

Discrete Euclidean Curvature Flow

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Isothermal Coordinates

Relation between conformal structure and Riemannian metric

Isothermal Coordinates

A surface Σ with a Riemannian metric \mathbf{g} , a local coordinate system (u, v) is an isothermal coordinate system, if

$$\mathbf{g} = e^{2\lambda(u,v)}(du^2 + dv^2).$$



Gaussian Curvature

Gaussian Curvature

Suppose $\bar{\mathbf{g}} = e^{2\lambda} \mathbf{g}$ is a conformal metric on the surface, then the Gaussian curvature on interior points are

$$K = -\Delta_{\bar{\mathbf{g}}} \lambda = -\frac{1}{e^{2\lambda}} \Delta \lambda,$$

where

$$\Delta = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$$

Conformal Metric Deformation

Definition

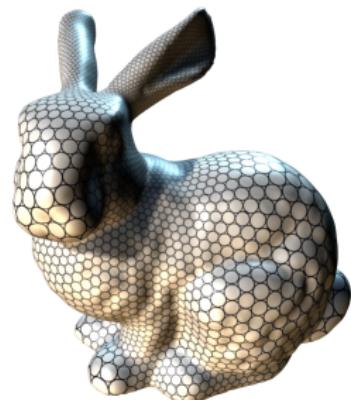
Suppose Σ is a surface with a Riemannian metric,

$$\mathbf{g} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

Suppose $\lambda : \Sigma \rightarrow \mathbb{R}$ is a function defined on the surface, then $e^{2\lambda}\mathbf{g}$ is also a Riemannian metric on Σ and called a **conformal metric**. λ is called the conformal factor.

$$\mathbf{g} \rightarrow e^{2\lambda}\mathbf{g}$$

Conformal metric deformation.



Angles are invariant measured by conformal metrics.

Curvature and Metric Relations

Yamabi Equation

Suppose $\bar{\mathbf{g}} = e^{2\lambda} \mathbf{g}$ is a conformal metric on the surface, then the Gaussian curvature on interior points are

$$\bar{K} = e^{-2\lambda}(-\Delta_{\mathbf{g}}\lambda + K),$$

geodesic curvature on the boundary

$$\bar{k}_{\mathbf{g}} = e^{-\lambda}(-\partial_n \lambda + k_{\mathbf{g}}).$$

Theorem (Poincaré Uniformization Theorem)

Let (Σ, \mathbf{g}) be a compact 2-dimensional Riemannian manifold. Then there is a metric $\tilde{\mathbf{g}} = e^{2\lambda} \mathbf{g}$ conformal to \mathbf{g} which has constant Gauss curvature.

Surface Uniformization

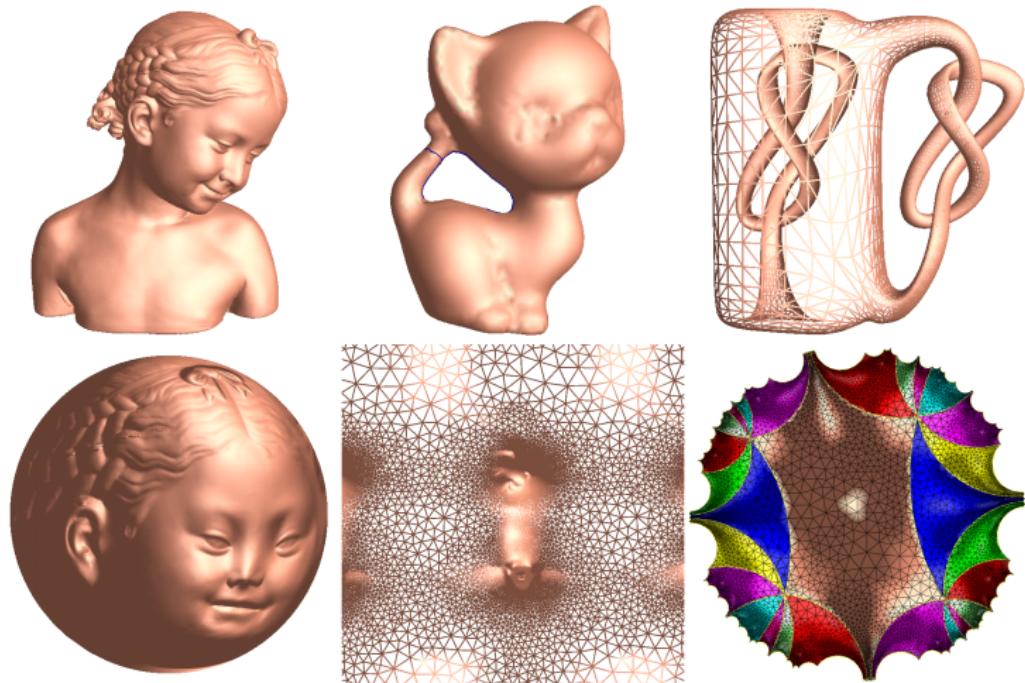


Figure: Closed surface uniformization.

