

Surface Differential Geometry, Movable Frame Method

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Compute Geodesics

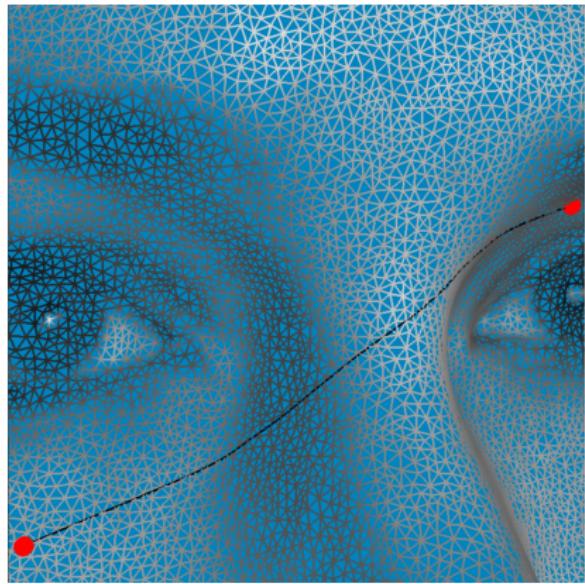
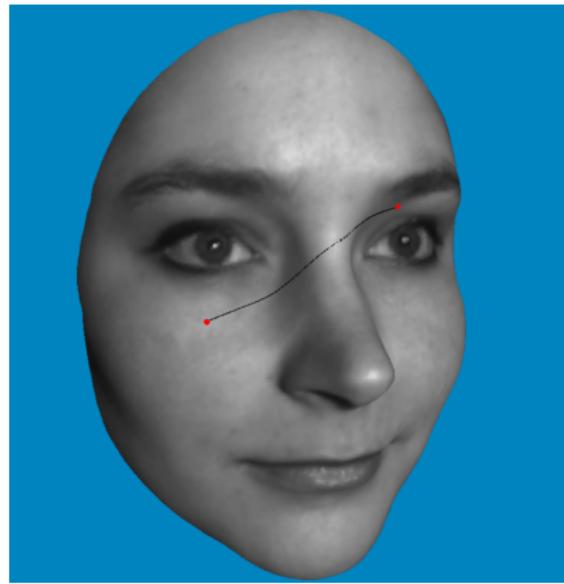


Figure: Geodesic on polyhedral surfaces.

Geodesic on a surface $\gamma : [0, 1] \rightarrow (S, \mathbf{g})$:

$$D_{\dot{\gamma}} \dot{\gamma} \equiv 0.$$

Compute Geodesics

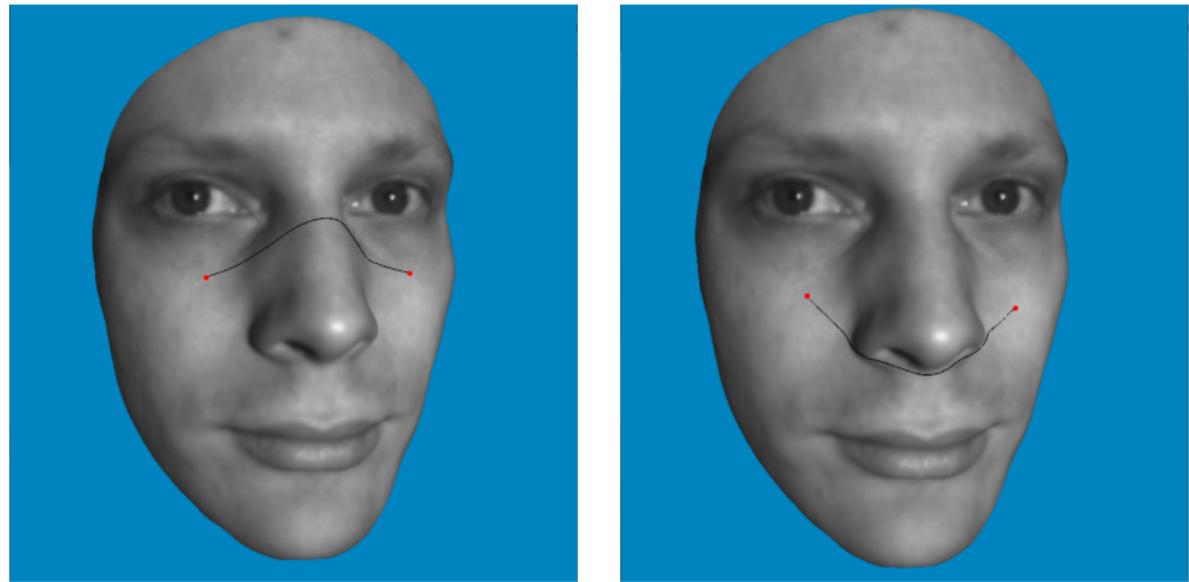
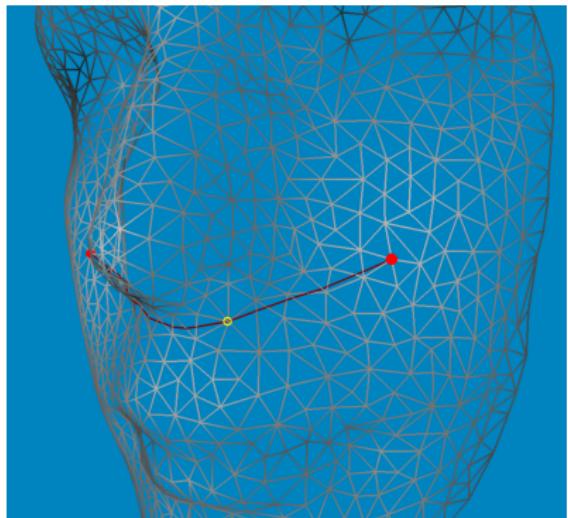
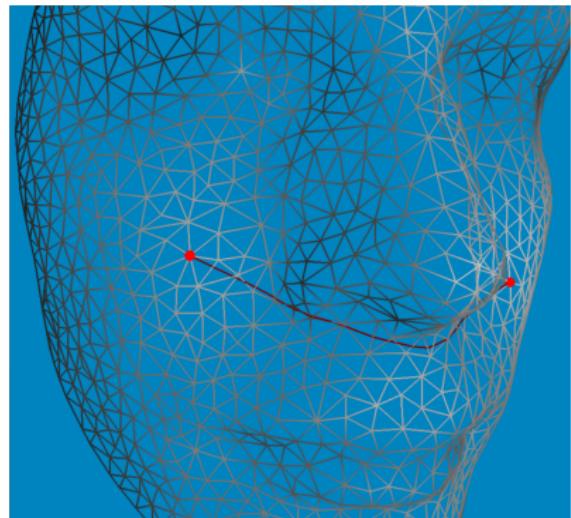


Figure: Conjugate point of geodesics.

