MATH1231 CALCULUS

Chapter 1A

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Overview

Functions of Several Variables

Integration Techniques

Ordinary Differential Equations

Taylor Series

Sequences and Series

Averages, Arc Length, Surface Area

1. Functions of Several Variables

Most functions which arise in real world applications depend on more than one variable.

Example

- The volume of a right circular cylinder is a function of its radius and height.
- $f(x, y) = \sqrt{x^2 + y^2}$ gives the distance of a point (x, y) from the origin
- $f(x, y) = xy$ gives the area of a rectangle of dimensions x, y.
- $f(x, y, z) = xyz$ gives the volume of a rectangular solid of dimensions x, y, z.

- Example: Set up a mathematical problem for finding the dimensions of a closed cylindrical tank with a fixed volume $C > 0$ which has the minimum surface area.

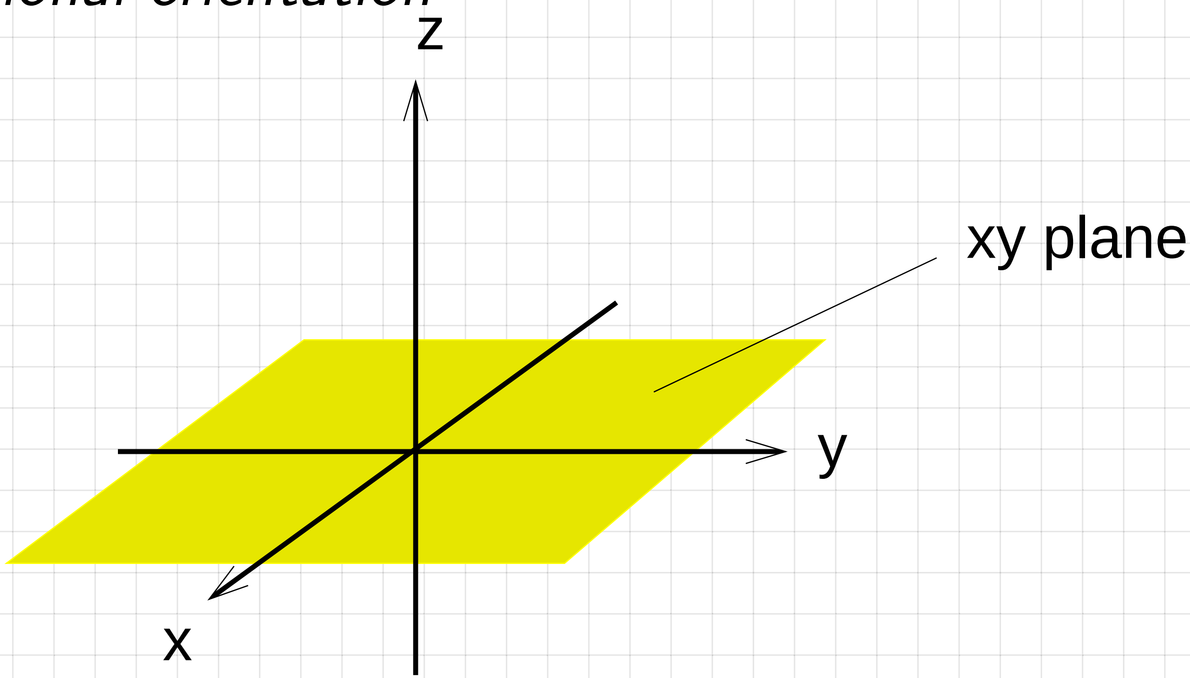
- Geometrically
  $y = f(x)$ – a curve in $\mathbb{R}^2$

  $z = f(x, y)$ – a surface in $\mathbb{R}^3$

(1.1) Sketching surfaces in $\mathbb{R}^3$

$$\boxed{z = F(x, y)}$$

Conventional orientation



z *is the distance above* $(+)$ *or below* $(-)$ *the* xy *plane.*

Past Mobius Exam Question

- Example: Consider the surface S with the equation $Z = F(x, y)$ where $F(x, y) = x^2 + y^2 + 4$. Which of the following points lies on the surface?

(i) $A = (-1, 3, 11)$.

(ii) $B = (1, 2, 9)$.

(iii) $C = (-3, -3, 22)$.

Level Curves and Profiles

- A **level curve** of a function $F : \mathbb{R}^2 \to \mathbb{R}$
  is a curve in $\mathbb{R}^2$ defined by $F(x, y) = C$ where C is a constant. ($F(x, y) = C$ means $z = F(x, y)$ and $z = C$).

  We can think of them as **horizontal slices** at constant z through the function surface.

- A **profile** is obtained from a plot of
  z versus y (with $x = 0$) or
  z versus x (with $y = 0$).

