MATH1081 – Discrete Mathematics

Topic 1 – Set theory and functions
Lecture 1.06 – Properties of functions and inverses

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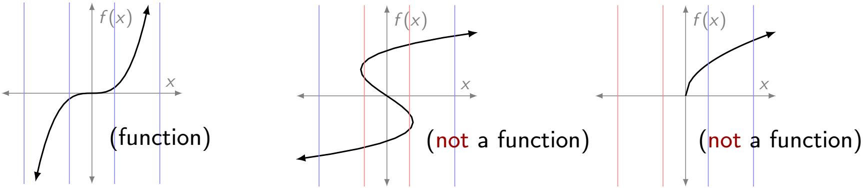
Functions (again)

Recall that a set $f \subseteq X \times Y$ is a function if and only if there is exactly one ordered pair $(x, y) \in f$ for each $x \in X$.

Equivalently, we could say that a set $f \subseteq X \times Y$ is a function if and only if every possible input value in X has exactly one corresponding output value in Y. That is, a set $f \subseteq X \times Y$ is a function if and only if for every $x \in X$, there is exactly one $y \in Y$ such that $f(x) = y$.

In terms of arrow diagrams, a set $f \subseteq X \times Y$ is a function if and only if each element of the domain X has exactly one outgoing arrow.

In terms of coordinate graphs, a set $f \subseteq \mathbb{R} \times \mathbb{R}$ is a function if and only if every possible vertical line touches the graph at exactly one point.



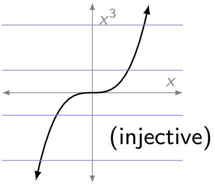
Injectivity

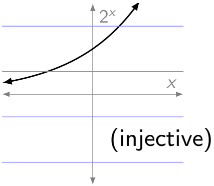
Definition. A function $f : X \to Y$ is injective (or one-to-one) if and only if every possible output value in Y has at most one corresponding input value in X. Equivalently, a function $f : X \to Y$ is injective if and only if any of the following are true:

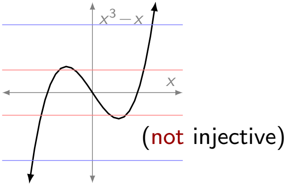
- For every $y \in Y$, there is at most one $x \in X$ such that $f(x) = y$.
- For every $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$ then we must have that $x_1 = x_2$.
- For every $x_1, x_2 \in X$, if $x_1 \neq x_2$ then we must have that $f(x_1) \neq f(x_2)$.

In terms of arrow diagrams, a function $f : X \to Y$ is injective if and only if each element of the codomain Y has at most one incoming arrow.

In terms of coordinate graphs, a function $f : \mathbb{R} \to \mathbb{R}$ is injective if and only if every possible horizontal line touches the graph at at most one point.







Injectivity – examples

Example. Decide whether each function below is injective, and prove your claim.

- $f : \{1, 2, 3\} \to \{1, 2, 3, 4\}$, $f = \{(1, 2), (2, 3), (3, 4)\}$
- $f : \{1, 2, 3\} \to \{1, 2, 3, 4\}$, $f = \{(1, 2), (2, 4), (3, 2)\}$
- $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = x^2$
- $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = 2x + 1$

Notice that to prove a function is injective when its domain is infinite (or very large), we need to give a proof for arbitrary domain elements x_1 and x_2.

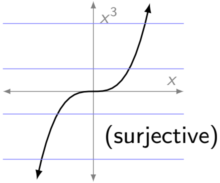
Surjectivity

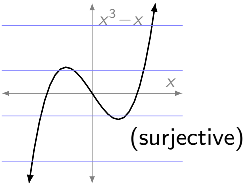
Definition. A function $f : X \to Y$ is surjective (or onto) if and only if every possible output value in Y has at least one corresponding input value in X.
Equivalently, a function $f : X \to Y$ is surjective if and only if any of the following are true:

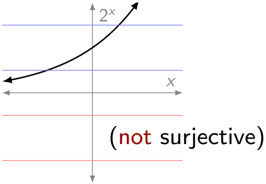
- For every $y \in Y$, there is at least one $x \in X$ such that $f(x) = y$.
- The range of f and the codomain of f are equal, that is, $f(X) = Y$.

In terms of arrow diagrams, a function $f : X \to Y$ is surjective if and only if each element of the codomain Y has at least one incoming arrow.

In terms of coordinate graphs, a function $f : \mathbb{R} \to \mathbb{R}$ is surjective if and only if every possible horizontal line touches the graph at at least one point.







Surjectivity – examples

Example. Decide whether each function below is surjective, and prove your claim.

- $f : \{1, 2, 3, 4\} \to \{1, 2, 3\}$, $f = \{(1, 2), (2, 3), (3, 1), (4, 2)\}$
- $f : \{1, 2, 3, 4\} \to \{1, 2, 3\}$, $f = \{(1, 1), (2, 2), (3, 1), (4, 2)\}$
- $f : \mathbb{N} \to \mathbb{N}$, $f(x) = x^2$
- $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = 2 - x$

Notice that to prove a function is surjective when its codomain is infinite (or very large), we need to give a proof for an arbitrary codomain element y.

