MATH1081 – Discrete Mathematics

Topic 2 – Number theory and relations
Lecture 2.06 – Partial orders

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Antisymmetry

Definition. A relation R on a set X is antisymmetric if and only if for all $x, y \in X$, whenever $x \, R \, y$ and $y \, R \, x$, we have $x = y$.

In terms of arrow diagrams, a relation R is antisymmetric if and only if it contains no arrows pointing in both directions.

Diagrammatically, this property can be represented as:

$$\bullet \qquad \bullet \;\text{ implies we do not have }\; \bullet \longleftrightarrow \bullet$$

Example. Let $X = \{1, 2, 3, 4\}$. Which of the following relations on X are antisymmetric?

- $R = \{(1, 1), (1, 3), (2, 2), (2, 3), (3, 3), (4, 4)\}$

- $R = \{(1, 1), (1, 3), (2, 4), (3, 1), (3, 3), (4, 2), (4, 4)\}$

Notice that symmetry and antisymmetry are not opposite properties! A given relation can be either, neither, or both symmetric and antisymmetric.

Partial order relations

Definition. A relation R on a set X is a partial order relation if and only if it is reflexive, antisymmetric, and transitive.

Definition. If two mathematical objects are related by a partial order relation, we say they are comparable. If $\preceq$ is a partial order relation and $x \preceq y$, we can say "x precedes y" or "y succeeds x".

For example, a very familiar partial order relation is $\leq$ on any subset of $\mathbb{R}$. We can easily check that $\leq$ on $\mathbb{R}$ is reflexive, antisymmetric, and transitive.

Definition. If a relation $\preceq$ is a partial order relation on a set X, we call X a partially ordered set or poset. We sometimes write a partially ordered set as an ordered pair containing the set and the relation, for example $(X, \preceq)$.

Definition. If a partially ordered set has the property that all of its elements are comparable with each other, it is called a totally ordered set.

For example, $(\mathbb{R}, \leq)$ is a totally ordered set.

Examples of partial order relations

Example. Let S be a set. Show that the relation $\preceq$ on $\mathcal{P}(S)$ defined by setting $X \preceq Y$ if and only if $X \subseteq Y$ is a partial order relation.

Solution.

Examples of partial order relations

Example. Show that the relation $\preceq$ on $\mathbb{N}$ defined by setting $x \preceq y$ if and only if $x \mid y$ is a partial order relation.

Solution.

Notice that if this relation $\preceq$ acted on the set $\mathbb{Z}$ instead of $\mathbb{N}$, the relation would no longer be antisymmetric and therefore $(\mathbb{Z}, \mid)$ would not be a partially ordered set. (For example, $1 \mid -1$ and $-1 \mid 1$ but $1 \neq -1$.)

Hasse diagrams

Notation. Given $\preceq$ is a partial order relation on some set X, we use the symbol $\prec$ to mean "precedes but does not equal". That is, $x \prec y$ means $x \preceq y$ and $x \neq y$.

Definition. A Hasse diagram is a simplified way of representing a partially ordered set. Given a partial order relation $\preceq$ on a set X, the Hasse diagram for the partially ordered set $(X, \preceq)$ is constructed as follows:

- Draw a labelled dot for each element of X.
- For each pair of elements $x, y \in X$, draw a line from x to y with x placed lower than y if and only if both
    - $x \prec y$, and
    - there does not exist $z \in X$ such that $x \prec z$ and $z \prec y$.

A Hasse diagram contains all the information of a usual arrow diagram, but in a simpler format.

- Arrowheads are not needed since direction is implied by element position.
- Loops are not included since they must exist at every element.
- Lines implied by transitivity are not included since they must exist for every upward path.

Hasse diagrams – subset example

Example. Draw the Hasse diagram for the partially ordered sets $(\mathcal{P}(\{a, b\}), \subseteq)$ and $(\mathcal{P}(\{a, b, c\}), \subseteq)$.

Solution.

Example. Draw the Hasse diagram for the partially ordered set $(\{1, 2, 3, 4, 6, 8, 9, 12, 18\}, \mid)$.

Solution.

Least and greatest elements

Definition. A minimal element of a partially ordered set $(X, \preceq)$ is any element $x \in X$ such that there is no $y \in X$ where $y \prec x$.

