MATH1081 – Discrete Mathematics

Topic 1 – Set theory and functions
Lecture 1.04 – Laws of set algebra

Lecturer: Dr Sean Gardiner – sean.gardiner@unsw.edu.au

Proofs involving set operations

There are three main ways to justify or prove a statement involving set operations:

- Using Venn diagrams.
  - Venn diagrams are a useful visual aide, but are not generally considered valid tools for rigorous proofs.
- Using set operation definitions, and thinking in terms of arbitrary elements.
  - This method is reliable and a good tool for most proofs. Using this method to prove equivalence of sets can sometimes become unwieldy, since two containment proofs are required.
- Using the laws of set algebra.
  - We will soon introduce the laws of set algebra, which are especially helpful in simplifying expressions.

Example 1

Example. Is $A \cap B$ a subset of $A \cup B$ for all sets A and B?

<u>Working:</u>

Proof.

Example 2

Example. Prove that for any sets A and B, we have $A - B = A \cap B^c$.

Proof.

Alternative proof.

Note that this second style of proof is more efficient, but may not always be applicable depending on the problem.

Example 3

Example. Are the sets $(A \cap B)^c$ and $A^c \cap B^c$ equal for all sets A and B?

Working:

Proof.

Example 4

Example. Are the sets $(A \cap B)^c$ and $A^c \cup B^c$ equal for all sets A and B?

Working:

Example 4 (continued)

Example. Are the sets $(A \cap B)^c$ and $A^c \cup B^c$ equal for all sets A and B?

Proof.

Example 5

Example. Are $A \cap (B \cup C)$ and $(A \cap B) \cup C$ equal for all sets A, B, and C?

Working:

Proof.

Example 6

Example. Are $A \cap (B \cup C)$ and $(A \cap B) \cup (A \cap C)$ equal for all sets A, B, C?

<u>Working:</u>

Proof. Try this yourself! (Use similar methods to Examples 2 and 4.)

Laws of set algebra

For any sets A, B, C with universal set $\mathcal{U}$ and empty set $\varnothing$, we have the following laws of set algebra:

Commutativity:

$$A \cup B = B \cup A,$$
$$A \cap B = B \cap A.$$

Associativity:

$$A \cup (B \cup C) = (A \cup B) \cup C,$$
$$A \cap (B \cap C) = (A \cap B) \cap C.$$

Distributivity:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C),$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

Absorption:

$$A \cup (A \cap B) = A,$$
$$A \cap (A \cup B) = A.$$

Idempotence:

$$A \cup A = A,$$
$$A \cap A = A.$$

Identity:

$$A \cup \varnothing = A,$$
$$A \cap \mathcal{U} = A.$$

Domination:

$$A \cup \mathcal{U} = \mathcal{U},$$
$$A \cap \varnothing = \varnothing.$$

Complement law:

$$A \cup A^c = \mathcal{U},$$
$$A \cap A^c = \varnothing.$$

Double complement law:

$$(A^c)^c = A.$$

De Morgan's law:

$$(A \cup B)^c = A^c \cap B^c,$$
$$(A \cap B)^c = A^c \cup B^c.$$

We also have the following definitions:

Difference:

$$A - B = A \cap B^c.$$

Symmetric difference:

$$A \ominus B = (A \cup B) - (A \cap B).$$

Comments on the laws of set algebra

The laws of set algebra completely describe the behaviour of sets under the basic set operations. It is possible to verify any statements involving set expressions by using only these laws, though doing so can take a lot of work.

While there are many laws to learn here, almost all of them are easily justified by considering them in terms of Venn diagrams.

When simplifying expressions or proving statements using the laws of set algebra, we should always state which laws are being used at each step. If you do not remember the name of a particular law, you may instead describe it in words and/or provide its general definition.

Definition. The dual of a set expression is the expression obtained by replacing every instance of $\cup$ with $\cap$, $\cap$ with $\cup$, $\varnothing$ with $\mathcal{U}$, and $\mathcal{U}$ with $\varnothing$.

Theorem. (Duality principle)
Any statement involving only sets and the union, intersection, and complement operations is true if and only if its dual statement is true.

Proof.

Example – Simplifying set expressions

Example. Simplify the set expression $A \cap (A \cap B^c)^c$.

Solution.

The dual of this result must also be true, so we now also know that:

Example – Proving equivalence of set expressions

Example. Show that $(A - B) \cap (A - C) = A - (B \cup C)$ for all sets A, B, C.

Solution.