

# Fundamental Group and Covering Space

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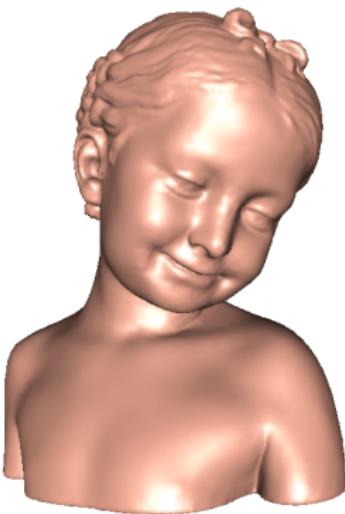
# Algebraic Topology: Fundamental Group

# Orientability-Möbius Band

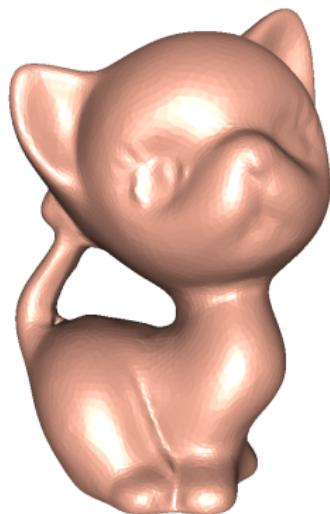


Figure: Escher. Ants

# Surface Genus



Topological Sphere



Topological Torus

**Figure:** How to differentiate the above two surfaces.

# Key Idea

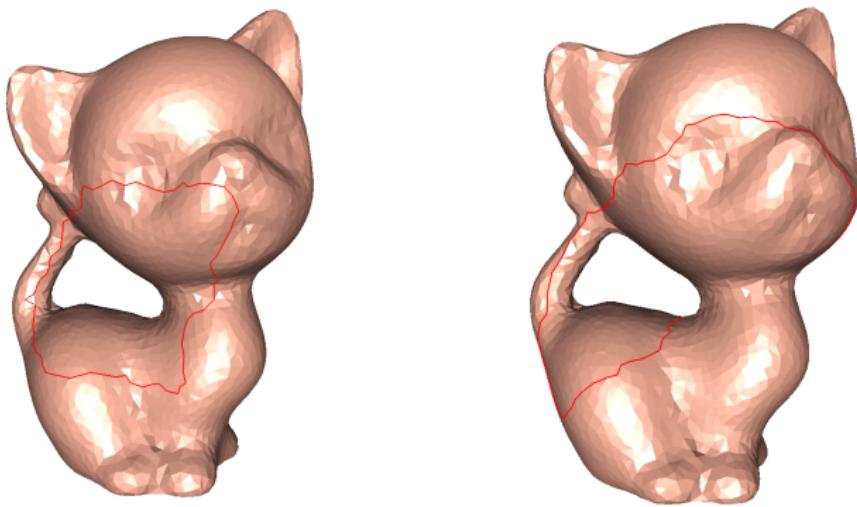


Figure: Check whether all loops on the surface can shrink to a point.

All oriented compact surfaces can be classified by their genus  $g$  and number of boundaries  $b$ . Therefore, we use  $(g, b)$  to represent the topological type of an oriented surface  $S$ .

# Application

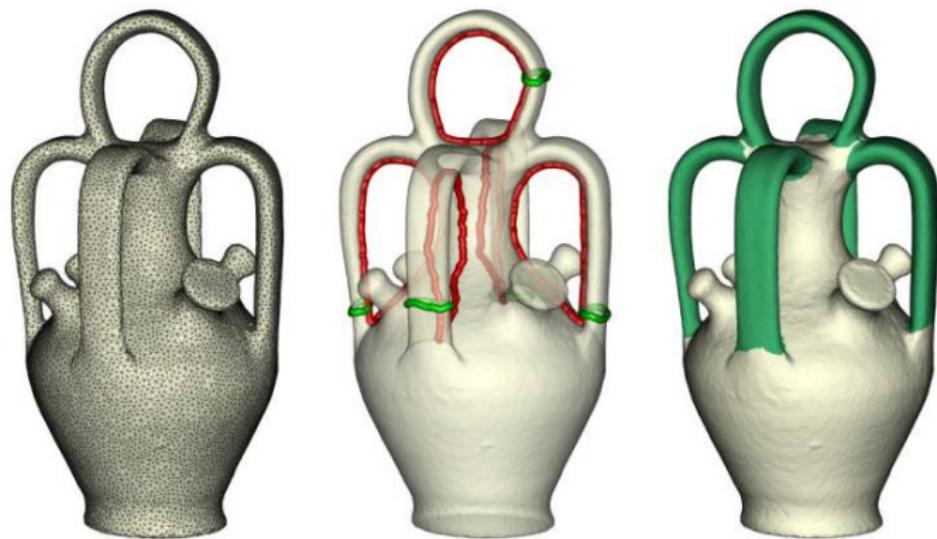


Figure: Handle detection by finding the handle loops and the tunnel loops.

# Application

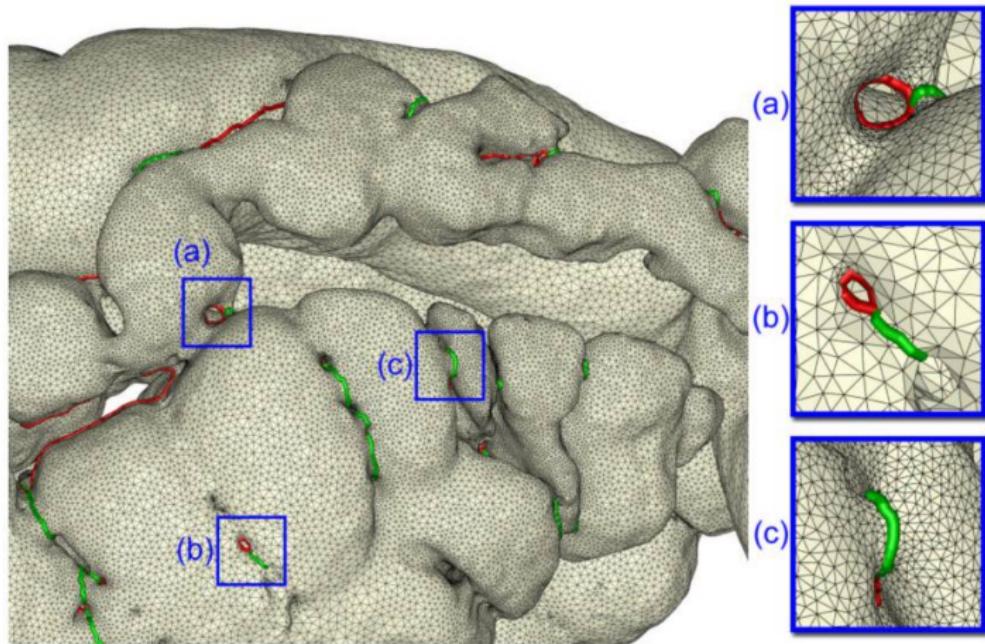


Figure: Topological Denoise in medical imaging.

# Surface Topology

## Philosophy

Associate groups with manifolds, study the topology by analyzing the group structures.

$$\begin{aligned}\mathfrak{C}_1 &= \{Topological\ Spaces, Homeomorphisms\} \\ \mathfrak{C}_2 &= \{Groups, Homomorphisms\} \\ \mathfrak{C}_1 &\rightarrow \mathfrak{C}_2\end{aligned}$$

Functor between categories.

# Fundamental group

Suppose  $q$  is a base point, all the oriented closed curves (loops) through  $q$  can be classified by homotopy. All the homotopy classes form the so-called *fundamental group* of  $S$ , or *the first homotopy group*, denoted as  $\pi_1(S, q)$ . The group structure of  $\pi_1(S, q)$  determines the topology of  $S$ .