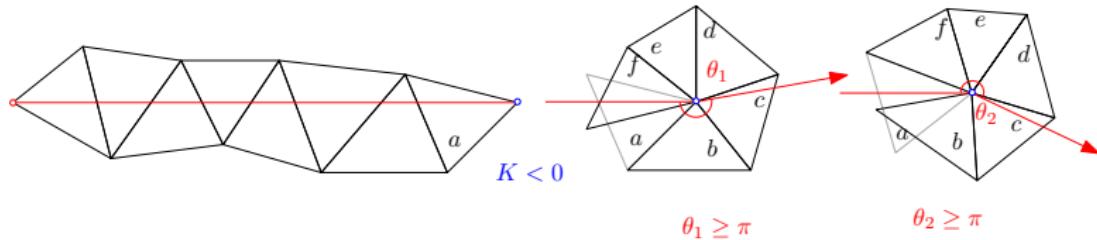
Geodesic on a surface $\gamma : [0, 1] \rightarrow (S, g)$:

$$D_{\dot{\gamma}} \dot{\gamma} \equiv 0.$$

Discrete Geodesics



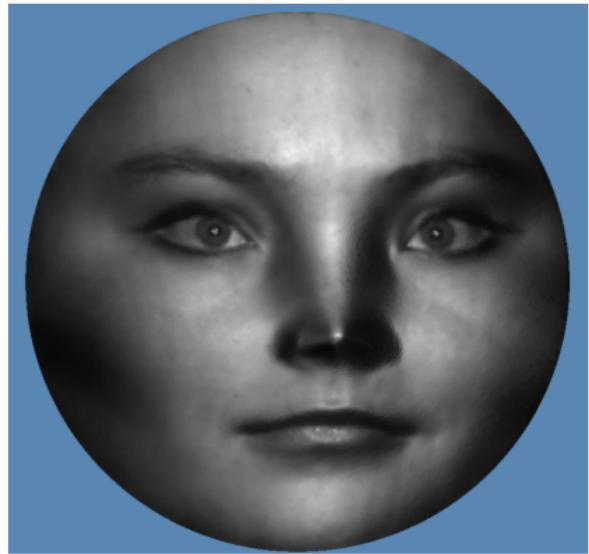
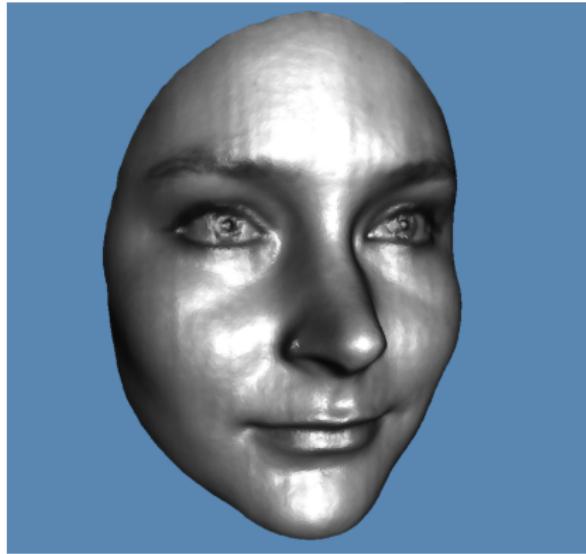
Discrete Geodesics



Suppose γ is a discrete geodesic:

- ① isometrically flatten the strip of curve γ onto the plane;
- ② when the γ crosses an edge, it is straight;
- ③ γ never crosses any convex vertex;
- ④ when γ crosses a concave vertex, if we flatten the neighborhood from right, then $\theta_1 \geq \pi$; flatten from left, $\theta_2 \geq \pi$.

Discrete Harmonic Map



Smooth surface harmonic map $\varphi : (S, \mathbf{g}) \rightarrow \mathbb{D}^2$, $\Delta_{\mathbf{g}}\varphi \equiv 0$, with Dirichlet boundary condition $\varphi|_{\partial S} = f$. A discrete harmonic map satisfies

$$\sum_{v_i \sim v_j} w_{ij}(\varphi(v_i) - \varphi(v_j)) = 0, \forall v_i \notin \partial M.$$

Compute Minimal Surface

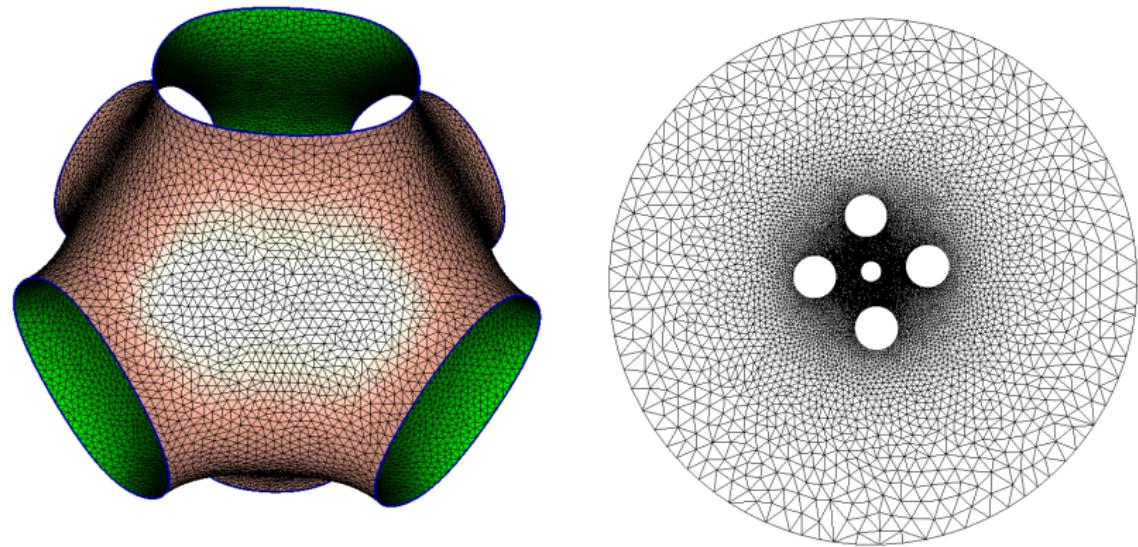
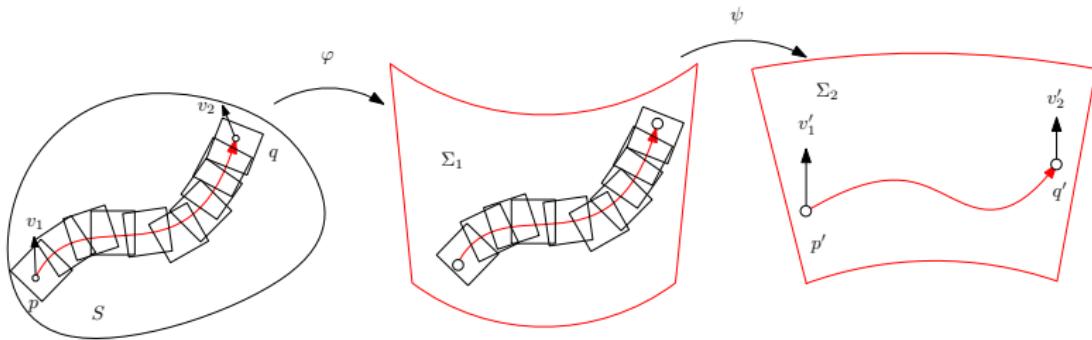


Figure: Minimal surface.

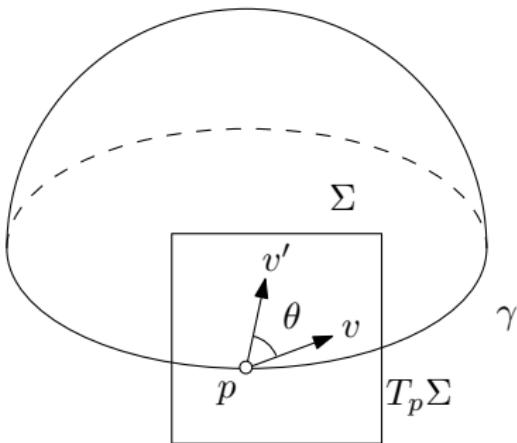
Smooth minimal surface satisfies $\Delta_{\mathbf{g}} r \equiv 0$, equivalently $H(p) \equiv 0$. A discrete minimal surface satisfies $\sum_{v_i \sim v_j} w_{ij}(\mathbf{r}(v_i) - \mathbf{r}(v_j)) = 0, \forall v_i \notin \partial M$.

