MATH1081 – Discrete Mathematics

# Topic 1 – Set theory and functions
# Lecture 1.04 – Laws of set algebra

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# Proofs involving set operations

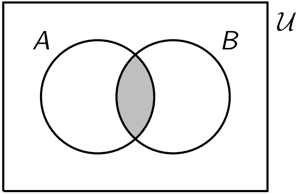
There are three main ways to justify or prove a statement involving set operations:

- Using Venn diagrams.
  - Venn diagrams are a useful visual aide, but are not generally considered valid tools for rigorous proofs.
- Using set operation definitions, and thinking in terms of arbitrary elements.
  - This method is reliable and a good tool for most proofs. Using this method to prove equivalence of sets can sometimes become unwieldy, since two containment proofs are required.
- Using the laws of set algebra.
  - We will soon introduce the laws of set algebra, which are especially helpful in simplifying expressions.

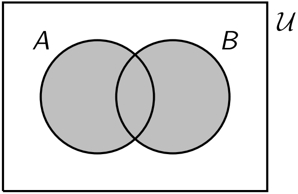
# Example 1

**Example.** Is $A \cap B$ a subset of $A \cup B$ for all sets $A$ and $B$?

Working: Considering the Venn diagrams for both sets, we have:




$A \cap B$




$A \cup B$

Since the shaded region on the left is completely contained within the shaded region on the right, this indicates that it is true in general that $A \cap B \subseteq A \cup B$.

**Proof.** Let $x \in A \cap B$ be an arbitrary element of $A \cap B$. Then $x \in A$ and $x \in B$. Since $x \in A$, we can certainly say $x \in A$ or $x \in B$. So $x \in A \cup B$. Thus since any element of $A \cap B$ is an element of $A \cup B$, we have that $A \cap B \subseteq A \cup B$.

# Example 2

**Example.** Prove that for any sets $A$ and $B$, we have $A - B = A \cap B^c$.

**Proof.** We need to show both that $A - B \subseteq A \cap B^c$, and that $A \cap B^c \subseteq A - B$.

First, let $x \in A - B$. Then $x \in A$ and $x \notin B$. So $x \in A$ and $x \in B^c$. Thus $x \in A \cap B^c$, meaning that $A - B \subseteq A \cap B^c$.

Next, let $x \in A \cap B^c$. Then $x \in A$ and $x \in B^c$. So $x \in A$ and $x \notin B$. Thus $x \in A - B$, meaning that $A \cap B^c \subseteq A - B$.

Hence since the sets are subsets of each other, we know $A - B = A \cap B^c$.

**Alternative proof.** We have

$$\begin{aligned} A - B &= \{x \in \mathcal{U} \ : \ x \in A \ \text{ and } \ x \notin B\} \\ &= \{x \in \mathcal{U} \ : \ x \in A \ \text{ and } \ x \in B^c\} \\ &= A \cap B^c, \end{aligned}$$

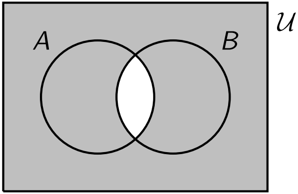
as required.

Note that this second style of proof is more efficient, but may not always be applicable depending on the problem.

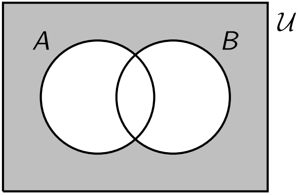
# Example 3

**Example.** Are the sets $(A \cap B)^c$ and $A^c \cap B^c$ equal for all sets $A$ and $B$?

Working: Considering the Venn diagrams for both sets, we have:




$(A \cap B)^c$




$A^c \cap B^c$

Since the shaded regions on the left and right are not identical, this indicates that in general $(A \cap B)^c \neq A^c \cap B^c$. To prove this, we just need to provide an example where a non-matching section contains at least one element.

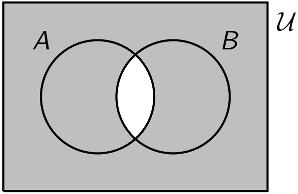
**Proof.** Consider the case where $A = \{1\}$ and $B = \{2\}$, with universal set $\mathcal{U} = \{1, 2, 3\}$. Then we have $(A \cap B)^c = (\{\})^c = \{1, 2, 3\}$, while $A^c \cap B^c = \{2, 3\} \cap \{1, 3\} = \{3\}$. Since these two sets are not the same, we know $(A \cap B)^c \neq A^c \cap B^c$ in general.