- yz-profile gives intersection of the surface with the yz-plane.

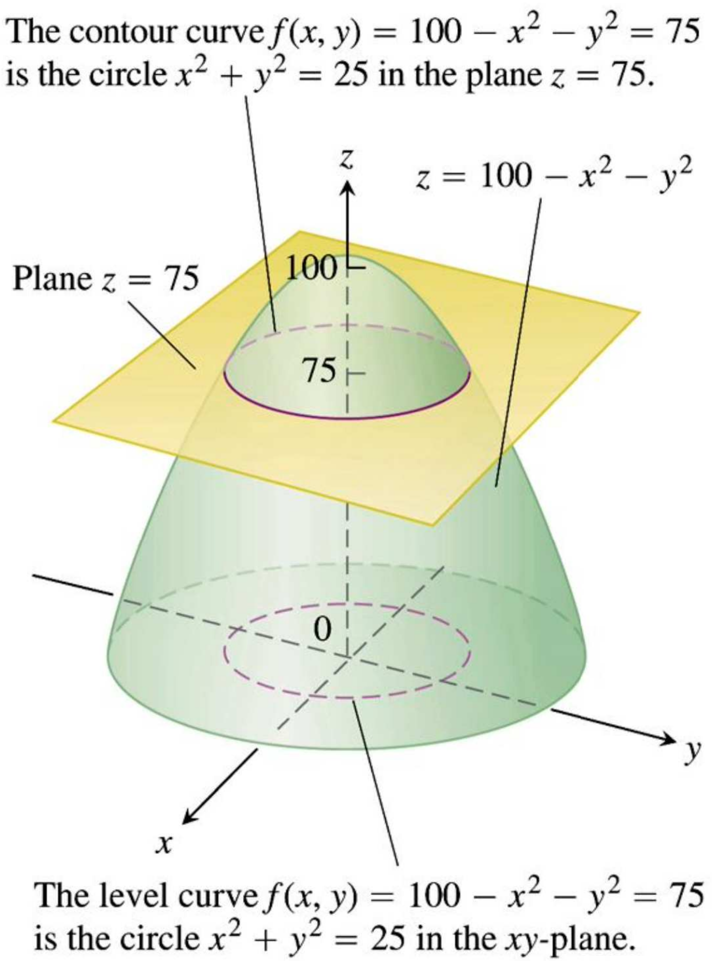
  We can think of these as **vertical cuts** through the function surface.

- We can combine level curves and profiles to visualize and sketch the surface.

Level curves can be obtained by intersecting the graph of F with the horizontal plane $z = C$ and then projecting this intersection onto the xy-plane