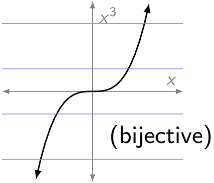
Bijectivity

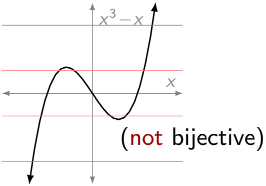
Definition. A function $f : X \to Y$ is bijective (or a one-to-one correspondence) if and only if every possible output value in Y has exactly one corresponding input value in X. Equivalently, a function $f : X \to Y$ is bijective if and only if any of the following are true:

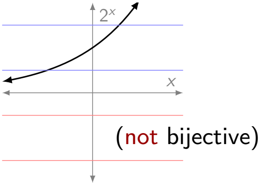
- For every $y \in Y$, there is exactly one $x \in X$ such that $f(x) = y$.
- The function f is both injective and surjective.

In terms of arrow diagrams, a function $f : X \to Y$ is bijective if and only if each element of the codomain Y has exactly one incoming arrow.

In terms of coordinate graphs, a function $f : \mathbb{R} \to \mathbb{R}$ is bijective if and only if every possible horizontal line touches the graph at exactly one point.







Cardinalities of domain and codomain

Theorem. Suppose $f : X \to Y$ is a function for some sets X and Y.

- If f is injective, then $|X| \leq |Y|$.
- If f is surjective, then $|X| \geq |Y|$.
- If f is bijective, then $|X| = |Y|$.

Proof.

Notice that the converse (opposite direction) statements are not true in general. For example, if $|X| \leq |Y|$ then this does not guarantee that $f : X \to Y$ is injective.

Function composition

Definition. Given two functions $f : X \to Y$ and $g : Y \to Z$ for any sets X, Y, Z, the composition of f and g is the function $g \circ f : X \to Z$ defined by

$$(g \circ f)(x) = g(f(x)) \quad \text{for all } x \in X.$$

More generally, the composition of functions f and g is defined whenever the range of f is a subset of the domain of g.

Notice that the order in which the component functions are applied is from right to left, since applying $g \circ f$ to x returns the result of applying f to x and then g to this output.

Example. Suppose $f : \mathbb{Z} \to \mathbb{Z}$ and $g : \mathbb{Z} \to \mathbb{Z}$ are functions defined by $f(x) = x^2$ and $g(x) = 2 - x$. Find $g \circ f$ and $f \circ g$.

Solution.

Theorem. Given two functions $f : X \to Y$ and $g : Y \to Z$:

- if f and g are both injective, then $g \circ f$ is also injective.
- if f and g are both surjective, then $g \circ f$ is also surjective.

Proof. Try this yourself! (See tutorial Problem Set 1, Question 34.)

Identity and inverse functions

Definition. For any set X, the identity function for X, denoted ι_X (Greek letter iota) or id_X, is the function $\iota_X : X \to X$ defined by $\iota_X(x) = x$ for all $x \in X$.

Theorem. For any function $f : X \to Y$, we have $f \circ \iota_X = f$ and $\iota_Y \circ f = f$.

Proof.

Definition. The inverse of a function $f : X \to Y$, if it exists, is the function $g : Y \to X$ satisfying $g \circ f = \iota_X$ and $f \circ g = \iota_Y$. That is, $(g \circ f)(x) = x$ for all $x \in X$, and $(f \circ g)(y) = y$ for all $y \in Y$.

Notation. We write f^{-1} for the inverse of f (if it exists). This looks the same as our notation for the pre-image of a set, but we can distinguish the two notations by the fact that the inverse function takes elements of Y as inputs, whereas the pre-image takes subsets of Y as inputs.

Properties of inverses

In terms of sets of ordered pairs, if $f \subseteq X \times Y$ is a function and its inverse f^{-1} exists, then $f^{-1} = \{(y, x) \in Y \times X \ : \ (x, y) \in f\}$. This justifies the following properties of the inverse function:

Lemma. If a function f has an inverse, its inverse must be unique.

Proof.

Lemma. If a function f has an inverse f^{-1}, then the inverse of f^{-1} is f. That is, $(f^{-1})^{-1} = f$.

Proof.

Theorem. Suppose $f : X \to Y$ and $g : Y \to Z$ are functions such that their composition $g \circ f$ has an inverse. Then $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Proof.

Finding inverses

If the output values of $f : X \to Y$ are defined by a formula $y = f(x)$ for all $x \in X$, then if the inverse f^{-1} exists, it satisfies the formula $x = f^{-1}(y)$ for all $y \in Y$.

Example. Find the inverse, if it exists, of the following functions.

- $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x + 1$.

  Solution.

- $g : \mathbb{Z} \to \mathbb{Z}$, $g(x) = 2x + 1$.

  Solution.

- $h : \mathbb{R} \to \mathbb{R}^+$, $h(x) = x^2$.

  Solution.

Inverses and bijective functions

Theorem. The inverse of a function $f : X \to Y$ exists if and only if f is bijective.

Proof.