Linear Algebra MATH 2501

Mircea Voineagu

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Linear transformations

Objective Know what is meant by a linear transformation (linear map, linear function). Determine whether a given function is linear or not. Understand clearly the difference between a “linear transformation” proof and a “vector subspace” proof.

- 1. How do you prove that a function is linear?
- 2. Give some short cuts for proving that a function is linear.

Definition A linear map $T : V \to V$ is a map with the following two properties:

- 1. $T(v + w) = T(v) + T(w)$ for any $v, w \in V$ and
- 2. $T(\lambda v) = \lambda T(v)$ for any $v \in V$ and $\lambda \in \mathbb{R}$(or $\mathbb{C}$).

In particular $T(0) = 0$. The two conditions can be written together as

$$T(\lambda v + \mu w) = \lambda T(v) + \mu T(w).$$

for any $\lambda, \mu \in R$ (or $\mathbb{C}$) and $v, w \in V$.

Example Do the following two problems in parallel columns, making sure that you understand the differences:

- (a) Show that $W = \{x \in \mathbb{R}^3 | -x_1 - 5x_2 + 2x_3 = 0\}$ is a subspace of $\mathbb{R}^3$.
- (b) A function $T : \mathbb{R}^3 \to \mathbb{R}$ is defined by $T(x) = -x_1 - 5x_2 + 2x_3$.

Prove that T is a linear transformation.

Example Let $T : P_2 \to P_2$, $T(p)(x) = p(x - 2)$. Show that T, ”the shift map”, is a linear map.

Example Prove that the following functions are not linear.

- 1.$T : \mathbb{R}^3 \to \mathbb{R}$ defined by $T(x) = x_1 x_2 x_3$;
- 2.$T : \mathbb{R}^2 \to \mathbb{R}$ given by $T(x) = |x|$; and
- 3.$T : M_{22} \to M_{22}$ given by $T(X) = X - 4I$.

Formulae for linear transformations Given the values of a linear transformation T at a basis of its domain, find $T(x)$ for all x.

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be linear. Find $T(x)$, given that

- (a) $T(1,0) = (-2,-1,3)$ and $T(0,1) = (4,3,-4)$;

- (b) $T(4,-7) = (-2,-1,3)$ and $T(-5,9) = (4,3,-4)$.

Matrices of linear transformations

There is a very close and important relationship between linear transformations and matrix multiplication.

Exercise Find the matrices with respect to standard bases of the linear transformations

- 1. $T : \mathbb{R}^2 \to \mathbb{R}^3$, where $T(x_1, x_2) = (x_1 + 2x_2, 3x_1 - 7x_2, x_2)$;

- 2. the differentiation map $T : P_3 \to P_2$ given by $T(p) = p'$;

- 3. Let $T : P_2 \to P_2$, $T(p)(x) = p(x-2)$.

The formal definition of “matrix of a linear transformation” is as follows.

Definition Let $T : V \to W$ be a linear transformation, where V and W are finite–dimensional vector spaces. Then A is the matrix of T with respect to ordered bases B for V and C for W if

$$[T(v)]_C = A[v]_B$$

for all vectors v in V. Here $[v]_B$ denotes the coordinate vector of v with respect to the basis B, and $[T(v)]_C$ denotes the coordinate vector of $T(v)$ with respect to the basis C.

The Diagram is the following:

Drawing an appropriate diagram in each case, use the preceding method to find the matrix of T with respect to the given bases.

- 1. $T : \mathbb{R}^2 \to \mathbb{R}^3$ given by $T(x) = (x_1, x_1 + x_2, x_1 - x_2)$, with the bases $\{(1,2),(3,4)\}$ for $\mathbb{R}^2$ and $\{(0,2,1),(1,-1,0),(0,3,2)\}$ for $\mathbb{R}^3$;

- 2.$T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(x) = Ax$, with respect to the basis $\{(2,3),\ (5,7)\}$ in both domain and codomain. Here A is a fixed 2×2 matrix.

- 3.$T : P_2 \to \mathbb{R}^2$ given by $T(p) = (p(1), p(2))$, with bases $\{1 - t, 2 - t, t^2\}$ for P_2 and $\{(2, 3), (4, 5)\}$ for $\mathbb{R}^2$.

A harder problem (Not hard as regards calculation, but you will need to understand the concepts properly!) Find the matrices with respect to standard bases of the linear transformations $T : M_{22} \to M_{22}$ given by

$$T(X) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} X$$

Nullspace and image of a linear transformation

Objective Know what is meant by the nullspace (kernel) and the image of a linear transformation; find bases and dimensions for these spaces. Know the connection between the kernel of a linear transformation and that of a matrix.

