

# Surface Differential Geometry, Movable Frame Method

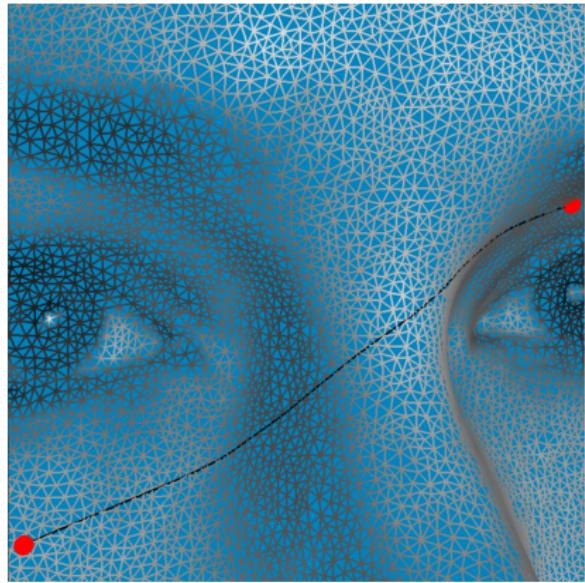
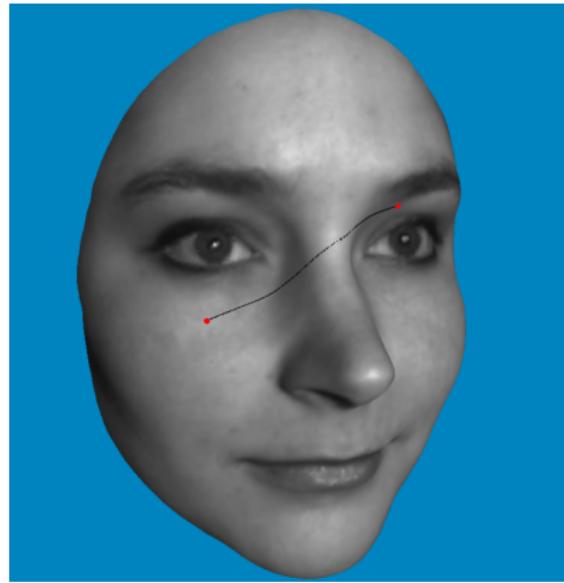
David Gu

Yau Mathematics Science Center  
Tsinghua University  
Computer Science Department  
Stony Brook University

*gu@cs.stonybrook.edu*

July 18, 2020

# Compute Geodesics

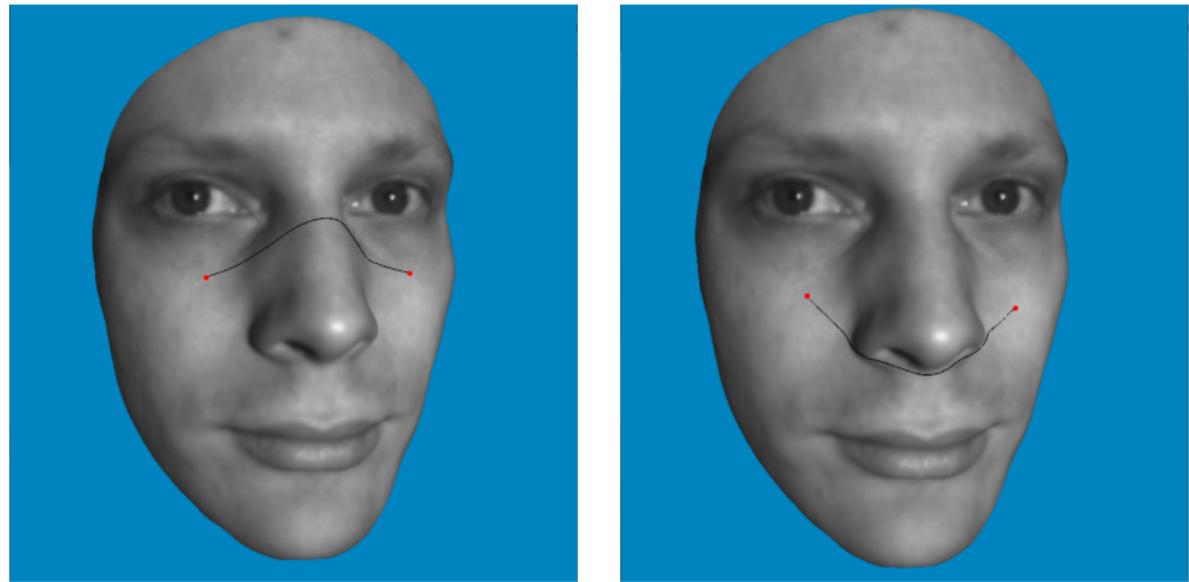


**Figure:** Geodesic on polyhedral surfaces.

Geodesic on a surface  $\gamma : [0, 1] \rightarrow (S, \mathbf{g})$ :

$$D_{\dot{\gamma}} \dot{\gamma} \equiv 0.$$

# Compute Geodesics

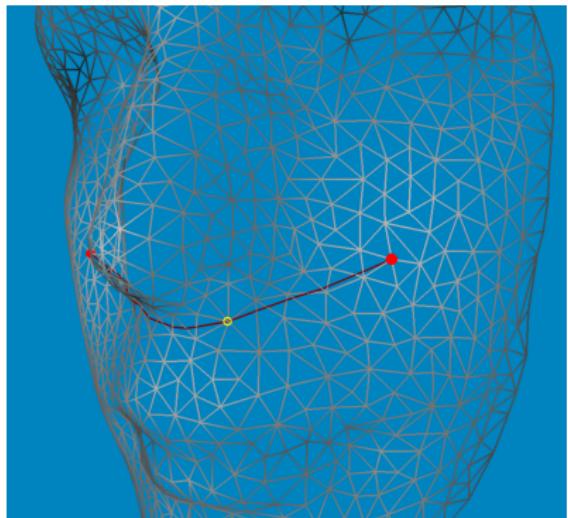
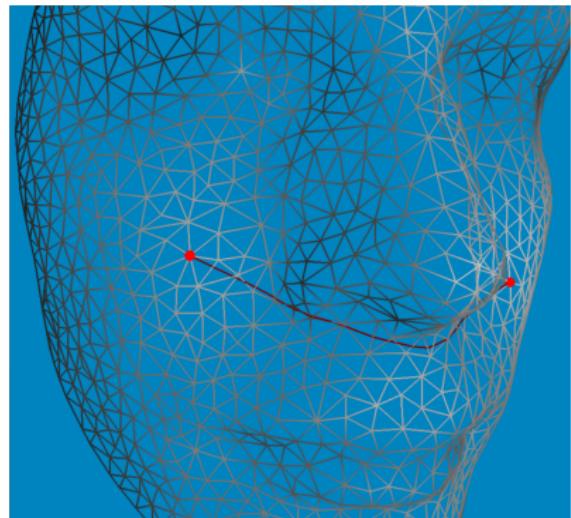


**Figure:** Conjugate point of geodesics.

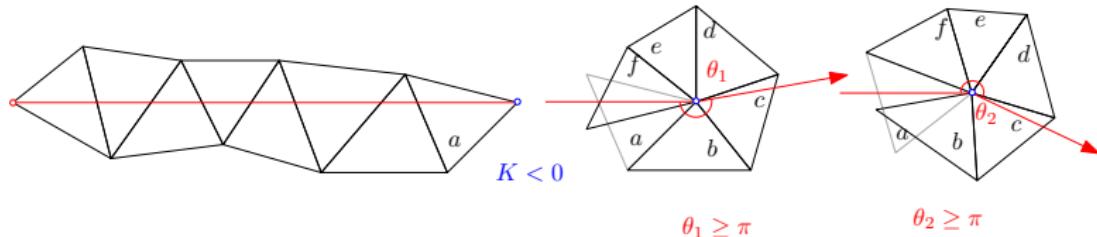
Geodesic on a surface  $\gamma : [0, 1] \rightarrow (S, g)$ :

$$D_{\dot{\gamma}} \dot{\gamma} \equiv 0.$$

# Discrete Geodesics



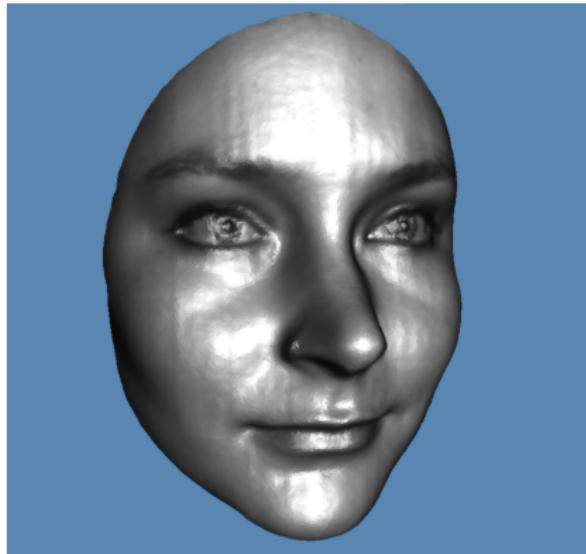
# Discrete Geodesics



Suppose  $\gamma$  is a discrete geodesic:

