

# Circle Domain Mapping: Koebe's Theorem

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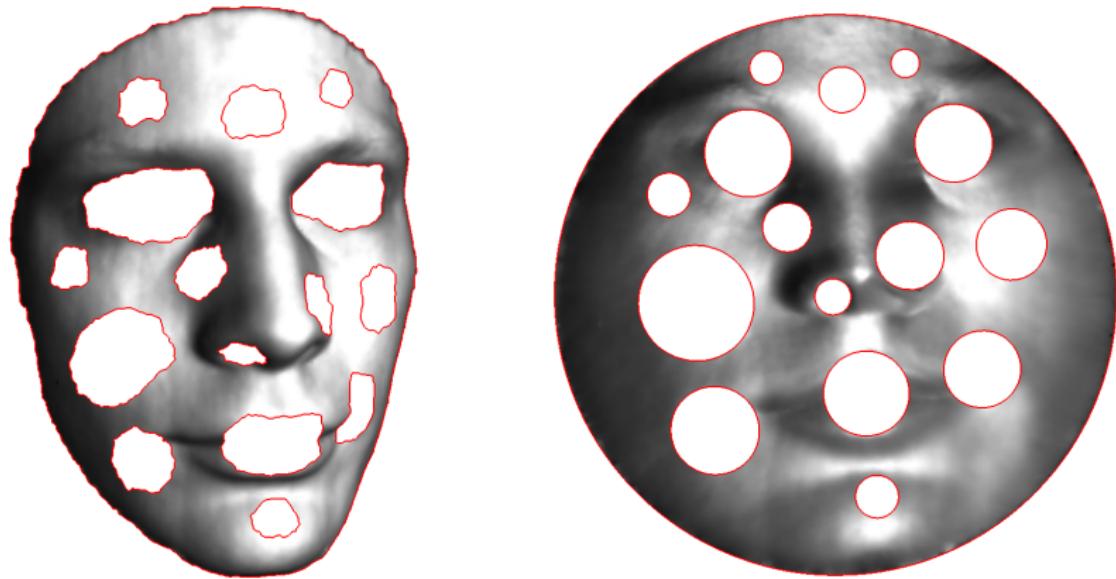
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# Motivation

# Conformal Module for Poly-annulus



**Figure:** Conformal mapping from a poly-annulus to a circle domain.

# Circle Domain

## Definition (Circle Domain)

Suppose  $\Omega \subset \hat{\mathbb{C}}$  is a planar domain, if  $\partial\Omega$  has finite number of connected components, each of them is either a circle or a point, then  $\Omega$  is called a circle domain.

## Theorem (Koebe)

*Suppose  $S$  is of genus zero,  $\partial S$  has finite number of connected components, then  $S$  is conformal equivalent to a circle domain. Furthermore, all such conformal mappings differ by a Möbius transformation.*

# Schwartz Reflection Principle

## Definition (Mirror Reflection)

Given a circle  $\Gamma : |z - z_0| = \rho$ , the reflection with respect to  $\Gamma$  is defined as:

$$\varphi_\Gamma : re^{i\theta} + z_0 \mapsto \frac{\rho^2}{r}e^{i\theta} + z_0. \quad (1)$$

Two planar domains  $S$  and  $S'$  are symmetric about  $\Gamma$ , if  $\varphi_\Gamma(S) = S'$ .

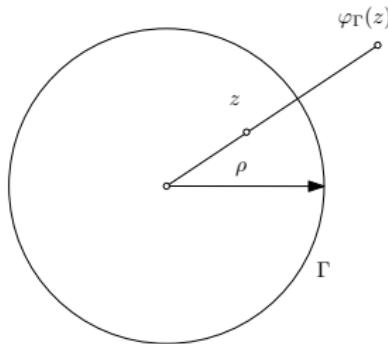


Figure: Reflection about a circle.

# Schwartz Reflection Principle

## Definition (Reflection)

Suppose  $\Gamma$  is an analytic curve, domain  $S, S'$  and  $\Gamma$  are included in a planar domain  $\Omega$ . There is a conformal map  $f : \Omega \rightarrow \hat{\mathbb{C}}$ , such that  $f(\Gamma)$  is a canonical circle,  $f(S)$  and  $f(S')$  are symmetric about  $f(\Gamma)$ , then we say  $S$  and  $S'$  are symmetric about  $\Gamma$ , and denoted as

$$S|S' \quad (\Gamma).$$

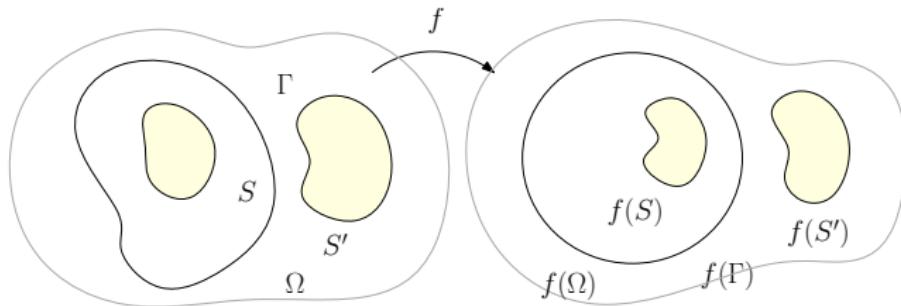


Figure: General symmetry.

# Schwartz Reflection Principle

## Theorem (Schwartz Reflection Principle)

Assume  $f$  is an analytic function, defined on the upper half disk  $\{|z| < 1, \Im(z) > 0\}$ . If  $f$  can be extended to a real continuous function on the real axis, then  $f$  can be extended to an analytic function  $F$  defined on the whole disk, satisfying

$$F(z) = \begin{cases} f(z), & \Im(z) \geq 0 \\ \overline{f(\bar{z})}, & \Im(z) < 0 \end{cases}$$

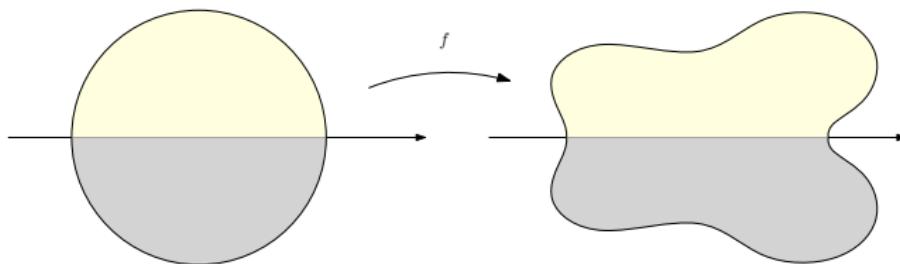
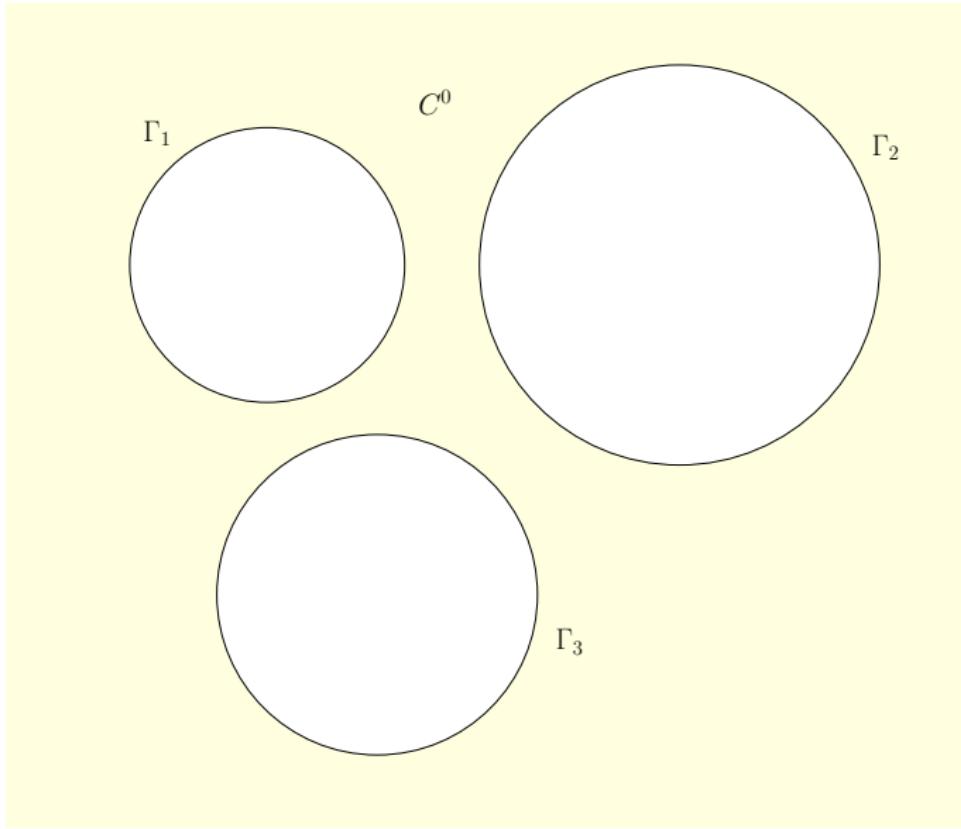
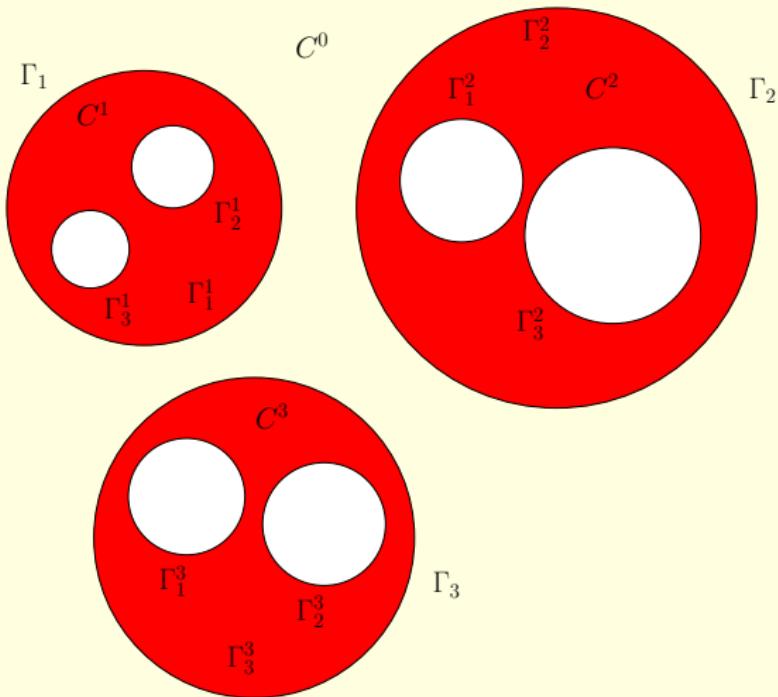


Figure: Schwartz reflection principle.