Surface Uniformization

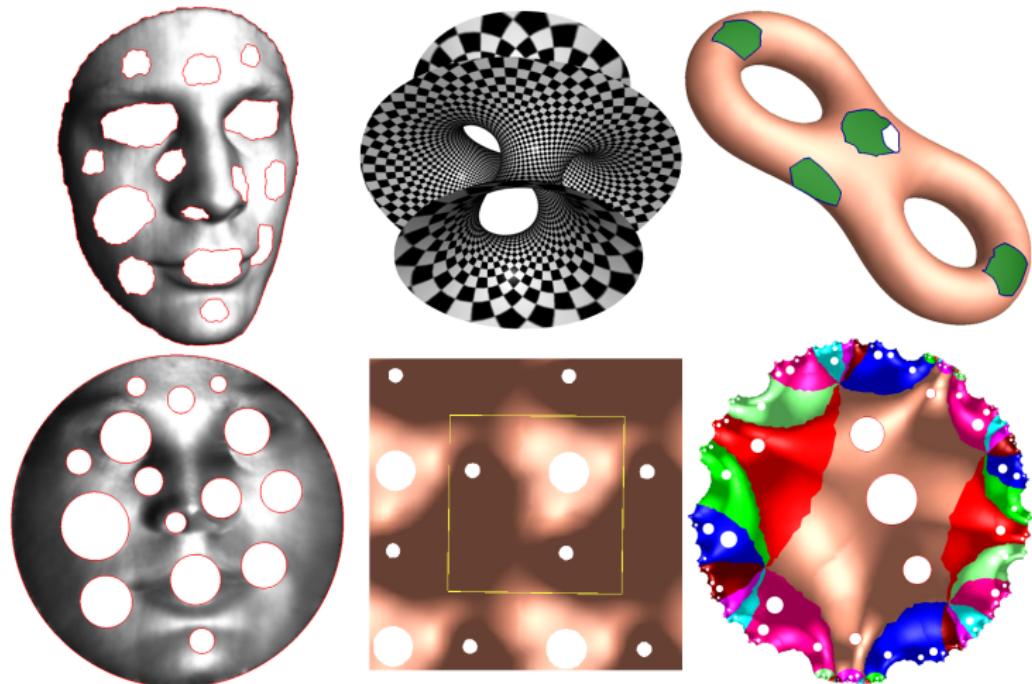


Figure: Open surface uniformization.

Surface Ricci Flow

Proposition

During the curvature flow $\frac{d\lambda}{dt} = -K$, then

$$\frac{d}{dt} K = 2K^2 + \Delta_g K.$$

$$\begin{aligned}\frac{d}{dt} K &= \frac{d}{dt} (-e^{-2\lambda} \Delta \lambda) \\&= - \left(-2 \frac{d\lambda}{dt} \right) e^{-2\lambda} \Delta \lambda - e^{-2\lambda} \Delta \frac{d\lambda}{dt} \\&= \left(-2 \frac{d\lambda}{dt} \right) \boxed{-e^{-2\lambda} \Delta \lambda} - \boxed{e^{-2\lambda} \Delta} \frac{d\lambda}{dt} \\&= \left(-2 \frac{d\lambda}{dt} \right) K - \Delta_g \frac{d\lambda}{dt} \\&= 2K^2 + \Delta_g K\end{aligned}$$

Surface Ricci Flow

Key Idea

$$K = -\Delta_{\mathbf{g}} \lambda,$$

Roughly speaking,

$$\frac{dK}{dt} = \frac{d}{dt} \Delta_{\mathbf{g}} \lambda$$

Let $\frac{d\lambda}{dt} = -K$,

$$\frac{dK}{dt} = \Delta_{\mathbf{g}} K + 2K^2$$

Diffusion and reaction equation!

Surface Ricci Flow

Definition (**Hamilton's Surface Ricci Flow**)

A closed surface with a Riemannian metric \mathbf{g} , the Ricci flow on it is defined as

$$\frac{dg_{ij}}{dt} = -2Kg_{ij}.$$

The normalized surface Ricci flow,

$$\frac{dg_{ij}}{dt} = \frac{2\pi\chi(S)}{A(0)} - 2Kg_{ij},$$

where $A(0)$ is the initial surface area.

The normalized surface Ricci flow is area-preserving, the Ricci flow will converge to a metric such that the Gaussian curvature is constant $\frac{2\pi\chi(S)}{A(0)}$ every where.

Theorem (Hamilton 1982)

For a closed surface of non-positive Euler characteristic, if the total area of the surface is preserved during the flow, the Ricci flow will converge to a metric such that the Gaussian curvature is constant (equals to \bar{K}) every where.

Theorem (Bennett Chow)

For a closed surface of positive Euler characteristic, if the total area of the surface is preserved during the flow, the Ricci flow will converge to a metric such that the Gaussian curvature is constant (equals to \bar{K}) every where.

Summary

Surface Ricci Flow

- Conformal metric deformation

$$\mathbf{g} \rightarrow e^{2u} \mathbf{g}$$

- Curvature Change - heat diffusion

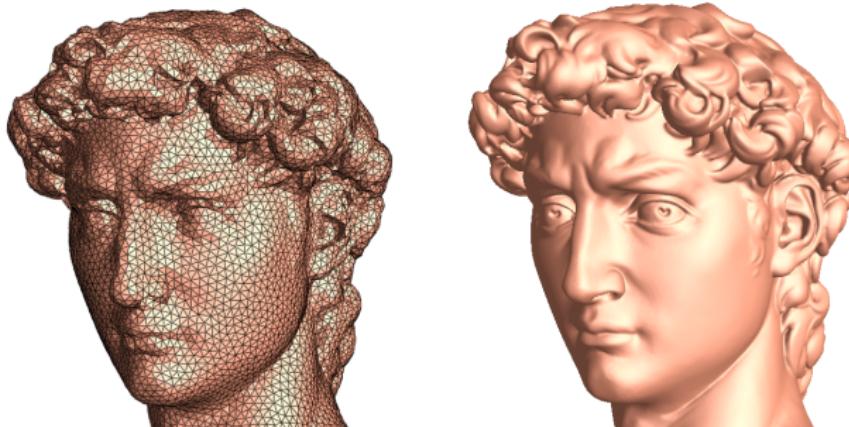
$$\frac{dK}{dt} = \Delta_{\mathbf{g}} K + 2K^2$$

