

COMP9334

# Capacity Planning for Computer Systems and Networks

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Week 2A: Operational Analysis (2).  
Workload Characterisation

# Last lecture

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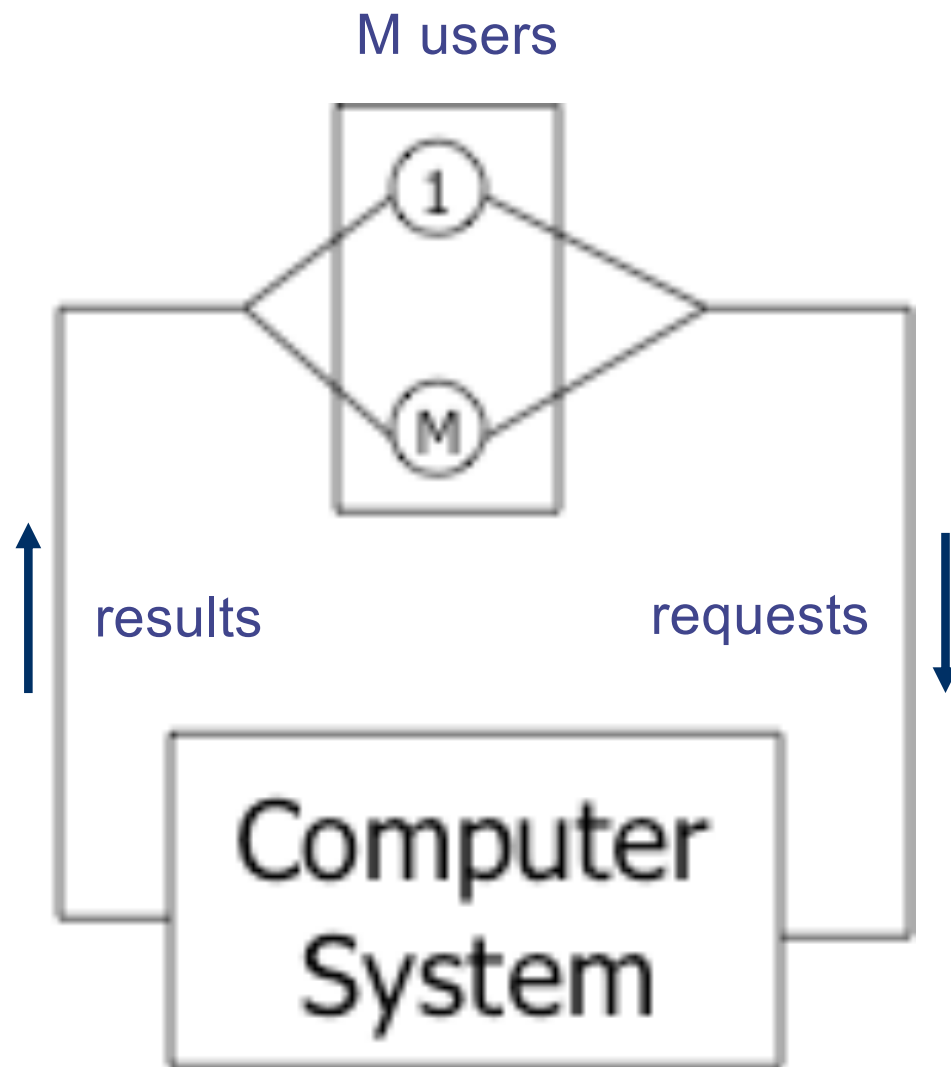
- Modelling a computer system as a queueing network
- Operational analysis on queueing networks
- We have derived these operational laws
  - Utilisation law  $U(j) = X(j) S(j)$
  - Forced flow law  $X(j) = V(j) X(0)$
  - Service demand law  $D(j) = V(j) S(j) = U(j) / X(0)$
  - Little's law  $N = X R$

# This lecture

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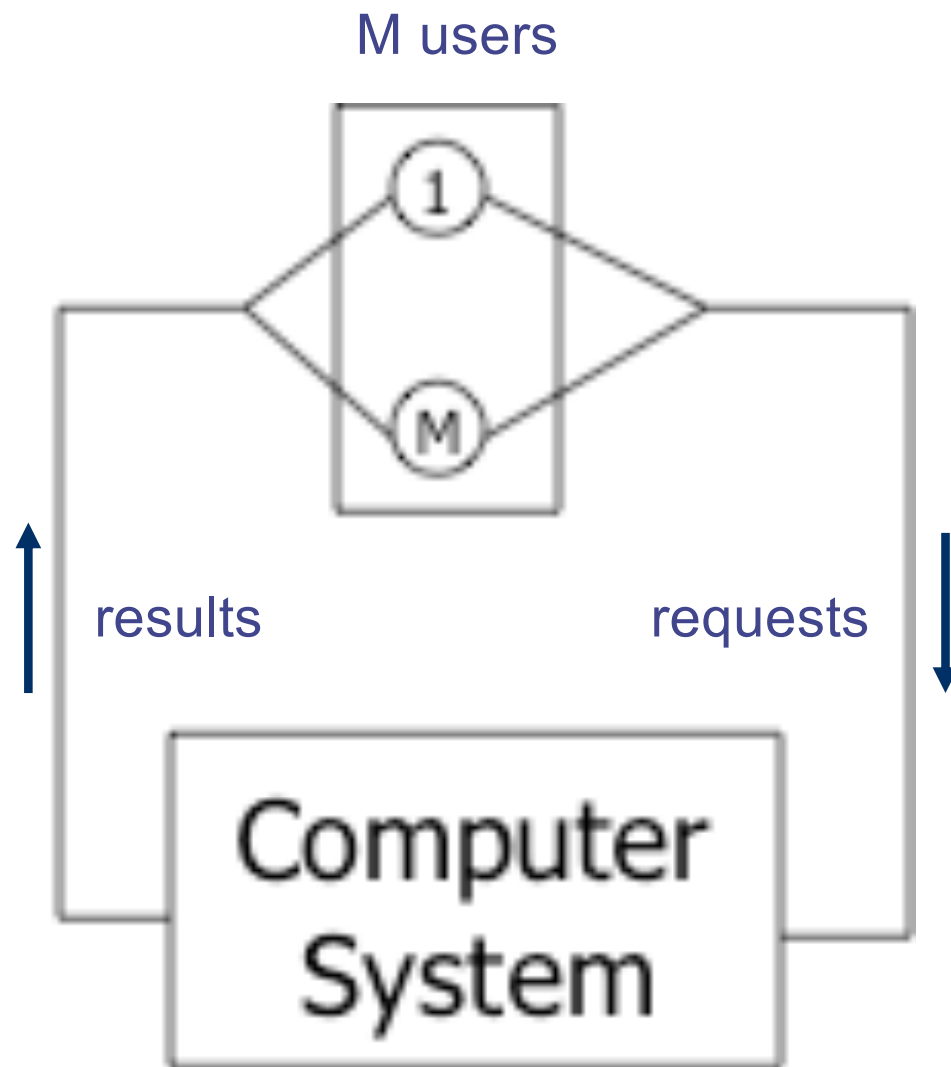
- Operational analysis (Continued)
  - Using operational law for
    - Performance analysis
    - Bottleneck analysis
- Workload characterisation
  - Poisson process and its properties