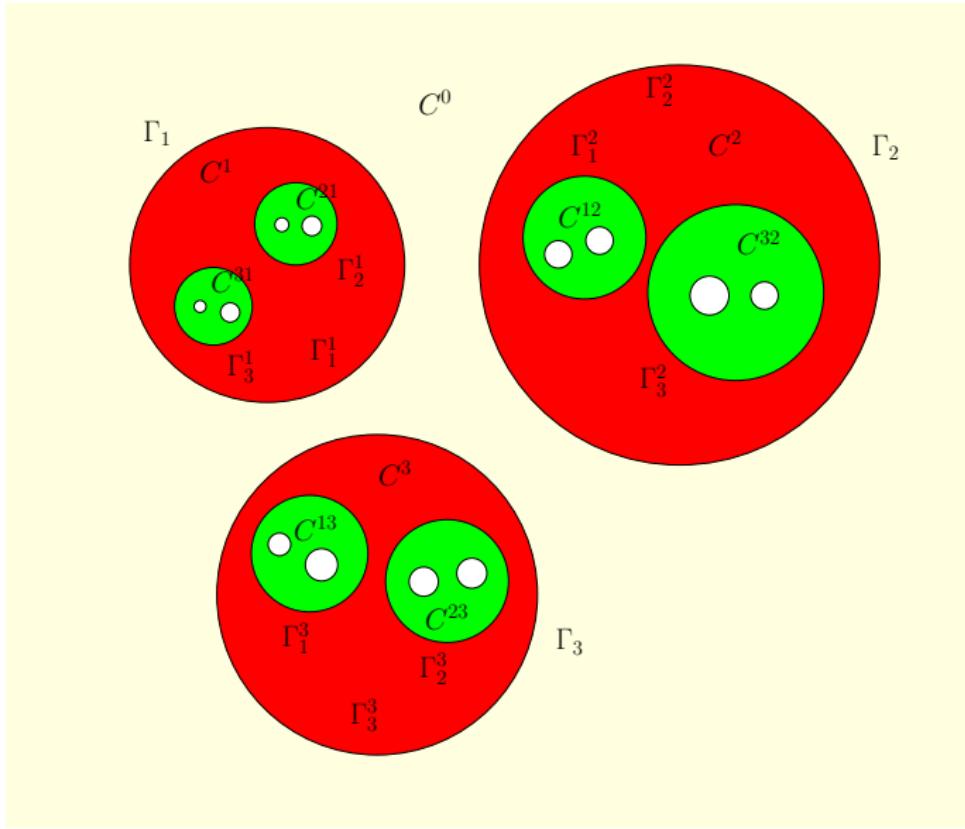
# Multiple Reflection



# Multiple Reflection



# Multiple Reflection



# Multiple Reflection

- ① Initial circle domain  $C^0$ : complex plane remove three disks, its boundary is  $\{\Gamma_1, \Gamma_2, \Gamma_3\}$ ;
- ② First level reflection:  $C^0$  is reflected about  $\Gamma_{i_1}$  to  $C^{i_1}$ ,  $i_1 = 1, 2, 3$ ;

$$\partial C^{i_1} = \Gamma_{i_1}^{i_1} - \sum_{j \neq i_1} \Gamma_j^{i_1},$$

where  $\Gamma_{i_1}^{i_1} = \Gamma_{i_1}$ .

- ③ Second level reflection:  $C^{i_1}$  is reflected about  $\Gamma_{i_2}$  to  $C^{i_1 i_2}$ ,  $i_1 \neq i_2$ ; the boundary of  $C^{i_1 i_2}$  are  $\Gamma_j^{i_1 i_2}$ , when  $j \neq i_1$ ,  $\Gamma_j^{i_1 i_2}$  is an interior boundary; when  $j = i_1$ ,  $\Gamma_j^{i_1 i_2}$  is the exterior boundary,  $\Gamma_{i_1}^{i_1 i_2} = \Gamma_{i_1}^{i_2}$ .

$$\partial C^{i_1 i_2} = \Gamma_{i_1}^{i_2} - \sum_{j \neq i_1} \Gamma_j^{i_1 i_2}$$

when  $j = i_1$ ,  $\Gamma_{i_1}^{i_1 i_2} = \Gamma_{i_1}^{i_2}$ ;

# Multiple Reflection

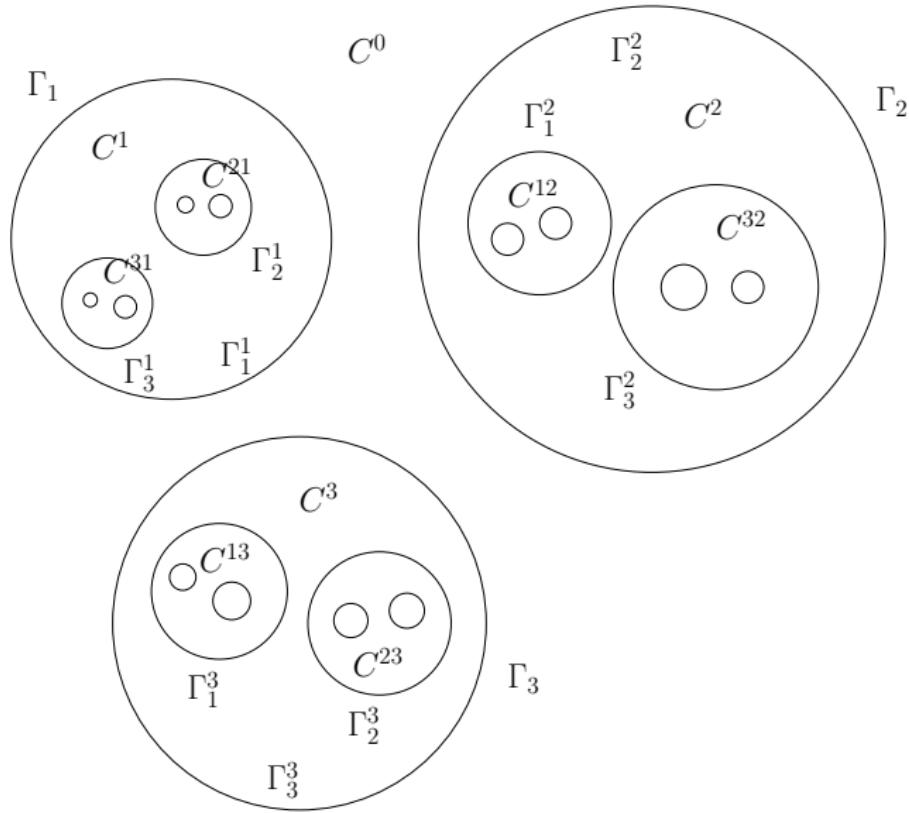
- ④ Third level reflection:  $C^{i_1 i_2}$  is reflected about  $\Gamma_{i_3}$  to  $C^{i_1 i_2 i_3}$ ,  $i_1 \neq i_2$ ,  $i_2 \neq i_3$ ; the boundary of  $C^{i_1 i_2 i_3}$  are  $\Gamma_j^{i_1 i_2 i_3}$ , when  $j \neq i_1$ ,  $\Gamma_j^{i_1 i_2 i_3}$  is an interior boundary; when  $j = i_1$ ,  $\Gamma_j^{i_1 i_2 i_3}$  is the exterior boundary,  $\Gamma_{i_1}^{i_1 i_2 i_3} = \Gamma_{i_1}^{i_2 i_3}$ .

$$\partial C^{i_1 i_2 i_3} = \Gamma_{i_1}^{i_2 i_3} - \sum_{j \neq i_1} \Gamma_j^{i_1 i_2 i_3}.$$

- ⑤ The  $m$ -level reflection:  $C^{i_1 i_2 \dots i_{m-1}}$  is reflected about  $\Gamma_{i_m}$  to  $C^{i_1 i_2 \dots i_{m-1} i_m}$ ,  $i_k \neq i_{k+1}$ ; the boundary of  $C^{i_1 i_2 \dots i_{m-1} i_m}$ ,  $i_k \neq i_{k+1}$  are  $\Gamma_j^{i_1 i_2 \dots i_{m-1} i_m}$ , when  $j \neq i_1$ ,  $\Gamma_j^{i_1 i_2 \dots i_{m-1} i_m}$  is an interior boundary; when  $j = i_1$ ,  $\Gamma_j^{i_1 i_2 \dots i_{m-1} i_m}$  is the exterior boundary,  $\Gamma_{i_1}^{i_1 i_2 \dots i_{m-1} i_m} = \Gamma_{i_1}^{i_2 \dots i_{m-1} i_m}$  is an interior boundary,

$$\partial C^{i_1 i_2 \dots i_m} = \Gamma_{i_1}^{i_2 i_3 \dots i_m} - \sum_{j \neq i_1} \Gamma_j^{i_1 i_2 \dots i_m}.$$

# Multiple Reflection



# Multiple Reflection

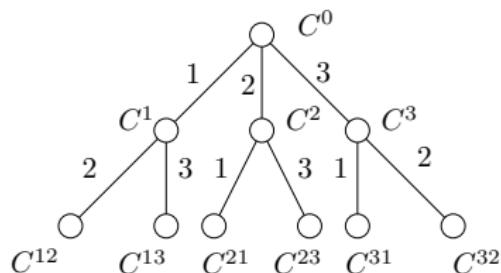


Figure: Reflection tree.