# Homotopy

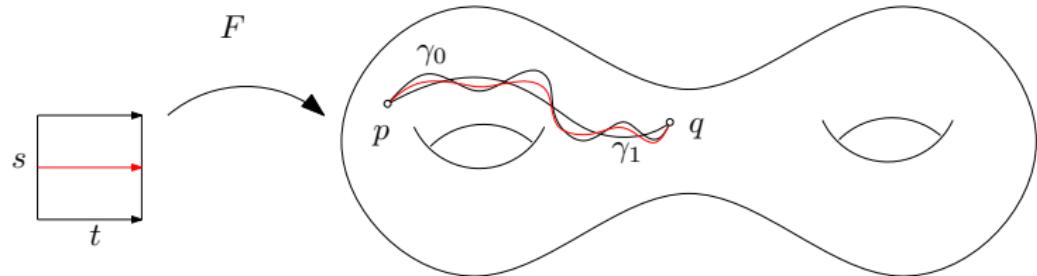


Figure: Path homotopy.

# Homotopy

Let  $S$  be a two manifold with a base point  $p \in S$ ,

## Definition (Curve)

A curve is a continuous mapping  $\gamma : [0, 1] \rightarrow S$ .

## Definition (Loop)

A closed curve through  $p$  is a curve, such that  $\gamma(0) = \gamma(1) = p$ .

## Definition (Homotopy)

Let  $\gamma_1, \gamma_2 : [0, 1] \rightarrow S$  be two curves. A homotopy connecting  $\gamma_1$  and  $\gamma_2$  is a continuous mapping  $F : [0, 1] \times [0, 1] \rightarrow S$ , such that

$$f(0, t) = \gamma_1(t), f(1, t) = \gamma_2(t).$$

We say  $\gamma_1$  is homotopic to  $\gamma_2$  if there exists a homotopy between them.

# Homotopy

## Lemma

*Homotopy relation is an equivalence relation.*

## Proof.

$\gamma \sim \gamma$ ,  $F(s, t) = \gamma(t)$ . If  $\gamma_1 \sim \gamma_2$ ,  $F(s, t)$  is the homotopy, then  $F(1 - s, t)$  is the homotopy from  $\gamma_2$  to  $\gamma_1$ .



## Corollary

*All the loops through the base point can be classified by homotopy relation. The homotopy class of a loops  $\gamma$  is denoted as  $[\gamma]$ .*

# Fundamental Group

## Definition (Loop product)

Suppose  $\gamma_1, \gamma_2$  are two loops through the base point  $p$ , the product of the two loops is defined as

$$\gamma_1 \cdot \gamma_2(t) = \begin{cases} \gamma_1(2t) & 0 \leq t \leq \frac{1}{2} \\ \gamma_2(2t - 1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

## Definition (Loop inverse)

$$\gamma^{-1}(t) = \gamma(1 - t).$$

# Loop Inversion

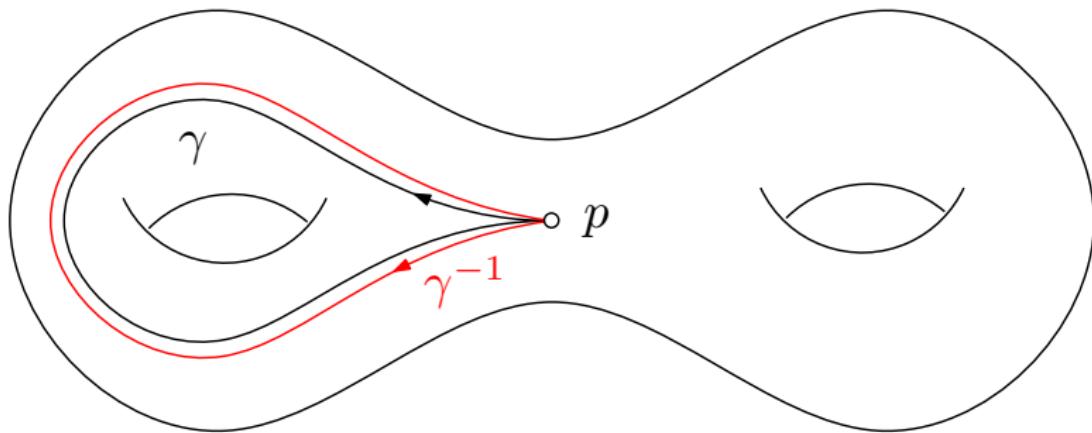


Figure: Loop inversion

# Loop Product

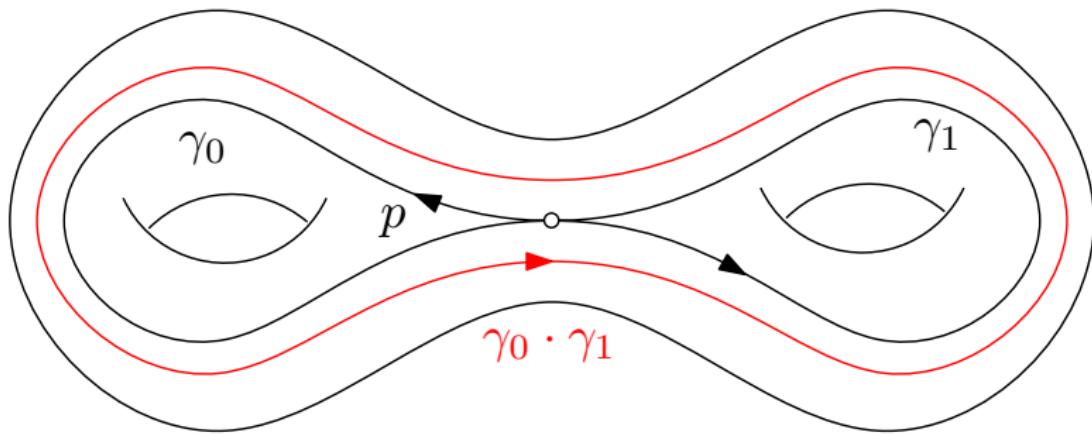


Figure: Loop product

# Fundamental Group

## Definition (Fundamental Group)

Given a topological space  $S$ , fix a base point  $p \in S$ , the set of all the loops through  $p$  is  $\Gamma$ , the set of all the homotopy classes is  $\Gamma / \sim$ . The product is defined as:

$$[\gamma_1] \cdot [\gamma_2] := [\gamma_1 \cdot \gamma_2],$$

the unit element is defined as  $[e]$ , the inverse element is defined as

$$[\gamma]^{-1} := [\gamma^{-1}],$$

then  $\Gamma / \sim$  forms a group, the fundamental group of  $S$ , and is denoted as  $\pi_1(S, p)$ .

# Fundamental Group Representation

Let  $G = \{g_1, g_2, \dots, g_n\}$  be  $n$  symbols, a word generated by  $G$  is a sequence

$$w = g_{i_1}^{e_1} g_{i_2}^{e_2} \cdots g_{i_k}^{e_k}, g_{i_j} \in G, e_j \in \mathbb{Z}.$$

- The empty word  $\emptyset$  is also treated as the unit element.
- Given two words  $w_1 = \alpha_1 \cdots \alpha_{n_1}$  and  $w_2 = \beta_1 \cdots \beta_{n_2}$ , the product is defined as concatenation:

$$w_1 \cdot w_2 = \alpha_1 \cdots \alpha_{n_1} \beta_1 \cdots \beta_{n_2}.$$

- The inverse of a work is defined as

$$(g_{i_1}^{e_1} g_{i_2}^{e_2} \cdots g_{i_k}^{e_k})^{-1} = g_{i_k}^{-e_k} g_{i_{k-1}}^{-e_{k-1}} \cdots g_{i_1}^{-e_1}.$$

All words form a group, freely generated by  $G$ ,

$$\langle g_1, g_2, \dots, g_n \rangle.$$

# Word Group

The relations  $R = \{R_1, R_2, \dots, R_m\}$  are  $m$  words, such that we can replace  $R_k$  by the empty word.