Parallel Transport



Given $\gamma \subset S$, find an envelope surface Σ_1 of all the tangent planes along γ , $\varphi : \gamma \rightarrow \Sigma_1$ isometrically maps γ to Σ_1 . Σ_1 is developable, flatten Σ_1 to obtain a planar domain Σ_2 , $\psi : \Sigma_1 \rightarrow \Sigma_2$. The composition $\psi \circ \varphi$ maps $p, q, v_1 \in T_p S$, $v_2 \in T_q S$ to p', q', v'_1, v'_2 . On the plane, translate a tangent vector v'_1 from starting point p to the ending point q to get v'_2 , maps back v'_2 , $v_2 = (\psi \circ \varphi)^{-1}(v'_2)$. Then v_1 is parallelly transported along γ to get v_2 .

Gaussian Curvature



Parallel transport v along $\partial\Sigma$, to get v' when returned to the original point p , then the angle difference between v and v' equals to the total Gaussian curvature,

$$\theta = \int_{\Sigma} K dA.$$

Gaussian Curvature

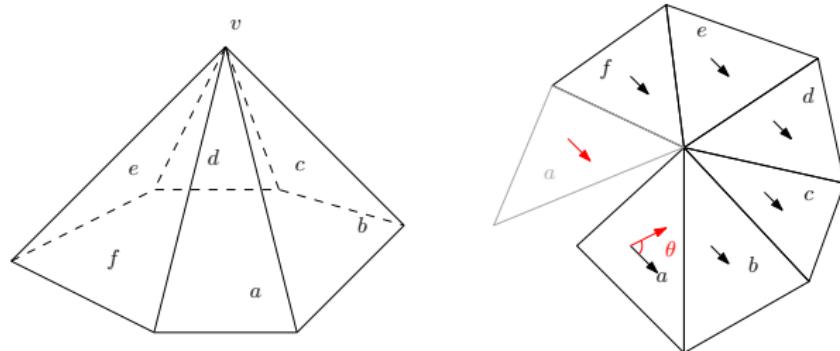


Figure: Discrete parallel transport, $K(v) = \theta$.

Parallel transport a vector, when return to the original position, the difference angle equals to the discrete Gaussian curvature of the interior vertices.

Gaussian Curvature

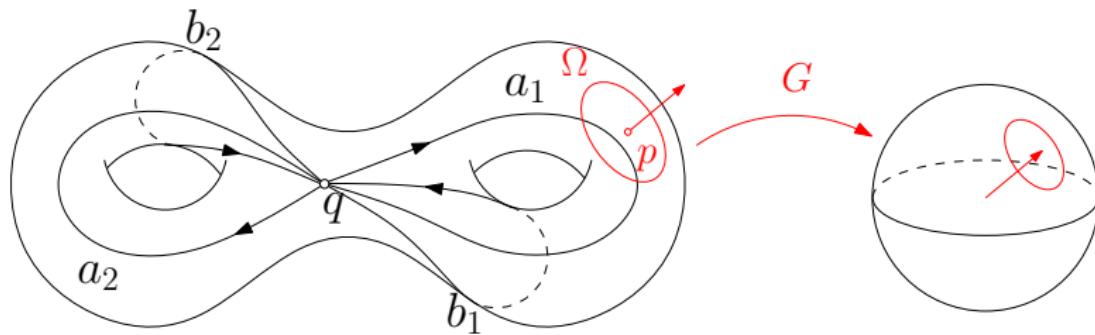


Figure: Gaussian curvature.

Gauss map: $\mathbf{r}(p) \mapsto \mathbf{n}(p)$,

$$K(p) := \lim_{\Omega \rightarrow \{p\}} \frac{|G(\Omega)|}{|\Omega|}$$

Gaussian Curvature

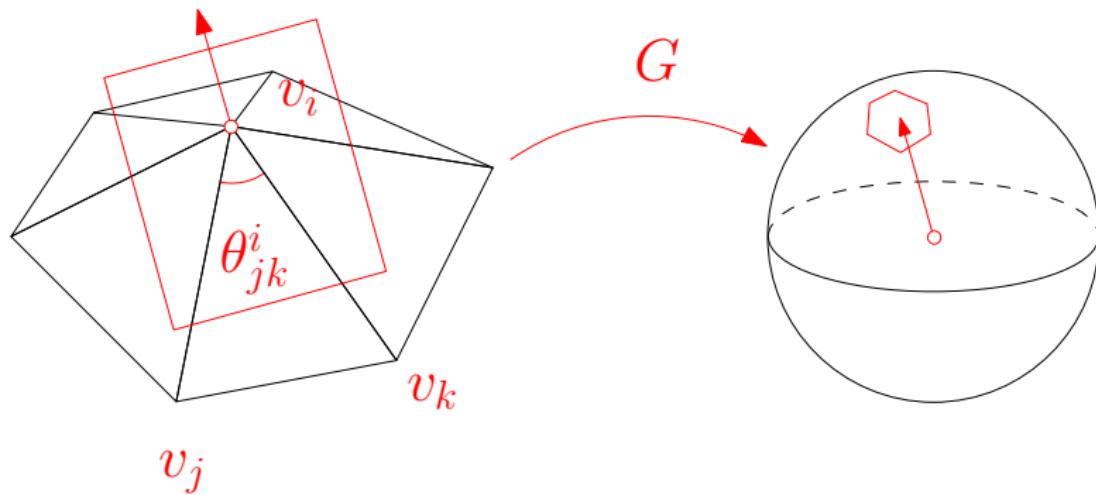


Figure: Discrete Gaussian curvature.

$$G(v_i) := \{ \mathbf{n} \in \mathbb{S}^2 \mid \exists \text{Support plane with normal } \mathbf{n} \}.$$

Gaussian Curvature

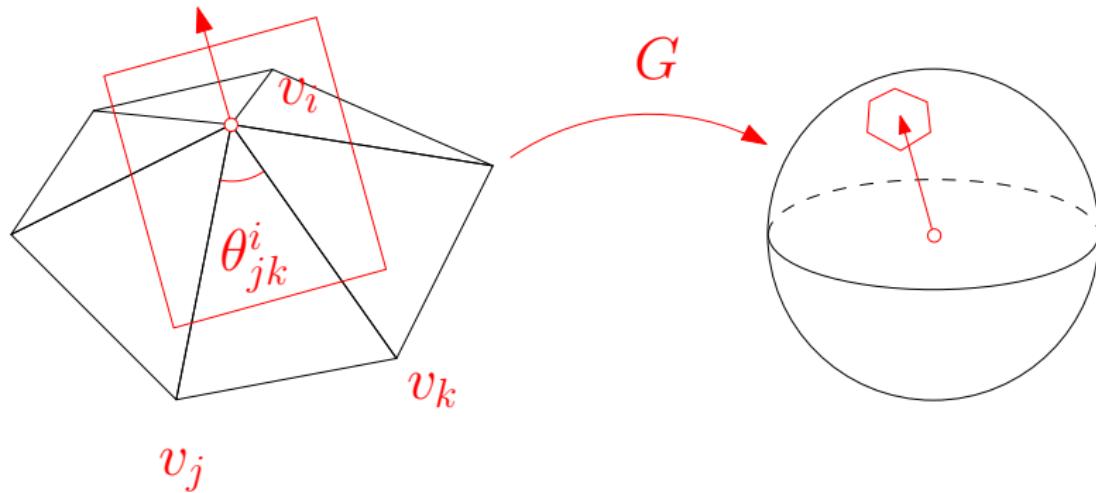


Figure: Discrete Gaussian curvature for convex vertex.

