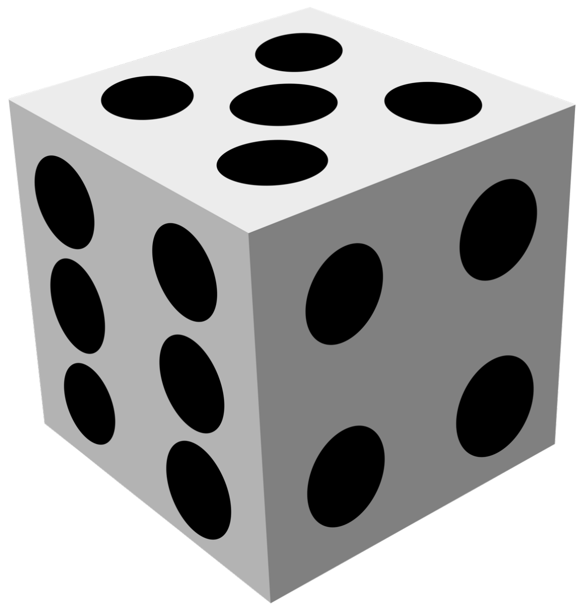
COMP9334
Capacity Planning for Computer Systems and Networks

Week 3A: Queues with Poisson arrivals (2)

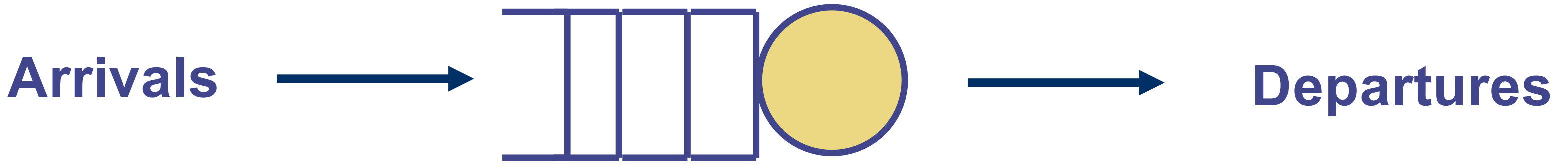
Pre-lecture exercise

- You have a loaded die with 6 faces with values 1, 2, 3, 4, 5 and 6
- The probability that you can get each face is given in the table below
- What is the mean value that you can get?

