MATH1081 – Discrete Mathematics

Topic 1 – Set theory and functions
Lecture 1.02 – Subsets and power sets

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Subsets

Note: The phrase "if and only if" is used in mathematical definitions to indicate equivalence of statements.

Definition. A set S is a subset of a set T if and only if every element of S is also an element of T.

Notation. We write $S \subseteq T$ to mean "S is a subset of T". Similarly, we write $S \not\subseteq T$ to indicate S is not a subset of T.

For example, we have $\{1, 3\} \subseteq \{1, 2, 3\}$ and $\{1, 2, 3\} \subseteq \{1, 2, 3\}$, but $\{1, 2, 3, 4\} \not\subseteq \{1, 2, 3\}$. We can also see that $\mathbb{Z}^+ \subseteq \mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$.

Definition. Two sets S and T are equal (written $S = T$) if and only if they contain exactly the same elements. This means that every element of S is an element of T, and every element of T is an element of S. So we have that $S = T$ if and only if $S \subseteq T$ and $T \subseteq S$.

Definition. A set S is a proper subset of a set T if and only if $S \subseteq T$ and $S \neq T$. (Equivalently, $S \subseteq T$ and $T \not\subseteq S$.)

Notation. We write $S \subset T$ or $S \subsetneq T$ to mean "S is a proper subset of T".

For example, we have $\{1, 3\} \subset \{1, 2, 3\}$, but $\{1, 2, 3\} \not\subset \{1, 2, 3\}$.

Example – Identifying elements and subsets

Example. Decide whether each of the statements below is true or false.

- $1 \in \{1, \{1\}\}$

- $1 \subseteq \{1, \{1\}\}$

- $\{1\} \in \{1, \{1\}\}$

- $\{1\} \subseteq \{1, \{1\}\}$

- $\{\{1\}\} \in \{1, \{1\}\}$

- $\{\{1\}\} \subseteq \{1, \{1\}\}$

Theorem. For any set S, we have $S \subseteq S$ and $\{\} \subseteq S$.
Proof.

Proving statements about set containment

Suppose that S and T are sets. To prove that S is a subset of T, we must show that every element of S is also an element of T. If S is very large, it might be impractical to check each element of S individually. So we typically prove that any arbitrary element of S also belongs to T by working with a general element $x \in S$.

That is, to prove that $S \subseteq T$, we write a proof with the following structure:

> Let $x \in S$ be an arbitrary element of S.
> $\vdots$
> Then $x \in T$.
> Thus $S \subseteq T$.

To prove that $S \nsubseteq T$, we only have to show that there is some particular element in S that is not in T. Such a proof would have the structure:

> Choose $x \in S$ to be the particular element...
> $\vdots$
> Then $x \notin T$.
> Thus $S \nsubseteq T$.

Example – Proving set containment

Example. Let $S = \{2k + 1 \ : \ k \in \mathbb{Z}\}$ and $T = \{4n - 1 \ : \ n \in \mathbb{Z}\}$.
Prove that $S \nsubseteq T$.

Solution.

Example. Let $S = \{6k + 1 \ : \ k \in \mathbb{Z}\}$ and $T = \{3n - 2 \ : \ n \in \mathbb{Z}\}$.
Prove that $S \subseteq T$.

Solution.

Proving further statements about set containment

To prove that $S = T$, we must prove that $S \subseteq T$ and that $T \subseteq S$. Note that this means we must provide two separate proofs (using the structure shown on the previous slide).

To prove that $S \neq T$, we must prove that $S \nsubseteq T$ or that $T \nsubseteq S$. Note that this just means we have to show there is some particular element that belongs to one of the sets but not the other.

To prove that $S \subset T$, we must prove that $S \subseteq T$ and that $S \neq T$. Note that this means after proving that $S \subseteq T$, we just have to find some particular element of T that is not in S.

(To prove that $S \not\subset T$, we must prove that $S \nsubseteq T$ or that $S = T$.)

Example. Let $S = \{6k + 1 \ : \ k \in \mathbb{Z}\}$ and $T = \{3n - 2 \ : \ n \in \mathbb{Z}\}$.
We have already shown that $S \subseteq T$. Prove that S is a proper subset of T.

Solution.

Example – Proving set equality

Example. Let $S = \{6k + 1 \: : \: k \in \mathbb{Z}\}$ and $T = \{6n - 5 \: : \: n \in \mathbb{Z}\}$.
Prove that $S = T$.

Solution.

Power sets

Definition. The power set of a set S, written as $\mathcal{P}(S)$ or just $P(S)$, is the set of all possible subsets of S.

For example, the subsets of $\{1, 2\}$ are $\{\}$, $\{1\}$, $\{2\}$, and $\{1, 2\}$, so the power set of $\{1, 2\}$ is $\mathcal{P}(\{1, 2\}) = \{\, \{\}, \{1\}, \{2\}, \{1, 2\} \,\}$.

Example. Find $\mathcal{P}(\{a, b, c\})$.

Solution.

Notice that $|\mathcal{P}(\{1, 2\})| = 4 = 2^2$, while $|\mathcal{P}(\{a, b, c\})| = 8 = 2^3$. We can also easily check that $|\mathcal{P}(\{1\})| = 2^1$ and $|\mathcal{P}(\{\})| = 2^0$. This seems to indicate a connection between cardinalities of power sets and powers of 2...

Cardinality of power sets

Theorem. For any finite set S, the cardinality of its power set is given by

$$|\mathcal{P}(S)| = 2^{|S|}.$$

Proof.

Example. Find the following cardinalities:

- $|\mathcal{P}(\{1, 2, 3, 4\})| =$
- $|\mathcal{P}(\mathcal{P}(\{1\}))| =$

Proofs involving power sets

The following fact follows straight from our definition of a power set, and can be useful when proving properties of power sets.

Fact. For any sets S and T, we have $S \subseteq T$ if and only if $S \in \mathcal{P}(T)$.

In order to prove a statement involving power sets, we first introduce a lemma about sets. ("Lemma" in mathematics means a minor theorem.)

Lemma. Suppose A, B, and C are sets such that $A \subseteq B$ and $B \subseteq C$. Then $A \subseteq C$.

Proof. Try this yourself! (See tutorial Problem Set 1, Question 7.)

Example. Suppose that S and T are sets such that $S \subseteq T$.
Prove that $\mathcal{P}(S) \subseteq \mathcal{P}(T)$.

Solution.