MATH1081 – Discrete Mathematics

Topic 1 – Set theory and functions
Lecture 1.05 – Cartesian product and functions

Lecturer: Dr Sean Gardiner – sean.gardiner@unsw.edu.au

Cartesian product

Definition. A tuple is a finite, ordered collection of objects. Unlike for a set, the order of the elements in a tuple is important, and elements may be repeated. A tuple with exactly n elements is sometimes called an n-tuple. A tuple with exactly 2 elements is also called an ordered pair.

Notation. A tuple can be represented by writing its elements surrounded by parentheses. For example, $(1, 2, 1)$ is a 3-tuple, and it is different to $(1, 1, 2)$.

Definition. The Cartesian product of two sets A and B, denoted $A \times B$, is the set containing all ordered pairs (2-tuples) for which the first element is an element of A, and the second element is an element of B. That is,

$$A \times B = \{\, (a, b) \;:\; a \in A \;\text{ and }\; b \in B \,\}.$$

Notation. We sometimes refer to the Cartesian product $A \times A$ as A^2. For example, the real coordinate plane is often referred to as $\mathbb{R}^2$.

Example. Suppose $A = \{1, 2\}$ and $B = \{x, y, z\}$. Evaluate the following.

- $A \times B = \{\, (1, x), (1, y), (1, z), (2, x), (2, y), (2, z) \,\}$.
- $A^2 = \{\, (1, 1), (1, 2), (2, 1), (2, 2) \,\}$.

Fact. For any sets A and B, we have $|A \times B| = |A| \times |B|$.

Functions

We typically think of a function as a rule that converts input values to output values. To properly define what this means, we will use the language of set theory.

Definition. Given sets X and Y, a function from X to Y is a subset of $X \times Y$ which contains exactly one ordered pair (x, y) for each $x \in X$.

Notation. A function f from a set X to a set Y is declared as $f : X \to Y$. If $(x, y) \in f$, we can say "f maps x to y". Instead of writing $(x, y) \in f$, we can also write $f : x \mapsto y$, or (more commonly) $f(x) = y$. We sometimes refer to x as an input value of f, and y as the output value of x under f.

For example, if $X = \{a, b, c\}$ and $Y = \{1, 2\}$, then a valid function $f : X \to Y$ is given by $f = \{(a, 1), (b, 2), (c, 2)\}$. This means we can write that $f(a) = 1$, $f(b) = 2$, and $f(c) = 2$.

We can also define a function using a formula. For example, consider the function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = x^2$ for all $x \in \mathbb{R}$. This function when interpreted as a subset of $\mathbb{R}^2$ has infinitely many elements, including $(1, 1)$, $(2, 4)$, $(-2, 4)$, and $(\sqrt{2}, 2)$.

Example – Identifying functions

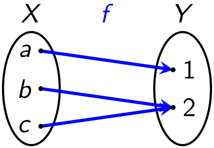
Example. Suppose $X = \{1, 2, 3\}$ and $Y = \{a, b, c, d\}$. Which of the following represent functions from X to Y?

- $\{(1, a), (2, b), (3, a)\}$ is a function, since there is exactly one output value for each possible input value. The fact that c and d are never returned as output values has no effect on whether f is a function.
- $\{(1, a), (2, b), (3, c), (d, 1)\}$ is not a function, since it includes the element $(d, 1)$, but $d \notin X$ is not a valid possible input and $1 \notin Y$ is not a valid possible output.
- $\{(1, a), (2, b)\}$ is not a function, since it does not define an output for the input value $3 \in X$; that is, it does not include the element $(3, y)$ for any $y \in Y$.
- $\{(1, a), (2, b), (3, c), (3, d)\}$ is not a function, since it defines more than one output for the input value $3 \in X$; that is, it includes the elements $(3, c)$ and $(3, d)$ where the first element is the same but the second element is different.

