Conformal Modules via Geometric Complex Analysis

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Surface Uniformization

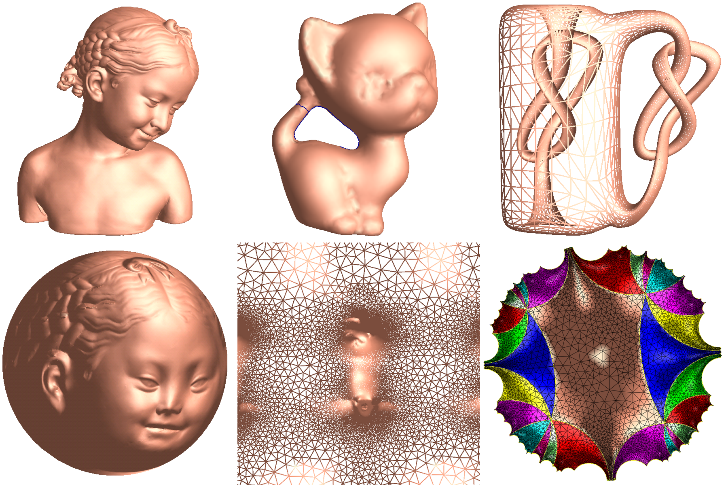


Figure: Closed surface uniformization.

Surface Uniformization

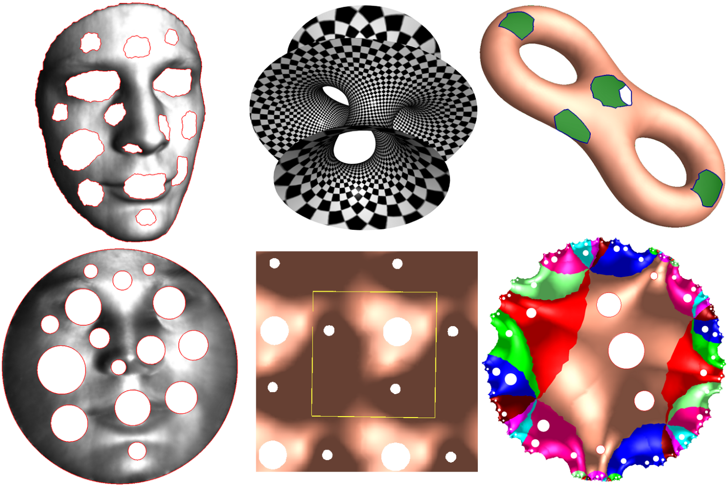


Figure: Open surface uniformization.

Geometric Complex Analysis

Strategy

1. Define a family of complex functions, with some constraints;
2. Show the family is a normal family;
3. Estimate some geometric or analytic bounds, such as distortions;
4. Maximize some coefficient of an item in the Laurent series;
5. Show the limit exists by normal family property;
6. Show the limit is the desired mapping.

Examples include Riemann mapping, slit mapping and so on.

Basic Concepts in Geometric Complex Analysis

Normal Family Definition

Definition (Uniform Convergence)

Assume $\{f_n : \Omega \to \mathbb{C}\}$ is a sequence of holomorphic functions defined on an open set Ω. We say the functions uniformly converge to a function $f : E \to \mathbb{C}$, if for any $\varepsilon > 0$, there is a n_0, such that for any $n > n_0$ and any $z \in E$, we have

$$|f_n(z) - f(z)| < \varepsilon.$$

Definition (Normal Family)

Let $\Omega \subset \mathbb{C}$ be an open set on $\mathbb{C}$, $\mathcal{F}$ is a normal family, if any subsequence $\{f_n\}$ in $\mathcal{F}$ uniformly converge on any compact subset in Ω.

Normal Family Properties

Theorem (Weierstrass)

Let $\{f_n : \Omega \to \mathbb{C}\}$ be a sequence of holomorphic functions defined on an open set $\Omega \subset \mathbb{C}$, assume $\{f_n\}$ uniformly converges to $f : \Omega \to \mathbb{C}$ on compact subsets in Ω, then f is holomorphic and $\{f_n' : \Omega \to \mathbb{C}\}$ uniformly converges to $f' : \Omega \to \mathbb{C}$.

Normal Family Properties

Definition (Univalent Map)

Let $U \subset \mathbb{C}$ be open subset on $\mathbb{C}$, if holomorphic map $f : U \to \mathbb{C}$ is injective, namely $z_1 \neq z_2$ implies $f(z_1) \neq f(z_2)$, then f is called a univalent map or univalent function.

Theorem (Hurwitz)

Let $\{f_n : \Omega \to \mathbb{C}\}$ be a family of holomorphic functions defined on an open set $\Omega \subset \mathbb{C}$, such that for any n and $z \in \Omega$, $f_n(z) \neq 0$. If $\{f_n\}$ uniformly converges to $f : \Omega \to \mathbb{C}$ on compact sets of Ω, then either $f \equiv 0$ or for any $z \in \Omega$, $f(z) \neq 0$.

Corollary

Let Ω be an open set in $\mathbb{C}$, let $\{f_n : \Omega \to \mathbb{C}\}$ be a holomorphic function series, and uniformly converges to $f : \Omega \to \mathbb{C}$ on compact sets. If each f_n on Ω is univalent, then either f is constant, or f is univalent on Ω.

Normal Family

Definition (Uniformly Bounded on Compact Sets)

Let $\mathcal{F}$ be a family of holomorphic functions, if for any compact set $E \subset \Omega$, there exists a constant M, such that for any $z \in E$ and any function $f \in \mathcal{F}$, we have $|f(z)| \leq M$, then we say $\mathcal{F}$ is uniformly bounded on compact sets.

Definition (equicontinuous)

Let $\mathcal{F}$ be a family of holomorphic functions defined on open set $\Omega \subset \mathcal{F}$. We say $\mathcal{F}$ is equicontinuous, if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for any distinct points z and z', $|z - z'| \leq \delta$ implies $|f(z) - f(z')| < \varepsilon$ for any $f \in \mathcal{F}$.

Normal Family

Theorem (Montel)

Let $\mathcal{F}$ be a family of holomorphic functions defined on an open set $\Omega \subset \mathbb{C}$, if $\mathcal{F}$ is uniformly bounded on compact sets in Ω, then

1. *$\mathcal{F}$ is equicontinuous on each compact set in Ω;*
2. *$\mathcal{F}$ is a normal family.*

1. Fix a point $p \in \Omega$, a family of univalent holomorphic functions $\mathcal{F}$ is a normal family, if for any $f \in \mathcal{F}$, $|f(p)| < M$ and $|f'(p)| < N$.
2. A family of holomorphic functions $\mathcal{F}$, if there are three points $\{z_1, z_2, z_3\}$, such that for any $f \in \mathcal{F}$, the image of f doesn't include them, then $\mathcal{F}$ is a normal family.
3. If $\mathcal{F}$ is a normal family, then
$$\mathcal{F}^{-1} = \{f^{-1} | f \in \mathcal{F}\}$$
is also a normal family.

