Week 3: Autoregressive Moving Average(ARMA) Models

- Shumway, R.H. & Stoffer, D.S. (2016). Time series analysis and its applications with R examples. springer.
  - Chapter 3: ARIMA Models
- Brockwell, P.J., & Davis, R.A. (2009). Time series: theory and methods. Springer.
  - Chapter 3: Stationary ARMA Processes

Chapter 3

Autoregressive Moving Average(ARMA) Models

Classical regression is often insufficient for explaining all of the interesting dynamics of a time series. Instead, the introduction of correlation that may be generated through lagged linear relations leads to proposing the autoregressive (AR) and autoregressive moving average (ARMA) models.

3.1 What are ARMA Models?

- The classical regression model was developed for the static case, namely, we only allow the dependent variable to be influenced by current values of the independent variables.

- In the time series case, it is desirable to allow the dependent variable to be influenced by the past values of the independent variables and possibly by its own past values.

- If the present can be plausibly modeled in terms of only the past values of the independent inputs, we have the enticing prospect that forecasting will be possible.

3.1.1 Introduction to Autoregressive Models

Autoregressive models are based on the idea that the current value of the series, x_t, can be explained as a function of p past values, $x_{t-1}, x_{t-2}, \ldots, x_{t-p}$, where p determines the number of steps into the past needed to forecast the current value.

Definition 3.1 (AR(p)) *An autoregressive model of order p, abbreviated AR(p), is of the form*

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \ldots + \phi_p x_{t-p} + w_t, \tag{3.1.1}$$

where x_t is stationary, $w_t \sim wn(0, \sigma_w^2)$, and $\phi_1, \phi_2, \ldots, \phi_p$ are constants ($\phi_p \neq 0$). The mean of x_t in (3.1.1) *is zero.*

Note: If the mean, μ, of x_t is not zero, replace x_t by $x_t - \mu$:

$$x_t - \mu = \phi_1(x_{t-1} - \mu) + (\overset{\phi_2}{\cancel{\phi_2}} x_{t-2} - \mu) + \ldots + (\overset{\phi_p}{\cancel{\phi_p}} x_{t-p} - \mu) + w_t,$$

or write

$$x_t = \alpha + \underbrace{\phi_1 x_{t-1} + \phi_2 x_{t-2} + \ldots + \phi_p x_{t-p}} + w_t, \tag{3.1.2}$$

where $\alpha = \mu(1 - \phi_1 - \ldots - \phi_p)$.

- (3.1.2) is similar to the regression model, and hence the term auto (or self) regression.
- In regression models, the regressors, $x_{t-1}, \cdots, x_{t-p}$, are random components, whereas the predictors z_t was assumed to be fixed.

regression $\underbrace{x_t}_{\text{variable}} = f(\underbrace{z_t}_{\text{fixed}}) + \varepsilon_t$

A useful form follows by using the backshift operator to write (3.1.1) as

$Bx_t = x_{t-1}$

$B^k x_t = x_{t-k}$

$$(1 - \phi_1 B - \phi_2 B^2 - \ldots, \phi_p B^p) x_t = w_t, \tag{3.1.3}$$

or even more concisely as

$$\phi(B) x_t = w_t, \tag{3.1.4}$$

The properties of $\phi(B)$ are important in solving (3.1.4) for x_t. This leads to the following definition.

Definition 3.2 (Autoregressive operator) *The autoregressive operator is defined to be*

$$\phi(B) = (1 - \phi_1 B - \phi_2 B^2 - \ldots, \phi_p B^p). \tag{3.1.5}$$

Example 3.1 (The AR(1) Model) *The AR model of order 1 is given by* $x_t = \phi x_{t-1} + w_t$. *Iterating backwards k times, we get*

$$x_t = \ldots = \phi^k x_{t-k} + \sum_{j=0}^{k} \phi^j w_{t-j}$$

This method suggests that, by continuing to iterate backward, and provided that $|\phi| < 1$ *and* $\sup_t Var(x_t) < \infty$, *we can represent an AR(1) model as a linear process given by*

to make sure that $\sum_{j=0}^{\infty} |\phi^j| < \infty$

$$x_t = \sum_{j=0}^{\infty} \phi^j w_{t-j} \tag{3.1.6}$$

Representation (3.1.6) *is called the* **stationary solution of the model**. *In fact, by simple substitution,*

$$x_t = \sum_{j=0}^{\infty} \phi^j w_{t-j} = \phi \left(\sum_{k=0}^{\infty} \phi^k w_{t-1-k} \right) + w_t = \phi x_{t-1} + w_t$$