In terms of Hasse diagrams, a minimal element is an element with no lines connecting to it from below.

Definition. A maximal element of a partially ordered set $(X, \preceq)$ is any element $x \in X$ such that there is no $y \in X$ where $x \prec y$.

In terms of Hasse diagrams, a maximal element is an element with no lines connecting to it from above.

Definition. The least element of a partially ordered set $(X, \preceq)$ (if it exists) is the element $x \in X$ for which $x \preceq y$ for all $y \in X$.

In terms of Hasse diagrams, the least element is the element that can reach all other elements in X by following lines in a strictly upwards path.

Definition. The greatest element of a partially ordered set $(X, \preceq)$ (if it exists) is the element $x \in X$ for which $y \preceq x$ for all $y \in X$.

In terms of Hasse diagrams, the greatest element is the element that can reach all other elements in X by following lines in a strictly downwards path.

Least and greatest elements – Examples and properties

Example. Find the minimal, maximal, least, and greatest elements of $(\mathcal{P}(\{a, b, c\}), \subseteq)$.

Solution.

Example. Find the minimal, maximal, least, and greatest elements of $(\{1, 2, 3, 4, 6, 8, 9, 12, 18\}, \mid)$.

Solution.

Fact. The following statements are true for any partially ordered set $(X, \preceq)$.

- If it exists, the least element is unique.
- If it exists, the greatest element is unique.
- If X is finite, the poset has a least element if and only if it has exactly one minimal element.
- If X is finite, the poset has a greatest element if and only if it has exactly one maximal element.

Lower and upper bounds

Definition. Given a partially ordered set $(X, \preceq)$, a lower bound of two elements $x, y \in X$ is any element $z \in X$ such that $z \preceq x$ and $z \preceq y$.

In terms of Hasse diagrams, a lower bound of x and y is any element that can reach both x and y by following lines in strictly upwards paths.

Definition. Given a partially ordered set $(X, \preceq)$, the greatest lower bound of two elements $x, y \in X$, denoted $\operatorname{glb}(x, y)$, is the greatest element amongst the set of all lower bounds (if it exists). That is, $\operatorname{glb}(x, y)$ is the greatest element (if it exists) of $\{z \in X : z \preceq x \text{ and } z \preceq y\}$.

Definition. Given a partially ordered set $(X, \preceq)$, an upper bound of two elements $x, y \in X$ is any element $z \in X$ such that $x \preceq z$ and $y \preceq z$.

In terms of Hasse diagrams, an upper bound of x and y is any element that can reach both x and y by following lines in strictly downwards paths.

Definition. Given a partially ordered set $(X, \preceq)$, the least upper bound of two elements $x, y \in X$, denoted $\operatorname{lub}(x, y)$, is the least element amongst the set of all upper bounds (if it exists). That is, $\operatorname{lub}(x, y)$ is the least element (if it exists) of $\{z \in X : x \preceq z \text{ and } y \preceq z\}$.

Lower and upper bounds – Examples and properties

Example. In the poset $(\mathcal{P}(\{a, b, c\}), \subseteq)$, find $\text{glb}(\{a, b\}, \{b, c\})$ and $\text{lub}(\{a\}, \{a, b\})$ if they exist.

Solution.

Example. In the poset $(\{1, 2, 3, 4, 6, 8, 9, 12, 18\}, \mid)$, find $\text{glb}(12, 18)$ and $\text{lub}(4, 9)$ if they exist.

Solution.

Fact. For any set S, in the partially ordered set $(\mathcal{P}(S), \subseteq)$, for any two elements $A, B \in \mathcal{P}(S)$ we have $\text{glb}(A, B) = A \cap B$ and $\text{lub}(A, B) = A \cup B$.

Fact. In the partially ordered set $(\mathbb{Z}^+, \mid)$, for any two elements $a, b \in \mathbb{Z}^+$ we have $\text{glb}(a, b) = \text{gcd}(a, b)$ and $\text{lub}(a, b) = \text{lcm}(a, b)$.

Note that the least element of $(\mathbb{Z}^+, \mid)$ is 1, but it has no greatest element. If we use the "alternate definition" given for the GCD and LCM, the above facts also hold for the poset $(\mathbb{N}, \mid)$, whose greatest element is 0.