# Interactive systems



- An interactive system is used to model the interaction between humans (users) and computers
- The system consists of
  - A number of users
  - A computer system

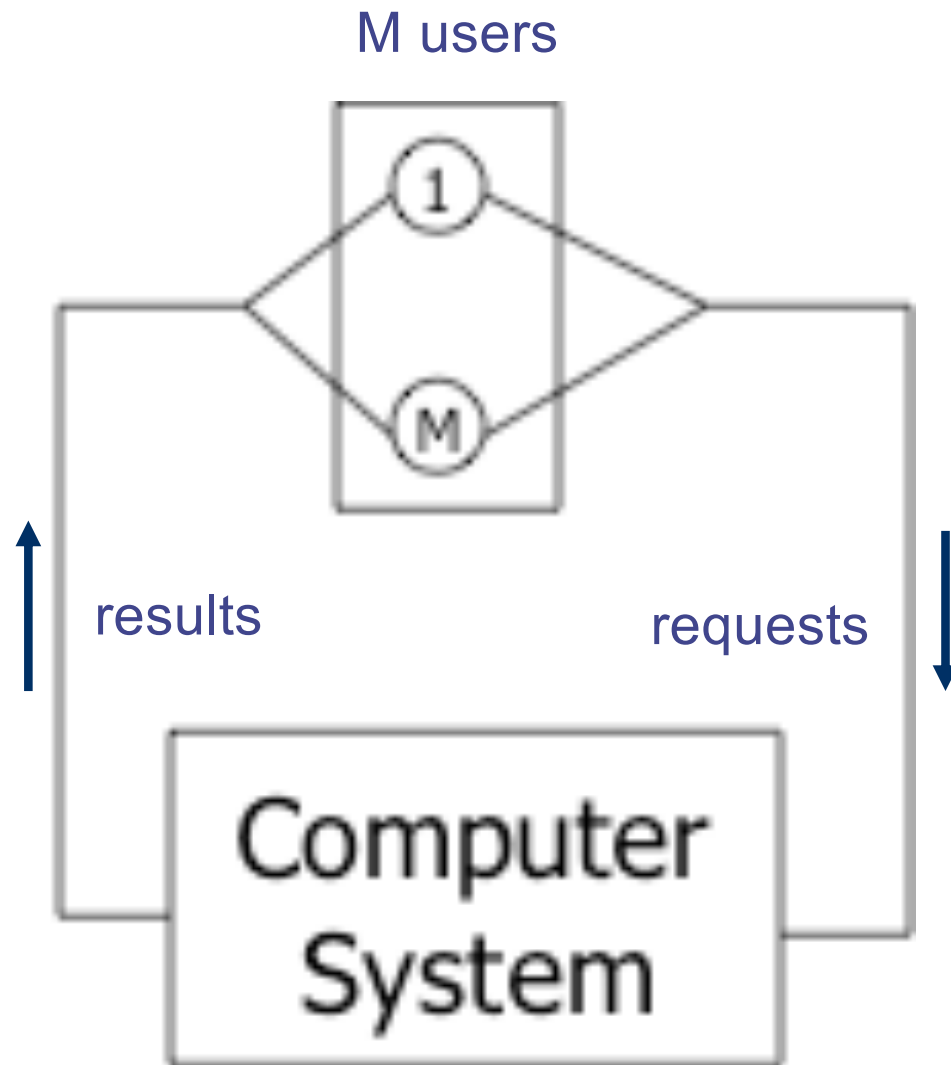
# Interactive systems (Cont'd)



- Interactions

- Users send requests to the computer system
- After finish processing a request, the computer system returns the result to the user
- A user, after inspecting the results from the computer system, will send another request to the system

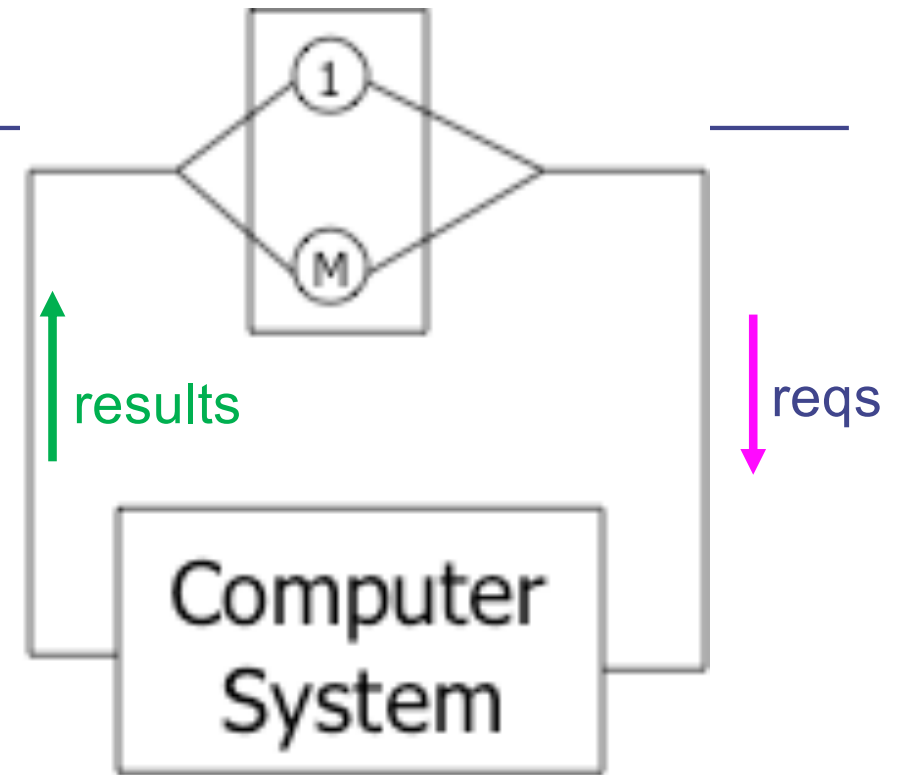
# Interactive systems: Modelling assumptions