$$K(v_i) := |G(v_i)| = 2\pi - \sum_{jk} \theta_{jk}^i.$$

Gauss-Bonnet

For a closed oriented metric surface (S, \mathbf{g}) ,

$$\int_S K dA = 2\pi\chi(S).$$

For a closed oriented discrete polygonal surface M ,

$$\sum_{v_i} K(v_i) = 2\pi\chi(M).$$

Gaussian Curvature

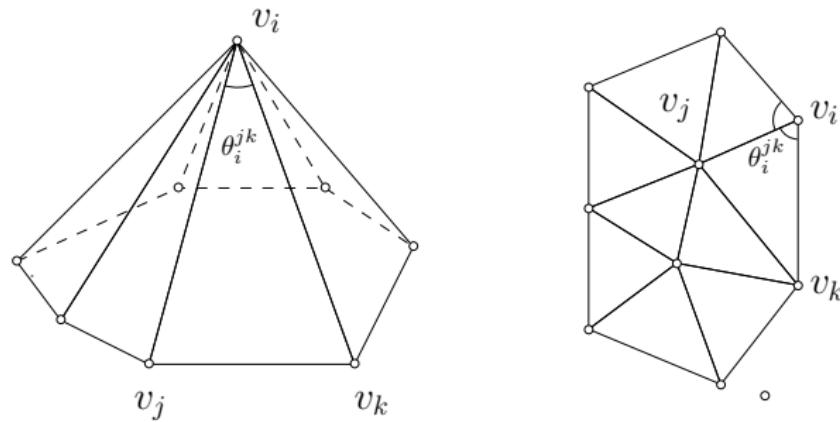


Figure: Discrete Gaussian curvature.

$$K(v_i) = \begin{cases} 2\pi - \sum_{jk} \theta_i^{jk} & v_i \notin \partial M \\ \pi - \sum_{jk} \theta_i^{jk} & v_i \in \partial M \end{cases} \quad (1)$$

Gauss-Bonnet

Theorem (Discrete Gauss-Bonnet Theorem)

Given polyhedral surface (S, V, \mathbf{d}) , the total discrete curvature is

$$\sum_{v \notin \partial M} K(v) + \sum_{v \in \partial M} K(v) = 2\pi\chi(S),$$

where $\chi(S)$ is the Euler characteristic number of S .

Proof.

We denote the polyhedral surface $M = (V, E, F)$, if M is closed, then

$$\sum_{v_i \in V} K(v_i) = \sum_{v_i \in V} \left(2\pi - \sum_{jk} \theta_i^{jk} \right) = \sum_{v_i \in V} 2\pi - \sum_{v_i \in V} \sum_{jk} \theta_i^{jk} = 2\pi|V| - \pi|F|.$$

Since M is closed, $3|F| = 2|E|$,

$$\chi(S) = |V| + |F| - |E| = |V| + |F| - \frac{3}{2}|F| = |V| - \frac{1}{2}|F|.$$

□

Discrete Guass-Bonnet

continued.

Assume M has boundary ∂M . Assume the interior vertex set is V_0 , boundary vertex set is V_1 , then $|V| = |V_0| + |V_1|$; assume interior edge set is E_0 , boundary edge set is E_1 , then $|E| = |E_0| + |E_1|$. Furthermore, all boundaries are closed loops, hence boundary vertex number equals to the boundary edge number, $|V_1| = |E_1|$. Every interior edge is adjacent to two faces, every boundary edge is adjacent to one face, we have

$3|F| = 2|E_0| + |E_1| = 2|E_0| + |V_1|$. We compute the Euler number

$$\chi(M) = |V| + |F| - |E| = |V_0| + |V_1| + |F| - |E_0| - |E_1| = |V_0| + |F| - |E_0|,$$

by $|E_0| = 1/2(3|F| - |V_1|)$

$$\chi(M) = |V_0| - \frac{1}{2}|F| + \frac{1}{2}|V_1|$$

Discrete Guass-Bonnet

continued.

we have:

$$\begin{aligned} \sum_{v_i \in V_0} K(v_i) + \sum_{v_j \in V_1} K(v_j) &= \sum_{v_i \in V_0} \left(2\pi - \sum_{jk} \theta_i^{jk} \right) + \sum_{v_i \in V_1} \left(\pi - \sum_{jk} \theta_i^{jk} \right) \\ &= 2\pi|V_0| + \pi|V_1| - \pi|F| \\ &= 2\pi \left(|V_0| - \frac{1}{2}|F| + \frac{1}{2}|V_1| \right) \\ &= 2\pi\chi(M). \end{aligned} \tag{2}$$

□.

Movable Frame

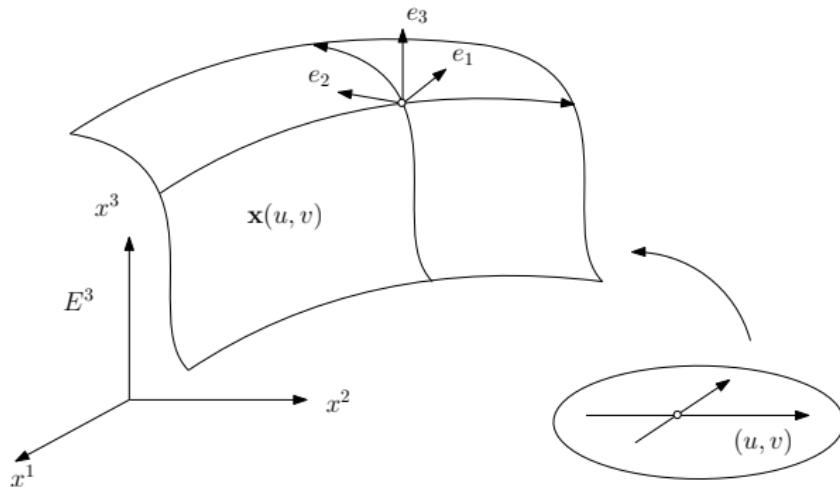


Figure: A parametric surface.

Orthonormal Movable frame

Movable Frame

Suppose a regular surface S is embedded in \mathbb{R}^3 , a parametric representation is $\mathbf{r}(u, v)$. Select two vector fields $\mathbf{e}_1, \mathbf{e}_2$, such that

$$\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta_{ij}.$$

Let \mathbf{e}_3 be the unit normal field of the surface. Then

$$\{\mathbf{r}; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$$

form the *orthonormal frame field* of the surface.

Orthonormal Movalbe frame

Tangent Vector

The tangent vector is the linear combination of the frame bases,

$$d\mathbf{r} = \omega_1 \mathbf{e}_1 + \omega_2 \mathbf{e}_2$$

where $\omega_k(\mathbf{v}) = \langle \mathbf{e}_k, \mathbf{v} \rangle$. $d\mathbf{r}$ is orthogonal to the normal vector \mathbf{e}_3 .

Motion Equation

$$d\mathbf{e}_i = \omega_{i1} \mathbf{e}_1 + \omega_{i2} \mathbf{e}_2 + \omega_{i3} \mathbf{e}_3,$$

where $\omega_{ij} = \langle d\mathbf{e}_i, \mathbf{e}_j \rangle$. Because

$$\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta_{ij}, \quad 0 = d\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \langle d\mathbf{e}_i, \mathbf{e}_j \rangle + \langle \mathbf{e}_i, d\mathbf{e}_j \rangle$$

we get

$$\omega_{ij} + \omega_{ji} = 0, \omega_{ii} = 0.$$

Motion Equation

Motion Equation

$$d\mathbf{r} = \omega_1 \mathbf{e}_1 + \omega_2 \mathbf{e}_2,$$

$$\begin{pmatrix} d\mathbf{e}_1 \\ d\mathbf{e}_2 \\ d\mathbf{e}_3 \end{pmatrix} = \begin{pmatrix} 0 & \omega_{12} & \omega_{13} \\ -\omega_{12} & 0 & \omega_{23} \\ -\omega_{13} & -\omega_{23} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}$$

Fundamental Forms

The first fundamental form is

$$I = \langle d\mathbf{r}, d\mathbf{r} \rangle = \omega_1 \omega_1 + \omega_2 \omega_2.$$

