MATH1231 CALCULUS
Chapter 1B

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(1.3) **Tangent Planes and Surface Normals**

Tangent Lines and Normals in one-dimension

Suppose (x_0, y_0) is a point on the curve $y = f(x)$.

- The **tangent line** through (x_0, y_0) has equation

$$y = y_0 + f'(x_0)(x - x_0)$$

- A **normal vector** to the curve at (x_0, y_0) is

$$\begin{pmatrix} f'(x_0) \\ -1 \end{pmatrix}.$$

The slope of the normal line in $\frac{-1}{f'(x_0)}$.

The equation of the normal line is given by

$$y - y_0 = \frac{-1}{f'(x_0)}(x - x_0).$$

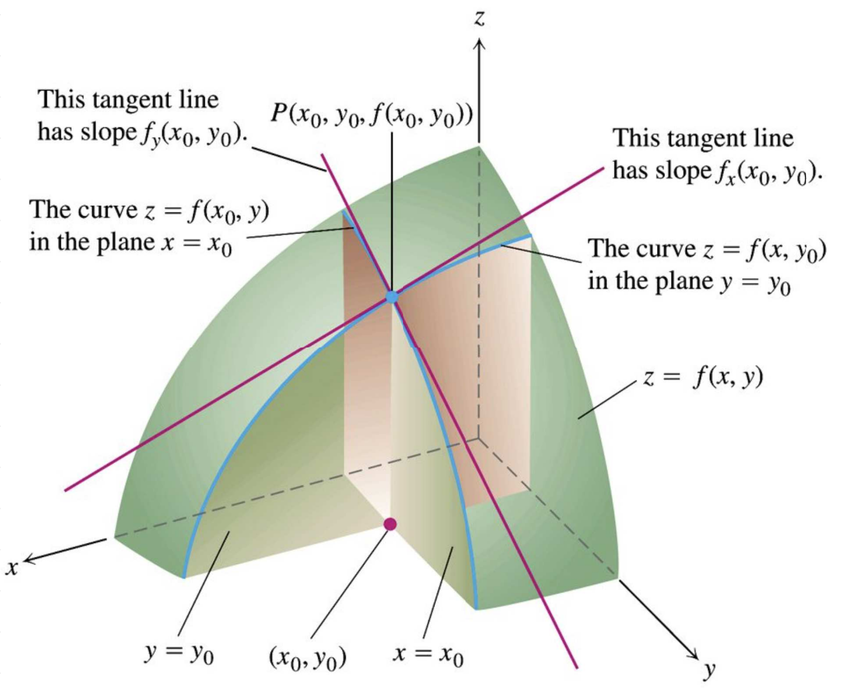
That is, $x - x_0 = f'(x_0)(y_0 - y)$.

If we let $\lambda = y_0 - y$ then the equation of the normal line in parametric vector form is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \lambda \begin{pmatrix} f'(x_0) \\ -1 \end{pmatrix}.$$

So, a normal vector to the curve $y = f(x)$ at (x_0, y_0) is $\begin{pmatrix} f'(x_0) \\ -1 \end{pmatrix}$.

Tangent Planes and Surface Normals



$\vec{u} = (1, 0, f_x(x_0, y_0))$ is tangent to cut through f with $y = y_0$ at $(x_0, y_0, f(x_0, y_0))$

$\vec{v} = (0, 1, f_y(x_0, y_0))$ is tangent to cut through f with $x = x_0$ at $(x_0, y_0, f(x_0, y_0))$

vectors $\vec{u}$ and $\vec{v}$ span the tangent plane touching f at $(x_0, y_0, f(x_0, y_0))$.

Normal Vectors

A **normal vector** to the tangent plane is $\vec{n} = \vec{v} \times \vec{u}$: The cross product of $\vec{v}$ and $\vec{u}$ is defined by

$$\begin{aligned}
\vec{n} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & f_y \\ 1 & 0 & f_x \end{vmatrix} \\
&= \hat{i}\,(f_x - 0) - \hat{j}(0 - f_y) + \hat{k}(0 - 1) \\
&= (f_x, f_y, -1)
\end{aligned}$$

Note that $-\vec{n}$ is also normal to the tangent plane. Now that we have a normal vector, we can easily find the equation of the tangent plane:

Tangent Planes and Surface Normals

Suppose (x_0, y_0, z_0) is a point on the surface $z = F(x, y)$.

