COMP3411/9814: Artificial Intelligence

1d. Constraint Satisfaction Problems

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February 15, 2024

Outline

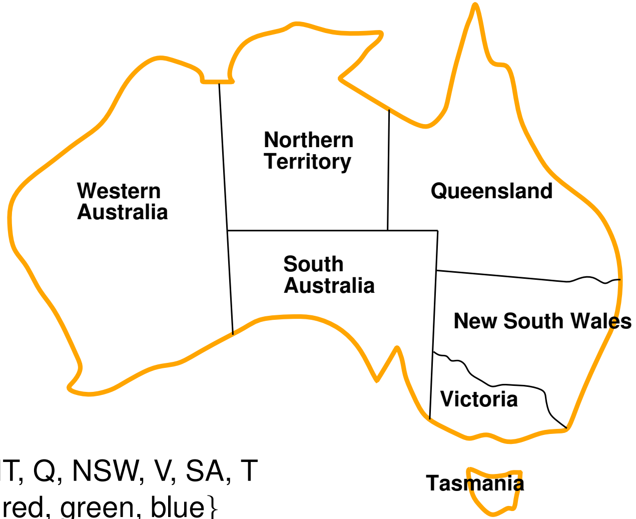
- ➢ Constraint Satisfaction Problems
- ➢ CSP examples
- ➢ backtracking search and heuristics
- ➢ forward checking and arc consistency
- ➢ local search
  - → hill climbing
  - → simulated annealing

Constraint Satisfaction Problems (CSPs)

Constraint Satisfaction Problems are defined by a set of *variables* X_i, each with a *domain* D_i of possible values, and a set of *constraints* C.

The aim is to find an assignment of the variables X_i from the domains D_i in such a way that none of the constraints C are violated.

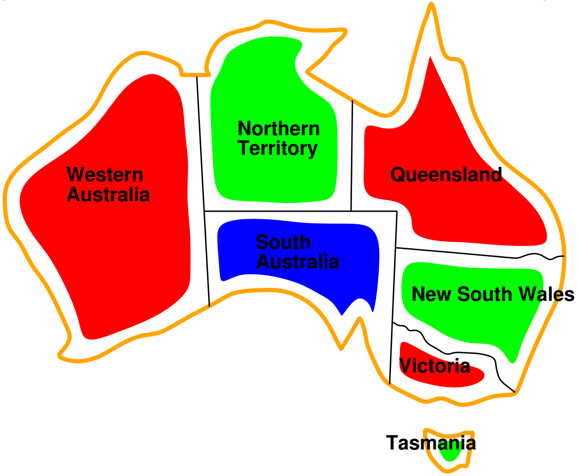
Example: Map-Coloring



Variables WA, NT, Q, NSW, V, SA, T
Domains $D_i = \{$red, green, blue$\}$
Constraints: adjacent regions must have different colors

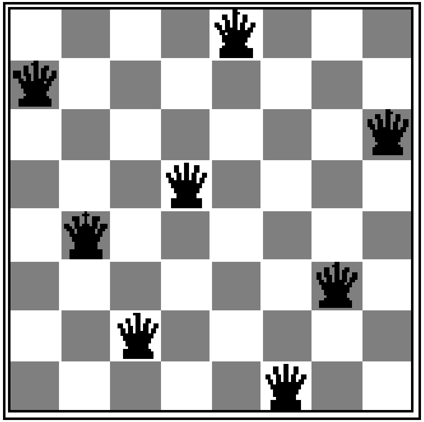
Example: Map-Coloring

Solution is an assignment that satisfies all the constraints, e.g.



{WA=red, NT=green, Q=red, NSW=green, V=red, SA=blue, T=green}

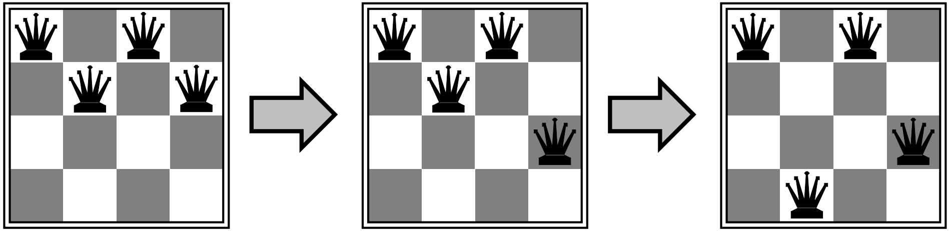
Example: n-Queens Puzzle



Put n queens on an n-by-n chess board so that no two queens are attacking each other.

n-Queens Puzzle as a CSP

Assume one queen in each column. Which row does each one go in?