- ① isometrically flatten the strip of curve  $\gamma$  onto the plane;
- ② when the  $\gamma$  crosses an edge, it is straight;
- ③  $\gamma$  never crosses any convex vertex;
- ④ when  $\gamma$  crosses a concave vertex, if we flatten the neighborhood from right, then  $\theta_1 \geq \pi$ ; flatten from left,  $\theta_2 \geq \pi$ .

# Discrete Harmonic Map



Smooth surface harmonic map  $\varphi : (S, \mathbf{g}) \rightarrow \mathbb{D}^2$ ,  $\Delta_{\mathbf{g}}\varphi \equiv 0$ , with Dirichlet boundary condition  $\varphi|_{\partial S} = f$ . A discrete harmonic map satisfies

$$\sum_{v_i \sim v_j} w_{ij}(\varphi(v_i) - \varphi(v_j)) = 0, \forall v_i \notin \partial M.$$

# Compute Minimal Surface

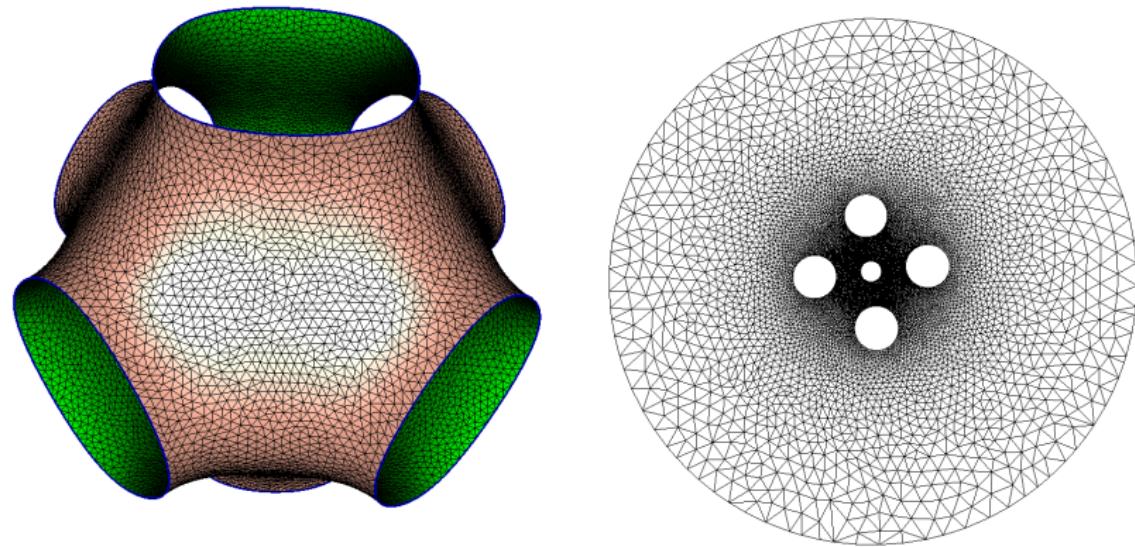
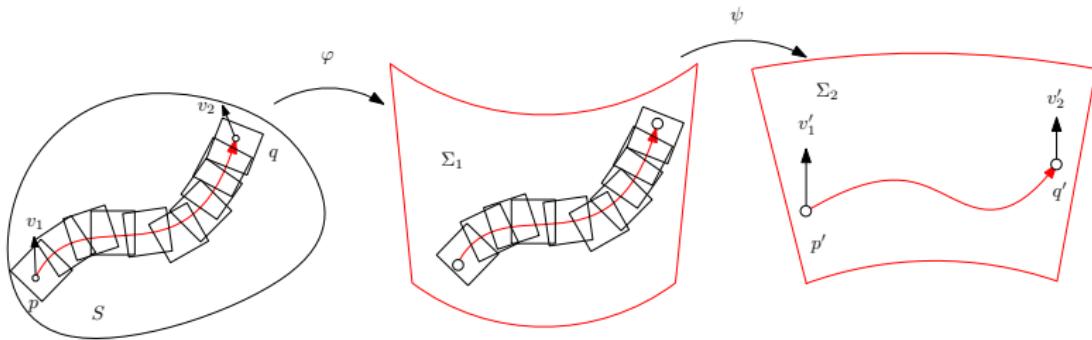


Figure: Minimal surface.

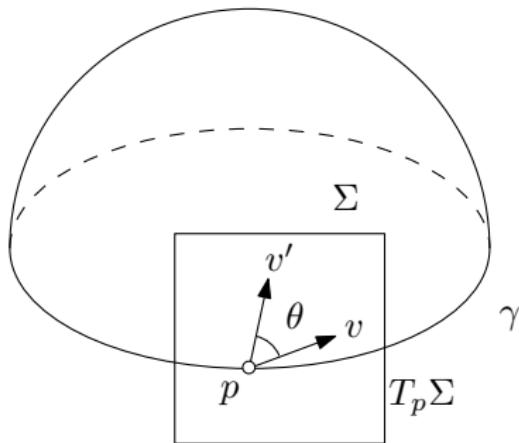
Smooth minimal surface satisfies  $\Delta_{\mathbf{g}} r \equiv 0$ , equivalently  $H(p) \equiv 0$ . A discrete minimal surface satisfies  $\sum_{v_i \sim v_j} w_{ij}(\mathbf{r}(v_i) - \mathbf{r}(v_j)) = 0, \forall v_i \notin \partial M$ .

# Parallel Transport



Given  $\gamma \subset S$ , find an envelope surface  $\Sigma_1$  of all the tangent planes along  $\gamma$ ,  $\varphi : \gamma \rightarrow \Sigma_1$  isometrically maps  $\gamma$  to  $\Sigma_1$ .  $\Sigma_1$  is developable, flatten  $\Sigma_1$  to obtain a planar domain  $\Sigma_2$ ,  $\psi : \Sigma_1 \rightarrow \Sigma_2$ . The composition  $\psi \circ \varphi$  maps  $p, q, v_1 \in T_p S$ ,  $v_2 \in T_q S$  to  $p', q', v'_1, v'_2$ . On the plane, translate a tangent vector  $v'_1$  from starting point  $p$  to the ending point  $q$  to get  $v'_2$ , maps back  $v'_2$ ,  $v_2 = (\psi \circ \varphi)^{-1}(v'_2)$ . Then  $v_1$  is parallelly transported along  $\gamma$  to get  $v_2$ .

# Gaussian Curvature



Parallel transport  $v$  along  $\partial\Sigma$ , to get  $v'$  when returned to the original point  $p$ , then the angle difference between  $v$  and  $v'$  equals to the total Gaussian curvature,

$$\theta = \int_{\Sigma} K dA.$$

# Gaussian Curvature

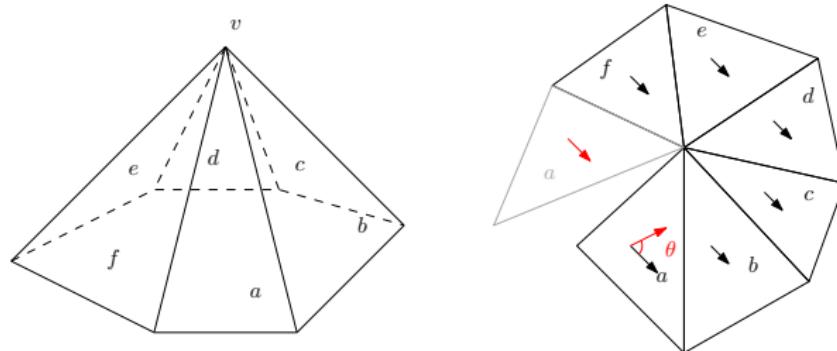


Figure: Discrete parallel transport,  $K(v) = \theta$ .

