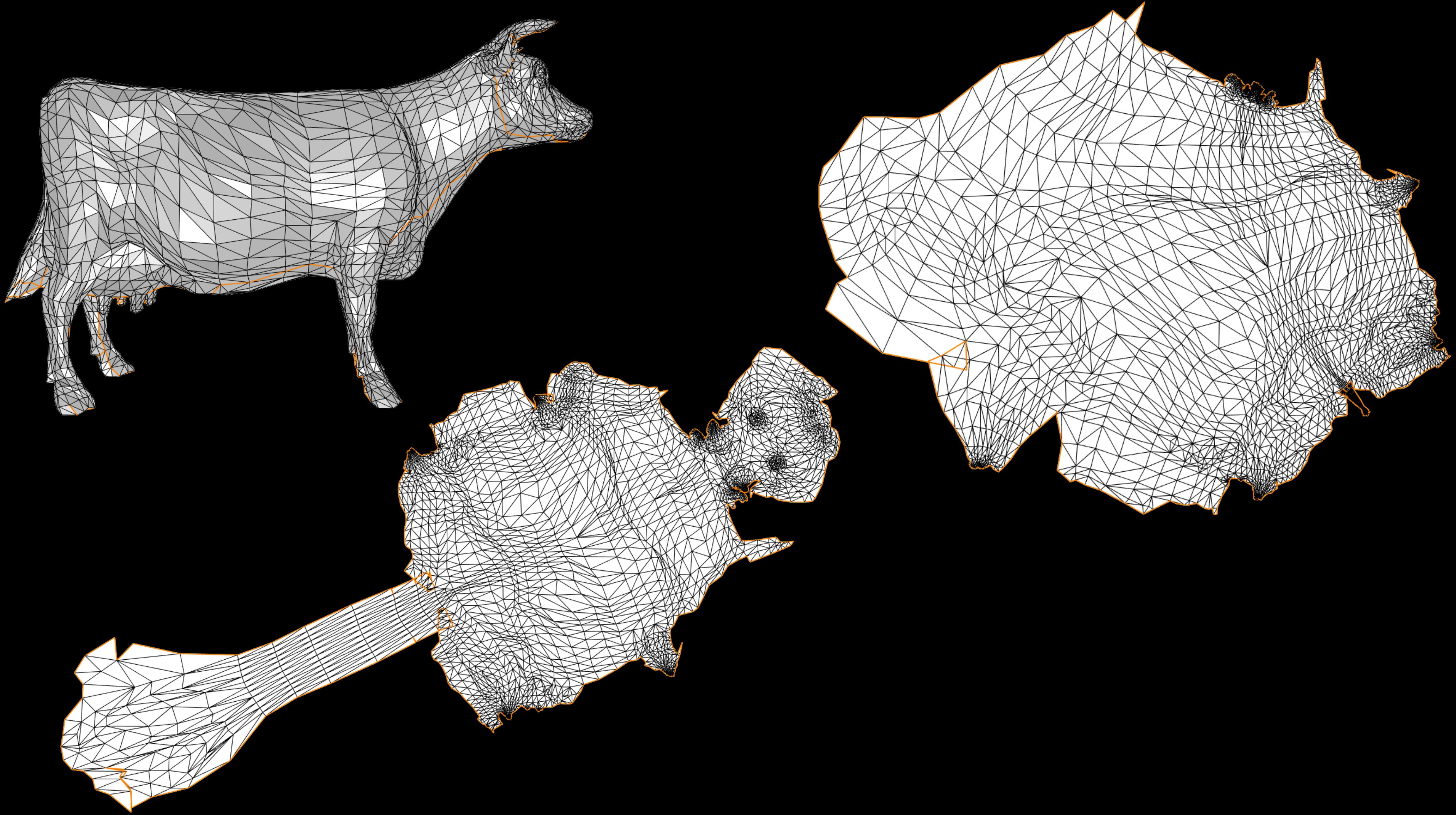
Similar transform

- 2D case: for one triangle t
  - $J_t = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \mathrm{s}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$
  - $\Longrightarrow \begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$
  - Cauchy-Riemann Equations.

$$J_t = \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{pmatrix}$$

Least squares conformal maps(LSCM, ASAP)

- Energy
  - $E_{LSCM} = \sum_t A_t \left(\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 \right)$
  - measure non-conformality
  - It is invariant with respect to arbitrary translations and rotations.
  - E_{LSCM} does not have a unique minimizer.
  - Fixing at least two vertices. Significantly affect the results.



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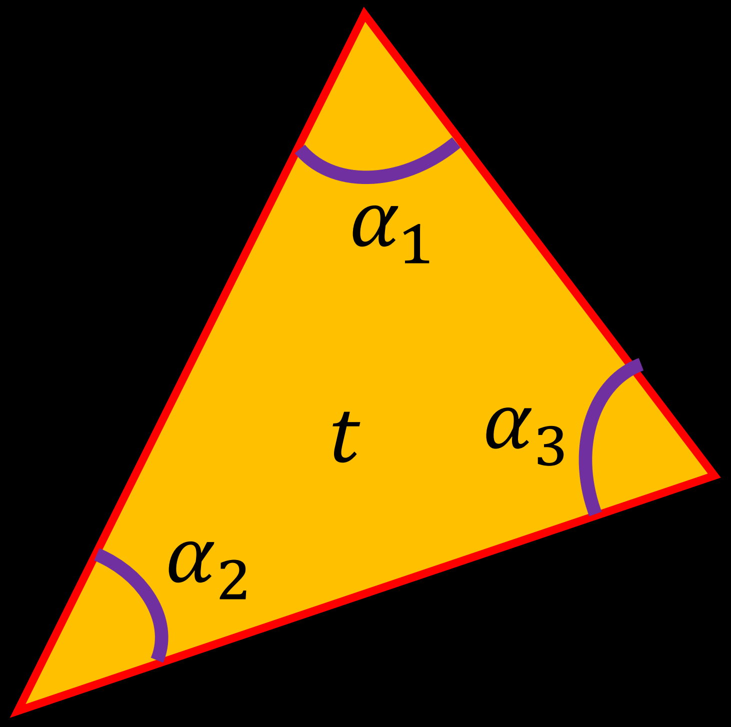
Angle-Based Flattening (ABF)

- Key observation: the parameter space is a 2D triangulation, uniquely defined by all the angles at the corners of the triangles.
  - Find angles instead of (u_i, v_i) coordinates.
  - Use angles to reconstruct the resulting parameterization.
- Optimization goal:



$$E_{ABF} = \sum_t \sum_{i=1}^{3} \omega_i^t \left(\alpha_i^t - \beta_i^t\right)^2$$

β_i^t: Optimal angles for α_i^t.