The second fundamental form is

$$II = -\langle d\mathbf{r}, d\mathbf{e}_3 \rangle = -\omega_1 \omega_{31} - \omega_2 \omega_{32} = \omega_1 \omega_{13} + \omega_2 \omega_{23}.$$

Weingarten Mapping

Definition (Weingarten Mapping)

The Gauss mapping is

$$\mathbf{r} \rightarrow \mathbf{e}_3,$$

its derivative map is called the Weingarten mapping,

$$d\mathbf{r} \rightarrow d\mathbf{e}_3, \omega_1 \mathbf{e}_1 + \omega_2 \mathbf{e}_2 \rightarrow \omega_{31} \mathbf{e}_1 + \omega_{32} \mathbf{e}_2.$$

Definition (Gaussian Curvature)

The area ratio (Jacobian of the Weingarten mapping) is the Gaussian curvature

$$K \omega_1 \wedge \omega_2 = \omega_{31} \wedge \omega_{32}.$$

Gaussian curvature

Weigarten Mapping

$\{\omega_1, \omega_2\}$ form the basis of the cotangent space, therefore ω_{13}, ω_{23} can be represented as the linear combination of them,

$$\begin{pmatrix} \omega_{13} \\ \omega_{23} \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$

therefore

$$\omega_{13} \wedge \omega_{23} = \begin{vmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{vmatrix} \omega_1 \wedge \omega_2$$

so $K = h_{11}h_{22} - h_{12}h_{21}$, the mean curvature $H = \frac{1}{2}(h_{11} + h_{22})$.

Gauss's theorem Egregium

Theorem (Gauss' Theorem Egregium)

The Gaussian curvature is intrinsic, solely determined by the first fundamental form.

Proof.

$$\begin{aligned}0 &= d^2 \mathbf{e}_1 \\&= d(\omega_{12} \mathbf{e}_2 + \omega_{13} \mathbf{e}_3) \\&= d\omega_{12} \mathbf{e}_2 - \omega_{12} \wedge d\mathbf{e}_2 + d\omega_{13} \mathbf{e}_3 - \omega_{13} \wedge d\mathbf{e}_3 \\&= d\omega_{12} \mathbf{e}_2 - \omega_{12} \wedge (\omega_{21} \mathbf{e}_1 + \omega_{23} \mathbf{e}_3) + \\&\quad d\omega_{13} \mathbf{e}_3 - \omega_{13} \wedge (\omega_{31} \mathbf{e}_1 + \omega_{32} \mathbf{e}_2) \\&= (d\omega_{12} - \omega_{13} \wedge \omega_{32}) \mathbf{e}_2 + (d\omega_{13} - \omega_{12} \wedge \omega_{23}) \mathbf{e}_3\end{aligned}$$

therefore

$$d\omega_{12} = -\omega_{13} \wedge \omega_{32} = -K \omega_1 \wedge \omega_2.$$

Gauss's theorem Egregium

Lemma

$$\omega_{12} = \frac{d\omega_1}{\omega_1 \wedge \omega_2} \omega_1 + \frac{d\omega_2}{\omega_1 \wedge \omega_2} \omega_2$$

Proof.

$$\begin{aligned} 0 &= d^2 \mathbf{r} \\ &= d(\omega_1 \mathbf{e}_1 + \omega_2 \mathbf{e}_2) \\ &= d\omega_1 \mathbf{e}_1 - \omega_1 \wedge d\mathbf{e}_1 + d\omega_2 \mathbf{e}_2 - \omega_2 \wedge d\mathbf{e}_2 \\ &= d\omega_1 \mathbf{e}_1 - \omega_1 \wedge (\omega_{12} \mathbf{e}_2 + \omega_{13} \mathbf{e}_3) + \\ &\quad d\omega_2 \mathbf{e}_2 - \omega_2 \wedge (\omega_{21} \mathbf{e}_1 + \omega_{23} \mathbf{e}_3) \\ &= (d\omega_1 - \omega_2 \wedge \omega_{21}) \mathbf{e}_1 + (d\omega_2 - \omega_1 \wedge \omega_{12}) \mathbf{e}_2 + \\ &\quad - (\omega_1 \wedge \omega_{13} + \omega_2 \wedge \omega_{23}) \mathbf{e}_3. \end{aligned}$$

Therefore $d\omega_1 = \omega_2 \wedge \omega_{21}$, $d\omega_2 = \omega_1 \wedge \omega_{12}$ and $h_{12} = h_{21}$.



Gaussian Curvature

Lemma (Gaussian curvature)

Under the isothermal coordinates, the Gaussian curvature is given by

$$K = -\frac{1}{e^{2u}} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u.$$

Proof.

Let (S, \mathbf{g}) be a metric surface, use isothermal coordinates

$$\mathbf{g} = e^{2u(x,y)}(dx^2 + dy^2).$$

Then

$$\begin{cases} \omega_1 = e^u dx \\ \omega_2 = e^u dy \end{cases} \quad \begin{cases} \mathbf{e}_1 = e^{-u} \frac{\partial}{\partial x} \\ \mathbf{e}_2 = e^{-u} \frac{\partial}{\partial y} \end{cases}$$



Gaussian Curvature

Continued.

By direct computation,

$$\begin{aligned} d\omega_1 &= de^u \wedge dx & d\omega_2 &= de^u \wedge dy \\ &= e^u(u_x dx + u_y dy) \wedge dx & &= e^u(u_x dx + u_y dy) \wedge dy \\ &= e^u u_y dy \wedge dx & &= e^u u_x dx \wedge dy. \end{aligned}$$

therefore

$$\begin{aligned} \omega_{12} &= \frac{d\omega_1}{\omega_1 \wedge \omega_2} \omega_1 + \frac{d\omega_2}{\omega_1 \wedge \omega_2} \omega_2 \\ &= \frac{e^u u_y dy \wedge dx}{e^{2u} dx \wedge dy} e^u dx + \frac{e^u u_x dx \wedge dy}{e^{2u} dx \wedge dy} e^u dy \\ \omega_{12} &= -u_y dx + u_x dy. \end{aligned}$$

Gaussian Curvature

Continued.

$$K = -\frac{d\omega_{12}}{\omega_1 \wedge \omega_2} = -\frac{(u_{xx} + u_{yy})dx \wedge dy}{e^{2u}dx \wedge dy} = -\frac{1}{e^{2u}}\Delta u.$$

Gaussian Curvature

Example

The unit disk $|z| < 1$ equipped with the following metric

$$ds^2 = \frac{4dzd\bar{z}}{(1 - z\bar{z})^2},$$

the Gaussian curvature is -1 everywhere.

Proof.

$e^{2u} = \frac{4}{1-x^2-y^2}$, then $u = \log 2 - \log(1-x^2-y^2)$.

$$u_x = -\frac{-2x}{1-x^2-y^2} = \frac{2x}{1-x^2-y^2}.$$



Gaussian Curvature

Proof.

then

$$u_{xx} = \frac{2(1 - x^2 - y^2) - 2x(-2x)}{(1 - x^2 - y^2)^2} = \frac{2 + 2x^2 - 2y^2}{(1 - x^2 - y^2)^2}$$

similarly

$$u_{yy} = \frac{2 + 2y^2 - 2x^2}{(1 - x^2 - y^2)^2}$$

so

$$u_{xx} + u_{yy} = \frac{4}{(1 - x^2 - y^2)} = e^{2u}, K = -\frac{1}{e^{2u}}(u_{xx} + u_{yy}) = -1.$$



Yamabe Equation

Lemma (Yamabe Equation)

Conformal metric deformation $\mathbf{g} \rightarrow e^{2\lambda}\mathbf{g} = \tilde{\mathbf{g}}$, then

$$\tilde{K} = \frac{1}{e^{2\lambda}}(K - \Delta_{\mathbf{g}}\lambda).$$

Proof.