- The **tangent plane** through (x_0, y_0, z_0) has equation

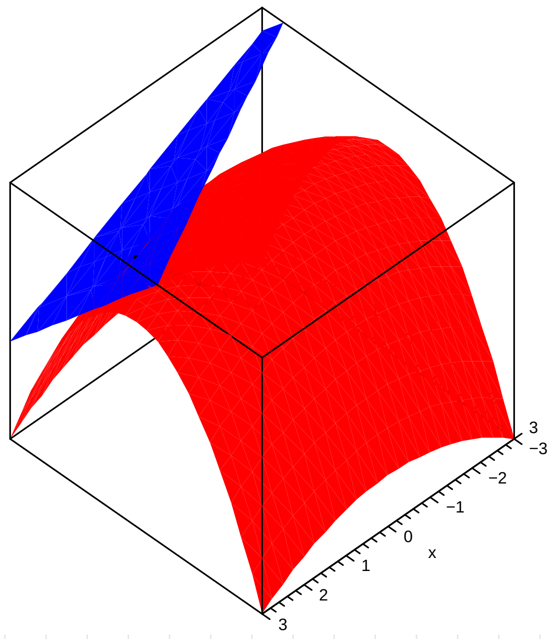
$$z = z_0 + F_x(x_0, y_0)(x - x_0) + F_y(x_0, y_0)(y - y_0)$$

- A **normal vector** to the surface at (x_0, y_0, z_0) is

$$\begin{pmatrix} F_x(x_0, y_0) \\ F_y(x_0, y_0) \\ -1 \end{pmatrix}.$$

Example

Find the tangent plane and the normal vector to the surface $z = F(x,y) = -\frac{x^2}{4} - y^2$ at $(2,-1,-2)$.



(1.4) **Total Differential Approximation**

If $f = f(x)$ then near a given point x_0

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0)$$

$$\Delta f = f(x) - f(x_0)$$

- The differential approximation to the difference is $\boxed{\Delta f \approx f'(x_0)(x - x_0)}$
- Geometrically this approximates points on the curve $y = f(x)$ near x_0 by points on the tangent line near this point.

If $F = F(x, y)$ then near a given point (x_0, y_0)

$$F(x, y) \approx F(x_0, y_0) + F_x(x_0, y_0)(x - x_0) + F_y(x_0, y_0)(y - y_0)$$

$$\Delta F = F(x, y) - F(x_0, y_0)$$

- The total differential approximation to the difference is $\boxed{\Delta F \approx F_x(x_0, y_0)(x - x_0) + F_y(x_0, y_0)(y - y_0)}$
- Geometrically this approximates points on the surface $z = F(x, y)$ near (x_0, y_0) by points on the tangent plane near this point.

Uses of the total differential approximation

- To estimate changes in output given changes in input
- To estimate upper bounds on absolute errors:

$$
\begin{aligned}
|\Delta F| \approx & \left|\frac{\partial F}{\partial x}\Delta x + \frac{\partial F}{\partial y}\Delta y\right| \\
\leq & \left|\frac{\partial F}{\partial x}\right| |\Delta x| + \left|\frac{\partial F}{\partial y}\right| |\Delta y|
\end{aligned}
$$

- Given a function F of x and y, one can interpret ΔF as the error in the output given errors Δx and Δy in the inputs.

Example

A right circular cone is measured to have a height of 10cm and a base radius of 30cm. Use the total differential approximation to estimate the maximum error in the volume if the maximum absolute error in each measurement is .1cm.

Example:

The dimensions of a cylinder are measured to the nearest millimeter using a measuring tape. The radius is measured to be 5cm and the height is measured to be 12cm. What should we expect the maximum percentage error in calculating the volume V to be?

(1.5) **Chain Rules**

If $F = F(x, y)$ and $x = x(t)$, $y = y(t)$ then

$$\boxed{\frac{dF}{dt} = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt}}$$

$$\begin{tikzcd}
 & x \arrow[dr, "\frac{dx}{dt}"] & \\
F \arrow[ur, "\frac{\partial F}{\partial x}"] \arrow[dr, "\frac{\partial F}{\partial y}"'] & & t \\
 & y \arrow[ur, "\frac{dy}{dt}"'] &
\end{tikzcd}$$

To compute $\frac{dF}{dt}$ sum the paths (left to right) from F to t and multiply derivatives on a path.