Example

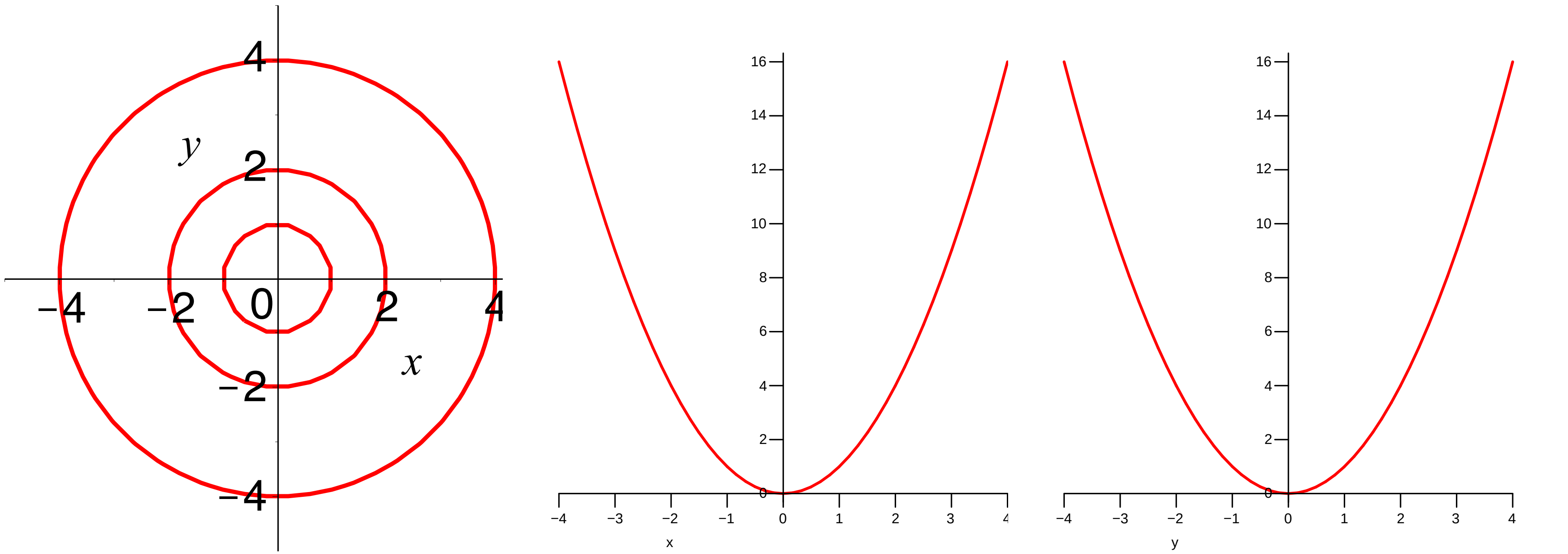
$$z = 100 - x^2 - y^2$$



Example: Sketch the level curves and profiles of $z = x^2 + y^2$.

Example

$$z = x^2 + y^2$$



$z = 0, 1, 4, 16$ $\qquad$ $z = F(x, 0) = x^2$ $\qquad$ $z = F(0, y) = y^2$

Example: Sketch the surface described by $z = x^2 + y^2$.

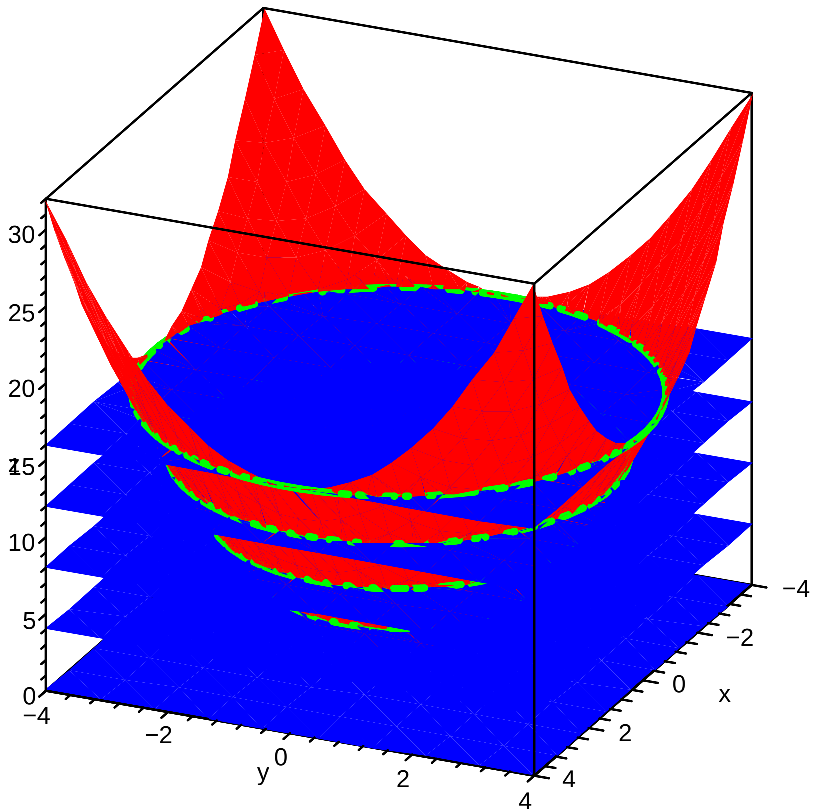
Step 1: Sketch the three co-ordinate axes.

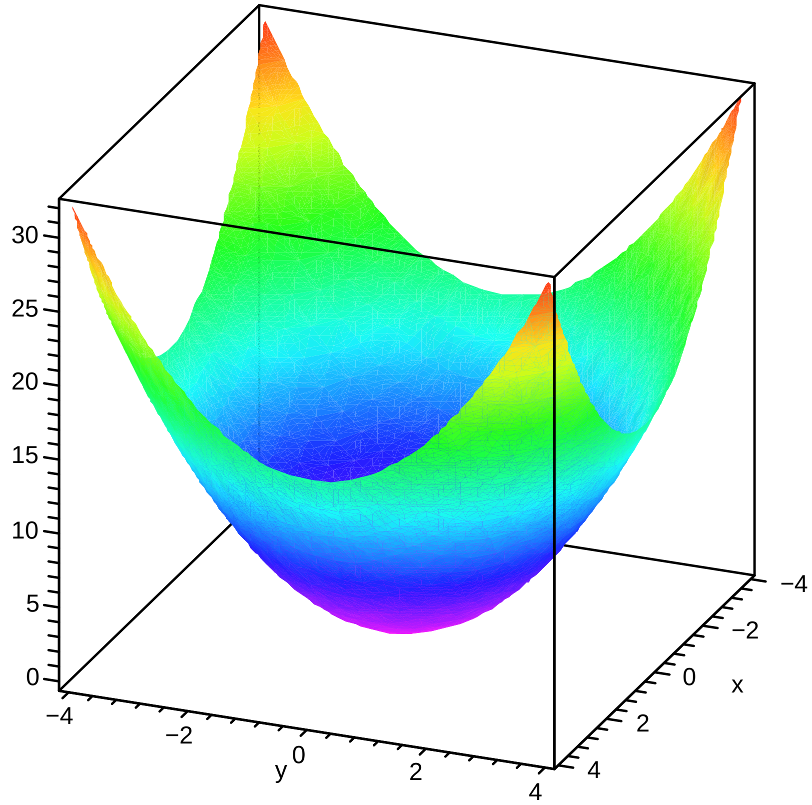
Step 2: Mark the intercepts on the axes.