- Each node represents a domain  $C^{i_1 i_2 \dots i_m}$ ;
- Each edge represents a circle  $\Gamma_k$ ,  $k = 1, \dots, n$ ;
- Father and Son share an edge  $i_1$

$$\Gamma_{i_1}^{i_1 i_2 \dots i_m} = \Gamma_{i_1}^{i_2 \dots i_m}.$$

- Each node  $C^{(i)}$ ,  $(i) = i_1 i_2 \dots i_m$  is the path from the root to  $C^{(i)}$ ,

$$C^{(i)} = \varphi_{\Gamma_{i_m}} \circ \varphi_{\Gamma_{i_{m-1}}} \cdots \varphi_{\Gamma_{i_1}}(C^0).$$

# Multiple Reflection

- Father node  $C^{i_2 \dots i_m}$  and child node  $C^{i_1 i_2 \dots i_m}$  is connected by edge  $i_1$ , the exterior boundary of child equals to an interior boundary of the father

$$\Gamma_{i_1}^{i_1 i_2 \dots i_m} = \Gamma_{i_1}^{i_2 \dots i_m}.$$

- From the root  $C^0$  to  $C^{i_1 \dots i_m}$ , the path is inverse to the index

$$(i)^{-1} = i_m i_{m-1} \cdots i_2 i_1,$$

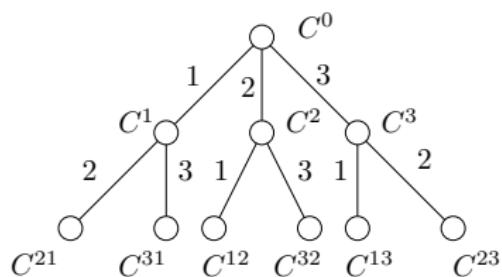


Figure: Embedding tree.

starting from  $C^0$  crosses  $\Gamma_{i_m}^{i_m}$  to  $C^{i_m}$ , crosses  $\Gamma_{i_{m-1}}^{i_m}$  to  $C^{i_{m-1} i_m}$ ; when arrives at  $C^{i_{k+1} \dots i_1}$ , crosses  $\Gamma_{i_k}^{i_{k+1} \dots i_1}$  to  $C^{i_k i_{k+1} \dots i_1}$ ; and eventually reach  $C^{(i)}$ .

# Hole Area Estimation

## Lemma

Suppose  $C^{(i)}$  is an interior node in the reflection tree,

$$(i) = i_1 i_2 \cdots i_m,$$

its exterior boundary is  $\Gamma_{i_1}^{(i)}$ , interior boundaries are  $\Gamma_j^{(i)}$ ,  $j \neq i_1$ , we have the estimate:

$$\sum_{j \neq i_1} \alpha(\Gamma_j^{(i)}) \leq \mu^4 \alpha(\Gamma_{i_1}^{(i)}).$$

# Hole Area Estimation

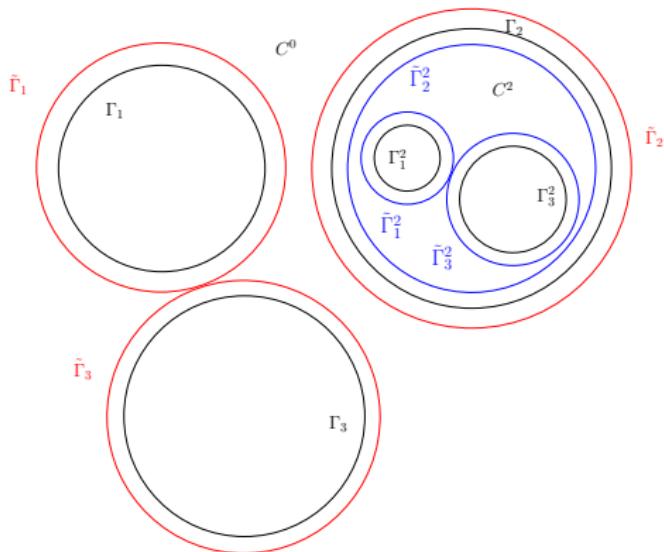


Figure: Hole area estimation.

Enlarge all  $\Gamma_k$ 's by factor  $\mu^{-1}$  to  $\tilde{\Gamma}_k$ ,  $\tilde{\Gamma}_1$  and  $\tilde{\Gamma}_3$  touch each other; reflect  $C^0$  about  $\Gamma_2$

- $\Gamma_k | \Gamma_k^2 (\Gamma_2)$ .
- $\tilde{\Gamma}_k | \Gamma_k^2 (\Gamma_2)$ .

$$\alpha(\tilde{\Gamma}_1^2) = \mu^{-2} \alpha(\Gamma_1^2)$$

$$\alpha(\tilde{\Gamma}_3^2) = \mu^{-2} \alpha(\Gamma_3^2)$$

$$\alpha(\tilde{\Gamma}_2^2) = \mu^2 \alpha(\Gamma_2)$$

$$\alpha(\Gamma_1^2) + \alpha(\Gamma_3^2) = \mu^2 (\alpha(\tilde{\Gamma}_1^2) + \alpha(\tilde{\Gamma}_3^2)) \leq \mu^2 \alpha(\tilde{\Gamma}_2^2) = \mu^4 \alpha(\Gamma_2).$$

# Hole Area Estimation

## Lemma

Suppose the boundaries of the initial circle domain  $C^0$  are  $\Gamma_1, \Gamma_2, \dots, \Gamma_n$ , consider the reflection tree with  $m$  layers, then the total area of the holes bounded by the interior boundaries of leaf nodes is no greater than  $\mu^{4m}$  times the area bounded by  $\Gamma_k$ 's,

$$\sum_{(i)=i_1 i_2 \dots i_m} \sum_{k \neq i_1} \alpha(\Gamma_k^{(i)}) \leq \mu^{4m} \sum_{i=1}^n \alpha(\Gamma_i). \quad (2)$$

## Proof.

By induction on  $m$ . The area bounded by the exterior boundaries of the nodes in the  $k + 1$ -layer is no greater than  $\mu^4$  times that of the  $k$ -layer. The total area of the interior boundaries of leaf nodes is no greater than the area bounded by the exterior boundaries of leaf nodes. □

# Uniqueness

## Theorem (Uniqueness)

*Given two circle domains  $C_1, C_2 \subset \hat{\mathbb{C}}$ ,  $f : C_1 \rightarrow C_2$  is a univalent holomorphic function, then  $f$  is a linear rational, namely a Möbius transformation.*

### Proof.

Assume both  $C_1$  and  $C_2$  include  $\infty$ , and  $f(\infty) = \infty$ . Since  $f$  is holomorphic, it maps the boundary circles of  $C_1$  to those of  $C_2$ . By Schwartz reflection principle,  $f$  can be extended to the multiple reflected domains. By the area estimation of the holes Eqn. 2, the multiple reflected domains cover the whole  $\hat{\mathbb{C}}$ , hence  $f$  can be extended to the whole  $\hat{\mathbb{C}}$ , since  $f(\infty) = \infty$ ,  $f$  is a linear function. If  $f(\infty) \neq \infty$ , we can use a Möbius map to transform  $f(\infty)$  to  $\infty$ .



# Existence

## Definition (Kernel)

Suppose  $\{B_n\}$  is a family of domains on the complex plane,  $\infty \in B_k$  for all  $k$ . Suppose  $B$  is the maximal set:  $\infty \in B$ , and for any closed set  $K \subset B$ , there is an  $N$ , such that for any  $n > N$ ,  $K \subset B_n$ . Then  $B$  is called the kernel of  $\{B_n\}$ .

## Definition (Domain Convergence)

We say a sequence  $\{B_n\}$  converges to its kernel  $B$ , if any sub-sequence  $\{B_{n_k}\}$  of  $\{B_n\}$  has the same kernel  $B$ . We denote  $B_n \rightarrow B$ .

## Theorem (Goluzin)

Let  $\{A_n\}$  be a sequence of domains on the complex domain. Any domain  $A_n$  includes  $\infty$ ,  $n = 1, 2, \dots$ . Assume  $\{A_n\}$  converges to its kernel  $A$ . Let  $\{f_n(z)\}$  be a family of analytic function, for all  $n$ ,  $f_n(z)$  maps  $A_n$  to  $B_n$  surjectively, such that  $f_n(\infty) = \infty$ ,  $f'_n(\infty) = 1$ . Then  $\{f_n(z)\}$  uniformly converges to a univalent analytic function  $f(z)$  in the interior of  $A$ , if and only if  $\{B_n\}$  converges to its kernel  $B$ , then the univalent analytic function  $f(z)$  maps  $A$  to  $B$  surjectively.

