COMP9020

Foundations of Computer Science

Lecture 9a: Functions

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Administrivia

- **No lectures next week (flex week)**
- Feedback
  - Speed and amount of content
  - Timing of assignments
- Second assignment due today 6pm.
  - To make a late submission:
    https://www.cse.unsw.edu.au/~cs9020/extension_request.html
- First assignment - Marks released later today

Applications of Functions and Big-O notation

- Functions, methods, procedures in programming
- Computer programs “are” functions
- Graphical transformations
- Algorithmic analysis

Outline

Outline

Properties of Binary Relations $R \subseteq S \times T$

A binary relation $R \subseteq S \times T$ is:

Definition

(Fun)	functional	For all $s \in S$ there is at most one $t \in T$ such that $(s, t) \in R$
(Tot)	total	For all $s \in S$ there is at least one $t \in T$ such that $(s, t) \in R$
(Inj)	injective	For all $t \in T$ there is at most one $s \in S$ such that $(s, t) \in R$
(Sur)	surjective	For all $t \in T$ there is at least one $s \in S$ such that $(s, t) \in R$
(Bij)	bijective	Injective and surjective

Functions

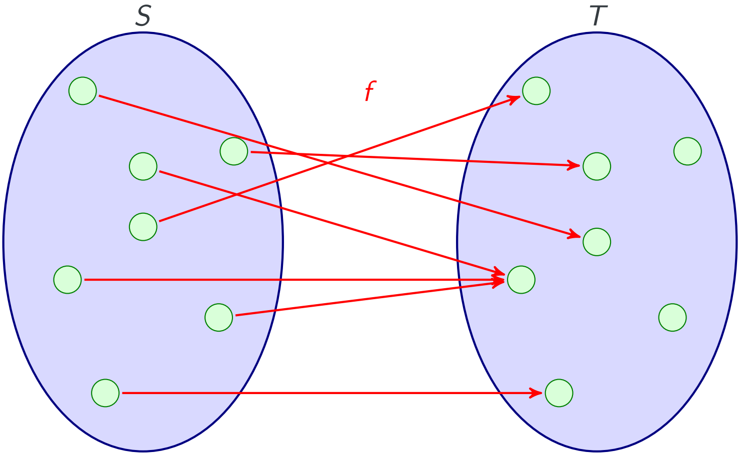
Definition

A **function**, $f : S \to T$, is a binary relation $f \subseteq S \times T$ that satisfies (Fun) and (Tot). That is, for all $s \in S$ there is *exactly one* $t \in T$ such that $(s, t) \in f$.

We write $f(s)$ for the unique element related to s.

We write T^S for the set of all functions from S to T.

Graphical representation



Functions

$f : S \longrightarrow T$ describes pairing of the sets: it means that f assigns to every element $s \in S$ a unique element $t \in T$. To emphasise where a specific element is sent, we can write $f : x \mapsto y$, which means the same as $f(x) = y$

		Symbol	
S	**domain** of f	$\mathrm{Dom}(f)$	(inputs)
T	**co-domain** of f	$\mathrm{Codom}(f)$	(*possible* outputs)
$f(S)$	**image** of f	$\mathrm{Im}(f)$	(*actual* outputs)
$= \{ f(x) : x \in \mathrm{Dom}(f) \}$			

Example

Example

The **identity** function on S

$$\mathrm{Id}_S(x) = x, x \in S$$

- $\mathrm{Dom}(\mathrm{Id}_S) = S$
- $\mathrm{Codom}(\mathrm{Id}_S) = S$
- $\mathrm{Im}(\mathrm{Id}_S) = S$

Important!

The domain and co-domain are critical aspects of a function's definition.

$$f : \mathbb{N} \to \mathbb{Z} \quad \text{given by} \quad f(x) = x^2$$

and

$$g : \mathbb{N} \to \mathbb{N} \quad \text{given by} \quad g(x) = x^2$$

are different functions even though they have the same behaviour!