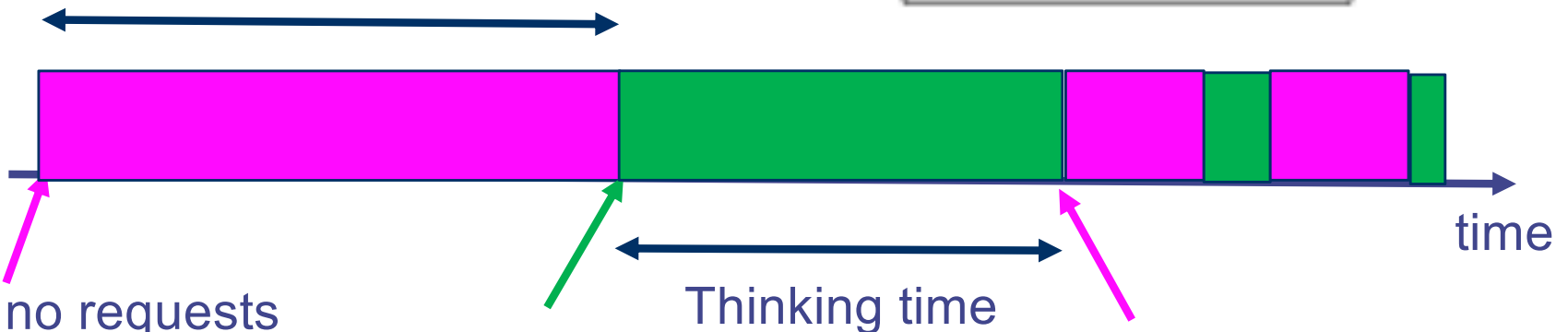
- Analyze interactive systems with specific assumptions
  - Fixed number of users denoted by  $M$
  - Each user can have at most 1 request at the computer system
  - Each user goes through a cycle consisting of
    - Thinking time
    - Waiting for result time

# Interactive cycle

- The time the request from User 1 spends in the computer system
- User 1 waiting for the results



User 1



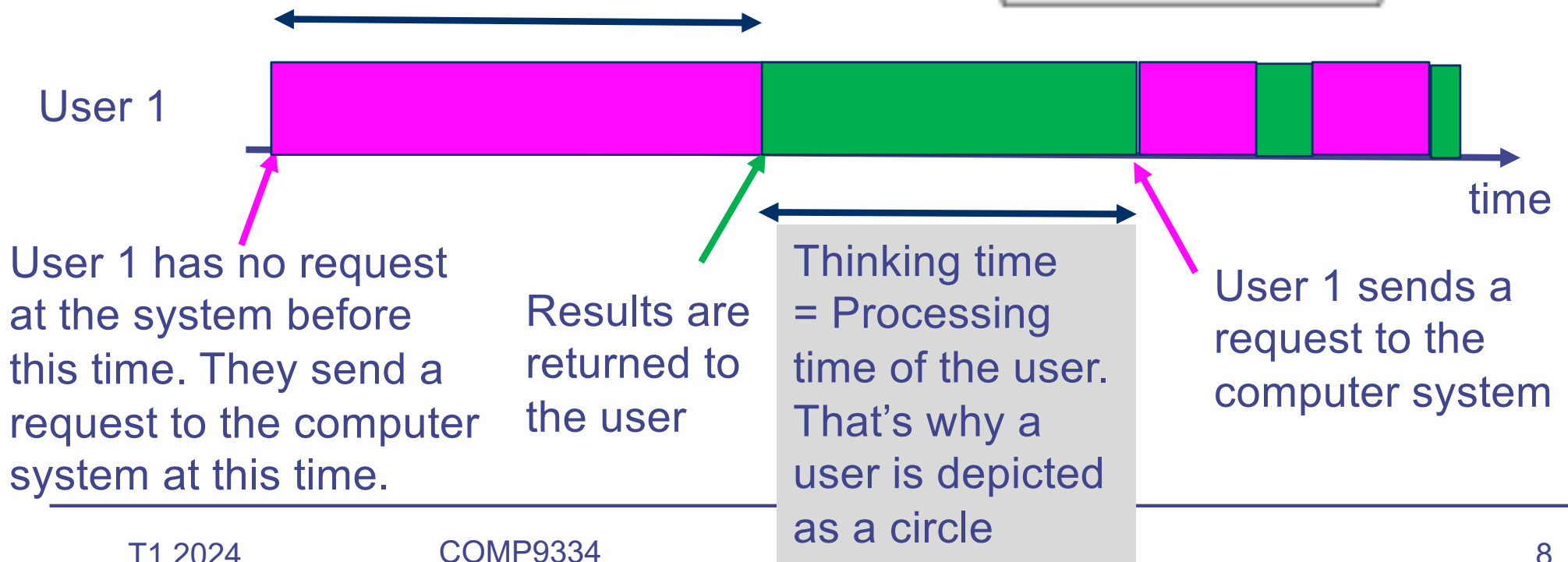
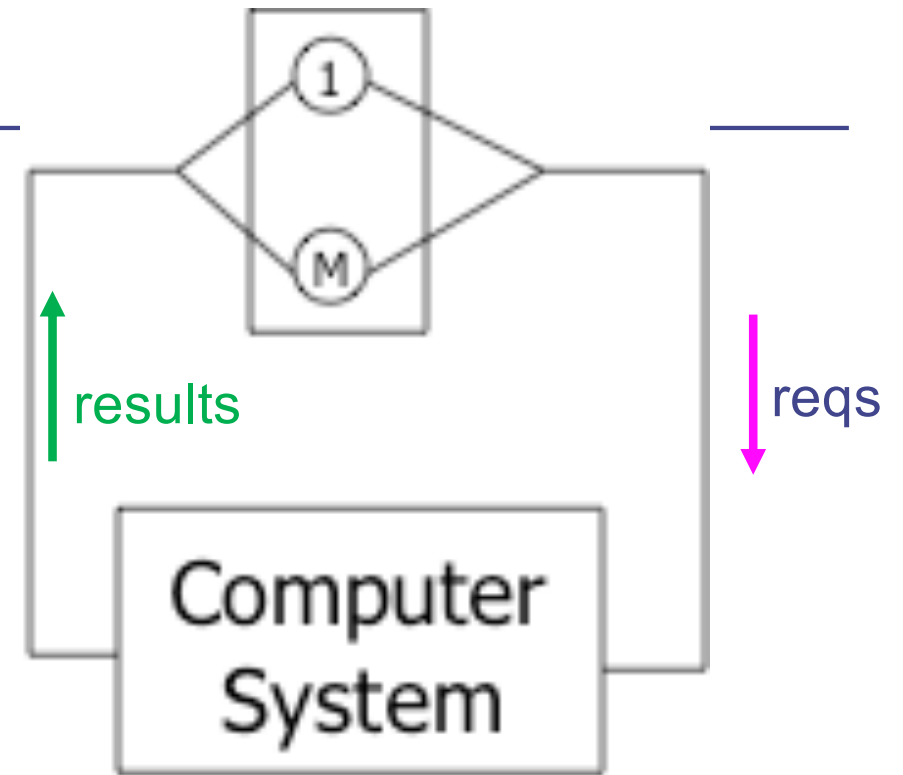
User 1 has no requests at the system before this time. They send a request to the computer system at this time.