Variables: Q_1, Q_2, Q_3, Q_4

Domains: $D_i = \{1, 2, 3, 4\}$

Constraints:

$Q_i \neq Q_j$ (cannot be in same row)

$|Q_i - Q_j| \neq |i - j|$ (or same diagonal)

Example: Cryptarithmetic

$$\begin{array}{ccccc} & \text{S} & \text{E} & \text{N} & \text{D} \\ + & \text{M} & \text{O} & \text{R} & \text{E} \\ \hline \text{M} & \text{O} & \text{N} & \text{E} & \text{Y} \end{array}$$

Variables:

D E M N O R S Y

Domains:

$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

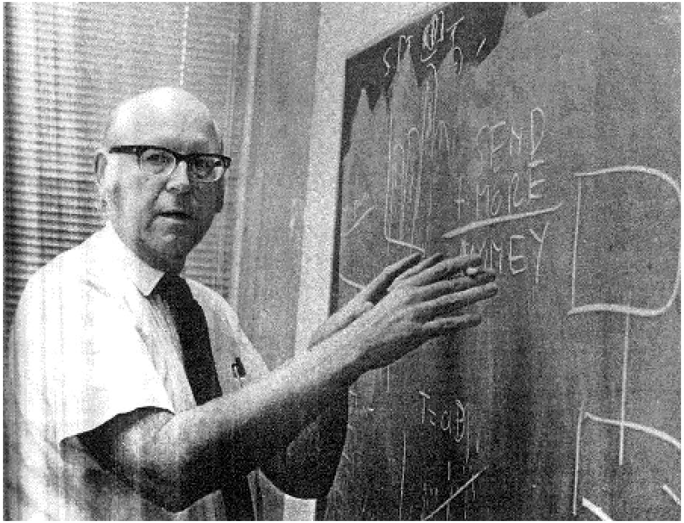
Constraints:

$M \neq 0$, $S \neq 0$ (*unary* constraints)

$Y = D + E$ or $Y = D + E - 10$, etc.

$D \neq E$, $D \neq M$, $D \neq N$, etc.

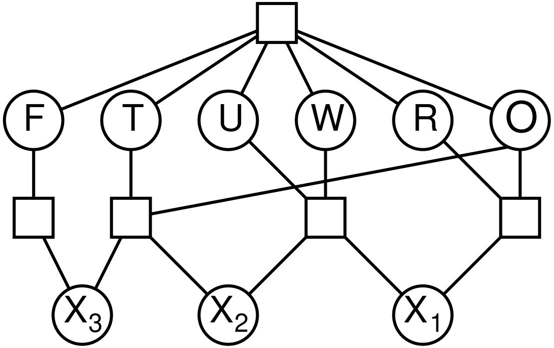
Cryptarithmetic with Allen Newell



Cryptarithmetic with Hidden Variables

We can add “hidden” variables to simplify the constraints.

```
    T W O
+   T W O
---------
F   O U R
```



Variables: F T U W R O $X_1 X_2 X_3$

Domains: $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

Constraints:

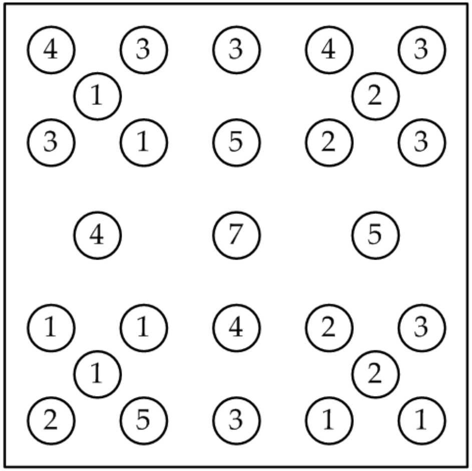
AllDifferent(F,T,U,W,R,O)