The AR(1) process defined by (3.1.6) *is stationary with mean*

$$E(x_t) = \sum_{j=0}^{\infty} \phi^j E(w_{t-j}) = 0,$$

$x_t = \sum_{j=0}^{\infty} \phi^j w_{t-j}$

and autocovariance function,

$$\gamma(h) = Cov(x_{t+h}, x_t) = \frac{\phi^h \sigma_w^2}{1 - \phi^2}, \qquad h \geq 0. \tag{3.1.7}$$

Recall that $\gamma(h) = \gamma(-h)$, so we will only exhibit the autocovariance function for $h \geq 0$.

From (3.1.7)*, the ACF of an AR(1) is*

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)} = \phi^h, \qquad h > 0, \tag{3.1.8}$$

and $\rho(h)$ *satisfies the recursion*

$$\rho(h) = \phi\rho(h-1), \qquad h = 1, 2, \ldots. \tag{3.1.9}$$

We will discuss the ACF of a general AR(p*) model later.*

Example 3.2 (The Sample Path of an AR(1) Process) *Figure 3.1 shows a time plot of two AR(1) processes, one with* ϕ *= .9 and one with* ϕ *= -.9; in both cases,* $\sigma_w^2 = 1$. *In the first case,* $\rho(h) = .9^h$, *for* $h \geq 0$, *so observations close together in time are positively correlated with each other. This result means that observations at contiguous time points will tend to be close in value to each other. This fact shows up in the top of Figure 3.1 as a very smooth sample path for* x_t.

Now, contrast this with the case in which $\phi = -.9$, *so that* $\rho(h) = (-.9)^h$, *for* $h \geq 0$. *This result means that observations at contiguous time points are negatively correlated but observations two time points apart are positively correlated. This fact shows up in the bottom of Figure 3.1. In this case, the sample path is very choppy.*

$x_t = \phi x_{t-1} + w_t$

$\rho(h)$

$= \phi^h$

$= \phi \rho(h-1)$

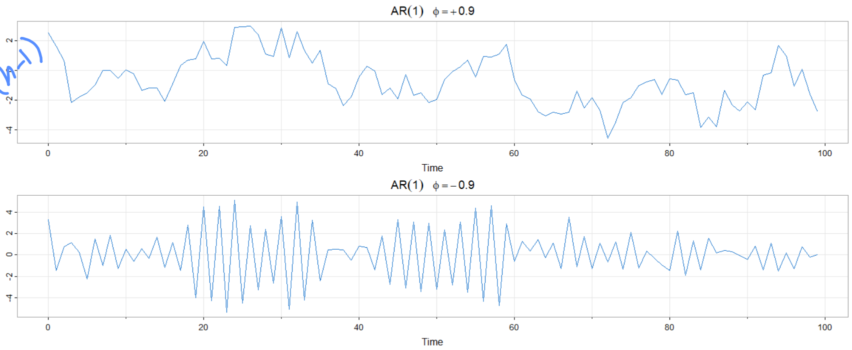

Figure 3.1: Simulated AR(1) models: ϕ = .9 (top); ϕ = -.9 (bottom)

- The technique of iterating backward to get an idea of the stationary solution of AR models works well when $p = 1$.

- A general technique is **matching coefficients**.

$x_t = \phi x_{t-1} + w_t$

$(1 - B\phi) x_t = w_t$

- Consider the AR(1) model in operator form $\phi(B)x_t = w_t$, where $\phi(B) = 1 - \phi B$ and $|\phi| < 1$. Also, write the model as

$$x_t = \psi(B)w_t \tag{3.1.10}$$

linear process

where $\psi(B) = \sum_{j=0}^{\infty} \psi_j B^j$ and $\psi_j = \phi^j$.

$$x_t = \sum_{j=0}^{\infty} \phi^j w_{t-j} = \sum_{j=0}^{\infty} \phi^j B^j w_t = \psi(B) w_t$$