Parallel transport a vector, when return to the original position, the difference angle equals to the discrete Gaussian curvature of the interior vertices.

# Gaussian Curvature

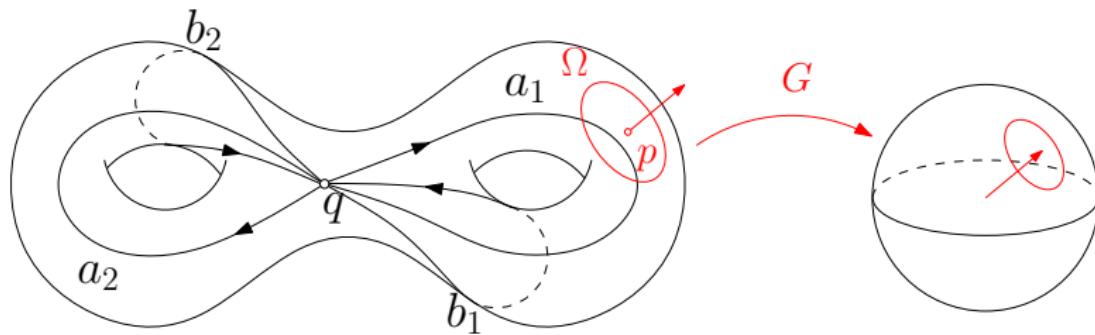


Figure: Gaussian curvature.

Gauss map:  $\mathbf{r}(p) \mapsto \mathbf{n}(p)$ ,

$$K(p) := \lim_{\Omega \rightarrow \{p\}} \frac{|G(\Omega)|}{|\Omega|}$$

# Gaussian Curvature

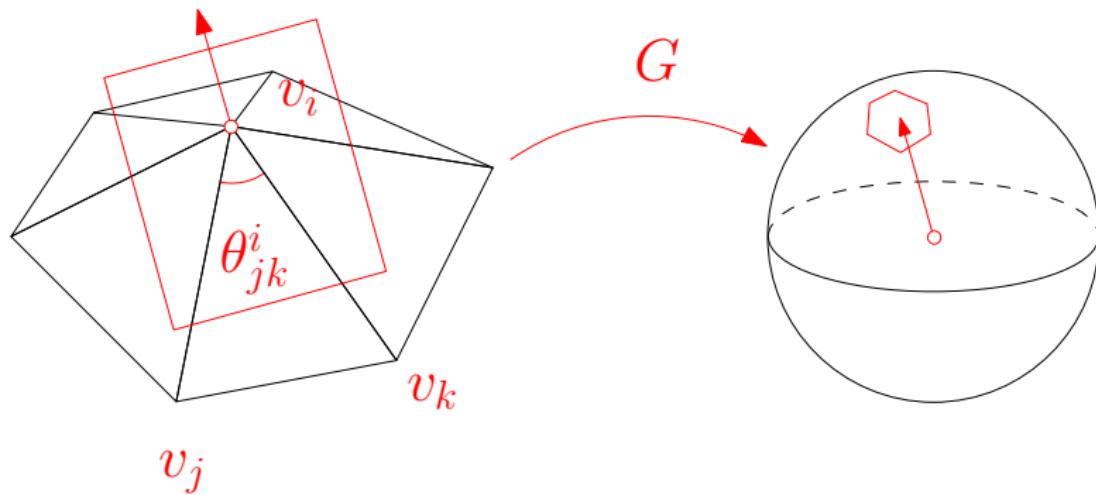


Figure: Discrete Gaussian curvature.

$$G(v_i) := \{ \mathbf{n} \in \mathbb{S}^2 \mid \exists \text{Support plane with normal } \mathbf{n} \}.$$

# Gaussian Curvature

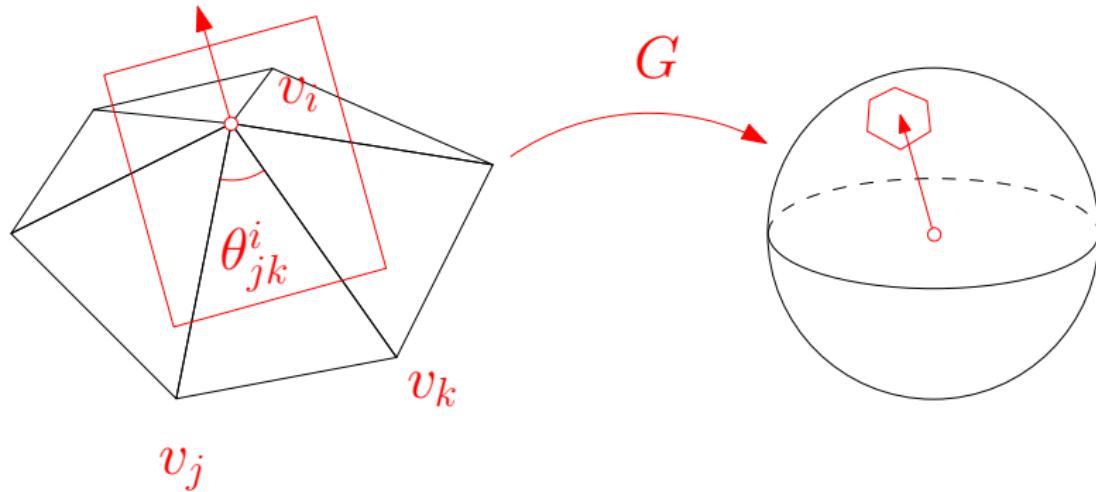


Figure: Discrete Gaussian curvature for convex vertex.

$$K(v_i) := |G(v_i)| = 2\pi - \sum_{jk} \theta_{jk}^i.$$

# Gauss-Bonnet

For a closed oriented metric surface  $(S, \mathbf{g})$ ,

$$\int_S K dA = 2\pi\chi(S).$$

For a closed oriented discrete polygonal surface  $M$ ,

$$\sum_{v_i} K(v_i) = 2\pi\chi(M).$$

# Gaussian Curvature

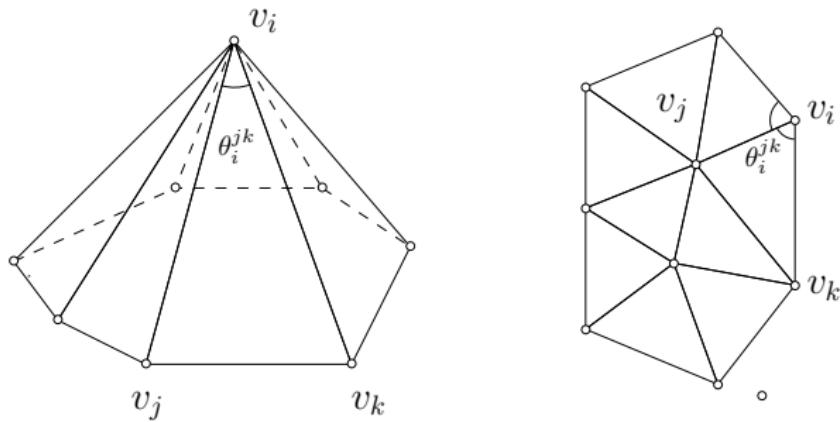


Figure: Discrete Gaussian curvature.

$$K(v_i) = \begin{cases} 2\pi - \sum_{jk} \theta_i^{jk} & v_i \notin \partial M \\ \pi - \sum_{jk} \theta_i^{jk} & v_i \in \partial M \end{cases} \quad (1)$$

# Gauss-Bonnet

## Theorem (Discrete Gauss-Bonnet Theorem)

Given polyhedral surface  $(S, V, \mathbf{d})$ , the total discrete curvature is

$$\sum_{v \notin \partial M} K(v) + \sum_{v \in \partial M} K(v) = 2\pi\chi(S),$$

where  $\chi(S)$  is the Euler characteristic number of  $S$ .