Arrow diagrams

It can sometimes be useful to represent a function $f : X \to Y$ visually. One way this can be done is by using arrow diagrams. Just like for Venn diagrams, the sets X and Y are represented as separate closed figures, and their elements are represented as labelled points. The function is then represented by a series of arrows, each pointing from an input value in X to its corresponding output value in Y.

For example, in the case with sets $X = \{a, b, c\}$ and $Y = \{1, 2\}$, and function $f : X \to Y$ given by $f = \{(a, 1), (b, 2), (c, 2)\}$, the function f can be represented with an arrow diagram as follows:

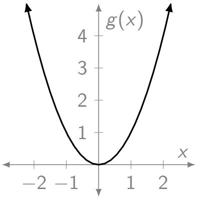


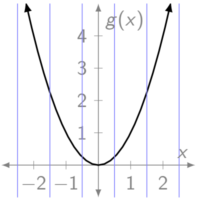
In the special case that $X = Y$ and we have $f : X \to X$, we can draw just one set of elements and represent the function as arrows pointing between elements in X. This is known as a directed graph, and we will investigate such structures further in Topic 5.

Coordinate graphs

Another way to represent functions visually is by using a coordinate graph. In a coordinate graph for a function $f : X \to Y$, the elements of X are listed along a horizontal axis, and the elements of Y are listed along a vertical axis. Elements (x, y) of the function are then marked as points at the coordinates corresponding with axis values x and y. This method of representation is particularly useful for cases where X and Y are each $\mathbb{R}$ or a subset of $\mathbb{R}$.

For example, for the function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = x^2$, the function g can be represented with a coordinate graph as follows:





When given a coordinate graph, we can determine whether it represents a function by using the vertical line test: the graph represents a function if and only if every possible vertical line touches the graph at exactly one point.

Domain, codomain, and range

Definition. For a function $f : X \to Y$, the set of all input values X is called the domain of f, and the set of potential output values Y is called the codomain of f.

Definition. For a function $f : X \to Y$, the set of all output values actually obtained when evaluating all input values is called the range or image of f. The range of f can be denoted as $f(X)$ or $\mathrm{range}(f)$ or $\mathrm{im}(f)$. So the range of f is given by

$$f(X) = \{f(x) \ : \ x \in X\} \subseteq Y.$$

For example, again consider the sets $X = \{a, b, c\}$ and $Y = \{1, 2\}$, with function $f : X \to Y$ given by $f = \{(a, 1), (b, 2), (c, 2)\}$. Clearly the domain of f is X and the codomain of f is Y. Furthermore, the range of f is $\{1, 2\}$, which happens to be the same as the codomain in this case.

Consider also the function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = x^2$ for all $x \in \mathbb{R}$. Clearly the domain and codomain of g are both $\mathbb{R}$. The range of g is the set of all non-negative real numbers, so we have

$$\mathrm{im}(g) = g(\mathbb{R}) = \{y \in \mathbb{R} \ : \ y \geq 0\}.$$

Image and pre-image

Definition. Suppose $f : X \to Y$ is a function and $A \subseteq X$. The image of A under f, written as $f(A)$, is the set of all output values attained by mapping all the input values in A under f. That is,

$$f(A) = \{f(x) \ : \ x \in A\} \subseteq Y.$$

Note that the image of the domain X under f is just the range (or image) of f, which justifies using the notation $f(X)$.

Definition. Suppose $f : X \to Y$ is a function and $B \subseteq Y$. The pre-image of B under f, written as $f^{-1}(B)$, is the set of all input values that map to the output values in B under f. That is,