Results are returned to the user

User 1 sends a new request to the computer system

# Interactive cycle

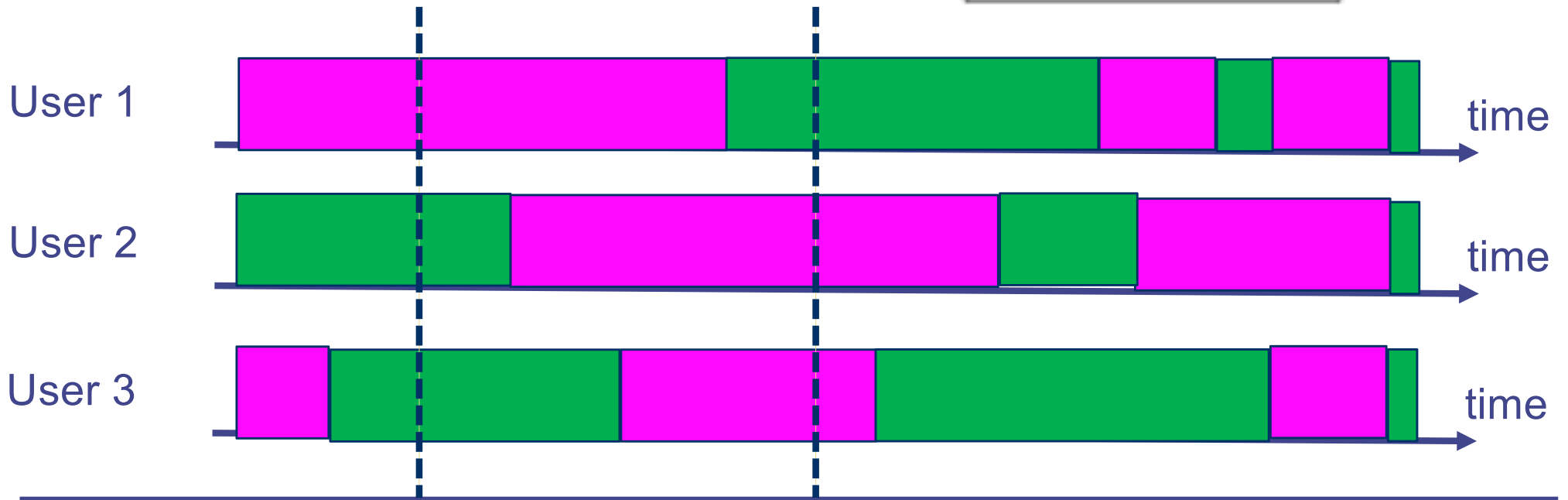
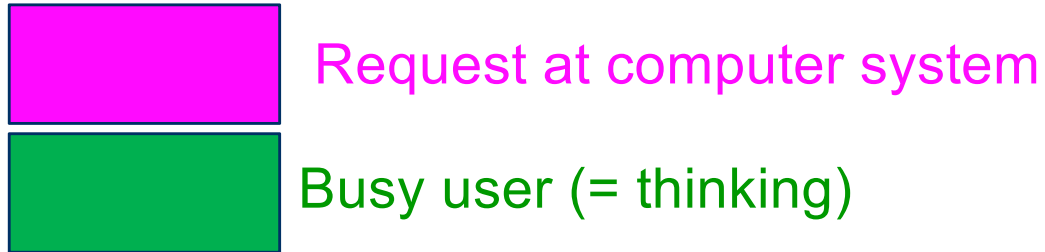
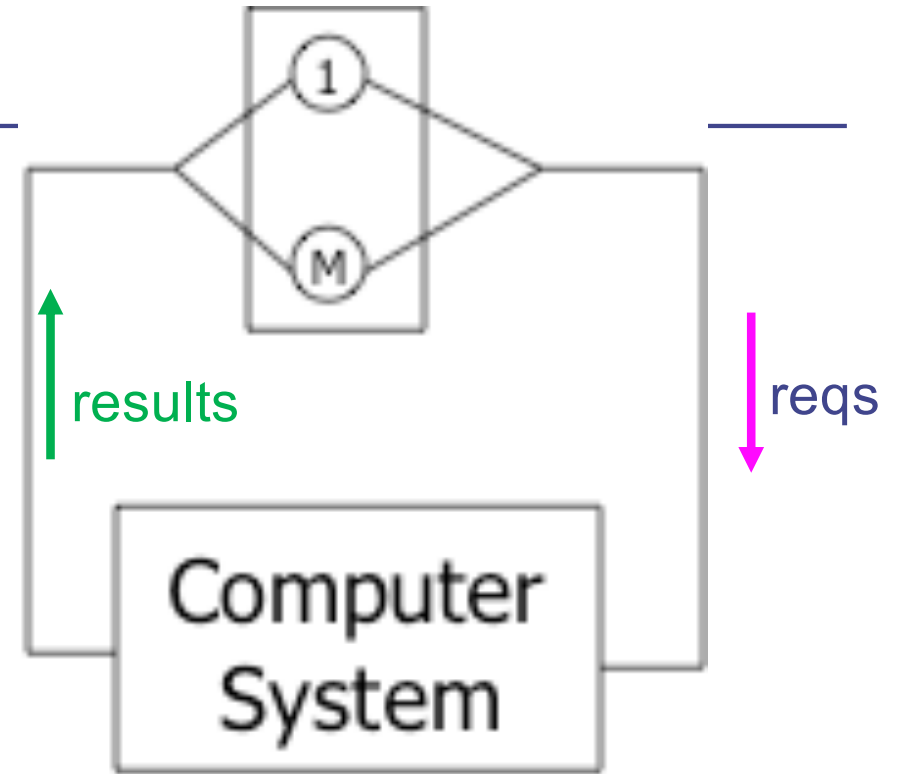
- User 1's perspective: waiting for the result of their job
- Computer system's perspective: Response time of the request from User 1



