Linear Algebra MATH 2501

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Systems of linear equations

Objective To be able to determine how many solutions a system of linear equations has, and to find all these solutions.

- 1.How many solutions can a system of linear equations have?
- 2.What is the most commonly used method of solving linear systems?
- 3.How do you use this method to determine the number of solutions of the system? Give full details.

Exercise Solve the following linear system: $\begin{cases} 2x_1 - x_2 - 3x_3 = -3 \\ -x_1 + x_2 + 5x_3 + x_4 = 4 \\ 5x_1 - x_2 + 3x_3 + 3x_4 = 0 \end{cases}$

Review The system of linear equations has no solutions if the RHS column of REF is leading.

The system of linear equations has a solution if the RHS column of REF is non-leading.

Examples Let's discuss the following REFs.

- 1. $\left(\begin{array}{cc|c} 1 & -2 & 8 \\ 0 & 2 & -1 \end{array}\right)$
- 2. $\left(\begin{array}{ccc|c} 1 & -2 & -8 & 8 \\ 0 & 0 & 0 & -1 \end{array}\right)$

Exercise Solve the following linear system:

$$\begin{cases} x_1 - 2x_2 - 2x_4 = -1 \\ -3x_1 + 5x_2 + x_3 + 7x_4 = 2 \\ 2x_1 + x_2 + 8x_3 - 9x_4 = 3 \end{cases}$$

Conditions for a solution to exist

Objective Given a matrix A, find conditions on b_1, b_2 such that the system $Ax = b$ has a solution.

Exercise Let $A = \begin{pmatrix} 1 & -1 & 3 \\ 0 & 2 & -1 \\ 2 & 0 & 5 \\ -2 & -4 & -3 \end{pmatrix}$

Find conditions on b_1, b_2, b_3, b_4 such that $Ax = b$ has a solution.

Exercise Find (for any c) the solutions of the system

$$\begin{cases} x_1 - x_2 = 3 \\ x_1 - cx_3 = 1 \\ -2x_1 + (c+2)x_2 - 4x_3 = -c - 8 \end{cases}$$

Matrix arithmetic

Objective Know when simple arithmetic operations are defined for matrices, and calculate them when they are defined.

- 1.Let A be an $m \times n$ matrix. What is the size of the matrix B if (a) the sum $A + B$ is defined? (b) the product AB is defined? What are the sizes of the sum and product if they are defined?
- 2.List at least two important differences between multiplication of matrices and multiplication of numbers.

Exercise: Let $A = \begin{pmatrix} 2 & 1 & 2 \\ -2 & 3 & 1 \end{pmatrix}$ $B = \begin{pmatrix} -1 & 1 \\ -2 & 3 \end{pmatrix}$ $C = \begin{pmatrix} -1 & 3 \\ 2 & 1 \\ 1 & -4. \end{pmatrix}$.

Calculate if possible $A - B$; $-2A + C^T$; $B - 4I$; BC; CB; A^2.

Matrix inverses

Objective Determine whether or not a given matrix is invertible, and find its inverse if so.

- 1.What does "B is the inverse of A" mean?
- 2.You can see immediately that certain matrices have no inverse. Which matrices are these?
- 3.How do you attempt to find the inverse of a given matrix? How do you know if the attempt fails?
- 4.State the "short cut" formula for the inverse of a 2×2 matrix.

Exercise Find the inverse (if any) of $\begin{pmatrix} 4 & 5 \\ 6 & 7 \end{pmatrix}$, $\begin{pmatrix} 1 & -2 & 2 \\ 0 & 3 & -1 \\ 2 & -1 & 4 \end{pmatrix}$

Exercise Show that the matrix $\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$ doesn't have an inverse.

Special matrices

Objective Recognise symmetric, skew-symmetric and orthogonal matrices, and simplify expressions involving such matrices.

- 1.Define *symmetric*, *skew-symmetric* and *orthogonal* matrices.
- 2.For any matrices A and B, expand $(AB)^{-1}$ and $(AB)^T$.