## Definition (word equivalence relation)

Two words are equivalent if we can transform one to the other by finite many steps of the following two elementary transformations:

- ① Insert a relation word anywhere.

$$\alpha_1 \cdots \alpha_i \alpha_{i+1} \cdots \alpha_l \mapsto \alpha_1 \cdots \alpha_i R_k \alpha_{i+1} \cdots \alpha_l$$

- ② If a subword is a relation word, remove it from the word.

$$\alpha_1 \cdots \alpha_i R_k \alpha_{i+1} \cdots \alpha_l \mapsto \alpha_1 \cdots \alpha_i \alpha_{i+1} \cdots \alpha_l.$$

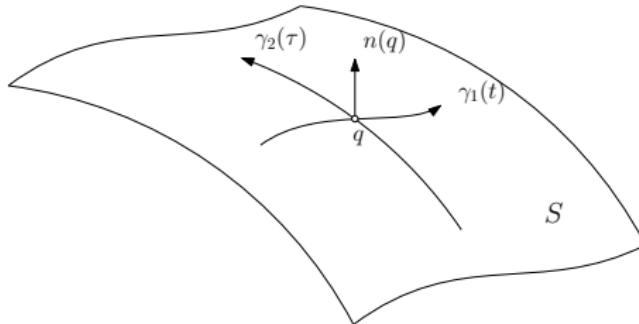
## Definition (Word Group)

Given a set of generators  $G$  and a set of relations  $R$ , all the equivalence classes of the words generated by  $G$  form a group under the concatenation, denoted as

$$\langle g_1, g_2, \dots, g_n | R_1, R_2, \dots, R_m \rangle.$$

If there is no relations, then the word group is called a free group.

# Intersection Index



## Definition (Intersection Index)

Suppose  $\gamma_1(t), \gamma_2(\tau) \subset S$  intersect at  $q \in S$ , the tangent vectors satisfy

$$\frac{d\gamma_1(t)}{dt} \times \frac{d\gamma_2(\tau)}{d\tau} \cdot \mathbf{n}(q) > 0,$$

then the index of the intersection point  $q$  of  $\gamma_1$  and  $\gamma_2$  is  $+1$ , denoted as  $\text{Ind}(\gamma_1, \gamma_2, q) = +1$ . If the mixed product is zero or negative, then the index is  $0$  or  $-1$ .

# Algebraic Intersection Number

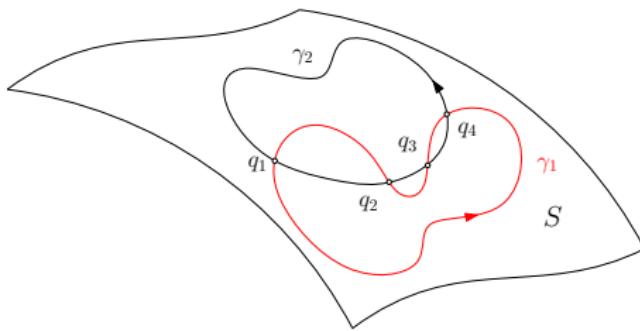


Figure: Algebraic intersection number

## Definition (Algebraic Intersection Number)

The algebraic intersection number of  $\gamma_1(t), \gamma_2(\tau) \subset S$  is defined as

$$\gamma_1 \cdot \gamma_2 := \sum_{q_i \in \gamma_1 \cap \gamma_2} \text{Ind}(\gamma_1, \gamma_2, q_i).$$

# Algebraic Intersection Number

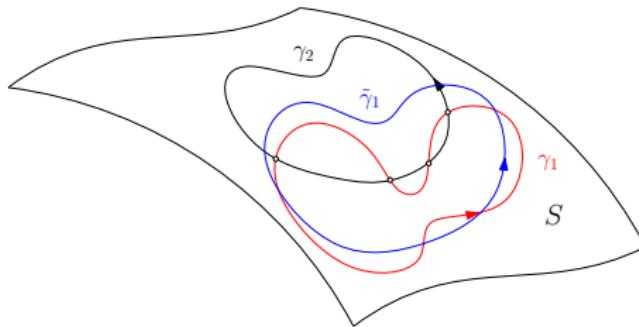


Figure: Algebraic intersection number

## Algebraic Intersection Number Homotopy Invariance

Suppose  $\gamma_1$  is homotopic to  $\tilde{\gamma}_1$ , then the algebraic intersection number

$$\gamma_1 \cdot \gamma_2 = \tilde{\gamma}_1 \cdot \gamma_2.$$

# Canonical Representation of $\pi_1(S, p)$

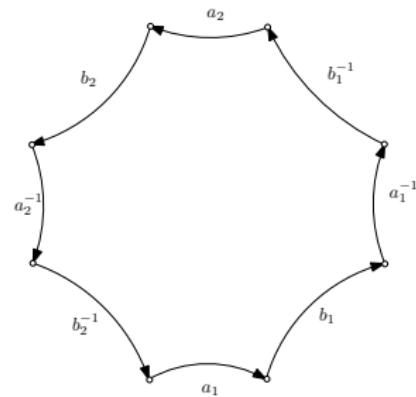
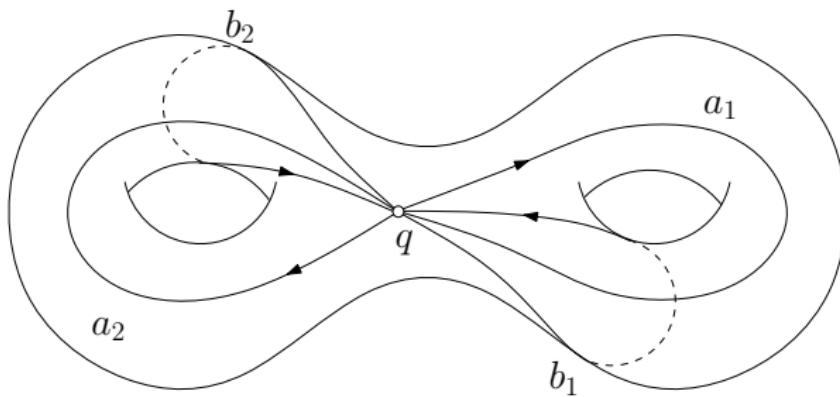


Figure: Canonical fundamental group representation.

# Canonical Representation of $\pi_1(S, p)$

## Definition (Canonical Basis)

Suppose  $S$  is a compact, oriented surface, there exists a set of generators of the fundamental group  $\pi_1(S, p)$ ,

$$G = \{[a_1], [b_1], [a_2], [b_2], \dots, [a_g], [b_g]\}$$

such that

$$a_i \cdot b_j = \delta_i^j, a_i \cdot a_j = 0, b_i \cdot b_j = 0,$$

where  $a_i \cdot b_j$  represents the algebraic intersection number of loops  $a_i$  and  $b_j$ ,  $\delta_{ij}$  is the Kronecker symbol, then  $G$  is called a set of canonical basis of  $\pi_1(S, p)$ .

# Canonical Representation of $\pi_1(S, p)$

Theorem (Surface Fundamental Group Canonical Representation)

Suppose  $S$  is a compact, oriented surface,  $p \in S$  is a fixed point, the fundamental group has a canonical representation,

$$\pi_1(S, p) = \langle a_1, b_1, a_2, b_2, \dots, a_g, b_g \mid \prod_{i=1}^g [a_i, b_i] \rangle,$$

where

$$[a_i, b_i] = a_i b_i a_i^{-1} b_i^{-1},$$

$g$  is the genus of the surface.