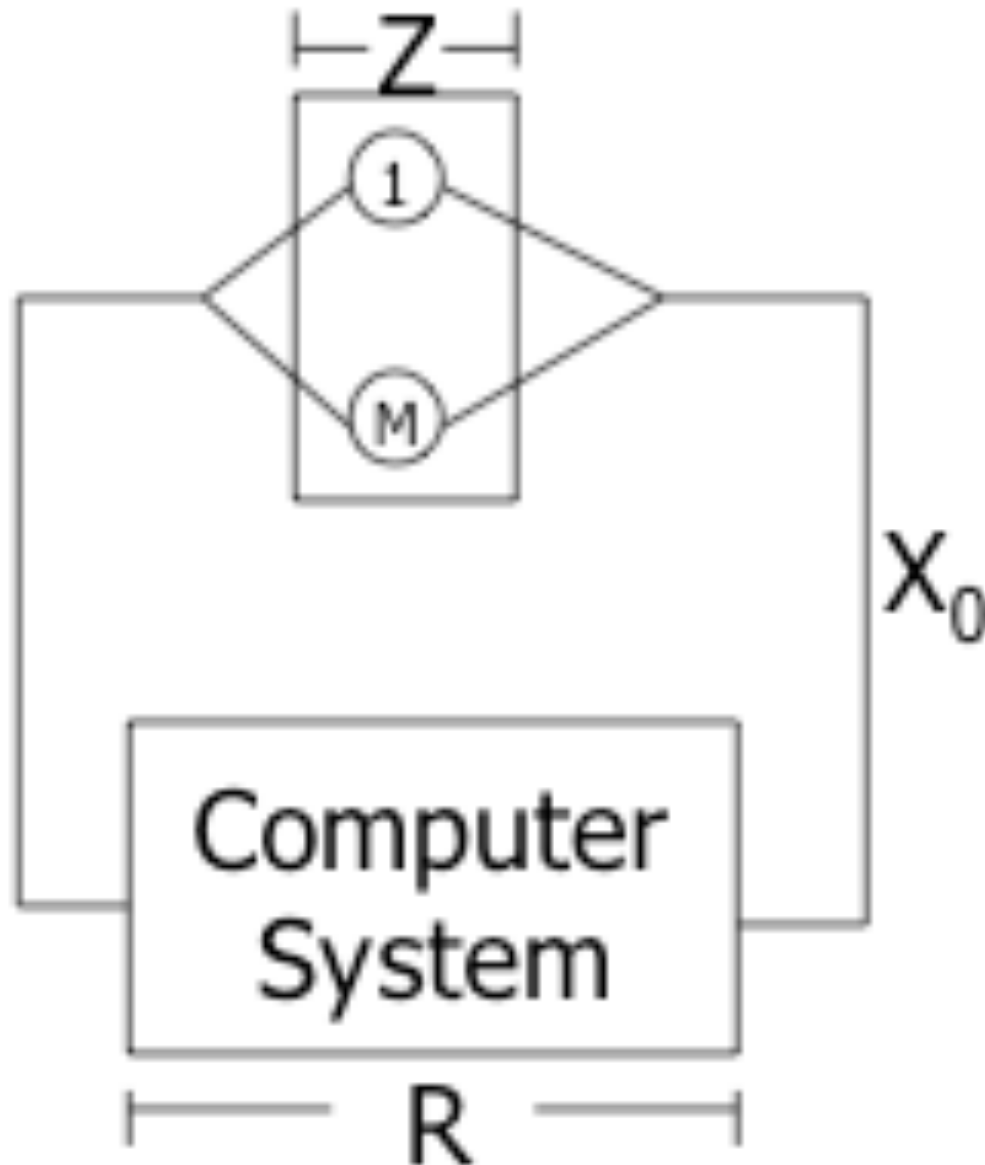


# Quiz

- Question: At any time, what is the sum of the number of busy users and the number of requests at the computer system?  $M$