Note that this is one of many valid cases that demonstrate inequality. An even simpler case is $A = \{1\}$, $B = \{\}$, and $\mathcal{U} = \{1\}$.

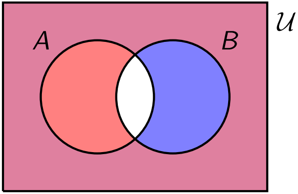
# Example 4

**Example.** Are the sets $(A \cap B)^c$ and $A^c \cup B^c$ equal for all sets $A$ and $B$?

Working: Considering the Venn diagrams for both sets, we have:




$(A \cap B)^c$




$A^c \cup B^c$

Since the shaded regions on the left and right are identical, this indicates that it is true in general that $(A \cap B)^c = A^c \cup B^c$.

**Example.** Are the sets $(A \cap B)^c$ and $A^c \cup B^c$ equal for all sets $A$ and $B$?

**Proof.** We claim that the sets are equal, so we want to show both that $A^c \cup B^c \subseteq (A \cap B)^c$, and that $(A \cap B)^c \subseteq A^c \cup B^c$.

First, let $x \in A^c \cup B^c$. Then $x \in A^c$ or $x \in B^c$, so we can say $x \notin A$ or $x \notin B$. So $x$ certainly can't be an element of both $A$ and $B$ at once, meaning $x \notin A \cap B$. So $x \in (A \cap B)^c$, and thus $A^c \cup B^c \subseteq (A \cap B)^c$.

Next, let $x \in (A \cap B)^c$. Then $x \notin A \cap B$. Either $x \in A$ or $x \notin A$, so we consider these two cases in turn.

Case 1: If $x \in A$, then since $x \notin A \cap B$, we must have that $x \notin B$. So $x \in B^c$, meaning that $x \in A^c \cup B^c$.

Case 2: If $x \notin A$, then $x \in A^c$, meaning that $x \in A^c \cup B^c$.

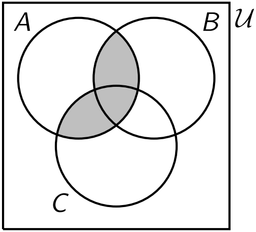
Since $x \in A^c \cup B^c$ in both cases, we can conclude that $(A \cap B)^c \subseteq A^c \cup B^c$.

Hence since the sets are subsets of each other, we can conclude that $(A \cap B)^c = A^c \cup B^c$.

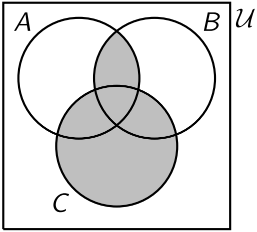
# Example 5

**Example.** Are $A \cap (B \cup C)$ and $(A \cap B) \cup C$ equal for all sets $A$, $B$, and $C$?

Working: Considering the Venn diagrams for both sets, we have:




$A \cap (B \cup C)$




$(A \cap B) \cup C$

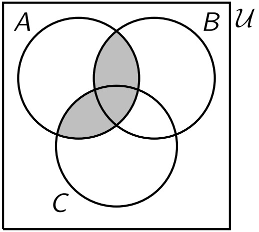
Since the shaded regions on the left and right are not identical, this indicates that in general $A \cap (B \cup C) \neq (A \cap B) \cup C$. To prove this, we just need to provide an example where a non-matching section contains an element.

**Proof.** Consider the case where $A = B = \{\}$ and $C = \{1\}$. Then we have $A \cap (B \cup C) = \{\} \cap \{1\} = \{\}$, while $(A \cap B) \cup C = \{\} \cup \{1\} = \{1\}$. Since these two sets are not the same, we conclude $A \cap (B \cup C) \neq (A \cap B) \cup C$ in general.

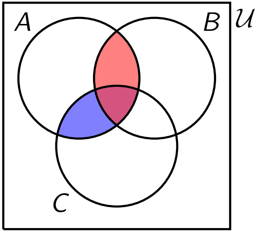
# Example 6

**Example.** Are $A \cap (B \cup C)$ and $(A \cap B) \cup (A \cap C)$ equal for all sets $A$, $B$, $C$?

Working: Considering the Venn diagrams for both sets, we have:




$A \cap (B \cup C)$




$(A \cap B) \cup (A \cap C)$

Since the shaded regions on the left and right are identical, this indicates that it is true in general that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

**Proof.** Try this yourself! (Use similar methods to Examples 2 and 4.)

# Laws of set algebra

For any sets $A, B, C$ with universal set $\mathcal{U}$ and empty set $\varnothing$, we have the following laws of set algebra:

Commutativity:

$$A \cup B = B \cup A,$$
$$A \cap B = B \cap A.$$

Associativity:

$$A \cup (B \cup C) = (A \cup B) \cup C,$$
$$A \cap (B \cap C) = (A \cap B) \cap C.$$

Distributivity:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C),$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

Absorption:

$$A \cup (A \cap B) = A,$$
$$A \cap (A \cup B) = A.$$

Idempotence:

$$A \cup A = A,$$
$$A \cap A = A.$$

Identity:

$$A \cup \varnothing = A,$$
$$A \cap \mathcal{U} = A.$$

Domination:

$$A \cup \mathcal{U} = \mathcal{U},$$
$$A \cap \varnothing = \varnothing.$$

Complement law:

$$A \cup A^c = \mathcal{U},$$
$$A \cap A^c = \varnothing.$$

Double complement law:

$$(A^c)^c = A.$$

De Morgan's law:

$$(A \cup B)^c = A^c \cap B^c,$$
$$(A \cap B)^c = A^c \cup B^c.$$

We also have the following definitions:

Difference:

$$A - B = A \cap B^c.$$

Symmetric difference:

$$A \ominus B = (A \cup B) - (A \cap B).$$

# Comments on the laws of set algebra

The laws of set algebra completely describe the behaviour of sets under the basic set operations. It is possible to verify any statements involving set expressions by using only these laws, though doing so can take a lot of work.

While there are many laws to learn here, almost all of them are easily justified by considering them in terms of Venn diagrams.

When simplifying expressions or proving statements using the laws of set algebra, we should always state which laws are being used at each step. If you do not remember the name of a particular law, you may instead describe it in words and/or provide its general definition.

**Definition.** The dual of a set expression is the expression obtained by replacing every instance of $\cup$ with $\cap$, $\cap$ with $\cup$, $\varnothing$ with $\mathcal{U}$, and $\mathcal{U}$ with $\varnothing$.

**Theorem.** (Duality principle)
Any statement involving only sets and the union, intersection, and complement operations is true if and only if its dual statement is true.

**Proof.** This is a consequence of the fact that every law of set algebra consists of a pair of dual statements (except for the double complement law, which is self-dual).

# Example – Simplifying set expressions

**Example.** Simplify the set expression $A \cap (A \cap B^c)^c$.

**Solution.** We proceed using the laws of set algebra:

$$\begin{aligned}
A \cap (A \cap B^c)^c &= A \cap (A^c \cup (B^c)^c) && \text{(De Morgan's law)}\\
&= A \cap (A^c \cup B) && \text{(double complement law)}\\
&= (A \cap A^c) \cup (A \cap B) && \text{(distributivity)}\\
&= \varnothing \cup (A \cap B) && \text{(complement law)}\\
&= (A \cap B) \cup \varnothing && \text{(commutativity)}\\
&= A \cap B && \text{(identity).}
\end{aligned}$$

Notice that some of these steps could have been performed at the same time, for example the commutativity and identity applications.

We could also have checked this by using a Venn diagram, though remember that a Venn diagram explanation would not constitute a rigorous proof.

The dual of this result must also be true, so we now also know that: $A \cup (A \cup B^c)^c = A \cup B$.

# Example – Proving equivalence of set expressions

**Example.** Show that $(A - B) \cap (A - C) = A - (B \cup C)$ for all sets $A$, $B$, $C$.

**Solution.** We proceed using the laws of set algebra. Simplifying the left-hand side yields:

$$\begin{aligned}
(A - B) \cap (A - C) &= (A \cap B^c) \cap (A \cap C^c) && \text{(def'n of difference)}\\
&= A \cap (B^c \cap A) \cap C^c && \text{(associativity)}\\
&= A \cap (A \cap B^c) \cap C^c && \text{(commutativity)}\\
&= (A \cap A) \cap B^c \cap C^c && \text{(associativity)}\\
&= A \cap B^c \cap C^c && \text{(idempotence).}
\end{aligned}$$

Simplifying the right-hand side yields:

$$\begin{aligned}
A - (B \cup C) &= A \cap (B \cup C)^c && \text{(def'n of difference)}\\
&= A \cap (B^c \cap C^c) && \text{(De Morgan's law)}\\
&= A \cap B^c \cap C^c && \text{(associativity).}
\end{aligned}$$

Since both expressions simplify to give the same set, they must be equal. So we have shown that $(A - B) \cap (A - C) = A - (B \cup C)$.