# Canonical Representation of $\pi_1(S, p)$

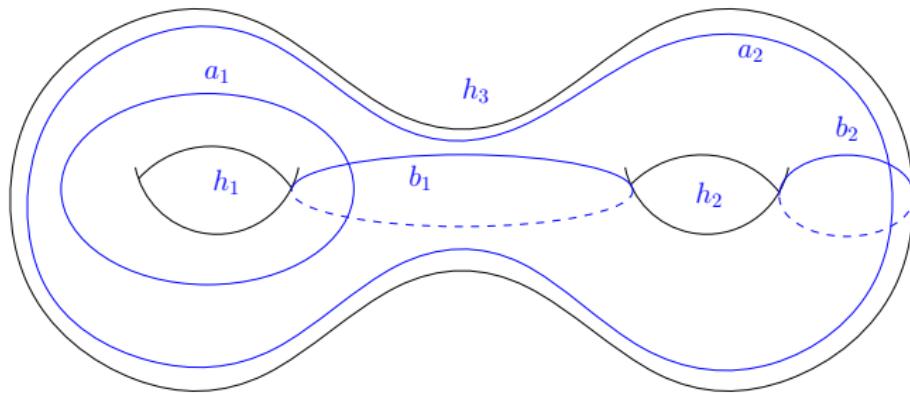


Figure: Canonical representation of  $\pi_1(S)$ .

## Non-uniqueness

The canonical representation of the fundamental group of the surface is not unique. It is NP hard to verify if two given representations are isomorphic.

# Fundamental Group Representation

## Theorem

*Suppose  $\pi_1(S_1, p_1)$  is isomorphic to  $\pi_2(S_2, p_2)$ , then  $S_1$  is homeomorphic to  $S_2$ , and vice versa.*

## Proof.

For each surface, find a canonical basis, slice the surface along the basis to get a  $4g$  polygonal scheme, then construct a homeomorphism between the polygonal schema with consistent boundary condition. □

# Seifert-Van Kampen Theorem

## Theorem (Seifert-Van Kampen)

*Topological space  $M$  is decomposed into the union of  $U$  and  $V$ , the intersection of  $U$  and  $V$  is  $W$ ,  $M = U \cup V$ ,  $W = U \cap V$ , where  $U, V$  and  $W$  are path connected.  $i : W \rightarrow U, j : W \rightarrow V$  are the inclusions. Pick a base point  $p \in W$ , the fundamental groups*

$$\pi_1(U, p) = \langle u_1, \dots, u_k | \alpha_1, \dots, \alpha_l \rangle$$

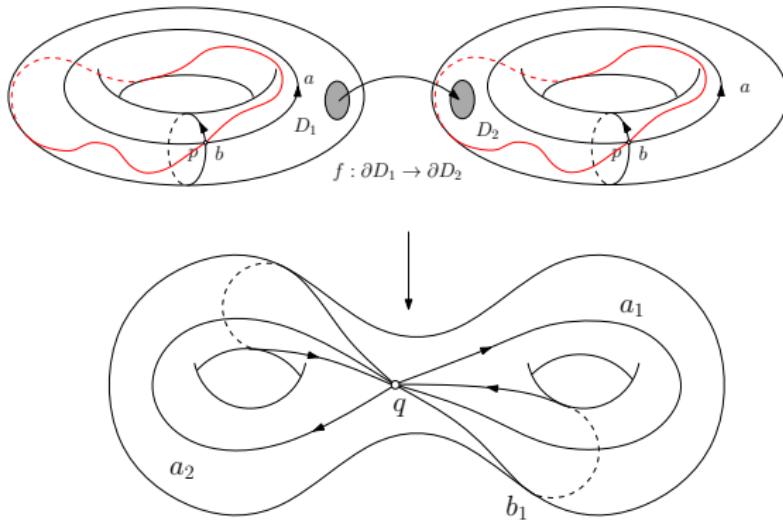
$$\pi_1(V, p) = \langle v_1, \dots, v_m | \beta_1, \dots, \beta_n \rangle$$

$$\pi_1(W, p) = \langle w_1, \dots, w_p | \gamma_1, \dots, \gamma_q \rangle$$

*then the  $\pi_1(M, p)$  is given by*

$$\pi_1(M, p) = \langle u_1, \dots, u_k, v_1, \dots, v_m | \alpha_i, \beta_j, i(w_1)j(w_1)^{-1}, \dots, i(w_p)j(w_p)^{-1} \rangle$$

# Canonical Representation of $\pi_1(S, p)$



## Definition (Connected Sum)

Let  $S_1$  and  $S_2$  be two surfaces,  $D_1 \subset S_1$  and  $D_2 \subset S_2$  are two topological disks.  $f : \partial D_1 \rightarrow \partial D_2$  is a homeomorphism between the boundaries of the disks. The connected sum is  $S_1 \oplus S_2 := S_1 \cup S_2 / \{p \sim f(p)\}$ .

# Surface Topology

## Theorem (Surface Topological Classification)

*All the compact closed surfaces can be represented as*

$$S \cong T^2 \oplus T^2 \oplus \cdots \oplus T^2$$

*for oriented surfaces, or*

$$S \cong RP^2 \oplus RP^2 \oplus \cdots \oplus RP^2.$$

$RP^2$  is gluing a Möbius band with a disk along its single boundary.

# Canonical Representation of $\pi_1(S, p)$

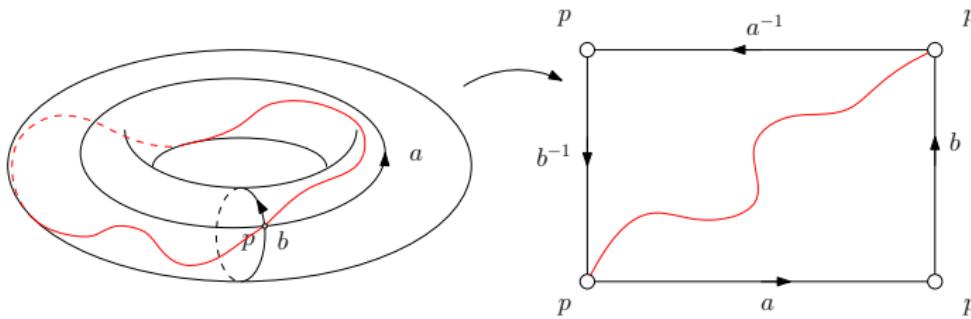


Figure:  $\pi_1(T, p) = \langle a, b | aba^{-1}b^{-1} \rangle$ .