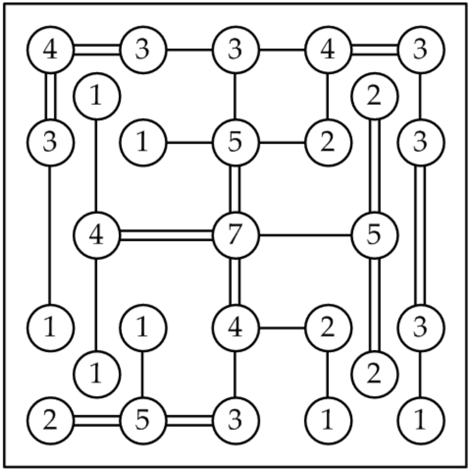
O + O = R + 10·X_1, etc.

Example: Sudoku

9				6				3
1		5		9	3	2		6
	4			5				9
8						4	7	1
		4	8	7				
7		2	6		1			8
2								
5				3	2		9	4
	8	7		1	6	3	5	

Example: Hashiwokakero





Real-World CSPs

- Assignment problems (e.g. who teaches what class)
- Timetabling problems (e.g. which class is offered when and where?)
- Hardware configuration
- Transport scheduling
- Factory scheduling

Varieties of constraints

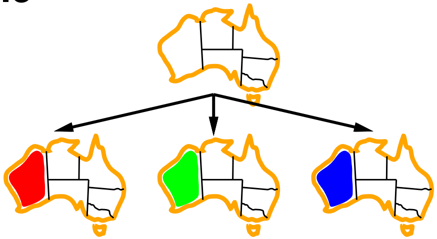
- *Unary* constraints involve a single variable
  - M $\neq 0$
- *Binary* constraints involve pairs of variables
  - SA $\neq$ WA
- *Higher-order* constraints involve 3 or more variables
  - $Y = D + E$ or $Y = D + E - 10$
- Inequality constraints on *Continuous* variables
  - $\text{EndJob}_1 + 5 \leq \text{StartJob}_3$
- *Soft constraints* (Preferences)
  - 11am lecture is better than 8am lecture!

Backtracking Search

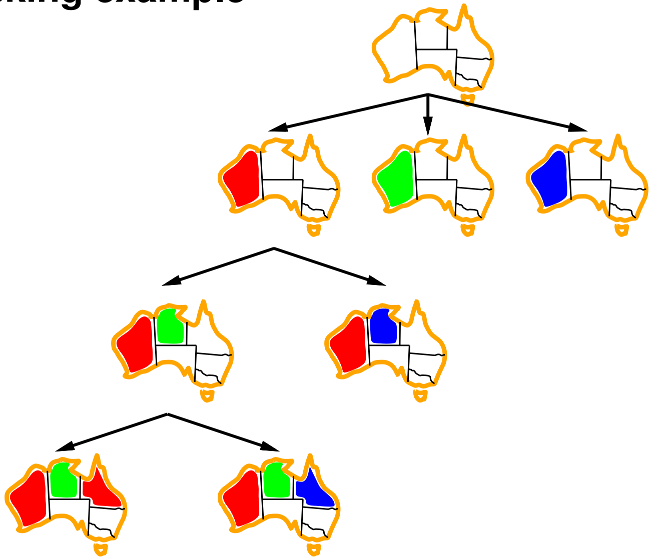
CSPs can be solved by assigning values to variables one by one, in different combinations. Whenever a constraint is violated, we go back to the most recently assigned variable and assign it a new value.
This can be achieved by a Depth First Search on a special kind of state space, where states are defined by the values assigned so far:

- ➢ *Initial state*: the empty assignment.
- ➢ *Successor function*: assign a value to an unassigned variable that does not conflict with previously assigned values of other variables. (If no legal values remain, the successor function fails.)
- ➢ *Goal test*: all variables have been assigned a value, and no constraints have been violated.