Suppose we did not know that $\psi_j = \phi^j$. We could substitute $\psi(B)w_t$ from (3.1.10) for x_t to obtain

$$\phi(B)x_t = w_t \qquad \Rightarrow \qquad \phi(B)\psi(B)w_t = w_t \tag{3.1.11}$$

The coefficients of B on the left-hand side of (3.1.11) must be equal to those on right-hand side of (3.1.11), which means

$$(1 - \phi B)(1 + \psi_1 B + \psi_2 B^2 + \ldots + \psi_j B^j + \ldots) = 1 \tag{3.1.12}$$

or equivalently,

$$1 + (\psi_1 - \phi)B + (\psi_2 - \psi_1\phi)B^2 + \ldots + (\psi_j - \psi_{j-1}\phi)B^j + \ldots = 1.$$

Therefore,

$$
\begin{aligned}
\psi_1 - \phi = 0 \quad &\Rightarrow \quad \psi_1 = \phi \\
\psi_2 - \psi_1\phi = 0 \quad &\Rightarrow \quad \psi_2 = \phi^2 \\
&\vdots \\
\psi_j - \psi_{j-1}\phi = 0 \quad &\Rightarrow \quad \psi_j = \psi_{j-1}\phi = \phi^j.
\end{aligned}
$$

Note that $\psi_0 = 1$.

this idea can be extended to AR(p)

Another way to think about the operations:

- Consider the AR(1) model in operator form, $\phi(B)x_t = w_t$.
- Multiply both sides by $\phi^{-1}(B)$ (assuming the inverse operator exists) to get:

$$\cancel{\phi^{-1}(B)}\cancel{\phi(B)}x_t = \phi^{-1}(B)w_t \qquad \Rightarrow \qquad \boxed{x_t = \phi^{-1}(B)w_t}$$

- We know already that

$$\phi^{-1}(B) = 1 + \phi B + \phi^2 B^2 + \ldots + \phi^j B^j + \ldots,$$

that is, $\phi^{-1}(B) = \psi(B)$.

- Working with operators is like working with polynomials.

in AR(1)

$$\phi(B) = 1 - \phi B \Rightarrow \phi^{-1}(B) = \frac{1}{1-\phi B} = \sum_{j=0}^{\infty} (\phi B)^j$$

Therefore, consider the polynomial $\phi(z) = 1 - \phi z$, where z is a complex number and $|\phi| < 1$. Then,

$$\phi^{-1}(z) = 1 + \phi z + \phi^2 z^2 + \ldots + \phi^j z^j + \ldots, \qquad |z| \leq 1.$$

and the coefficients of B^j in $\phi^{-1}(B)$ are the same as the coefficients of z^j in $\phi^{-1}(z)$.

These results will be generalized in our discussion of ARMA models. We will find the polynomials corresponding to the operators useful in exploring the general properties of ARMA models.

$x_t = \phi x_{t-1} + w_t \rightarrow \phi = 1 \Rightarrow$ Random walk

Note 3.1 • *Consider the random walk $x_t = x_{t-1} + w_t$. This process is not stationary. In general. we might wonder if there is an stationary AR(1) process with $|\phi| \geq 1$. Such processes are called* **explosive** *because the values of the time series quickly become large in magnitude.*

- *An explosive process $x_t = \phi x_{t-1} + w_t$ with $|\phi| > 1$ can be rewritten as*

$$x_t = -\sum_{j=1}^{\infty} \phi^{-j} w_{t+j}.$$

- *Unfortunately, this model is useless. Why?* X

- *When a process does not depend on the future, such as the AR(1) when $|\phi| < 1$, we will say the process is causal. In the explosive case, the process is stationary, but it is also future dependent, and not causal.*

3.1.2 Introduction to Moving Average Models

The moving average model of order q, abbreviated as MA(q), assumes the white noise w_t on the right-hand side of the defining equation are combined linearly to form the observed data.

Definition 3.3 (MA(q)) *The moving average model of order q is defined to be*

$$x_t = w_t + \theta_1 w_{t-1} + \theta_2 w_{t-2} + \ldots + \theta_q w_{t-q}, \tag{3.1.13}$$

where $w_t \sim wn(0, \sigma_w^2)$, *and* $\theta_1, \theta_2, \ldots, \theta_q$ $(\theta_q \neq 0)$ *are parameters.*

$\theta_q = 0 \Rightarrow MA(q-1)$

The system is the same as the infinite moving average defined as the linear process (3.1.10), where $\psi_0 = 1$, $\psi_j = \theta_j$, for $j = 1, \ldots, q$, and $\psi_j = 0$ for other values. We may also write the MA(q) process in the equivalent form

$$x_t = \theta(B)w_t, \tag{3.1.14}$$

using the following definition.