Exercise Prove that if A is invertible then $(A^{-1})^T = (A^T)^{-1}$.

Exercise Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$. Compute A^{2024}.

Vector spaces and subspaces

Objective Understand what is meant by a vector space. Given a vector space V and a subset W of V, determine whether or not W is a subspace of V.

- 1.What is a vector space? (Just give a brief explanation.)
- 2.Given that V is a vector space and $W \subset V$, how do you show that W is a vector space?
- 3.Define precisely the statements "W is closed under addition" and "W is closed under scalar multiplication".
- 4.Give a short cut for showing that a set W is not a vector space.

Exercise Prove that

- 1.$W = \{x \in \mathbb{R}^4 | 3x_1 - x_2 + 7x_4 = 0\}$ is a vector space;
- 2.if A is a fixed 3×5 matrix, then $W = \{Ax | x \in \mathbb{R}^5\}$ is a vector space.

Exercise Show that

- 1.$W = \{x \in \mathbb{R}^2 | x_1 = x_2^2\}$ is not a subspace of $\mathbb{R}^2$;
- 2.$W = \{x \in \mathbb{R}^2 | x_1 = \pm x_2\}$ is not a subspace of $\mathbb{R}^2$;
- 3.$W = \{x \in \mathbb{R}^4 | 3x_1 - x_2 + 7x_4 = 5\}$ is not a subspace of $\mathbb{R}^4$.

Linear independence

Objective Determine whether given vectors are linearly independent or not.

- 1.What is meant by a linear combination of the vectors $v_1, v_2, \ldots, v_n$?
- 2.Give a precise definition of the statement "the vectors $v_1, v_2, \ldots, v_n$ are linearly independent".
- 3.What is the general method of determining whether a set of vectors is linearly dependent?
- 4.Give some short cuts for proving linear dependence or independence.

Example Are the vectors

$$(1,4,-1,3),(-2,-7,1,2),(0,-2,1,-9)$$

linearly independent? Are the following sets of vectors linearly independent?

- 1.$\{(1,4,-1,3),(-2,-7,1,2),(0,-1,1,-8)\}$.
- 2.$\{(1,4,-1),(3,-2,-7),(1,2,0),(-1,1,-8)\}$.
- 3.$\{1-2t^2, 3-t-t^2, -1+2t+5t^2\}$.

Spanning sets

Objective Given a set S of vectors in a vector space V, determine whether a specific vector is in $Span(S)$, and whether or not S is a spanning set for V.

Let V be a vector space, $v \in V$ and $S \subseteq V$.

- 1.Define what is meant by $Span(S)$, and by the statement "S is a spanning set for V".
- 2.How do you tell whether $v \in Span(S)$?
- 3.How do you normally decide whether or not S is a spanning set for a vector space V?

Exercise

- 1.Let

$$S = \{(1,-1,-1),(3,-1,5),(-1,2,1),(1,-3,-6)\}.$$

Is $(-3,6,2)$ in $Span(S)$? Is S a spanning set for $\mathbb{R}^3$?

- 2.Is $\{(1,-1,-1),(3,-1,5),(-1,2,1)\}$ a spanning set for $\mathbb{R}^3$?

- 3.Is $4 + t - 3t^2$ in the span of the polynomials $1 + 2t - t^2$, $-2 + 3t + t^2$, $1 + 9t - 2t^2$ and $5 - 4t - 3t^2$? Do these four polynomials span $\mathbb{P}_2$?

Nullspace and column space of a matrix

Any $m \times n$ matrix has two important vector spaces associated with it.

Exercise Let A be an $m \times n$ matrix. Prove that $V = \{x|Ax = 0\}$ is a vector space.

Definition Let A be an $m \times n$ matrix. Then

- 1. $\{x \in \mathbb{R}^n | Ax = 0\}$ is called the **nullspace** or **kernel** of A;
- 2. the span of the columns of A is called the **column space** of A.

These spaces are denoted by $NS(A)$ and $CS(A)$ respectively.