Backtracking example



Backtracking example



Path Search vs. Constraint Satisfaction

Important difference between Path Search Problems and CSP’s:

- ➛ Constraint Satisfaction Problems (e.g. n-Queens)
  - → difficult part is knowing the final state
  - → how to get there is easy
- ➛ Path Search Problems (e.g. Rubik’s Cube)
  - → knowing the final state is easy
  - → difficult part is how to get there

Backtracking search

The search space for this Depth First Search has certain very specific properties:

- ➛ if there are n variables, every solution will occur at exactly depth n.
- ➛ variable assignments are *commutative*

[WA = red then NT = green] same as [NT = green then WA = red]

Backtracking search can solve n-Queens for $n \approx 25$

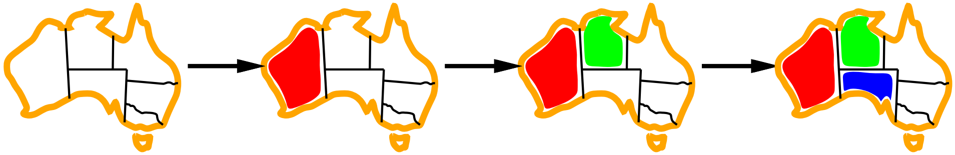
Improvements to Backtracking search

General-purpose heuristics can give huge gains in speed:

- ➛ which variable should be assigned next?
- ➛ in what order should its values be tried?
- ➛ can we detect inevitable failure early?

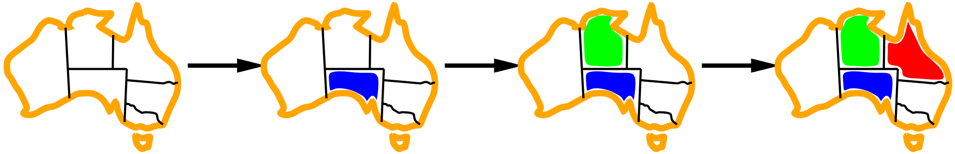
Minimum Remaining Values

Minimum Remaining Values (MRV):
Choose the variable with the fewest legal values.



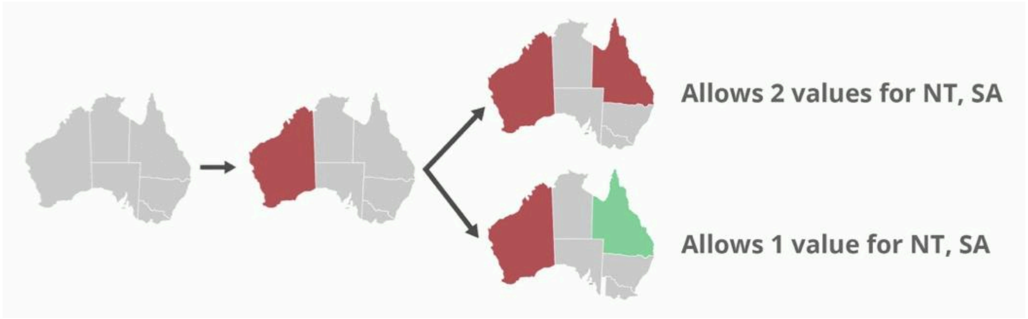
Degree Heuristic

Tie-breaker among MRV variables
Degree heuristic:
Choose the variable with the most constraints on remaining variables.



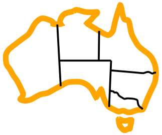
Least Constraining Value

Given a variable, choose the least constraining value:
the one that rules out the fewest values in the remaining variables



Forward Checking

Idea: Keep track of remaining legal values for unassigned variables
Terminate search when any variable has no legal values



WA	NT	Q	NSW	V	SA	T
🟥 🟩 🟦	🟥 🟩 🟦	🟥 🟩 🟦	🟥 🟩 🟦	🟥 🟩 🟦	🟥 🟩 🟦	🟥 🟩 🟦

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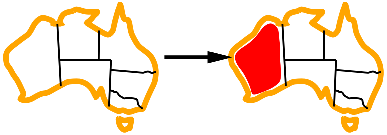

WA	NT	Q	NSW	V	SA	T
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🟥	🟩 🟦	🟥 🟩 🟦	🟥 🟩 🟦	🟥 🟩 🟦	🟩 🟦	🟥 🟩 🟦

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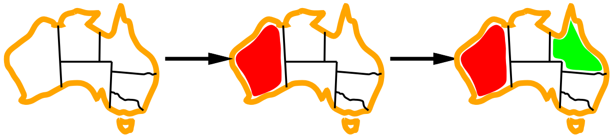

