MATH1081 – Discrete Mathematics

Topic 1 – Set theory and functions
Lecture 1.03 – Venn diagrams and set operations

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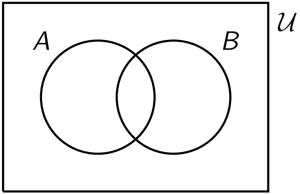
Venn diagrams

Definition. When working with related sets, it can be useful to define a universal set which contains all elements relevant to these sets. The universal set is usually denoted by $\mathcal{U}$ or just U. In particular, all the related sets are subsets of $\mathcal{U}$.

For example, if our sets are MATH1081 tutorials, then an appropriate universal set might be the set of all MATH1081 students. If our sets are intervals on the real number line, then the universal set would be $\mathbb{R}$.

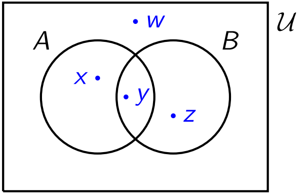
Definition. A Venn diagram is a diagrammatic tool for visualising relationships between related sets. Sets are represented as overlapping closed figures (usually circles) within a larger figure (usually a rectangle) representing the universal set. Elements can be represented as points placed within different sections of the diagram to show which sets they belong to.

For example, here is a general Venn diagram for two sets A and B with universal set $\mathcal{U}$:



Venn diagram example

Example. Consider the following Venn diagram:



For each of the elements $w, x, y, z \in \mathcal{U}$, we can make the following conclusions:

- $w \notin A$ and $w \notin B$.
- $x \in A$ and $x \notin B$.
- $y \in A$ and $y \in B$.
- $z \notin A$ and $z \in B$.

In order to more easily refer to the different sections in a Venn diagram, we shall introduce new notation that allows us to perform operations on sets.

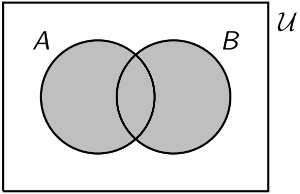
Union

Definition. The union of two sets A and B, written as $A \cup B$, is the set of all elements in A or B (or both). That is,

$$A \cup B = \{x \in \mathcal{U} \ : \ x \in A \ \text{ or } \ x \in B\}.$$

(Note: The word "or" in mathematics is always treated as inclusive.)

In a Venn diagram, $A \cup B$ is represented by this shaded region:



Example. Given that $R = \{a, c, e\}$, $S = \{b, d\}$, and $T = \{d, e, f\}$, find the following:

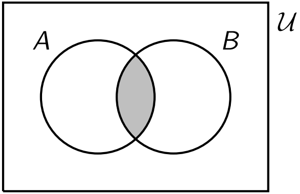
- $R \cup S = \{a, b, c, d, e\}$.
- $R \cup T = \{a, c, d, e, f\}$.
- $S \cup T = \{b, d, e, f\}$.
- $R \cup S \cup T = \{a, b, c, d, e, f\}$.

Intersection

Definition. The intersection of two sets A and B, written as $A \cap B$, is the set of all elements in both A and B. That is,

$$A \cap B = \{x \in \mathcal{U} \ : \ x \in A \ \text{ and } \ x \in B\}.$$

In a Venn diagram, $A \cap B$ is represented by this shaded region:



Example. Given that $R = \{a, c, e\}$, $S = \{b, c, d\}$, and $T = \{a, b, d, e\}$, find the following:

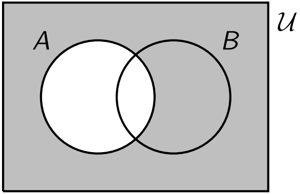
- $R \cap S = \{c\}$.
- $R \cap T = \{a, e\}$.
- $S \cap T = \{b, d\}$.
- $R \cap S \cap T = \{\}$.

Complement

Definition. The complement of a set A, written as A^c (or sometimes $\overline{A}$), is the set of all elements in the universal set that are not in A. That is,

$$A^c = \{x \in \mathcal{U} \ : \ x \notin A\}.$$

In a Venn diagram, A^c is represented by this shaded region:



Example. Given that $\mathcal{U} = \{a, b, c, d, e\}$, $R = \{a, c, e\}$, $S = \{b\}$, and $T = \{\}$, find the following:

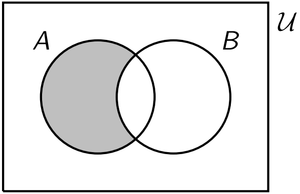
- $R^c = \{b, d\}$.
- $S^c = \{a, c, d, e\}$.
- $T^c = \{a, b, c, d, e\}$.

Set difference

Definition. The difference of two sets A and B, written as $A - B$ (or sometimes $A \setminus B$), is the set of all elements in A which are not in B. That is,

$$A - B = \{x \in \mathcal{U} \;:\; x \in A \;\text{ and }\; x \notin B\}.$$

In a Venn diagram, $A - B$ is represented by this shaded region:



Example. Given that $R = \{a, b, c, d\}$, $S = \{b, c, d\}$, and $T = \{b, d, e\}$, find the following:

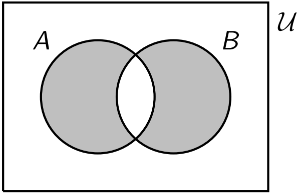
- $R - S = \{a\}$.
- $R - T = \{a, c\}$.
- $S - T = \{c\}$.
- $S - R = \{\}$. (Notice that in general, $A - B \neq B - A$ for sets A and B.)

