MATH1081 – Discrete Mathematics

Topic 1 – Set theory and functions
Lecture 1.06 – Properties of functions and inverses

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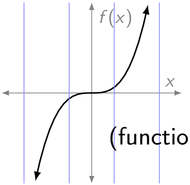
Functions (again)

Recall that a set $f \subseteq X \times Y$ is a function if and only if there is exactly one ordered pair $(x, y) \in f$ for each $x \in X$.

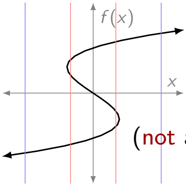
Equivalently, we could say that a set $f \subseteq X \times Y$ is a function if and only if every possible input value in X has exactly one corresponding output value in Y. That is, a set $f \subseteq X \times Y$ is a function if and only if for every $x \in X$, there is exactly one $y \in Y$ such that $f(x) = y$.

In terms of arrow diagrams, a set $f \subseteq X \times Y$ is a function if and only if each element of the domain X has exactly one outgoing arrow.

In terms of coordinate graphs, a set $f \subseteq \mathbb{R} \times \mathbb{R}$ is a function if and only if every possible vertical line touches the graph at exactly one point.



(function)



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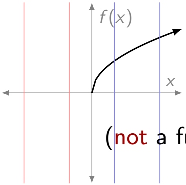


(not a function)

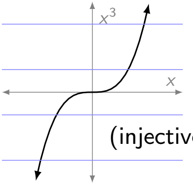
Injectivity

Definition. A function $f : X \to Y$ is injective (or one-to-one) if and only if every possible output value in Y has at most one corresponding input value in X. Equivalently, a function $f : X \to Y$ is injective if and only if any of the following are true:

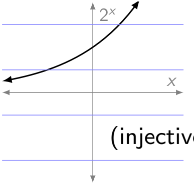
- For every $y \in Y$, there is at most one $x \in X$ such that $f(x) = y$.
- For every $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$ then we must have that $x_1 = x_2$.
- For every $x_1, x_2 \in X$, if $x_1 \neq x_2$ then we must have that $f(x_1) \neq f(x_2)$.

In terms of arrow diagrams, a function $f : X \to Y$ is injective if and only if each element of the codomain Y has at most one incoming arrow.

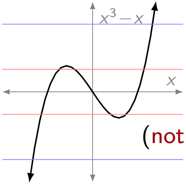
In terms of coordinate graphs, a function $f : \mathbb{R} \to \mathbb{R}$ is injective if and only if every possible horizontal line touches the graph at at most one point.



(injective)



(injective)



(not injective)

Injectivity – examples

Example. Decide whether each function below is injective, and prove your claim.

- $f : \{1,2,3\} \to \{1,2,3,4\}$, $f = \{(1,2),(2,3),(3,4)\}$ is injective since for every possible output value (1, 2, 3, 4), there is at most one corresponding input value (none, 1, 2, 3 respectively).
- $f : \{1,2,3\} \to \{1,2,3,4\}$, $f = \{(1,2),(2,4),(3,2)\}$ is not injective since there is an output value with more than one corresponding input value. Specifically, $f(1) = 2 = f(3)$ but $1 \neq 3$.
- $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = x^2$ is not injective since there exists an output value with more than one corresponding input value. For example, $2 \neq -2$ but $f(2) = 4 = f(-2)$.
- $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = 2x + 1$ is injective since for every $x_1, x_2 \in \mathbb{Z}$, if $f(x_1) = f(x_2)$ then we have $2x_1 + 1 = 2x_2 + 1$, which after rearrangement implies $x_1 = x_2$.

  Notice that to prove a function is injective when its domain is infinite (or very large), we need to give a proof for arbitrary domain elements x_1 and x_2.

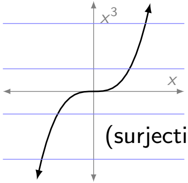
Surjectivity

Definition. A function $f : X \to Y$ is surjective (or onto) if and only if every possible output value in Y has at least one corresponding input value in X. Equivalently, a function $f : X \to Y$ is surjective if and only if any of the following are true:

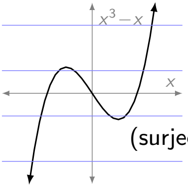
- For every $y \in Y$, there is at least one $x \in X$ such that $f(x) = y$.
- The range of f and the codomain of f are equal, that is, $f(X) = Y$.

In terms of arrow diagrams, a function $f : X \to Y$ is surjective if and only if each element of the codomain Y has at least one incoming arrow.

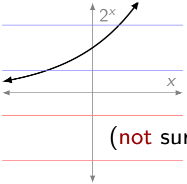
In terms of coordinate graphs, a function $f : \mathbb{R} \to \mathbb{R}$ is surjective if and only if every possible horizontal line touches the graph at at least one point.



(surjective)



(surjective)



(not surjective)

Surjectivity – examples

Example. Decide whether each function below is surjective, and prove your claim.