Value	**Probability**
1	0.1
2	0.1
3	0.2
4	0.1
5	0.3
6	0.2

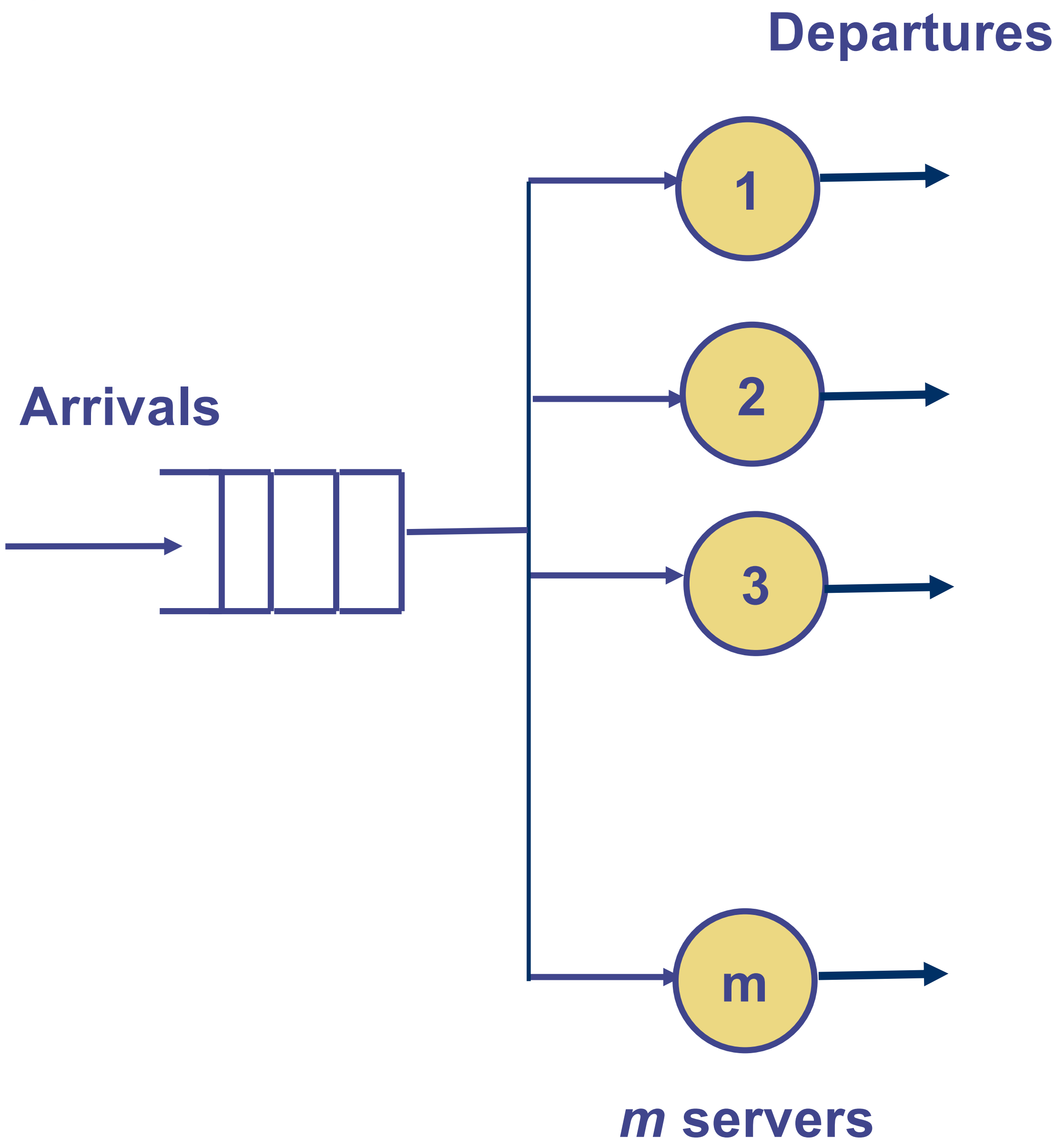


Single-server queue



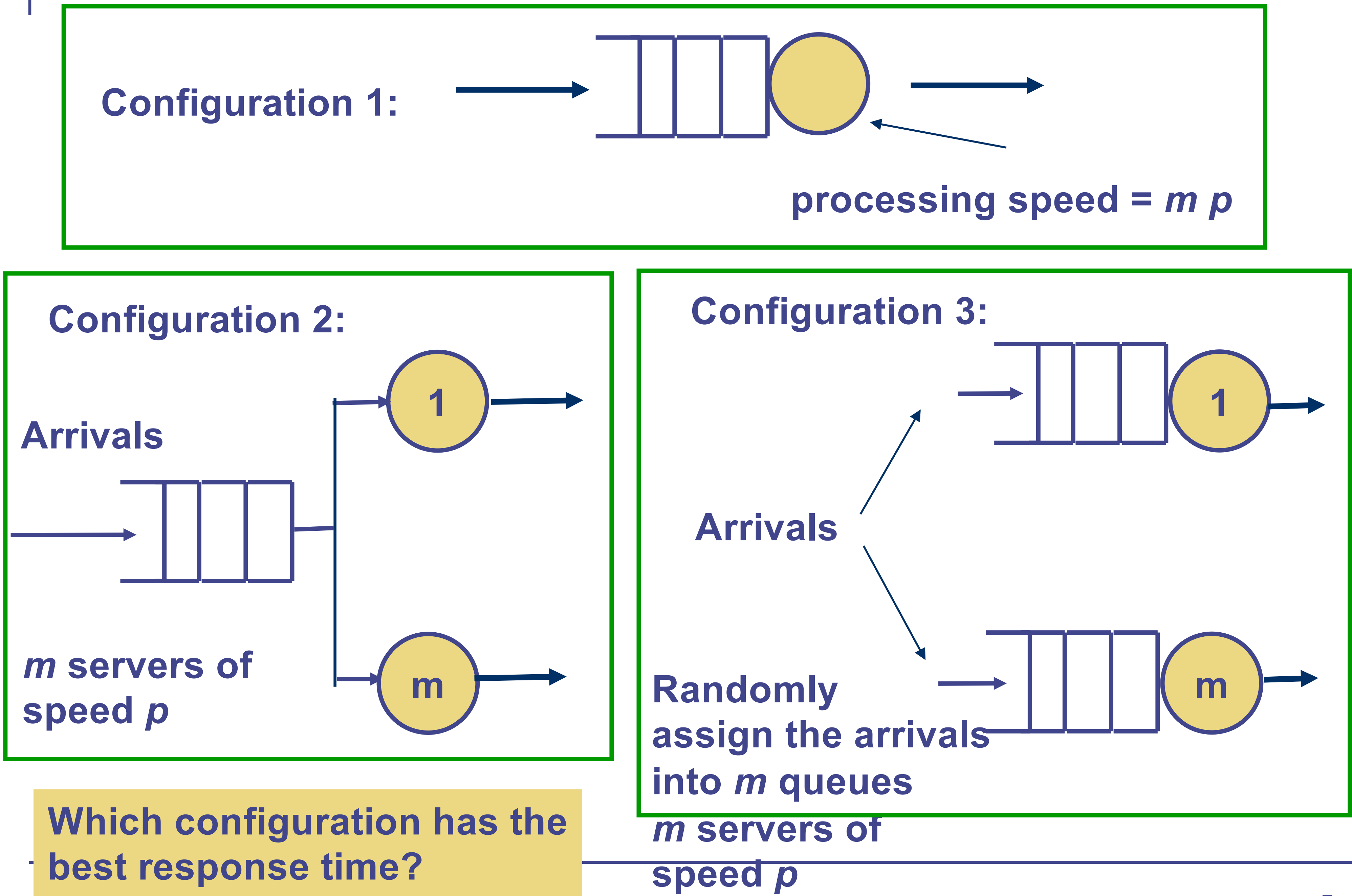
- Open, single server queues
- How to find:
  - Waiting time
  - Response time
  - Mean queue length etc.
- The technique to find waiting time etc. is called *Queueing Theory*

Multiple server queue



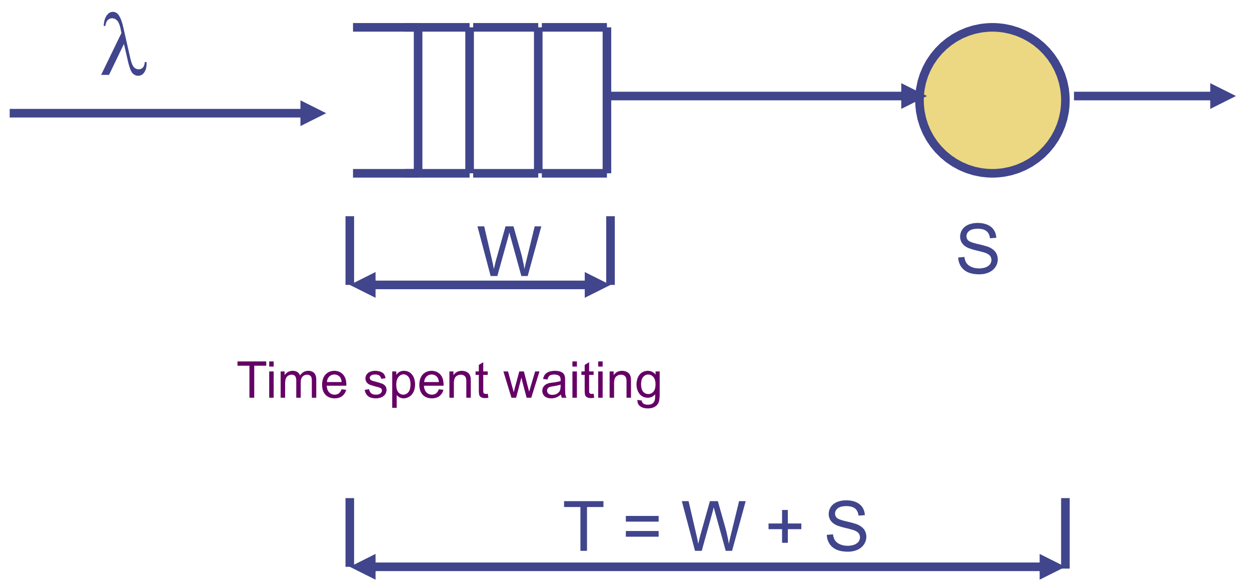
- Open, multi-server queue
- How to find:
  - Waiting time
  - Response time
  - Mean queue length etc.

What will you be able to do with the results?



Which configuration has the best response time?

Single Server Queue: Terminology



Response Time T
= Waiting time W + Service time S

Note: We use T for response time because this is the notation in many queueing theory books. For a similar reason, we will use ρ for utilisation rather than U.

Call centre analogy from Week 2B

- Consider a call centre
  - Calls are arriving according to Poisson distribution with rate λ
  - The length of each call is exponentially distributed with parameter μ
    - Mean length of a call is $1/\mu$

Call centre:

Arrivals →

***m* operators**
If all operators are busy, the centre can put at most *n* additional calls on hold.
If a call arrives when all operators and holding slots are used, the call is rejected.

- We solved the problems for
  - ($m = 1$ and $n = 0$), and ($m = 1$ and $n = 1$)
- How about other values of *m* and *n*? What about response time?

Kendall's notation

- To represent different types of queues, queueing theorists use the Kendall's notation
- The call centre example on the previous page can be represented as:

$$M \,/\, M \,/\, s \; (/\, B)$$

Inter-arrival distribution is Markovian i.e. Exponential

Service time distribution is Markovian i.e. Exponential

Number of servers

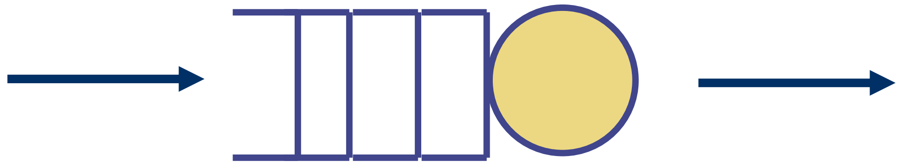
Buffer Positions (wait room)

The call centre example on the last page is a M/M/m/(m+n) queue
If n = ∞, we simply write M/M/m

M/M/1 queue

Exponential Inter-arrivals (λ)

Exponential Service time (μ)



Infinite buffer **One server**

- Consider a call centre analogy
  - Calls are arriving according to Poisson distribution with rate λ
  - The length of each call is exponentially distributed with parameter μ
    - Mean length of a call is $1/\mu$

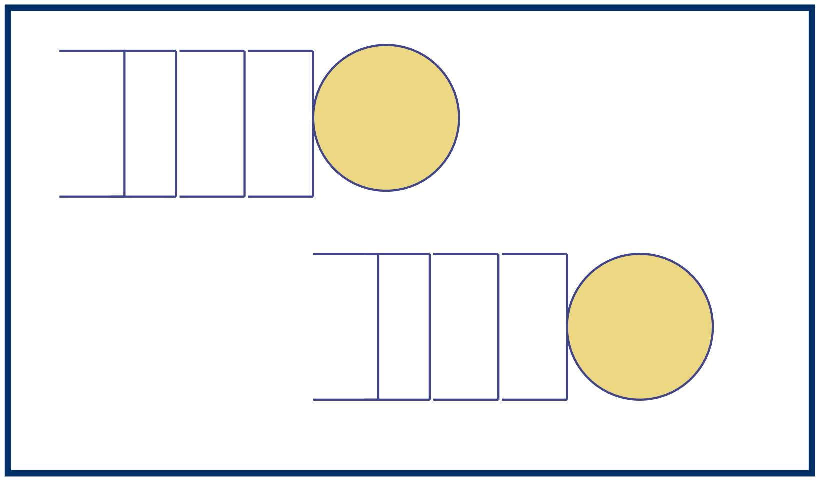
Arrivals

Call centre with *1* operator
If the operator is busy, the centre will put the call on hold.
A customer will wait until his call is answered.