- 1.Define the terms nullspace (kernel), image, nullity and rank for a linear transformation $T : V \to W$.
- 2.If A is a matrix, and a linear mapping is defined by $T(x) = Ax$, what is the relation between the kernel of T and the kernel of A?
- 3.What equation connects the rank and nullity of a linear map?

Example Find kernel, image, nullity and rank for the linear mappings

- 1.$T : \mathbb{R}^5 \to \mathbb{R}^3$ defined by $T(x) = Ax$, where A is
$$\begin{pmatrix} 1 & -1 & 4 & 0 & 4 \\ 2 & 1 & 7 & -1 & 11 \\ -1 & -8 & -1 & 3 & -13. \end{pmatrix}$$

- 2.$T : P_3 \to \mathbb{R}$ given by $T(p) = p(1)$;

Dot product, length and projections

Objective Know how to find the dot product of two vectors and the length of a vector. Understand the geometric meaning of these concepts. Be able to find the projection of a vector onto another.

- 1.Write down a formula relating dot products, lengths and angles.

- 2.How do you tell if two vectors are orthogonal?

- 3.Draw a diagram illustrating the projection of a vector u onto a non-zero vector v. The formula for the projection in terms of the vectors u and v is

$$proj_v u = \frac{u.v}{||v||^2} v.$$

Example Let $u = (3, -1, 1)$, $v = (1, 4, 1)$ and $w = (3, 2, -3)$. Find a unit vector in the direction of u, show that u and v are orthogonal, find the angle between u and w, and find the projection of v onto w.

Example Repeat the above for $u = (2, 3, -1)$, $v = (-1, 2, 4)$ and $w = (1, 5, -4)$.

Orthogonal complement and projection onto subspaces

Definition Let W be a subspace of a vector space V. Use properties of dot products to show that

$$W^{\perp} = \{v \in V | v \cdot w = 0 \text{ for all } w \in W\}$$

is also a subspace of V.

The subspace $W^{\perp}$ in the above exercise is called the **orthogonal complement** of W. Geometrically, it consists of those vectors in V which are perpendicular to every vector in W. In $\mathbb{R}^3$,

- the orthogonal complement of a plane through the origin is a line through the origin;
- the orthogonal complement of a line through the origin is a plane through the origin.

- 1.Let $W = \{x \in \mathbb{R}^3 | 3x_1 - x_2 + 7x_3 = 0\}$. Find $W^{\perp}$.

- 2.In $\mathbb{R}^3$, let $W = Span\{(1,4,-1),(1,2,0)\}$. Find $W^\perp$.

- 3. An example in $\mathbb{R}^4$. Let $W = Span\{(1, -1, 2, 0), (-2, 1, 0, 1)\}$. Find $W^{\perp}$.

We know how to find the projection of a vector into a given direction, that is, the projection onto a line. But a line is simply a subspace of dimension 1, and sometimes we want to project onto a subspace of larger dimension.

For example, in physics or engineering we may need to find the horizontal component of a force or velocity – that is, the projection into a horizontal plane.

Exercise Find the projection of the vector $v = (6, 1, -5)$ onto the plane $W =< \{w_1, w_2\} >$, where $w_1 = (1, 2, 1)$ and $w_2 = (-1, 1, 0)$.

Remark. The difference between v and the projection is

$$v - proj_W v = (2, 2, -6)$$

and this is perpendicular to both $w_1 = (1, 2, 1)$ and $w_2 = (-1, 1, 0)$.

Now suppose that $w_1 \perp w_2$, that is, the basis vectors for the subspace W are perpendicular. Then the above calculations become much easier: we have

$$v = \lambda_1 w_1 + \lambda_2 w_2 + w,$$

so

$$v \cdot w_1 = \lambda_1 w_1 \cdot w_1$$

and

$$\lambda_1 = \frac{v \cdot w_1}{w_1 \cdot w_1}.$$

Calculating a similar formula for λ_2 gives the projection

$$proj_W v = \frac{v \cdot w_1}{w_1 \cdot w_1} w_1 + \frac{v \cdot w_2}{w_2 \cdot w_2} w_2.$$

The resemblance to the formula for projection onto a line should be clear. **Note** carefully that this formula is only true when w_1 and w_2 are perpendicular.

Definition A set of vectors is **orthonormal** if the vectors are all of unit length and perpendicular to each other. That is, $\{v_1, v_2, ..., v_n\}$ is orthonormal if $v_i.v_j = \begin{cases} 1 & \text{if} i = j \\ 0 & \text{if} i \neq j. \end{cases}$