Injective functions

Function $f : S \to T$ is called an **injection** or **1-1** (**one-to-one**) if it satisfies (Inj)

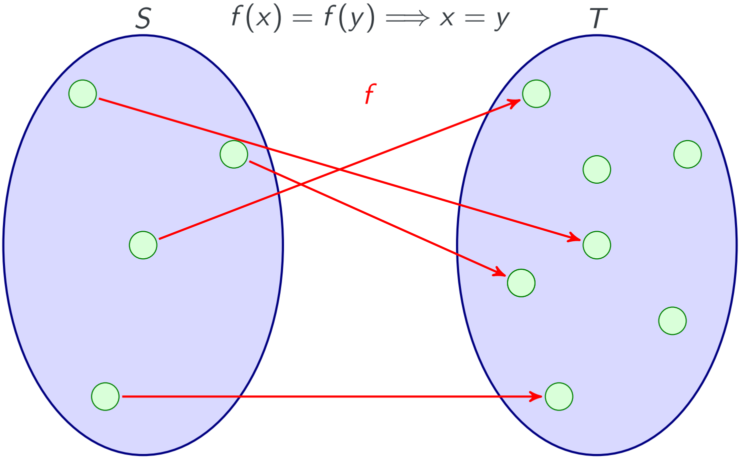
Examples (of functions that are injective)

- $f : \mathbb{N} \longrightarrow \mathbb{N}$ with $f(x) \mapsto x$
- set complement (for a fixed universe)

Examples (of functions that are not injective)

- absolute value, floor, ceiling
- length of a word

Graphical representation: Injective



Surjective functions

Function $f : S \to T$ is called a **surjection** or **onto** if it satisfies (Sur). That is, if

$$\mathrm{Im}(f) = \mathrm{Codom}(f)$$

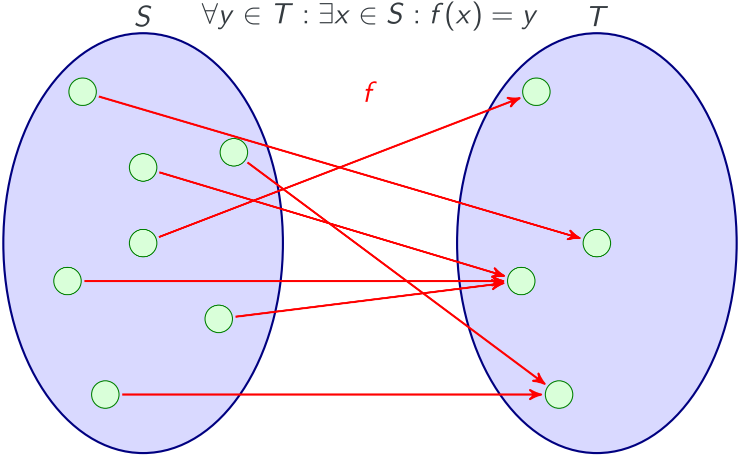
Examples (of functions that are surjective)

- $f : \mathbb{N} \longrightarrow \mathbb{N}$ with $f(x) \mapsto x$
- Floor, ceiling

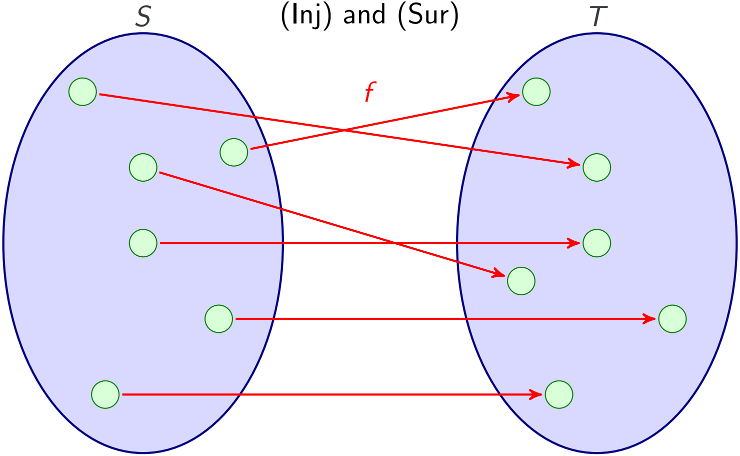
Examples (of functions that are not surjective)