Geometric Distortion Estimate

Definition ($\mathcal{S}$ Family)

All univalent holomorphic functions defined on the unit disk, with normalization condition form a normal family:

$$\mathcal{S} = \{f : \mathbb{D} \to \mathbb{C} : f \text{ univalent}, f(0) = 0,\ f'(0) = 1\}$$

any $f \in \mathcal{F}$ has Taylor expansion in a neighborhood of 0,

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots ,$$

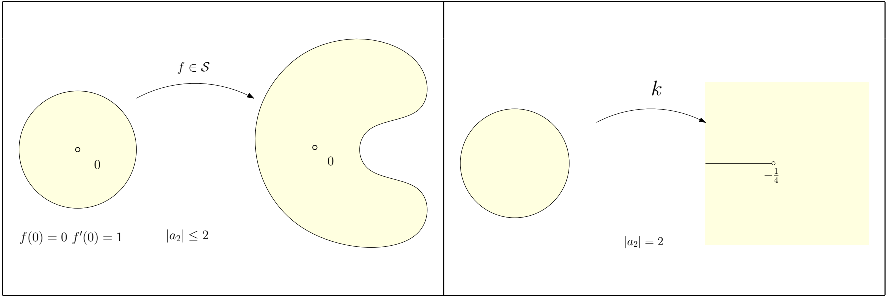
Taylor series converge in the unit disk $|z| < 1$.

Definition (Koebe Function)

The holomorphic function $k(z) \in \mathcal{S}$,

$$k(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + 4z^4 + \cdots$$

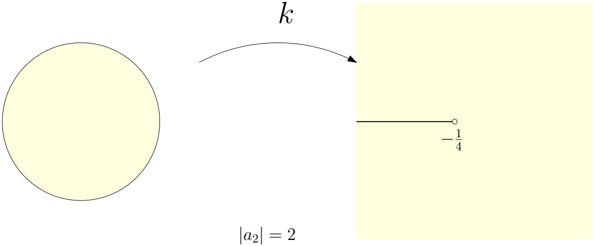
is called the Koebe fuction, which maps $\mathbb{D}$ to $\mathcal{C} \setminus (-\infty, -1/4]$.



Geometric Distortion Estimate

Theorem (Bieberbach a_2 of $\mathcal{S}$)

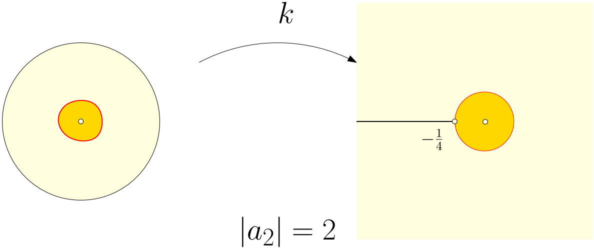
If $f \in \mathcal{S}$, then $|a_2| \leq 2$, equality holds if and only if f is a rotation of the Koebe function.



Geometric Distortion Estimate

Theorem (Koebe 1/4)

For any $f \in \mathcal{S}$, $f(\mathbb{D})$ includes an open disk $|w| < 1/4$. If there exists a $|w| = 1/4$ and $w \notin f(\mathbb{D})$, then f is a rotation of Koebe function.



Geometric Distortion Estimate

Proof.

Let $f(z) = z + a_2 z^2 + a_3 z^3 \cdots$ be a function of $\mathcal{S}$, $w \notin f(\mathbb{D})$. Construct a holomorphic function

$$h(z) = \frac{wf(z)}{w - f(z)} = z + \left(a_2 + \frac{1}{w}\right) z^2 + \cdots$$

then $h(z)$ is in $\mathcal{S}$, by Bieberbach theorem,

$$\left| a_2 + \frac{1}{w} \right| \leq 2 \tag{1}$$

and $|a_2| \leq 2$, therefore $|1/w| \leq 4$, $|w| \geq 1/4$. Equality holds if and only if f is a rotation of Koebe function. □

Geometric Distortion Estimate

Definition (Σ Family)

All holomorphic functions defined on $\Delta = \{|w| > 1\}$ with normalization condition form a normal family,

$$\Sigma = \{g : \Delta \to \mathbb{C} : g \text{ univalent}, \ g(\infty) = \infty, \ g'(\infty) = 1\},$$

for any $g \in \Sigma$, it ias Laurent power series in a neighborhood of ∞,

$$g(z) = z + b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \cdots$$

the series converges in Δ.

Geometric Distortion Estimate

Definition (Full Mapping Family)

The family of holmorphic functions

$$\tilde{\Sigma} := \{f : \Delta \to \mathbb{C} : f \in \Sigma, \mathbb{C} \setminus f(\Delta) \text{ has zero Lebesgure Measure}\}$$

Theorem (Gronwall Area)

Suppose $g \in \Sigma$, and

$$g(z) = z + b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \frac{b_3}{z^3} + \cdots$$

then

$$\sum_{n=1}^{\infty} n|b_n|^2 \leq 1,$$

equality holds if and only if g is a full mapping, $g \in \tilde{\Sigma}$.

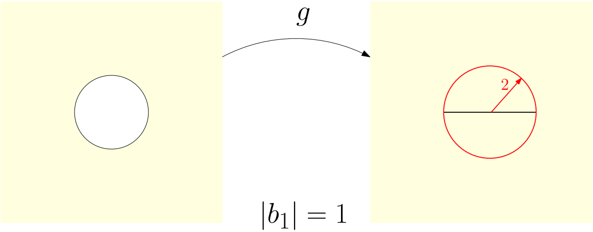
Geometric Distortion Estimate

Corollary (b_1 of Σ)

If $g \in \Sigma$, then $|b_1| \leq 1$, equality holds if and only if

$$g(z) = z + b_0 + \frac{b_1}{z}, |b_1| = 1 \tag{2}$$

g maps Δ to the complement of a length segment with length 4.