- Ricci flow

$$\frac{du}{dt} = \bar{K} - K.$$

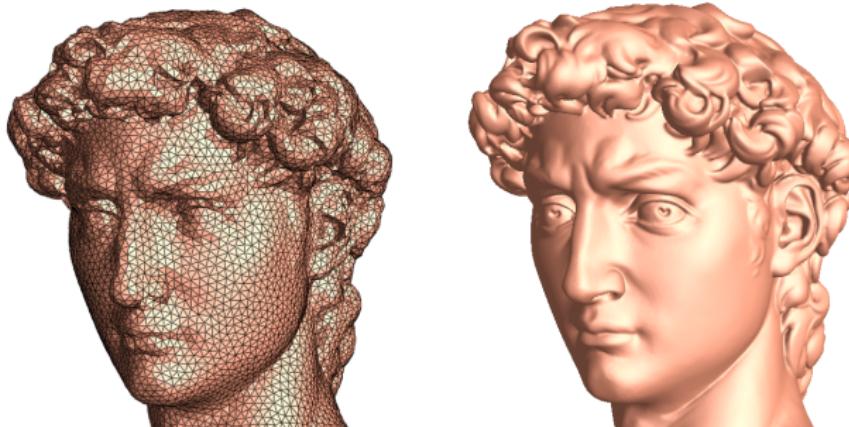
Generic Surface Model - Triangular Mesh

- Surfaces are represented as polyhedron triangular meshes.



Generic Surface Model - Triangular Mesh

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- Isometric gluing of triangles in \mathbb{E}^2 .



Generic Surface Model - Triangular Mesh

- Surfaces are represented as polyhedron triangular meshes.
- Isometric gluing of triangles in \mathbb{E}^2 .
- Isometric gluing of triangles in $\mathbb{H}^2, \mathbb{S}^2$.



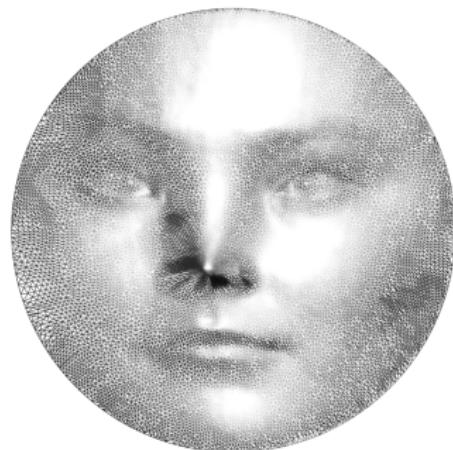
Concepts

- ① Discrete Riemannian Metric
- ② Discrete Curvature
- ③ Discrete Conformal Metric Deformation

Definition (Discrete Metric)

A Discrete Metric on a triangular mesh is a function defined on the vertices, $\ell : E = \{ \text{all edges} \} \rightarrow \mathbb{R}^+$, satisfies triangular inequality.

A mesh has infinite metrics.



Discrete Curvature

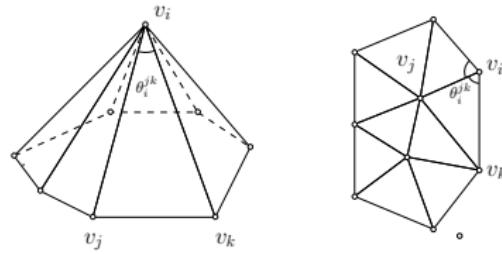
Definition (Discrete Curvature)

Discrete curvature: $K : V = \{vertices\} \rightarrow \mathbb{R}^1$.

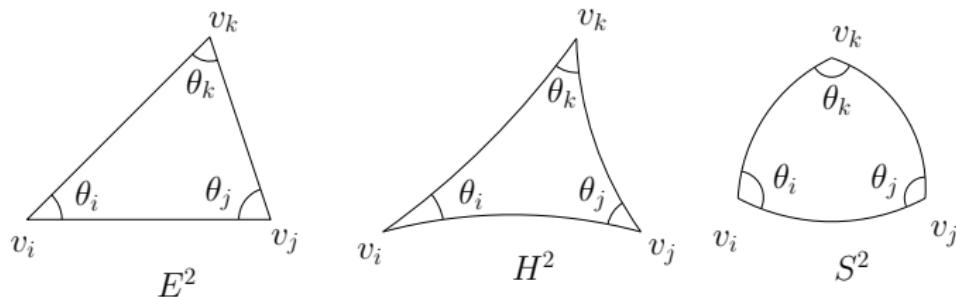
$$K(v_i) = 2\pi - \sum_{jk} \theta_i^{jk}, v_i \notin \partial M; K(v_i) = \pi - \sum_{jk} \theta_{jk}, v_i \in \partial M$$

Theorem (Discrete Gauss-Bonnet theorem)

$$\sum_{v \notin \partial M} K(v) + \sum_{v \in \partial M} K(v) = 2\pi\chi(M).$$