$$f^{-1}(B) = \{x \in X \ : \ f(x) \in B\} \subseteq X.$$

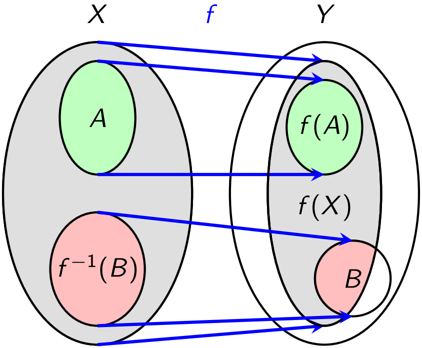
Consider again the example with sets $X = \{a, b, c\}$ and $Y = \{1, 2\}$, and function $f : X \to Y$ given by $f = \{(a, 1), (b, 2), (c, 2)\}$.
The image of $\{a, b\}$ under f is $f(\{a, b\}) = \{1, 2\}$, while $f(\{b, c\}) = \{2\}$.
The pre-image of $\{1\}$ under f is $f^{-1}(\{1\}) = \{a\}$, while $f^{-1}(\{2\}) = \{b, c\}$.

Consider also the function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = x^2$ for all $x \in \mathbb{R}$.
Then $g(\{-1, 0, 1\}) = \{0, 1\}$, while $g^{-1}(\{4, 9\}) = \{-3, -2, 2, 3\}$.

Venn and arrow diagrams

We can visualise images and pre-images for functions by adapting characteristics of Venn diagrams to our arrow diagrams.

For example, for a function $f : X \to Y$ with a subset of the domain $A \subseteq X$ and a subset of the codomain $B \subseteq Y$, we can draw the following:



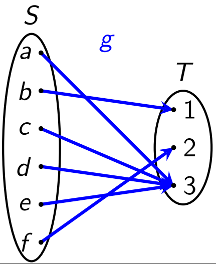
Example 1

Example. Suppose the students Altair, Bayek, Connor, Desmond, Ezio, and Frye are studying MATH1081 this term. Bayek is in tutorial 1, Frye is in tutorial 2, and the other four students are in tutorial 3.

(i) Define a function g that maps the students to their tutorials.
(ii) Draw the arrow diagram representing g.
(iii) Find the range of g.
(iv) Find the pre-image of each of the tutorials under g.

Solution. Labelling the students by their initials, we can define the set of students $S = \{a, b, c, d, e, f\}$ and the set of tutorials $T = \{1, 2, 3\}$. Then the function $g : S \to T$ that maps students to tutorials is given by the set $g = \{\,(a, 3), (b, 1), (c, 3), (d, 3), (e, 3), (f, 2)\,\}$.

The arrow diagram for g is provided to the right.



We can see from the diagram that the range of g is $g(S) = \{1, 2, 3\} = T$.

We can also see that the pre-images are given by

- $g^{-1}(\{1\}) = \{b\}$,
- $g^{-1}(\{2\}) = \{f\}$, and
- $g^{-1}(\{3\}) = \{a, c, d, e\}$.

Example 2

Example. Let the set $S = \{a, b, c, d, e, f\}$ represent the set of students Altair, Bayek, Connor, Desmond, Ezio, and Frye respectively. The function $h : S \to S$ mapping each student to their best friend is given by $h = \{(a, e), (b, f), (c, e), (d, e), (e, e), (f, b)\}$.

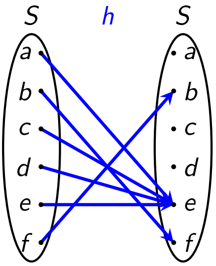
(i) Interpret this function h.
(ii) Draw the arrow diagram representing h.
(iii) Find the range of h.
(iv) Find $h(\{a, c\})$ and $h^{-1}(\{a, c\})$.

Solution. Ezio is the best friend of Altair, Connor, Desmond, and himself. Bayek and Frye are each other's best friends.

The arrow diagram for h is provided to the right.

We can see from the diagram that the range of h is $h(S) = \{b, e, f\}$.

We can also see that $h(\{a, c\}) = \{e\}$ while $h^{-1}(\{a, c\}) = \{\}$.



Floor and ceiling functions

Definition. The floor function is a function with domain $\mathbb{R}$ and codomain $\mathbb{Z}$ defined as follows: for any $x \in \mathbb{R}$, the floor of x is written as $\lfloor x \rfloor$ and is given by the largest integer less than or equal to x.