We can prove that the nullspace of a matrix is a vector space; $CS(A)$ is a vector space since the span of set is a vector space. An alternative argument for the column space: suppose that A has columns $c_1, \dots, c_n$. Then

$$CS(A) = Span\{c_1, \dots, c_n\} = \{x_1 c_1 + \cdots + x_n c_n | x_1, \dots, x_n \in \mathbb{R}\} =$$

$$= \{Ax | x \in \mathbb{R}^n\}$$

and this is a vector space. Note that when A is an $m \times n$ matrix, $NS(A)$ is a subspace of $\mathbb{R}^n$, but $CS(A)$ is a subspace of $\mathbb{R}^m$.

Basis and dimension

Objective Determine whether a given set is a basis for a vector space. Find the dimension of a vector space.

- 1.Define basis and dimension of a vector space.
- 2.Give examples of bases for $\mathbb{R}^3$, $\mathbb{P}_3$ and $M_{3,3}$, and state the dimensions of these vector spaces.

Exercise:

- 1.Is $\{(1,4,-1,3),(-2,-7,1,2),(0,-2,1,-9)\}$ a basis for $\mathbb{R}^4$?
- 2.Is $\{1-2t^2, 3-t-t^2, -1+2t+5t^2\}$ a basis for $\mathbb{P}_2$?

Exercise:

- 1.Is $\{(1,-1,-1),(3,-1,5),(-1,2,1)\}$ a basis for $\mathbb{R}^3$?
- 2.Is $\{1+2t-t^2,-2+3t+t^2,1+9t-2t^2,5-4t-3t^2\}$ a basis for $\mathbb{P}_2$?

Exercise: Find a basis for

$$P = \{x \in \mathbb{R}^3 | x_1 - x_2 + 8x_3 = 0\},$$

a plane through the origin in $\mathbb{R}^3$.

Exercise Find a basis in the space of 2×2 matrices that has at least two elements of the set S given by $\begin{pmatrix} 1 & 4 \\ -1 & 3 \end{pmatrix}, \begin{pmatrix} -2 & -7 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & -8 \end{pmatrix}$.

Coordinates

Recall that a basis $B = \{b_1, b_2, \dots, b_n\}$ for a vector space V is a set of vectors in V with the following properties: B is a spanning set for V, and B is linearly independent. Let v be a vector in V. The fact that B spans V tells us that

$$v = x_1 b_1 + x_2 b_2 + \cdots + x_n b_n$$

for certain scalars $x_1, x_2, \dots, x_n$; the fact that B is linearly independent tells us that for any specific v, there is only one possible choice of these scalars: prove it. The scalars are referred to as the coordinates of the vector v with respect to the basis B.

Definition Let V be a real vector space, let $B = \{b_1, b_2, \ldots, b_n\}$ be a basis for V, and let v be a vector in V. The **coordinate vector** of v with respect to the ordered basis B is the vector $[v]_B = (x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$ such that

$$v = x_1 b_1 + x_2 b_2 + \cdots + x_n b_n.$$

Exercise Show that

$$B = \{(-1, 2, 1), (2, -5, -3), (5, -7, -3)\}$$

is a basis of $\mathbb{R}^3$, and find the coordinates of $(9, 1, 5)$ with respect to this (ordered) basis. Find the vector which has coordinates $(5, 2, -1)$ with respect to the ordered basis B.

Exercise Prove that the matrices $M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, $M_2 = \begin{pmatrix} 0 & 1 \\ 2 & 2 \end{pmatrix}$, $M_3 = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$, $M_4 = \begin{pmatrix} 1 & 0 \\ 3 & -4 \end{pmatrix}$ form a basis for $M_{2,2}$, and find the coordinate vector with respect to this basis of $A = \begin{pmatrix} 3 & -2 \\ 8 & 1 \end{pmatrix}$.

