MATH1081 – Discrete Mathematics

Topic 2 – Number theory and relations
Lecture 2.06 – Partial orders

Lecturer: Dr Sean Gardiner – sean.gardiner@unsw.edu.au

Antisymmetry

Definition. A relation R on a set X is antisymmetric if and only if for all $x, y \in X$, whenever $x \, R \, y$ and $y \, R \, x$, we have $x = y$.

In terms of arrow diagrams, a relation R is antisymmetric if and only if it contains no arrows pointing in both directions.

Diagrammatically, this property can be represented as:

$$\bullet \qquad \bullet \quad \text{implies we do not have} \quad \bullet \longleftrightarrow \bullet$$

Example. Let $X = \{1, 2, 3, 4\}$. Which of the following relations on X are antisymmetric?

- $R = \{(1,1), (1,3), (2,2), (2,3), (3,3), (4,4)\}$ is antisymmetric since for every pair of elements $x, y \in X$, whenever $x \, R \, y$ and $y \, R \, x$, we have that $x = y$.
- $R = \{(1,1), (1,3), (2,4), (3,1), (3,3), (4,2), (4,4)\}$ is not antisymmetric since for the pair of elements $1, 3 \in X$, we have both $1 \, R \, 3$ and $3 \, R \, 1$, but $1 \neq 3$.

Notice that symmetry and antisymmetry are not opposite properties! A given relation can be either, neither, or both symmetric and antisymmetric.

Partial order relations

Definition. A relation R on a set X is a partial order relation if and only if it is reflexive, antisymmetric, and transitive.

Definition. If two mathematical objects are related by a partial order relation, we say they are comparable. If $\preceq$ is a partial order relation and $x \preceq y$, we can say "x precedes y" or "y succeeds x".

For example, a very familiar partial order relation is $\leq$ on any subset of $\mathbb{R}$. We can easily check that $\leq$ on $\mathbb{R}$ is reflexive, antisymmetric, and transitive.

Definition. If a relation $\preceq$ is a partial order relation on a set X, we call X a partially ordered set or poset. We sometimes write a partially ordered set as an ordered pair containing the set and the relation, for example $(X, \preceq)$.

Definition. If a partially ordered set has the property that all of its elements are comparable with each other, it is called a totally ordered set.

For example, $(\mathbb{R}, \leq)$ is a totally ordered set.

Examples of partial order relations

Example. Let S be a set. Show that the relation $\preceq$ on $\mathcal{P}(S)$ defined by setting $X \preceq Y$ if and only if $X \subseteq Y$ is a partial order relation.

Solution. We proceed by proving $\preceq$ is reflexive, antisymmetric, and transitive.

<u>Reflexivity:</u> Let $X \in \mathcal{P}(S)$. Since every element of X is of course an element of X, we know $X \subseteq X$, so $X \preceq X$ and thus $\preceq$ is reflexive.

<u>Antisymmetry:</u> Let $X, Y \in \mathcal{P}(S)$, and suppose $X \preceq Y$ and $Y \preceq X$. Then $X \subseteq Y$ and $Y \subseteq X$, so by definition, we have $X = Y$. Thus $\preceq$ is antisymmetric.

<u>Transitivity:</u> Let $X, Y, Z \in \mathcal{P}(S)$, and suppose $X \preceq Y$ and $Y \preceq Z$. Then $X \subseteq Y$ and $Y \subseteq Z$, so for all $x \in X$ we have $x \in Y$ since $X \subseteq Y$, and therefore $x \in Z$ since $Y \subseteq Z$. So $X \subseteq Z$, meaning $X \preceq Z$, and thus $\preceq$ is transitive.

Hence $\preceq$ acting on $\mathcal{P}(S)$ is a partial order relation.

Examples of partial order relations

Example. Show that the relation $\preceq$ on $\mathbb{N}$ defined by setting $x \preceq y$ if and only if $x \mid y$ is a partial order relation.

Solution. We proceed by proving $\preceq$ is reflexive, antisymmetric, and transitive.

Reflexivity: Let $x \in \mathbb{N}$. Since $x = x \times 1$ and $1 \in \mathbb{Z}$, we can write that $x \mid x$. So $x \preceq x$, and thus $\preceq$ is reflexive.

Antisymmetry: Let $x, y \in \mathbb{N}$, and suppose $x \preceq y$ and $y \preceq x$. Then $y = xk$ and $x = yl$ for some natural numbers k and l. So $x = (xk)l$, implying $kl = 1$, whose only solution over the natural numbers is $k = l = 1$. So $x = y$, and thus $\preceq$ is antisymmetric.

