MATH1081 – Discrete Mathematics

Topic 1 – Set theory and functions
Lecture 1.01 – Set notation

Lecturer: Dr Sean Gardiner – sean.gardiner@unsw.edu.au

Introduction to set theory and functions

The mathematical study of set theory began in 1874, founded by Georg Cantor. Mathematical sets underlie almost all branches of mathematics, and set theory provides an important framework for describing and understanding formal logic.

In MATH1081, we will encounter sets throughout all 5 topics of the course. In this first topic, we will also use sets to motivate the investigation of functions, another fundamental mathematical concept that appears in almost all branches of mathematics.

Much of our study of Topic 1 will be focused on definitions and results about sets and functions. Probably the most difficult part of this topic is understanding and applying the different methods of proof regarding properties of sets and functions. We will revisit these ideas in a more generalised sense in Topic 3 (Proofs and logic).

Set notation

Definition. A set is a well-defined, unordered collection of distinct objects. The objects contained in a set are called its elements.

Notation. A set can be represented by writing its elements surrounded by braces (curly brackets). For example, the set S with just the elements 1, 2, and 3 can be written as $S = \{1, 2, 3\}$.

Since sets are unordered collections, this set S can also be written as $S = \{3, 1, 2\} = \{1, 2, 3\}$.

Since the elements of a set are distinct, repetition of elements is ignored, so the set S can also be written as $S = \{1, 2, 2, 3, 3, 3\} = \{1, 2, 3\}$.

Example. Simplify each of the following sets.

- $\{x, a, y, x, a, y\} = \{a, x, y\}$.
- $\{\text{even numbers between 1 and 9}\} = \{2, 4, 6, 8\}$.
- $\{\text{letters in } \textbf{BANANA}\} = \{\textbf{A}, \textbf{B}, \textbf{N}\}$.

Notice that the elements of a set can be numbers, letters, or any other object, and can be given explicitly or descriptively.

Defining sets

Notation. The set membership symbol $\in$ is used to indicate that an object is an element of a set. We write $x \in S$ to mean "x is an element of S". We can also use the symbol $\notin$ to indicate non-membership of a set.

For example, writing $S = \{1, 2, 3\}$, we have $1 \in S$ but $0 \notin S$. Similarly, we have $a \in \{a, x, y\}$ while $b \notin \{a, x, y\}$.

We can properly define a set by writing out all its elements, or by giving a careful description of its elements. So long as there is no ambiguity, we can also use the ellipsis symbol ($\dots$) to help describe a set. For example:

- $\{\text{letters in the English alphabet}\} = \{\mathbf{A}, \mathbf{B}, \mathbf{C}, \dots, \mathbf{Z}\}$.
- $\{\text{positive even numbers}\} = \{2, 4, 6, 8, \dots\}$.

Notation. A colon ($:$) or vertical bar ($|$) symbol can be used to introduce additional properties that define the elements of a set. We write

$$\{x \in S : (\text{some property of } x)\} \quad \text{or} \quad \{x \in S \mid (\text{some property of } x)\}$$

to mean "the set of elements in S that satisfy the property", or literally, "all x in S such that x satisfies the property".

For example, writing $S = \{1, 2, 3\}$, we have $\{x \in S : x > 1\} = \{2, 3\}$.

Numeric sets

Notation. The following are some important and commonly-used sets of numbers.

- $\mathbb{N} = \{0, 1, 2, 3, \dots\}$, the set of natural numbers. Note here that $0 \in \mathbb{N}$.
- $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$, the set of integers.
- $\mathbb{Z}^+ = \{x \in \mathbb{Z} \ : \ x > 0\} = \{1, 2, 3, \dots\}$, the set of positive integers.
- $\mathbb{Q} = \left\{ \frac{p}{q} \ : \ p, q \in \mathbb{Z},\ q \neq 0 \right\} = \left\{ \frac{p}{q} \ : \ p \in \mathbb{Z},\ q \in \mathbb{Z}^+ \right\}$, the set of rational numbers. The rational numbers include all integers and fractions.
- $\mathbb{R} = \{\text{all points on the real number line}\}$, the set of real numbers. The real numbers include all rational numbers as well as all irrational numbers like $\sqrt{2}$, π, and e.
- $\mathbb{C} = \{a + bi \ : \ a, b \in \mathbb{R},\ i^2 = -1\}$, the set of complex numbers. The complex numbers include all real numbers as well as imaginary numbers (real multiples of the imaginary unit i) and their sums.