- $f : \mathbb{N} \longrightarrow \mathbb{N}$ with $f(x) \mapsto x^2$
- $f : \{a, \ldots, e\}^* \longrightarrow \{a, \ldots, e\}^*$ with $f(w) \mapsto awe$

Graphical representation: Surjective



Graphical representation: Bijection



Functions on finite sets

NB

For a **finite** *set* S *and* $f : S \longrightarrow S$ *the properties*

1. *surjective, and*
2. *injective*

are equivalent.

Outline

Composition of functions

Question

If $f : S \to T$ and $g : T \to U$ are functions, then $f\,;g$ is a relation. When is it a function?

Composition of functions

Question

If $f : S \to T$ *and* $g : T \to U$ *are functions, then* $f ; g$ *is a relation. When is it a function?*

Answer

If $\mathrm{Im}(f) \subseteq \mathrm{Dom}(g)$ *– so always!*

Composition of Functions

Definition

If $f : S \to T$ and $g : T \to U$ then the **composition of** f **and** g, written $g \circ f$, is the function given by

$$(g \circ f)(x) = g(f(x)).$$

That is, $g \circ f = f; g$.

Facts

- Composition is associative

$$h \circ (g \circ f) = (h \circ g) \circ f$$

- For $g : S \to T$

$$g \circ \mathrm{Id}_S = g \quad \text{and} \quad \mathrm{Id}_T \circ g = g.$$

Iteration of Functions

If a function maps a set into itself, i.e. when $\mathrm{Dom}(f) = \mathrm{Codom}(f)$, the function can be composed with itself — **iterated**

$$f \circ f, f \circ f \circ f, \ldots, \quad \text{also written } f^2, f^3, \ldots$$

Exercises

Exercises

Let $f, g : \mathbb{Z} \to \mathbb{Z}$ be given by $f(n) = n^2 + 3$ and $g(n) = 5n - 11$. What is:

- $f \circ g(n) =$
- $g \circ f(n) =$
- $g^2(n) =$

Exercises

Exercises

Let $f, g : \mathbb{Z} \to \mathbb{Z}$ be given by $f(n) = n^2 + 3$ and $g(n) = 5n - 11$.
What is:

- $f \circ g(n) = (5n - 11)^2 + 3 = 25n^2 - 110n - 118$
- $g \circ f(n) = 5(n^2 + 3) - 11 = 5n^2 + 4$
- $g^2(n) = 5(5n - 11) - 11 = 25n - 66$

Outline

Converse of a function

Question

$f^{\leftarrow}$ *is a relation; when is it a function?*

Converse of a function

Question

$f^{\leftarrow}$ is a relation; when is it a function?

Answer

When f is a bijection.

Inverse Functions

Definition

If $f^{\leftarrow}$ is a function then it is called the **inverse function**; denoted f^{-1}.

NB

f^{-1} *only exists if* f *is a bijection.*
$f^{\leftarrow}$ *always exists.*

f^{-1} is the procedure of "undoing" f.

Properties of the inverse

Fact

If $f : S \to T$ *and* f^{-1} *exists then:*

$$f^{-1} \circ f = Id_S \quad \text{and} \quad f \circ f^{-1} = Id_T.$$

Conversely, if $f : S \to T$ *and* $g : T \to S$ *and*

$$g \circ f = Id_S \quad \text{and} \quad f \circ g = Id_T$$

then f^{-1} *exists and is equal to* g.

Exercises

$\boxed{\text{RW: 1.7.5}}$ f and g are 'shift' functions $\mathbb{N} \longrightarrow \mathbb{N}$ defined by $f(n) = n + 1$, and $g(n) = \max(0, n - 1)$

(c) Is f injective? surjective?