## Proof.

We denote the polyhedral surface  $M = (V, E, F)$ , if  $M$  is closed, then

$$\sum_{v_i \in V} K(v_i) = \sum_{v_i \in V} \left( 2\pi - \sum_{jk} \theta_i^{jk} \right) = \sum_{v_i \in V} 2\pi - \sum_{v_i \in V} \sum_{jk} \theta_i^{jk} = 2\pi|V| - \pi|F|.$$

Since  $M$  is closed,  $3|F| = 2|E|$ ,

$$\chi(S) = |V| + |F| - |E| = |V| + |F| - \frac{3}{2}|F| = |V| - \frac{1}{2}|F|.$$

□

# Discrete Guass-Bonnet

continued.

Assume  $M$  has boundary  $\partial M$ . Assume the interior vertex set is  $V_0$ , boundary vertex set is  $V_1$ , then  $|V| = |V_0| + |V_1|$ ; assume interior edge set is  $E_0$ , boundary edge set is  $E_1$ , then  $|E| = |E_0| + |E_1|$ . Furthermore, all boundaries are closed loops, hence boundary vertex number equals to the boundary edge number,  $|V_1| = |E_1|$ . Every interior edge is adjacent to two faces, every boundary edge is adjacent to one face, we have

$3|F| = 2|E_0| + |E_1| = 2|E_0| + |V_1|$ . We compute the Euler number

$$\chi(M) = |V| + |F| - |E| = |V_0| + |V_1| + |F| - |E_0| - |E_1| = |V_0| + |F| - |E_0|,$$

by  $|E_0| = 1/2(3|F| - |V_1|)$

$$\chi(M) = |V_0| - \frac{1}{2}|F| + \frac{1}{2}|V_1|$$

# Discrete Guass-Bonnet

continued.

we have:

$$\begin{aligned}\sum_{v_i \in V_0} K(v_i) + \sum_{v_j \in V_1} K(v_j) &= \sum_{v_i \in V_0} \left( 2\pi - \sum_{jk} \theta_i^{jk} \right) + \sum_{v_i \in V_1} \left( \pi - \sum_{jk} \theta_i^{jk} \right) \\ &= 2\pi|V_0| + \pi|V_1| - \pi|F| \\ &= 2\pi \left( |V_0| - \frac{1}{2}|F| + \frac{1}{2}|V_1| \right) \\ &= 2\pi\chi(M).\end{aligned}\tag{2}$$

□.

# Movable Frame

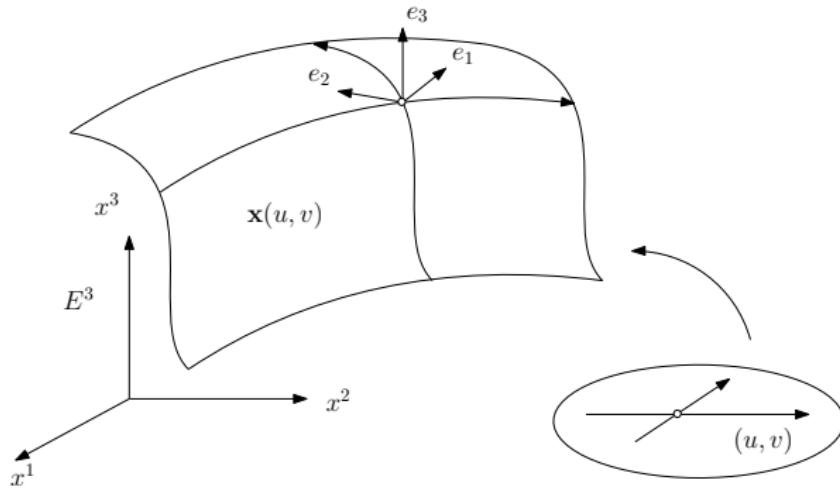


Figure: A parametric surface.

# Orthonormal Movable frame

## Movable Frame

Suppose a regular surface  $S$  is embedded in  $\mathbb{R}^3$ , a parametric representation is  $\mathbf{r}(u, v)$ . Select two vector fields  $\mathbf{e}_1, \mathbf{e}_2$ , such that

$$\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta_{ij}.$$

Let  $\mathbf{e}_3$  be the unit normal field of the surface. Then

$$\{\mathbf{r}; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$$

form the *orthonormal frame field* of the surface.

# Orthonormal Movalbe frame

## Tangent Vector

The tangent vector is the linear combination of the frame bases,

$$d\mathbf{r} = \omega_1 \mathbf{e}_1 + \omega_2 \mathbf{e}_2$$

where  $\omega_k(\mathbf{v}) = \langle \mathbf{e}_k, \mathbf{v} \rangle$ .  $d\mathbf{r}$  is orthogonal to the normal vector  $\mathbf{e}_3$ .

## Motion Equation

$$d\mathbf{e}_i = \omega_{i1} \mathbf{e}_1 + \omega_{i2} \mathbf{e}_2 + \omega_{i3} \mathbf{e}_3,$$

where  $\omega_{ij} = \langle d\mathbf{e}_i, \mathbf{e}_j \rangle$ . Because

$$\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \delta_{ij}, \quad 0 = d\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \langle d\mathbf{e}_i, \mathbf{e}_j \rangle + \langle \mathbf{e}_i, d\mathbf{e}_j \rangle$$

we get

$$\omega_{ij} + \omega_{ji} = 0, \omega_{ii} = 0.$$

# Motion Equation

## Motion Equation

$$d\mathbf{r} = \omega_1 \mathbf{e}_1 + \omega_2 \mathbf{e}_2,$$

$$\begin{pmatrix} d\mathbf{e}_1 \\ d\mathbf{e}_2 \\ d\mathbf{e}_3 \end{pmatrix} = \begin{pmatrix} 0 & \omega_{12} & \omega_{13} \\ -\omega_{12} & 0 & \omega_{23} \\ -\omega_{13} & -\omega_{23} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}$$

## Fundamental Forms

The first fundamental form is

$$I = \langle d\mathbf{r}, d\mathbf{r} \rangle = \omega_1 \omega_1 + \omega_2 \omega_2.$$

The second fundamental form is

$$II = -\langle d\mathbf{r}, d\mathbf{e}_3 \rangle = -\omega_1 \omega_{31} - \omega_2 \omega_{32} = \omega_1 \omega_{13} + \omega_2 \omega_{23}.$$

# Weingarten Mapping

## Definition (Weingarten Mapping)

The Gauss mapping is

$$\mathbf{r} \rightarrow \mathbf{e}_3,$$

its derivative map is called the Weingarten mapping,

$$d\mathbf{r} \rightarrow d\mathbf{e}_3, \omega_1 \mathbf{e}_1 + \omega_2 \mathbf{e}_2 \rightarrow \omega_{31} \mathbf{e}_1 + \omega_{32} \mathbf{e}_2.$$

## Definition (Gaussian Curvature)

The area ratio (Jacobian of the Weingarten mapping) is the Gaussian curvature

$$K \omega_1 \wedge \omega_2 = \omega_{31} \wedge \omega_{32}.$$

# Gaussian curvature

## Weigarten Mapping

$\{\omega_1, \omega_2\}$  form the basis of the cotangent space, therefore  $\omega_{13}, \omega_{23}$  can be represented as the linear combination of them,

$$\begin{pmatrix} \omega_{13} \\ \omega_{23} \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$

therefore

$$\omega_{13} \wedge \omega_{23} = \begin{vmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{vmatrix} \omega_1 \wedge \omega_2$$

so  $K = h_{11}h_{22} - h_{12}h_{21}$ , the mean curvature  $H = \frac{1}{2}(h_{11} + h_{22})$ .