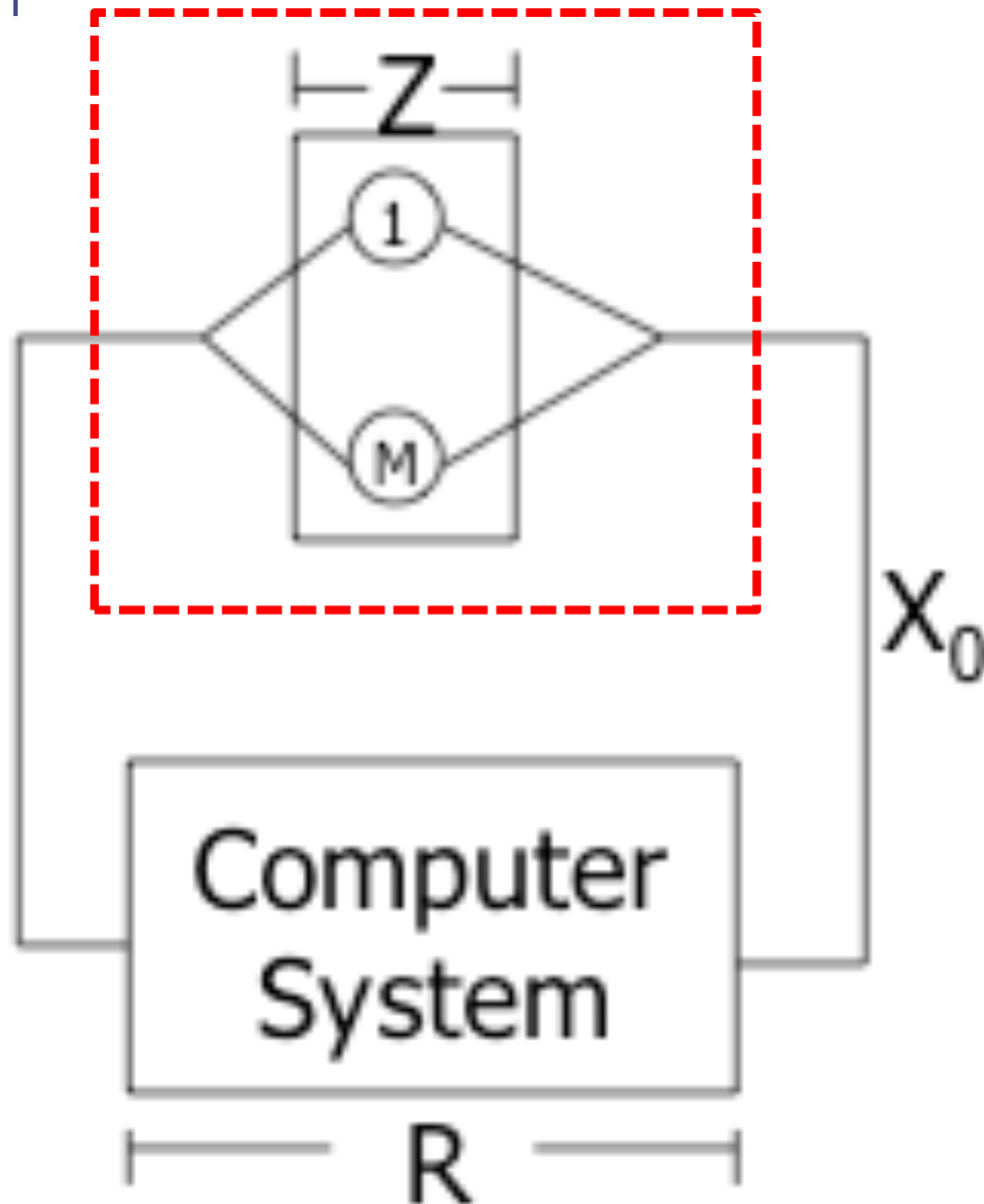


# Interactive system: Parameters



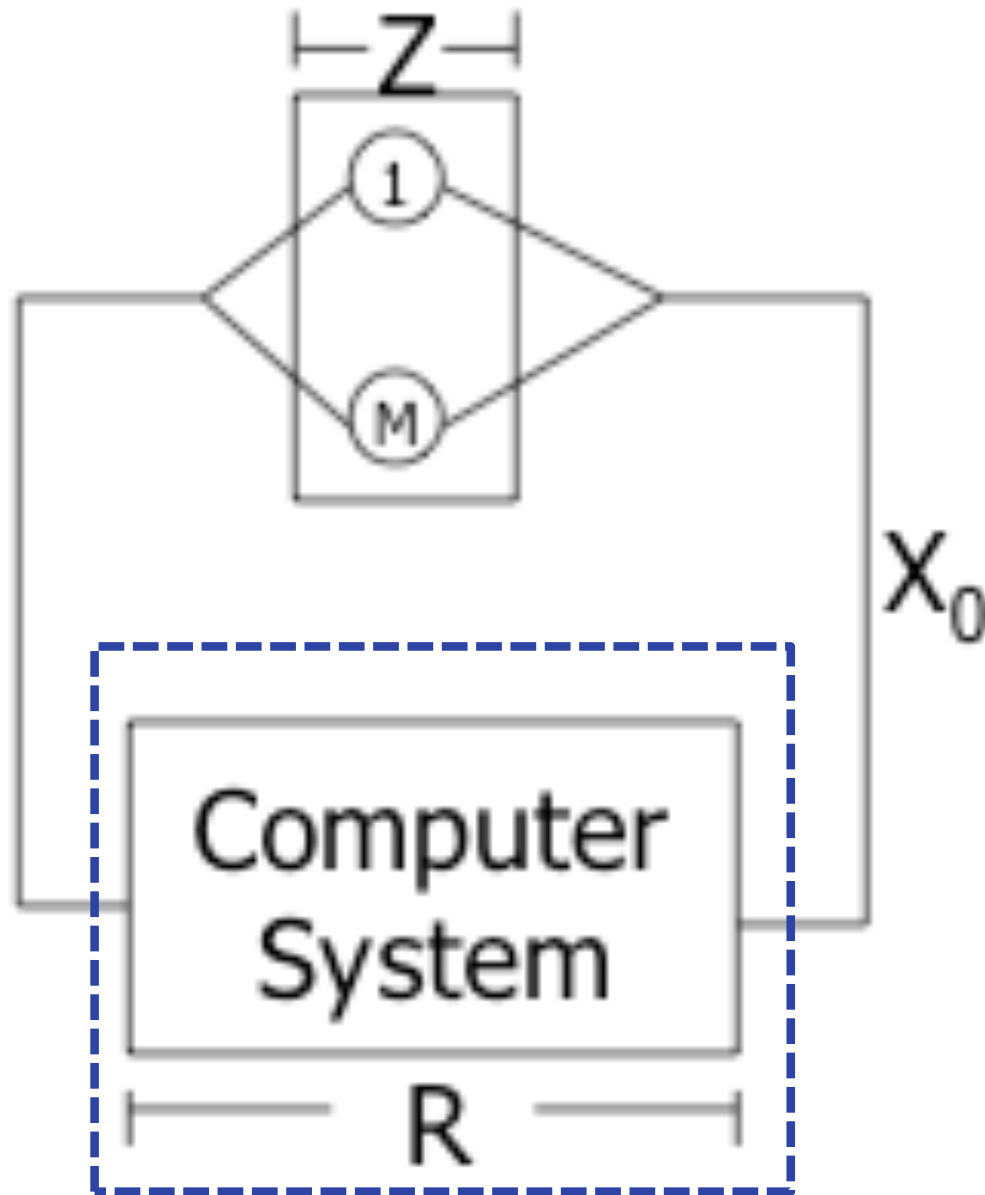
- $M$  interactive users
- $Z$  = mean thinking time
- $R$  = mean response time of the computer system
- $X_0$  = throughput