Discrete Metrics Determines the Curvatures



cosine laws

$$\begin{aligned}\cos l_i &= \frac{\cos \theta_i + \cos \theta_j \cos \theta_k}{\sin \theta_j \sin \theta_k} & \mathbb{S}^2 \\ \cosh l_i &= \frac{\cosh \theta_i + \cosh \theta_j \cosh \theta_k}{\sinh \theta_j \sinh \theta_k} & \mathbb{H}^2 \\ 1 &= \frac{\cos \theta_i + \cos \theta_j \cos \theta_k}{\sin \theta_j \sin \theta_k} & \mathbb{E}^2\end{aligned}$$

Discrete Conformal Metric Deformation

Conformal maps Properties

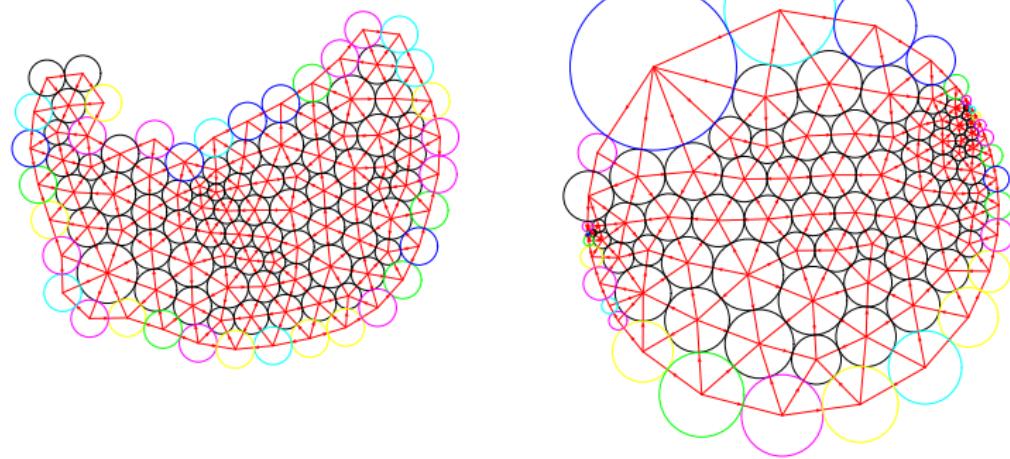
- transform infinitesimal circles to infinitesimal circles.
- preserve the intersection angles among circles.



Idea - Approximate conformal metric deformation

Replace infinitesimal circles by circles with finite radii.

Discrete Conformal Metric Deformation vs CP



Circle Packing Metric

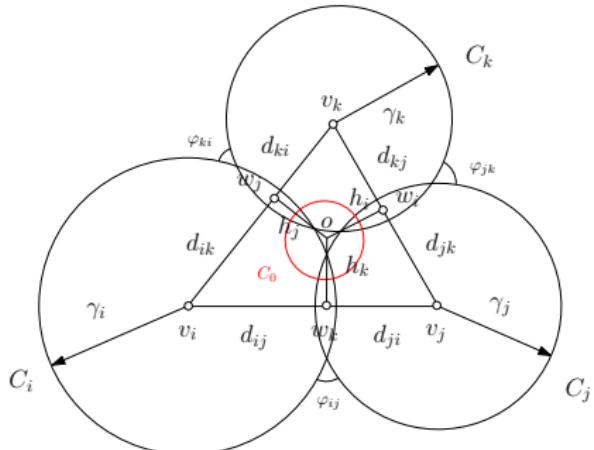
CP Metric

We associate each vertex v_i with a circle with radius γ_i . On edge e_{ij} , the two circles intersect at the angle of Φ_{ij} . The edge lengths are

$$l_{ij}^2 = \gamma_i^2 + \gamma_j^2 + 2\gamma_i\gamma_j \cos \varphi_{ij}$$

CP Metric (Σ, Γ, Φ) , Σ triangulation,

$$\Gamma = \{\gamma_i | \forall v_i\}, \Phi = \{\varphi_{ij} | \forall e_{ij}\}$$



Discrete Conformal Factor

Conformal Factor

Defined on each vertex $\mathbf{u} : V \rightarrow \mathbb{R}$,

$$u_i = \begin{cases} \log \gamma_i & \mathbb{R}^2 \\ \log \tanh \frac{\gamma_i}{2} & \mathbb{H}^2 \\ \log \tan \frac{\gamma_i}{2} & \mathbb{S}^2 \end{cases}$$

Properties

- Symmetry

$$\frac{\partial K_i}{\partial u_j} = \frac{\partial K_j}{\partial u_i}$$

- Discrete Laplace Equation

$$d\mathbf{K} = \Delta d\mathbf{u},$$

Δ is a discrete Laplace-Beltrami operator.

Analogy

- Curvature flow

$$\frac{du}{dt} = \bar{K} - K,$$

- Energy

$$E(\mathbf{u}) = \int \sum_i (\bar{K}_i - K_i) du_i,$$

- Hessian of E denoted as Δ ,

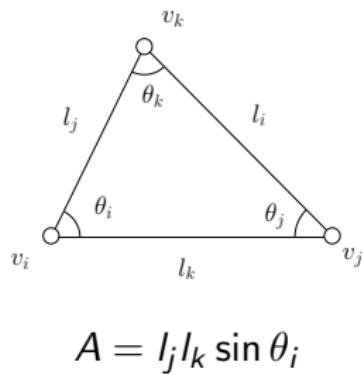
$$d\mathbf{K} = \Delta d\mathbf{u}.$$

Criteria for Discretization

Key Points

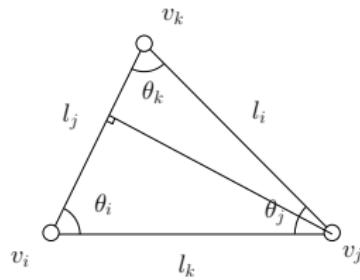
- Convexity of the energy $E(\mathbf{u})$
- Convexity of the metric space (\mathbf{u} -space)
- Admissible curvature space (\mathbf{K} -space)
- Preserving or reflecting richer structures
- Conformality