$\omega_i^t = \left(\beta_i^t\right)^{-2}$.

$$\beta_i^t = \begin{cases} \dfrac{\tilde{\beta}_i^t \cdot 2\pi}{\sum_i \tilde{\beta}_i^t}, Interior\ vertex \\ \tilde{\beta}_i^t, Boundary\ verterx \end{cases}$$

Constraints

- Positive resulting angles:

$$\alpha_i^t > 0$$

- The three triangle angles have to sum to π:

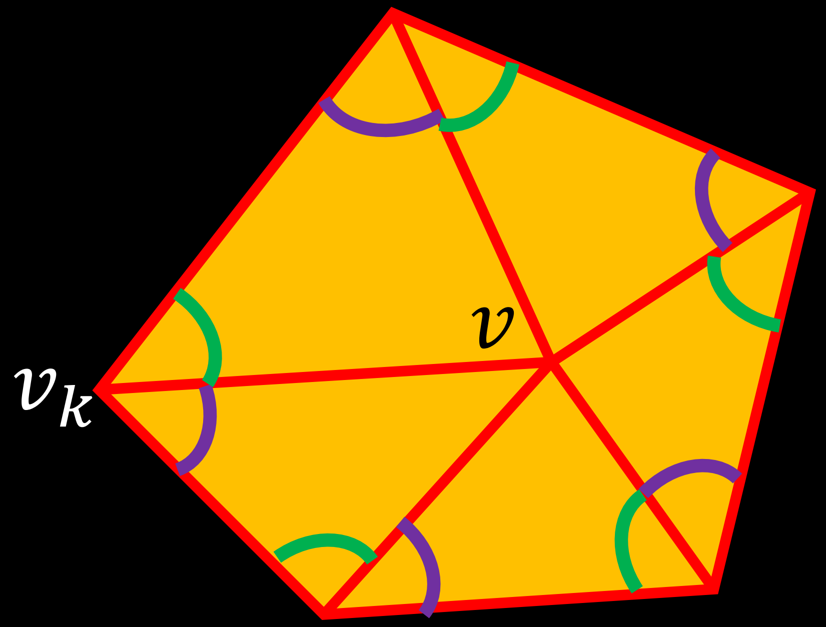
$$\alpha_i^t + \alpha_2^t + \alpha_3^t = \pi$$

- For each internal vertex the incident angles have to sum to 2π:

$$\sum_{t \in \Omega(v)} \alpha_k^t = 2\pi$$

- Reconstruction constraints:

$$\prod_{t \in \Omega(v)} \sin \alpha_{k \oplus 1}^t = \prod_{t \in \Omega(v)} \sin \alpha_{k \ominus 1}^t \; ???$$



Linear ABF

- Reconstruction constraints are nonlinear and hard to solve.
- Initial estimation + estimation error
  - $\alpha_i^t = \gamma_i^t + e_i^t$

$$\log\left(\prod_{t\in\Omega(v)} \sin\alpha_{k\oplus 1}^t\right) = \log\left(\prod_{t\in\Omega(v)} \sin\alpha_{k\ominus 1}^t\right)$$

$$\sum_{t\in\Omega(v)} \log\left(\sin\alpha_{k\oplus 1}^t\right) = \sum_{t\in\Omega(v)} \log\left(\sin\alpha_{k\ominus 1}^t\right)$$

- Taylor expansion:

$$\log\left(\sin\alpha_{k\oplus 1}^t\right) = \log\left(\sin\gamma_{k\oplus 1}^t + e_{k\oplus 1}^t\right)$$
$$= \log\left(\sin\gamma_{k\oplus 1}^t\right) + e_{k\oplus 1}^t \cot\gamma_{k\oplus 1}^t + \cdots$$

It is linear with estimation error.

Solver

- Set $\gamma_i^t = \beta_i^t$
- Problem:

$$\min_e E_{ABF} = \sum_t \sum_{i=1}^{3} \omega_i^t \left(e_i^t\right)^2$$

$$\text{subject to} \quad Ae = b$$

$\Longrightarrow$

$$\begin{pmatrix} D & A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} e \\ \lambda \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix}$$

$\Longrightarrow$

$$e = D^{-1}A^T\left(AD^{-1}A^T\right)^{-1}b\ ???$$

Reconstruct parameterization

- Greed method.
    - constructs the triangles one by one using a depth-first traversal.

- Least squares method.
    - an angle based least squares formulation which solves a set of linear equations relating angles to coordinates.

Greed method

- Choose a mesh edge $e^1 = (v_a^1, v_b^1)$.
- Project v_a^1 to $(0,0,0)$ and v_b^1 to $(\|e^1\|, 0,0)$.
- Push e^1 on the stack S.
- While S not empty, pop an edge $e = (v_a, v_b)$. For each face $f_i = (v_a, v_b, v_c)$ containing e:
  - If f_i is marked as set, continue.
  - If v_c is not projected, compute its position based on v_a, v_b and the face angles of f_i.
  - Mark f_i as set, push edge (v_b, v_c) and (v_a, v_c) on the stack.

- Accumulate numerical error.

Least squares method

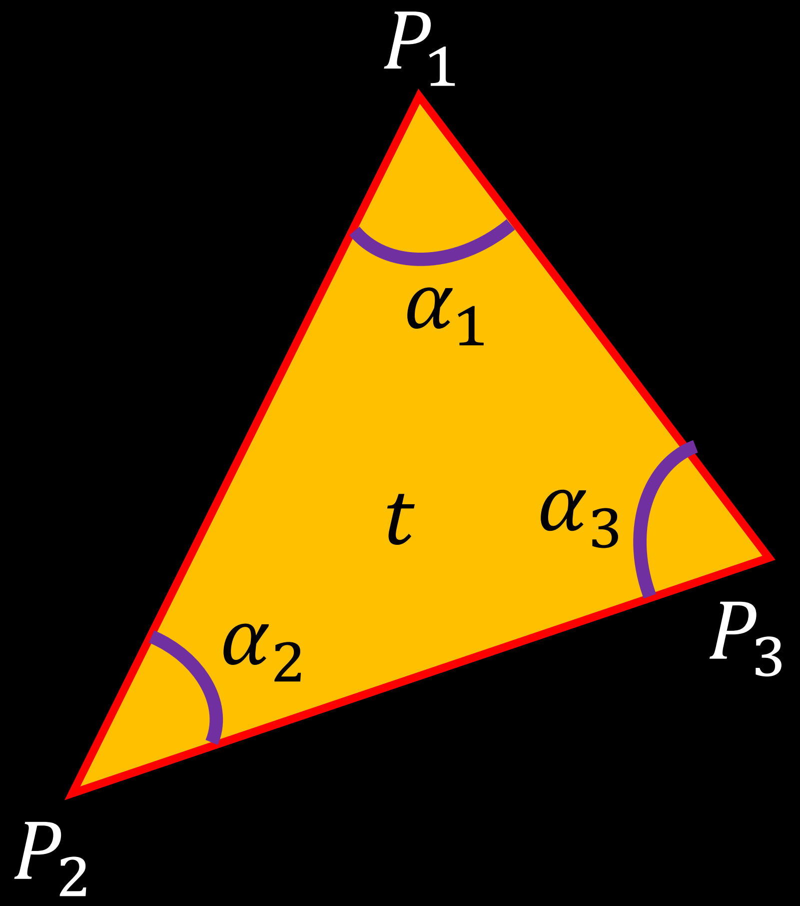
- The ratio of triangle edge lengths $\|\overrightarrow{P_1P_3}\|$ and $\|\overrightarrow{P_1P_2}\|$ is

$$\frac{\|\overrightarrow{P_1P_3}\|}{\|\overrightarrow{P_1P_2}\|} = \frac{\sin\alpha_2}{\sin\alpha_3}$$

$\Longrightarrow$

$$\overrightarrow{P_1P_3} = \frac{\sin\alpha_2}{\sin\alpha_3}\begin{pmatrix}\cos\alpha_1 & -\sin\alpha_1\\ \sin\alpha_1 & \cos\alpha_1\end{pmatrix}\overrightarrow{P_1P_2}$$

- Thus for each triangle, given the position of two vertices and the angles, the position of the third vertex can be uniquely derived.
  - greedy method.