- $f : \{1, 2, 3, 4\} \to \{1, 2, 3\}$, $f = \{(1, 2), (2, 3), (3, 1), (4, 2)\}$ is surjective since for every possible output value (1, 2, 3), there is at least one corresponding input value (3, 1 and 4, 2 respectively).
- $f : \{1, 2, 3, 4\} \to \{1, 2, 3\}$, $f = \{(1, 1), (2, 2), (3, 1), (4, 2)\}$ is not surjective since there is an output value with less than one corresponding input value. Specifically, 3 is in the codomain but $f(x) = 3$ has no solution for any $x \in \{1, 2, 3, 4\}$.
- $f : \mathbb{N} \to \mathbb{N}$, $f(x) = x^2$ is not surjective since there exists an output value with less than one corresponding input value. For example, $2 \in \mathbb{Z}$ but $f(x) = 2$ has no solution for any $x \in \mathbb{Z}$.
- $f : \mathbb{Z} \to \mathbb{Z}$, $f(x) = 2 - x$ is surjective since for every $y \in \mathbb{Z}$, the equation $f(x) = y$ has a solution for some $x \in \mathbb{Z}$, namely $x = 2 - y$.

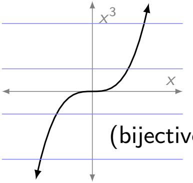
  Notice that to prove a function is surjective when its codomain is infinite (or very large), we need to give a proof for an arbitrary codomain element y.

Definition. A function $f : X \to Y$ is bijective (or a one-to-one correspondence) if and only if every possible output value in Y has exactly one corresponding input value in X. Equivalently, a function $f : X \to Y$ is bijective if and only if any of the following are true:

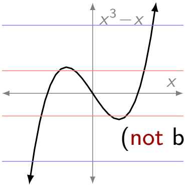
- For every $y \in Y$, there is exactly one $x \in X$ such that $f(x) = y$.
- The function f is both injective and surjective.

In terms of arrow diagrams, a function $f : X \to Y$ is bijective if and only if each element of the codomain Y has exactly one incoming arrow.

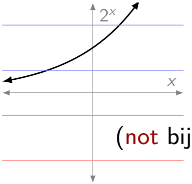
In terms of coordinate graphs, a function $f : \mathbb{R} \to \mathbb{R}$ is bijective if and only if every possible horizontal line touches the graph at exactly one point.



(bijective)



(not bijective)



(not bijective)

Cardinalities of domain and codomain

Theorem. Suppose $f : X \to Y$ is a function for some sets X and Y.

- If f is injective, then $|X| \leq |Y|$.
- If f is surjective, then $|X| \geq |Y|$.
- If f is bijective, then $|X| = |Y|$.

Proof. For ease of reference, we describe f in terms of its arrow diagram representation. Since f is a function, we know there must be exactly one outgoing arrow for each element in the domain, so there are exactly $|X|$ arrows in the diagram.

If f is injective, then there must be at most one incoming arrow for each element in the codomain, so there are at least $|X|$ elements in the codomain.

If f is surjective, then there must be at least one incoming arrow for each element in the codomain, so there are at most $|X|$ elements in the codomain.

If f is bijective, then f is both injective and surjective, meaning both $|X| \leq |Y|$ and $|X| \geq |Y|$, so we must have that $|X| = |Y|$.

Notice that the converse (opposite direction) statements are not true in general. For example, if $|X| \leq |Y|$ then this does not guarantee that $f : X \to Y$ is injective.

Function composition

Definition. Given two functions $f : X \to Y$ and $g : Y \to Z$ for any sets X, Y, Z, the composition of f and g is the function $g \circ f : X \to Z$ defined by

$$(g \circ f)(x) = g(f(x)) \quad \text{for all } x \in X.$$

More generally, the composition of functions f and g is defined whenever the range of f is a subset of the domain of g.

Notice that the order in which the component functions are applied is from right to left, since applying $g \circ f$ to x returns the result of applying f to x and then g to this output.

Example. Suppose $f : \mathbb{Z} \to \mathbb{Z}$ and $g : \mathbb{Z} \to \mathbb{Z}$ are functions defined by $f(x) = x^2$ and $g(x) = 2 - x$. Find $g \circ f$ and $f \circ g$.

Solution. For all $x \in \mathbb{Z}$, we have $(g \circ f)(x) = g(f(x)) = g(x^2) = 2 - x^2$, while $(f \circ g)(x) = f(g(x)) = f(2 - x) = (2 - x)^2 = 4 - 4x + x^2$.

Theorem. Given two functions $f : X \to Y$ and $g : Y \to Z$:

- if f and g are both injective, then $g \circ f$ is also injective.
- if f and g are both surjective, then $g \circ f$ is also surjective.

Proof. Try this yourself! (See tutorial Problem Set 1, Question 34.)

Identity and inverse functions

Definition. For any set X, the identity function for X, denoted ι_X (Greek letter iota) or id_X, is the function $\iota_X : X \to X$ defined by $\iota_X(x) = x$ for all $x \in X$.

Theorem. For any function $f : X \to Y$, we have $f \circ \iota_X = f$ and $\iota_Y \circ f = f$.