Definition 3.4 (Moving average operator) *The moving average operator is*

$$\theta(B) = 1 + \theta_1 B + \theta_2 B^2 + \ldots + \theta_q B^q, \tag{3.1.15}$$

Unlike the autoregressive process, the moving average process is stationary for any values of the parameters $\theta_1, \ldots, \theta_q$.

Example 3.3 (The MA(1) Process) *Consider the MA(1) model* $x_t = w_t + \theta w_{t-1}$. *Then,* $E(x_t) = 0$,

$Cov(x_{t+h}, x_t)$

- use the auto-cov of MA

$$\gamma(h) = \begin{cases} (1+\theta^2)\sigma_w^2 & h = 0, \\ \theta\sigma_w^2 & h = 1, \\ 0 & h > 1 \end{cases}$$

$\theta = 5 \Rightarrow \rho(1) = 5/26$

$\theta = 1/5 \Rightarrow \rho(1) = 5/26$

and the ACF is

$$\frac{\gamma(h)}{\gamma(0)} = \rho(h) = \begin{cases} 1 & h = 0, \\ \frac{\theta}{(1+\theta^2)} & h = 1, \\ 0 & h > 1 \end{cases}$$

$x_t = w_t + \theta w_{t-1}$

$x_{t-1} = w_{t-1} + \theta w_{t-2}$

$x_{t-2} = w_{t-2} + \theta w_{t-3}$

- $|\rho(1)| \leq 1/2$ *for all values of* θ.

- x_t *is correlated with* x_{t-1}, *but not with* $x_{t-2}, x_{t-3}, \ldots$. *Contrast this with the case of the AR(1) model in which the correlation between* x_t *and* x_{t-k} *is never zero.*

- *When* $\theta = .9$, *for example,* x_t *and* x_{t-1} *are positively correlated, and* $\rho(1) = .497$. *When* $\theta = -.9$, x_t *and* x_{t-1} *are negatively correlated,* $\rho(1) = -.497$.

Figure 3.2 shows a time plot of these two processes with $\sigma_w^2 = 1$. *The series for which* $\theta = .9$ *is smoother than the series for which* $\theta = .9$.

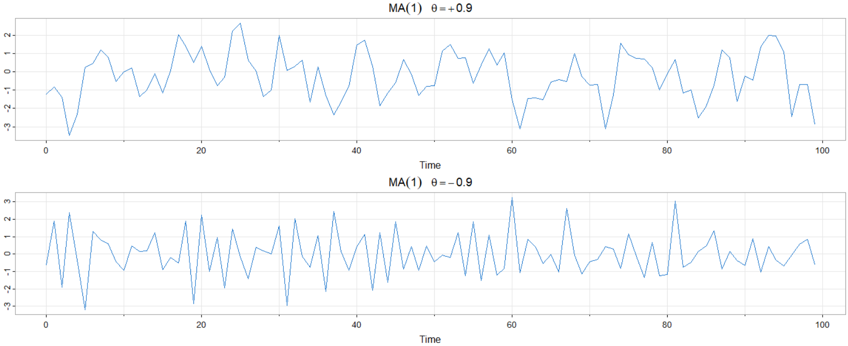

Figure 3.2: Simulated MA(1) models: θ = .9 (top); θ = -.9 (bottom)

Note 3.2 • *In general, there is no restriction on the value of θ in MA(1) process. But, it can be shown that without any restriction different MA(1) processes will end up with the same ACF. (Check the previous example with $\theta = 5$ and $\theta = 1/5$.)*

- *To get a unique representation for the process, we will choose the model with an infinite AR representation. Such a process is called an* **invertible** *process. In general, a time series is invertible if errors can be inverted into a representation of past observations.*

- *To discover which model is the invertible model, we can reverse the roles of x_t and w_t and write the MA(1) model as $w_t = -\theta w_{t-1} + x_t$. It can be shown that, if $|\theta| < 1$, then $w_t = -\sum_{j=0}^{\infty}(-\theta)^j x_{t-j}$, which is the desired infinite AR representation of the model.*