Geometric Distortion Estimate

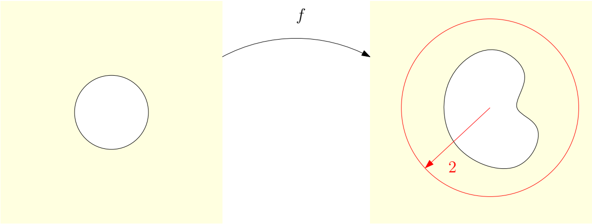
Corollary

For any $f \in \Sigma$, $f : \{|z| > 1\} \to \mathbb{C}$, $f(\infty) = \infty$, $f'(\infty) = 1$,

$$f(z) = z + b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \cdots,$$

we have

$$\partial f(|z| > 1) = f(|z| = 1) \subset \{|w - b_0| \le 2\}. \tag{3}$$



Geometric Distortion Estimate

Proof.

If $f(z) \in \Sigma$, then $f(z^{-1})^{-1} \in \mathcal{S}$,

$$f(z) = z + b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \cdots$$

therefore

$$\begin{aligned} f(z^{-1}) &= \frac{1}{z} + b_0 + b_1 z + b_2 z^2 + \cdots \\ f(z^{-1})^{-1} &= z(1 + b_0 z + b_1 z^2 + \cdots)^{-1} \\ &= z(1 - (b_0 z + b_1 z^2 + \cdots) + \cdots) \\ &= z - b_0 z^2 - b_1 z^3 + \cdots \end{aligned}$$

let $g(z) = f(z^{-1})^{-1}$, then $g(0) = 0$, $g'(0) = 1$, hence $g \in \mathcal{S}$. $\square$

Geometric Distortion Estimate

Continued.

Given any point $\zeta \in \partial\mathbb{D}$, $|\zeta| = 1$, then $w = g(\zeta) \notin g(\mathbb{D})$, by Bieberbach inequality (1),

$$\left| -b_0 + \frac{1}{w} \right| \leq 2,$$

by $w = g(\zeta) = 1/f(\zeta^{-1})$, we obtain $1/w = f(1/zeta)$. Set $\zeta' = 1/\zeta \in \partial\mathbb{D}$, we obtain

$$| -b_0 + f(\zeta')| \leq 2.$$

Riemann Mapping

Riemann Mapping

Theorem (Riemann)

Givne a non-empty, simply connected, open subset $\Omega \subset \mathbb{C}$, Ω is not the entire complex plane $\mathbb{C}$, for any point $z_0 \in \Omega$, there exits a unique biholomorphic mapping from Ω to the unit disk $\mathbb{D}$, $f : \Omega \to \mathbb{D}$, such that $f(z_0) = 0$ and $f'(z_0) > 0$.

Riemann Mapping

Uniqueness

If we don't require $f(z_0) = 0$ and $f'(z_0) > 0$, then conformal mapping is not unique. All such kind of mappings differ by a Möbius transformation, $\varphi : \mathbb{D} \to \mathbb{D}$,

$$\varphi(z) = e^{i\theta} \frac{z - z_0}{1 - \bar{z}_0 z}, \quad z_0 \in \mathbb{D}, \theta \in [0, 2\pi)$$

Extendibility

If Ω is a Jordan domain, the boundary $\partial\Omega$ is a piecewise anylatical curves, then the conformal mapping φ can be extended to the boundary $\varphi : \partial\Omega \to \partial\mathbb{D}$.

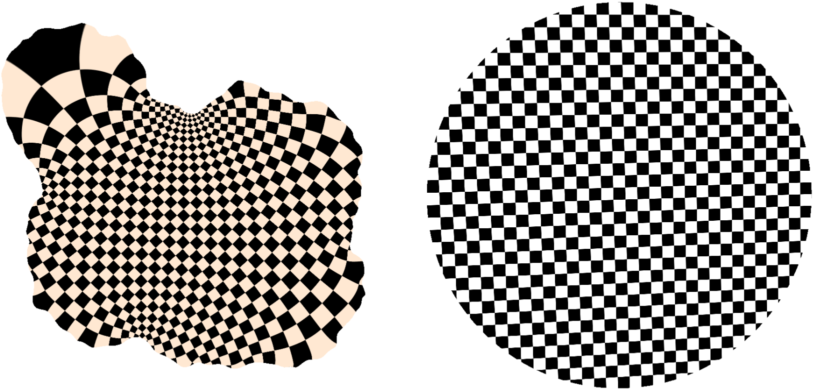


Figure: Riemann Mapping

Riemann Mapping

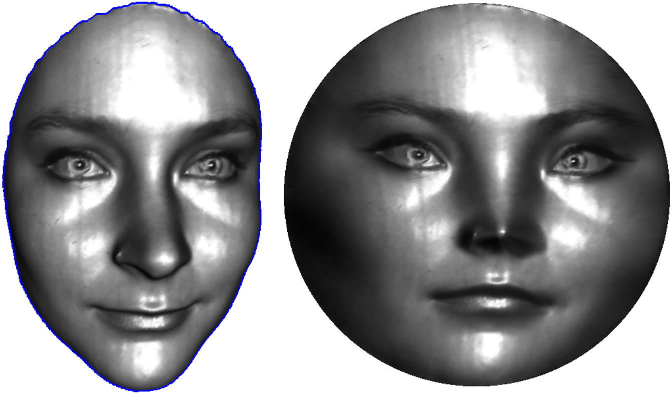


Figure: Riemann Mapping

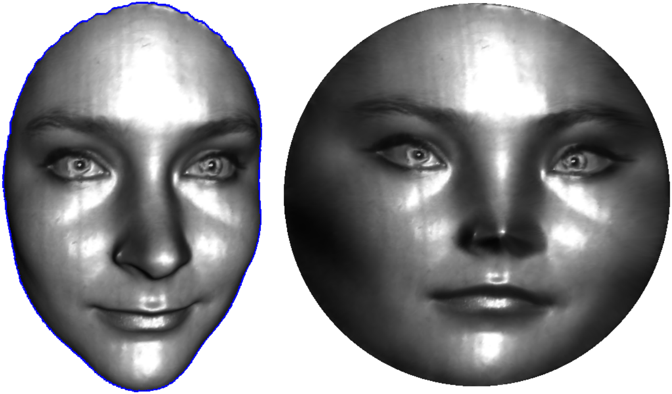


Figure: Riemann Mapping

Riemann Mapping

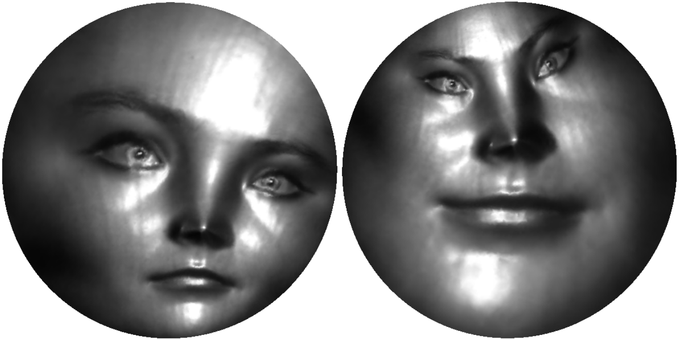


Figure: Mobius Transformation.