- Queueing theory will be able to answer these questions:
  - What are the mean waiting time, mean response time for a call?

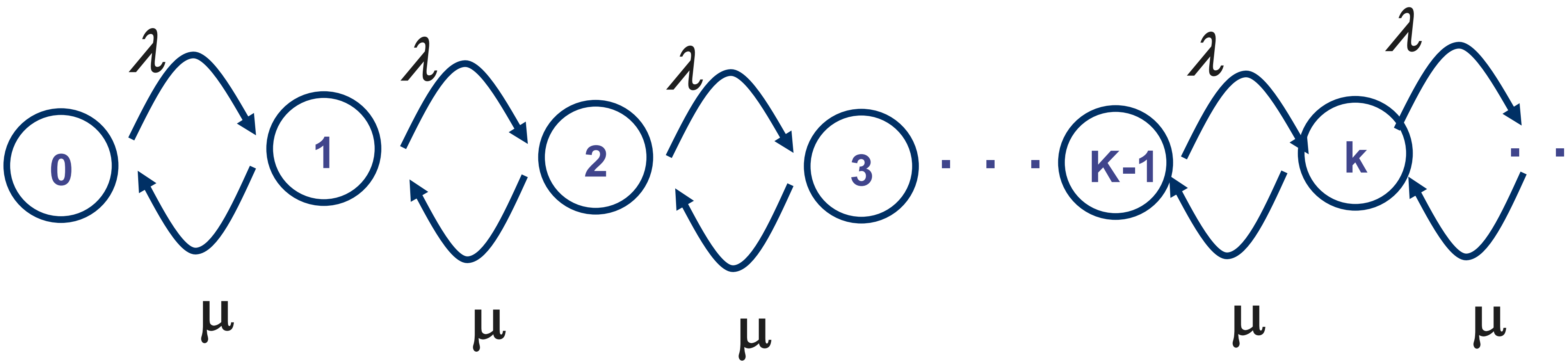
Little's Law

- Applicable to any "box" that contains some queues or servers
- Mean number of requests in the "box" =
  Mean response time x Throughput
- We will use Little's Law in this lecture to derive the mean response time
  - We first compute the mean number of requests in the "box" and throughput
- For the computation of throughput, we have from week 1A:
  - Throughput = min(offered load, maximum processing rate)
    - Note: the offered load will need to be adjusted if jobs can be rejected



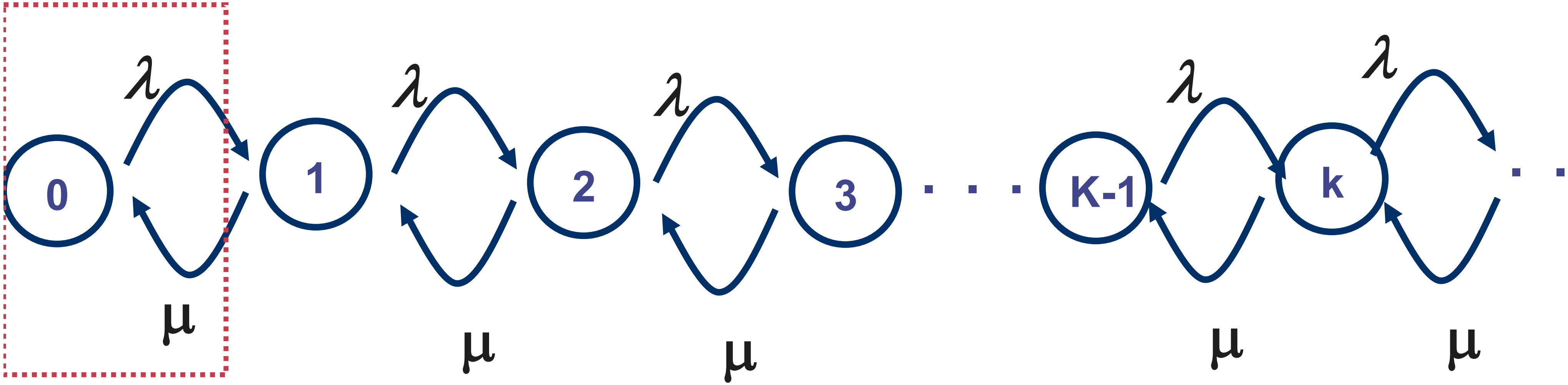
M/M/1: State and transition diagram

- We will solve for the steady state response
- Define the states of the queue
  - State 0 = There are zeros job in the system (= The server is idle)
  - State 1 = There is 1 job in the system (= 1 job at the server, no job queueing)
  - State 2 = There are 2 jobs in the system (= 1 job at the server, 1 job queueing)
  - State *k* = There are *k* jobs in the system (= 1 job at the server, *k-1* job queueing)
- The state transition diagram



M/M/1 state balance:

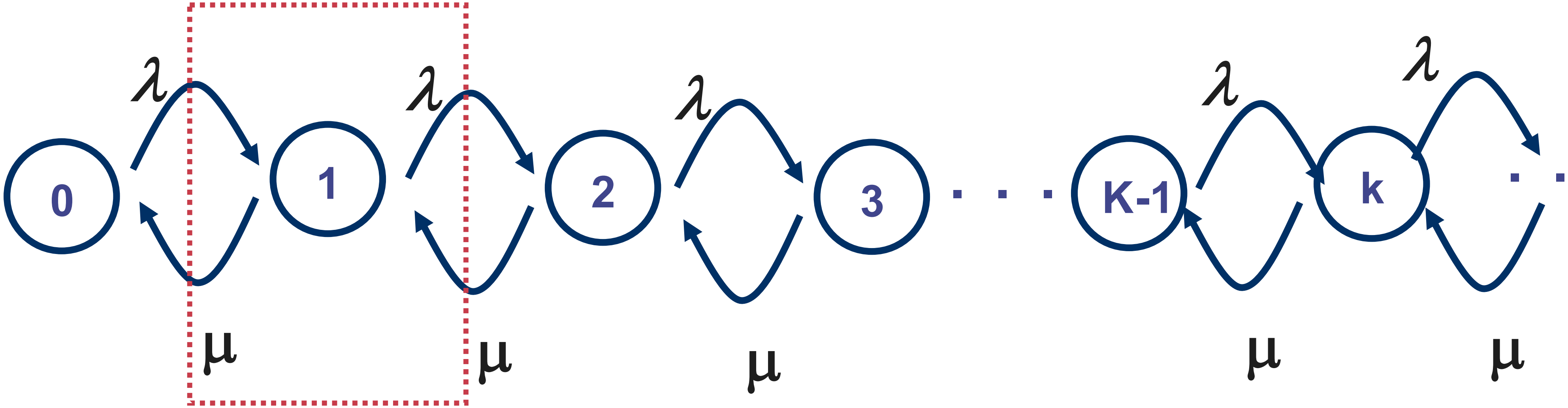
$$P_k = \text{Prob. } k \text{ jobs in system}$$