Step 3: Sketch cross-sections in the coordinate planes and in a parallel planes.

Example

$$z = F(x, y) = x^2 + y^2$$





Example: A surface in $\mathbb{R}^3$ is described by the equation

$$z^2 = 1 - x^2 - y^2.$$

Sketch some level curves and hence sketch the surface in $\mathbb{R}^3$.

Example

$$x^2 + y^2 - z^2 = 1$$

$z = F(x, y)$ is not defined explicitly; but use same principles;

level curves with $z = 0$ are circles with radius 1
level curves with $z = \pm 1$ are circles with radius $\sqrt{2}$.
level curves with $z = \pm 2$ are circles with radius $\sqrt{5}$.

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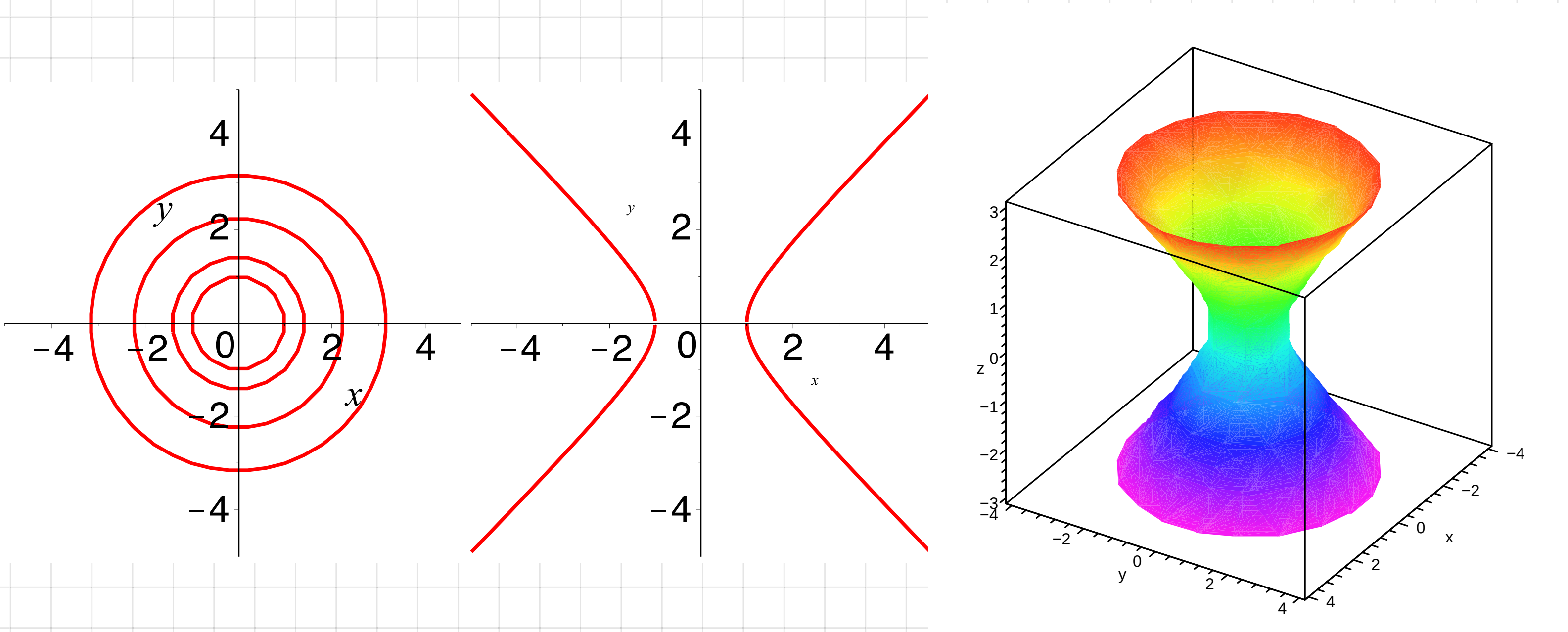
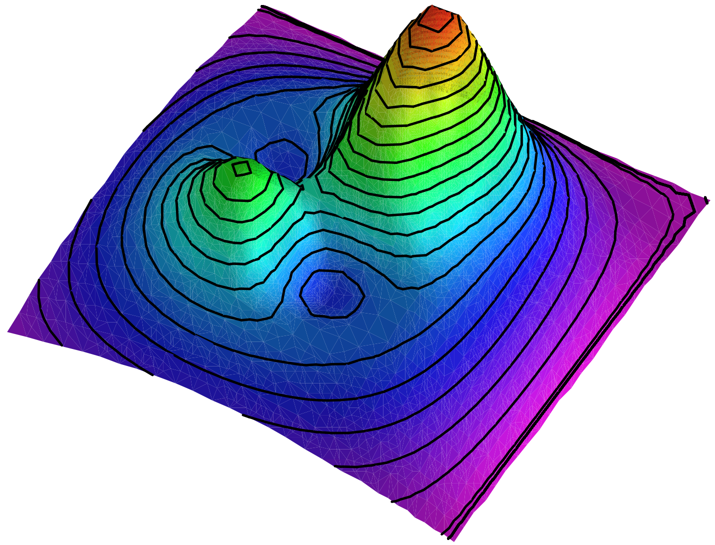