Riemann Mapping

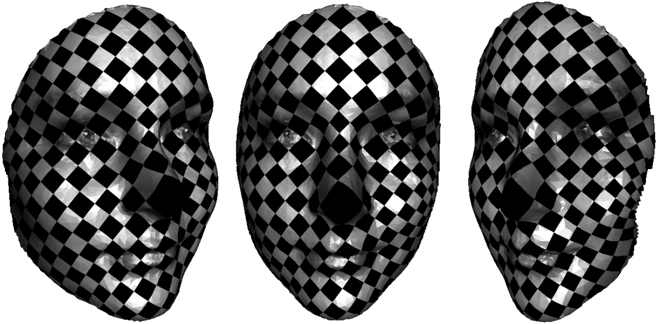


Figure: Texture mapping.

Schwartz Lemma

Lemma (Schwartz)

Assume $f(z)$ *is analytic on* $\mathbb{D} = \{|z| < 1\}$, *satisfying* $|f(z)| \leq 1$, *and* $f(0) = 0$, *then* $|f'(0)| \leq 1$ *and for* $\forall z \in \mathbb{D}$,

$$|f(z)| \leq |z|.$$

If $|f'(0)| = 1$, *or* $\exists 0 \neq z_0 \in \mathbb{D}$, *such that* $|f(z_0)| = |z_0|$, *then* f *is a rotation,*

$$f(z) = e^{i\theta} z.$$

Schwartz Lemma

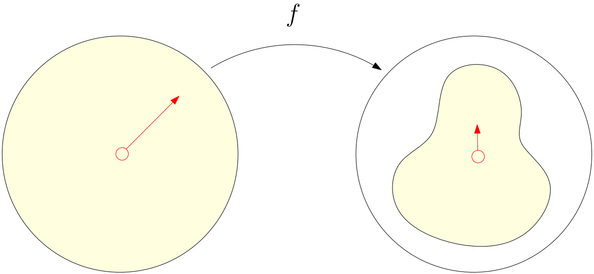


Figure: Schwartz lemma.

Schwartz Lemma

Proof.

Since f is holomorphic, it can be represented as power series in a neighborhood of 0,

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$$

Because $f(0) = 0$, $a_0 = 0$, hence

$$f(z) = a_1 z + a_2 z^2 + \cdots = z(a_1 + a_2 z + a_3 z^2 + \cdots),$$

the power series in the parenthesis converge. Construct auxiliary holomorphic function,

$$g(z) = \begin{cases} f(z)/z & z \neq 0 \\ f'(0) & z = 0 \end{cases}$$

□

Schwartz Lemma

Proof.

here the auxiliary function has power series $g(z) = a_1 + a_2 z + a_3 z^3 + \cdots$ converges in $\mathbb{D}$, where $g(0) = a_1 = f'(0)$. On every circle $|z| = r < 1$, $|f(z)| < 1$, the norm of the function

$$|g(z)| = \frac{|f(z)|}{|z|} < 1/r.$$

By maximal value principle, on the entire disk $|z| < r$, $|g(z)| < 1/r$, let $r \to 1$, we obtain on the unit disk $\mathbb{D}$,

$$|g(z)| \leq 1,$$

namely $|f(z)| \leq |z|$. If at some interior point z_0, $|g(z_0)| = 1$, by maximal value principle, $g(z)$ must be a constant a. By $|a| = 1$, we get $a = e^{i\theta}$,

$$f(z) = e^{i\theta} z.$$

Uniqueness of Riemann Mapping

Lemma

Assume $f : \mathbb{D} \to \mathbb{D}$ is a conformal automorphism of the unit disk, then $f(z)$ must be a Möbius transformation.

Proof.

We construct a Möbius transformation

$$\varphi(z) = \frac{z - f(0)}{1 - \overline{f(0)}z},$$

then $g = \varphi \circ f$ is a conformal automorophism of $\mathbb{D}$, and $g(0) = 0$. By Schwarz lemma, for all $z \in \mathbb{D}$, $|g(z)| \leq |z|$. Similarly, $w = g(z)$, then $|g^{-1}(w)| \leq |w|$, therefore, for all $z \in \mathbb{D}$, $|g(z)| = |z|$. By Schwartz lemma, we get $g(z) = e^{i\theta} z$. Hence $f(z) = \varphi^{-1}(z) \circ g(z)$ is a Möbius transformation. □

Existence Proof

Consider the functions family $\mathcal{F}$, consisting all functions $g(z) : \Omega \to \mathbb{D}$ satisfying the following 3 conditions:

1. $g(z)$ is analytic and univalent on Ω;
2. $\forall z \in \Omega$, $|g(z)| < 1$;
3. $g(z_0) = 0$ and $g'(z_0) > 0$.

The whole proof has three steps:

1. the function family $\mathcal{F}$ is non-empty, $\mathcal{F} \neq \emptyset$;
2. there exists a function $f \in \mathcal{F}$, such that $f'(z_0)$ is maximized;
3. this function f is the desired conformal mapping.

Existence Proof

Step 1 $\mathcal{F} \neq \emptyset$

There is a point $a \neq \infty$, $a \notin \Omega$. Since Ω is simply connected, we can define a single-valued branch of $\sqrt{z-a}$, denoted as $h(z)$. $h(z)$ won't take the same value twice, or take the opposite value: if $w \in h(\Omega)$, then $-w \notin h(\Omega)$. Choose a small disk $|w - h(z_0)| < \rho$ inside $h(\Omega)$, then $|w + h(z_0)| < \rho$ has no intersection point with $h(\Omega)$. Therefore for any $z \in \Omega$, $|h(z) + h(z_0)| > \rho$,

$$h_0(z) := \frac{\rho}{h(z) + h(z_0)}$$

is univalent on Ω, and $\forall z \in \Omega$, $|h_0(z)| < 1$. Choose $\theta_0 \in [0, 2\pi)$, such that $h_1'(z_0) > 0$, where

$$h_1(z) := e^{i\theta_0} \frac{h_0(z) - h_0(z_0)}{1 - \overline{h_0(z_0)} h_0(z)}, \quad h_1 \in \mathcal{F}.$$

$\mathcal{F} \neq \emptyset$

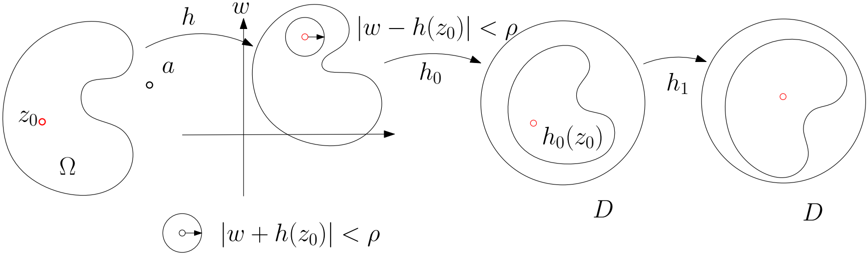


Figure: $\mathcal{F}$ is non-empty.