$$\lambda P_0 = \mu P_1$$

$$\Rightarrow P_1 = \frac{\lambda}{\mu} P_0$$

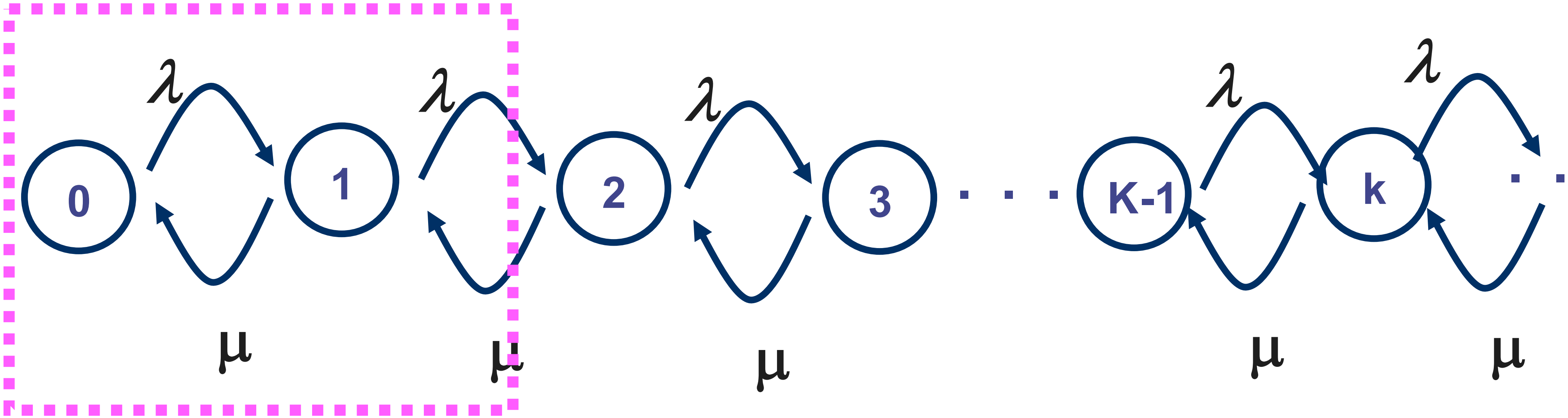
M/M/1 state balance: Exercise 1



- Exercise: Write the state balance equation for State 1

$$\lambda P_0 + \mu P_2 = (\lambda + \mu) P_1$$

M/M/1 state balance: Exercise 2

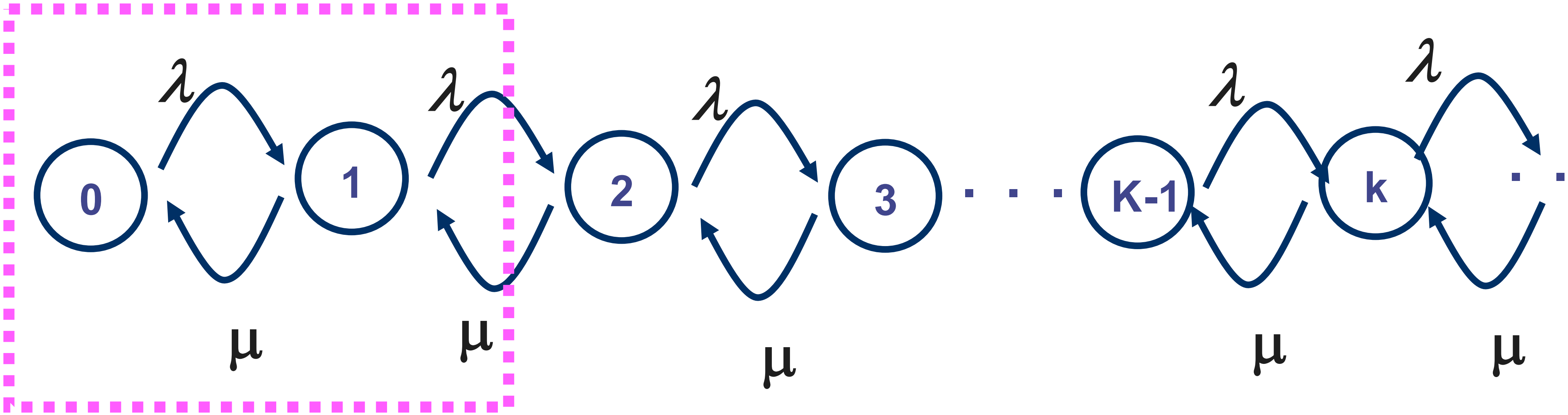


- Exercise: Write the state balance equation for magenta box, i.e.

Rate of transiting out of the magenta box
= Rate of transiting into the magenta box

$$\lambda P_1 = \mu P_2$$

Which state balance is easier to work with?