WA	NT	Q	NSW	V	SA	T
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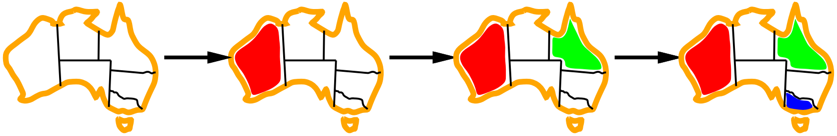

WA	NT	Q	NSW	V	SA	T
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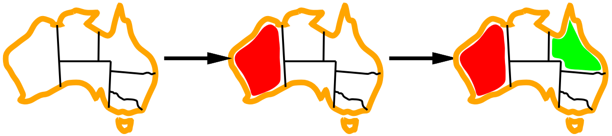
Constraint propagation

Forward Checking propagates information from assigned to unassigned variables, but doesn't provide early detection for all failures:



WA	NT	Q	NSW	V	SA	T
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🟥	🟦	🟩	🟥🟦	🟥🟩🟦	🟦	🟥🟩🟦

NT and *SA* cannot both be blue!

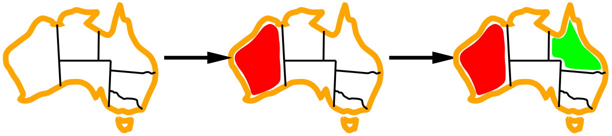
Constraint propagation repeatedly enforces constraints locally.

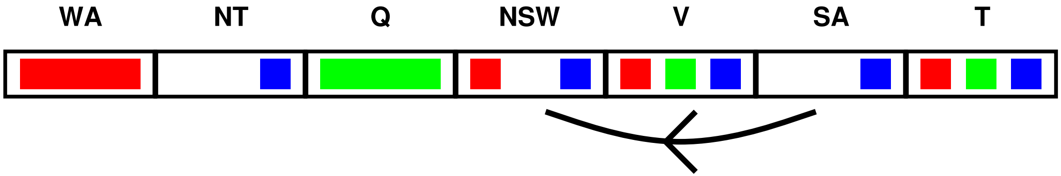
Arc consistency

Simplest form of constraint propagation makes each arc *consistent*

$X \rightarrow Y$ is consistent if

for *every* value x of X there is *some* allowed y





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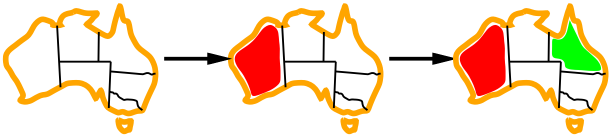

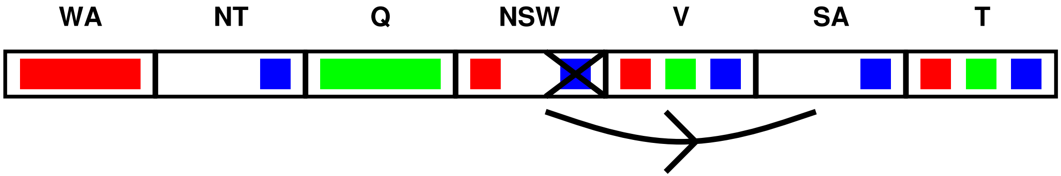

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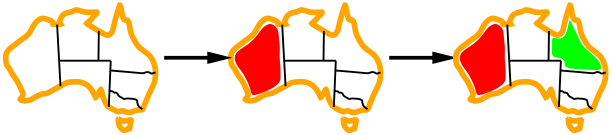

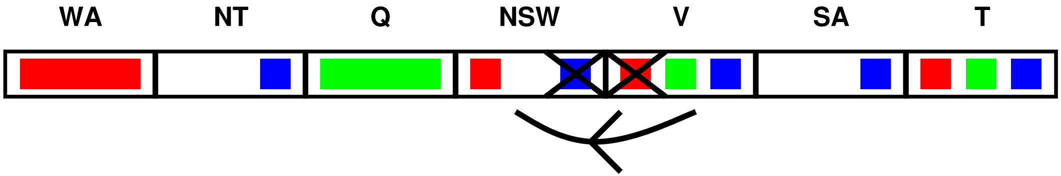

If X loses a value, neighbors of X need to be rechecked.