# Gauss's theorem Egregium

## Theorem (Gauss' Theorem Egregium)

*The Gaussian curvature is intrinsic, solely determined by the first fundamental form.*

## Proof.

$$\begin{aligned}0 &= d^2 \mathbf{e}_1 \\&= d(\omega_{12} \mathbf{e}_2 + \omega_{13} \mathbf{e}_3) \\&= d\omega_{12} \mathbf{e}_2 - \omega_{12} \wedge d\mathbf{e}_2 + d\omega_{13} \mathbf{e}_3 - \omega_{13} \wedge d\mathbf{e}_3 \\&= d\omega_{12} \mathbf{e}_2 - \omega_{12} \wedge (\omega_{21} \mathbf{e}_1 + \omega_{23} \mathbf{e}_3) + \\&\quad d\omega_{13} \mathbf{e}_3 - \omega_{13} \wedge (\omega_{31} \mathbf{e}_1 + \omega_{32} \mathbf{e}_2) \\&= (d\omega_{12} - \omega_{13} \wedge \omega_{32}) \mathbf{e}_2 + (d\omega_{13} - \omega_{12} \wedge \omega_{23}) \mathbf{e}_3\end{aligned}$$

therefore

$$d\omega_{12} = -\omega_{13} \wedge \omega_{32} = -K \omega_1 \wedge \omega_2.$$

# Gauss's theorem Egregium

## Lemma

$$\omega_{12} = \frac{d\omega_1}{\omega_1 \wedge \omega_2} \omega_1 + \frac{d\omega_2}{\omega_1 \wedge \omega_2} \omega_2$$

## Proof.

$$\begin{aligned} 0 &= d^2 \mathbf{r} \\ &= d(\omega_1 \mathbf{e}_1 + \omega_2 \mathbf{e}_2) \\ &= d\omega_1 \mathbf{e}_1 - \omega_1 \wedge d\mathbf{e}_1 + d\omega_2 \mathbf{e}_2 - \omega_2 \wedge d\mathbf{e}_2 \\ &= d\omega_1 \mathbf{e}_1 - \omega_1 \wedge (\omega_{12} \mathbf{e}_2 + \omega_{13} \mathbf{e}_3) + \\ &\quad d\omega_2 \mathbf{e}_2 - \omega_2 \wedge (\omega_{21} \mathbf{e}_1 + \omega_{23} \mathbf{e}_3) \\ &= (d\omega_1 - \omega_2 \wedge \omega_{21}) \mathbf{e}_1 + (d\omega_2 - \omega_1 \wedge \omega_{12}) \mathbf{e}_2 + \\ &\quad - (\omega_1 \wedge \omega_{13} + \omega_2 \wedge \omega_{23}) \mathbf{e}_3. \end{aligned}$$

Therefore  $d\omega_1 = \omega_2 \wedge \omega_{21}$ ,  $d\omega_2 = \omega_1 \wedge \omega_{12}$  and  $h_{12} = h_{21}$ . □

# Gaussian Curvature

## Lemma (Gaussian curvature)

Under the isothermal coordinates, the Gaussian curvature is given by

$$K = -\frac{1}{e^{2u}} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u.$$

## Proof.

Let  $(S, \mathbf{g})$  be a metric surface, use isothermal coordinates

$$\mathbf{g} = e^{2u(x,y)}(dx^2 + dy^2).$$

Then

$$\begin{cases} \omega_1 = e^u dx \\ \omega_2 = e^u dy \end{cases} \quad \begin{cases} \mathbf{e}_1 = e^{-u} \frac{\partial}{\partial x} \\ \mathbf{e}_2 = e^{-u} \frac{\partial}{\partial y} \end{cases}$$



# Gaussian Curvature

Continued.

By direct computation,

$$\begin{aligned} d\omega_1 &= de^u \wedge dx & d\omega_2 &= de^u \wedge dy \\ &= e^u(u_x dx + u_y dy) \wedge dx & &= e^u(u_x dx + u_y dy) \wedge dy \\ &= e^u u_y dy \wedge dx & &= e^u u_x dx \wedge dy. \end{aligned}$$

therefore

$$\begin{aligned} \omega_{12} &= \frac{d\omega_1}{\omega_1 \wedge \omega_2} \omega_1 + \frac{d\omega_2}{\omega_1 \wedge \omega_2} \omega_2 \\ &= \frac{e^u u_y dy \wedge dx}{e^{2u} dx \wedge dy} e^u dx + \frac{e^u u_x dx \wedge dy}{e^{2u} dx \wedge dy} e^u dy \\ \omega_{12} &= -u_y dx + u_x dy. \end{aligned}$$

# Gaussian Curvature

Continued.

$$K = -\frac{d\omega_{12}}{\omega_1 \wedge \omega_2} = -\frac{(u_{xx} + u_{yy})dx \wedge dy}{e^{2u}dx \wedge dy} = -\frac{1}{e^{2u}}\Delta u.$$

# Gaussian Curvature

## Example

The unit disk  $|z| < 1$  equipped with the following metric

$$ds^2 = \frac{4dzd\bar{z}}{(1 - z\bar{z})^2},$$

the Gaussian curvature is  $-1$  everywhere.

## Proof.

$e^{2u} = \frac{4}{1-x^2-y^2}$ , then  $u = \log 2 - \log(1-x^2-y^2)$ .

$$u_x = -\frac{-2x}{1-x^2-y^2} = \frac{2x}{1-x^2-y^2}.$$



# Gaussian Curvature

Proof.

then

$$u_{xx} = \frac{2(1 - x^2 - y^2) - 2x(-2x)}{(1 - x^2 - y^2)^2} = \frac{2 + 2x^2 - 2y^2}{(1 - x^2 - y^2)^2}$$

similarly

$$u_{yy} = \frac{2 + 2y^2 - 2x^2}{(1 - x^2 - y^2)^2}$$

so

$$u_{xx} + u_{yy} = \frac{4}{(1 - x^2 - y^2)} = e^{2u}, K = -\frac{1}{e^{2u}}(u_{xx} + u_{yy}) = -1.$$



# Yamabe Equation

## Lemma (Yamabe Equation)

Conformal metric deformation  $\mathbf{g} \rightarrow e^{2\lambda}\mathbf{g} = \tilde{\mathbf{g}}$ , then

$$\tilde{K} = \frac{1}{e^{2\lambda}}(K - \Delta_{\mathbf{g}}\lambda).$$

## Proof.

Use isothermal parameters,  $\mathbf{g} = e^{2u}(dx^2 + dy^2)$ ,  $K = -e^{2u}\Delta u$ , similarly  $\tilde{\mathbf{g}} = e^{2\tilde{u}}(dx^2 + dy^2)$ ,  $\tilde{K} = -e^{2\tilde{u}}\Delta \tilde{u}$ ,  $\tilde{u} = u + \lambda$ ,

$$\begin{aligned}\tilde{K} &= -\frac{1}{e^{2(u+\lambda)}}\Delta(u + \lambda) \\ &= \frac{1}{e^{2\lambda}}\left(-\frac{1}{e^{2u}}\Delta u - \frac{1}{e^{2u}}\Delta\lambda\right) \\ &= \frac{1}{e^{2\lambda}}(K - \Delta_{\mathbf{g}}\lambda).\end{aligned}$$

# Gauss-Bonnet Theorem

## Theorem (Gauss-Bonnet)

Suppose  $M$  is a closed orientable  $C^2$  surface, then

$$\int_M A dA = 2\pi\chi(M),$$

where  $dA$  is the area element of the surface,  $\chi(M)$  is the Euler characteristic number of  $M$ .

## Proof.

Construct a smooth vector field  $v$ , with isolated zeros  $\{p_1, p_2, \dots, p_n\}$ . Choose a small disk  $D(p_i, \varepsilon)$ . On the surface

$$\bar{M} = M \setminus \bigcup_{i=1}^n D(p_i, \varepsilon)$$



# Gauss-Bonnet Theorem

Proof.

construct orthonormal frame  $\{p, e_1, e_2, e_3\}$ , where

$$e_1(p) = \frac{v(p)}{|v(p)|}, \quad e_3(p) = n(p).$$

The integration

$$\int_{\bar{M}} K dA = \int_{\bar{M}} K \omega_1 \wedge \omega_2 = - \int_{\bar{M}} d\omega_{12}$$

by Stokes theorem and Poincar  re-Hopf theorem, we obtain

$$- \sum_{i=1}^n \int_{\partial D(p_i, \varepsilon)} \omega_{12} = 2\pi \sum_{i=1}^n \text{Index}(p_i, v) = 2\pi \chi(M).$$

Here by  $\omega_{12} = \langle de_1, e_2 \rangle$ ,  $\omega_{12}$  is the rotation speed of  $e_1$ . Let  $\varepsilon \rightarrow 0$ , the equation holds.