- Tricky example - use MAPLE – plot3d

$$z = 2\cos(\frac{2x}{5})\cos(\frac{2y}{5}) + 5xy\exp(-(x^2+y^2)) + 3\exp(-((x-2)^2+(y-2)^2))$$



Partial Differentiation

- In finding the rate of change of $f(x) = \sin(ax)$ with respect to x we differentiate with respect to x keeping a constant and get $\frac{df}{dx} = a\cos(ax)$.

- Equally, we could find the rate of change of $g(a) = \sin(ax)$ with respect to a we differentiate with respect to a keeping x constant and get $\frac{dg}{da} = x\cos(ax)$.

- In both cases we have effectively found the partial derivatives of the function of two variables,

$$F(x, a) = \sin(ax).$$

-

$$\frac{\partial F}{\partial x} = a\cos(ax) \ \& \ \frac{\partial F}{\partial a} = x\cos(ax)$$

Partial Differentiation

- Differentiate with respect to one of the variables by holding the other variables fixed

Example - Find $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$

1. $$F(x, y) = x^2 + y^2$$

2. $$F(x, y) = 3x + 4y + 9$$

Example - Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for $z = yx^2$

Example: Consider again the function $z = F(x, y) = yx^2$.

(1.2) **Partial Differentiation**

- Definition:
  If $F = F(x, y)$ then define partial derivatives

$$\frac{\partial F}{\partial x} = \lim_{h \to 0} \frac{F(x+h, y) - F(x, y)}{h}$$

$$\frac{\partial F}{\partial y} = \lim_{h \to 0} \frac{F(x, y+h) - F(x, y)}{h}$$

Notation:

- The partial derivative of F w.r.t. x

$$\frac{\partial F}{\partial x} \equiv F_x(x, y) \equiv D_1 F$$

- The partial derivative of F w.r.t. y

$$\frac{\partial F}{\partial y} \equiv F_y(x, y) \equiv D_2 F$$

- $$F_y(a, b) = \lim_{h \to 0} \frac{F(a, b+h) - F(a, b)}{h}$$

Rules of the Game:

If $F = F(x, y), \quad G = G(x, y), \quad H = H(y), \quad C = const.$

$$
\begin{aligned}
\frac{\partial}{\partial x}(F + G) &= \frac{\partial F}{\partial x} + \frac{\partial G}{\partial x} \\
\frac{\partial}{\partial x}(H\,F) &= H(y)\frac{\partial F}{\partial x} \\
\frac{\partial}{\partial x}(FG) &= \frac{\partial F}{\partial x}G + F\frac{\partial G}{\partial x} \\
\frac{\partial}{\partial x}\left(\frac{F}{G}\right) &= \frac{\frac{\partial F}{\partial x}G - F\frac{\partial G}{\partial x}}{G^2} \qquad \text{if} \quad G(x, y) \neq 0 \\
\text{if} \quad F(x, y) &= C \quad \forall x, y \quad \text{then} \quad \frac{\partial F}{\partial x} = 0 = \frac{\partial F}{\partial y}
\end{aligned}
$$

Examples - Find $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$

1. $$F(x, y) = 3y^2x + \cos(y) + y\sin(x)$$

2. $$F(x, y) = e^{xy}$$

3. $$F(x, y) = e^{xy}\sin(y)$$

Example: Given that $F(x, y) = \sin(\pi x y^2)$, calculate $F_x(2, -1)$ and $F_y(2, -1)$

Geometrical Interpretation

$\dfrac{\partial F}{\partial x}$ is the slope of the surface $z = F(x, y)$ in the x direction.