Existence Proof

Step 2

Define supreme

$$\beta = \sup_{g \in \mathcal{F}} g'(z_0),$$

there is a sequence $\{g_n\} \subset \mathcal{F}$, such that

$$\lim_{n \to \infty} g_n'(z_0) = \beta.$$

Based on Montel theorem, $\mathcal{F}$ is a normal function family, hence there is a subsequence $\{g_{n_k}\} \subset \{g_n\}$, which converges to an analytic function f on Ω, and uniformly converges on any compact subset on Ω. By Weierstrass theorem, $\beta = f'(z_0)$. Because β is finite, and $\beta > 0$, we obtain f is not constant.

Existence Proof

Step 3

Because $\{g_n\}$ on Ω is univalent, by Hurwitz the limit function f is also univalent. f is analytic, therefore conformal. We need to show f is surjective. Because f is bijective, Ω is simply connected, hence $f(\Omega)$ is simply connected. Assume there is an interior point $w_0 \in \mathbb{D}$, such that $w_0 \notin f(\Omega)$. Define a function $f_2 : \Omega \to \mathbb{D}$,

$$f_2(z) := \sqrt{\frac{f(z) - w_0}{1 - \overline{w}_0 f(z)}}$$

has an analytic branch, restricted on the image set $f_2 : \Omega \to f_2(\Omega)$ is bijective, $f_2(\Omega) \subset \mathbb{D}$. Let

$$F(z) = \frac{f_2(z) - f_2(z_0)}{1 - \overline{f_2(z_0)} f_2(z)},$$

then $F : \Omega \to \mathbb{D}$ is injective.

Existence Proof

Step 3

By $f(z_0) = 0$, we obtain $|f_2(z_0)| = \sqrt{|w_0|}$,

$$\begin{aligned}
|F'(z)| &= \left| \frac{1 - \overline{f_2(z_0)} f_2(z_0)}{[1 - \overline{f_2(z_0)} f_2(z)]^2} \right| |f_2'(z)| \\
&= \left| \frac{1 - \overline{f_2(z_0)} f_2(z_0)}{[1 - \overline{f_2(z_0)} f_2(z)]^2} \right| \frac{1}{2\sqrt{\frac{f(z)-w_0}{1-\overline{w}_0 f(z)}}} \left| \frac{1 - \overline{w}_0 w_0}{[1 - \overline{w}_0 f(z)]^2} \right| |f'(z)|
\end{aligned} \tag{4}$$

Step 3

plug in $z = z_0$

$$
\begin{aligned}
|F'(z_0)| &= \left|\frac{1-|f_2(z_0)|^2}{[1-|f_2(z_0)|^2]^2}\right| \frac{1}{2\sqrt{\left|\frac{f(z_0)-w_0}{1-\overline{w}_0 f(z_0)}\right|}} \left|\frac{1-|w_0|^2}{[1-\overline{w}_0 f(z_0)]^2}\right| |\beta| \\
&= \frac{1}{1-|w_0|}\frac{1}{2\sqrt{|w_0|}}|1-|w_0|^2|\cdot|\beta| \\
&= \frac{1+|w_0|}{2\sqrt{|w_0|}}|\beta| > |\beta|
\end{aligned}
\tag{5}
$$

Construct the function

$$
g(z) = \frac{|F'(z_0)|}{F'(z_0)}F(z),
$$

then $g \in \mathcal{F}$ and $g'(z_0) > \beta$. Contradiction. Hence $f : \Omega \to \mathbb{D}$ is surjective.

Topological Annulus

Conformal Mapping for Annulus

(a)Topological annulus

(b) Conformal module

Figure: Canonical conformal mapping for topological annulus.

Conformal Module for Topological Annulus

Theorem

Suppose Ω is a doubly connected domain on $\mathbb{C}$, then Ω is conformally equivalent to a canonical annulus.

Conformal Module for Topological Annulus

Step 1.

Assume $\partial\Omega = \gamma_1 - \gamma_2$, both γ_1 and γ_2 include more than 1 point, and γ_1 is finite. Suppose the complementary of γ_1 has two connected components, the one containing Ω is dentoed as Ω_1. By Riemann mapping theorem, we can conformally map Ω_1 onto the unit planar disk $|z'| < 1$, Ω is mapped to Ω', γ_2 to γ_2' inside the unit disk.

Step 2.

The complementary of γ_2' has two connected components, the one containing Ω' is denoted as Ω_2'. We conformally map Ω_2' onto the exterior to the unit disk $|z''| > 1$, mapping $z' = \infty$ to $z'' = \infty$. $\gamma_1' \mapsto \gamma_1''$, $\Omega' \mapsto \Omega''$,$\infty \notin \Omega''$, $\partial\Omega'' = \gamma_1'' - \gamma_2''$.

Conformal Module for Topological Annulus

Step 3.

Use the map $t = \log z''$, map Ω'' to B_1, B_1 is included in the right half plane $\{t|\Re t > 0\}$. The mapping is not one-to-one, B_1 is a infinite stripe, B_1 is periodic, for any $t \in B_1$, $t + 2k\pi i$, $k \in \mathbb{Z}$ is also in B_1.

Step 4.

By Riemann mapping theorem, there is a map $\omega = f(t)$, which maps B_1 to the vertical stripe region

$$B_2 := \{\omega|0 < \Re\omega < h\},$$

the mapping $f : B_1 \to B_2$ maps

$$f : \{-\sqrt{-1}\infty, 0 + \sqrt{-1}\infty\} \mapsto \{-\sqrt{-1}\infty, 0 + \sqrt{-1}\infty\}.$$

Because both B_1 and B_2 are simply connected, the boundaries are with more than one point, since they are conformal equivalent.