Cosine law



$$\begin{aligned} 2l_j l_k \cos \theta_i &= l_j^2 + l_k^2 - l_i^2 \\ -2l_j l_k \sin \theta_i \frac{d\theta_i}{dl_i} &= -2l_i \\ \frac{d\theta_i}{dl_i} &= \frac{l_i}{A} \end{aligned}$$

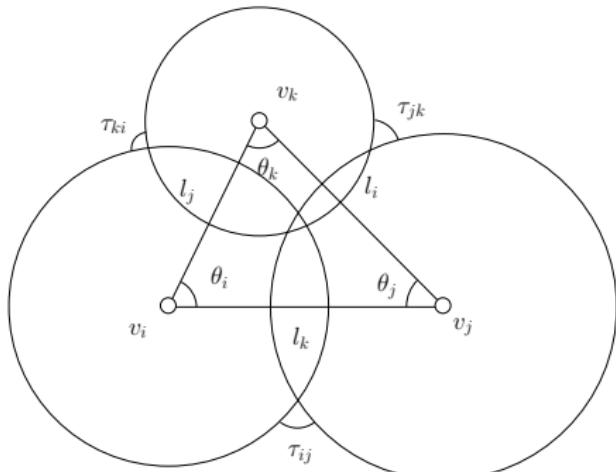
Cosine law



$$l_j = l_i \cos \theta_k + l_k \cos \theta_i$$

$$\begin{aligned} 2l_j l_k \cos \theta_i &= l_j^2 + l_k^2 - l_i^2 \\ 2l_j &= 2l_k \cos \theta_i - 2l_j l_k \sin \theta_i \frac{d\theta_i}{dl_j} \\ \frac{d\theta_i}{dl_j} &= \frac{l_k \cos \theta_i - l_j}{A} \\ &= -\frac{l_i \cos \theta_k}{A} \\ &= -\frac{d\theta_i}{dl_i} \cos \theta_k \end{aligned}$$

Cosine law



$$l_k^2 = r_i^2 + r_j^2 + 2 \cos \tau_{ij} r_i r_j$$

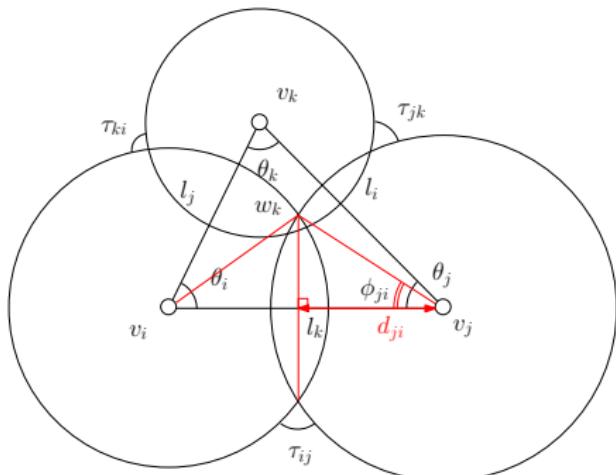
$$\begin{aligned} l_i^2 &= r_j^2 + r_k^2 + 2r_j r_k \cos \tau_{jk} \\ 2l_i \frac{dl_i}{dr_j} &= 2r_j + 2r_k \cos \tau_{jk} \\ \frac{dl_i}{dr_j} &= \frac{2r_j^2 + 2r_j r_k \cos \tau_{jk}}{2l_i r_j} \\ &= \frac{r_j^2 + r_k^2 + 2r_j r_k \cos \tau_{jk} + r_j^2}{2l_i r_j} \\ &= \frac{l_i^2 + r_j^2 - r_k^2}{2l_i r_j} \end{aligned}$$

Cosine law

Let $u_i = \log r_i$, then

$$\begin{pmatrix} d\theta_1 \\ d\theta_2 \\ d\theta_3 \end{pmatrix} = \frac{-1}{A} \begin{pmatrix} l_1 & 0 & 0 \\ 0 & l_2 & 0 \\ 0 & 0 & l_3 \end{pmatrix} \begin{pmatrix} -1 & \cos \theta_3 & \cos \theta_2 \\ \cos \theta_3 & -1 & \cos \theta_1 \\ \cos \theta_2 & \cos \theta_1 & -1 \end{pmatrix}$$
$$\begin{pmatrix} 0 & \frac{l_1^2 + r_2^2 - r_3^2}{2l_1r_2} & \frac{l_1^2 + r_3^2 - r_2^2}{2l_1r_3} \\ \frac{l_2^2 + r_1^2 - r_3^2}{2l_2r_1} & 0 & \frac{l_2^2 + r_3^2 - r_1^2}{2l_2r_3} \\ \frac{l_3^2 + r_1^2 - r_2^2}{2l_3r_1} & \frac{l_3^2 + r_2^2 - r_1^2}{2l_3r_2} & 0 \end{pmatrix} \begin{pmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_3 \end{pmatrix} \begin{pmatrix} du_1 \\ du_2 \\ du_3 \end{pmatrix}$$