Exercise Let $(V, +, *)$ a vector space with two bases $B = \{b_1, b_2\}$ and $C = \{c_1, c_2\}$. Suppose

$$c_1 = b_1 + b_2$$

and

$$c_2 = b_1 - b_2.$$

For a vector $x \in V$ we know that $[x]_C = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$. Find $[x]_B$.

The **nullity** of a matrix is the dimension of its nullspace, and the **rank** is the dimension of the column space.

Exercise: Find the nullspace, column space, nullity and rank of the matrix
$$A = \begin{pmatrix} 1 & -1 & 4 & 0 & 4 \\ 2 & 1 & 7 & -1 & 11 \\ -1 & -8 & -1 & 3 & -13 \end{pmatrix}$$

We have for a matrix A with a row echelon form U that

$$dimNS(A) = \#\{\text{parameters in the solution of } Ax = 0\} =$$

$$= \#\{\text{non-leading columns in U}\}$$

and

$$dimCS(A) = \#\{\text{leading columns in U}\}.$$

The previous exercise is an example for the following theorem:

Theorem (The Rank–Nullity Theorem). For any matrix A we have

$$rank(A) + nullity(A) = \#\{\text{columns of A}\}.$$

Exercise Let A be an $m \times n$ matrix. Prove that if u is a row of A and v is in $NS(A)$, then u and v are perpendicular.

Linear transformations

Objective Know what is meant by a linear transformation (linear map, linear function). Determine whether a given function is linear or not. Understand clearly the difference between a "linear transformation" proof and a "vector subspace" proof.

- 1. How do you prove that a function is linear?
- 2. Give some short cuts for proving that a function is linear.

Definition A linear map $T : V \to V$ is a map with the following two properties:

- 1. $T(v + w) = T(v) + T(w)$ for any $v, w \in V$ and
- 2. $T(\lambda v) = \lambda T(v)$ for any $v \in V$ and $\lambda \in \mathbb{R}$(or $\mathbb{C}$).

In particular $T(0) = 0$. The two conditions can be written together as

$$T(\lambda v + \mu w) = \lambda T(v) + \mu T(w).$$

for any $\lambda, \mu \in R$ (or $\mathbb{C}$) and $v, w \in V$.

Example Do the following two problems in parallel columns, making sure that you understand the differences:

- (a) Show that $W = \{x \in \mathbb{R}^3 | -x_1 - 5x_2 + 2x_3 = 0\}$ is a subspace of $\mathbb{R}^3$.
- (b) A function $T : \mathbb{R}^3 \to \mathbb{R}$ is defined by $T(x) = -x_1 - 5x_2 + 2x_3$.

Prove that T is a linear transformation.

Example Let $T : P_2 \to P_2$, $T(p)(x) = p(x-2)$. Show that T, "the shift map", is a linear map.

Example Prove that the following functions are not linear.

- 1. $T : \mathbb{R}^3 \to \mathbb{R}$ defined by $T(x) = x_1 x_2 x_3$;
- 2. $T : \mathbb{R}^2 \to \mathbb{R}$ given by $T(x) = |x|$; and
- 3. $T : M_{22} \to M_{22}$ given by $T(X) = X - 4I$.

Formulae for linear transformations Given the values of a linear transformation T at a basis of its domain, find $T(x)$ for all x.

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be linear. Find $T(x)$, given that

- (a) $T(1,0) = (-2,-1,3)$ and $T(0,1) = (4,3,-4)$;

- (b) $T(4,-7) = (-2,-1,3)$ and $T(-5,9) = (4,3,-4)$.

Matrices of linear transformations

There is a very close and important relationship between linear transformations and matrix multiplication.

Exercise Find the matrices with respect to standard bases of the linear transformations

- 1. $T : \mathbb{R}^2 \to \mathbb{R}^3$, where $T(x_1, x_2) = (x_1 + 2x_2, 3x_1 - 7x_2, x_2)$;

- 2. the differentiation map $T : P_3 \to P_2$ given by $T(p) = p'$;

- 3. Let $T : P_2 \to P_2$, $T(p)(x) = p(x-2)$.