Use isothermal parameters, $\mathbf{g} = e^{2u}(dx^2 + dy^2)$, $K = -e^{2u}\Delta u$, similarly $\tilde{\mathbf{g}} = e^{2\tilde{u}}(dx^2 + dy^2)$, $\tilde{K} = -e^{2\tilde{u}}\Delta \tilde{u}$, $\tilde{u} = u + \lambda$,

$$\begin{aligned}\tilde{K} &= -\frac{1}{e^{2(u+\lambda)}}\Delta(u + \lambda) \\ &= \frac{1}{e^{2\lambda}}\left(-\frac{1}{e^{2u}}\Delta u - \frac{1}{e^{2u}}\Delta\lambda\right) \\ &= \frac{1}{e^{2\lambda}}(K - \Delta_{\mathbf{g}}\lambda).\end{aligned}$$

Gauss-Bonnet Theorem

Theorem (Gauss-Bonnet)

Suppose M is a closed orientable C^2 surface, then

$$\int_M A dA = 2\pi\chi(M),$$

where dA is the area element of the surface, $\chi(M)$ is the Euler characteristic number of M .

Proof.

Construct a smooth vector field v , with isolated zeros $\{p_1, p_2, \dots, p_n\}$. Choose a small disk $D(p_i, \varepsilon)$. On the surface

$$\bar{M} = M \setminus \bigcup_{i=1}^n D(p_i, \varepsilon)$$



Gauss-Bonnet Theorem

Proof.

construct orthonormal frame $\{p, e_1, e_2, e_3\}$, where

$$e_1(p) = \frac{v(p)}{|v(p)|}, \quad e_3(p) = n(p).$$

The integration

$$\int_{\bar{M}} K dA = \int_{\bar{M}} K \omega_1 \wedge \omega_2 = - \int_{\bar{M}} d\omega_{12}$$

by Stokes theorem and Poincar  re-Hopf theorem, we obtain

$$-\sum_{i=1}^n \int_{\partial D(p_i, \varepsilon)} \omega_{12} = 2\pi \sum_{i=1}^n \text{Index}(p_i, v) = 2\pi \chi(M).$$

Here by $\omega_{12} = \langle de_1, e_2 \rangle$, ω_{12} is the rotation speed of e_1 . Let $\varepsilon \rightarrow 0$, the equation holds.



Computing Geodesics

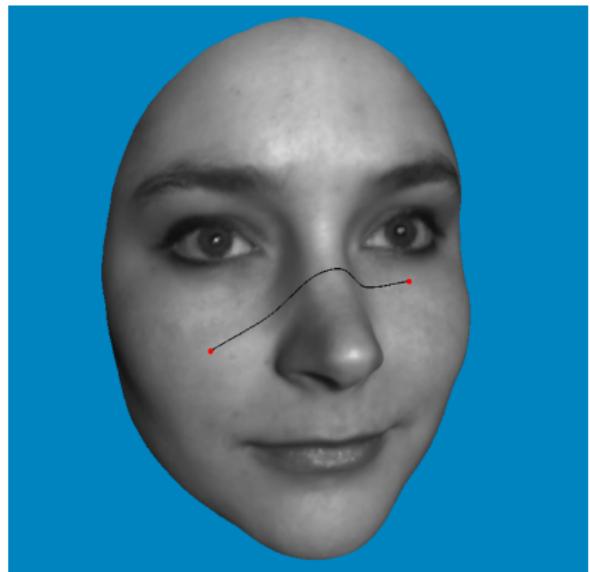
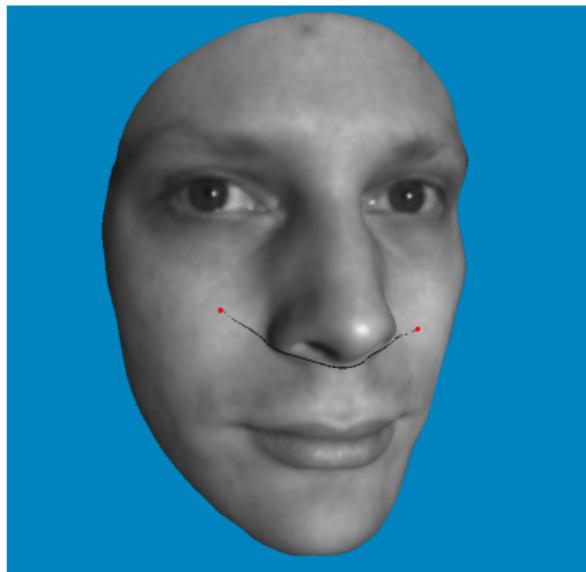


Figure: Geodesics.

Covariant Differential

Definition (Covariant Differentiation)

Covariant differentiation is the generalization of directional derivatives, satisfies the following properties: assume v and w are tangent vector fields on a surface, $f : S \rightarrow \mathbb{R}$ is a C^1 function, then

- ① $D(v + w) = D(v) + D(w),$
- ② $D(fv) = df(v) + fDv,$
- ③ $D\langle v, w \rangle = \langle Dv, w \rangle + \langle v, Dw \rangle.$

By movable framework, the motion equation of the surface is

$$d\mathbf{e}_1 = \omega_{12}\mathbf{e}_2 + \omega_{13}\mathbf{e}_3, \quad d\mathbf{e}_2 = \omega_{21}\mathbf{e}_1 + \omega_{23}\mathbf{e}_3,$$

We only keep tangential component, and delete the normal part to obtain covariant differential

$$D\mathbf{e}_1 = \omega_{12}\mathbf{e}_1, \quad D\mathbf{e}_2 = \omega_{21}\mathbf{e}_1.$$

Covariant Differential

Definition (Parallel transport)

Suppose S is a metric surface, $\gamma : [0, 1] \rightarrow S$ is a smooth curve, $v(t)$ is a vector field along γ , if

$$\frac{Dv}{dt} \equiv 0,$$

then we say the vector field $v(t)$ is parallel transportation along γ .

Given a tangent vector field $v = f_1\mathbf{e}_1 + f_2\mathbf{e}_2$, then

$$\begin{aligned} Dv &= df_1\mathbf{e}_1 + f_1D\mathbf{e}_1 + df_2\mathbf{e}_2 + f_2D\mathbf{e}_2 \\ &= (df_1 - f_2\omega_{12})\mathbf{e}_1 + (df_2 + f_1\omega_{12})\mathbf{e}_2. \end{aligned}$$

and

$$\frac{Dv}{dt} = \left(\frac{df_1}{dt} - f_2 \frac{\omega_{12}}{dt} \right) \mathbf{e}_1 + \left(\frac{df_2}{dt} + f_1 \frac{\omega_{12}}{dt} \right) \mathbf{e}_2.$$

where $\frac{\omega_{12}}{dt} = \langle \omega_{12}, \dot{\gamma} \rangle$. If $\omega_{12} = \alpha dx + \beta dy$, then $\frac{\omega_{12}}{dt} = \alpha \dot{x} + \beta \dot{y}$.



Parallel Transport

Parallel Transport Equation

Therefore parallel vector field satisfies the ODE

$$\begin{cases} \frac{df_1}{dt} - f_2 \frac{\omega_{12}}{dt} = 0 \\ \frac{df_2}{dt} + f_1 \frac{\omega_{12}}{dt} = 0 \end{cases}$$

Given an initial condition $v(0)$, the solution uniquely exists.

Geodesic Curvature

Definition (Geodesic Curvature)

Assume $\gamma : [0, 1] \rightarrow S$ is a C^2 curve on a surface S , s is the arc length parameter. Construct orthonormal frame field along the curve $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, where \mathbf{e}_1 is the tangent vector field of γ , \mathbf{e}_3 is the normal field of the surface,

$$k_g := \frac{D\mathbf{e}_1}{ds} = k_g \mathbf{e}_2$$

is called geodesic curvature vector,

$$k_g = \left\langle \frac{D\mathbf{e}_1}{ds}, \mathbf{e}_2 \right\rangle = \frac{\omega_{12}}{ds}$$

is called geodesic curvature.