$F_x(a, b)$ is the gradient of the tangent to the cross section at (a, b) when the surface $z = F(x, y)$ is intersected with the plane $y = b$.

$\dfrac{\partial F}{\partial y}$ is the slope of the surface $z = F(x, y)$ in the y direction.

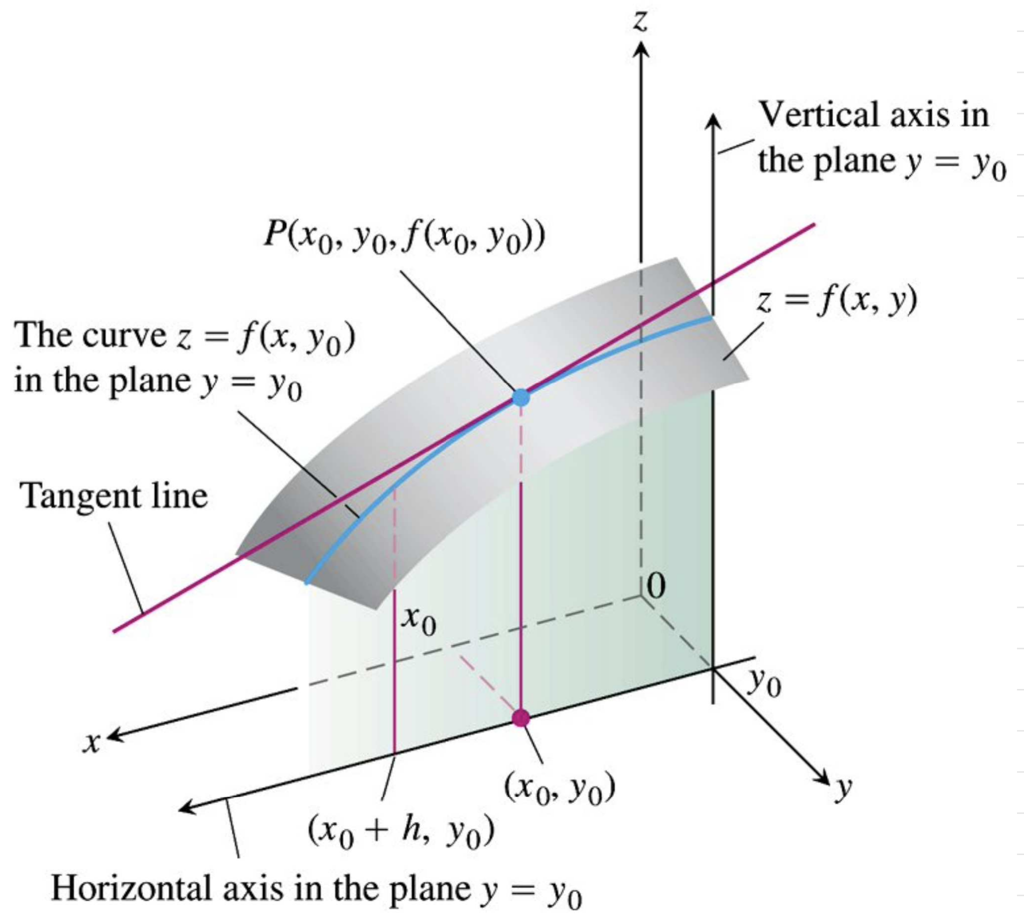
$F_y(a, b)$ is the gradient of the tangent to the cross section at (a, b) when the surface $z = F(x, y)$ is intersected with the plane $x = a$.

Geometrical Interpretation

The **partial derivative** of $f(x, y)$ with respect to x at the point (x_0, y_0) is

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \lim_{h\to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