Transitivity: Let $x, y, z \in \mathbb{N}$, and suppose $x \preceq y$ and $y \preceq z$. Then $y = xk$ and $z = yl$ for some natural numbers k and l. So $z = (xk)l$ where $kl \in \mathbb{N}$, which means $x \mid z$. So $x \preceq z$, and thus $\preceq$ is transitive.

Hence $\preceq$ acting on $\mathbb{N}$ is a partial order relation.

Notice that if this relation $\preceq$ acted on the set $\mathbb{Z}$ instead of $\mathbb{N}$, the relation would no longer be antisymmetric and therefore $(\mathbb{Z}, \mid)$ would not be a partially ordered set. (For example, $1 \mid -1$ and $-1 \mid 1$ but $1 \neq -1$.)

Hasse diagrams

Notation. Given $\preceq$ is a partial order relation on some set X, we use the symbol $\prec$ to mean "precedes but does not equal". That is, $x \prec y$ means $x \preceq y$ and $x \neq y$.

Definition. A Hasse diagram is a simplified way of representing a partially ordered set. Given a partial order relation $\preceq$ on a set X, the Hasse diagram for the partially ordered set $(X, \preceq)$ is constructed as follows:

- Draw a labelled dot for each element of X.
- For each pair of elements $x, y \in X$, draw a line from x to y with x placed lower than y if and only if both
  - $x \prec y$, and
  - there does not exist $z \in X$ such that $x \prec z$ and $z \prec y$.

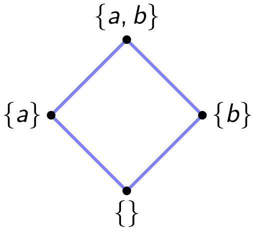
A Hasse diagram contains all the information of a usual arrow diagram for a poset, but in a simpler format.

- Arrowheads are not needed since direction is implied by element position.
- Loops are not included since they must exist at every element.
- Lines implied by transitivity are not included since they must exist for every upward path.

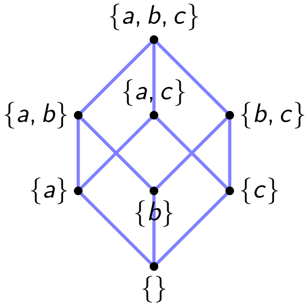
Example. Draw the Hasse diagram for the partially ordered sets $(\mathcal{P}(\{a, b\}), \subseteq)$ and $(\mathcal{P}(\{a, b, c\}), \subseteq)$.

Solution.

The Hasse diagram for the poset $(\mathcal{P}(\{a, b\}), \subseteq)$ is:



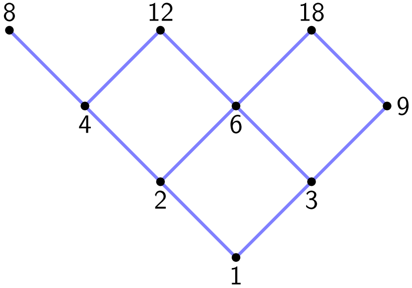
The Hasse diagram for the poset $(\mathcal{P}(\{a, b, c\}), \subseteq)$ is:



Hasse diagrams – divisibility example

Example. Draw the Hasse diagram for the partially ordered set $(\{1, 2, 3, 4, 6, 8, 9, 12, 18\}, \mid)$.

Solution. The Hasse diagram for the poset $(\{1, 2, 3, 4, 6, 8, 9, 12, 18\}, \mid)$ is:



Least and greatest elements

Definition. A minimal element of a partially ordered set $(X, \preceq)$ is any element $x \in X$ such that there is no $y \in X$ where $y \prec x$.

In terms of Hasse diagrams, a minimal element is an element with no lines connecting to it from below.

Definition. A maximal element of a partially ordered set $(X, \preceq)$ is any element $x \in X$ such that there is no $y \in X$ where $x \prec y$.

In terms of Hasse diagrams, a maximal element is an element with no lines connecting to it from above.

Definition. The least element of a partially ordered set $(X, \preceq)$ (if it exists) is the element $x \in X$ for which $x \preceq y$ for all $y \in X$.

In terms of Hasse diagrams, the least element is the element that can reach all other elements in X by following lines in a strictly upwards path.