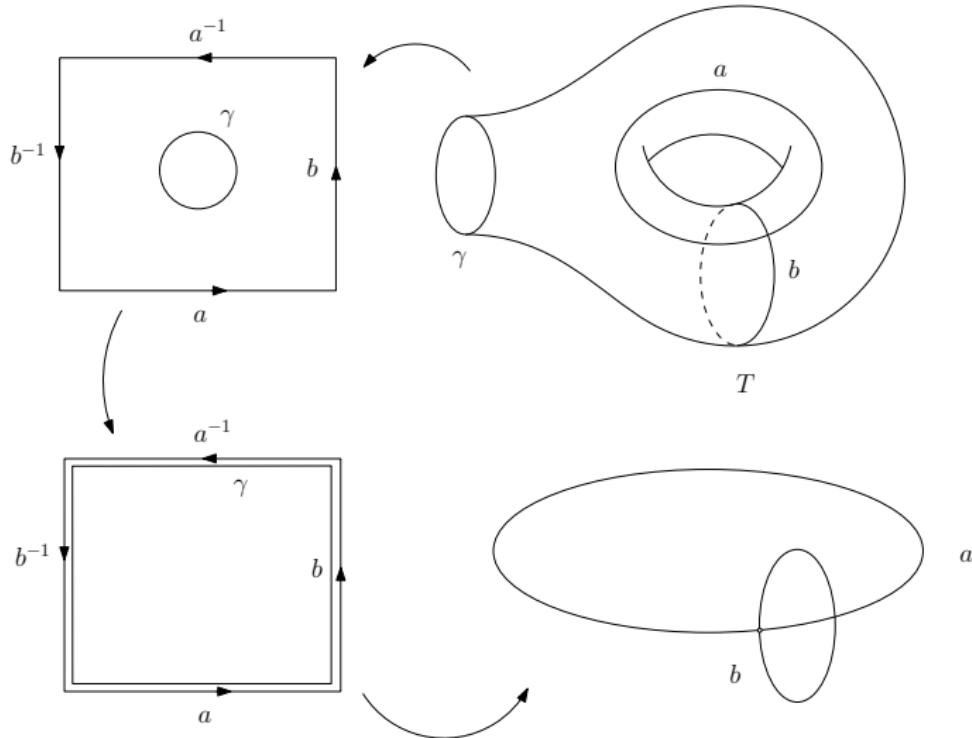
## Lemma

The fundamental group of a torus is  $\pi_1(T, p) = \langle a, b | aba^{-1}b^{-1} \rangle$ .

## Proof.

Homotopic deform a loop  $\gamma$ , such that  $\gamma$  intersects  $a$  and  $b$  only at  $p$ ; decompose  $\gamma$  to  $\gamma_1\gamma_2\dots\gamma_k$ , such that  $\gamma_i$  starts and ends at  $p$ , the interior doesn't intersect  $a$  and  $b$ ; each  $\gamma_i$  is generated by  $a, b$ . □

# Canonical Representation of $\pi_1(S, p)$



**Figure:** Punctured torus, fundamental group  $\pi_1(T \setminus \{q\}, p) = \langle a, b \rangle$ .

# Canonical Representation of $\pi_1(S, p)$

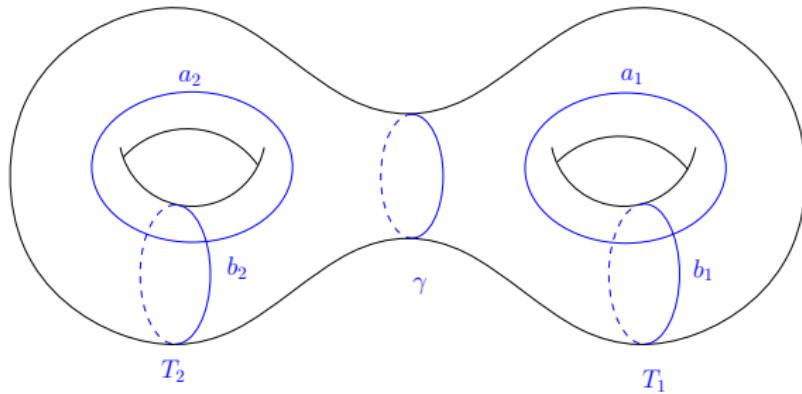


Figure: Divide conquer method.

## Fundamental Groups

$$\pi_1(T_1, p) = \langle a_1, b_1 \rangle, \quad \pi_1(T_2, p) = \langle a_2, b_2 \rangle, \quad \pi_1(T_1 \cap T_2, p) = \langle \gamma \rangle$$

# Canonical Representation of Fundamental Group

## Theorem

Show that  $\pi_1(S)$  is  $\langle a_1, b_1, \dots, a_g, b_g | \prod_{i=1}^g [a_i, b_i] \rangle$  for a surface  $S = \bigoplus_{i=1}^g T^2$ .

## Proof.

By induction. If  $g = 1$ , obvious. Let  $g = 2$ ,

$$\begin{aligned}\pi_1(T_1) &= \langle a_1, b_1 \rangle \\ \pi_1(T_2) &= \langle a_2, b_2 \rangle \\ \pi_1(T_1 \cap T_2) &= \langle \gamma \rangle\end{aligned}$$

$[\gamma] = a_1 b_1 a_1^{-1} b_1^{-1}$  in  $\pi_1(T_1)$ ,  $[\gamma] = a_2 b_2 a_2^{-1} b_2^{-1}$  in  $\pi_1(T_2)$ , so

$$\pi_1(T_1 \cup T_2) = \langle a_1, b_1, a_2, b_2 | [a_1, b_1][a_2, b_2]^{-1} \rangle.$$

where  $[a_k, b_k] = a_k b_k a_k^{-1} b_k^{-1}$ . □

# Canonical Representation of Fundamental Group

continued.

Suppose it is true for  $g - 1$  case. Then for  $g$  case, the intersection is an annulus,

$$\begin{aligned}\pi_1(T_1 \cup T_2 \dots T_{g-1}) &= \langle a_1, b_1, \dots a_{g-1}, b_{g-1} | \prod_{k=1}^{g-1} [a_k, b_k] \rangle \\ \pi_1(T_g) &= \langle a_g, b_g | [a_g, b_g] \rangle \\ \pi_1(S \cap T_g) &= \langle \gamma \rangle\end{aligned}$$

$[\gamma] = \pi_{k=1}^{g-1} [a_k, b_k]$  in  $\pi_1(T_1 \cup T_2 \dots T_{g-1})$  and  $[a_g, b_g] \in \pi_1(T_g)$ . □

# Computational Topology: Fundamental Group

# Cut Graph

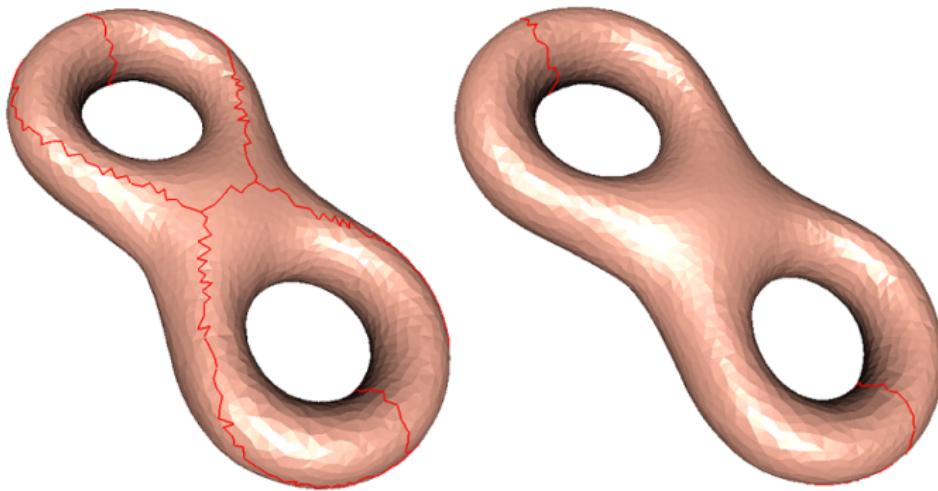


Figure: Cut graph of a genus two surface.

## Definition (Cut Graph)

$\Gamma$  is a graph on the surface  $S$ , such that  $S \setminus \Gamma$  is a topological disk, then  $\Gamma$  is a cut graph of  $S$ .

# Algorithm for Cut Graph

## Cut Graph Algorithm

Input : A closed triangle mesh  $M$ ;

Output: A cut graph  $\Gamma$  of  $M$ .