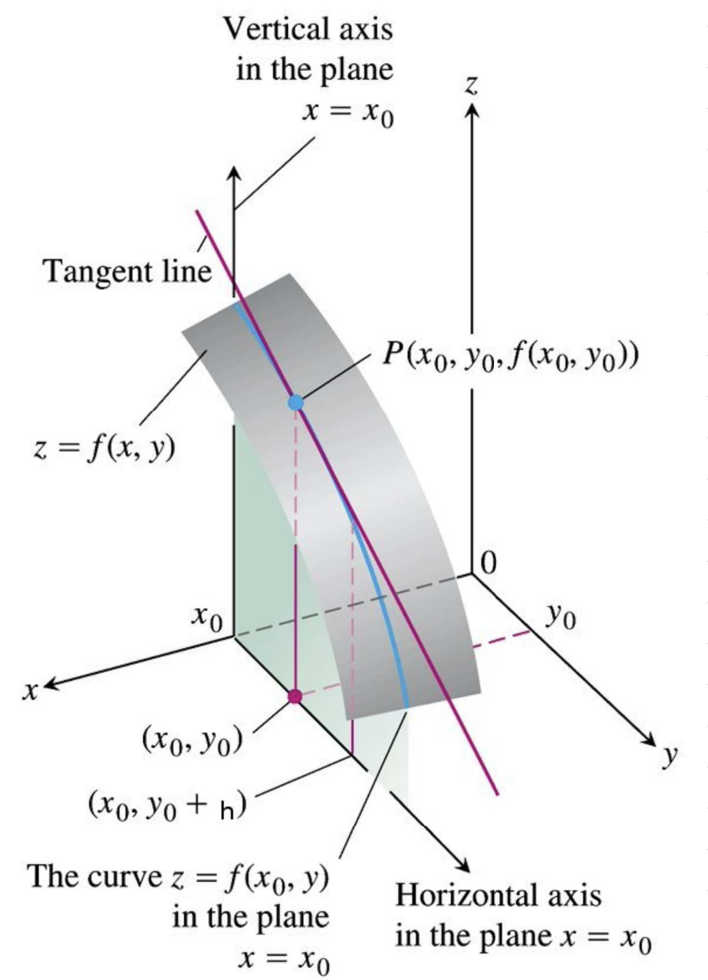
provided the limit exists.



The **partial derivative** of $f(x, y)$ with respect to y at the point (x_0, y_0) is

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = \lim_{h\to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

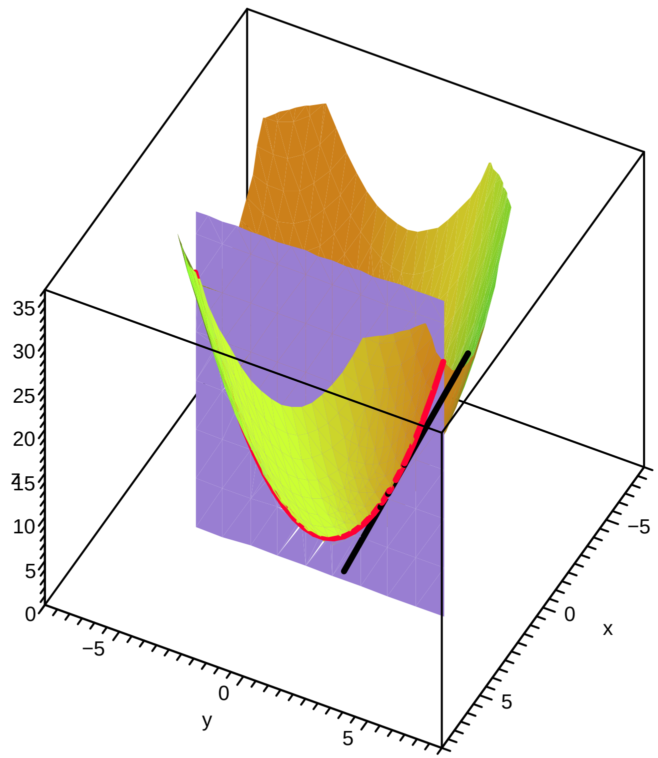
provided the limit exists.



Example $\qquad z = F(x, y) = x^2 + y^2$

Find the slope of the tangent to the cross section at $(2, 3, 13)$ when the surface $z = x^2 + y^2$ intersects with the plane $x = 2$.

The cross-section of the surface $z = x^2 + y^2$ *with the plane* $x = 2$ *is the line* $z = 4 + y^2$ *and this has slope* $\frac{dz}{dy} = 2y$ *so that at* $y = 3$ *we have slope* 6.



$$F_y(2, 3) = \frac{\partial F}{\partial y}\Big|_{y=3} = 2y\,\Big|_{y=3} = 6$$

Equation of the tangent line to the curve $z = 4 + y^2$ (i.e. the cross-section of $z = x^2 + y^2$ with $x = 2$) at the point $(2, 3, 13)$ is

$$z - 13 = 6(y - 3), \quad x = 2.$$

In vector form

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 13 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 6 \end{pmatrix}$$