AR(p) $\Rightarrow$ $x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \cdots + \phi_p x_{t-p} + w_t$

$\Rightarrow$ $w_t = x_t - \phi_1 x_{t-1} - \cdots - \phi_p x_{t-p}$ $\Rightarrow$ are invertible

As in the AR case, the polynomial, $\theta(z)$, will be useful in exploring general properties of MA processes. $w_t + \theta w_{t-1} = (1+\theta B) w_t = \theta(B) w_t$

For example, following the steps of equations (3.1.10)-(3.1.12), we can write the MA(1) model as $x_t = \theta(B)w_t$, where $\theta(B) = 1 + \theta B$. If $|\theta| < 1$, then we can write the model as $\pi(B)x_t = w_t$, where $\pi(B) = \theta^{-1}(B)$. Let $\theta(z) = 1 + \theta z$, for $|z| \leq 1$, then

$$\pi(z) = \theta^{-1}(z) = \frac{1}{1 + \theta z} = \sum_{j=0}^{\infty} (-\theta)^j z^j,$$

3.1.3 Autoregressive Moving Average Models

Now, mix autoregressive and moving average models to form ARMA models for stationary time series.

Definition 3.5 *[ARMA(p, q)] A time series $\{x_t;\ t = 0, \pm 1, \pm 2, \ldots\}$ is ARMA(p, q) if it is stationary and*

$$x_t = \overbrace{\phi_1 x_{t-1} + \ldots + \phi_p x_{t-p}}^{AR(p)} + w_t + \overbrace{\theta_1 w_{t-1} + \ldots + \theta_q w_{t-q}}^{MA(q)}, \tag{3.1.16}$$

with $\phi_p \neq 0, \theta_q \neq 0$ and $\sigma_w^2 > 0$. The parameters p and q are called the autoregressive and the moving average orders, respectively. If x_t has a nonzero mean μ, we set $\alpha = \mu(1 - \phi_1 - \ldots - \phi_p)$ and write the model as

$$x_t = \alpha + \phi_1 x_{t-1} + \ldots + \phi_p x_{t-p} + w_t + \theta_1 w_{t-1} + \ldots + \theta_q w_{t-q}, \tag{3.1.17}$$

where $w_t \sim wn(0, \sigma_w^2)$

- When $q = 0$, the model is called an autoregressive model of order p, AR(p).

- When $p = 0$, the model is called a moving average model of order q, MA(q).

- The ARMA(p, q) model in (3.1.16) can then be written in concise form as

$$\phi(B)x_t = \theta(B)w_t. \tag{3.1.18}$$

There are some problems with the general definition of ARAM(p, q) process, such as,

(i) parameter redundant models,

(ii) stationary AR models that depend on the future,

(iii) MA models that are not unique.

To overcome these problems, we will require some additional restrictions on the model parameters.

Definition 3.6 *The AR and MA polynomials are defined as*

$$\phi(z) = 1 - \phi_1 z - \ldots - \phi_p z^p, \qquad \phi_p \neq 0 \tag{3.1.19}$$

and

$$\theta(z) = 1 + \theta_1 z + \ldots + \theta_q z_q, \qquad \theta_q \neq 0, \tag{3.1.20}$$

respectively, where z is a complex number.

solution to parameter redundant models:

To address the first problem, we will henceforth refer to an ARMA(p, q) model to mean that it is in its simplest form. That is, in addition to the original definition given in equation (3.1.16), we will also require that $\phi(z)$ and $\theta(z)$ have no common factors.

To address the problem of future-dependent models, we formally introduce the concept of **causality**.

Definition 3.7 (Causality) *An ARMA(p, q) model is said to be causal, if the time series $\{x_t;\ t = 0, \pm 1, \pm 2, \ldots\}$ can be written as a one-sided linear process:*

$$x_t = \sum_{j=0}^{\infty} \psi_j w_{t-j} = \psi(B) w_t \qquad (3.1.21)$$

just depends on past

where $\psi(B) = \sum_{j=0}^{\infty} \psi_j B^j$ *and* $\sum_{j=0}^{\infty} |\psi_j| \overset{<}{\cancel{>}} \infty$.; *we set* $\psi_0 = 1$.