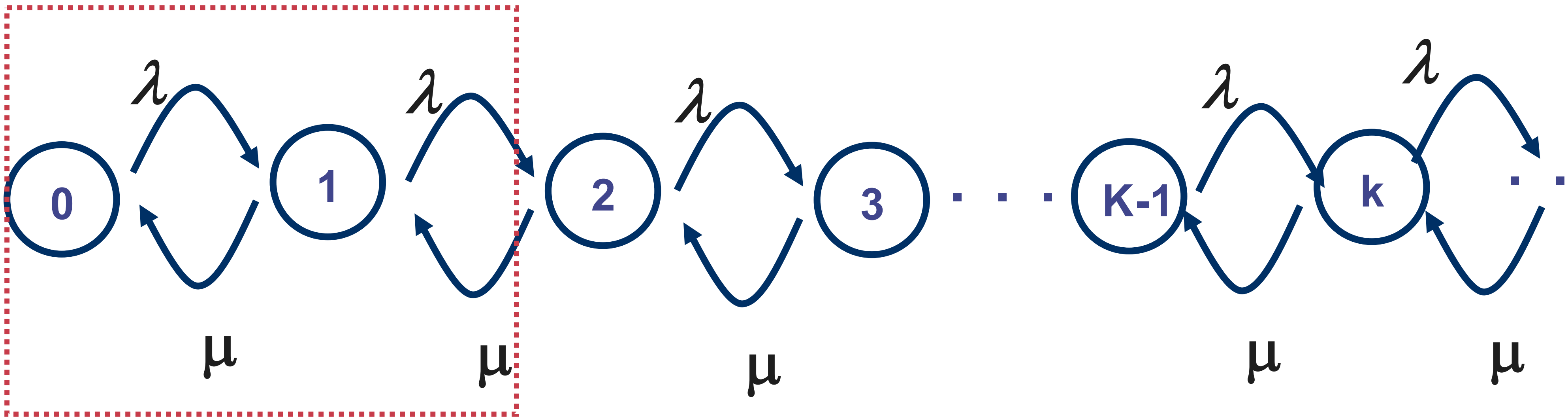
State balance for State 1

$$\lambda P_0 + \mu P_2 = (\lambda + \mu) P_1$$

State balance for State 1 and State 2 combined

$$\lambda P_1 = \mu P_2$$

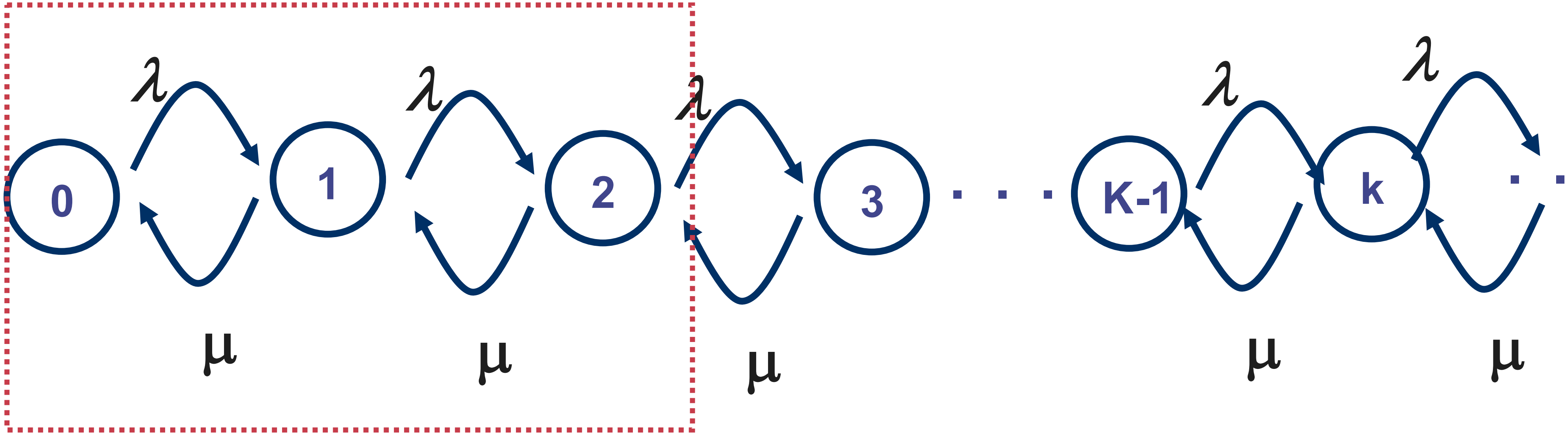
M/M/1 state balance: Relating P_2 and P_0



$$\lambda P_0 = \mu P_1 \qquad \lambda P_1 = \mu P_2$$

$$\Rightarrow P_2 = \frac{\lambda}{\mu} P_1 \quad \Rightarrow P_2 = \left(\frac{\lambda}{\mu}\right)^2 P_0$$

M/M/1 state balance: Relating P_3 and P_0



$$\lambda P_2 = \mu P_3$$

$$\Rightarrow P_3 = \frac{\lambda}{\mu} P_2 \quad \Rightarrow P_3 = \left(\frac{\lambda}{\mu}\right)^3 P_0$$

M/M/1 state balance: Relating P_k and P_0

In general $$P_k = \left(\frac{\lambda}{\mu}\right)^k P_0$$

$$\text{Let } \rho = \frac{\lambda}{\mu}$$

We have $$P_k = \rho^k P_0$$

Solving for P_k

With $P_k = \rho^k P_0$ **and**

$$P_0 + P_1 + P_2 + P_3 + \ldots = 1$$

$$\Rightarrow (1 + \rho + \rho^2 + \ldots)P_0 = 1$$

$$\Rightarrow P_0 = 1 - \rho \text{ if } \rho < 1$$

$$\Rightarrow P_k = (1 - \rho)\rho^k$$

Claim: ρ is the utilisation
$\rho = 1 - P_0$
= 1 - Prob server is idle
= Prob server is busy

Since $\rho = \dfrac{\lambda}{\mu}$ $, \rho < 1 \Rightarrow \lambda < \mu$

Arrival rate < service rate

Exercise: Mean number of jobs

Recall that $P_k = \text{Prob. } k \text{ jobs in system}$

and we have calculated that $P_k = (1-\rho)\rho^k$

Determine the mean number of jobs in the system.

Hint 1: Look at pre-lecture exercise 1.

You can use the following formula to help you.

For $0 \leq x < 1$,

$$p + x(p+q) + x^2(p+2q) + x^3(p+3q) + \ldots = \frac{p}{1-x} + \frac{xq}{(1-x)^2}$$

Mean number of jobs

$$P_k = \text{Prob. } k \text{ jobs in system}$$

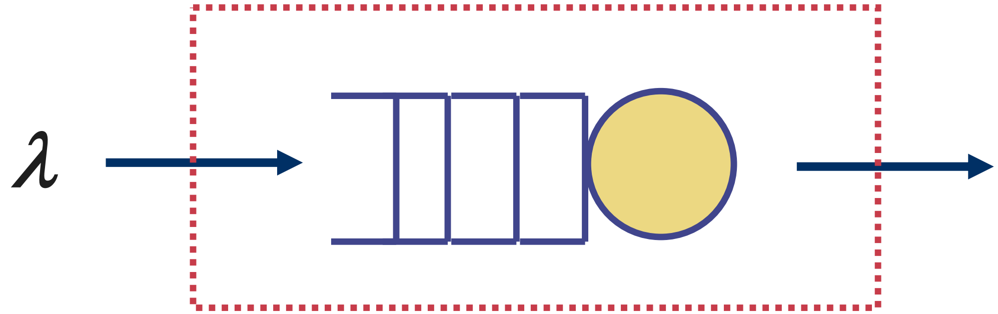
$$P_k = (1-\rho)\rho^k$$

The mean number of jobs in the system =

$$\sum_{k=0}^{\infty} kP_k = \sum_{k=0}^{\infty} k(1-\rho)\rho^k$$

$$= \frac{\rho}{1-\rho}$$

M/M/1: mean response time

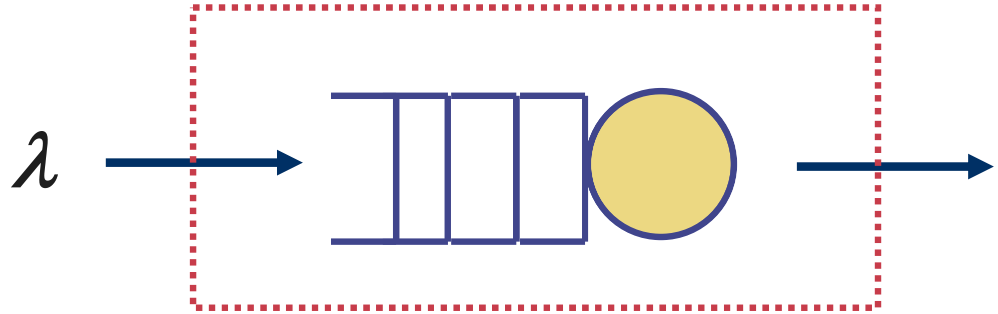


Little's law:
mean number of customers = throughput x response time

Throughput is λ ***(why?)***

$$\text{Response time } T = \frac{\rho}{\lambda(1-\rho)} = \frac{1}{\mu - \lambda}$$

Exercise: M/M/1 mean waiting time



What is the mean waiting time at the queue?