## Theorem (Existence)

*On the  $z$ -plane, every  $n$ -connected domain  $\Omega$  can be mapped to a circle domain on the  $\zeta$ -plane by a univalent holomorphic function. Choose a point  $a \in \Omega$ , there is a unique map which maps  $a$  to  $\zeta = \infty$ , and in a neighborhood of  $z = a$ , the map has the power series*

$$\frac{1}{z - a} + a_1(z - a) + \cdots \text{ if } a \neq \infty$$

$$z + \frac{a_1}{z} + \cdots \text{ if } a = \infty$$

# Existence

## Proof.

According to Hilbert theorem, all  $n$ -connected domains are conformally equivalent to slit domains. We can assume  $\Omega$  is a slit domain. We use  $\mathcal{S}$  represent all the  $n$ -connected slit domains with horizontal slits, and  $\mathcal{C}$  the  $n$ -connected circle domains. We label all the boundaries of the domains,  $\partial\Omega = \bigcup_{k=1}^n \gamma_k$ . For each slit  $\gamma_k$ , we represent it by the starting point  $p_k$  and the length  $l_k$ , then we get the coordinates of the slit domain  $\Omega$

$$(p_1, l_1, p_2, l_2, \dots, p_n, l_n).$$

Hence  $\mathcal{S}$  is a connected open set in  $\mathbb{R}^{3n}$ . Similarly, consider a circle domain  $\mathcal{D} \in \mathcal{C}$ , we use the center and the radius to represent each circle  $(q_k, r_k)$ , and the coordinates of  $\mathcal{D}$  are given by,

$$(q_1, r_1, q_2, r_2, \dots, q_n, r_n).$$

$\mathcal{C}$  is also a connected open set in  $\mathbb{R}^{3n}$ .



continued

Consider a normalized univalent holomorphic function  $f : \Omega \rightarrow \mathcal{D}$ ,  $\Omega \in \mathcal{S}$  and  $\mathcal{D} \in \mathcal{C}$ ,  $f$  maps the  $k$ -th boundary curve  $\gamma_k$  to the  $k$ -th circular boundary of  $\mathcal{D}$ . By the existence of slit mapping and the uniqueness of circle domain mapping, we have

- ① Every circle domain  $\mathcal{D} \in \mathcal{C}$  corresponds to a unique slit domain  $\Omega \in \mathcal{S}$ ;
- ② Every slit domain  $\Omega \in \mathcal{S}$  corresponds to at most one circle domain  $\mathcal{D} \in \mathcal{C}$ .

Then we establish a mapping from circle domains to slit domains  
 $\varphi : \mathcal{C} \rightarrow \mathcal{S}$ .

## continued

Assume  $\{\mathcal{D}_n\}$  is a family of circle domains, converge to the kernel  $\mathcal{D}^*$ . The domain convergence definition is consistent with the convergence of coordinates, namely, the boundary circles of  $\mathcal{D}_n$  converge to the corresponding boundary circles of  $\mathcal{D}^*$ , denoted as  $\lim_{n \rightarrow \infty} \mathcal{D}_n = \mathcal{D}^*$ . The convergence of slit domains can be similarly defined. By Goluzin's theorem, we obtain the mapping  $\varphi : \mathcal{C} \rightarrow \mathcal{S}$  is continuous:

$$\varphi\left(\lim_{n \rightarrow \infty} \mathcal{D}_n\right) = \lim_{n \rightarrow \infty} \varphi(\mathcal{D}_n).$$

By the uniqueness of circle domain mapping, we obtain  $\varphi$  is injective. We will prove the mapping  $\varphi$  is surjective.

## continued

$\mathcal{C}$  is an open set in Euclidean space  $\varphi : \mathcal{C} \rightarrow \mathcal{S}$  is injective continuous map. According to invariance of domain theorem,  $\varphi(\mathcal{C})$  is an open set,  $\varphi : \mathcal{C} \rightarrow \varphi(\mathcal{C})$  is a homeomorphism.

Choose a circle domain  $\mathcal{D}_0 \in \mathcal{C}$ , its corresponding slit domain is  $\varphi(\mathcal{D}_0) = \Omega_0 \in \mathcal{S}$ , then  $\Omega_0 \in \varphi(\mathcal{C})$ . Choose another slit map  $\Omega_1 \in \mathcal{S}$ , we don't know if  $\Omega_1$  is in  $\varphi(\mathcal{C})$  or not. We draw a path  $\Gamma : [0, 1] \rightarrow \mathcal{S}$ ,  $\Gamma(0) = \Omega_0$  and  $\Gamma(1) = \Omega_1$ . Let

$$t^* = \sup\{t \in [0, 1] | \forall 0 \leq \tau \leq t, \Gamma(\tau) \in \varphi(\mathcal{C})\},$$

namely  $\Gamma$  from starting point to  $t^*$  belongs to  $\varphi(\mathcal{C})$ .

# Existence

continued

By the definition of domain convergence,

$$\lim_{n \rightarrow \infty} \Gamma(t_n) \rightarrow \Gamma(t^*).$$

By  $\{\Gamma(t_n)\} \subset \varphi(\mathcal{C})$ , there is a family of circle domains  $\{\mathcal{D}_n\} \subset \mathcal{C}$ ,  $\varphi(\mathcal{D}_n) = \Gamma(t_n)$ . Let  $\lim_{n \rightarrow \infty} \mathcal{D}_n = \mathcal{D}^*$ , by domain limit theorem, we have

$$\varphi(\mathcal{D}^*) = \varphi\left(\lim_{n \rightarrow \infty} \mathcal{D}_n\right) = \lim_{n \rightarrow \infty} \varphi(\mathcal{D}_n) = \lim_{n \rightarrow \infty} \Gamma(t_n) = \Gamma(t^*),$$

namely  $\varphi(\mathcal{D}^*) = \Gamma(t^*)$ , hence  $\Gamma(t^*) \in \varphi(\mathcal{C})$ . But  $\varphi(\mathcal{C})$  is an open set, hence if  $t^* < 1$ ,  $t^*$  can be further extended. This contradicts to the choice of  $t^*$ , hence  $t^* = 1$ . Therefore  $\Omega_1 \in \varphi(\mathcal{C})$ . Since  $\Omega_1$  is arbitrarily chosen, hence  $\varphi : \mathcal{C} \rightarrow \mathcal{S}$  is surjective. This proves the existence of the circle domain mapping.

## Convergence of Koebe Iteration Method

# Koebe Iteration Algorithm

Input: Poly annulus  $M$ ,  $\partial M = \gamma_0 - \gamma_1 - \cdots - \gamma_n$ ;

Output: Conformal map  $\varphi : M \rightarrow \mathbb{D}$ , where  $\mathbb{D}$  is a circle domain.

- ① Compute a slit map, map the surface to the circular slit domain  $f : M \rightarrow \mathbb{C}$ ,  $\gamma_0$  and  $\gamma_k$  are mapped to the exterior and interior circular boundary of  $\mathbb{C}$ ;
- ② Fill the inner circle using Delaunay refinement mesh generation;
- ③ Repeat step 1 and 2, fill all the holes step by step;

# Koebe Iteration Method

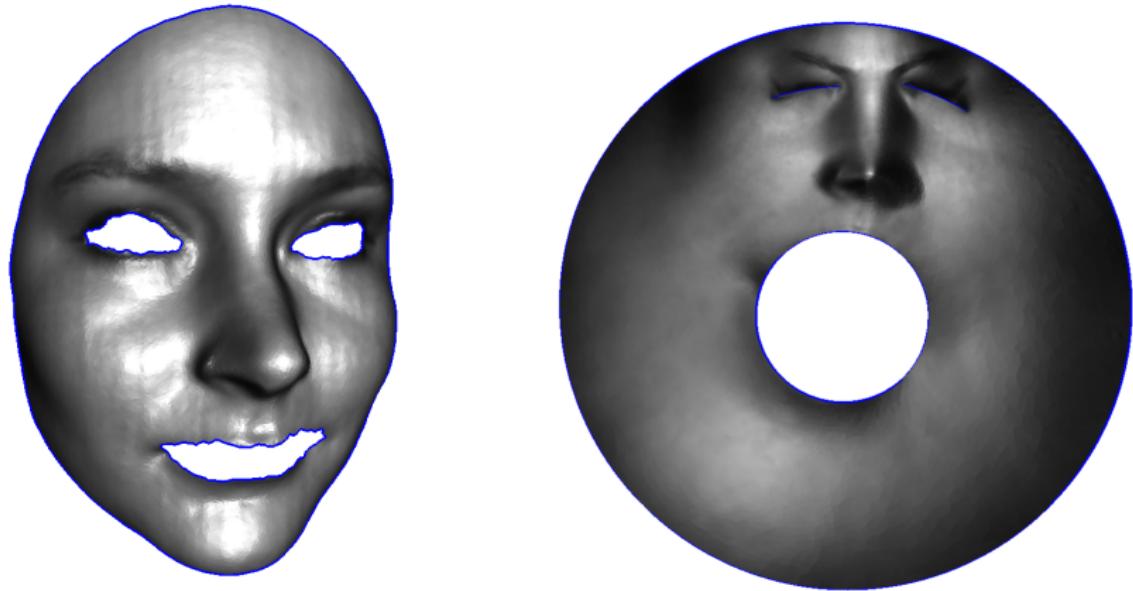


Figure: Slit map.