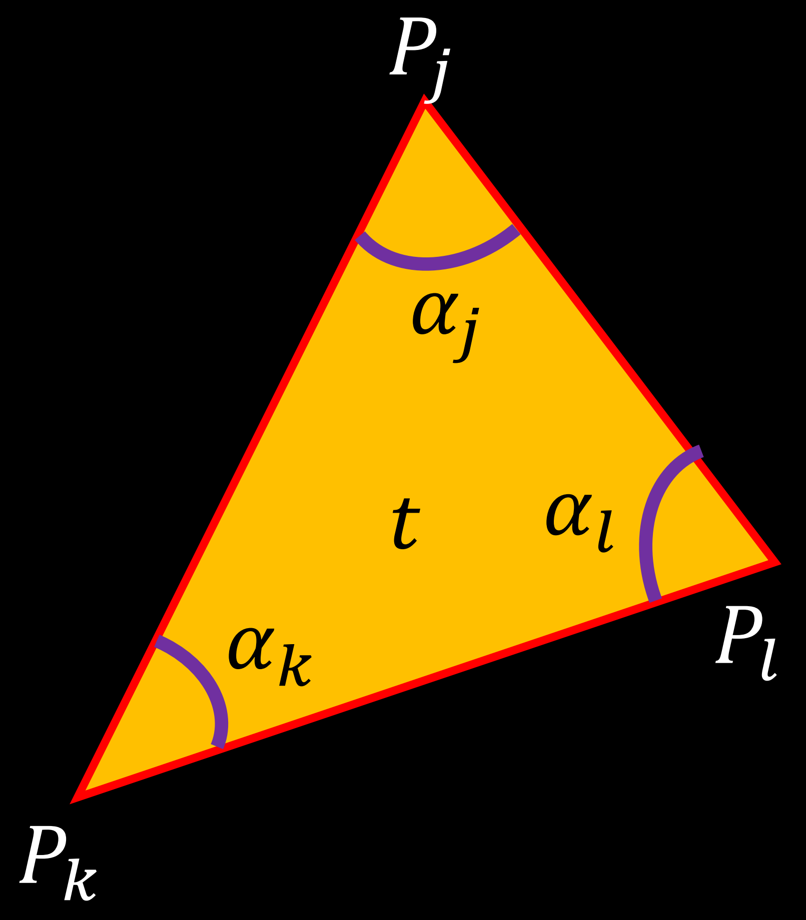


Least squares method

$$\forall t = (j, k, j), \qquad M^t(P_k - P_j) + P_j - P_l = 0$$

$$M^t = \frac{\sin \alpha_k}{\sin \alpha_l} \begin{pmatrix} \cos \alpha_j & -\sin \alpha_j \\ \sin \alpha_j & \cos \alpha_j \end{pmatrix}$$

1. Two equations per triangle for the x and y coordinates of the vertices.
2. The angles of a planar triangulation define it uniquely up to rigid transformation and global scaling.
   - Introduce four constraints which eliminate these degrees of freedom.
   - Fix two vertices sharing a common edge.

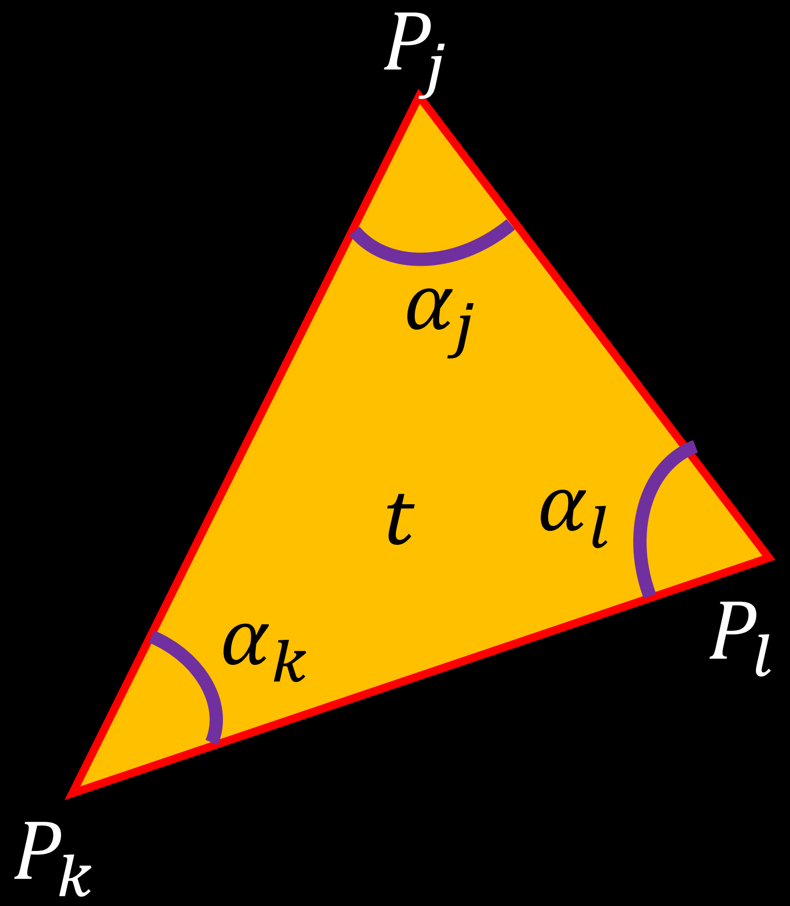


Least squares method

- Choose one edge $e^1 = (v_a^1, v_b^1)$.
- Project v_a^1 to $(0,0,0)$ and v_b^1 to $(\|e^1\|, 0,0)$.

- Solve following energy to compute positions of other vertices:

$$E = \sum_t \left\| M^t \left(P_k - P_j\right) + P_j - P_l \right\|^2$$



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As-rigid-as-possible method

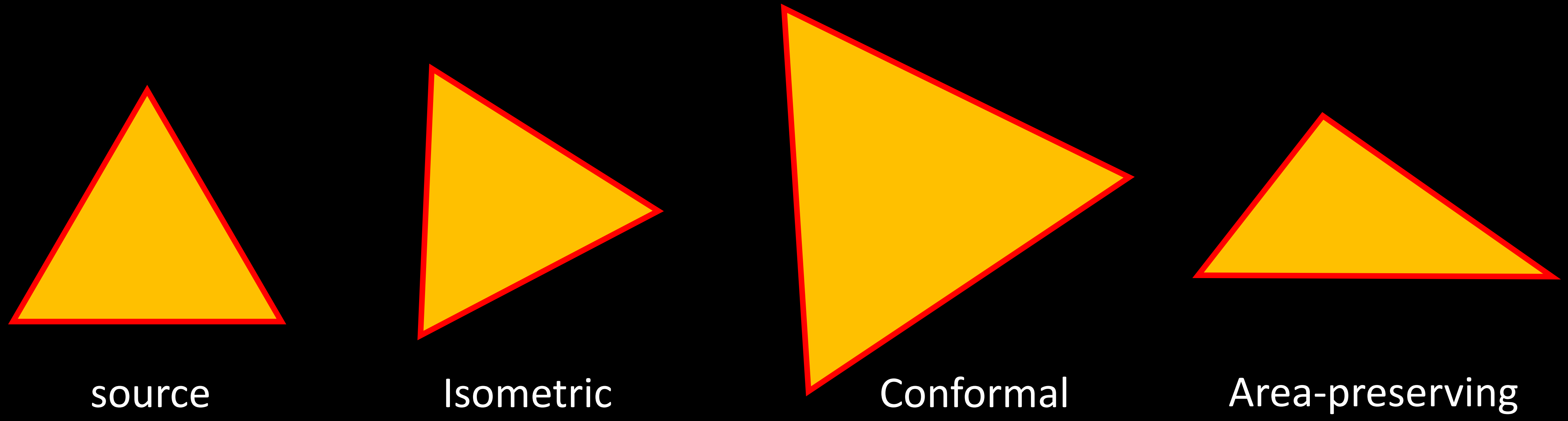
Paper: A Local/Global Approach to Mesh Parameterization

(a) ASAP (b) ARAP (c) LABF (d) IC (e) CP

Figure 1: *Parameterization of the Gargoyle model using (a) our As-Similar-As-Possible (ASAP) procedure, (b) As-Rigid-As-Possible (ARAP) procedure, (c) Linear ABF [ZLS07], (d) inverse curvature approach [YKL*08], and (e) curvature prescription approach [BCGB08]. The pink lines are the seams of the closed mesh when cut to a disk.*

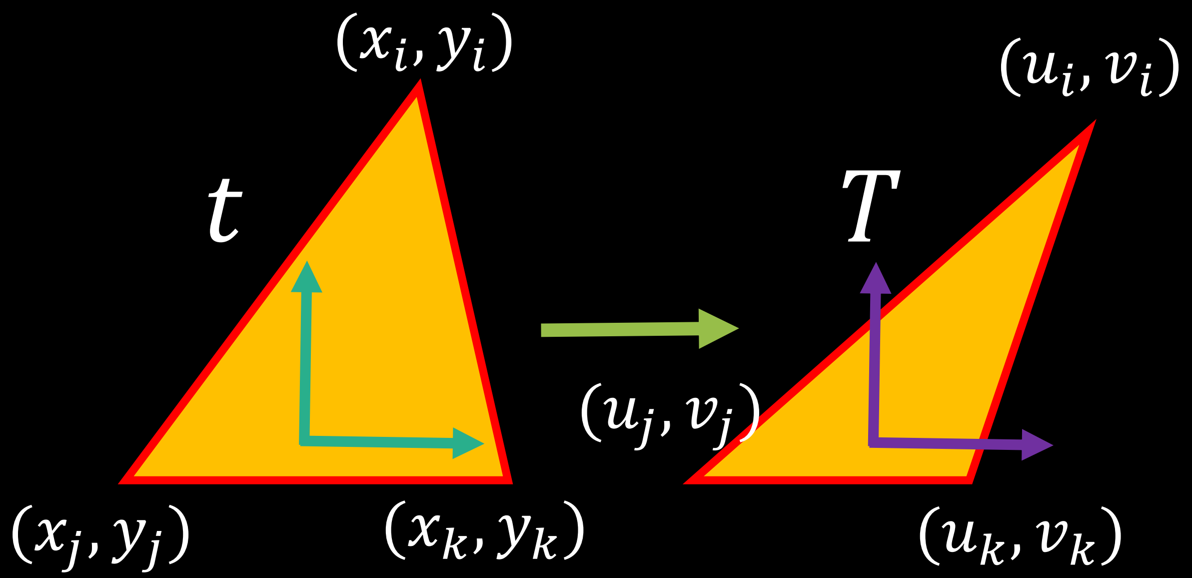
Distortion type

- Three common distortion types:
  - Isometric mapping: rotation + translation
  - Conformal mapping: similarity + translation
  - Area-preserving mapping: area-preserving + translation
  - Conformal + Area-preserving $\Longleftrightarrow$ Isometric



Singular values

- Isometric mapping
  - $J_t \Longrightarrow$ rotation matrix
  - $\sigma_1 = \sigma_2 = 1$
- Conformal mapping
  - $J_t \Longrightarrow$ similar matrix
  - $\sigma_1 = \sigma_2$
- Area-preserving mapping
  - $\det J_t = 1$
  - $\sigma_1 \sigma_2 = 1$



$$f_t(\boldsymbol{x}) = J_t \boldsymbol{x} + \boldsymbol{b}_t$$

$$J_t = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

σ_1, σ_2 are the two singular values of J_t.

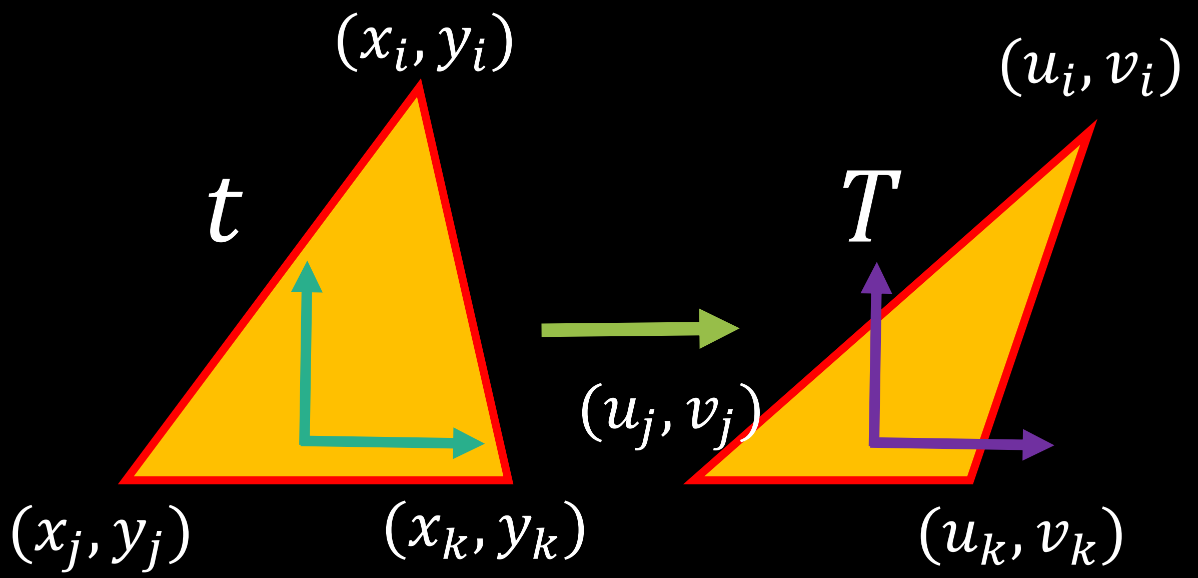
Goal

$$E(u, L) = \sum_t A_t \|J_t - L_t\|_F^2$$

L_t: target transformation

- Isometric mapping: rotation matrix
- Conformal mapping: similar matrix

- Variables:
  - 2D parameterization coordinate
  - Target transformation
- **How to optimize?**



$$\boldsymbol{f}_t(\boldsymbol{x}) = J_t \boldsymbol{x} + \boldsymbol{b}_t$$

$$J_t = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

σ_1, σ_2 are the two singular values of J_t.