Geodesic Curvature

Geodesic curvature, normal curvature

Given a spacial curve, its curvature vector satisfies

$$\frac{d^2\gamma}{ds^2} = k_g \mathbf{e}_2 + k_n \mathbf{e}_3,$$

where k_n is the normal curvature of the curve. The curvature of the curve, geodesic curvature and normal curvature satisfy

$$k^2 = k_g^2 + k_n^2.$$

Geodesic curvature k_g only depends on the Riemannian metric of the surface, is independent of the 2nd fundamental form. Therefore k_g is intrinsic, k_n is extrinsic.

Gauss-Bonnet

Theorem

Suppose (S, \mathbf{g}) is an oriented metric surface with boundaries, then

$$\int_S K dA + \int_{\partial S} k_g ds = 2\pi\chi(S).$$

Proof.

Construct a vector field with isolated zeros $\{p_i\}$, \mathbf{e}_1 is tangent to ∂S , small disks $D(p_i, \varepsilon)$. Define $\bar{S} := S \setminus \bigcup_i D(p_i, \varepsilon)$,

$$\begin{aligned}\int_{\bar{S}} K dA &= - \int_{\bar{S}} \frac{d\omega_{12}}{\omega_1 \wedge \omega_2} dA = - \int_{\bar{S}} d\omega_{12} = - \int_{\partial \bar{S}} \omega_{12} \\ &= - \int_{\partial S - \bigcup_i \partial D(p_i, \varepsilon)} \omega_{12} = - \int_{\partial S} \frac{\omega_{12}}{ds} ds + \sum_i \int_{\partial D(p_i, \varepsilon)} \omega_{12} \\ &= - \int_{\partial S} k_g ds + 2\pi \sum_i \text{Index}(p_i) = - \int_{\partial S} k_g ds + 2\pi\chi(S).\end{aligned}$$

Geodesic Curvature

We use isothermal parameter (u, v) of (S, \mathbf{g}) , given a curve $\gamma(s)$ with arc length parameter s . Construct orthonormal frame $\{p; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, where \mathbf{e}_3 is the normal field of S . The tangent vector of γ is $\bar{\mathbf{e}}_1$, $\bar{\mathbf{e}}_2$ is orthogonal to $\bar{\mathbf{e}}_1$ everywhere. The angle between $\bar{\mathbf{e}}_1$ and \mathbf{e}_1 is $\theta(s)$,

$$\begin{cases} \bar{\mathbf{e}}_1 &= \cos \theta \mathbf{e}_1 + \sin \theta \mathbf{e}_2 \\ \bar{\mathbf{e}}_2 &= -\sin \theta \mathbf{e}_1 + \cos \theta \mathbf{e}_2 \end{cases}$$

Direct computation

$$\begin{aligned} D\bar{\mathbf{e}}_1 &= D(\cos \theta \mathbf{e}_1 + \sin \theta \mathbf{e}_2) = d \cos \theta \mathbf{e}_1 + \cos \theta D\mathbf{e}_1 + d \sin \theta \mathbf{e}_2 + \sin \theta D\mathbf{e}_2 \\ &= -\sin \theta d\theta \mathbf{e}_1 + \cos \theta \omega_{12} \mathbf{e}_2 + \cos \theta d\theta \mathbf{e}_2 - \sin \theta \omega_{12} \mathbf{e}_1 \\ &= -\sin \theta(d\theta + \omega_{12}) \mathbf{e}_1 + \cos \theta(\omega_{12} + d\theta) \mathbf{e}_2 \end{aligned}$$

$$k_g = \left\langle \frac{D\bar{\mathbf{e}}_1}{ds}, \bar{\mathbf{e}}_2 \right\rangle = \frac{d\theta}{ds} + \frac{\omega_{12}}{ds}$$

Geodesic Curvature

Under the isothermal coordinates, we have $\omega_{12} = -u_y dx + u_x dy$. Suppose on the parameter domain, the planar curve arc length is dt , then $ds = e^u dt$. The parameterization preserves angle, therefore

$$\begin{aligned} k_g &= \frac{d\theta}{ds} + \frac{-u_y dx + u_x dy}{ds} \\ &= \frac{d\theta}{dt} \frac{dt}{ds} + \frac{-u_y dx + u_x dy}{dt} \frac{dt}{ds} \\ &= e^{-u}(k - \langle \nabla u, n \rangle) \\ &= e^{-u}(k - \partial_n u) \end{aligned}$$

where k is the curvature of the planar curve, n is the normal to the planar curve.

Geodesic Curvature

Lemma

Given a metric surface (S, \mathbf{g}) , under conformal deformation, $\bar{\mathbf{g}} = e^{2\lambda} \mathbf{g}$, the geodesic curvature satisfies

$$k_{\bar{\mathbf{g}}} = e^{-\lambda} (k_{\mathbf{g}} - \partial_{\mathbf{n}, \mathbf{g}} \lambda).$$

Proof.

$$\begin{aligned} k_{\mathbf{g}} &= e^{-(u+\lambda)} (k - \partial_{\mathbf{n}}(u + \lambda)) \\ &= e^{-\lambda} (e^{-u}(k - \partial_{\mathbf{n}} u) - e^{-u} \partial_{\mathbf{n}} \lambda) \\ &= e^{-\lambda} (k_{\mathbf{g}} - \partial_{\mathbf{n}, \mathbf{g}} \lambda) \end{aligned}$$



Definition (geodesic)

Given a metric surface (S, \mathbf{g}) , a curve $\gamma : [0, 1] \rightarrow S$ is a geodesic if $k_{\mathbf{g}}$ is zero everywhere.

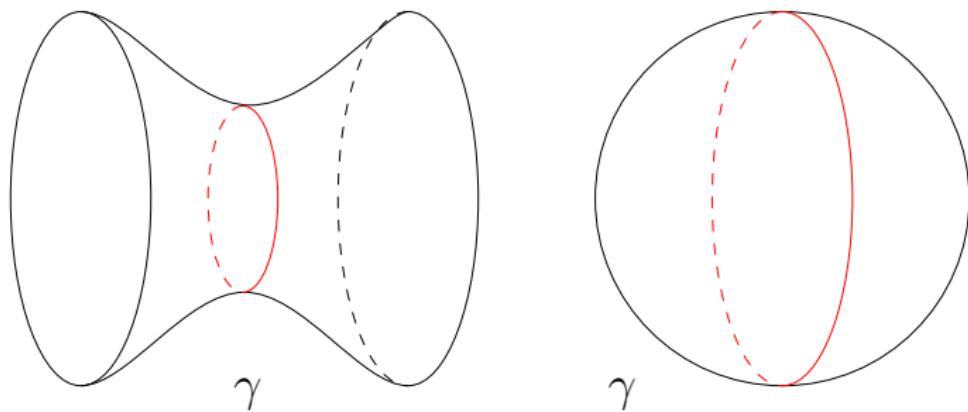


Figure: Stable and unstable geodesics.

Geodesics

Lemma (geodesic)

If γ is the shortest curve connecting p and q , then γ is a geodesic.

Proof.

Consider a family of curves, $F : (-\varepsilon, \varepsilon) \rightarrow S$, such that $F(0, t) = \gamma(t)$, and

$$F(s, 0) = p, F(s, 1) = q, \frac{\partial F(s, t)}{\partial s} = f(t)\mathbf{e}_2(t),$$

where $f : [0, 1] \rightarrow \mathbb{R}$, $f(0) = f(1) = 0$. Fix parameter s , curve $\gamma_s := F(s, \cdot)$, $\{\gamma_s\}$ for a variation. Define an energy,

$$L(s) = \int_0^1 \left| \frac{d\gamma_s(t)}{dt} \right|^2 dt, \quad \frac{\partial L(s)}{\partial s} = - \int_0^1 fk_{\mathbf{g}}(\tau) d\tau.$$



Geodesics

The second derivative of the length variation $L(s)$ depends on the Gaussian curvature of the underlying surface. If $K < 0$, then the second derivative is positive, the geodesic is stable; if $K > 0$, then the secondary derivative is negative, the geodesic is unstable.