# Analyzing interactive system: Quiz 1



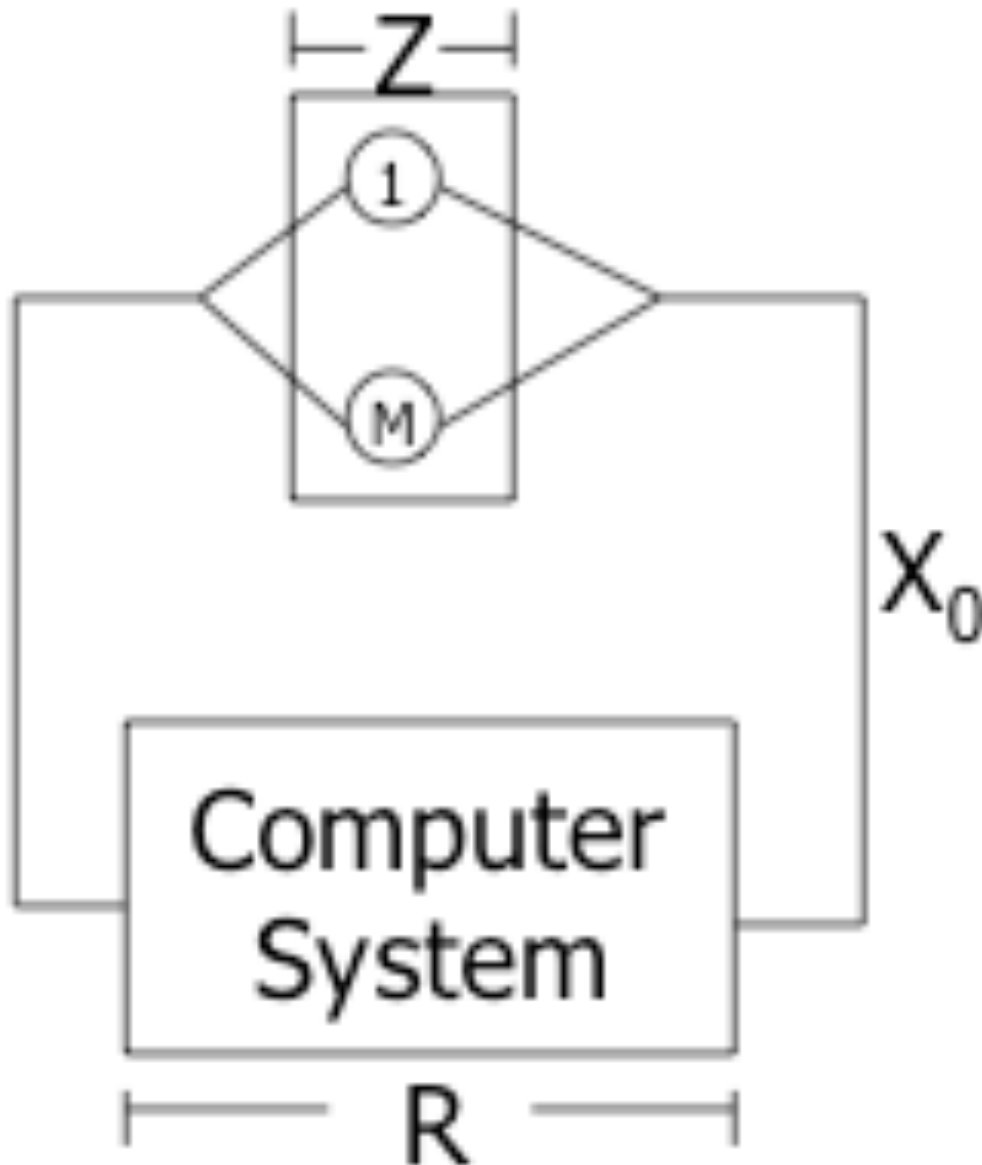
- $M_{avg}$  = mean # busy users
- $Z$  = mean thinking time
- $X_0$  = throughput
- Apply Little's Law to the red box. What do you get?
  - $M_{avg} = Z * X_0$

## Analyzing interactive system: Quiz 2



- $N_{avg}$  = average # requests in the computer system
- $R$  = mean response time at the computer system
- $X_0$  = throughput
- Apply Little's Law to the computer system (i.e. the blue box), what do you get?
  - $N_{avg} = R * X_0$

# Analyzing interactive system: Quiz 3



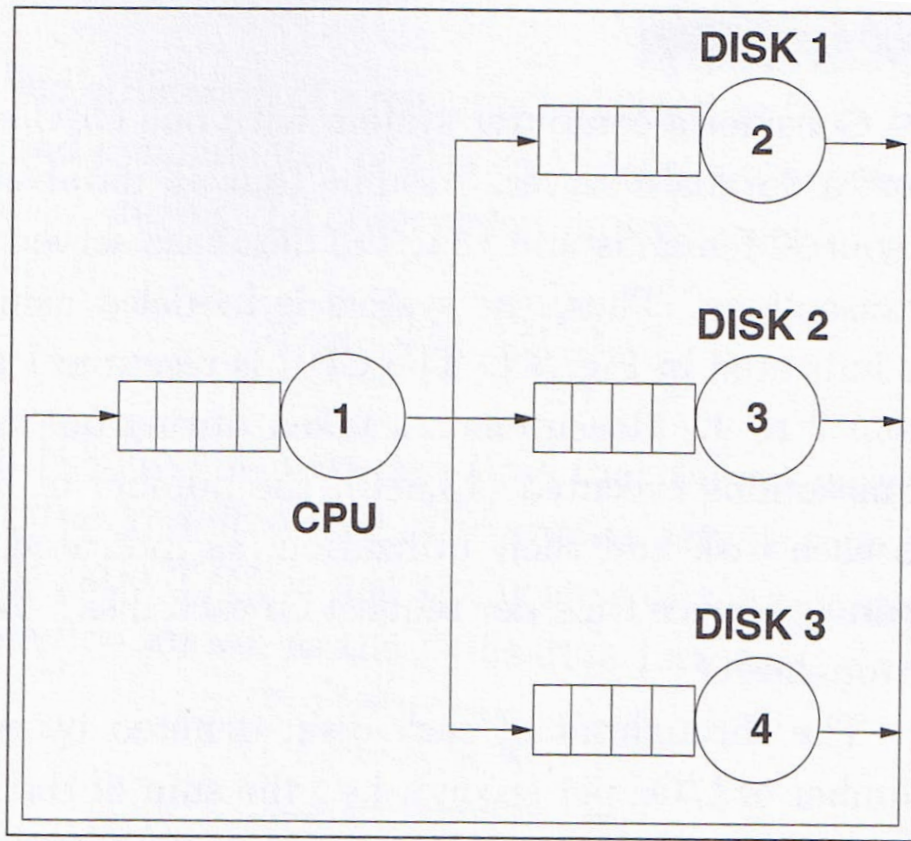
- Quiz 1:  $M_{avg} = X0 * Z$
- Quiz 2:  $N_{avg} = X0 * R$
- What is  $M_{avg} + N_{avg}$ ?
  - $M = M_{avg} + N_{avg}$
- Interactive response time law
  - $M = X0 * (Z+R)$