(d) Is g injective? surjective?

(e) Do f and g commute, i.e. $\forall n\,((f \circ g)(n) = (g \circ f)(n))$?

Exercises

Exercises

$\boxed{\text{RW: 1.7.5}}$ f and g are 'shift' functions $\mathbb{N} \longrightarrow \mathbb{N}$ defined by $f(n) = n + 1$, and $g(n) = \max(0, n - 1)$

(c) Is f injective? surjective? injective, not surjective

(d) Is g injective? surjective? surjective, not injective

(e) Do f and g commute, i.e. $\forall n\,((f \circ g)(n) = (g \circ f)(n))$?

f and g do not commute:

$g \circ f : n \mapsto (n + 1) - 1 = n$, thus $g \circ f = \mathrm{Id}_{\mathbb{N}}$

$f \circ g : 0 \mapsto 1$, hence $f \circ g \neq \mathrm{Id}_{\mathbb{N}}$

Exercises

$\boxed{\text{RW: 1.7.6}}$ $\Sigma = \{a, b, c\}$

(c) Is $\mathrm{length} : \Sigma^* \longrightarrow \mathbb{N}$ surjective?

(d) $\mathrm{length}^{\leftarrow}(2) \stackrel{?}{=}$

$\boxed{\text{RW: 1.7.12}}$ Verify that $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R} \times \mathbb{R}$ defined by $f(x, y) = (x + y, x - y)$ is invertible.

Exercises

$\boxed{\text{RW: 1.7.6}}$ $\Sigma = \{a, b, c\}$

(c) Is length : $\Sigma^* \longrightarrow \mathbb{N}$ surjective? Yes

(d) $\text{length}^{\leftarrow}(2) \stackrel{?}{=}$ $\{aa, ab, ac, ba, bb, bc, ca, cb, cc\}$

$\boxed{\text{RW: 1.7.12}}$ Verify that $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R} \times \mathbb{R}$ defined by $f(x, y) = (x + y, x - y)$ is invertible.

Let $g(x, y) = (\frac{x+y}{2}, \frac{x-y}{2})$. Then

$$\begin{aligned} (f \circ g)(x, y) &= f(\tfrac{x+y}{2}, \tfrac{x-y}{2}) = (x, y) \\ (g \circ f)(x, y) &= g(x + y, x - y) \\ &= (\tfrac{(x+y)+(x-y)}{2}, \tfrac{(x+y)-(x-y)}{2}) = (x, y) \end{aligned}$$

Outline

Matrices

An $\mathbf{m} \times \mathbf{n}$ **matrix** is a rectangular array with m horizontal rows and n vertical columns.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

NB

Matrices are important objects in Computer Science, e.g. for

- *optimisation*
- *graphics and computer vision*
- *cryptography*
- *information retrieval and web search*
- *machine learning*

Matrix Motivation

Solving linear equations:

$$5x = 15$$

$$\begin{aligned} 5x + 3y &= 15 \\ 4x - 2y &= 12 \end{aligned}$$

$$A = \begin{pmatrix} 5 & 3 \\ 4 & -2 \end{pmatrix} \qquad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 15 \\ 12 \end{pmatrix}$$

$$A\mathbf{x} = \mathbf{b}$$

$$\begin{aligned} x' &= 5x + 3y \qquad & x'' &= 2x' + y' \\ y' &= 4x - 2y \qquad & y'' &= 3x' + 3y' \end{aligned}$$

$$A = \begin{pmatrix} 5 & 3 \\ 4 & -2 \end{pmatrix} \qquad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \mathbf{x}' = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$B = \begin{pmatrix} 2 & 1 \\ 3 & 3 \end{pmatrix} \qquad \mathbf{x}'' = \begin{pmatrix} x'' \\ y'' \end{pmatrix}$$

Basic Matrix Operations

The **transpose** $\mathbf{A}^\mathsf{T}$ of an $m \times n$ matrix $\mathbf{A} = [a_{ij}]$ is the $n \times m$ matrix whose entry in the ith row and jth column is a_{ji}.