- ① Compute the dual mesh  $\bar{M}$  of the input mesh  $M$ ;
- ② Compute a spanning tree  $\bar{T}$  of  $\bar{M}$ ;
- ③ The cut graph is given by

$$\Gamma := \{e \in M \mid \bar{e} \notin \bar{T}\}.$$

# Fundamental Group Generators

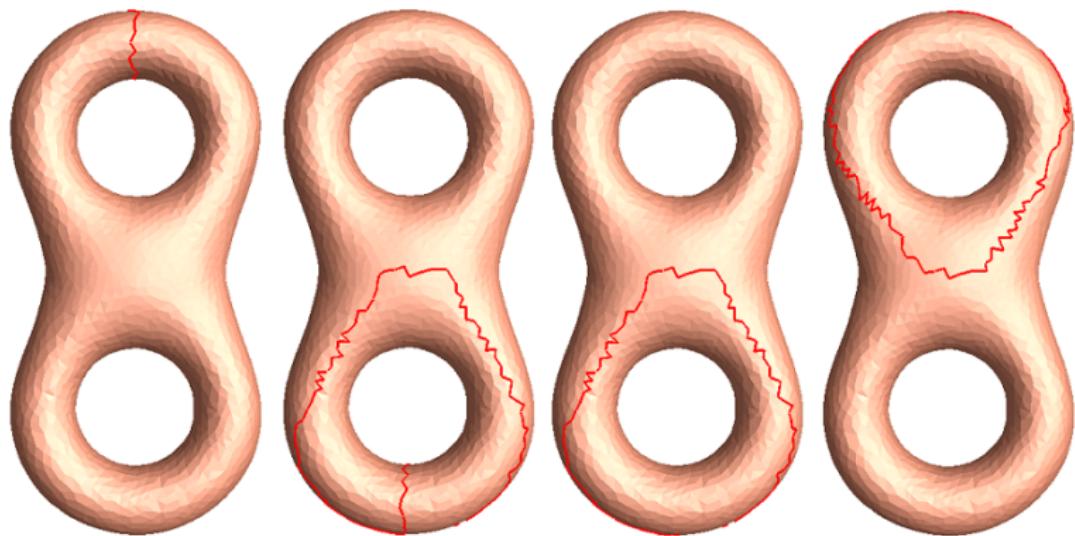


Figure: Fundamental group generators of a genus two surface.

# Algorithm for Fundamental Group Generators

## Fundamental Group Generators Algorithm

Input : A closed triangle mesh  $M$ ;

Output: A set of generators of  $\pi_1(M, p)$ .

- ① Compute a cut graph  $\Gamma$  of the input mesh  $M$ ;
- ② Compute a spanning tree  $T$  of  $\Gamma$ ;
- ③ Select an edge  $e_i \in \Gamma \setminus T$ ,  $e_i \cup T$  has a unique loop  $\gamma_i$ ;
- ④  $\{\gamma_1, \gamma_2, \dots, \gamma_k\}$  is a set of generators of the fundamental group of  $M$ .

# Algorithm for Fundamental Group Relations

## Fundamental Group Relations Algorithm

Input : A closed triangle mesh  $M$ ;

Output: The relations in  $\pi_1(M, p)$ .

- ① Compute a cut graph  $\Gamma$  of the input mesh  $M$ ;
- ② Compute a spanning tree  $T$  of  $\Gamma$ ,  $\Gamma \setminus T = \{e_1, e_2, \dots, e_k\}$ ;
- ③ For each oriented edge,  $e_i \cup T$  has an oriented loop  $\gamma_i$ ,  
 $\{\gamma_1, \gamma_2, \dots, \gamma_k\}$ ;
- ④ Cut the mesh  $M$  along  $\Gamma$  to obtain  $\bar{M}$ ;
- ⑤ Set Let  $\gamma = \partial \bar{M}$ , traverse  $\gamma$ . Set  $w = \emptyset$ , once  $e_i^{\pm 1}$  is encountered,  
append  $\gamma_i^{\pm 1}$  to  $w$ ,  $w \leftarrow w\gamma_i^{\pm 1}$ .

# Algebraic Topology: Universal Covering Space

# Universal Covering Space

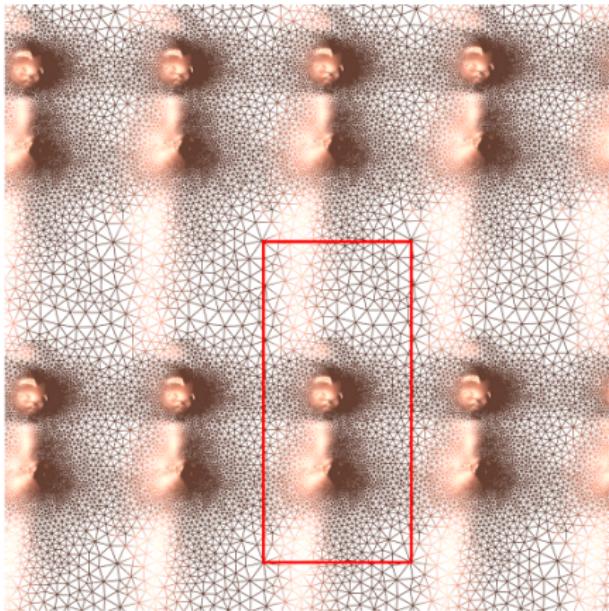
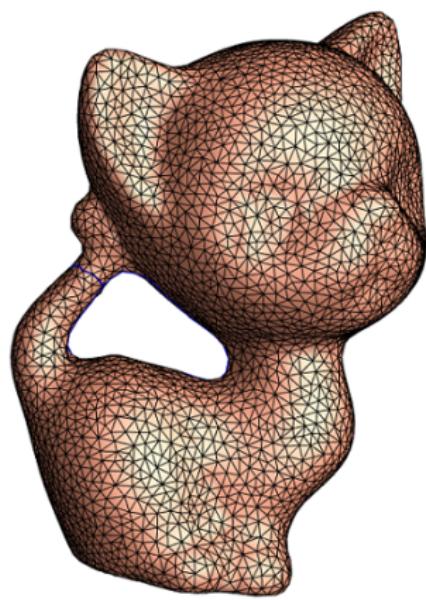


Figure: Universal Covering Space

# Covering Space

## Definition (Covering Space)

Given topological spaces  $\tilde{S}$  and  $S$ , a continuous map  $p : \tilde{S} \rightarrow S$  is surjective, such that for each point  $q \in S$ , there is a neighborhood  $U$  of  $q$ , its preimage  $p^{-1}(U) = \cup_i \tilde{U}_i$  is a disjoint union of open sets  $\tilde{U}_i$ , and the restriction of  $p$  on each  $\tilde{U}_i$  is a local homeomorphism, then  $(\tilde{S}, p)$  is a *covering space* of  $S$ ,  $p$  is called a *projection map*.

## Definition (Deck Transformation)

The automorphisms of  $\tilde{S}$ ,  $\tau : \tilde{S} \rightarrow \tilde{S}$ , are called *deck transformations*, if they satisfy  $p \circ \tau = p$ . All the deck transformations form a group, the *covering group*, and denoted as  $\text{Deck}(\tilde{S})$ .

# Covering Group

Suppose  $\tilde{q} \in \tilde{S}$ ,  $p(\tilde{q}) = q$ . The projection map  $p : \tilde{S} \rightarrow S$  induces a homomorphism between their fundamental groups,  
 $p_* : \pi_1(\tilde{S}, \tilde{q}) \rightarrow \pi_1(S, q)$ , if  $p_*\pi_1(\tilde{S}, \tilde{q})$  is a normal subgroup of  $\pi_1(S, q)$  then

## Theorem (Covering Group Structure)

*The quotient group of  $\frac{\pi_1(S)}{p_*\pi_1(\tilde{S}, \tilde{q})}$  is isomorphic to the deck transformation group of  $\tilde{S}$ .*

$$\frac{\pi_1(S, q)}{p_*\pi_1(\tilde{S}, \tilde{q})} \cong \text{Deck}(\tilde{S}).$$

# universal covering space

## Definition (Universal Covering Space)

If a covering space  $\tilde{S}$  is simply connected (i.e.  $\pi_1(\tilde{S}) = \{e\}$ ), then  $\tilde{S}$  is called a *universal covering space* of  $S$ .