Cosine law



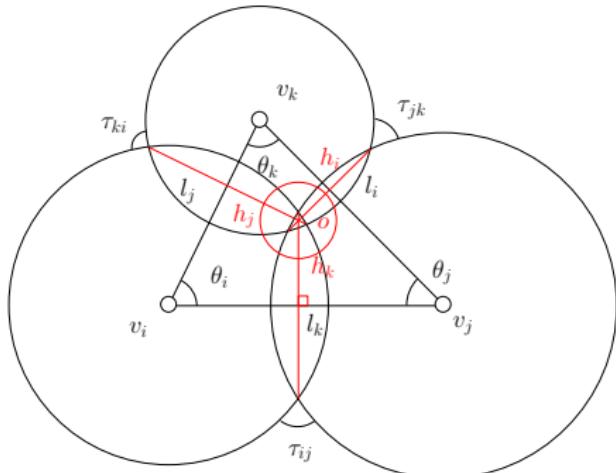
$$l_k^2 = r_i^2 + r_j^2 + 2 \cos \tau_{ij} r_i r_j$$

$$\begin{aligned} 2l_k \frac{dl_k}{dr_j} &= 2r_j + 2r_i \cos \tau_{ij} \\ r_j \frac{dl_k}{dr_j} &= \frac{2r_j^2 + 2r_i r_j \cos \tau_{ij}}{2l_k} \\ &= \frac{l_k^2 + r_j^2 - r_i^2}{2l_k} \end{aligned}$$

In triangle $[v_i, v_j, w_k]$,

$$\frac{dl_k}{du_j} = 2 \frac{l_k r_j \cos \phi_{ji}}{2l_k} = r_j \cos \phi_{ji} = d_{ji}$$

Cosine law

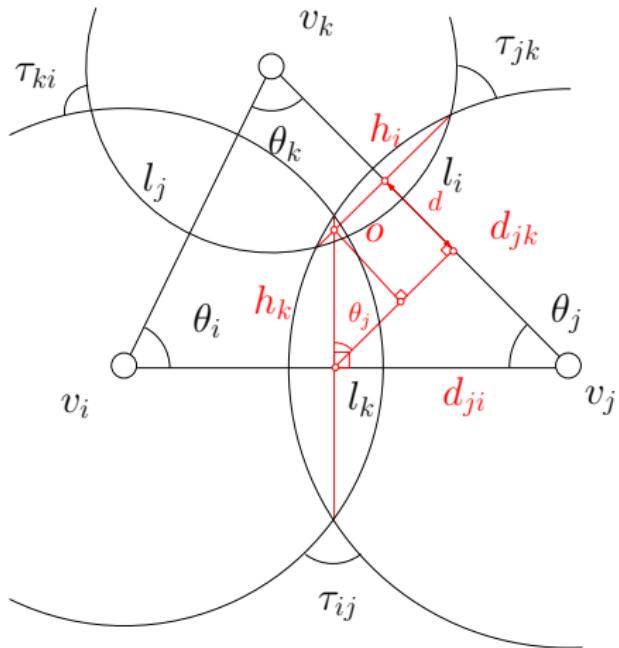


There is a unique circle orthogonal to three circles (v_i, r_i) , the center is o , the distance from o to edge $[v_i, v_j]$ is h_k .

Theorem

$$\begin{aligned}\frac{d\theta_i}{du_j} &= \frac{d\theta_j}{du_i} = \frac{h_k}{l_k} \\ \frac{d\theta_j}{du_k} &= \frac{d\theta_k}{du_j} = \frac{h_i}{l_i} \\ \frac{d\theta_k}{du_i} &= \frac{d\theta_i}{du_k} = \frac{h_j}{l_j}\end{aligned}$$

Cosine law



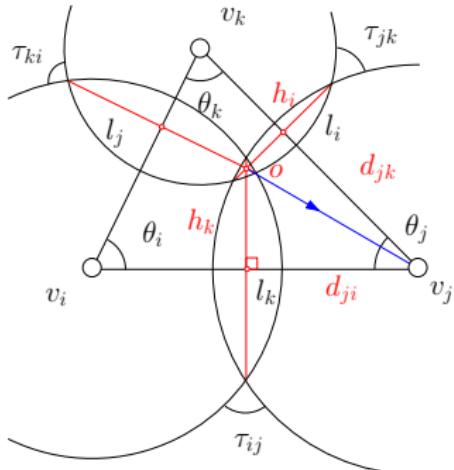
$$\frac{d\theta_i}{du_i} = \frac{h_k}{l_k}$$

Proof.

$$\begin{aligned}\frac{\partial \theta_i}{\partial u_j} &= \frac{\partial \theta_i}{\partial l_i} \frac{\partial l_i}{\partial u_j} + \frac{\partial \theta_i}{\partial l_k} \frac{\partial l_k}{\partial u_j} \\&= \frac{\partial \theta_i}{\partial l_i} \left(\frac{\partial l_i}{\partial u_j} - \frac{\partial l_k}{\partial u_j} \cos \theta_j \right) \\&= \frac{l_i}{A} (d_{jk} - d_{ji} \cos \theta_j) \\&= \frac{dl_i}{l_i l_k \sin \theta_j} \\&= \frac{h_k \sin \theta_j}{l_k \sin \theta_j} \\&= \frac{h_k}{l_k}\end{aligned}$$