Conformal Module for Topological Annulus

Stpe 4. Continued

Assume $f(2\pi i) = \omega_0$, by scaling map, we can assume $f(2\pi i) = 2\pi i$. We prove the mapping has the property:

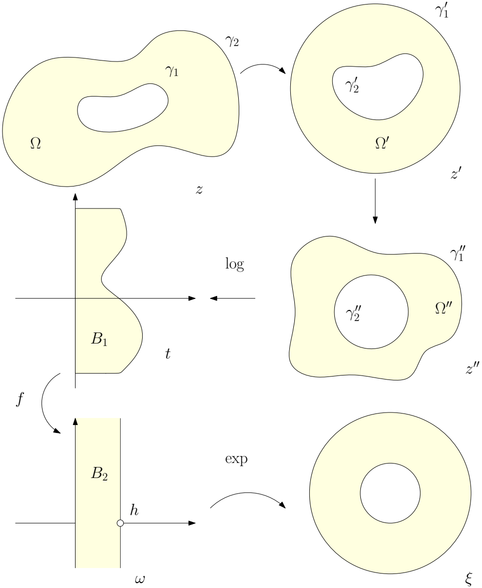
$$f(t + 2\pi i) = f(t) + 2\pi i.$$

Since both two conformal mappings $f(t + 2\pi i) - 2\pi i$ and $f(t)$ map B_1 to B_2, and maps $-\infty i, 0, +\infty i$ to $-\infty i, 0, +\infty i$, therefore by the uniqueness of Riemann mapping $f(t + 2\pi i) - 2\pi i = f(t)$.

Stpe 5.

The map $\xi = \exp(\omega)$ maps B_2 to the canonical annulus $1 < |\xi| < e^h$, the composition $\xi = \exp(f(\log z''))$ maps Ω'' to the annulus $1 < |\xi| < e^h$, which is conformal injective. So the composition of all the mappings together is the conformal mapping between Ω to $1 < |\xi| < e^h$.

Conformal Module for Topological Annulus



Slit Map Topological Poly-annulus

Slit Map

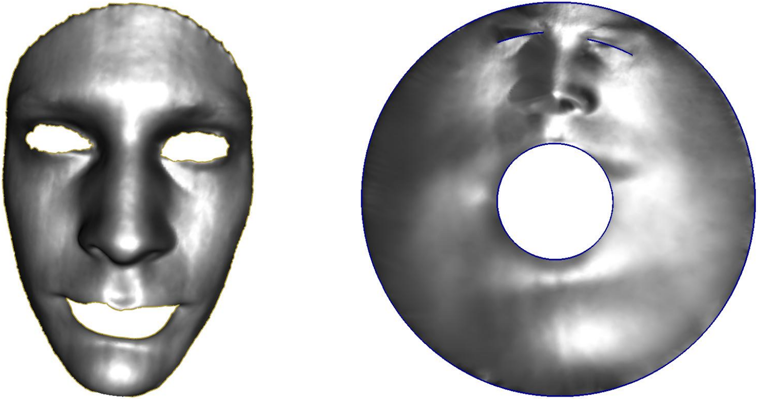


Figure: Slit map.

Slit Map

Definition (slit domain)

A connected open set (domain) $\Omega \subset \mathbb{C}$ is called a slit domain, if every connected component of its boundary $\partial\Omega$ is either a point or a horizontal closed interval.

Theorem (Hilbert)

Given any domain $\Omega \subset \mathbb{C}$, its boundary has finite number of connected components, then Ω is conformal equivalent to a slit domain.

Hilbert Theorem

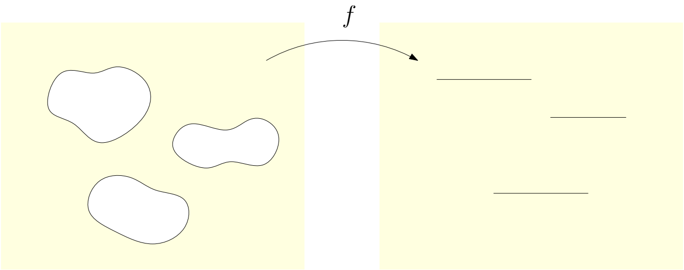


Figure: Hilbert theorem.

Lemma

In a neighborhood of ∞, given analytic functions

$$\alpha(z) = z + \frac{k_1}{z} + \cdots, \quad \beta(z) = z + \frac{l_1}{z} + \cdots,$$

then

$$\beta \circ \alpha(z) = z + \frac{k_1 + l_1}{z} + \cdots \tag{6}$$

Proof.

By direct computation. □

Slit Map

Proof.

Given a planar domain $\Omega \subset \hat{\mathbb{C}}$, by a Möbius transformation, we can assume $\infty \in \Omega$ and $\Omega \subset \{|z| > 1\}$, let univalent holomorphic mapping family be

$$\Sigma = \left\{ f : \Omega \to \hat{\mathbb{C}} \middle| f(z) = z + b_0 + \frac{b_1}{z} + \frac{b_2}{z^2} + \cdots, |z| > 1 \right\},$$

if $f \in \Sigma$, then $f(\infty) = \infty$ and $f'(\infty) = 1$. Let $f(z) = z$, then $f \in \Sigma$, $\Sigma \neq \emptyset$.
Consider the function family $\Sigma^{-1} = \{f^{-1} | f \in \Sigma\}$, by Corollary (3), we have

$$\{|z| < 1\} \subset [f^{-1}(|w - b_0| > 2)]^c,$$

hence $f^{-1}(|w - b_0| > 2)$ excludes three points $\{-1 + \varepsilon, 0, 1 - \varepsilon\}$, therefore Σ^{-1} is a normal function family, hence Σ is a normal functional family. $\square$

Normal family

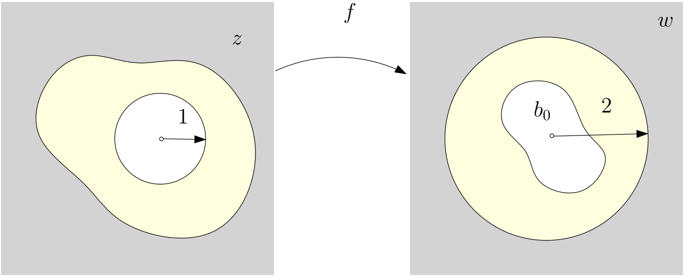


Figure: Estimate of image.

Slit Map

Proof.