Arc consistency

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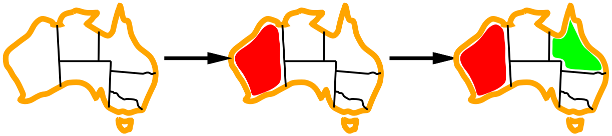
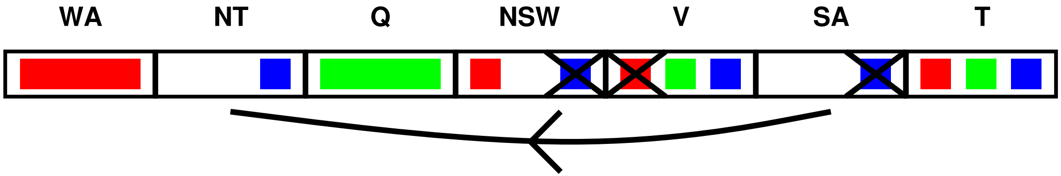


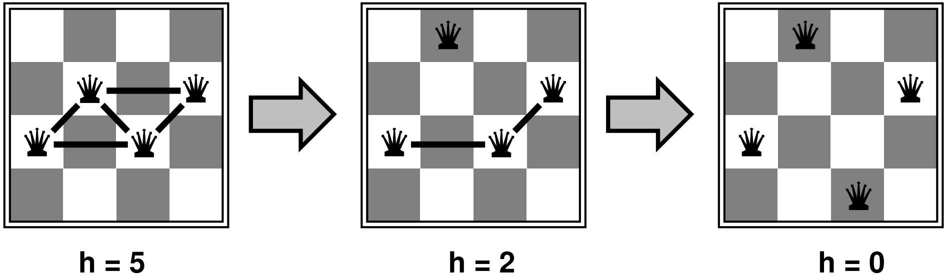


Arc consistency detects failure earlier than forward checking.
For some problems, it can speed up the search enormously.
For others, it may slow the search due to computational overheads.

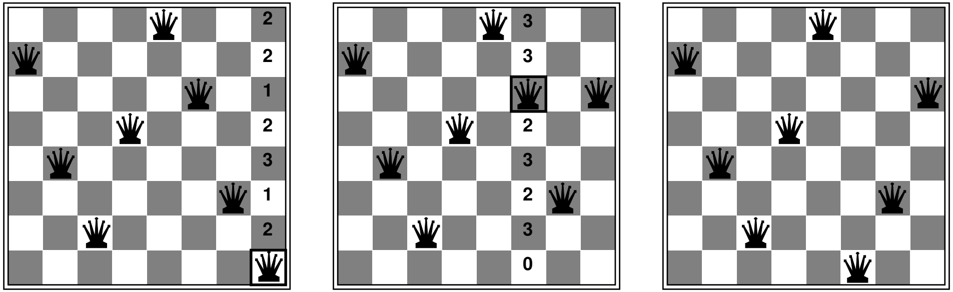
Local Search

There is another class of algorithms for solving CSP’s, called “Iterative Improvement” or “Local Search”.

These algorithms assign all variables randomly in the beginning (thus violating several constraints), and then change one variable at a time, trying to reduce the number of violations at each step.