Lemma (Uniqueness of geodesics)

Suppose (S, \mathbf{g}) is a closed oriented metric surface, \mathbf{g} induces negative Gaussian curvature everywhere, then each homotopy class has a unique geodesic.

Proof.

The existence can be obtained by variational method. The uniqueness is by Gauss-Bonnet theorem. Assume two geodesics $\gamma_1 \sim \gamma_2$, then they bound a topological annulus Σ , by Gauss-Bonnet,

$$\int_{\Sigma} K dA + \int_{\partial\Sigma} k_g ds = \chi(\Sigma),$$

The first term is negative, the second is along the geodesics, hence 0, $\chi(\Sigma) = 0$. Contradiction. □

Homotopy Detection

Input: A high genus closed mesh M , two loops γ_1 and γ_2 ;

Output: Whether $\gamma_1 \sim \gamma_2$;

- ① Compute a hyperbolic metric of M , using Ricci flow;
- ② Homotopically deform γ_k to geodesics, $k = 1, 2$;
- ③ if two geodesics coincide, return true; otherwise, return false;

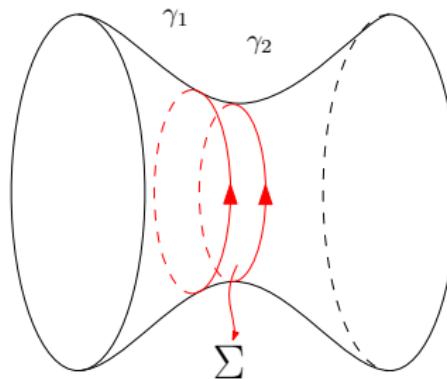


Figure: Geodesics uniqueness.

Compute Minimal Surface

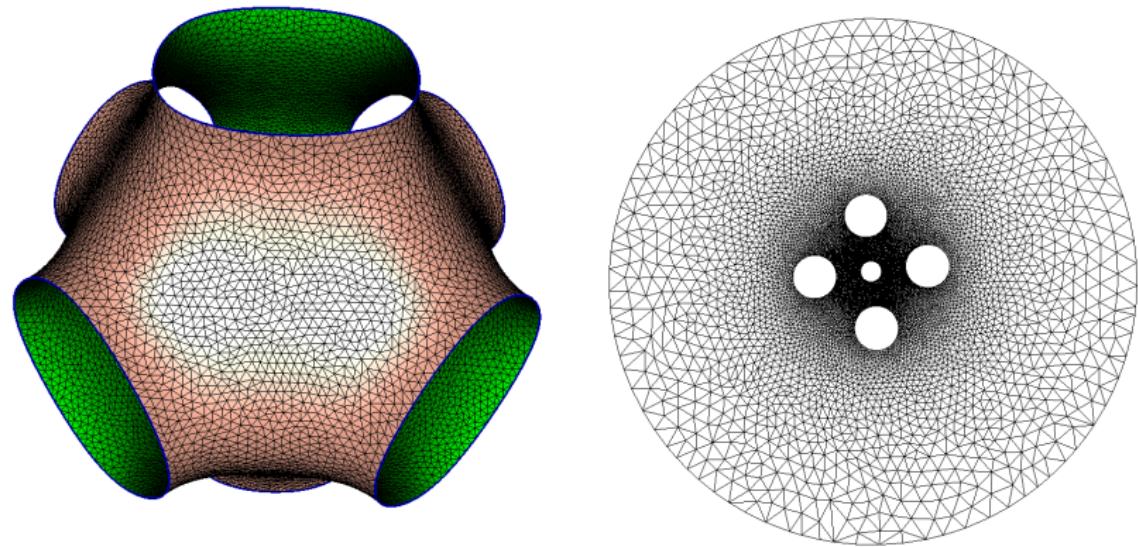


Figure: Minimal surface.

Smooth minimal surface satisfies $\Delta_{\mathbf{g}} r \equiv 0$, equivalently $H(p) \equiv 0$. A discrete minimal surface satisfies $\sum_{v_i \sim v_j} w_{ij}(\mathbf{r}(v_i) - \mathbf{r}(v_j)) = 0, \forall v_i \notin \partial M$.

Minimal Surface

Lemma

Given a metric surface (S, \mathbf{g}) embedded in \mathbb{R}^3 , then $\Delta_{\mathbf{g}} \mathbf{r} = 2H(p)\mathbf{n}$, where \mathbf{r}, \mathbf{n} are the position and normal vectors.

Proof.

We choose isothermal coordinates (x, y) . Then $\mathbf{g} = e^{2\lambda} \mathbf{I}$,
 $\omega_{12} = -\lambda_y dx + \lambda_x dy$, $\omega_{13} = h_{11}\omega_1 + h_{12}\omega_2$, $\omega_{23} = h_{12}\omega_1 + h_{22}\omega_2$,
 $\omega_1 = e^\lambda dx$, $\omega_2 = e^\lambda dy$,

$$\begin{aligned}\frac{\partial}{\partial x} \mathbf{r}_x &= \frac{\partial}{\partial x} e^\lambda \mathbf{e}_1 = e^\lambda \lambda_x \mathbf{e}_1 + e^\lambda \frac{\partial}{\partial x} \mathbf{e}_1 \\&= e^\lambda \lambda_x \mathbf{e}_1 + e^\lambda \langle d\mathbf{e}_1, \frac{\partial}{\partial x} \rangle = e^\lambda \lambda_x \mathbf{e}_1 + e^\lambda \langle \omega_{12} \mathbf{e}_2 + \omega_{13} \mathbf{e}_3, \partial_x \rangle \\&= e^\lambda \lambda_x \mathbf{e}_1 + e^\lambda (-\lambda_y) \mathbf{e}_2 + e^\lambda \mathbf{e}_3 \langle h_{11} \omega_1, \partial_x \rangle \\&= e^\lambda \lambda_x \mathbf{e}_1 - e^\lambda \lambda_y \mathbf{e}_2 + e^{2\lambda} h_{11} \mathbf{e}_3\end{aligned}$$

Minimal Surface

Proof.

Similarly,

$$\begin{aligned}\frac{\partial}{\partial y} \mathbf{r}_y &= \frac{\partial}{\partial y} e^\lambda \mathbf{e}_2 = e^\lambda \lambda_y \mathbf{e}_2 + e^\lambda \frac{\partial}{\partial y} \mathbf{e}_2 \\&= e^\lambda \lambda_y \mathbf{e}_2 + e^\lambda \langle d\mathbf{e}_2, \frac{\partial}{\partial y} \rangle = e^\lambda \lambda_y \mathbf{e}_2 + e^\lambda \langle \omega_{21} \mathbf{e}_1 + \omega_{23} \mathbf{e}_3, \partial_y \rangle \\&= e^\lambda \lambda_y \mathbf{e}_2 + e^\lambda (-\lambda_y) \mathbf{e}_2 + e^\lambda \mathbf{e}_3 \langle h_{22} \omega_2, \partial_y \rangle \\&= e^\lambda \lambda_y \mathbf{e}_2 - e^\lambda \lambda_x \mathbf{e}_1 + e^{2\lambda} h_{22} \mathbf{e}_3\end{aligned}$$

Therefore

$$\Delta_g \mathbf{r} = \frac{1}{e^{2\lambda}} (\mathbf{r}_{xx} + \mathbf{r}_{yy}) = (h_{11} + h_{22}) \mathbf{e}_3 = 2H \mathbf{e}_3.$$