Property 3.1 *[Causality of an ARMA(p, q) Process] An ARMA(p, q) model is causal if and only if $\phi(z) \neq 0$ for $|z| \leq 1$. The coefficients of the linear process given in 3.1.21 can be determined by solving*

$$\psi(z) = \sum_{j=0}^{\infty} \psi_j z^j = \frac{\theta(z)}{\phi(z)}, \qquad |z| \leq 1.$$

Another way to phrase Property 3.1 is that an ARMA process is causal only when the roots of $\phi(z)$ lie outside the unit circle; that is, $\phi(z) = 0$ only when $|z| > 1$.

ARMA(p,q) $\rightarrow$ $\phi(B)x_t = \theta(B)w_t \Rightarrow x_t = \frac{\theta(B)}{\phi(B)} w_t$

$\sum_{j=0}^{\infty} \psi_j B^j$

Finally, to address the problem of uniqueness problem, we choose the model that allows an infinite autoregressive representation.

Definition 3.8 (Invertibility) *An ARMA*(p, q) *model is said to be invertible, if the time series* $\{x_t; t = 0, \pm 1, \pm 2, \ldots\}$ *can be written as*

$$\pi(B)x_t = \sum_{j=0}^{\infty} \pi_j x_{t-j} = w_t, \tag{3.1.22}$$

where $\pi(B) = \sum_{j=0}^{\infty} \pi_j B^j$ *and* $\sum_{j=0}^{\infty} |\pi_j| < \infty$*; we set* $\pi_0 = 1$.

Property 3.2 *[Invertibility of an ARMA(p, q) Process] An ARMA(p, q) model is invertible if and only if $\theta(z) \neq 0$ for $|z| \leq 1$. The coefficients π_j of $\pi(B)$ given in* (3.1.22) *can be determined by solving*

$$\pi(z) = \sum_{j=0}^{\infty} \pi_j z^j = \frac{\phi(z)}{\theta(z)}, \qquad |z| \leq 1.$$

Another way to phrase Property 3.2 is that an ARMA process is invertible only when the roots of $\theta(z)$ lie outside the unit circle; that is, $\theta(z) = 0$ only then $|z| > 1$.

$$\phi(B) x_t = \theta(B) w_t$$

$$\Rightarrow w_t = \frac{\phi(B)}{\theta(B)} x_t \rightarrow \sum_{j=0}^{\infty} \pi_j B^j$$

Example 3.4 (Parameter Redundancy, Causality, Invertibility) *Consider the process*

$$x_t = .4x_{t-1} + .45x_{t-2} + w_t + w_{t-1} + .25w_{t-2},$$

ARMA(2,2)

Bx_t B^2x_t Bw_t B^2w_{t-2}

or, in operator form,

$$(1 - .4B - .45B^2)x_t = (1 + B + .25B^2)w_t.$$

AR polynomial $1-0.4z-0.45z^2$

$1+z+0.25z^2$ MA polynomial

- *At first, x_t appears to be an ARMA(2, 2) process. But notice that*

  $$\phi(B) = 1 - .4B - .45B^2 = (1 + .5B)(1 - .9B)$$

  and

  $$\theta(B) = (1 + B + .25B2) = (1 + .5B)^2$$

  have a common factor that can be canceled.

$(1-0.9B)x_t = (1+0.5B)w_t$

- *After cancellation, the operators are $\phi(B) = (1 - .9B)$ and $\theta(B) = (1 + .5B)$, so the model is an ARMA(1, 1) model, $(1 - .9B)x_t = (1 + .5B)w_t$, or*

$$x_t = .9x_{t-1} + .5w_{t-1} + wt. \tag{3.1.23}$$

The model is causal because $\phi(z) = (1 - .9z) = 0$ when $z = 10/9$, which is outside the unit circle. AR polynomial outside the unit circle

- *The model is also invertible because the root of $\theta(z) = (1 + .5z)$ is $z = -2$, which is outside the unit circle.*

MA polynomial

- *To write the model as a linear process, we can obtain the ψ-weights using Property 3.1, $\phi(z)\psi(z) = \theta(z)$, or*

$$(1 - .9z)(1 + \psi_1 z + \psi_2 z^2 + \ldots + \psi_j z^j + \ldots) = 1 + .5z.$$