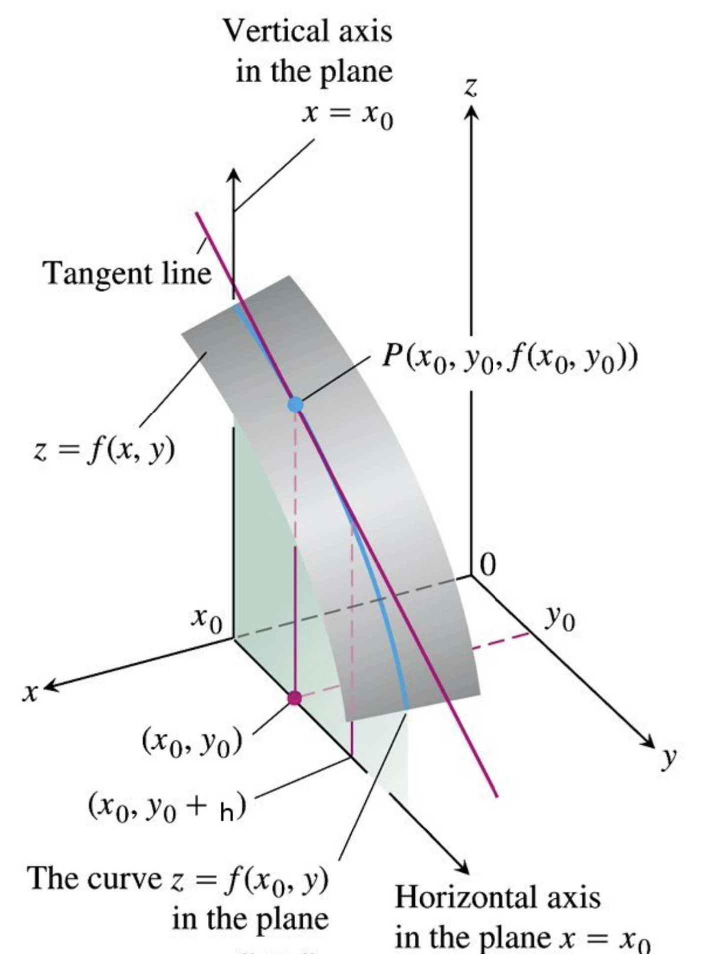
The tangent vector along the tangent line is

$$v = \begin{pmatrix} 0 \\ 1 \\ 6 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ F_y(2, 3) \end{pmatrix}$$

The tangent vector along the tangent line to the cross-section at $(x_0, y_0, f(x_0, y_0))$ when the surface $z = f(x, y)$ intersects with the plane $x = x_0$ is

$$v = \begin{pmatrix} 0 \\ 1 \\ f_y(x_0, y_0) \end{pmatrix}.$$

The slope of this tangent line at $(x_0, y_0, f(x_0, y_0))$ is $f_y(x_0, y_0)$.



Second order partial derivatives are partial derivatives of first order partial derivatives

$$\frac{\partial^2 F}{\partial x^2} \equiv \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial x}\right) \equiv F_{xx} \equiv D_1^2 F$$

$$\frac{\partial^2 F}{\partial y^2} \equiv \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial y}\right) \equiv F_{yy} \equiv D_2^2 F$$

$$\frac{\partial^2 F}{\partial x \partial y} \equiv \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right) \equiv F_{yx} \equiv D_1 D_2 F$$

$$\frac{\partial^2 F}{\partial y \partial x} \equiv \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) \equiv F_{xy} \equiv D_2 D_1 F$$

WARNING: The F_{xy} and F_{yx} notation is not standard

Example - Let $F(x, y) = x^2 \cos(y) + y^2 \sin(x)$. Find $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$, $\frac{\partial^2 F}{\partial x \partial y}$ and $\frac{\partial^2 F}{\partial y \partial x}$

- Theorem: If $F = F(x, y)$ is a continuous function of two variables (x, y) and all of its first and second order partial derivatives are continuous then

$$\boxed{\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x}}$$

- *Proof: See comments in notes. A proper proof (not given) requires a formal treatment of continuity.*
- *Continuity:*
  $G = G(x, y)$ *is continuous at* (x_0, y_0) *if for every* $\epsilon > 0$ *there exists a* $\delta > 0$ *such that* $|G(x, y) - G(x_0, y_0)| < \epsilon$ *whenever* $0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$.

Example $F(x, y) = \sin(xy)$

Example $F(x,y) = e^x \sin(xy)$

$$
\begin{aligned}
\frac{\partial F}{\partial x} &= e^x \sin(xy) + e^x y \cos(xy) \\
\frac{\partial^2 F}{\partial x^2} &= e^x \sin(xy) + e^x y \cos(xy) + e^x y \cos(xy) - e^x y^2 \sin(xy) \\
\frac{\partial^2 F}{\partial y \partial x} &= e^x x \cos(xy) + e^x \cos(xy) - e^x yx \sin(xy)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial F}{\partial y} &= e^x x \cos(xy) \\
\frac{\partial^2 F}{\partial y^2} &= -e^x x^2 \sin(xy) \\
\frac{\partial^2 F}{\partial x \partial y} &= e^x x \cos(xy) + e^x \cos(xy) - e^x xy \sin(xy)
\end{aligned}
$$