MATH1081 – Discrete Mathematics

Topic 2 – Number theory and relations
Lecture 2.05 – Relations and equivalences

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Relations

Definition. Given sets X and Y, a relation from X to Y is a subset of $X \times Y$.

Notation. A relation R from a set X to a set Y is declared as $R \subseteq X \times Y$. If $(x, y) \in R$, we can say "x is related to y". Instead of writing $(x, y) \in R$, we can also write $x \mathrel{R} y$. If $(x, y) \notin R$, we can write $x \not{R}\, y$.

A function is a relation with the additional condition that each element of x has exactly one corresponding y value. That is, $R \subseteq X \times Y$ is a function if and only if for every $x \in X$, there is exactly one $y \in Y$ such that $x \mathrel{R} y$.

Notation. Relations can be represented by capital letters like R, but also by symbols like $\sim$ or $\preceq$. In fact, we have already encountered many relations including $=$, $<$, $\leq$, $\in$, $\subseteq$, and $\mid$.

For example, the divisibility relation $R \subseteq \mathbb{Z} \times \mathbb{Z}$ is defined by the statement

$$a \mathrel{R} b \text{ if and only if } b = ak \text{ for some } k \in \mathbb{Z}.$$

So we have $R = \{(0, 0), (1, 0), (1, 1), (1, 2), \dots, (2, 0), (2, 2), (2, 4), \dots\}$.

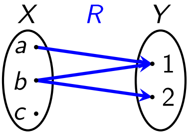
Equivalently, we could just define the divisibility relation $\mid$ on $\mathbb{Z}$ by

$$a \mid b \text{ if and only if } b = ak \text{ for some } k \in \mathbb{Z}.$$

Representations of relations

Just like for functions, we can represent relations using arrow diagrams.

For example, in the case with sets $X = \{a, b, c\}$ and $Y = \{1, 2\}$, and relation $R \subseteq X \times Y$ given by $R = \{(a, 1), (b, 1), (b, 2)\}$, the relation R can be represented with an arrow diagram as follows:



Note we no longer require each element of X has exactly one outgoing arrow.

We can also represent relations using a relation matrix. For a relation from X to Y, we can construct a matrix whose rows are indexed by elements of X and whose columns are indexed by elements of Y such that each matrix entry is either 1 if its corresponding row element is related to its corresponding column element, or 0 otherwise. Note that the appearance of a relation matrix is dependent on the chosen order of rows and columns.

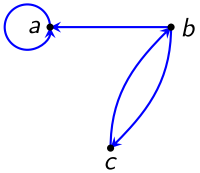
Using the above example, a relation matrix for R is $\begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 0 \end{pmatrix}$.

Arrow diagrams as directed graphs

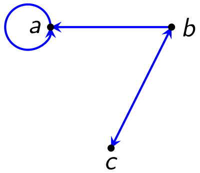
Most of the time, we are interested in relations from a set to itself. We refer to a relation $R \subseteq X \times X$ for some set X as a relation on X.

To represent a relation R on X, we can write out the elements of X just once and represent the relation as arrows pointing between these elements. Recall this is known as a directed graph, which will appear again in Topic 5.

For example, in the case with set $X = \{a, b, c\}$ and relation R on X given by $R = \{(a, a), (b, a), (b, c), (c, b)\}$, the relation R can be represented with a directed graph arrow diagram as follows:



or



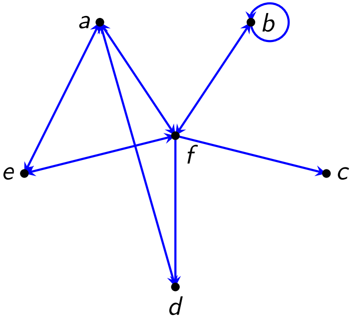
Notice that if two elements are related to each other, we can draw two separate arrows representing these two relations, or just one arrow pointing in both directions.

Example – Representing relations

Example. Consider the set of people $S = \{a, b, c, d, e, f\}$ representing students Alyx, Barney, Chell, Dog, Eli, and Freeman. The relation R on the set S is defined by $x \, R \, y$ if and only if x considers y to be their friend. Alyx's friends are Dog, Eli, and Freeman. Barney's friends are Freeman and himself. Dog's friend is Alyx. Eli's friends are Alyx and Freeman. Freeman is friends with everyone else. Represent this relation by a relation matrix and an arrow diagram.