Example

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 & 4 \\ 3 & 2 & -1 & 2 \\ 4 & 0 & 1 & 3 \end{bmatrix} \qquad \mathbf{A}^\mathsf{T} = \begin{bmatrix} 2 & 3 & 4 \\ -1 & 2 & 0 \\ 0 & -1 & 1 \\ 4 & 2 & 3 \end{bmatrix}$$

NB

A matrix $\mathbf{M}$ *is called symmetric if* $\mathbf{M}^\mathsf{T} = \mathbf{M}$

Matrix Sum

The **sum** of two $m \times n$ matrices $\mathbf{A} = [a_{ij}]$ and $\mathbf{B} = [b_{ij}]$ is the $m \times n$ matrix whose entry in the ith row and jth column is $a_{ij} + b_{ij}$.

Example

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 & 4 \\ 3 & 2 & -1 & 2 \\ 4 & 0 & 1 & 3 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 0 & 5 & 3 \\ 2 & 3 & -2 & 1 \\ 4 & -2 & 0 & 2 \end{bmatrix}$$

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} 3 & -1 & 5 & 7 \\ 5 & 5 & -3 & 3 \\ 8 & -2 & 1 & 5 \end{bmatrix}$$

Fact

$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$ *and* $(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C})$

Scalar Product

Given $m \times n$ matrix $\mathbf{A} = [a_{ij}]$ and $c \in \mathbb{R}$, the **scalar product** $c\mathbf{A}$ is the $m \times n$ matrix whose entry in the ith row and jth column is $c \cdot a_{ij}$.

Example

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 0 & 4 \\ 3 & 2 & -1 & 2 \\ 4 & 0 & 1 & 3 \end{bmatrix} \qquad 2\mathbf{A} = \begin{bmatrix} 4 & -2 & 0 & 8 \\ 6 & 4 & -2 & 4 \\ 8 & 0 & 2 & 6 \end{bmatrix}$$

Matrix Product

The **product** of an $m \times n$ matrix $\mathbf{A} = [a_{ij}]$ and an $n \times p$ matrix $\mathbf{B} = [b_{jk}]$ is the $m \times p$ matrix $\mathbf{C} = [c_{ik}]$ defined by

$$c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk} \qquad \text{for } 1 \leq i \leq m \text{ and } 1 \leq k \leq p$$

Example

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

NB

The rows of $\mathbf{A}$ *must have the same number of entries as the columns of* $\mathbf{B}$.
The product of a $1 \times n$ *matrix and an* $n \times 1$ *matrix is usually called the* **inner product** *of two* **n-dimensional vectors.**

Example

Example

Consider

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 2 & -1 \\ -6 & 3 \end{bmatrix}$$

Calculate $\mathbf{AB}$, $\mathbf{BA}$

$$\mathbf{AB} = \begin{bmatrix} -10 & 5 \\ -20 & 10 \end{bmatrix} \qquad \mathbf{BA} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

NB

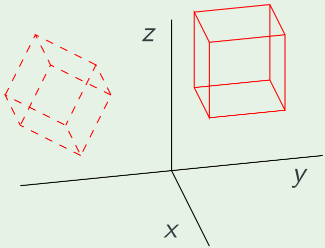
In general, $\mathbf{A} \cdot \mathbf{B} \neq \mathbf{B} \cdot \mathbf{A}$

Example: Computer Graphics

Example

Rotating an object w.r.t. the x axis by degree α:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} \cdot \begin{bmatrix} 5 & 5 & 7 & 7 & 5 & 7 & 5 & 7 \\ 1 & 1 & 1 & 1 & 3 & 3 & 3 & 3 \\ 9 & 7 & 7 & 9 & 7 & 7 & 9 & 9 \end{bmatrix}$$



Outline

Motivation

Want to compare functions, particularly functions from $\mathbb{N}$ to $\mathbb{R}$

Options:

- Equality: $f(n) = g(n)$ for all n
- (Pointwise) comparison: $f(n) \leq g(n)$ for all n
- (Almost all) comparison: $f(n) \leq g(n)$ for all but finitely many n
- Asymptotic growth: $\lim_{n\to\infty} \frac{f(n)}{g(n)}$

Motivating example: Algorithmic analysis

Example

Want to compare algorithms – particularly ones that can solve *arbitrarily large* instances.