# Computing Geodesics

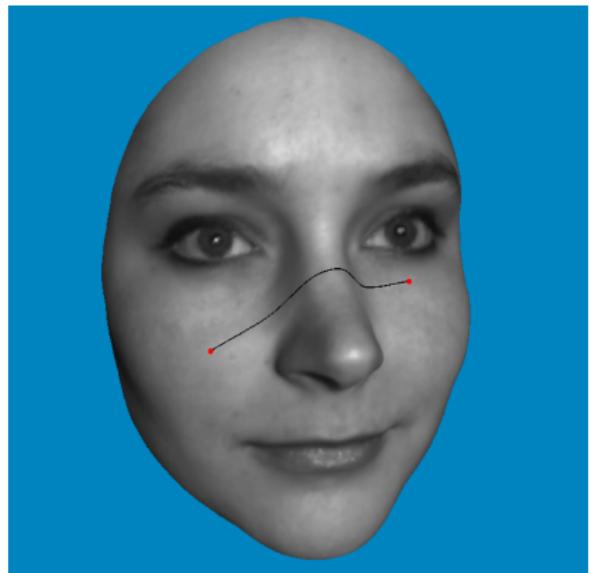
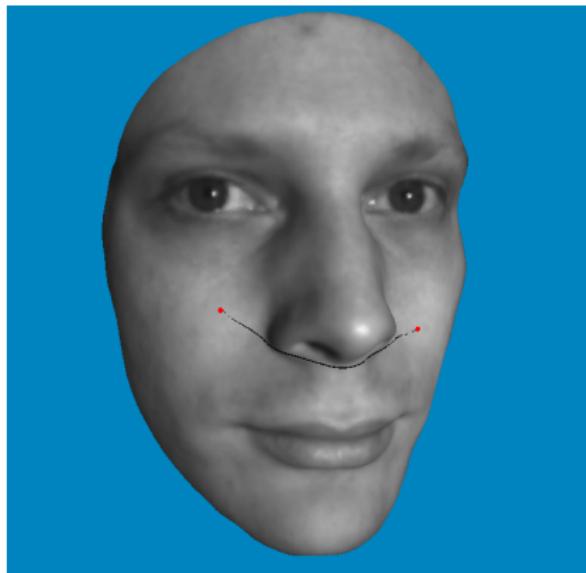


Figure: Geodesics.

# Covariant Differential

## Definition (Covariant Differentiation)

Covariant differentiation is the generalization of directional derivatives, satisfies the following properties: assume  $v$  and  $w$  are tangent vector fields on a surface,  $f : S \rightarrow \mathbb{R}$  is a  $C^1$  function, then

- ①  $D(v + w) = D(v) + D(w),$
- ②  $D(fv) = df(v) + fDv,$
- ③  $D\langle v, w \rangle = \langle Dv, w \rangle + \langle v, Dw \rangle.$

By movable framework, the motion equation of the surface is

$$d\mathbf{e}_1 = \omega_{12}\mathbf{e}_2 + \omega_{13}\mathbf{e}_3, \quad d\mathbf{e}_2 = \omega_{21}\mathbf{e}_1 + \omega_{23}\mathbf{e}_3,$$

We only keep tangential component, and delete the normal part to obtain covariant differential

$$D\mathbf{e}_1 = \omega_{12}\mathbf{e}_1, \quad D\mathbf{e}_2 = \omega_{21}\mathbf{e}_1.$$

# Covariant Differential

## Definition (Parallel transport)

Suppose  $S$  is a metric surface,  $\gamma : [0, 1] \rightarrow S$  is a smooth curve,  $v(t)$  is a vector field along  $\gamma$ , if

$$\frac{Dv}{dt} \equiv 0,$$

then we say the vector field  $v(t)$  is parallel transportation along  $\gamma$ .

Given a tangent vector field  $v = f_1\mathbf{e}_1 + f_2\mathbf{e}_2$ , then

$$\begin{aligned} Dv &= df_1\mathbf{e}_1 + f_1D\mathbf{e}_1 + df_2\mathbf{e}_2 + f_2D\mathbf{e}_2 \\ &= (df_1 - f_2\omega_{12})\mathbf{e}_1 + (df_2 + f_1\omega_{12})\mathbf{e}_2. \end{aligned}$$

and

$$\frac{Dv}{dt} = \left( \frac{df_1}{dt} - f_2 \frac{\omega_{12}}{dt} \right) \mathbf{e}_1 + \left( \frac{df_2}{dt} + f_1 \frac{\omega_{12}}{dt} \right) \mathbf{e}_2.$$

where  $\frac{\omega_{12}}{dt} = \langle \omega_{12}, \dot{\gamma} \rangle$ . If  $\omega_{12} = \alpha dx + \beta dy$ , then  $\frac{\omega_{12}}{dt} = \alpha \dot{x} + \beta \dot{y}$ .

# Parallel Transport

## Parallel Transport Equation

Therefore parallel vector field satisfies the ODE

$$\begin{cases} \frac{df_1}{dt} - f_2 \frac{\omega_{12}}{dt} = 0 \\ \frac{df_2}{dt} + f_1 \frac{\omega_{12}}{dt} = 0 \end{cases}$$

Given an initial condition  $v(0)$ , the solution uniquely exists.

# Geodesic Curvature

## Definition (Geodesic Curvature)

Assume  $\gamma : [0, 1] \rightarrow S$  is a  $C^2$  curve on a surface  $S$ ,  $s$  is the arc length parameter. Construct orthonormal frame field along the curve  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ , where  $\mathbf{e}_1$  is the tangent vector field of  $\gamma$ ,  $\mathbf{e}_3$  is the normal field of the surface,

$$k_g := \frac{D\mathbf{e}_1}{ds} = k_g \mathbf{e}_2$$

is called geodesic curvature vector,

$$k_g = \left\langle \frac{D\mathbf{e}_1}{ds}, \mathbf{e}_2 \right\rangle = \frac{\omega_{12}}{ds}$$

is called geodesic curvature.

# Geodesic Curvature

## Geodesic curvature, normal curvature

Given a spacial curve, its curvature vector satisfies

$$\frac{d^2\gamma}{ds^2} = k_g \mathbf{e}_2 + k_n \mathbf{e}_3,$$

where  $k_n$  is the normal curvature of the curve. The curvature of the curve, geodesic curvature and normal curvature satisfy

$$k^2 = k_g^2 + k_n^2.$$

Geodesic curvature  $k_g$  only depends on the Riemannian metric of the surface, is independent of the 2nd fundamental form. Therefore  $k_g$  is intrinsic,  $k_n$  is extrinsic.

# Gauss-Bonnet

## Theorem

Suppose  $(S, \mathbf{g})$  is an oriented metric surface with boundaries, then

$$\int_S K dA + \int_{\partial S} k_g ds = 2\pi\chi(S).$$

## Proof.

Construct a vector field with isolated zeros  $\{p_i\}$ ,  $\mathbf{e}_1$  is tangent to  $\partial S$ , small disks  $D(p_i, \varepsilon)$ . Define  $\bar{S} := S \setminus \bigcup_i D(p_i, \varepsilon)$ ,

$$\begin{aligned}\int_{\bar{S}} K dA &= - \int_{\bar{S}} \frac{d\omega_{12}}{\omega_1 \wedge \omega_2} dA = - \int_{\bar{S}} d\omega_{12} = - \int_{\partial \bar{S}} \omega_{12} \\ &= - \int_{\partial S - \bigcup_i \partial D(p_i, \varepsilon)} \omega_{12} = - \int_{\partial S} \frac{\omega_{12}}{ds} ds + \sum_i \int_{\partial D(p_i, \varepsilon)} \omega_{12} \\ &= - \int_{\partial S} k_g ds + 2\pi \sum_i \text{Index}(p_i) = - \int_{\partial S} k_g ds + 2\pi\chi(S).\end{aligned}$$

# Geodesic Curvature

We use isothermal parameter  $(u, v)$  of  $(S, \mathbf{g})$ , given a curve  $\gamma(s)$  with arc length parameter  $s$ . Construct orthonormal frame  $\{p; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ , where  $\mathbf{e}_3$  is the normal field of  $S$ . The tangent vector of  $\gamma$  is  $\bar{\mathbf{e}}_1$ ,  $\bar{\mathbf{e}}_2$  is orthogonal to  $\bar{\mathbf{e}}_1$  everywhere. The angle between  $\bar{\mathbf{e}}_1$  and  $\mathbf{e}_1$  is  $\theta(s)$ ,

$$\begin{cases} \bar{\mathbf{e}}_1 &= \cos \theta \mathbf{e}_1 + \sin \theta \mathbf{e}_2 \\ \bar{\mathbf{e}}_2 &= -\sin \theta \mathbf{e}_1 + \cos \theta \mathbf{e}_2 \end{cases}$$