Solution. A relation matrix and arrow diagram for R are provided below.

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$



Reflexivity

Definition. A relation R on a set X is reflexive if and only if for all $x \in X$, we have $x \, R \, x$.

In an arrow diagram, a loop is an arrow pointing from one element to itself. In terms of arrow diagrams, a relation R is reflexive if and only if every element has a loop.

Diagrammatically, this property can be represented as:

$$\bullet \qquad \text{implies} \qquad \bullet$$

Example. Which of the following relations on $X = \{1, 2, 3, 4\}$ are reflexive?

- $R = \{(1, 1), (1, 3), (2, 2), (3, 1), (3, 3), (4, 4)\}$ is reflexive since for every element $x \in X$, we have $x \, R \, x$.
- $R = \{(1, 1), (1, 3), (2, 4), (3, 1), (3, 3), (4, 2), (4, 4)\}$ is not reflexive since for the element $2 \in X$, we do not have $2 \, R \, 2$.
- $R = \{(1, 1), (2, 2), (3, 3)\}$ is not reflexive since for the element $4 \in X$, we do not have $4 \, R \, 4$.

Symmetry

Definition. A relation R on a set X is symmetric if and only if for all $x, y \in X$, whenever $x \, R \, y$ we have $y \, R \, x$.

In terms of arrow diagrams, a relation R is symmetric if and only if every non-loop arrow points in both directions.

Diagrammatically, this property can be represented as:

$$\bullet \longrightarrow \bullet \quad \text{implies} \quad \bullet \longleftrightarrow \bullet$$

Example. Which of the following relations on $X = \{1, 2, 3, 4\}$ are symmetric?

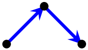
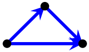
- $R = \{(1,1), (1,3), (2,2), (3,1), (3,3), (4,4)\}$ is symmetric since for every pair of elements $x, y \in X$, whenever $x \, R \, y$, we have that $y \, R \, x$.
- $R = \{(1,1), (1,3), (2,4), (3,1), (3,3), (4,4)\}$ is not symmetric since for the elements $2, 4 \in X$, we have $2 \, R \, 4$ but $4 \not R \, 2$.
- $R = \{(1,1), (2,2), (3,3)\}$ is symmetric since for every pair of elements $x, y \in X$, whenever $x \, R \, y$, we have that $y \, R \, x$.
- $R = \{\}$ is symmetric since the symmetry condition is satisfied vacuously.

Transitivity

Definition. A relation R on a set X is transitive if and only if for all $x, y, z \in X$, whenever $x \, R \, y$ and $y \, R \, z$, we have $x \, R \, z$.

In terms of arrow diagrams, a relation R is transitive if and only if every pair of points connected by a path of arrows also has a single arrow from the first point to the last.

Diagrammatically, this property can be represented as:

 implies  , and  implies 

Example. Which of the following relations on $X = \{1, 2, 3, 4\}$ are transitive?

- $R = \{(1,1), (1,2), (1,3), (2,3), (3,3), (4,4)\}$ is transitive since for all elements $x, y, z \in X$, whenever $x \, R \, y$ and $y \, R \, z$, we have that $x \, R \, z$.
- $R = \{(1,2), (2,3), (3,4), (4,1)\}$ is not transitive since for the elements $1, 2, 3 \in X$, we have $1 \, R \, 2$ and $2 \, R \, 3$ but $1 \not{R} \, 3$.
- $R = \{(1,1), (1,3), (2,2), (3,1), (4,4)\}$ is not transitive since for the elements $1, 3 \in X$, we have $3 \, R \, 1$ and $1 \, R \, 3$ but $3 \not{R} \, 3$.
- $R = \{\}$ is transitive since the transitivity condition is satisfied vacuously.

Equivalence relations

Definition. A relation R on a set X is an equivalence relation if and only if it is reflexive, symmetric, and transitive.

If two mathematical objects are related by an equivalence relation, we can interpret this as meaning they are the same in some particular sense.

For example, a very familiar equivalence relation is $=$ itself. We can easily check that $=$ on any appropriate set is reflexive, symmetric, and transitive.