We would like to be able to talk about the resources (running time, memory, energy consumption) required by a program/algorithm as a function $f(n)$ of some parameter n (e.g. the size) of its input.

e.g. How long does a given sorting algorithm take to run on a list of n elements?

Motivating example: Algorithmic analysis

Issues

- The exact resources required for an algorithm are difficult to pin down. Heavily dependent on:
  - Environment the program is run in (hardware, choice of language, external factors, etc)
  - Choice of inputs used
- Cost functions can be complex, e.g.

$$2n\log(n) + (n-100)\log(n)^2 + \frac{1}{2^n}\log(\log(n))$$

Need to identify the "important" aspects of the function.

Solution

Look at the **asymptotic growth**: how do the costs **scale** as n gets large?

"Big-O" Asymptotic Upper Bounds

Definition

Let $f, g : \mathbb{N} \to \mathbb{R}_{\geq 0}$. We say that g is *asymptotically less than* f (or: f **is an upper bound of** g) if there exists $n_0 \in \mathbb{N}$ and a real constant $c > 0$ such that for all $n \geq n_0$,

$$g(n) \leq c \cdot f(n)$$

Write $O(f(n))$ for the class of all functions g that are asymptotically less than f.

Example

$$g(n) = 3n + 1 \quad \Longrightarrow \quad g(n) \leq 4n\,, \text{ for all } n \geq 1$$

$$\text{Therefore, } 3n + 1 \in O(n)$$

Example

$$\frac{1}{10}n^2 \in O(n^2) \qquad 10n\log n \in O(n\log n) \qquad O(n\log n) \subsetneq O(n^2)$$

The traditional notation has been

$$g(n) = O(f(n))$$

instead of $g(n) \in O(f(n))$.

It allows one to use $O(f(n))$ or similar expressions as part of an equation; of course these 'equations' express only an approximate equality. Thus,

$$T(n) = 2 \cdot T\left(\frac{n}{2}\right) + O(n)$$

means

"There exists a function $f(n) \in O(n)$ such that $T(n) = 2T(\frac{n}{2}) + f(n)$."

Alternative definition

Fact

$$f(n) \in O(g(n)) \quad \textit{if and only if} \quad \lim_{n \to \infty} \frac{f(n)}{g(n)} < \infty.$$

Properties

Fact

Suppose $f(n) \in O(g(n))$, $g(n) \in O(h(n))$ *and* $j(n) \in O(k(n))$. *Then:*

- $f(n) \in O(h(n))$
- $f(n) + j(n) \in O(g(n) + k(n))$
- $f(n) \cdot j(n) \in O(g(n) \cdot k(n))$

Examples

Examples

$$5n^2 + 3n + 2 \in O(n^2)$$

$$n^3 + 2^{100}n^2 + 2n + 2^{2^{100}} \in O(n^3)$$

Generally, for constants $a_k \dots a_0$,

$$a_k n^k + a_{k-1} n^{k-1} + \dots + a_0 \in O(n^k)$$

“Big-Omega” Asymptotic Lower Bounds

Definition

Let $f, g : \mathbb{N} \to \mathbb{R}$. We say that g is *asymptotically greater than* f (or: f **is an lower bound of** g) if there exists $n_0 \in \mathbb{N}$ and a real constant $c > 0$ such that for all $n \geq n_0$,

$$g(n) \geq c \cdot f(n)$$

Write $\Omega(f(n))$ for the class of all functions g that are asymptotically greater than f.