General Local/Global Approach

- Alternatively optimization
  - Local step:
    - Fix 2D parameterization coordinates, optimize target transformations.
  - Global step:
    - Fix target transformations, optimize 2D parameterization coordinates.
- **Global step**:
  - Quadratic energy
  - Linear system
  - Eigen

$$E(u, L) = \sum_t A_t \|J_t - L_t\|_F^2$$

Local step: Procrustes analysis

- Approximate one 2×2 matrix J_t as best we can by another 2×2 matrix L_t.
- $d(J_t, L_t) = \|J_t - L_t\|_F^2 = trace\left((J_t - L_t)^T (J_t - L_t)\right)$
- Minimize $d(J_t, L_t)$ through Singular Value Decomposition (SVD)
  - $J_t = U\Sigma V^T, \quad \Sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}$
  - Signed SVD: U and V are rotation matrix, σ_2 maybe negative
  - Best rotation: UV^T
  - Best similar matrix: $U \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} V^T, s = \frac{\sigma_1 + \sigma_2}{2}$

Local/Global Approach summary

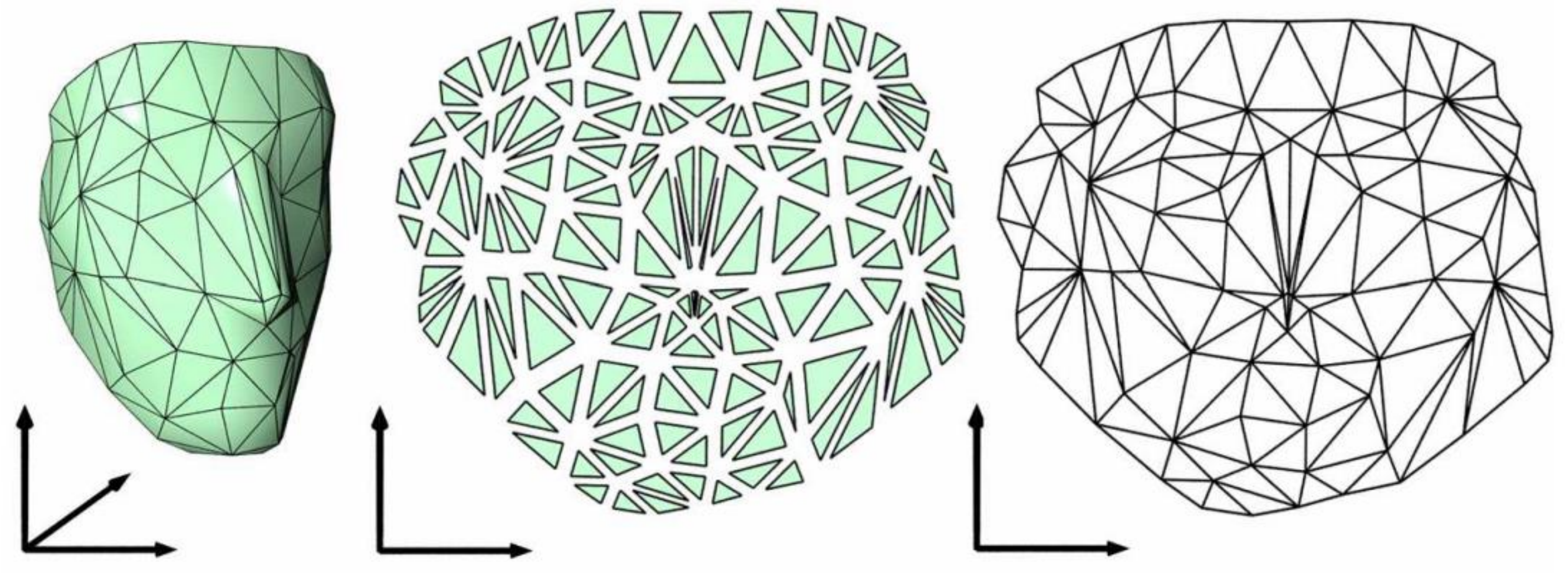


Figure 2: *Parameterizing a mesh by aligning locally flattened triangles. (Left) Original 3D mesh; (middle) flattened triangles; (right) 2D parameterization.*

Connection to singular values

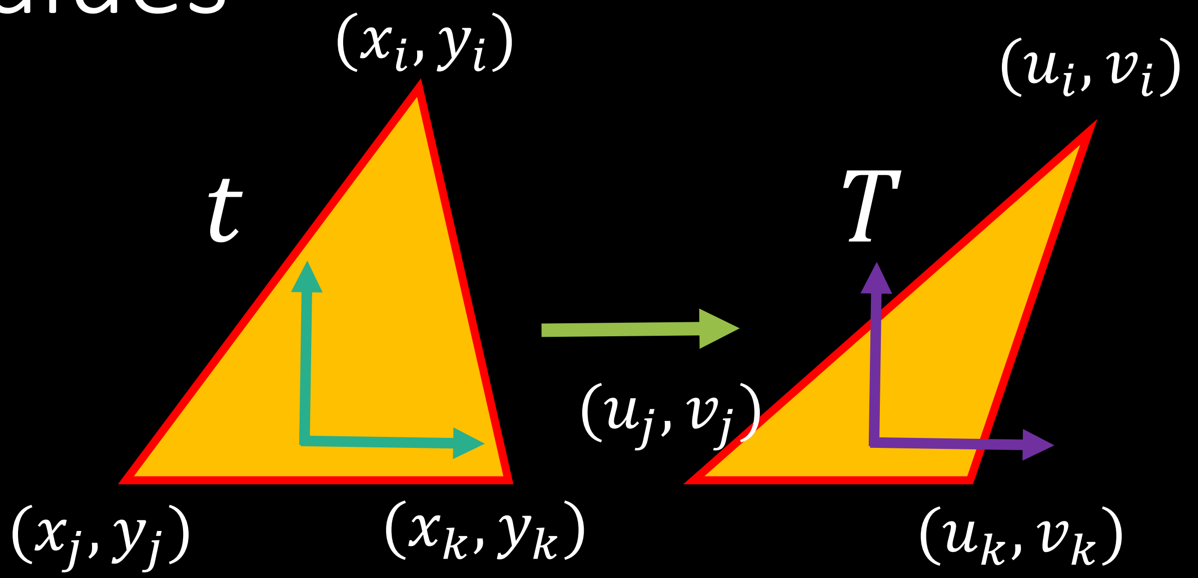
- Conformal

$$E(u) = \sum_t A_t(\sigma_t^1 - \sigma_t^2)^2$$

- Isometric

$$E(u) = \sum_t A_t\left((\sigma_t^1 - 1)^2 + (\sigma_t^2 - 1)^2\right)$$

???????



$$f_t(\boldsymbol{x}) = J_t\boldsymbol{x} + \boldsymbol{b}_t$$

σ_t^1, σ_t^2 are the two singular values of J_t.