For universal covering space

$$\pi_1(\pi) \cong \text{Deck}(\tilde{S}).$$

Namely, the fundamental group of the base space is isomorphic to the deck transformation group of the universal covering space.

# Universal Covering Space

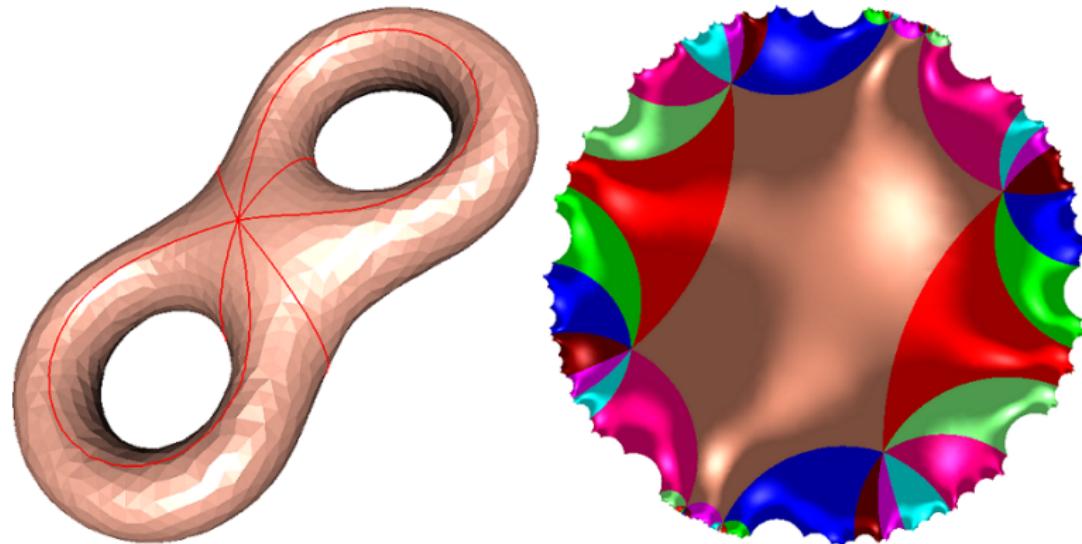


Figure: Universal Covering Space of a genus two surface.

# Universal Covering Space

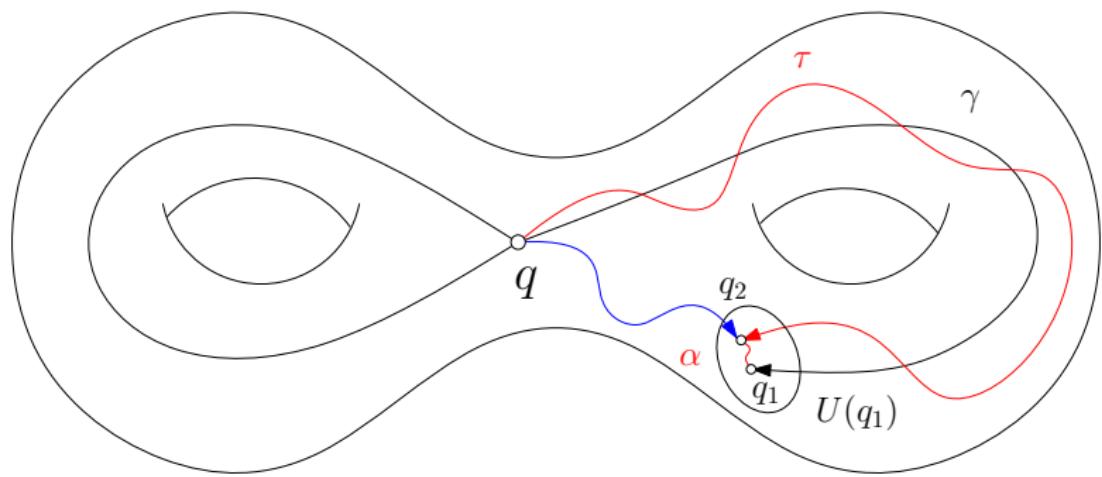


Figure: Universal Covering Space Construction.

Path homotopy classes form the universal covering space.

# Universal Covering Space

## Theorem

Suppose the topological manifold is path connected, then there is a universal covering space  $p : \tilde{S} \rightarrow S$ .

## Proof.

Fix a base point  $q \in S$ , consider all the paths starting from  $q$ ,  
 $\Gamma := \{\gamma : [0, 1] \rightarrow S | \gamma(0) = q\}$ . Define  $\tilde{S} := \Gamma / \sim$ , the homotopy classes  
of paths in  $\Gamma$ . Pick a path  $\gamma \in \Gamma$ ,  $\gamma(1) = q_0$ , let  $U \subset S$  be an open set of  
 $q_1$ . For each point  $q' \in U$ , there is a path  $\alpha(q') \subset U$  connecting  $q'$  to  $q_0$ .  
Then we define an open set  $\tilde{U} \subset \tilde{S}$  of  $[\gamma]$  as

$$\tilde{U} := \{[\tau] | \tau(1) \in U, \tau \cdot \alpha(\tau(1)) \sim \gamma\}.$$

The  $\{\tilde{U}\}$  define a topology of  $\tilde{\Gamma}$ .  $p : \tilde{\Gamma} \rightarrow S$ ,  $[\gamma] \mapsto \gamma(1)$  is a universal  
covering space of  $S$ . □

# Lifting to Universal Covering Space

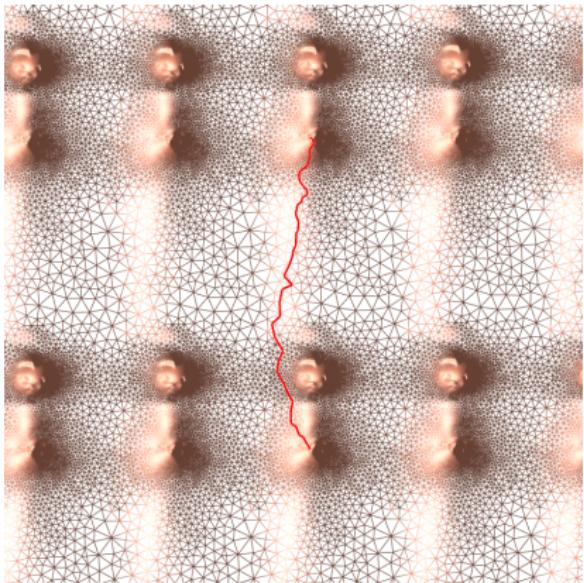
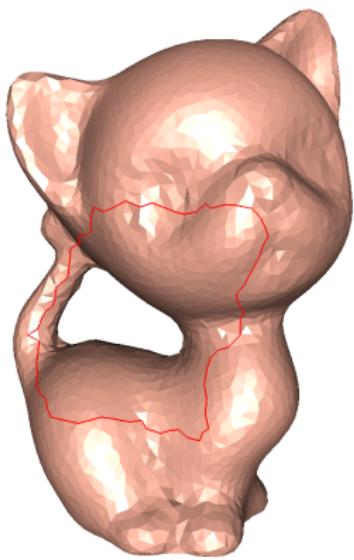