Example. Show that the relation $\sim$ on $\mathbb{Z}$ defined by setting $x \sim y$ if and only if $x \equiv y \pmod 3$ is an equivalence relation.

Solution. We proceed by proving $\sim$ is reflexive, symmetric, and transitive.

Reflexivity: Let $x \in \mathbb{Z}$. Since $x = x + 3 \times 0$ and $0 \in \mathbb{Z}$, we can write that $x \equiv x \pmod 3$. So $x \sim x$, and thus $\sim$ is reflexive.

Symmetry: Let $x, y \in \mathbb{Z}$, and suppose $x \sim y$. Then $x = y + 3k$ for some integer k. So $y = x + 3(-k)$ where $-k \in \mathbb{Z}$, which means $y \equiv x \pmod 3$. So $y \sim x$, and thus $\sim$ is symmetric.

Transitivity: Let $x, y, z \in \mathbb{Z}$, and suppose $x \sim y$ and $y \sim z$. Then $x = y + 3k$ and $y = z + 3l$ for some integers k and l. So $x = z + 3(l + k)$ where $l + k \in \mathbb{Z}$, which means $x \equiv z \pmod 3$. So $x \sim z$, and thus $\sim$ is transitive.

Equivalence classes

Definition. Given an equivalence relation $\sim$ on a set X, and some element $a \in X$, the equivalence class of a with respect to $\sim$, written as $[a]$, is the set of all elements in X that are related to a. That is,

$$[a] = \{x \in X \ : \ x \sim a\}.$$

For example, if $\sim$ is the equivalence relation on $\mathbb{Z}$ defined by setting $x \sim y$ if and only if $x \equiv y \pmod 2$, then the equivalence class of 0 with respect to $\sim$ is the set of all integers that have remainder 0 when divided by 2, that is, $[0] = \{\text{even numbers}\}$. Similarly, we have $[1] = \{\text{odd numbers}\}$. Notice that we could also write $[2k] = [0]$ and $[2k+1] = [1]$ for any integer k.

Indeed, whenever we write $a \equiv b \pmod m$, we are really saying $[a] = [b]$ where the equivalence classes are with respect to "congruence modulo m".

Example. For each of the following equivalence relations $\sim$ on $\mathbb{Z}$, find the equivalence class of 6.

- Given $x \sim y$ if and only if $x = y$, we have $[6] = \{6\}$.
- Given $x \sim y$ if and only if $x \equiv y \pmod 3$, we have $[6] = \{3k \ : \ k \in \mathbb{Z}\}$.
- Given $x \sim y$ if and only if x has the same number of letters as y when spelled out in English, we have $[6] = \{1, 2, 6, 10\}$.

Properties of equivalence classes

Lemma. Given an equivalence relation $\sim$ on a set X, for all $x \in X$ we have that $x \in [x]$.

Proof. Since $\sim$ is reflexive, we know $x \sim x$ for all $x \in X$. Thus $x \in [x]$.

Lemma. Given an equivalence relation $\sim$ on a set X, for all $x, y \in X$, whenever $x \sim y$ we have $[x] = [y]$.

Proof. For all $a \in [x]$, we have $a \sim x$, and since $x \sim y$, by the transitive property of $\sim$ we have $a \sim y$. So $a \in [y]$, implying $[x] \subseteq [y]$. Next, for all $a \in [y]$, we have $a \sim y$, and since $x \sim y$, by the symmetric property of $\sim$ we have $y \sim x$. So by the transitive property of $\sim$, we have $a \sim x$. So $a \in [x]$, implying $[y] \subseteq [x]$. Thus $[x] = [y]$.

Theorem. Given an equivalence relation $\sim$ on a set X, the equivalence classes with respect to $\sim$ partition X.

Proof. Clearly the union of all the equivalences classes equals X, since every element $x \in X$ is an element of its own equivalence class $[x]$ (by the first lemma). It remains to show that the different equivalence classes are pairwise disjoint. Suppose we have some $x, y \in X$ such that $[x] \cap [y] \neq \varnothing$. Then there is some $a \in X$ such that $a \in [x]$ and $a \in [y]$. So $a \sim x$ and $a \sim y$, meaning $[a] = [x]$ and $[a] = [y]$ by the second lemma, and so $[x] = [y]$. Thus any pair of non-equal equivalence classes must be disjoint.