Example

$$g(n) = 3n + 1 \quad \Longrightarrow \quad g(n) \geq 3n\,, \text{ for all } n \geq 1$$

$$\text{Therefore, } 3n + 1 \in \Omega(n)$$

“Big-Theta” Notation

Definition

Two functions f, g have the *same order of growth*, or are **asymptotically equivalent**, if they scale up in the same way: There exists $n_0 \in \mathbb{N}$ and real constants $c > 0, d > 0$ such that for all $n \geq n_0$,

$$c \cdot f(n) \leq g(n) \leq d \cdot f(n)$$

Write $\Theta(f(n))$ for the class of all functions g that have the same order of growth as f.

If $g \in O(f)$ (or $\Omega(f)$) we say that f is an *upper bound* (*lower bound*) on the order of growth of g; if $g \in \Theta(f)$ we call it a **tight bound**.

Properties

Observe that, somewhat symmetrically

$$g \in \Theta(f) \iff f \in \Theta(g)$$

We obviously have

$$\Theta(f(n)) \subseteq O(f(n)) \qquad \text{and} \qquad \Theta(f(n)) \subseteq \Omega(f(n)),$$

in fact

$$\Theta(f(n)) = O(f(n)) \cap \Omega(f(n)).$$

At the same time the 'Big-Oh' is *not* a symmetric relation

$$g \in O(f) \not\Rightarrow f \in O(g),$$

but

$$g \in O(f) \Leftrightarrow f \in \Omega(g)$$

Observations

Fact

- *For all* $k, \epsilon > 0$*:*

$$O((\log n)^k) \subsetneq O(n^\epsilon) \quad \textit{and} \quad O(n^k) \subsetneq O((1+\epsilon)^n).$$

- *All logarithms have the same order, irrespective of base:*

$$O(\log_2 n) = O(\log_3 n) = \ldots = O(\log_{10} n) = \ldots$$

- *Exponentials to different bases have different orders:*

$$O(r^n) \subsetneq O(s^n) \subsetneq O(t^n) \ldots \quad \textit{for} \quad r < s < t \ldots$$

- *Similarly for polynomials*

$$O(n^k) \subsetneq O(n^l) \subsetneq O(n^m) \ldots \quad \textit{for} \quad k < l < m \ldots$$

Examples

Examples

Here are some of the most common functions occurring in the analysis of the performance of programs (algorithm complexity), arranged in increasing asymptotic growth:

$$1,\ \log\log n,\ \log n,\ \sqrt{n},\ \sqrt{n}(\log n),\ n,\ n(\log\log n),\ n\log n, n\sqrt{n},$$
$$n^2,\ n^2\log n,\ n^3,\ n^{12},\ 2^{\sqrt{n}},\ 1.01^n,\ 2^n,\ 3^n,\ n!,\ n^n,\ 2^{n^2},\ldots$$

NB

$O(1) \equiv$ *const, although technically it could be any function that varies between two constants* c *and* d*.*

Exercises

Exercises

True or false?

$\boxed{\text{RW: 4.3.5}}$	(a)	$2^{n+1} \in O(2^n)$
	(b)	$(n+1)^2 \in O(n^2)$
	(c)	$2^{2n} \in O(2^n)$
	(d)	$(200n)^2 \in O(n^2)$
$\boxed{\text{RW: 4.3.6}}$	(b)	$\log(n^{73}) \in O(\log n)$
	(c)	$\log(n^n) \in O(\log n)$
	(d)	$(\sqrt{n}+1)^4 \in O(n^2)$

Exercises

Exercises

True or false?

RW: 4.3.5	(a)	$2^{n+1} \in O(2^n)$	True
	(b)	$(n+1)^2 \in O(n^2)$	True
	(c)	$2^{2n} \in O(2^n)$	False
	(d)	$(200n)^2 \in O(n^2)$	True
RW: 4.3.6	(b)	$\log(n^{73}) \in O(\log n)$	True
	(c)	$\log(n^n) \in O(\log n)$	False
	(d)	$(\sqrt{n}+1)^4 \in O(n^2)$	True