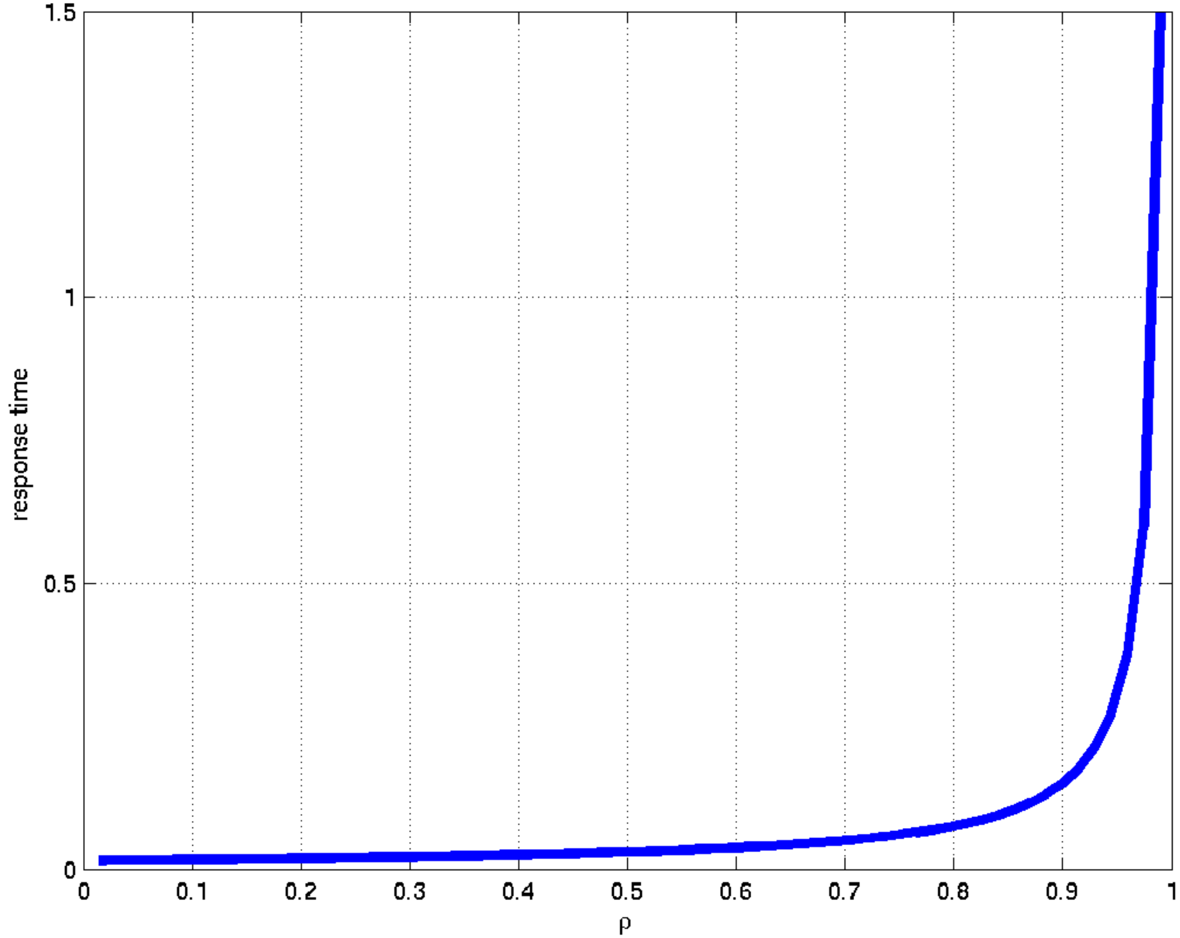
Mean waiting time = mean response time - mean service time

We know mean response time (from last slide)

Mean service time is = 1 / μ

Using the service time parameter ($1/\mu$ = 15ms) in the example, let us see how response time T varies with λ

$$T = \frac{1}{\mu(1-\rho)}$$



Observation: Response time increases sharply when ρ gets close to 1

Infinite queue assumption means $\rho \to 1$, $T \to \infty$

Non-linear effect on response time

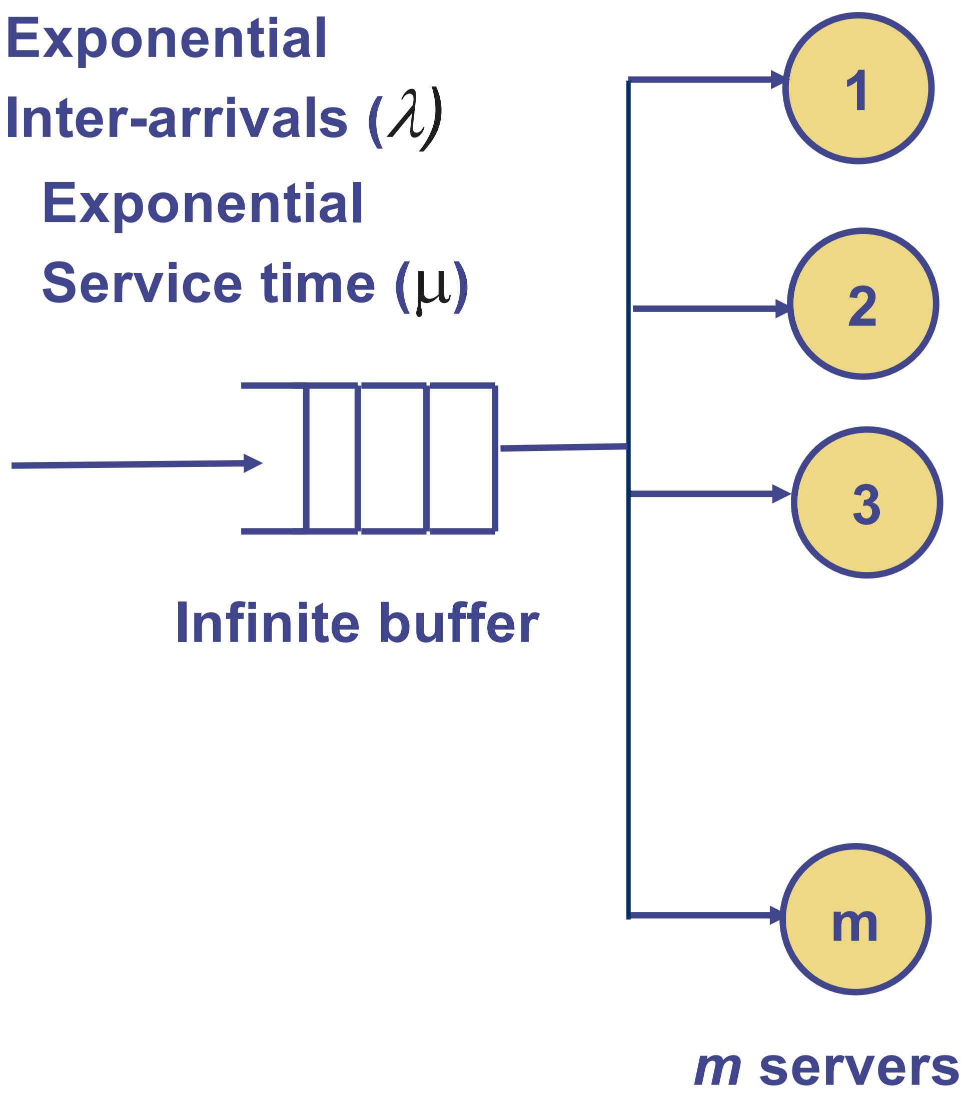
- The response time of an M/M/1 queue $= \dfrac{1}{\mu - \lambda}$

- Assuming the mean arrival rate is 10 requests/s
- We will calculate the effect of service rate on response time
- What can you conclude?

Service rate	Utilisation λ/μ	Response time
11	10/11 = 0.909	1/(11 - 10) = 1
22	10/22 = 0.454	1/(22 - 10) = 0.08

- Doubling the service rate can sometimes reduce by response time by a factor more than 2.

Multi-server queues M/M/m



All arrivals go into one queue.

Customers can be served by any one of the *m* servers.