Rearranging, we get

$$1 + \boxed{(\psi_1 - .9)}z + \underbrace{(\psi_2 - .9\psi_1)}_{=0}z^2 + \ldots + \underbrace{(\psi_j - .9\psi_{j-1})}_{=0}z^j + \ldots = 1 + \boxed{.5}z.$$

Matching the coefficients of z on the left and right sides we get $\psi_1 - .9 =$ $.5$ and $\psi_j - .9\psi_{j-1} = 0$ for $j > 1$. Thus, $\psi_j = 1.4(.9)^{j-1}$ for $j \geq 1$ and (3.1.23) *can be written as*

$$x_t = w_t + 1.4 \sum_{j=1}^{\infty} .9^{j-1} w_{t-j}.$$

linear representation for x_t

- *The invertible representation using Property 3.2 is obtained by matching coefficients in $\theta(z)\pi(z) = \phi(z)$,*

$$(1 + .5z)(1 + \pi_1 z + \pi_2 z^2 + \pi_3 z^3 + \ldots) = 1 - .9z.$$

In this case, the π-weights are given by $\pi_j = (-1)^j 1.4(.5)^{j-1}$, for $j \geq 1$, and hence, because $w_t = \sum_{j=0}^{\infty} \pi_j x_{t-j}$, we can also write (3.1.23) *as*

$$x_t = 1.4 \sum_{j=1}^{\infty} (-.5)^{j-1} x_{t-j} + w_j$$

invertible form for x_t

$$1 + \underbrace{(0.5 + \pi_1)}\, z + \underbrace{(\pi_2 + 0.5\pi_1)}_{=0} z^2 + \cdots = 1 - 0.9z$$

```
ARMAtoMA(ar = .9, ma = .5, 10)    # first 10 psi-weights
# [1] 1.4000000 1.2600000 1.1340000 1.0206000 0.9185400
    0.8266860 0.7440174 0.6696157
# [9] 0.6026541 0.5423887

ARMAtoAR(ar = .9, ma = .5, 10)    # first 10 pi-weights
# [1] -1.400000000  0.700000000 -0.350000000  0.175000000
    -0.087500000  0.043750000
# [7] -0.021875000  0.010937500 -0.005468750  0.002734375
```

Listing 3.1: The code to calculte ψ_j in (3.1.21)

$$x_t = \phi_1 x_{t-1} + \cdots + \phi_p x_{t-p} + w_t$$

$$\Rightarrow x_t = \phi_1 B x_t + \cdots + \phi_p B^p x_t + w_t$$

$$\Rightarrow \underbrace{(1 - \phi_1 B - \cdots - \phi_p B^p)}_{\phi(B)} x_t = w_t$$

$$\Rightarrow \phi(B) x_t = w_t$$

Example 3.1

$$x_t = \phi x_{t-1} + w_t = \phi(\phi x_{t-2} + w_{t-1}) + w_t$$

$$= \phi^2 x_{t-2} + \phi w_{t-1} + w_t$$

$$\vdots$$

$$= \phi^k x_{t-k} + \sum_{j=0}^{k} \phi^j w_{t-j} \checkmark$$

$$\xrightarrow{k \to \infty} \sum_{j=0}^{\infty} \phi^j w_{t-j}$$

<u>linear process</u>

$$x_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j w_{t-j} \quad \text{with} \quad \sum_{j=-\infty}^{\infty} |\psi_j| < \infty$$

$$\xrightarrow{\text{causal}} x_t = \mu + \sum_{j=0}^{\infty} \psi_j w_{t-j} \quad \text{with} \quad \sum |\psi_j| < \infty$$

AR(1) $x_t = \phi x_{t-1} + w_t$

$$\gamma(h) = Cov(x_{t+h}, x_t) = Cov\left(\sum_{j=0}^{\infty} \phi^j w_{t+h-j}, \sum_{k=0}^{\infty} \phi^k w_{t-k}\right)$$

$$= \sum_j \sum_k \phi^j \phi^k \, Cov(w_{t+h-j}, w_{t-k})$$

$w_t \sim WN(0, \sigma_w^2)$

$$= \sum_{k=0}^{\infty} \phi^{h+k} \phi^k \sigma_w^2 = \phi^h \sigma_w^2 \sum_{k=0}^{\infty} \phi^{2k} \hookrightarrow |\phi|<1$$

$$\sum_{j=b}^{\infty} a^j = \frac{a^b}{1-a} \text{ if } |a|<1$$

$$= \phi^h \sigma_w^2 \frac{1}{1-\phi^2} \checkmark$$

$$Cov(w_{t+h-j}, w_{t-k}) = \begin{cases} 0 & \text{if } t+h-j \neq t-k \\ \sigma_w^2 & \text{if } t+h-j = t-k \Rightarrow j = h+k \end{cases}$$