□

Cosine law



$$\frac{\partial v_j}{\partial u_j} = v_j - o$$

Similarly

$$d_{jk} = \left\langle \frac{\partial v_j}{\partial u_j}, \frac{v_j - v_k}{l_i} \right\rangle$$

$$\text{So } \frac{\partial v_j}{\partial u_j} = v_j - 0.$$

$$\frac{\partial \langle v_j - v_i, v_j - v_i \rangle}{\partial u_j} = 2 \left\langle \frac{\partial v_j}{\partial u_j}, v_j - v_i \right\rangle$$

$$\frac{\partial l_k^2}{\partial u_j} = 2 \left\langle \frac{\partial v_j}{\partial u_j}, v_j - v_i \right\rangle$$

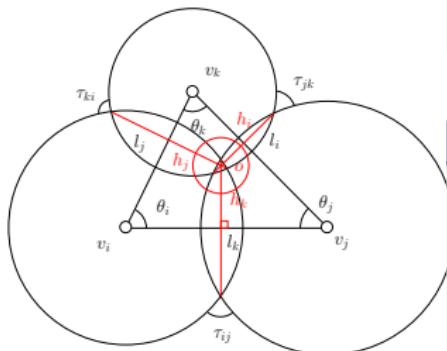
$$\frac{\partial l_k}{\partial u_j} = \left\langle \frac{\partial v_j}{\partial u_j}, \frac{v_j - v_i}{l_k} \right\rangle$$

$$d_{ji} = \left\langle \frac{\partial v_j}{\partial u_j}, \frac{v_j - v_i}{l_k} \right\rangle$$

Metric Space

Lemma

For any three obtuse angles $\tau_{ij}, \tau_{jk}, \tau_{ki} \in [0, \frac{\pi}{2})$ and any three positive numbers r_1, r_2 and r_3 , there is a configuration of 3 circles in Euclidean geometry, unique upto isometry, having radii r_i and meeting in angles τ_{ij} .



Proof.

$$\max\{r_i^2, r_j^2\} < r_i^2 + r_j^2 + 2r_i r_j \cos \tau_{ij} \leq (r_i + r_j)^2$$

$$\max\{r_i^2, r_j^2\} < l_k \leq r_i + r_j$$

so

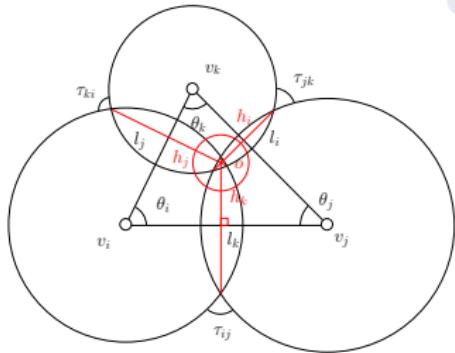
$$l_k \leq r_i + r_j < l_i + l_j.$$

Discrete Ricci Energy

Lemma

ω is closed 1-form in $\Omega := \{(u_1, u_2, u_3) \in \mathbb{R}^3\}$.

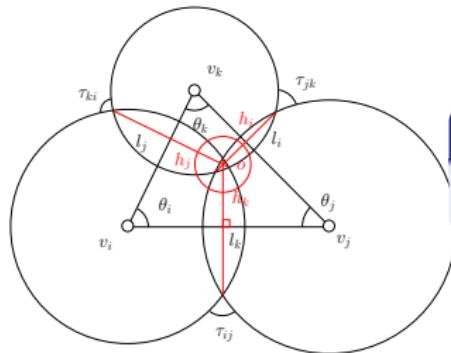
Because $\frac{\partial \theta_i}{\partial u_j} = \frac{\partial \theta_j}{\partial u_i}$, so



$$\omega = \theta_i du_i + \theta_j du_j + \theta_k du_k$$

$$\begin{aligned} d\omega &= \left(\frac{\partial \theta_i}{\partial u_j} - \frac{\partial \theta_j}{\partial u_i} \right) du_j \wedge du_i + \\ &\quad \left(\frac{\partial \theta_j}{\partial u_k} - \frac{\partial \theta_k}{\partial u_j} \right) du_k \wedge du_j + \\ &\quad \left(\frac{\partial \theta_k}{\partial u_i} - \frac{\partial \theta_i}{\partial u_k} \right) du_i \wedge du_k \\ &= 0. \end{aligned}$$

Discrete Ricci Energy



Lemma

The Ricci energy $E(u_1, u_2, u_3)$ is well defined.

Because $\Omega = \mathbb{R}^3$ is convex, closed 1-form is exact, therefore $E(u_1, u_2, u_3)$ is well defined.

$$E(u_1, u_2, u_3) = \int_{(0,0,0)}^{(u_1, u_2, u_3)} \omega$$

Discrete Ricci Energy

Lemma

The Ricci energy $E(u_1, u_2, u_3)$ is strictly concave on the subspace $u_1 + u_2 + u_3 = 0$.

The gradient $\nabla E = (\theta_1, \theta_2, \theta_3)$, the Hessian matrix is

$$H = \begin{pmatrix} \frac{\partial \theta_1}{\partial u_1} & \frac{\partial \theta_1}{\partial u_2} & \frac{\partial \theta_1}{\partial u_3} \\ \frac{\partial \theta_2}{\partial u_1} & \frac{\partial \theta_2}{\partial u_2} & \frac{\partial \theta_2}{\partial u_3} \\ \frac{\partial \theta_3}{\partial u_1} & \frac{\partial \theta_3}{\partial u_2} & \frac{\partial \theta_3}{\partial u_3} \end{pmatrix}$$

$$E(u_1, u_2, u_3) = \int_{(0,0,0)}^{(u_1, u_2, u_3)} \omega \text{ because of } \theta_1 + \theta_2 + \theta_3 = \pi,$$

$$\frac{\partial \theta_i}{\partial u_i} = -\frac{\partial \theta_i}{\partial u_j} - \frac{\partial \theta_i}{\partial u_k} = -\frac{\partial \theta_j}{\partial u_i} - \frac{\partial \theta_k}{\partial u_i}$$

Ricci energy

Proof.