$$E(u, L) = \sum_t A_t \|J_t - L_t\|_F^2$$

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Information

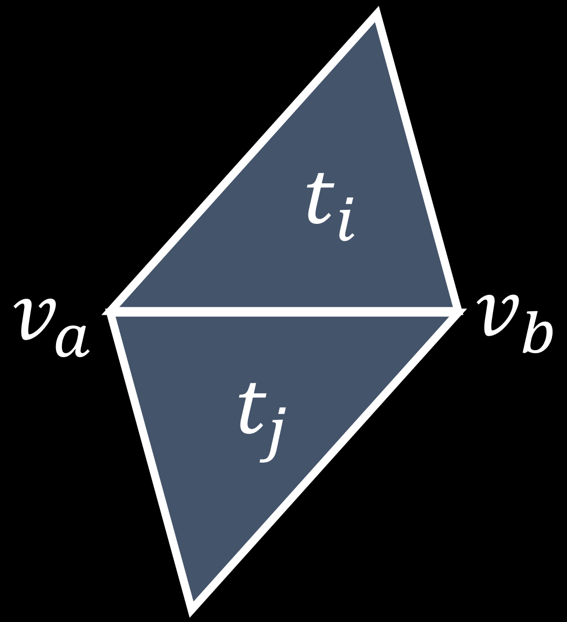
- **Computing Inversion-Free Mappings by Simplex Assembly**
- ***ACM Transactions on Graphics(SIGGRAPH Asia)* 35(6), 2016.**
- http://staff.ustc.edu.cn/~fuxm/projects/SimplexAssembly/index.html

Affine transformation

Key observation: the parameter space is a 2D triangulation, uniquely defined by all the AFFINE TRANSFORAMTIONS on the triangles.

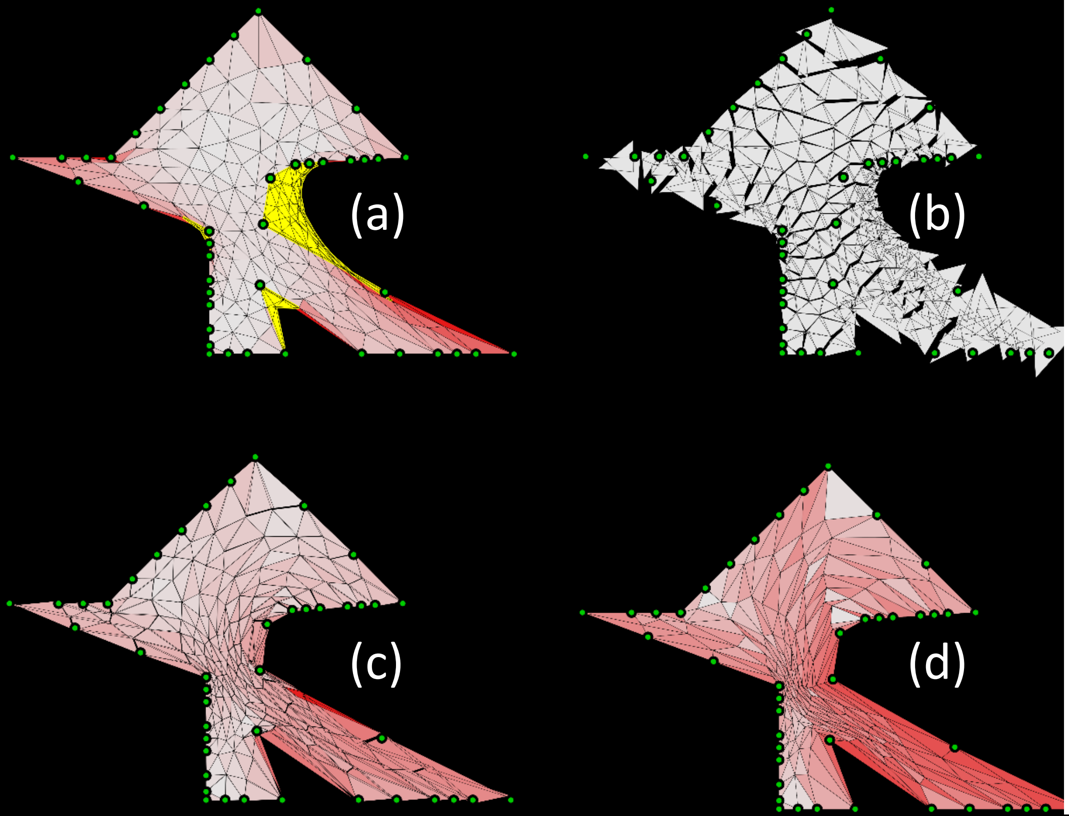
Edge assembly constraints:

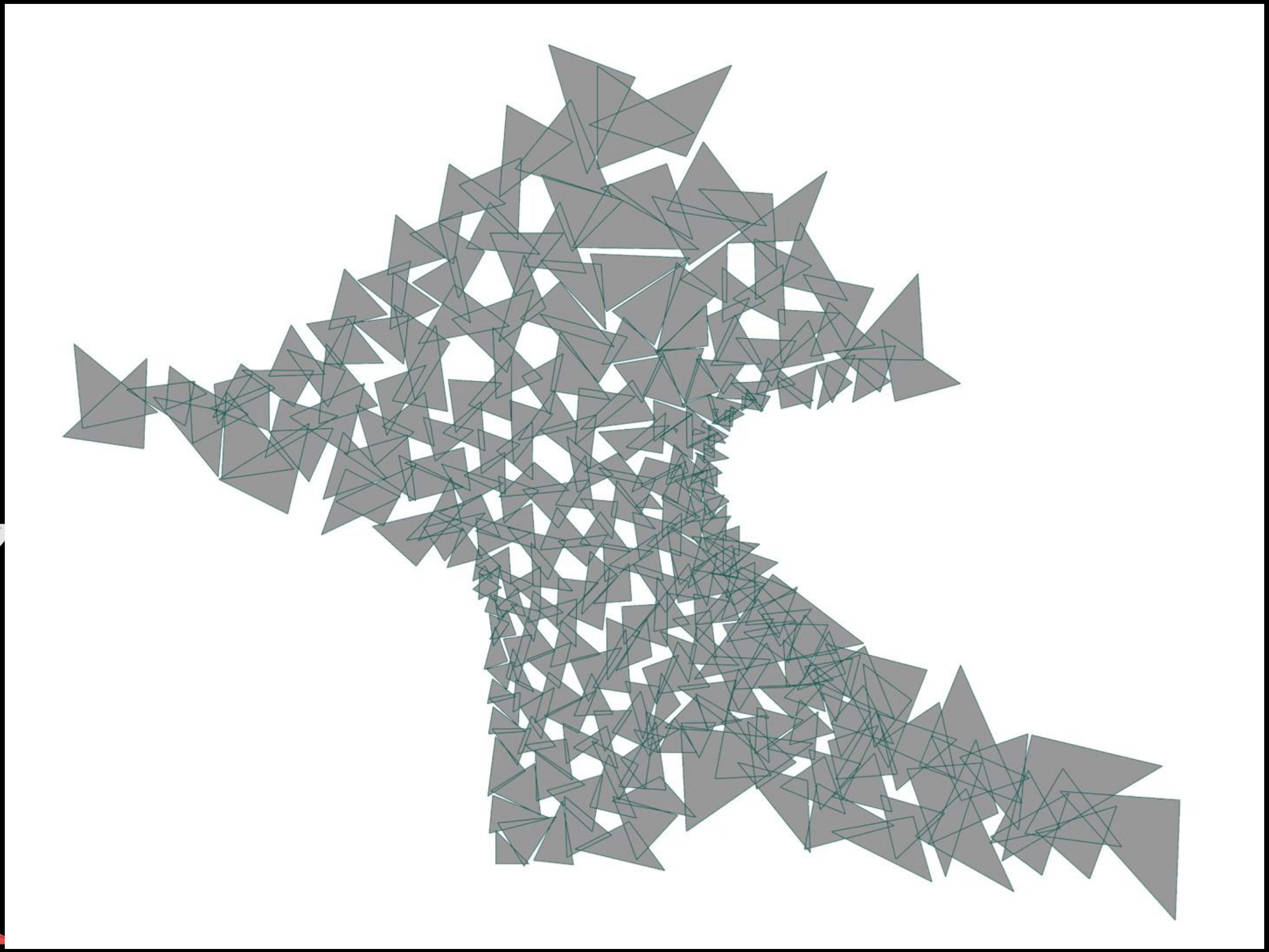
$$A_i(v_a - v_b) = A_j(v_a - v_b)$$



Key idea

- disassembly + assembly
  - Treat affine transformation as variables
  - Unconstrained optimization





Distortion control

Conformal: $d_i^c = \begin{cases} \frac{1}{2}\|A_i\|_F \left\|A_i^{-1}\right\|_F & , d = 2 \\ \frac{1}{8}\left(\|A_i\|_F^2 \left\|A_i^{-1}\right\|_F^2 - 1\right) & , d = 3 \end{cases}$

Volumetric: $d_i^{vol} = \frac{1}{2}\left(\det(A_i) + \frac{1}{\det(A_i)}\right)$

Isometric: $d_i^{iso} = 0.5 \cdot \left(d_i^c + d_i^{vol}\right)$

Barrier function on distortion:

1. The type of distortion and distortion bound K are given:

$$E_C^* = \sum_{i=1}^{N} \frac{e^{s \cdot d_i^*}}{K - d_i^*}$$

2. The type of distortion is not specified or distortion bound $K = \infty$:

$$E_C^* = \sum_{i=1}^{N} e^{s \cdot d_i^*}$$

Unconstrained optimization problem

Disassembly: project initial A_i^0 into feasible space.

Assembly: unconstrained optimization.

$E_{assembly}$: summation of squares of edge, assembly constraints.

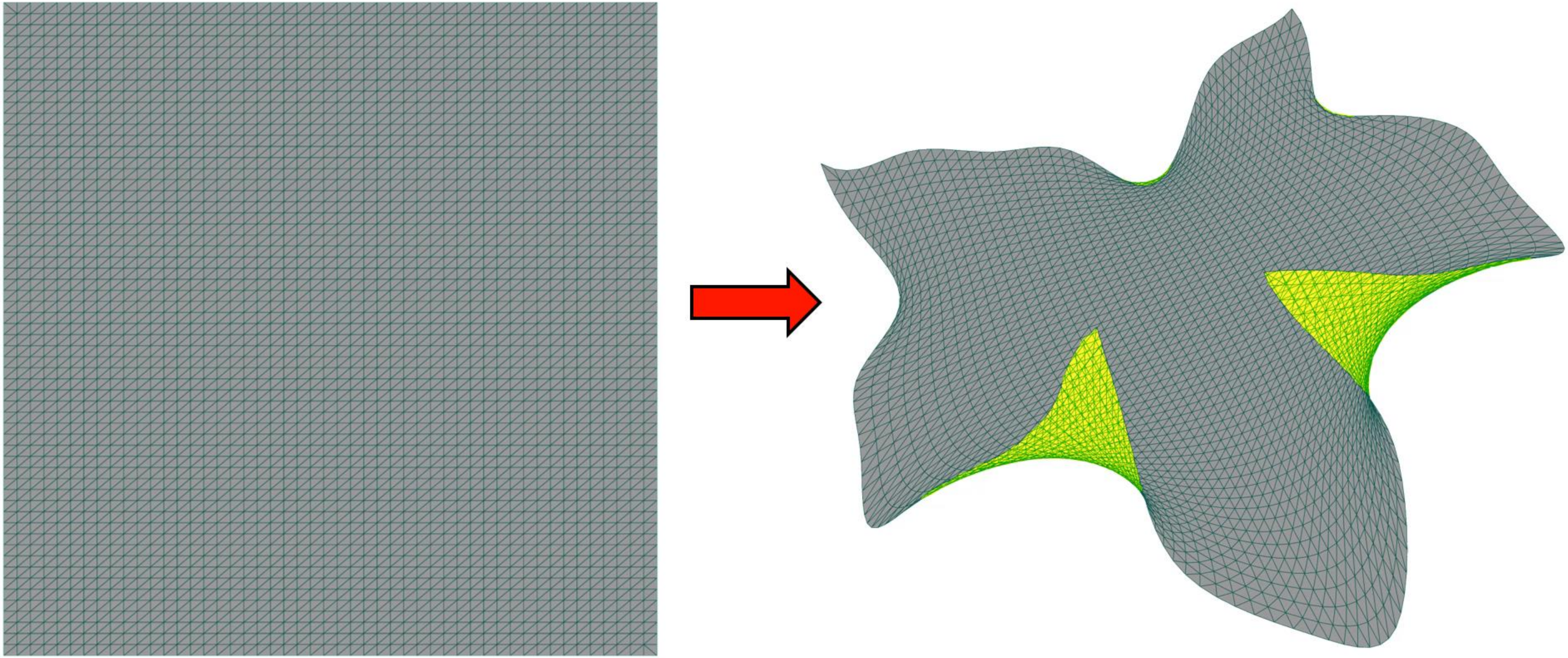
$$\min_{\substack{A_1,\ldots,A_N \\ T_1,\ldots,T_N}} \lambda E_{assembly} + E_C + \mu E_m$$