When a customer arrives

- If all servers are busy, it will join the queue
- Otherwise, it will be served by one of the available servers

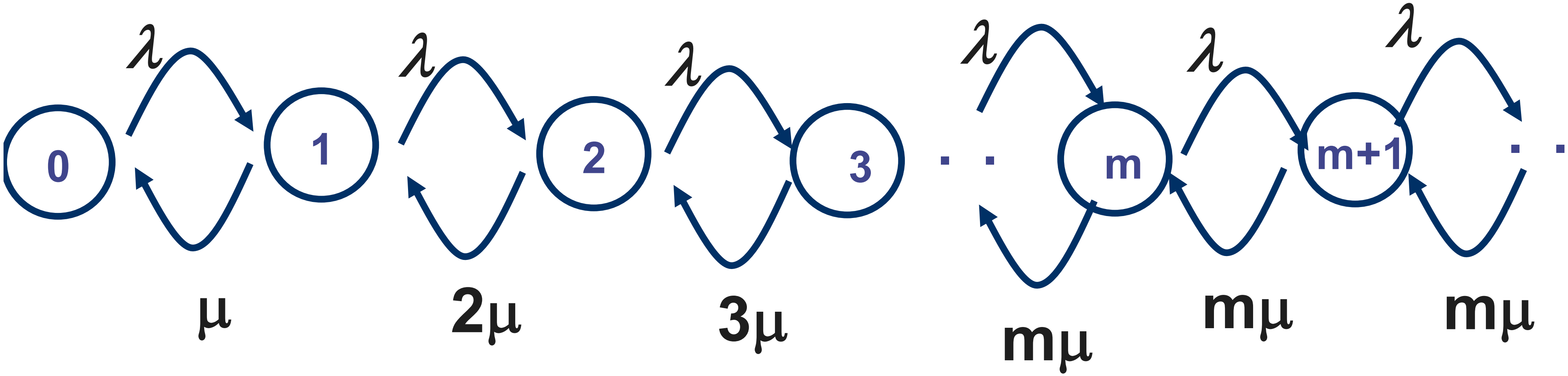
A call centre analogy of M/M/m queue

- Consider a call centre analogy
  - Calls are arriving according to Poisson distribution with rate λ
  - The length of each call is exponentially distributed with parameter μ
    - Mean length of a call is $1/\mu$

Arrivals →

Call centre with *m* operators
If all *m* operators are busy, the centre will put the call on hold.
A customer will wait until his call is answered.

State transition for M/M/m



M/M/m

- Following the same method, we have mean response time *T* is

$$T = \frac{C(\rho, m)}{m\mu(1-\rho)} + \frac{1}{\mu}$$

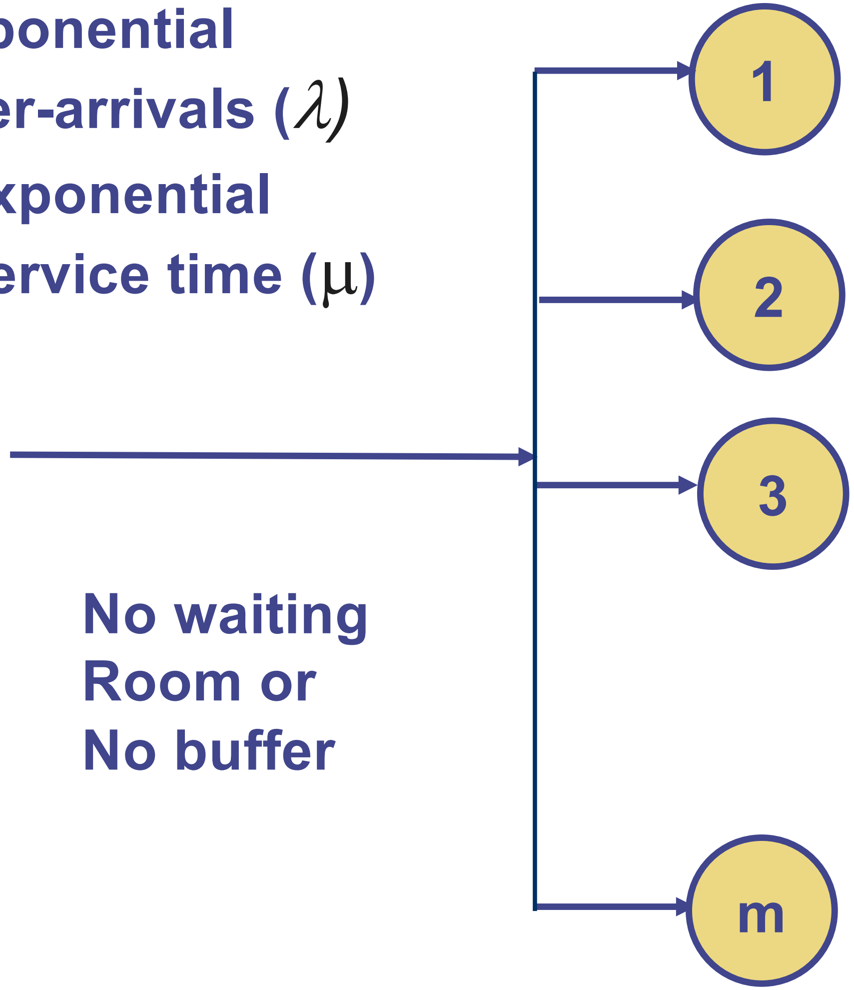
where $$\rho = \frac{\lambda}{m\mu}$$

$$C(\rho, m) = \frac{\frac{(m\rho)^m}{m!}}{(1-\rho)\sum_{k=0}^{m-1} \frac{(m\rho)^k}{k!} + \frac{(m\rho)^m}{m!}}$$

Multi-server queues M/M/m/m with no waiting room

Exponential Inter-arrivals (λ)

Exponential Service time (μ)



No waiting Room or No buffer

***m* servers**

An arrival can be served by any one of the *m* servers.

When a customer arrives

- **If all servers are busy, it will *depart* from the system**

- **Otherwise, it will be served by one of the available servers**

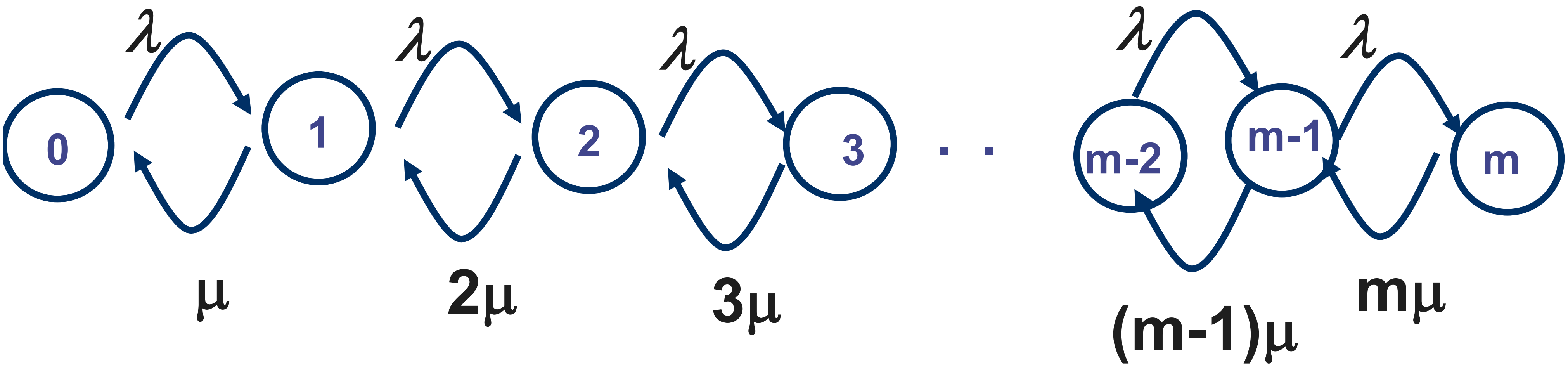
A call centre analogy of M/M/m/m queue

- Consider a call centre analogy
  - Calls are arriving according to Poisson distribution with rate λ
  - The length of each call is exponentially distributed with parameter μ
    - Mean length of a call is $1/\mu$

Arrivals

Call centre with *m* operators
If all *m* operators are busy, the call is dropped.