# Koebe Iteration Method

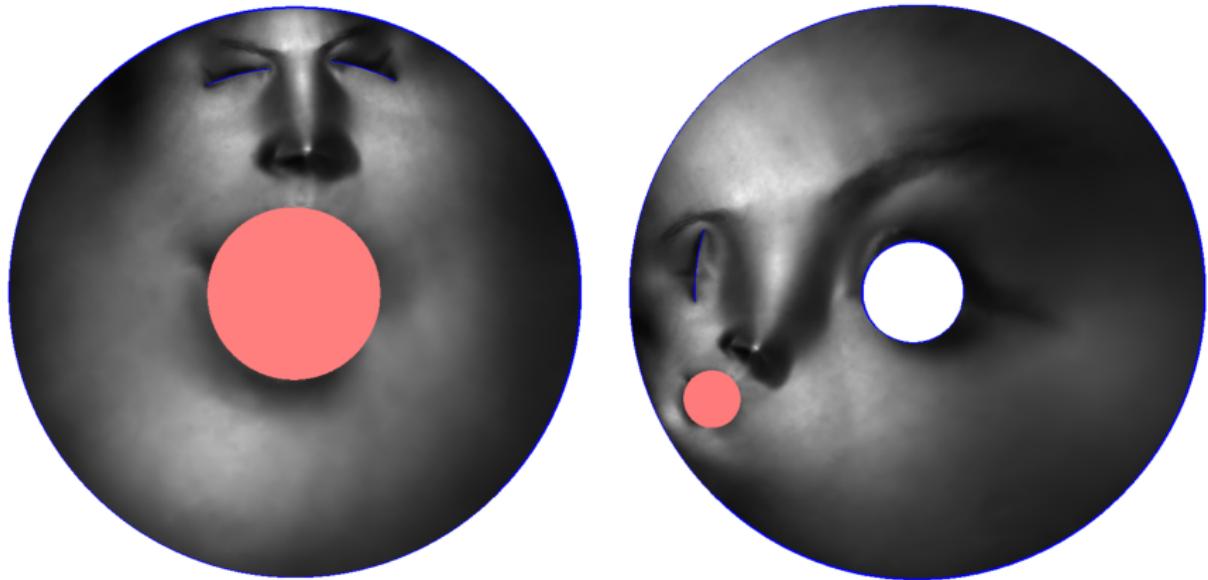


Figure: Hole filling and slit map.

# Koebe Iteration Method

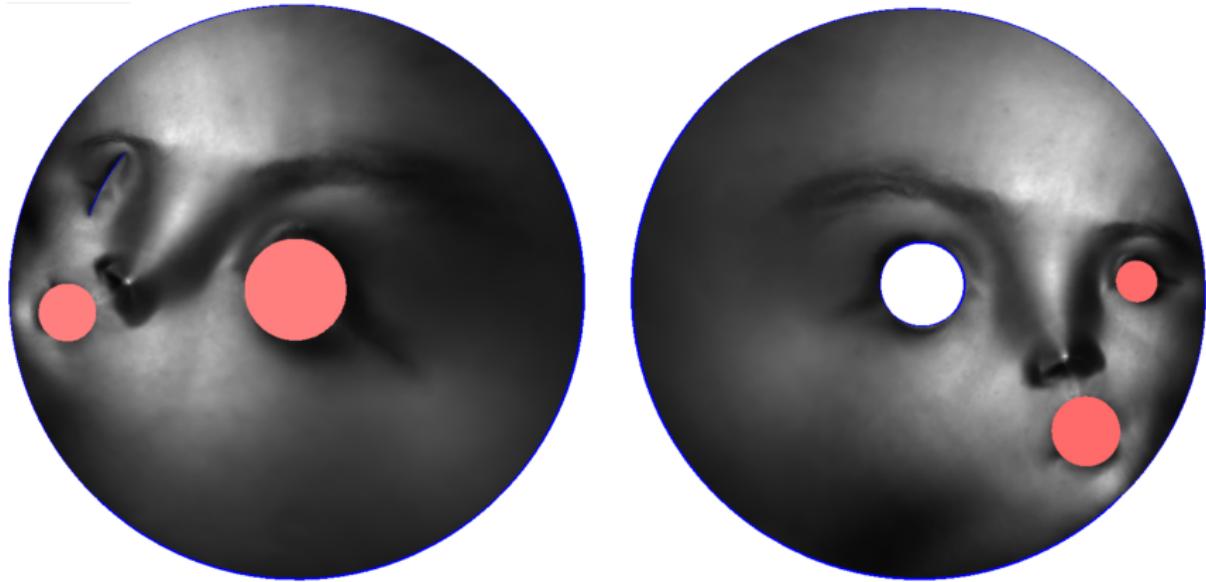


Figure: Hole filling and slit map.

# Koebe Iteration Method



Figure: All holes are filled.

# Koebe Iteration Algorithm

- ④ Punch a hole at the  $k$ -th inner boundary;
- ⑤ Compute a conformal map, to map the surface onto a canonical planar annulus;
- ⑥ Fill the inner circular hole;
- ⑦ Repeat step 4 through 6, each time punch a different hole, until the process converges.

# Koebe Iteration Method



# Koebe Iteration Method



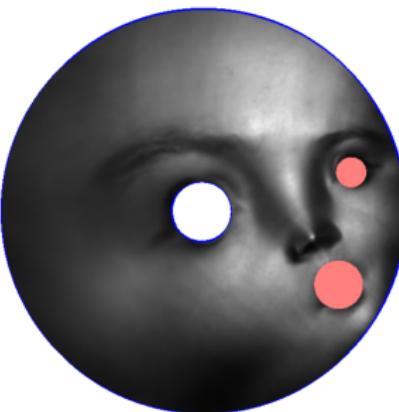
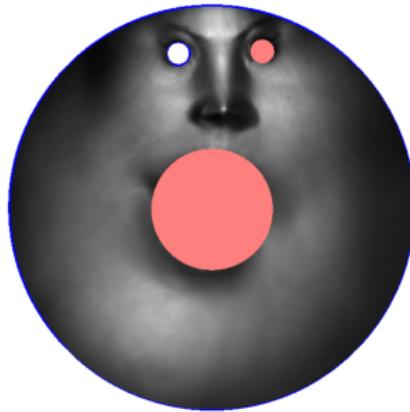
# Koebe Iteration Method



# Koebe Iteration Method



# Koebe Iteration Method



# Koebe Iteration Method



# Koebe Iteration Method



# Koebe Iteration Method



Figure: Final result.

# Area, Diameter Estimate

## Lemma

Suppose  $A$  is a topological annulus on  $\mathbb{C}$ , the conformal module of  $A$  is  $\mu^{-1} > 1$ , the exterior and interior boundaries of  $A$  are Jordan curves  $\Gamma_0$  and  $\Gamma_1$ ,  $\partial A = \Gamma_0 - \Gamma_1$ , then we have the area and diameter estimates:

$$\alpha(\Gamma_1) \leq \mu^2 \alpha(\Gamma_0), \quad (3)$$

and

$$[\operatorname{diam} \Gamma_1]^2 \leq \frac{\pi}{2 \log \mu^{-1}} \alpha(\Gamma_0), \quad (4)$$

where  $\alpha(\Gamma_k)$  is the area bounded by  $\Gamma_k$ ,  $k = 0, 1$ .

# Area, Diameter Estimate

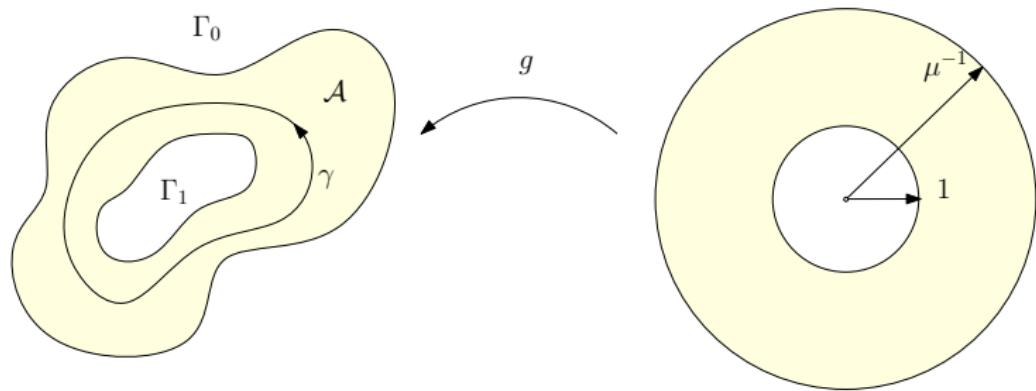


Figure: Topological annulus with conformal module  $\mu^{-1}$ .