Direct computation

$$\begin{aligned} D\bar{\mathbf{e}}_1 &= D(\cos \theta \mathbf{e}_1 + \sin \theta \mathbf{e}_2) = d \cos \theta \mathbf{e}_1 + \cos \theta D\mathbf{e}_1 + d \sin \theta \mathbf{e}_2 + \sin \theta D\mathbf{e}_2 \\ &= -\sin \theta d\theta \mathbf{e}_1 + \cos \theta \omega_{12} \mathbf{e}_2 + \cos \theta d\theta \mathbf{e}_2 - \sin \theta \omega_{12} \mathbf{e}_1 \\ &= -\sin \theta(d\theta + \omega_{12}) \mathbf{e}_1 + \cos \theta(\omega_{12} + d\theta) \mathbf{e}_2 \end{aligned}$$

$$k_g = \left\langle \frac{D\bar{\mathbf{e}}_1}{ds}, \bar{\mathbf{e}}_2 \right\rangle = \frac{d\theta}{ds} + \frac{\omega_{12}}{ds}$$

# Geodesic Curvature

Under the isothermal coordinates, we have  $\omega_{12} = -u_y dx + u_x dy$ . Suppose on the parameter domain, the planar curve arc length is  $dt$ , then  $ds = e^u dt$ . The parameterization preserves angle, therefore

$$\begin{aligned} k_g &= \frac{d\theta}{ds} + \frac{-u_y dx + u_x dy}{ds} \\ &= \frac{d\theta}{dt} \frac{dt}{ds} + \frac{-u_y dx + u_x dy}{dt} \frac{dt}{ds} \\ &= e^{-u}(k - \langle \nabla u, n \rangle) \\ &= e^{-u}(k - \partial_n u) \end{aligned}$$

where  $k$  is the curvature of the planar curve,  $n$  is the normal to the planar curve.

# Geodesic Curvature

## Lemma

Given a metric surface  $(S, \mathbf{g})$ , under conformal deformation,  $\bar{\mathbf{g}} = e^{2\lambda} \mathbf{g}$ , the geodesic curvature satisfies

$$k_{\bar{\mathbf{g}}} = e^{-\lambda} (k_{\mathbf{g}} - \partial_{\mathbf{n}, \mathbf{g}} \lambda).$$

## Proof.

$$\begin{aligned} k_{\mathbf{g}} &= e^{-(u+\lambda)} (k - \partial_{\mathbf{n}}(u + \lambda)) \\ &= e^{-\lambda} (e^{-u}(k - \partial_{\mathbf{n}} u) - e^{-u} \partial_{\mathbf{n}} \lambda) \\ &= e^{-\lambda} (k_{\mathbf{g}} - \partial_{\mathbf{n}, \mathbf{g}} \lambda) \end{aligned}$$



## Definition (geodesic)

Given a metric surface  $(S, \mathbf{g})$ , a curve  $\gamma : [0, 1] \rightarrow S$  is a geodesic if  $k_{\mathbf{g}}$  is zero everywhere.

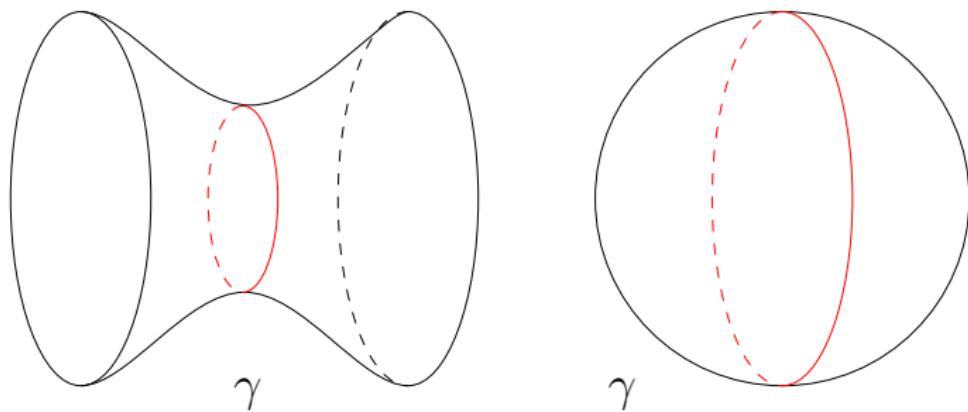


Figure: Stable and unstable geodesics.

# Geodesics

## Lemma (geodesic)

If  $\gamma$  is the shortest curve connecting  $p$  and  $q$ , then  $\gamma$  is a geodesic.

## Proof.

Consider a family of curves,  $F : (-\varepsilon, \varepsilon) \rightarrow S$ , such that  $F(0, t) = \gamma(t)$ , and

$$F(s, 0) = p, F(s, 1) = q, \frac{\partial F(s, t)}{\partial s} = f(t)\mathbf{e}_2(t),$$

where  $f : [0, 1] \rightarrow \mathbb{R}$ ,  $f(0) = f(1) = 0$ . Fix parameter  $s$ , curve  $\gamma_s := F(s, \cdot)$ ,  $\{\gamma_s\}$  for a variation. Define an energy,

$$L(s) = \int_0^1 \left| \frac{d\gamma_s(t)}{dt} \right|^2 dt, \quad \frac{\partial L(s)}{\partial s} = - \int_0^1 fk_{\mathbf{g}}(\tau) d\tau.$$



# Geodesics

The second derivative of the length variation  $L(s)$  depends on the Gaussian curvature of the underlying surface. If  $K < 0$ , then the second derivative is positive, the geodesic is stable; if  $K > 0$ , then the secondary derivative is negative, the geodesic is unstable.

## Lemma (Uniqueness of geodesics)

*Suppose  $(S, \mathbf{g})$  is a closed oriented metric surface,  $\mathbf{g}$  induces negative Gaussian curvature everywhere, then each homotopy class has a unique geodesic.*

### Proof.

The existence can be obtained by variational method. The uniqueness is by Gauss-Bonnet theorem. Assume two geodesics  $\gamma_1 \sim \gamma_2$ , then they bound a topological annulus  $\Sigma$ , by Gauss-Bonnet,

$$\int_{\Sigma} K dA + \int_{\partial\Sigma} k_g ds = \chi(\Sigma),$$

The first term is negative, the second is along the geodesics, hence 0,  $\chi(\Sigma) = 0$ . Contradiction. □

# Homotopy Detection

Input: A high genus closed mesh  $M$ , two loops  $\gamma_1$  and  $\gamma_2$ ;

Output: Whether  $\gamma_1 \sim \gamma_2$ ;

- ① Compute a hyperbolic metric of  $M$ , using Ricci flow;
- ② Homotopically deform  $\gamma_k$  to geodesics,  $k = 1, 2$ ;
- ③ if two geodesics coincide, return true; otherwise, return false;

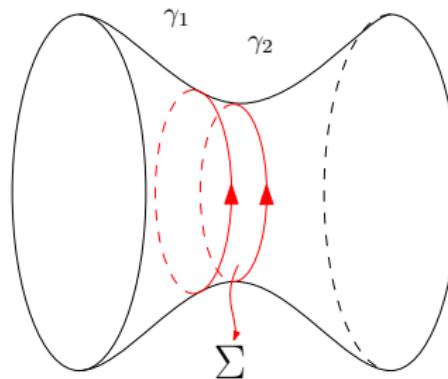


Figure: Geodesics uniqueness.