Justification

A small change Δt in t produces small changes Δx in x and Δy in y which produces a small change ΔF in F – use differential approximations.

$$\begin{aligned}
\Delta F &\approx \frac{\partial F}{\partial x}\Delta x + \frac{\partial F}{\partial y}\Delta y \\
\frac{\Delta F}{\Delta t} &\approx \frac{\partial F}{\partial x}\frac{\Delta x}{\Delta t} + \frac{\partial F}{\partial y}\frac{\Delta y}{\Delta t}
\end{aligned}$$

$$\begin{aligned}
\frac{\Delta F}{\Delta t} &= \frac{F(t+\Delta t) - F(t)}{\Delta t} \to \frac{dF}{dt} \\
\frac{\Delta x}{\Delta t} &= \frac{x(t+\Delta t) - x(t)}{\Delta t} \to \frac{dx}{dt} \\
\frac{\Delta y}{\Delta t} &= \frac{y(t+\Delta t) - y(t)}{\Delta t} \to \frac{dy}{dt}
\end{aligned}$$

Note: Never treat a partial derivative as a quotient.

Example: If $z = F(x, y) = x^2y + 3xy^4$ and $x = e^t$, $y = sint$ then use the chain rule to find $\frac{dz}{dt}$.

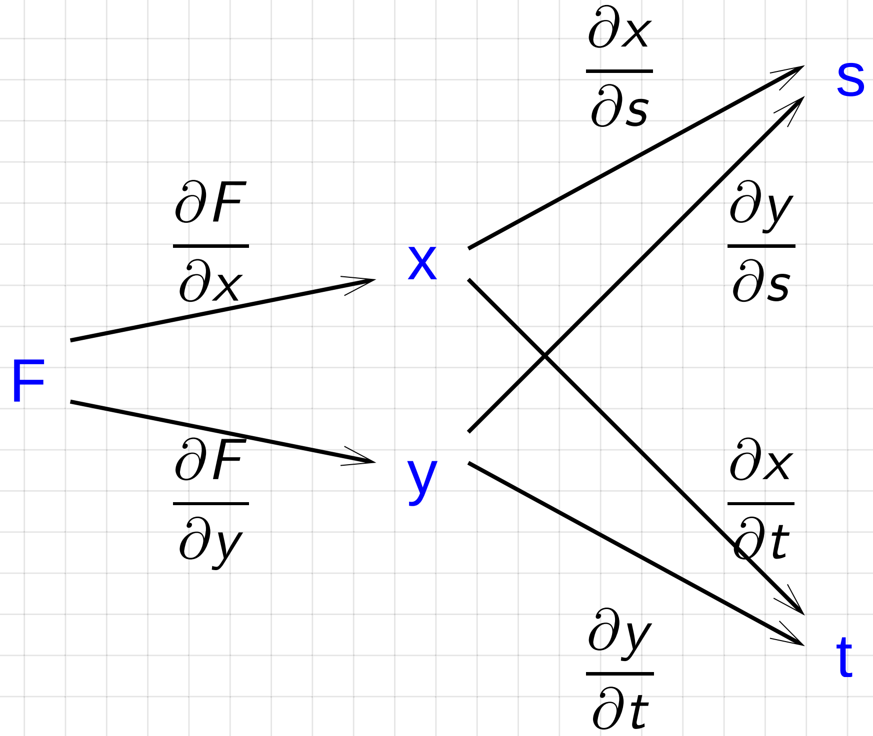
Example

A particle moves ona path, defined by the parametric equations $x = t$, $y = t^3 + 2$. The temperature at a point (x, y) on the path is given by $T = x^2 + y^2$. Find the rate of change of temperature T on the path.

Chain Rule

If $F = F(x, y)$ and $x = x(s, t), y = y(s, t)$ then

$$\boxed{\frac{\partial F}{\partial t} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial t}} \qquad \boxed{\frac{\partial F}{\partial s} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial s}}$$



To compute $\frac{\partial F}{\partial s}$ sum the paths (left to right) from F to s and multiply partial derivatives on a path.

Example: If $z = e^x \sin(y)$ and $x = st^2$, $y = s^2 t$ then use the chain rule to find $\frac{\partial z}{\partial s}$.

Chain Rule

If $F = F(u)$ and $u = u(x, y)$ then

$$\boxed{\frac{\partial F}{\partial x} = \frac{dF}{du}\frac{\partial u}{\partial x}} \qquad \boxed{\frac{\partial F}{\partial y} = \frac{dF}{du}\frac{\partial u}{\partial y}}$$

$$\begin{tikzcd}
 & & x \\
F \arrow[r, "\frac{dF}{du}"] & u \arrow[ur, "\frac{\partial u}{\partial x}"] \arrow[dr, "\frac{\partial u}{\partial y}"'] & \\
 & & y
\end{tikzcd}$$

To compute $\frac{\partial F}{\partial x}$ sum the paths (left to right) from F to x and multiply derivatives on a path.

Example: If $w = tan^{-1}(y/x)$ then calculate $\frac{\partial w}{\partial x}$ via the chain rule.