Proof. For all $x \in X$, we have $(f \circ \iota_X)(x) = f(\iota_X(x)) = f(x)$.
Similarly for all $x \in X$, we have $(\iota_Y \circ f)(x) = \iota_Y(f(x)) = f(x)$.

Definition. The inverse of a function $f : X \to Y$, if it exists, is the function $g : Y \to X$ satisfying $g \circ f = \iota_X$ and $f \circ g = \iota_Y$. That is, $(g \circ f)(x) = x$ for all $x \in X$, and $(f \circ g)(y) = y$ for all $y \in Y$.

Notation. We write f^{-1} for the inverse of f (if it exists). This looks the same as our notation for the pre-image of a set, but we can distinguish the two notations by the fact that the inverse function takes elements of Y as inputs, whereas the pre-image takes subsets of Y as inputs.

Properties of inverses

In terms of sets of ordered pairs, if $f \subseteq X \times Y$ is a function and its inverse f^{-1} exists, then $f^{-1} = \{(y,x) \in Y \times X \;:\; (x,y) \in f\}$. This justifies the following properties of the inverse function:

Lemma. If a function f has an inverse, its inverse must be unique.

Proof. It is clear that given any function $f \subseteq X \times Y$, its inverse f^{-1} is defined uniquely as a subset of $Y \times X$ in terms of f.

Lemma. If a function f has an inverse f^{-1}, then the inverse of f^{-1} is f. That is, $(f^{-1})^{-1} = f$.

Proof. Suppose $f \subseteq X \times Y$ has inverse $f^{-1} = \{(y,x) \;:\; (x,y) \in f\}$. Then $(f^{-1})^{-1} = \{(x,y) \;:\; (y,x) \in f^{-1}\} = \{(x,y) \;:\; (x,y) \in f\} = f$.

Theorem. Suppose $f : X \to Y$ and $g : Y \to Z$ are functions such that their composition $g \circ f$ has an inverse. Then $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Proof. Let $h = f^{-1} \circ g^{-1}$. Then we have

$$h \circ (g \circ f) = f^{-1} \circ (g^{-1} \circ g) \circ f = f^{-1} \circ \iota_Y \circ f = f^{-1} \circ f = \iota_X,$$

$$(g \circ f) \circ h = g \circ (f \circ f^{-1}) \circ g^{-1} = g \circ \iota_X \circ g^{-1} = g \circ g^{-1} = \iota_Y.$$

So indeed h is the unique inverse of $g \circ f$, that is, $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Finding inverses

If the output values of $f : X \to Y$ are defined by a formula $y = f(x)$ for all $x \in X$, then if the inverse f^{-1} exists, it satisfies the formula $x = f^{-1}(y)$ for all $y \in Y$.

Example. Find the inverse, if it exists, of the following functions.

- $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x + 1$.

  Solution. We write $y = 2x + 1$ and rearrange to make x the subject, giving $x = \frac{y-1}{2}$. So the inverse $f^{-1} : \mathbb{R} \to \mathbb{R}$ exists and is given by $f^{-1}(y) = \frac{y-1}{2}$. (Or equivalently, $f^{-1}(x) = \frac{x-1}{2}$.)

- $g : \mathbb{Z} \to \mathbb{Z}$, $g(x) = 2x + 1$.

  Solution. If the inverse exists, it is given by $g^{-1} : \mathbb{Z} \to \mathbb{Z}$, $g^{-1}(y) = \frac{y-1}{2}$. But this formula returns non-integer outputs for even integer inputs, so it is not well-defined. So the inverse function does not exist.

- $h : \mathbb{R} \to \mathbb{R}^+$, $h(x) = x^2$.

  Solution. Writing $y = x^2$ and rearranging to make x the subject gives $x = \pm\sqrt{y}$. So if the inverse of h exists, it is given by $h^{-1} : \mathbb{R}^+ \to \mathbb{R}$, $h^{-1}(x) = \pm\sqrt{x}$. But this formula returns more than one output for any positive real input, so it is not well-defined as a function. So the inverse function does not exist.

Theorem. The inverse of a function $f : X \to Y$ exists if and only if f is bijective.

Proof. For ease of reference, we describe f and f^{-1} in terms of their arrow diagram representations. Since f is a function, we know there must be exactly one outgoing arrow for each element in the domain X.

Proof that if its inverse function exists, then f must be bijective:
The arrow diagram representation for the inverse function f^{-1} is formed by reversing the direction of all the arrows in the arrow diagram for f. Since $f^{-1} : Y \to X$ is a function, every element of Y must have exactly one outgoing arrow. This means that in the arrow diagram for f, every element of Y must have exactly one incoming arrow. So f is bijective.

Proof that if f is bijective, then it must have an inverse function:
Since it is bijective, the arrow diagram for f must have exactly one incoming arrow for every element in Y. Reversing the direction of all the arrows in the diagram will create a representation for a new function $g : Y \to X$, since every element in its domain Y will have exactly one outgoing arrow. Furthermore, this function g must be the inverse of f, since $g(f(x)) = x$ for all $x \in X$ and $f(g(y)) = y$ for all $y \in Y$.