State transition for M/M/m/m

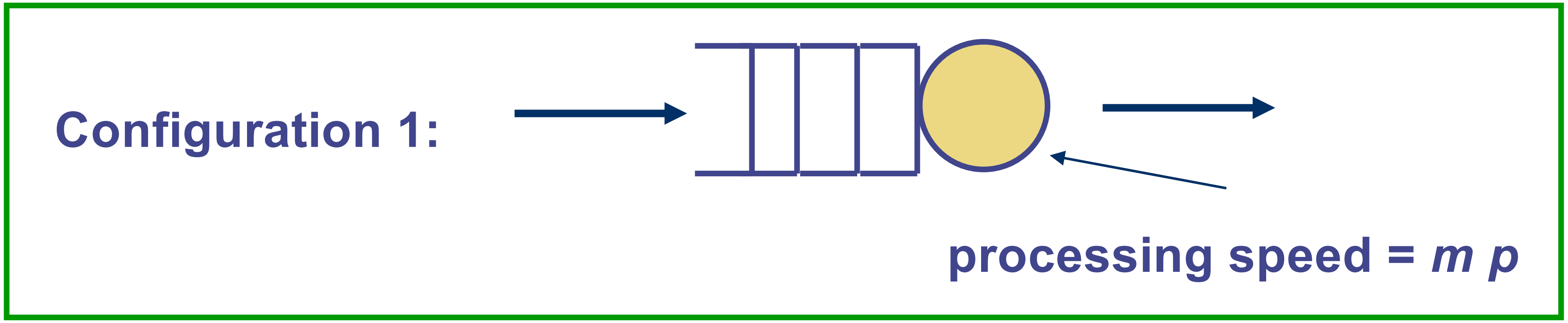


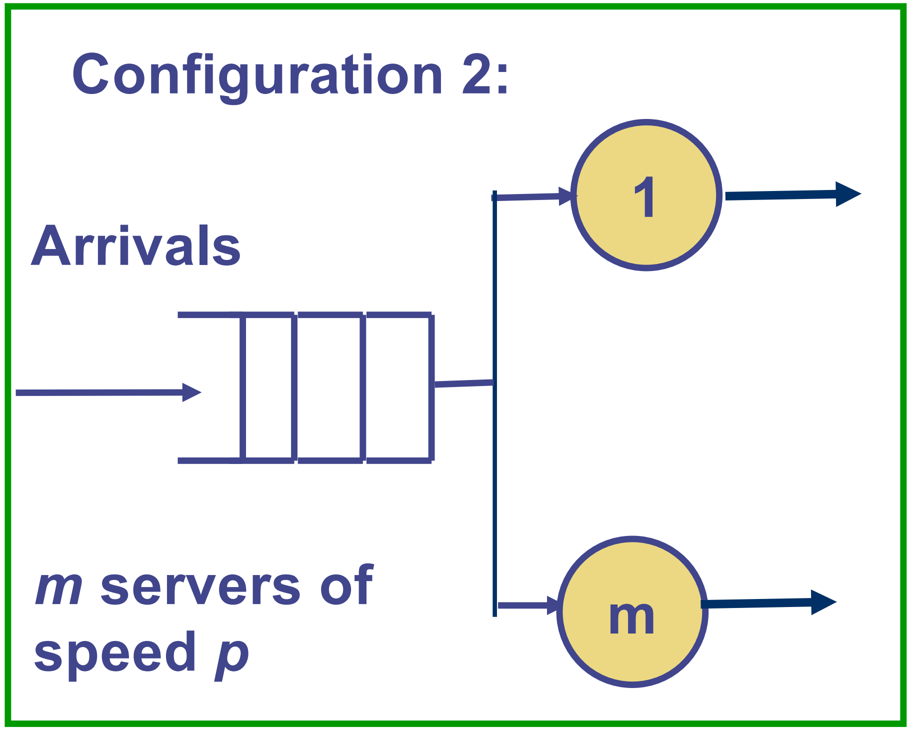
Probability that an arrival is blocked
= Probability that there are m customers in the system

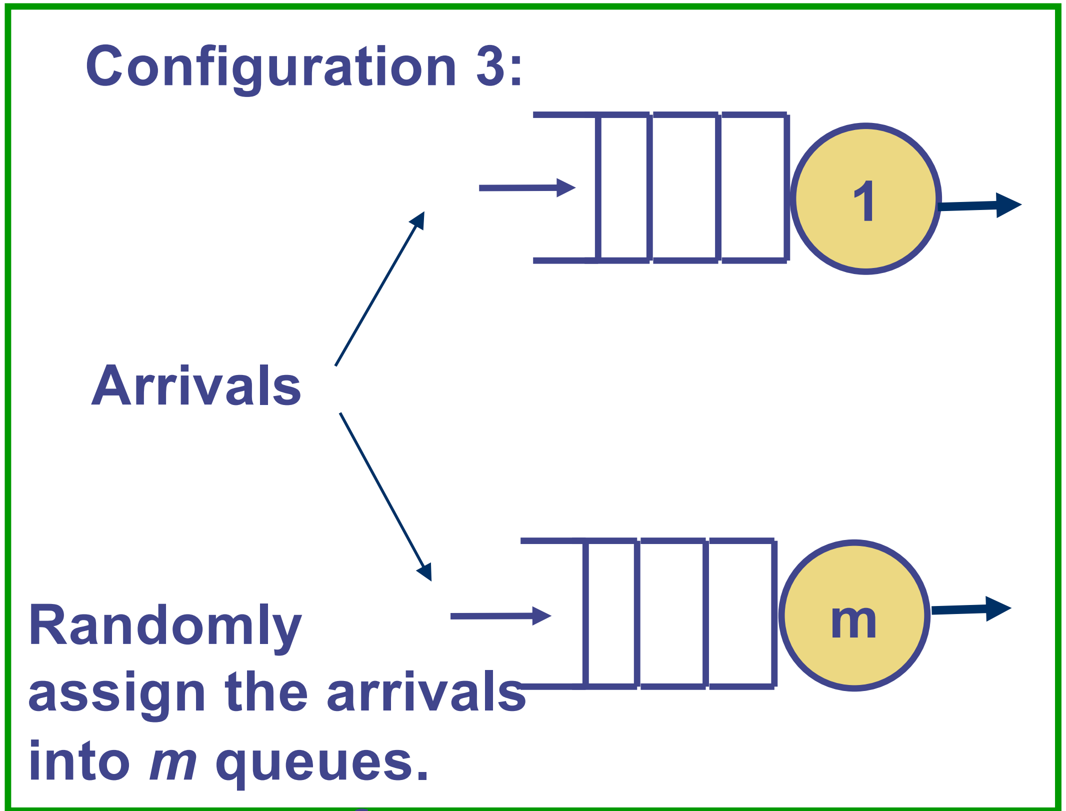
$$P_m = \frac{\frac{\rho^m}{m!}}{\sum_{k=0}^{m} \frac{\rho^k}{k!}} \quad \textbf{where} \quad \rho = \frac{\lambda}{\mu}$$

"Erlang B formula"

What configuration has the best response time?







m servers of speed *p*

Try out the tutorial question!

Summary

- Queues with Poisson arrivals and exponential service time
- Different configurations
  - Single server, multiple servers
  - Finite or infinite waiting places
- Analysis technique
  - Draw a diagram with the states
  - Add arcs between states with transition rates
  - Derive flow balance equation for each state, i.e.
    - Rate of entering a state = Rate of leaving a state
  - Solve the equation for steady state probability
- Little's Law to find mean response time

References

- Recommended reading
  - Queues with Poisson arrival are discussed in
  - Bertsekas and Gallager, *Data Networks*, Sections 3.3 to 3.4.3
  - Note: I derived the formulas here using continuous Markov chain but Bertsekas and Gallager used discrete Markov chain
  - Mor Harchal-Balter. Chapters 13 and 14