(1.6) **Functions of Three (or more) Variables**

If $F = F(x, y, z)$ then

$$\begin{aligned}
D_1 F \equiv F_x(x, y, z) \equiv \frac{\partial F}{\partial x} &= \lim_{h \to 0} \frac{F(x + h, y, z) - F(x, y, z)}{h} \\
D_2 F \equiv F_y(x, y, z) \equiv \frac{\partial F}{\partial y} &= \lim_{h \to 0} \frac{F(x, y + h, z) - F(x, y, z)}{h} \\
D_3 F \equiv F_z(x, y, z) \equiv \frac{\partial F}{\partial z} &= \lim_{h \to 0} \frac{F(x, y, z + h) - F(x, y, z)}{h}
\end{aligned}$$

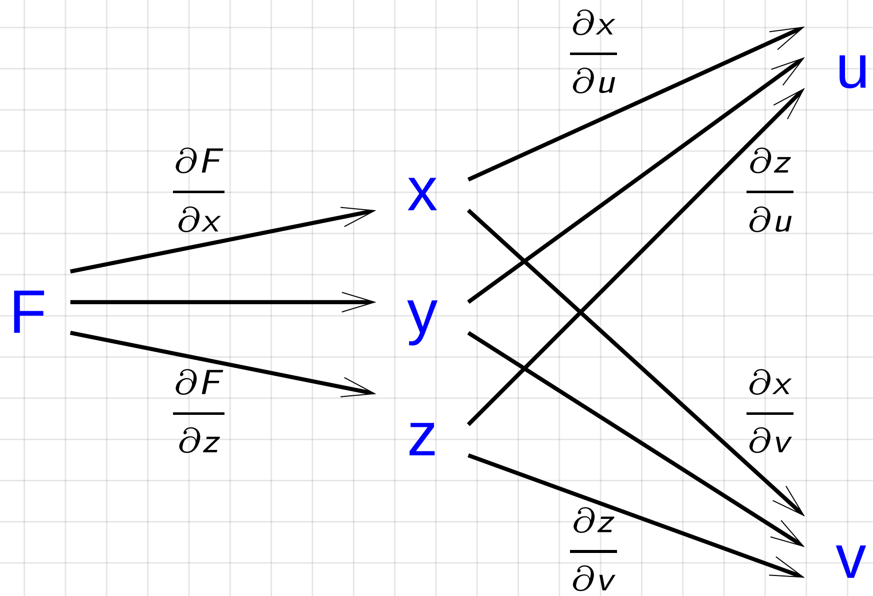
Differentiate w.r.t. one of the variables by holding the other variables fixed.

Example: If $w = xy + yz + zx$ then calculate $\frac{\partial w}{\partial y}$.

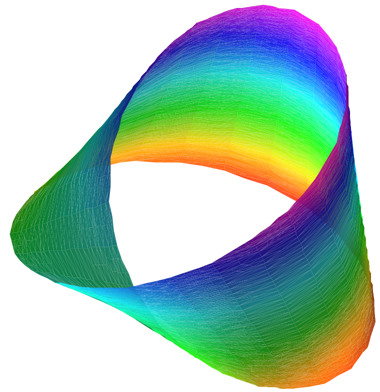
Chain Rule

If $F = F(x, y, z)$ and $x = x(u, v), y = y(u, v), z = z(u, v)$ then

$$\boxed{\frac{\partial F}{\partial u} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial u}}$$

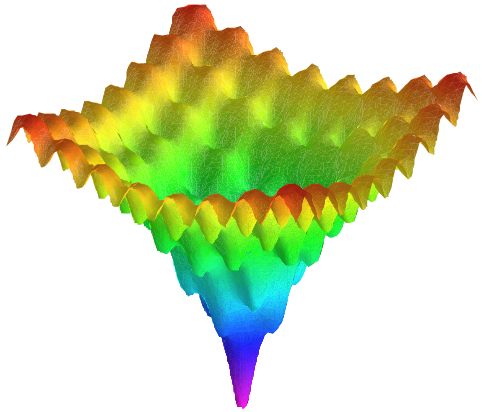


(1.7) **MAPLE NOTES**



The MAPLE `plot3d` command is useful for visualization.

MAPLE also has a useful package for several variable calculus. Click on `Tools` in the menu bar then drag down to `Tutors` and across to `Calculus - Multi-Variable` to plot level curves and surfaces in 3D.



The MAPLE `diff` command carries out partial differentiation. For example `diff(f(x,y),x)` computes $\frac{\partial f}{\partial x}$