# Area, Diameter Estimate

## Proof.

Let holomorphic function  $g$  maps  $\{1 \leq |w| \leq \mu^{-1}\}$  to  $A$ ,

$$g(w) = w + a_0 + \frac{a_1}{w} + \frac{a_2}{w^2} + \dots$$

By Gnowell area estimate, we have

$$\alpha(\Gamma_1) = \pi \left( 1 - \sum_{n=1}^{\infty} n|a_n|^2 \right)$$

$$\alpha(\Gamma_0) = \pi \left( \mu^{-2} - \sum_{n=1}^{\infty} n|a_n|^2 \mu^{2n} \right)$$

hence, this proves the area inequality (3)

$$\alpha(\Gamma_0) - \mu^{-2} \alpha(\Gamma_1) = \pi \sum_{n=1}^{\infty} n|a_n|^2 (\mu^{-2} - \mu^{2n}) \geq 0$$

# Area, Diameter Estimate

## Continued

The diameter  $\text{diam}\Gamma_1$  is determined by  $g(\{1 < |w| < \rho\})$ , where  $\rho \in (1, \mu^{-1})$ . The diameter is bounded by half of the boundary length  $g(|w| = \rho)$ , we have

$$2\text{diam}\Gamma_1 \leq \int_{|w|=\rho} |g'(w)| dw = \int_0^{2\pi} |g'(\rho e^{i\theta})| \rho d\theta = \int_0^2 \pi |g'(\rho e^{i\theta})| \sqrt{\rho} \sqrt{\rho} d\theta,$$

By Schwartz inequality, we have

$$[2\text{diam}\Gamma_1]^2 \leq \int_0^{2\pi} |g'(\rho e^{i\theta})|^2 \rho d\theta \int_0^{2\pi} \rho d\theta = 2\pi\rho \int_0^{2\pi} |g'(\rho e^{i\theta})|^2 \rho d\theta$$

# Area, Diameter Estimate

Continued

Equivalent

$$\frac{2}{\pi\rho}[\operatorname{diam}\Gamma_1]^2 \leq \int_0^{2\pi} |g'(\rho e^{i\theta})|^2 \rho d\theta$$

Integrate with respect to  $\rho$ ,

$$\int_1^{\mu^{-1}} \frac{2}{\pi\rho} [\operatorname{diam}\Gamma_1]^2 d\rho \leq \int_1^{\mu^{-1}} \int_0^{2\pi} |g'(\rho e^{i\theta})|^2 \rho d\theta d\rho = \alpha(\Gamma_0) - \alpha(\Gamma_1).$$

Calculate left

$$\frac{2 \log \mu^{-1}}{\pi} [\operatorname{diam}\Gamma_1]^2 \leq \alpha(\Gamma_0) - \alpha(\Gamma_1) \leq \alpha(\Gamma_0).$$

This proves inequality 4.

# Multiple Reflected Domain

## Definition (Multi-reflected circle domain)

Given an  $m$ -level embedding relation tree of a circle domain  $C$ , the union of all nodes in the tree is called a multiple-reflected circle domain,

$$\Omega_m = \bigcup_{k \leq m} (i) = i_1 i_2 \cdots i_k C^{(i)} = \hat{\mathbb{C}} \setminus \bigcup_{(i)=i_1 i_2 \cdots i_m} \bigcup_{k \neq i_1} \alpha(\Gamma_k^{(i)})$$

where  $\alpha(\Gamma)$  is the area bounded by  $\Gamma$ .

Suppose we have a holomorphic univalent map  $g_m : \Omega_m \rightarrow \hat{\mathbb{C}}$ , define

$$C_m = g_m(C^0)$$

$$C_m^{(i)} = g_m(C^{(i)})$$

$$\Gamma_{m,k} = g_m(\Gamma_k)$$

$$\Gamma_{m,k}^{(i)} = g_m(\Gamma_k^{(i)})$$

# Symmetric Relation

According to the reflection generation tree, we have the symmetry

$$C^{i_1 i_2 \dots i_{m-1} i_m} \mid C^{i_1 i_2 \dots i_{m-1} i_m} (\Gamma_{i_m})$$

this symmetric relation is preserved by the holomorphic map  $g_m$ :

$$C_m^{i_1 i_2 \dots i_{m-1} i_m} \mid C_m^{i_1 i_2 \dots i_{m-1} i_m} (\Gamma_{m,i_m})$$

therefore  $g_m$  maps the embedding relation tree of  $\{C^{(i)}\}$  to the embedding relation tree of  $\{C_m^{(i)}\}$ .

# Hole Area Estimation

## Lemma

Suppose the boundaries of  $C_m$  are  $\Gamma_{m,1}, \Gamma_{m,2}, \dots, \Gamma_{m,n}$ . In the  $m$ -level embedding relation tree of  $C_m$ , the total area of the holes bounded by the interior boundaries of leaf nodes is less than  $\mu^{4m}$  times the total area of holes bounded by  $\Gamma_{m,k}$ 's,

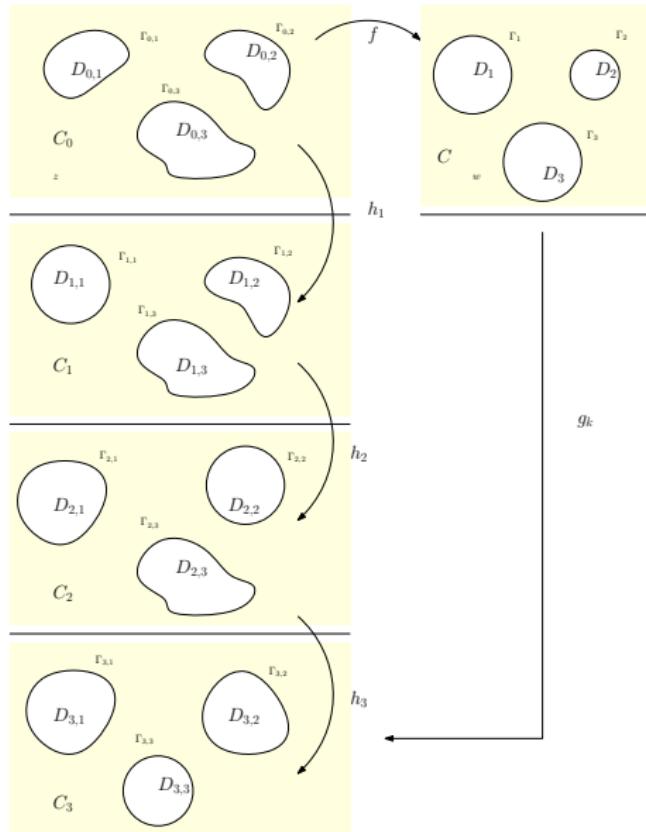
$$\sum_{(i)=i_1 i_2 \dots i_m} \sum_{k \neq i_1} \alpha(\Gamma_{m,k^{(i)}}) \leq \mu^{4m} \sum_{i=1}^n \alpha(\Gamma_{m,i}). \quad (5)$$

## Proof.

Using area estimate (3) and induction on  $m$ .



# Koebe's Iteration



# Koebe's Iteration

## Key Observation

Given a multi-annulus  $\mathcal{R}$ , there is a biholomorphic map  $g : \mathcal{C} \rightarrow \mathcal{R}$  maps a circle domain  $\mathcal{C}$  to  $\mathcal{R}$ . During the process of Koebe's iteration, the domain of the mapping  $\mathcal{C}$  can be extended to the image of the multiple reflection, (multiple reflected circle domain), which eventually covers the whole augmented complex plane  $\hat{\mathbb{C}}$ .

# Koebe's Iteration

## Lemma

During Koebe's iteration, at the  $mn$ -th step, the algorithm generates a univalent holomorphic function  $g_{mn}$ , its domain is extended to the  $m$ -level reflected circle domain,

$$g_{mn} : \Omega_m \rightarrow \hat{\mathbb{C}}.$$

## Proof.

Initial domain is  $C_0$ ,  $\infty \in C_0$ , the complementary sets are

$D_{0,1}, D_{0,2}, \dots, D_{0,n}$ ,  $\partial D_{0,i} = \Gamma_{0,i}$ ,  $i = 1, 2, \dots, n$ .

There is a biholomorphic function,  $f : C_0 \rightarrow \mathcal{C}$ , the complementary of  $\mathcal{C}$  is the set  $D_1, D_2, \dots, D_n$ , where  $D_i$ 's are disks,  $\partial D_i = \Gamma_i$  is a canonical circle. In the neighborhood of  $\infty$ ,  $f(z) = z + O(z^{-1})$ . □

# Koebe's Iteration

continued.

By Riemann mapping theorem, there is a Riemann mapping

$$h_1 : \hat{\mathbb{C}} \setminus D_{0,1} \rightarrow \hat{\mathbb{C}} \setminus \mathbb{D},$$

maps  $\Gamma_{0,1}$  to the unit circle  $\Gamma_{1,1}$ ,  $C_0$  to  $C_1$ , satisfying the normalization condition,

$$h_1(\infty) = \infty, \quad h'_1(\infty) = 1,$$

and

$$D_{1,k} = h_1(D_{0,k}), \quad k = 2, \dots, n.$$

Repeat this procedure, at  $k \leq n$  step, construct a Riemann mapping,

$$h_k : \hat{\mathbb{C}} \setminus D_{k-1,k} \rightarrow \hat{\mathbb{C}} \setminus \mathbb{D},$$

which maps  $\Gamma_{k-1,k}$  to the unit circle,  $C_{k-1}$  to  $C_k$ ,  $h_k(\infty) = \infty$  and  $h'(\infty) = 1$ .

# Koebe's Iteration

continued.

We recursively define the symbols as follows:

$$C_k = h_k(C_{k-1})$$

$$\Gamma_{k,i} = h_k(\Gamma_{k-1,i}), i \neq k$$

$$D_{k,i} = h_k(D_{k-1}, i), i \neq k$$

$D_{k,k}$  is the unit disk  $\mathbb{D}$ ,  $\Gamma_{k,k}$  the unit circle. We construct a biholomorphic map  $f_k : C_0 \rightarrow C_k$ :

$$f_k = h_k \circ h_{k-1} \circ \cdots \circ h_1$$

and the biholomorphic map from the circle domain  $\mathcal{C}$  to  $C_k$ ,  $g_k : \mathcal{C} \rightarrow C_k$ ,

$$g_k := f_k \circ f^{-1},$$

$g_k$  satisfies normalization condition  $g_k(\infty) = \infty$ ,  $g'_k(\infty) = 1$ .

# Koebe's Iteration

continued.

We generalize the domain of  $g_k$  to multiple reflected circle domain.