Much of this course is focused on the sets $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Z}^+$. In fact, the study of number theory (seen in Topic 2) is exclusively concerned with integers.

The empty set and cardinality

Example. Write out the following sets explicitly.

- $A = \{x \in \mathbb{N} \ \mid \ x < 5\} = \{0, 1, 2, 3, 4\}$.
- $B = \{n \in \mathbb{Z} \ \mid \ n^2 = 4\} = \{-2, 2\}$.
- $C = \{x \in \mathbb{Z}^+ \ : \ \frac{x}{2} \in \mathbb{Z}\} = \{2, 4, 6, 8, \dots\}$.
- $D = \{2k \ : \ k \in \mathbb{Z}^+\} = \{2, 4, 6, 8, \dots\} = C$.
- $E = \{x \in D \ : \ x \notin \mathbb{R}\} = \{\}$.

Definition. The set with no elements is called the empty set, which is written as $\{\}$ or $\varnothing$.

Definition. The cardinality or size of a set is the number of distinct elements it contains. We write $|S|$ to mean "the cardinality of the set S". Note that if S is finite, then $|S| \in \mathbb{N}$.

Example. Find the cardinality of each of the sets from the previous example.

- $|A| = 5$.
- $|B| = 2$.
- $|C| = |D| = \infty$.
- $|E| = 0$.

We have seen that sets can contain elements of any type. In particular, sets themselves can be contained in other sets. For example, if we think of tutorials as sets of students, then the set of all MATH1081 tutorials is a set containing sets.

For another example, consider the set $S = \{1, 2, \{3, 4\}\}$. It contains the elements 1, 2, and $\{3, 4\}$. This means we can write that $\{3, 4\} \in S$. This also tells us that the cardinality of S is given by $|S| = 3$. Notice that the elements of $\{3, 4\}$ are not related to the elements of S nor the cardinality of S. So in particular, in this case $3 \notin S$ and $4 \notin S$.

Example. Find the cardinality of each of the following sets.

- $|\,\{x, \{x\}\}\,| = 2.$
- $|\,\{x, y, \{x\}, \{y\}, \{x, y\}, \mathbb{N}\}\,| = 6.$
- $|\,\{x, \{x\}, \{x, x\}\}\,| = |\,\{x, \{x\}, \{x\}\}\,| = |\,\{x, \{x\}\}\,| = 2.$
- $|\,\{\{\}\}\,| = 1.$
- $|\,\{\{\{\}\}\}\,| = 1.$

Case study: Russell's paradox

Occasionally we will encounter "case studies", which are included to give more context to the topics we are studying.

These case studies are considered additional content and are not examinable.

In 1901, logician Bertrand Russell posed a problem that is now known as Russell's paradox (also attributed to Ernst Zermelo):

Problem. (Russell's paradox)
Let $S = \{\text{sets which are not elements of themselves}\}$. Is S an element of S?

- By the definition, any element $X \in S$ must satisfy $X \notin X$. Replacing X with S, we see that if $S \in S$, then $S \notin S$, which is a contradiction.
- Similarly, by the definition, any set such that $X \notin X$ must satisfy $X \in S$. Replacing X with S, we see that if $S \notin S$, then $S \in S$, which is again a contradiction.

So neither "$S \in S$" nor "$S \notin S$" can be true statements!

We mentioned earlier that a set must be well-defined. Russell's paradox provides an example of a set that is not well-defined, in this case due to its definition being self-referential.