Definition. The ceiling function is a function with domain $\mathbb{R}$ and codomain $\mathbb{Z}$ defined as follows: for any $x \in \mathbb{R}$, the ceiling of x is written as $\lceil x \rceil$ and is the smallest integer greater than or equal to x.

Example. Evaluate the following:

- $\lfloor 3.14 \rfloor = 3$.
- $\lceil 3.14 \rceil = 4$.
- $\lfloor -0.5 \rfloor = -1$.
- $\lceil -0.5 \rceil = 0$.
- $\lfloor 1 \rfloor = 1$.
- $\lceil 1 \rceil = 1$.

Example 3

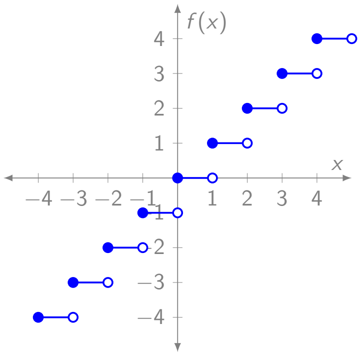
Example. Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by $f(x) = \lfloor x \rfloor$, and let $g : \mathbb{R} \to \mathbb{R}$ be the function given by $g(x) = \lceil x \rceil$.

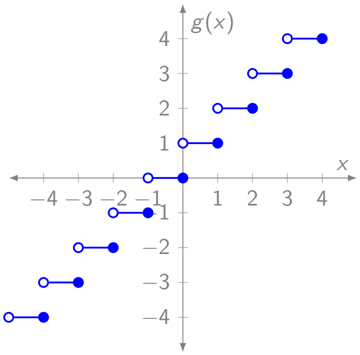
(i) Draw the coordinate graphs representing f and g.

(ii) Find the range of f and of g.

(iii) Let $S = \{\frac{1}{n} \ : \ n \in \mathbb{Z}^+\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$. Find each of $f(S)$, $g(S)$, $f^{-1}(S)$, and $g^{-1}(S)$.

Solution. The coordinate graphs are provided below.





Example. Let $f:\mathbb{R}\to\mathbb{R}$ be the function given by $f(x)=\lfloor x\rfloor$, and let $g:\mathbb{R}\to\mathbb{R}$ be the function given by $g(x)=\lceil x\rceil$.

(i) Draw the coordinate graphs representing f and g.

(ii) Find the range of f and of g.

(iii) Let $S=\{\frac{1}{n} \ : \ n\in\mathbb{Z}^+\}=\{1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\dots\}$. Find each of $f(S)$, $g(S)$, $f^{-1}(S)$, and $g^{-1}(S)$.

Solution. The coordinate graphs indicate that the range of both f and of g is $\mathbb{Z}$. To confirm this, notice that for any $k\in\mathbb{Z}$, we have $f(k)=g(k)=k$, so there is at least one input value that returns any integer k as an output.

Notice that $f(S)=\{f(1),f(\frac{1}{2}),f(\frac{1}{3}),f(\frac{1}{4}),\dots\}=\{1,0,0,0,\dots\}=\{0,1\}$.

Similarly, $g(S)=\{g(1),g(\frac{1}{2}),g(\frac{1}{3}),g(\frac{1}{4}),\dots\}=\{1,1,1,1,\dots\}=\{1\}$.

Since the range of both f and of g is $\mathbb{Z}$, no input value can return a non-integer output. So in particular, the pre-image of any non-integer is empty, that is, $f(\frac{1}{2})=g(\frac{1}{2})=\{\}$, $f(\frac{1}{3})=g(\frac{1}{3})=\{\}$, and so on.

So $f^{-1}(S)=f^{-1}(\{1,\frac{1}{2},\frac{1}{3},\dots\})=f^{-1}(\{1\})=\{x\in\mathbb{R} \ : \ 1\le x<2\}$.

Similarly, $g^{-1}(S)=g^{-1}(\{1,\frac{1}{2},\frac{1}{3},\dots\})=g^{-1}(\{1\})=\{x\in\mathbb{R} \ : \ 0<x\le 1\}$.