Figure: Universal Covering Space

# Lifting to Universal Covering Space

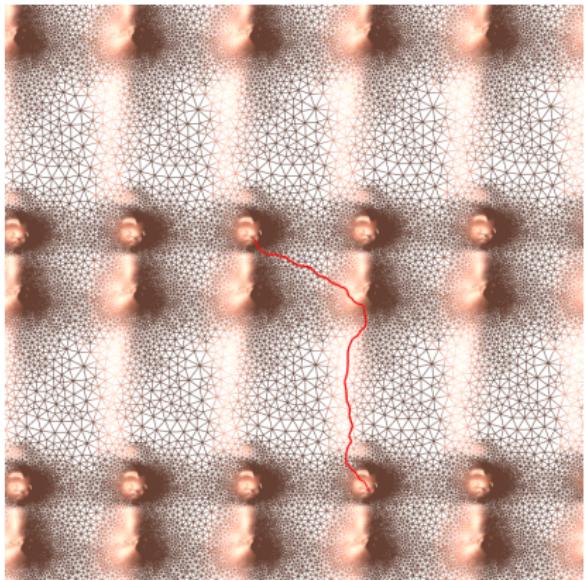
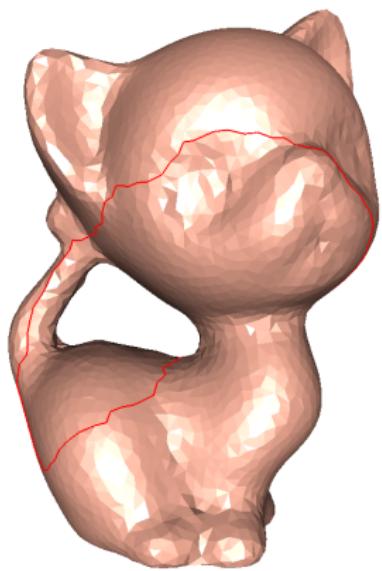


Figure: Universal Covering Space

# Lifting to Universal Covering Space

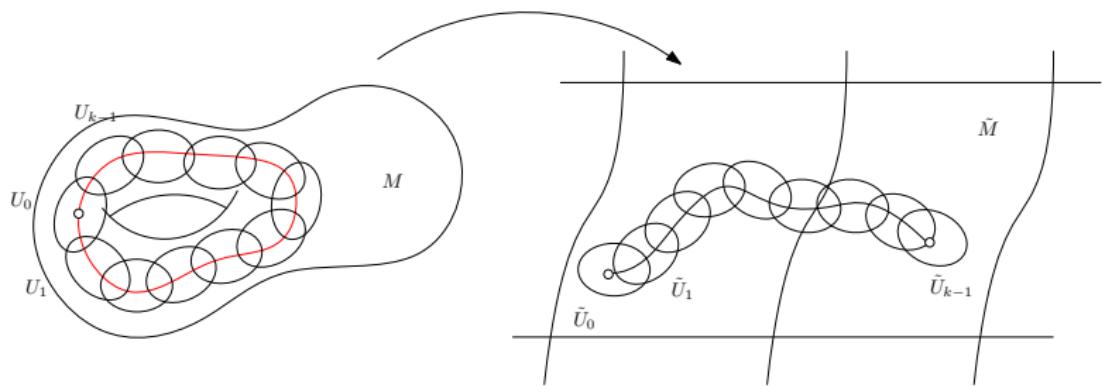


Figure: Lifting to the Universal Covering Space

# Lifting to Universal Covering Space

Let  $(\tilde{S}, p)$  be the universal covering space of  $S$ ,  $q$  be the base point. The orbit of base is  $p^{-1}(q) = \{\tilde{q}_k\}$ . Given a loop through  $q$ , there exists a unique lift of  $\gamma$ ,  $\tilde{\gamma} \subset \tilde{S}$ , starting from  $\tilde{q}_0$ .

## Lemma

$\gamma_1$  and  $\gamma_2$  are two loops through the base point, their lifts are  $\tilde{\gamma}_1$  and  $\tilde{\gamma}_2$ .  
 $\gamma_1 \sim \gamma_2$  if and only if the end points of  $\tilde{\gamma}_1$  and  $\tilde{\gamma}_2$  coincide.

$$\begin{array}{ccc}[0,1] & \xrightarrow{\tilde{\gamma}} & \tilde{M} \\ \downarrow id & & \downarrow p \\ [0,1] & \xrightarrow{\gamma} & M\end{array}$$

# Graph fundamental group

Let  $G$  be an unoriented graph,  $T$  is a spanning tree of  $G$ ,  
 $G - T = \{e_1, e_2, \dots, e_n\}$ , where  $e_k$  is an edge not in the tree. Then  
 $T \cup e_k$  has a unique loop  $\gamma_k$ . Choose one orientation of  $\gamma_k$ .

## Lemma

*The fundamental group of  $G$  is  $\pi_1(G) = \langle \gamma_1, \gamma_2 \dots, \gamma_n \rangle$ , which is a free group.*

# CW-cell decomposition

## Definition (CW-cell decomposition)

A  $k$  dimensional cell  $D_k$  is a  $k$  dimensional topological disk. Suppose  $M$  is a  $n$ -dimensional manifold.

- ① 0-skeleton  $S_0$  is the union of a set of 0-cells.
- ②  $k$ -skeleton  $S_k$

$$S_k = S_{k-1} \cup D_k^1 \cup D_k^2 \cdots \cup D_k^{n_k},$$

such that

$$\partial D_k^i \subset S_{k-1}.$$

The  $k$ -skeleton is constructed by gluing  $k$ -cells to the  $k - 1$  skeleton, all the boundaries of the cells are in the  $k - 1$  skeleton.

- ③  $S_n = M$ .

# CW-cell decomposition

Theorem (CW-cell decomposition)

$$\pi_1(S_2) = \pi_1(S_3) \cdots \pi_1(S_n) = \pi_1(M)$$

Proof.

using induction.  $S_2 \cap D_3^1$  is  $\partial D_3^1$ , which is a topological sphere.

$\pi_1(D_3^1) = \langle e \rangle$ ,  $\pi_1(\mathbb{S}^2)$  is  $\langle e \rangle$ .



# Computational Topology: Universal Covering Space

# Algorithm for Universal Covering Space

## Universal Covering Space Algorithm

**Input :** A closed triangle mesh  $M$ ;

**Output:** A finite portion of the universal covering space  $\tilde{M}$ .

- ① Compute a cut graph  $\Gamma$  of  $M$ , divide  $\Gamma$  into nodes and oriented segments,  $\{s_1, s_2, \dots, s_k\}$ ;
- ② Slice  $M$  along  $\Gamma$  to obtain one fundamental domain  $\bar{M}$ ;
- ③ Initialize  $\tilde{M} \leftarrow \bar{M}$
- ④ Choose an oriented segment  $s_i$  on the boundary of  $\tilde{M}$ , glue a copy of  $\bar{M}$  with  $\tilde{M}$  along  $s_i$ ,

$$\tilde{M} \leftarrow \tilde{M} \cup_{\partial \tilde{M} \supset s_i \sim s_i^{-1} \subset \partial \bar{M}} \bar{M}$$

- ⑤ Trace the boundary of  $\tilde{M}$ , if there are two adjacent segments  $s_i, s_{i+1} \subset \partial \tilde{M}$ , such that  $s_i^{-1} = s_{i+1}$ , then glue them together;
- ⑥ Repeat step 4 and step 5, until  $\tilde{M}$  is large enough.

# Algorithm for Homotopy Detection

## Homotopy Detection Algorithm

**Input :** A closed triangle mesh  $M$ , two loops  $\gamma_1$  and  $\gamma_2$  through a base point  $p$ ;

**Output:** Verify whether  $\gamma_1 \sim \gamma_2$ .

- ① Compute a finite portion of the universal covering space  $\tilde{M}$  of  $M$ ;
- ② Lift  $\gamma_1 \cdot \gamma_2^{-1}$  to  $\tilde{M}$ , the lifted path is  $\tilde{\gamma}$ ;
- ③ If  $\tilde{\gamma}$  is a closed loop, then return Yes; otherwise, return No.