AR(2)

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + w_t$$

$$= \phi_1(\phi_1 x_{t-2} + \phi_2 x_{t-3} + w_{t-1})$$

$$+ \phi_2(\phi_1 x_{t-3} + \phi_2 x_{t-4} + w_{t-2}) + w_t$$

matching coefficients

AR(1)

$$\phi(B)\,x_t = w_t \implies \phi(B)\,\psi(B)\,w_t = w_t \quad *$$

$(1+\psi_1 B + \dots)$, ψ_j are unknown

$$x_t = \psi(B)\,w_t$$

$$* \quad \phi(B)\,\psi(B) = 1$$

$$\implies (1-\phi B)(1+\psi_1 B + \psi_2 B^2 + \dots + \psi_j B^j + \dots) = 1$$

$$\implies 1 + (\underbrace{\psi_1 - \phi}_{=0})B + (\underbrace{\psi_2 - \phi\psi_1}_{=0})B^2 + \dots = 1$$

explosive processes : Do they have linear presentation ?!

$$x_t = \phi x_{t-1} + w_t \qquad |\phi| > 1$$

$$\implies x_{t-1} = \phi^{-1} x_t + \phi^{-1} w_t$$

$$\implies x_t = \phi^{-1} x_{t+1} + \phi^{-1} w_{t+1}$$

$$= \phi^{-1}\left[\phi^{-1} x_{t+2} + \phi^{-1} w_{t+2}\right] + \phi^{-1} w_{t+1}$$

$$\vdots$$

$$= \phi^{-k} x_{t+k} - \sum_{j=1}^{k} (\phi^{-j})\, w_{t+j} \xrightarrow{k\to\infty} -\sum_{j=1}^{\infty} \phi^{-j} w_{t+j}$$

MA(q)

linear process $\quad x_t = \sum_{j=0}^{\infty} \psi_j w_{t-j}$

MA(q) $\quad x_t = w_t + \theta_1 \underbrace{w_{t-1}}_{Bw_t} + \cdots + \theta_q \underbrace{w_{t-q}}_{B^q w_t}$ *

$\Rightarrow$ MA(q) is a linear process

$$\psi_0 = 1 \quad \psi_1 = \theta_1 , \ldots , \psi_q = \theta_q$$

$$\psi_k = 0 \quad , k > q$$

$$* \Rightarrow \quad x_t = \underbrace{(1 + \theta_1 B + \cdots + \theta_q B^q)}_{\text{moving average operator } \theta(B)} w_t$$

moving average operator $\theta(B)$

MA(1) invertible

$$x_t = w_t + \theta w_{t-1} \Rightarrow w_t = x_t - \theta w_{t-1}$$

$$\begin{aligned}
\Rightarrow w_t &= x_t - \theta w_{t-1} \\
&= x_t - \theta(x_{t-1} - \theta w_{t-2}) = x - \theta x_{t-1} + \theta^2 w_{t-2} \\
&\vdots \\
&= -\sum_{j=0}^{\infty} (-\theta)^j x_{t-j} \qquad |\theta| < 1 \checkmark
\end{aligned}$$

Parameter redundant models

$$x_t = w_t \quad \text{where} \quad w_t \sim WN(0, \sigma_w^2)$$

$$(1-B)\, x_t = (1-B)\, w_t \Rightarrow x_t - x_{t-1} = w_t - w_{t-1}$$

$$\Rightarrow x_t = x_{t-1} + w_t - w_{t-1} \Rightarrow ARMA(1,1)$$

Example 3.4

write the ARMA(1,1) as a linear process

$$\phi(B)\, x_t = \theta(B)\, w_t$$

$$x_t = \underbrace{\left(\frac{\theta(B)}{\phi(B)}\right)} w_t = \psi(B)\, w_t$$

$$\Rightarrow \theta(B) = \psi(B)\, \phi(B) \checkmark$$