# Compute Minimal Surface

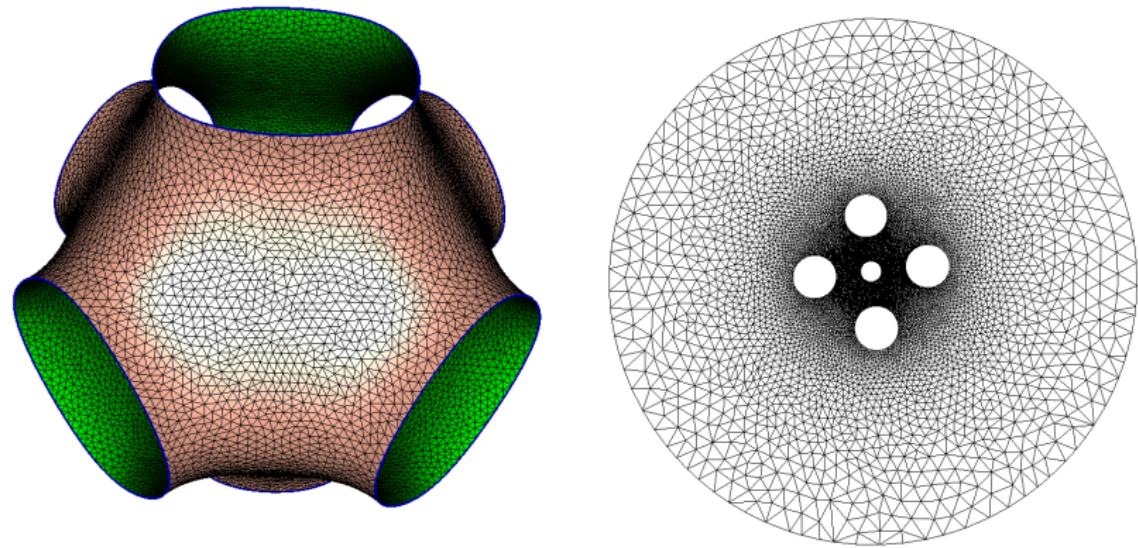


Figure: Minimal surface.

Smooth minimal surface satisfies  $\Delta_{\mathbf{g}} r \equiv 0$ , equivalently  $H(p) \equiv 0$ . A discrete minimal surface satisfies  $\sum_{v_i \sim v_j} w_{ij}(\mathbf{r}(v_i) - \mathbf{r}(v_j)) = 0, \forall v_i \notin \partial M$ .

# Minimal Surface

## Lemma

Given a metric surface  $(S, \mathbf{g})$  embedded in  $\mathbb{R}^3$ , then  $\Delta_{\mathbf{g}} \mathbf{r} = 2H(p)\mathbf{n}$ , where  $\mathbf{r}, \mathbf{n}$  are the position and normal vectors.

## Proof.

We choose isothermal coordinates  $(x, y)$ . Then  $\mathbf{g} = e^{2\lambda} \mathbf{I}$ ,  
 $\omega_{12} = -\lambda_y dx + \lambda_x dy$ ,  $\omega_{13} = h_{11}\omega_1 + h_{12}\omega_2$ ,  $\omega_{23} = h_{12}\omega_1 + h_{22}\omega_2$ ,  
 $\omega_1 = e^\lambda dx$ ,  $\omega_2 = e^\lambda dy$ ,

$$\begin{aligned}\frac{\partial}{\partial x} \mathbf{r}_x &= \frac{\partial}{\partial x} e^\lambda \mathbf{e}_1 = e^\lambda \lambda_x \mathbf{e}_1 + e^\lambda \frac{\partial}{\partial x} \mathbf{e}_1 \\&= e^\lambda \lambda_x \mathbf{e}_1 + e^\lambda \langle d\mathbf{e}_1, \frac{\partial}{\partial x} \rangle = e^\lambda \lambda_x \mathbf{e}_1 + e^\lambda \langle \omega_{12} \mathbf{e}_2 + \omega_{13} \mathbf{e}_3, \partial_x \rangle \\&= e^\lambda \lambda_x \mathbf{e}_1 + e^\lambda (-\lambda_y) \mathbf{e}_2 + e^\lambda \mathbf{e}_3 \langle h_{11} \omega_1, \partial_x \rangle \\&= e^\lambda \lambda_x \mathbf{e}_1 - e^\lambda \lambda_y \mathbf{e}_2 + e^{2\lambda} h_{11} \mathbf{e}_3\end{aligned}$$

# Minimal Surface

Proof.

Similarly,

$$\begin{aligned}\frac{\partial}{\partial y} \mathbf{r}_y &= \frac{\partial}{\partial y} e^\lambda \mathbf{e}_2 = e^\lambda \lambda_y \mathbf{e}_2 + e^\lambda \frac{\partial}{\partial y} \mathbf{e}_2 \\&= e^\lambda \lambda_y \mathbf{e}_2 + e^\lambda \langle d\mathbf{e}_2, \frac{\partial}{\partial y} \rangle = e^\lambda \lambda_y \mathbf{e}_2 + e^\lambda \langle \omega_{21} \mathbf{e}_1 + \omega_{23} \mathbf{e}_3, \partial_y \rangle \\&= e^\lambda \lambda_y \mathbf{e}_2 + e^\lambda (-\lambda_y) \mathbf{e}_2 + e^\lambda \mathbf{e}_3 \langle h_{22} \omega_2, \partial_y \rangle \\&= e^\lambda \lambda_y \mathbf{e}_2 - e^\lambda \lambda_x \mathbf{e}_1 + e^{2\lambda} h_{22} \mathbf{e}_3\end{aligned}$$

Therefore

$$\Delta_g \mathbf{r} = \frac{1}{e^{2\lambda}} (\mathbf{r}_{xx} + \mathbf{r}_{yy}) = (h_{11} + h_{22}) \mathbf{e}_3 = 2H \mathbf{e}_3.$$



# Surface Area Variation

## Lemma

*Given a surface  $S$  with position vector  $\mathbf{r}(x, y)$ , perturb the surface along the normal direction*

$$\mathbf{r}_\varepsilon(x, y) = \mathbf{r}(x, y) + \varepsilon \mathbf{n}(x, y),$$

*the area variation is given by*

$$\frac{d}{d\varepsilon} \Big|_{\varepsilon=0} \text{Area}(\mathbf{r}_\varepsilon) = \int_S 2H e^{2u(x,y)} dx dy = \int_S 2H dA.$$

# Surface Area Variation

## Proof.

We use isothermal coordinate, the first fundamental form:

$$E = \langle \mathbf{r}_x + \varepsilon \mathbf{n}_x, \mathbf{r}_x + \varepsilon \mathbf{n}_x \rangle = e^{2u} + 2\varepsilon \langle \mathbf{r}_x, \mathbf{n}_x \rangle + \varepsilon^2 |\mathbf{n}_x|^2$$

$$G = \langle \mathbf{r}_y + \varepsilon \mathbf{n}_y, \mathbf{r}_y + \varepsilon \mathbf{n}_y \rangle = e^{2u} + 2\varepsilon \langle \mathbf{r}_y, \mathbf{n}_y \rangle + \varepsilon^2 |\mathbf{n}_y|^2$$

$$F = \langle \mathbf{r}_x + \varepsilon \mathbf{n}_x, \mathbf{r}_y + \varepsilon \mathbf{n}_y \rangle = \varepsilon \langle \mathbf{r}_x, \mathbf{n}_y \rangle + \varepsilon \langle \mathbf{r}_y, \mathbf{n}_x \rangle + \varepsilon^2 \langle \mathbf{n}_x, \mathbf{n}_y \rangle$$

$$EG - F^2 = e^{4u} + 2\varepsilon e^{2u} (\langle \mathbf{r}_x, \mathbf{n}_x \rangle + \langle \mathbf{r}_y, \mathbf{n}_y \rangle) + O(\varepsilon^2)$$

$$\frac{d}{d\varepsilon} \Big|_{\varepsilon=0} \sqrt{EG - F^2} = \langle \mathbf{r}_x, \mathbf{n}_x \rangle + \langle \mathbf{r}_y, \mathbf{n}_y \rangle = 2H e^{2u}$$

where we use the mean curvature formula

$$2H = \text{Tr} \left( -\frac{\mathbf{II}}{I} \right) = -e^{-2u} (\langle \mathbf{r}_{xx}, \mathbf{n} \rangle + \langle \mathbf{r}_{yy}, \mathbf{n} \rangle) = e^{-2u} (\langle \mathbf{r}_x, \mathbf{n}_x \rangle + \langle \mathbf{r}_y, \mathbf{n}_x \rangle)$$

$$\frac{d}{d\varepsilon} \text{Area}(\varepsilon) = \frac{d}{d\varepsilon} \Big|_{\varepsilon=0} \int_S \sqrt{EG - F^2} dx dy = \int_S 2H e^{2u} dx dy.$$