By the compactness of normal function family, there exists a limit $f \in \Sigma$, such that

$$\Re_f(b_1) = \max_{g \in \Sigma} \Re_g(b_1),$$

we will show $f(\Sigma)$ is a slit domain. Otherwise, there is a connected component Γ of $\partial f(\Omega)$, Γ is neither a point or a horizontal line segment. Construct a map

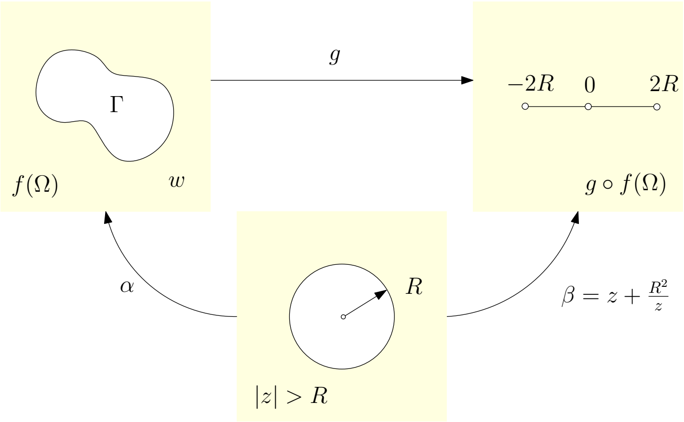
$$g : \hat{\mathbb{C}} \setminus \Gamma \to \hat{\mathbb{C}}[-2R, 2R]$$

as follows: first construct the inverse map of a Riemann mapping $\alpha : \{|z| > R\} \to \hat{\mathbb{C}} \setminus \Gamma$,

$$\alpha(z) = z + \frac{\varepsilon}{z} + \cdots$$

and slit map $\beta : \{|z| > R\} \to \hat{\mathbb{C}} \setminus [-2R, 2R]$, $\beta(z) = z + \frac{R^2}{z}$, □

Slit Map



Slit Map

continued.

The the composition map $g : \hat{\mathbb{C}} \setminus \Gamma \to \hat{\mathbb{C}} \setminus [-2R, 2R]$,

$$g(w) = \beta \circ \alpha^{-1}(w) = w + \frac{\lambda}{w} + \cdots,$$

by the corollary of Gronwall theorem (2), comapre α and β, they maps the complement of the disk to planar domains, the real part of b_1 of the slit map reaches the maximum, hence

$$R^2 = \Re_\beta(b_1) > \Re_\alpha(b_1) = \varepsilon.$$

By Eqn. (6), $\beta(z) = g \circ \alpha(z)$, we obtain

$$R^2 = \Re_\beta(b_1) = \Re_{g\circ\alpha}(b_1) = \Re_g(b_1) + \Re_\alpha(b_1) = \lambda + \varepsilon > \varepsilon.$$

Therefore $\Re_g(b_1) = \lambda > 0$.

Slit Map

continued.

By Eqn. (6), on $\{|z| > 1\}$, composition map

$$g \circ f(z) = z + \frac{\Re_f(b_1) + \lambda}{z} + \cdots,$$

by $\lambda > 0$, we obtain $\Re_{g\circ f}(b_1) > \Re_f(b_1)$, this contradicts to the choice of f. Hence the assumption is incorrect, the claim holds.

Slit Map Algorithm

Figure: Exact harmonic forms.

Slit Map Algorithm

Figure: Closed, non-exact harmonic forms.

Slit Map Algorithm

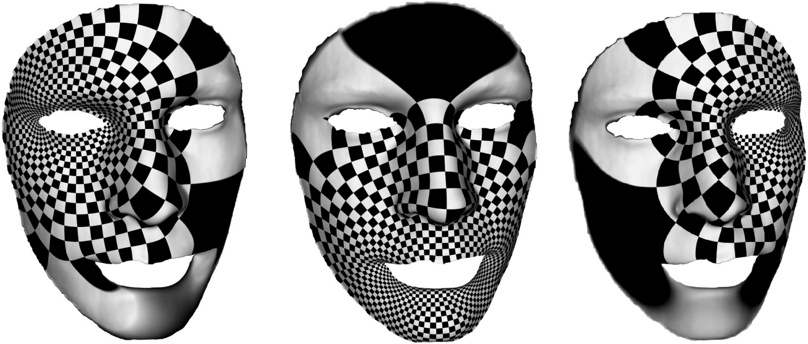


Figure: Holomorphic forms.

Slit Map Algorithm

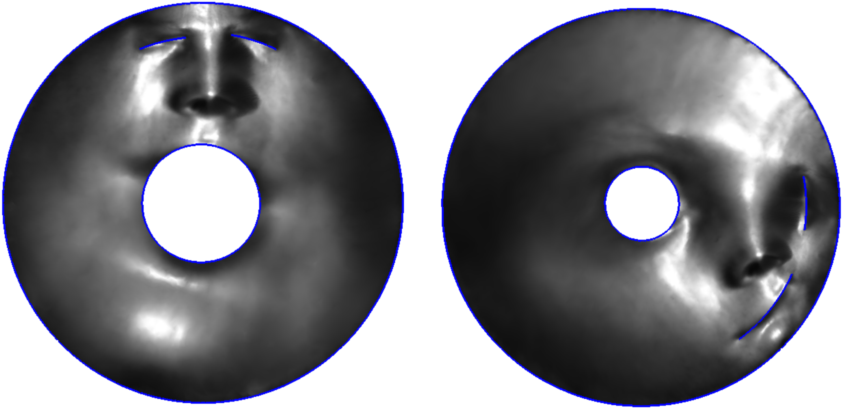


Figure: Slit maps.

Slit Map Algorithm

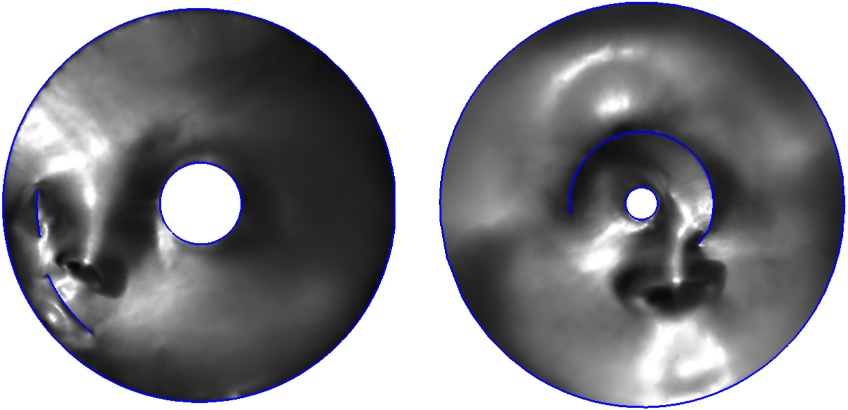


Figure: Slit maps.