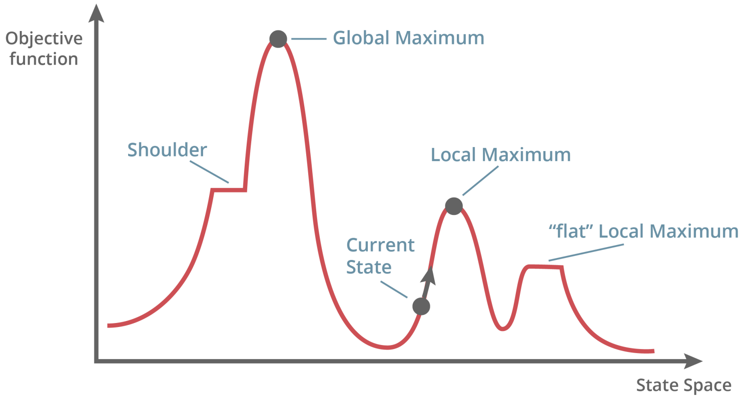


Hill-climbing by min-conflicts



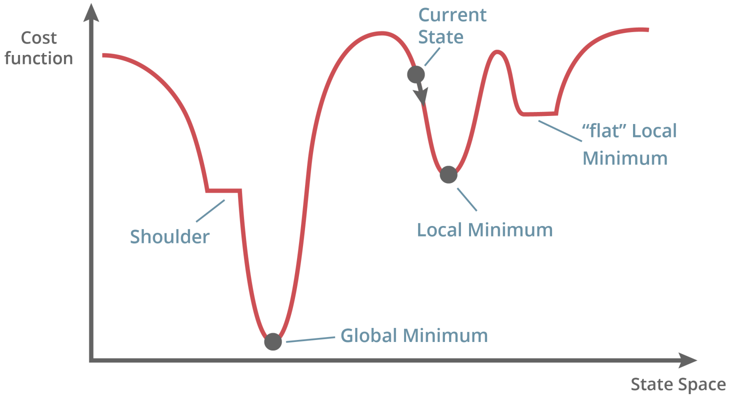
- ➛ Variable selection: randomly select any conflicted variable
- ➛ Value selection by *min-conflicts* heuristic
  - → choose value that violates the fewest constraints

Flat regions and local optima



Sometimes, have to go sideways or even backwards in order to make progress towards the actual solution.

Inverted View



When we are minimizing violated constraints, it makes sense to think of starting at the top of a ridge and climbing *down* into the valleys.

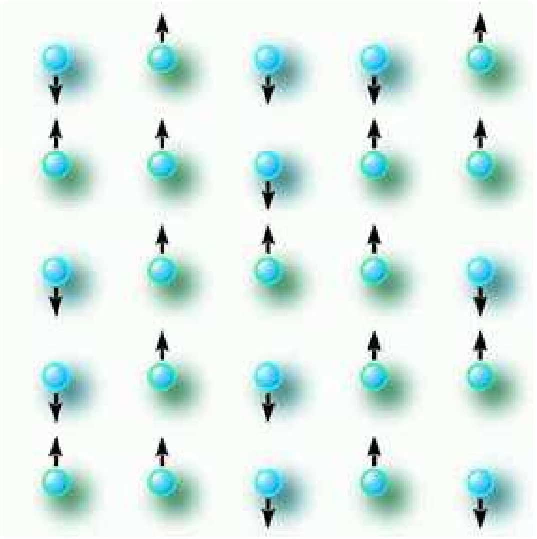
Simulated Annealing

- *stochastic* hill climbing based on difference between evaluation of previous state (h_0) and new state (h_1).
- if $h_1 < h_0$, definitely make the change
- otherwise, make the change with probability

$$e^{-(h_1 - h_0)/T}$$

  where T is a "temperature" parameter.
- reduces to ordinary hill climbing when $T = 0$
- becomes totally random search as $T \to \infty$
- sometimes, we gradually decrease the value of T during the search

Ising Model of Ferromagnetism

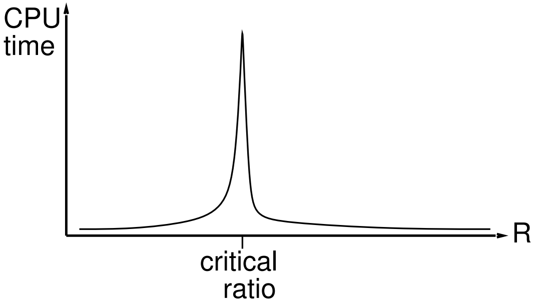


Phase transition in CSP's

Given random initial state, hill climbing by min-conflicts with random restarts can solve n-queens in almost constant time for arbitrary n with high probability (e.g., n = 10,000,000).

In general, randomly-generated CSP's tend to be easy if there are very few or very many constraints. They become extra hard in a narrow range of the ratio

$$R = \frac{\text{number of constraints}}{\text{number of variables}}$$



Summary

- Much interest in CSP’s for real-world applications
- Backtracking = depth-first search with one variable assigned per node
- Variable and Value ordering heuristics help significantly
- Forward Checking helps by detecting inevitable failure early
- Hill Climbing by min-conflicts often effective in practice
- Simulated Annealing can help to escape from local optima
- Which method(s) are best? It varies from one task to another!