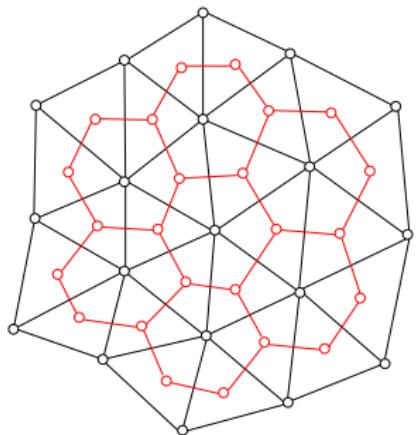
$$H = - \begin{pmatrix} \frac{h_3}{l_3} + \frac{h_2}{l_2} & -\frac{h_3}{l_3} & -\frac{h_2}{l_2} \\ -\frac{h_3}{l_3} & \frac{h_3}{l_3} + \frac{h_1}{l_1} & -\frac{h_1}{l_1} \\ -\frac{h_2}{l_2} & -\frac{h_1}{l_1} & \frac{h_2}{l_2} + \frac{h_1}{l_1} \end{pmatrix}$$

$-H$ is diagonal dominant, it has null space $(1, 1, 1)$, on the subspace $u_1 + u_2 + u_3 = 0$, it is strictly negative definite. Therefore the discrete Ricci energy $E(u_1, u_2, u_3)$ is strictly concave. □

Discrete Ricci Energy

Lemma

The Ricci energy $E(\mathbf{u})$ is strictly convex on the subspace $\sum_{v_i \in M} u_i = 0$.



The gradient $\nabla E = (K_1, K_2, \dots, K_n)$. The Ricci energy

$$E(\mathbf{u}) = 2\pi \sum_{v_i \in M} u_i - \sum_{[v_i, v_j, v_k] \in M} E_{ijk}(u_i, u_j, u_k)$$

where E_{ijk} is the ricci energy defined on the face $[v_i, v_j, v_k]$. The linear term won't affect the convexity of the energy. The null space of the Hessian is $(1, 1, \dots, 1)$. In the subspace $\sum u_i = 0$, the energy is strictly convex.

$$\omega = \sum_{v_i \in M} K_i du_i, E(\mathbf{u}) = \int_0^{\mathbf{u}} \omega$$

Uniqueness

Lemma

Suppose $\Omega \subset \mathbb{R}^n$ is a convex domain, $f : \Omega \rightarrow \mathbb{R}$ is a strictly convex function, then the map

$$\mathbf{x} \rightarrow \nabla f(\mathbf{x})$$

is one-to-one.

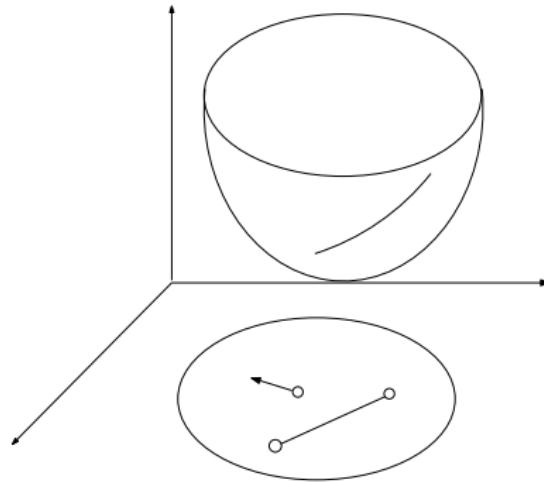
Proof.

Suppose $x_1 \neq x_2$, $\nabla f(x_1) = \nabla f(x_2)$. Because Ω is convex, the line segment $(1-t)x_1 + tx_2$ is contained in Ω . construct a convex function $g(t) = f((1-t)x_1 + tx_2)$, then $g'(t)$ is monotonous. But

$$g'(0) = \langle \nabla f(x_1), x_2 - x_1 \rangle = \langle \nabla f(x_2), x_2 - x_1 \rangle = g'(1),$$

contradiction. □

Uniqueness



Lemma

Suppose $\Omega \subset \mathbb{R}^n$ is a convex domain, $f : \Omega \rightarrow \mathbb{R}$ is a strictly convex function, then the map

$$\mathbf{x} \rightarrow \nabla f(\mathbf{x})$$

is one-to-one.

Uniqueness

Theorem

Suppose M is a mesh, with circle packing metric, all edge intersection angles are non-obtuse. Given the target curvature (K_1, K_2, \dots, K_n) , $\sum_i K_i = 2\pi\chi(M)$. If the solution $(u_1, u_2, \dots, u_n) \in \Omega(M)$, $\sum_i u_i = 0$ exists, then it is unique.

Proof.

The discrete Ricci energy E on $\Omega \cap \{\sum_i u_i = 0\}$ is convex,

$$\nabla E(u_1, u_2, \dots, u_n) = (K_1, K_2, \dots, K_n).$$

Use previous lemma. □