Definition. The greatest element of a partially ordered set $(X, \preceq)$ (if it exists) is the element $x \in X$ for which $y \preceq x$ for all $y \in X$.

In terms of Hasse diagrams, the greatest element is the element that can reach all other elements in X by following lines in a strictly downwards path.

Least and greatest elements – Examples and properties

Example. Find the minimal, maximal, least, and greatest elements of $(\mathcal{P}(\{a, b, c\}), \subseteq)$.

Solution. The only minimal element is $\{\}$, which is also the least element. The only maximal element is $\{a, b, c\}$, which is also the greatest element.

Example. Find the minimal, maximal, least, and greatest elements of $(\{1, 2, 3, 4, 6, 8, 9, 12, 18\}, \mid)$.

Solution. The only minimal element is 1, which is also the least element. The maximal elements are 8, 12, and 18. There is no greatest element, since for example, 8 and 12 are not comparable.

Fact. The following statements are true for any partially ordered set $(X, \preceq)$.

- If it exists, the least element is unique.
- If it exists, the greatest element is unique.
- If X is finite, the poset has a least element if and only if it has exactly one minimal element.
- If X is finite, the poset has a greatest element if and only if it has exactly one maximal element.

Lower and upper bounds

Definition. Given a partially ordered set $(X, \preceq)$, a lower bound of two elements $x, y \in X$ is any element $z \in X$ such that $z \preceq x$ and $z \preceq y$.

In terms of Hasse diagrams, a lower bound of x and y is any element that can reach both x and y by following lines in strictly upwards paths.

Definition. Given a partially ordered set $(X, \preceq)$, the greatest lower bound of two elements $x, y \in X$, denoted $\mathrm{glb}(x, y)$, is the greatest element amongst the set of all lower bounds (if it exists). That is, $\mathrm{glb}(x, y)$ is the greatest element (if it exists) of $\{z \in X \,:\, z \preceq x \ \text{ and } \ z \preceq y\}$.

Definition. Given a partially ordered set $(X, \preceq)$, an upper bound of two elements $x, y \in X$ is any element $z \in X$ such that $x \preceq z$ and $y \preceq z$.

In terms of Hasse diagrams, an upper bound of x and y is any element that can reach both x and y by following lines in strictly downwards paths.

Definition. Given a partially ordered set $(X, \preceq)$, the least upper bound of two elements $x, y \in X$, denoted $\mathrm{lub}(x, y)$, is the least element amongst the set of all upper bounds (if it exists). That is, $\mathrm{lub}(x, y)$ is the least element (if it exists) of $\{z \in X \,:\, x \preceq z \ \text{ and } \ y \preceq z\}$.

Lower and upper bounds – Examples and properties

Example. In the poset $(\mathcal{P}(\{a,b,c\}), \subseteq)$, find $\mathrm{glb}(\{a,b\},\{b,c\})$ and $\mathrm{lub}(\{a\},\{a,b\})$ if they exist.

Solution. The lower bounds of $\{a,b\}$ and $\{b,c\}$ are $\{b\}$ and $\{\}$, so the greatest lower bound exists and is $\{b\}$. The upper bounds of $\{a\}$ and $\{a,b\}$ are $\{a,b\}$ and $\{a,b,c\}$, so the least upper bound exists and is $\{a,b\}$.

Example. In the poset $(\{1,2,3,4,6,8,9,12,18\}, \mid)$, find $\mathrm{glb}(12,18)$ and $\mathrm{lub}(4,9)$ if they exist.

Solution. The lower bounds of 12 and 18 are 1, 2, 3, and 6, so the greatest lower bound exists and is 6. There is no upper bound for 4 and 9, so their least upper bound does not exist in this poset.

Fact. For any set S, in the partially ordered set $(\mathcal{P}(S), \subseteq)$, for any two elements $A, B \in \mathcal{P}(S)$ we have $\mathrm{glb}(A,B) = A \cap B$ and $\mathrm{lub}(A,B) = A \cup B$.

Fact. In the partially ordered set $(\mathbb{Z}^+, \mid)$, for any two elements $a, b \in \mathbb{Z}^+$ we have $\mathrm{glb}(a,b) = \gcd(a,b)$ and $\mathrm{lub}(a,b) = \mathrm{lcm}(a,b)$.

Note that the least element of $(\mathbb{Z}^+, \mid)$ is 1, but it has no greatest element. If we use the "alternate definition" given for the GCD and LCM, the above facts also hold for the poset $(\mathbb{N}, \mid)$, whose greatest element is 0.