Symmetric difference

Definition. The symmetric difference of two sets A and B, written as $A \ominus B$ (or sometimes $A \triangle B$ or $A \oplus B$), is the set of all elements in A or B, but not both. That is,

$$A \ominus B = \{x \in \mathcal{U} \ : \ x \in A \cup B \ \text{ and } \ x \notin A \cap B\}.$$

In a Venn diagram, $A \ominus B$ is represented by this shaded region:

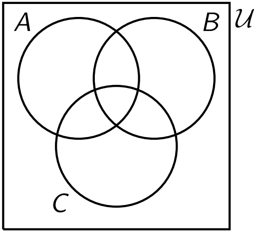


Example. Given that $R = \{a, b, c\}$, $S = \{b\}$, and $T = \{b, c, d\}$, find the following:

- $R \ominus S = \{a, c\}$.
- $R \ominus T = \{a, d\}$.
- $S \ominus T = \{c, d\}$.

Combining set operations

A general Venn diagram for three sets A, B, and C with universal set $\mathcal{U}$ is shown to the right.

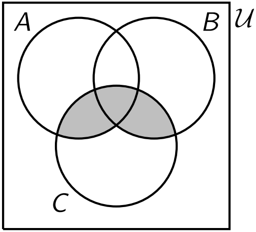


Venn diagrams for four or more sets can become unwieldy or impossible to draw, so they are normally avoided.

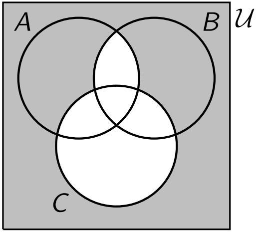
Set operations can be applied to multiple sets, so long as the order of operations is clearly indicated by the use of brackets.

Example. Shade the regions indicated by the following set expressions.

$(A \cup B) \cap C$



$(A \cap B)^c - C$



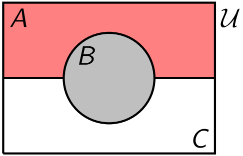
Disjoint sets

Definition. Two sets A and B are disjoint if their intersection is empty, that is, if $A \cap B = \varnothing$.

Definition. The sets $A_1, A_2, A_3, \dots, A_k$ are pairwise disjoint if every pair of sets is disjoint, that is, if $A_i \cap A_j = \varnothing$ for all $i \neq j$.

Definition. The sets $A_1, A_2, A_3, \dots, A_k$ partition the set B if they are pairwise disjoint and $A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_k = B$.

For example, in the Venn diagram on the right, $\mathcal{U}$ is partitioned by the sets A, B, and C.



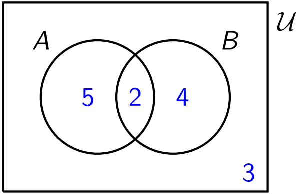
Example. Complete the following sentences regarding general sets A and B:

- A and A^c partition $\mathcal{U}$.
- A and $B - A$ partition $A \cup B$.
- $A \cap B$ and $A \ominus B$ partition $A \cup B$.
- $A \cap B$ and $A - B$ partition A.

Cardinalities in Venn diagrams

Another way to represent information in a Venn diagram is by writing the cardinality of each individual section in the diagram. Cardinalities can be represented as numbers placed within each section (without points, to distinguish set cardinalities from set elements).

Example. Consider the following Venn diagram:



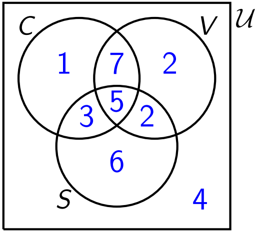
Find each of the following:

- $|A \cap B| = 2$.
- $|A| = 7$.
- $|A \cup B| = 11$.
- $|\mathcal{U}| = 14$.

Example – Finding cardinalities with a Venn diagram

Example. 30 people were asked to select their preferred ice-cream flavours from a list, where more than one selection was allowed. 16 people selected chocolate, 16 selected vanilla, and 16 selected strawberry. 12 people selected both chocolate and vanilla, 8 people selected both chocolate and strawberry, and 7 people selected both vanilla and strawberry. 5 people selected all three flavours. How many surveyed people selected none of the three flavours?

Solution. Let C, V, and S be the set of people who selected chocolate, vanilla, and strawberry respectively. Drawing up a Venn diagram, we can fill out the cardinalities by working from the innermost section outwards:



So the answer is 4.

Inclusion-exclusion principle

Theorem. (Inclusion-exclusion principle)
The cardinality of a union of sets can be expressed in terms of cardinalities of their intersections. For example,

- For any two sets A and B, we have
$$|A \cup B| = |A| + |B| - |A \cap B|.$$
- For any three sets A, B, and C, we have
$$|A \cup B \cup C| = |A| + |B| + |C| - (|A \cap B| + |A \cap C| + |B \cap C|) + |A \cap B \cap C|.$$

Proof (sketch). In the case of two sets, finding $|A| + |B|$ counts each element of $A \cup B$ at least once, but elements that belong to both A and B are counted twice. To account for this, we subtract the size of $A \cap B$. The cases for three or more sets work similarly.

Example. Solve the previous problem using the inclusion-exclusion principle.

Solution. Using the inclusion-exclusion principle, we have
$$\begin{aligned}|C \cup V \cup S| &= |C| + |V| + |S| - |C \cap V| - |C \cap S| - |V \cap S| + |C \cap V \cap S| \\ &= 16 + 16 + 16 - 12 - 8 - 7 + 5 = 26,\end{aligned}$$
so the number of people who chose none of the flavours is $30 - 26 = 4$.