Because  $\Gamma_{1,1}$  is a canonical circle,  $C_1$  can be reflected about  $\Gamma_{1,1}$  to  $C_1^1$ ,

$$C_1|C_1^1 \quad (\Gamma_{1,1})$$

$h_2 : \hat{\mathbb{C}} \setminus D_{1,2} \rightarrow \hat{\mathbb{C}} \setminus \mathbb{D}$ , hence  $h_2$  is well defined on  $D_{1,1}$ . we denote

$$C_2^1 := h_2(C_1^1), \quad C_2^1|C_2 \quad (\Gamma_{2,1}).$$

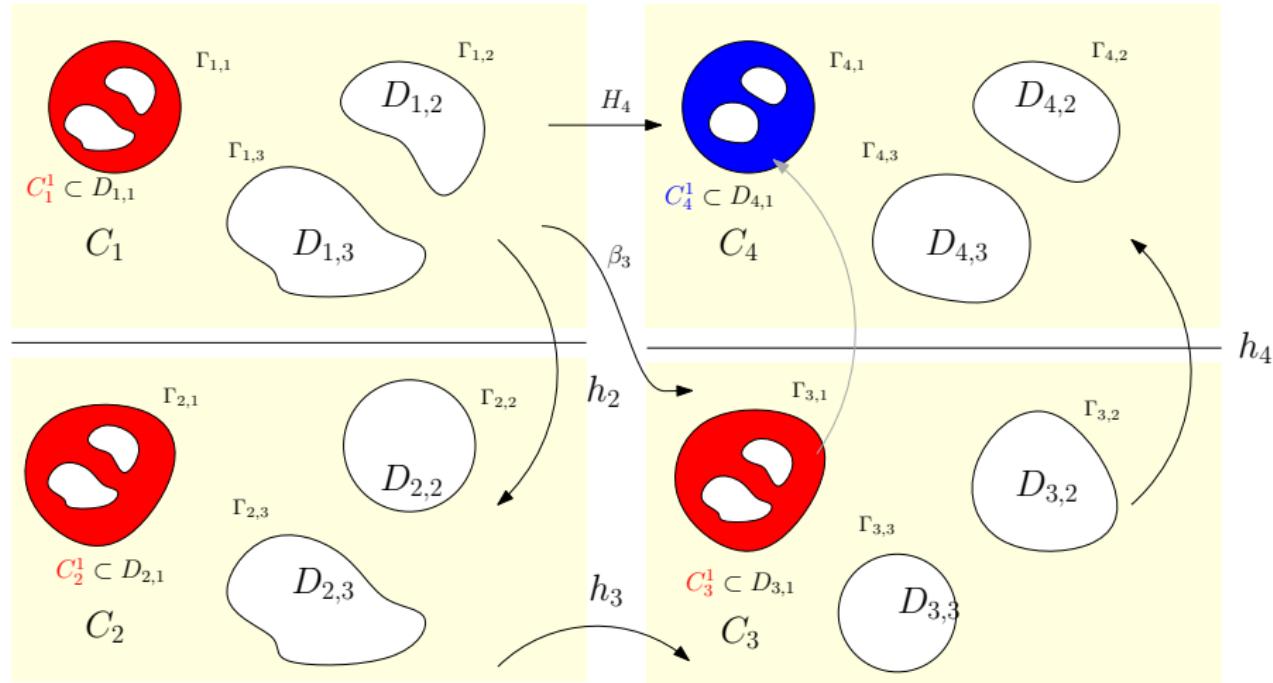
when  $k = 2, 3, \dots, n$ , the Riemann mapping  $h_k$  is well defined on  $C_k \cup D_{k,1}$ , domain

$$C_k^1 := h_k \circ h_{k-1} \circ \dots \circ h_1(C_1^1), \quad k = 2, \dots, n,$$

satisfying

$$C_k^1|C_k \quad (\Gamma_{k,1}).$$

# Koebe's Iteration



# Koebe's Iteration

continued.

But the map  $h_{n+1}$  on  $D_{n,1}$  is not defined. We can use Schwartz reflection to define  $C_{n+1}^1$ . Consider the composition:

$$\beta_n := h_n \circ h_{n-1} \circ \cdots \circ h_2 : C_1 \rightarrow C_n,$$

$\beta_n$  is well defined on  $D_{1,1}$ .

$$h_{n+1} \circ \beta_n : C_1 \rightarrow C_{n+1},$$

maps the circle  $\Gamma_{1,1}$  to the circle  $\Gamma_{n+1,1}$ , but is not defined on  $D_{1,1}$ . By Schwartz reflection principle, the map  $h_{n+1} \circ \beta_n$  can be extended to

$$H_{n+1} : C_1 \cup C_1^1 \rightarrow C_{n+1} \cup C_{n+1}^1,$$

where

$$C_{n+1}^1 | C_n \quad (\Gamma_{n+1,1}).$$

# Koebe's Iteration

Continued.

$$\begin{array}{ccc} C_1 \cup C_1^1 & \xrightarrow{\beta_n} & C_n \cup C_n^1 \\ H_{n+1} \downarrow & & \downarrow H_{n+1} \circ \beta_n^{-1} \\ C_{n+1} \cup C_{n+1}^1 & \xrightarrow{Id} & C_{n+1} \cup C_{n+1}^1 \end{array}$$

we obtain the composition map

$$H_{n+1} \circ \beta_n^{-1} : C_n \cup C_n^1 \rightarrow C_{n+1} \cup C_{n+1}^1.$$

for convenience, we still use  $h_{n+1}$  to represent  $H_{n+1} \circ \beta_n^{-1}$ . Hence, we extend the domain of  $h_{n+1}$  to  $C_n^1$ :  $h_{n+1} : C_n \cup C_n^1 \rightarrow C_{n+1} \cup C_{n+1}^1$ . Repeat this procedure, we conclude: for all  $k \geq 1$ ,  $C_k^1$  and  $C_k$  are symmetric,

$$C_k^1 | C_k \quad (\Gamma_{k,1}).$$

# Koebe's Iteration

Continued.

Similarly, when  $k = 2$ ,  $\Gamma_{2,2}$  is a circle,  $C_2^2$  and  $C_2$  are symmetric about  $\Gamma_{2,2}$ . When  $k > 2$ , we define

$$C_k^2 := h_k \circ h_{k-1} \circ \cdots \circ h_3(C_2^2),$$

similarly, for every  $h_{kn+2}$  map, we use Schwartz reflection principle to extend analytically. For all  $k \geq 2$ ,  $C_k 2$  and  $C_k$  are symmetric:

$$C_k^2 | C_k \quad (\Gamma_{k,2}).$$

Similarly, for any  $i = 3, \dots, n$ , we use Schwartz reflection principle to extend the domain and define  $C_k^i$  symmetric to  $C_k$ , for all  $k \geq i$ ,

$$C_k^i | C_k \quad (\Gamma_{k,i}).$$

Continued.

After the first round of iterations, all  $C_k^i$ ,  $i = 1, 2, \dots, n$  are defined. Since  $\Gamma_{n+1,1}$  is the unit circle, we define  $C_{n+1}^{i1}$  to be the mirror image of  $C_{n+1}^i$  with respect to  $\Gamma_{n+1,1}$ ,  $C_{n+1}^{11} = C_{n+1}$ , but all other  $C_{n+1}^{i1}$  are newly generated domains  $i \neq 1$ . Apply the extended Riemann mapping, we get a series of mirror images:

$$C_k^{i1} | C_k^i \quad (\Gamma_{k,1}), \quad \forall k \geq n + 1, i = 2, 3, \dots, n.$$

Similarly, we can define mirror image domains:

$$C_k^{ij} | C_k^i \quad (\Gamma_{k,j}), \quad \forall k \geq n + j.$$

# Koebe's Iteration

Continued.

After  $mn$  iterations, we obtain  $m$ -level mirror images  $C_k^{i_1 i_2 \dots i_m}$ , satisfying the symmetric relation:

$$C_k^{i_1 i_2 \dots i_m i_{m+1}} | C_k^{i_1 i_2 \dots i_m} \quad (\Gamma_k, i_{m+1}), \quad k \geq mn + i_{m+1},$$

Now the  $j$ -th boundary of  $C_k^{i_1 i_2 \dots i_m i_{m+1}}$  is denoted as  $\Gamma_{k,j}^{i_1 i_2 \dots i_m i_{m+1}}$ ,

$$\partial C_k^{i_1 i_2 \dots i_m i_{m+1}} = \Gamma_{k,i_1}^{i_1 i_2 \dots i_m i_{m+1}} - \bigcup_{j \neq i_1}^n \Gamma_{k,j}^{i_1 i_2 \dots i_m i_{m+1}}.$$

# Koebe's Iteration

Continued.