E_C: Barrier function on distortion

E_m: users' designed energy

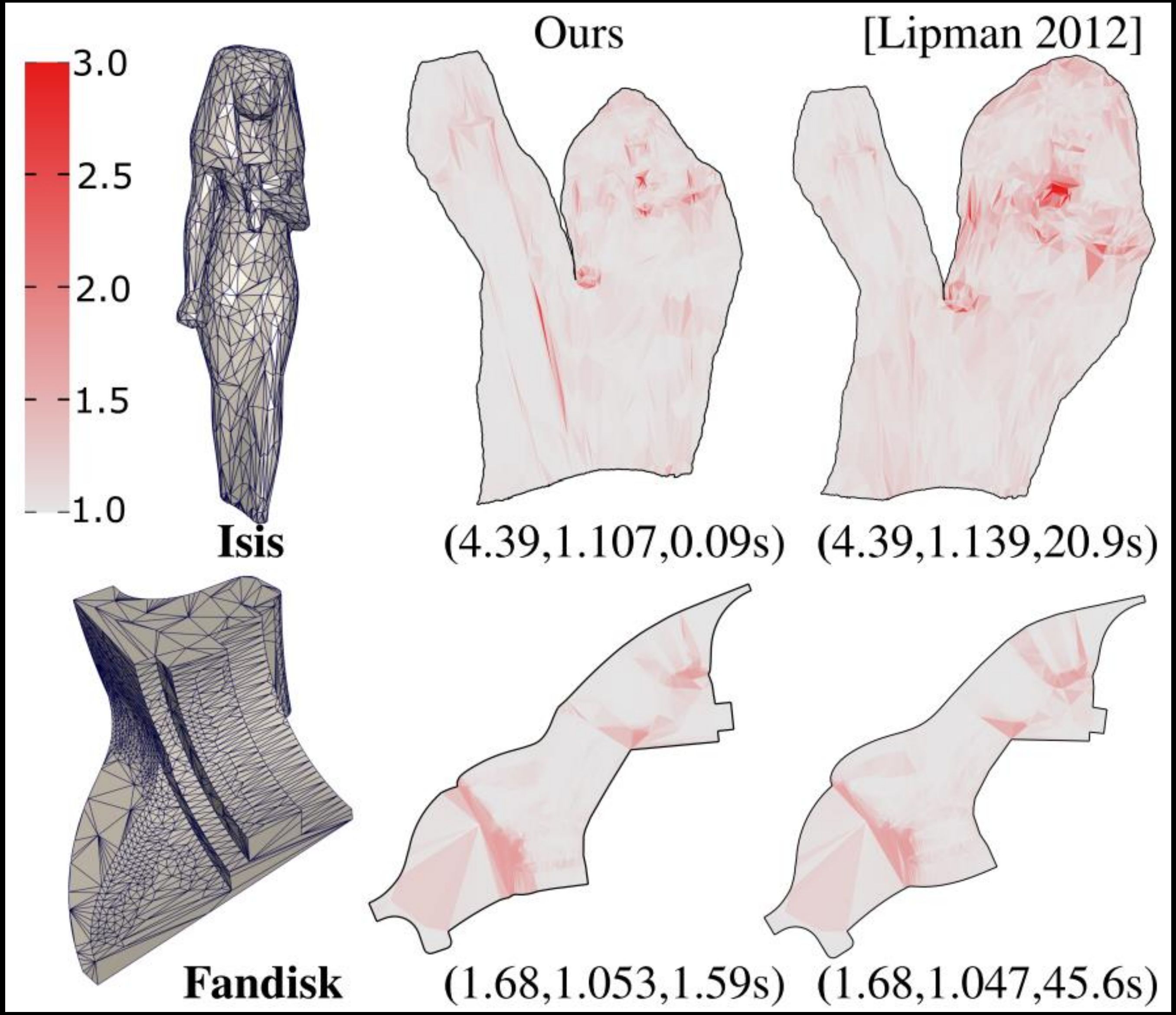
$$\lambda_{k+1} = \min\left(\lambda_{\min} \cdot \max\left(\frac{E_{C,k} + \mu E_{m,k}}{E_{assembly,k}}, 1\right), \lambda_{\max}\right)$$

1. $E_{assembly}$ dominates the energy, approach zero;
2. $\lambda_{\max}$: avoid large distortion.



Optimal bound

- Use the current maximal distortion as the bound for the next round of minimization.

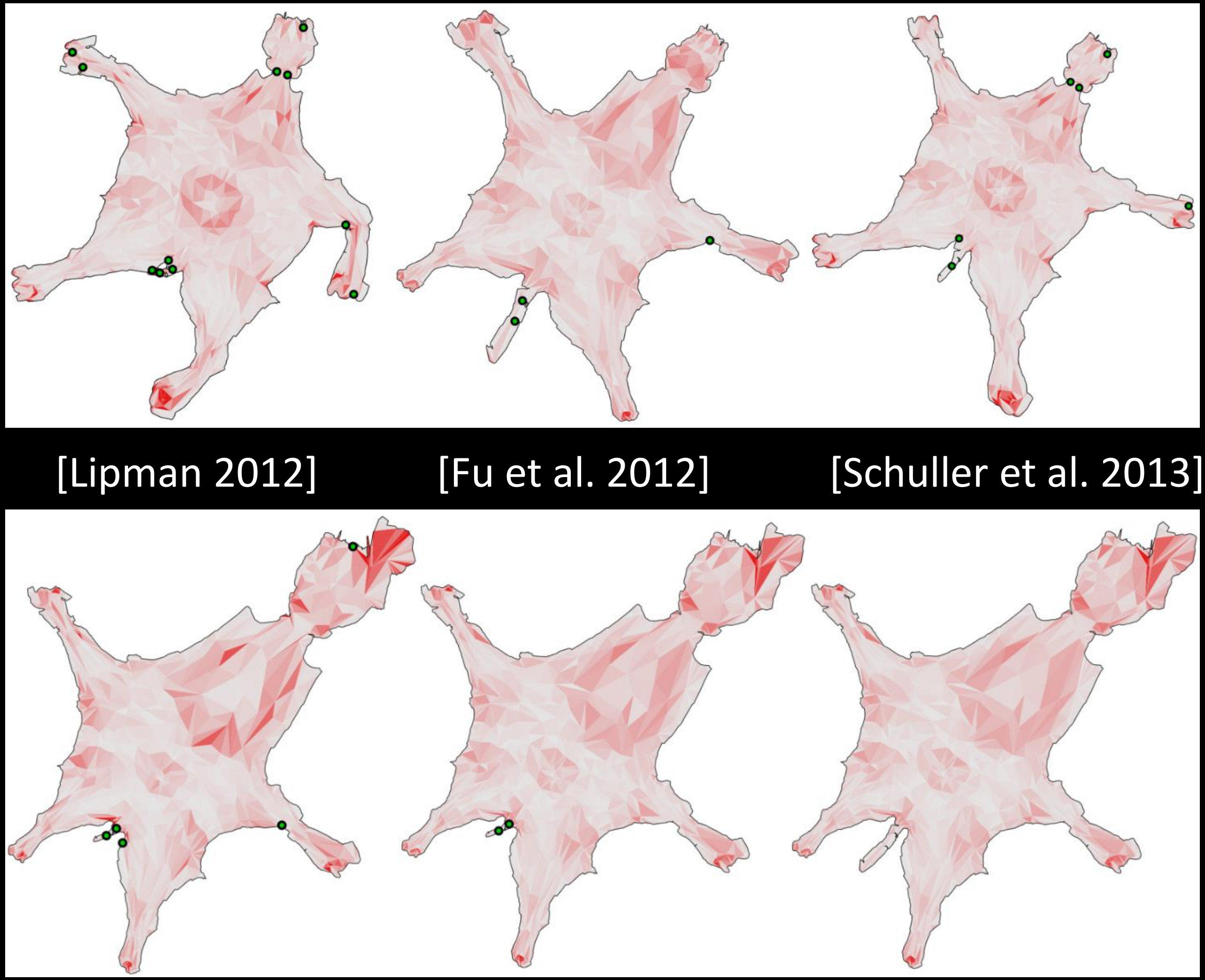


Locally injective mapping

- Requirements for locally injective mapping on triangle mesh:
  - 1. inversion-free;
  - 2. the sum of triangle angles θ_v around boundary vertex v is less than 2π.

- A barrier term:

$$E_\theta = \sum_{v \in \partial M} \frac{1}{2\pi - \theta_v}$$



[Lipman 2012] [Fu et al. 2012] [Schuller et al. 2013]

[Kovalsky et al. 2012] Ours without E_θ Ours with E_θ