Slit Map Algorithm

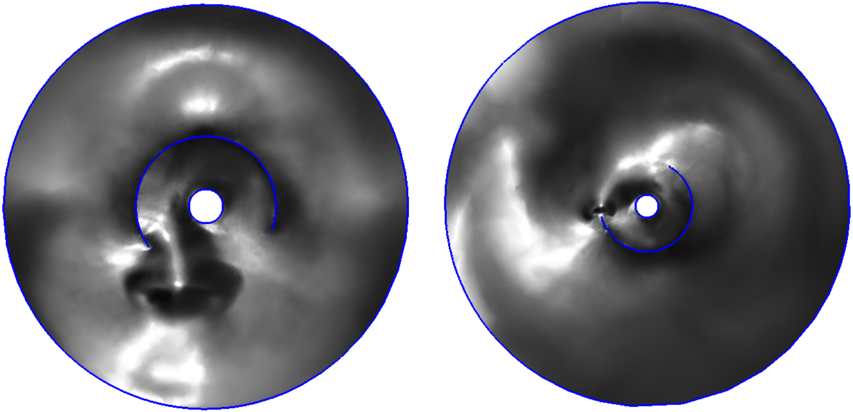


Figure: Slit maps.

Slit Map Algorithm

Input: A genus zero mesh with $n+1$ boundary components M, $\partial M = \gamma_0 - \gamma_1 - \cdots - \gamma_n$;

Output: A slit map : $M \to D$, D is a circular slit domain.

1. Compute exact harmonic 1-forms $\omega_1, \omega_2, \dots, \omega_n$;
2. Compute closed, non-exact harmonic 1-forms $h_1, h_2, \dots, h_n$;
3. Compute conjugate harmonic 1-forms ${}^*\omega_1, {}^*\omega_2, \dots, {}^*\omega_n$;
4. Find special holomorphic 1-form φ

$$\Im \int_{\gamma_0} \varphi = 2\pi, \Im \int_{\gamma_1} \varphi = -2\pi, \Im \int_{\gamma_k} \varphi = 0, k = 2, 3, \dots, n.$$

5. Slit map $f : M \to D$, choose a fixed based point $p \in M$,

$$f(q) := \exp \int_p^q \varphi$$

the integration path can be chosen arbitrarily.

Slit Map Algorithm

Exact Harmonic Forms

Construct n harmonic functions $f_1, f_2, \cdots, f_n$ with Dirichlet boundary condition, for each $1 \leq k \leq n$,

$$\left\{ \begin{array}{rcll} \Delta f_k(v_i) & = & 0 & v_i \notin \partial M \\ f_k(v_j) & = & -1 & v_j \in \gamma_k \\ f_k(v_l) & = & 0 & v_l \in \partial M \setminus \gamma_k \end{array} \right.$$

Exact harmonic 1-form group basis are given by

$$\omega_k = df_k, \quad 1 \leq k \leq n.$$

Slit Map Algorithm

Random Harmonic Forms

Generate a random 1-form ω, according to Hodge decomposition theorem,

$$\omega = df + \delta\eta + h,$$

where f is a 0-form, η a 2-form,

$$\delta\omega = \delta df, \quad d\omega = d\delta\eta,$$

the harmonic form is given by

$$h = \omega - df - \delta\eta.$$

Slit Map Algorithm

Gram–Schmidt Orthonormalization

1. for $k = 1, 2, \dots, n$,
    1. Generate a random harmonic form h_k,
    2. Decompose h_k with respect to the orthonormal frame $\{h_1, h_2, \dots, h_{k-1}\}$,

    $$h_k \leftarrow h_k - \sum_{i=1}^{k-1} (h_i, h_k) h_i, \quad (h_i, h_j) := \int_M h_i \wedge {}^* h_j,$$

    3. if $\|h_k\|^2 = (h_k, h_k) < \varepsilon$, then regenerate h_k and re-decompose h_k, until $\|h_k\|^2 > \varepsilon$
    4. normalize h_k

    $$h_k \leftarrow \frac{h_k}{\sqrt{(h_k, h_k)}}.$$

Slit Map Algorithm

Hodge Star Operator

Given an exact harmonic 1-form ω_k, then

$$^*\omega_k = \lambda_{k1}h_1 + \lambda_{k2}h_2 + \cdots + \lambda_{kn}h_n,$$

$$\begin{pmatrix} h_1 \wedge {}^*\omega_k \\ h_2 \wedge {}^*\omega_k \\ \vdots \\ h_n \wedge {}^*\omega_k \end{pmatrix} = \begin{pmatrix} h_1 \wedge h_1 & h_1 \wedge h_2 & \cdots & h_1 \wedge h_n \\ h_2 \wedge h_1 & h_2 \wedge h_2 & \cdots & h_2 \wedge h_n \\ \vdots & \vdots & & \vdots \\ h_n \wedge h_1 & h_n \wedge h_2 & \cdots & h_n \wedge h_n \end{pmatrix} \begin{pmatrix} \lambda_{k1} \\ \lambda_{k2} \\ \vdots \\ \lambda_{kn} \end{pmatrix}$$

Taking integration on M for every element, and solve the linear system.

Slit Map Algorithm

Special Holomorphic 1-form

Suppose

$$\partial M = \gamma_0 - \gamma_1 - \gamma_2 \cdots - \gamma_n,$$

choose a holomorphic 1-form

$$\varphi = \sum_{i=1}^{n} \mu_i \left(\omega_i + \sqrt{-1}^* \omega_i \right),$$

$$\begin{pmatrix} 2\pi \\ -2\pi \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \int_{\gamma_0} {}^*\omega_1 & \int_{\gamma_0} {}^*\omega_2 & \cdots & \int_{\gamma_0} {}^*\omega_n \\ \int_{\gamma_1} {}^*\omega_1 & \int_{\gamma_1} {}^*\omega_2 & \cdots & \int_{\gamma_1} {}^*\omega_n \\ \int_{\gamma_2} {}^*\omega_1 & \int_{\gamma_2} {}^*\omega_2 & \cdots & \int_{\gamma_2} {}^*\omega_n \\ \vdots & \vdots & & \vdots \\ \int_{\gamma_n} {}^*\omega_1 & \int_{\gamma_n} {}^*\omega_2 & \cdots & \int_{\gamma_n} {}^*\omega_n \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \vdots \\ \mu_n \end{pmatrix}$$

Slit Map Algorithm

Integration

Choose a vertex $p \in M$, use width first search to access all the vertices on M, and for each vertex $q \in M$, we obtain a path γ_q from p to q, the circular slit mapping is given by

$$f(q) := \exp\left(\int_{\gamma_q} \varphi\right),$$

where

$$\exp(a + \sqrt{-1}b) = e^a(\cos b + \sqrt{-1}\sin b).$$