Consider  $g_k^{-1} = f \circ f_k^{-1}$ , for all  $k$  we have

$$C = g_k^{-1}(C_k)$$

similarly,

$$C^{i_1 i_2 \dots i_m} = g_k^{-1}(C_k^{i_1 i_2 \dots i_m})$$

and its boundaries

$$\Gamma_j^{i_1 i_2 \dots i_m} = g_k^{-1}(\Gamma_{k,j}^{i_1 i_2 \dots i_m}).$$

# Error Estimate

The circle domain  $C = C^0$  is reflected about  $\Gamma_{i_1}, \Gamma_{i_2}, \dots, \Gamma_{i_k}$  sequentially, to a  $k$ -level mirror reflection image  $C^{i_1 i_2 \dots i_k}$ , its interior boundary is

$$\Gamma_j^{i_1 i_2 \dots i_k} = \Gamma_j \dots (i), \quad j \neq i_1,$$

such that  $i_l$  and  $i_{l+1}$  are not equal. After analytic extension,  $g_k$  is defined on the augmented complex plane with  $n(n-1)^{k-1}$  disks removed. The boundaries of these disks are

$$\bigcup_{i_1 i_2 \dots i_k, i_l \neq i_{l+1}} \bigcup_{j \neq i_1} \Gamma_j^{i_1 i_2 \dots i_k}$$

# Error Estimate

We choose a big circle  $\Gamma_\rho$ , enclosing all the initial boundaries  $\Gamma_j$ . For any point  $w \in C^0$ , by Cauchy formula

$$g(w) - w = \frac{1}{2\pi i} \oint_{\Gamma_\rho} \frac{g_k(s) - w}{s - w} ds - \sum_{(i),j} \frac{1}{2\pi i} \oint_{\Gamma_j^{(i)}} \frac{g_k(s) - w}{s - w} ds$$

at  $\infty$  neighborhood,  $g_k(w) - w = O(w^{-1})$ , when  $\rho \rightarrow \infty$

$$\frac{1}{2\pi i} \oint_{\Gamma_\rho} \frac{g_k(s) - w}{s - w} ds = \frac{1}{2\pi i} \oint_{\Gamma_\rho} \frac{g_k(s) - s}{s - w} + \frac{s - w}{s - w} ds \rightarrow 0.$$

# Error Estimate

Since  $w$  is outside all  $\Gamma_j^{(i)}$ , integration

$$\frac{1}{2\pi i} \oint_{\Gamma_j^{(i)}} \frac{w}{s-w} ds = 0,$$

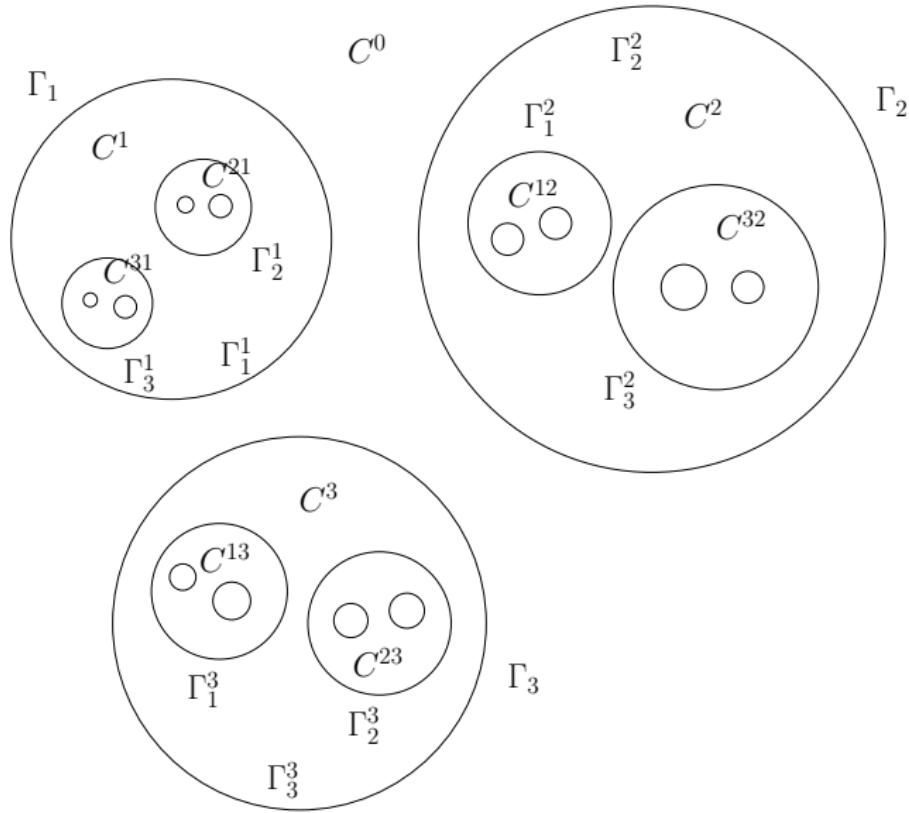
for any complex number  $c_j^{(i)}$ , integration

$$\frac{1}{2\pi i} \oint_{\Gamma_j^{(i)}} \frac{c_j^{(i)}}{s-w} ds = 0$$

we obtain

$$g_k(w) - w = - \sum_{(i),j} \frac{1}{2\pi i} \oint_{\Gamma_j^{(i)}} \frac{g_k(s) - c_j^{(i)}}{s-w} ds$$

# Multiple Reflection



# Error Estimate

In the initial circle domain  $C^0$ , let distance constant

$$\delta := \min_{i \neq j} \text{dist}(\Gamma_i, \Gamma_j^i),$$

we have  $\delta > 0$ . Since  $\Gamma_j^{(i)} \subset \Gamma_{i_{m-1}}^{i_m}$ ,  $|s - w| > \delta$ . Define

$$\delta_{k,j}^{(i)} := \text{diam}(\Gamma_{k,j}^{(i)}),$$

the curve  $\Gamma_{k,j}^{(i)} = g_k(\Gamma_j^{(i)})$  is enclosed by the circle centered as  $c_j^{(i)}$  and diameter  $\delta_{k,j}^{(i)}$ , then for all  $s \in \Gamma_j^{(i)}$ ,

$$|g_k(s) - c_j^{(i)}| \leq \delta_{k,j}^{(i)},$$

the length of the integration is  $\pi \delta_j^{(i)}$ , where  $\delta_j^{(i)} = \text{diam}(\Gamma_j^{(i)})$ .

# Error Estimate

$$\begin{aligned}|g_k(w) - w| &\leq \sigma_{(i),j} \frac{1}{2\pi} \oint_{\Gamma_j^{(i)}} \frac{|g_k(s) - c_j^{(i)}|}{|s - w|} |ds| \leq \sum_{(i),j} \frac{1}{2\pi} \frac{\delta_{k,j}^{(i)}}{\delta} \pi \delta_j^{(i)} \\&= \sum_{(i),j} \frac{1}{2\delta} \delta_{k,j}^{(i)} \delta_j^{(i)} \leq \sum_{(i),j} \frac{1}{4\delta} \left( [\delta_{k,j}^{(i)}]^2 + [\delta_j^{(i)}]^2 \right)\end{aligned}$$

For the first term,

$$\sum_{(i),j} [\delta_j^{(i)}]^2 = \frac{4}{\pi} \sum_{(i),j} \alpha(\Gamma_j^{(i)}) \leq \mu^{4m} \sum_j \alpha(\Gamma_j) = \frac{4}{\pi} \mu^{4m} \gamma_1,$$

where  $\sum_j \alpha(\Gamma_j) = \gamma_1$ .

# Error Estimate

For the second term, consider the topological annulus bounded by  $\tilde{\Gamma}_{k,j}^{(i)}$  and  $\Gamma_{k,j}^{(i)}$ , by the diameter estimation (4), we obtain

$$[\delta_{j,k}^{(i)}]^2 \leq \frac{\pi}{2 \log \mu^{-1}} \alpha(\tilde{\Gamma}_{k,j}^{(i)}),$$

By inequality (5), we obtain

$$\sum_{(i),j} [\delta_{j,k}^{(i)}]^2 \leq \frac{\pi}{2 \log \mu^{-1}} \sum_{(i),j} \alpha(\tilde{\Gamma}_{k,j}^{(i)}) \leq \frac{\pi}{2 \log \mu^{-1}} \sum_j \alpha(\tilde{\Gamma}_{k,j}) = \frac{\pi}{2 \log \mu^{-1}} \mu^{4m} \gamma_2,$$

where  $\gamma_2 = \sum_j \alpha(\tilde{\Gamma}_{k,j})$ .

# Error Estimate

We estimate  $\gamma_1$  and  $\gamma_2$ . The circle  $\Gamma_\rho$  enclose all the circles  $\tilde{\Gamma}_i$ , then  $\gamma_1 < \pi\rho^2$ . Using  $g_k(w)$ , we estimate  $\gamma_2$ .  $g_k$  is univalent on  $|w| > \rho$ , in the neighborhood of  $\infty$ ,  $g_k(w) = w + O(w^{-1})$ . Perform coordinate change  $\zeta = 1/w$ ,  $\eta = 1/z$ , construct univalent holomorphic function  $\varphi : \zeta \rightarrow \eta$ ,

$$\varphi(\zeta) = \frac{1}{g_k(1/\zeta)},$$

$\varphi$  is defined on the disk  $|\zeta| < \rho^{-1}$ ,  $\varphi(0) = 0$  and  $\varphi'(0) = 1$ . By Koebe 1/4 theorem,

$$\left\{ |\eta| < \frac{1}{4\rho} \right\} \subset \varphi \left( \left\{ |\zeta| < \frac{1}{\rho} \right\} \right),$$

equivalently

$$\{|z| > 4\rho\} \subset g_k(\{|w| > \rho\}),$$

hence all  $\tilde{\Gamma}_{k,j}$  are included in the interior of  $|z| < 4\rho$ , hence the total area of all holes

$$\gamma_2 = \sum_j \alpha(\tilde{\Gamma}_{k,j}) < 16\pi\rho^2.$$

# Error Estimate

We proved the convergence rate of Koebe's iteration.

## Theorem (Convergence Rate of Koebe's Iteration)

*In the Koebe's iteration, when  $k > mn$ ,*

$$|g_k(w) - w| \leq \frac{1}{4\delta} \left( \frac{4}{\pi} \pi \rho^2 + \frac{\pi}{2 \log \mu^{-1}} 16 \pi \rho^2 \right) \mu^{4m}.$$

This shows  $\mu$  controls the convergence rate.