# The operational laws

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- These are the operational laws
  - Utilisation law  $U(j) = X(j) S(j)$
  - Forced flow law  $X(j) = V(j) X(0)$
  - Service demand law  $D(j) = V(j) S(j) = U(j) / X(0)$
  - Little's law  $N = X R$
  - Interactive response time  $M = X(0) (R+Z)$
- Applications
  - Mean value analysis (later in the course)
  - Bottleneck analysis
  - Modification analysis

# Bottleneck analysis - motivation



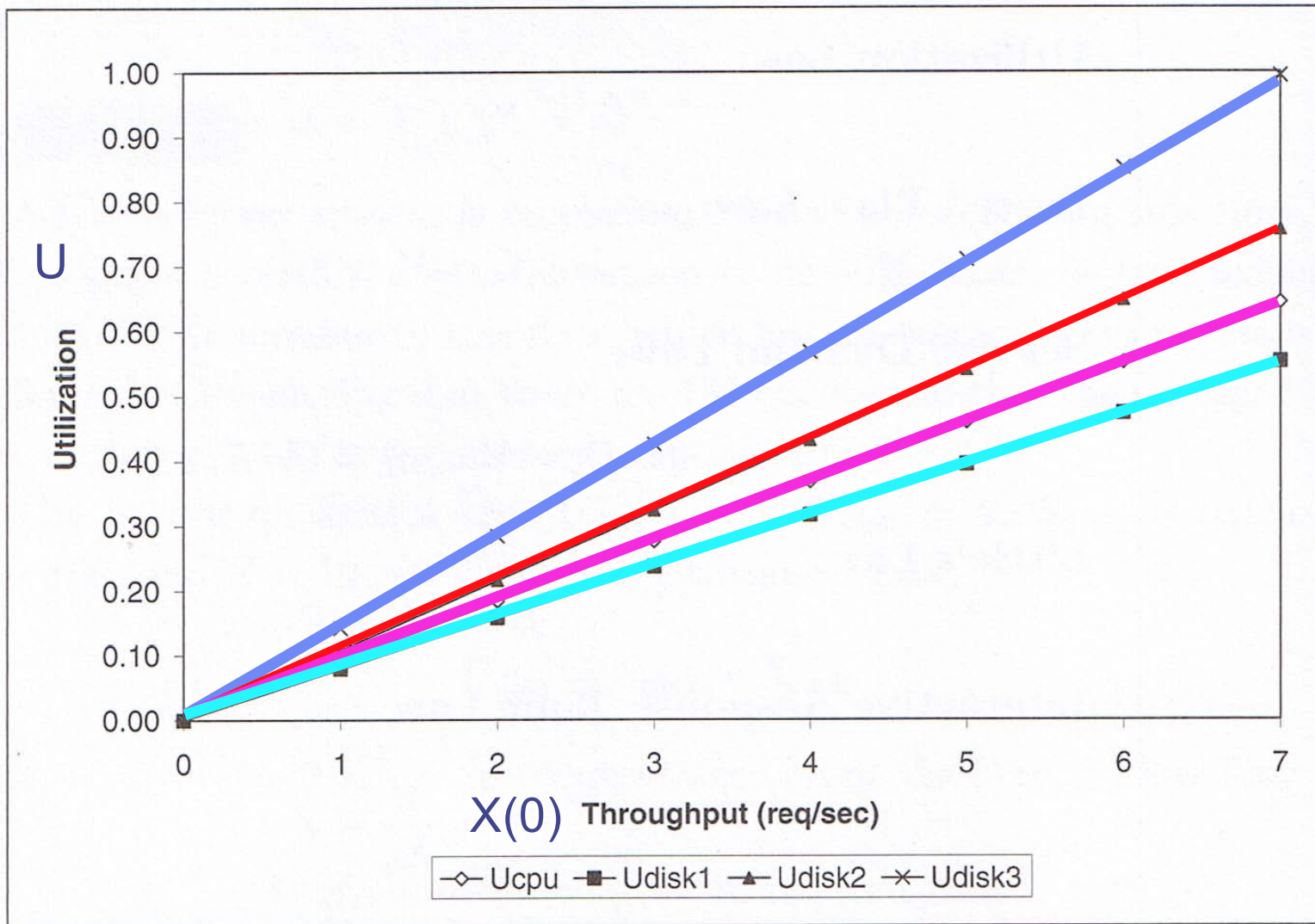
	D(j)	Utilisation
Disk 1	79ms	0.30
Disk 2	108ms	0.41
Disk 3	142ms	0.54
CPU	92ms	0.35

Service demand law:  $D(j) = U(j) / X(0)$

$\Rightarrow U(j) = D(j) X(0)$

Utilisation increases with increasing throughput and service demand

## Utilisation vs. throughput plot $U(j) = D(j) X(0)$



Disk 3

Disk 2

CPU

Disk 1

What determines this order?

Observation: For all system throughput:  
Utilisation of Disk 3 > Utilisation of Disk 2 >  
Utilisation of CPU > Utilisation of Disk 1



# Bottleneck analysis

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- Recall that utilisation is the busy time of a device divided by measurement time
  - What is the maximum value of utilisation?
- Based on the example on the previous slide, which device will reach the maximum utilisation first?

## Bottleneck (1)

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- Disk 3 has the highest service demand
- It is the bottleneck of the whole system

Operational law:  $X(0) = \frac{U(j)}{D(j)}$

Utilisation limit:  $U(j) \leq 1$

}  $X(0) \leq \frac{1}{D(j)}$

## Bottleneck (2)

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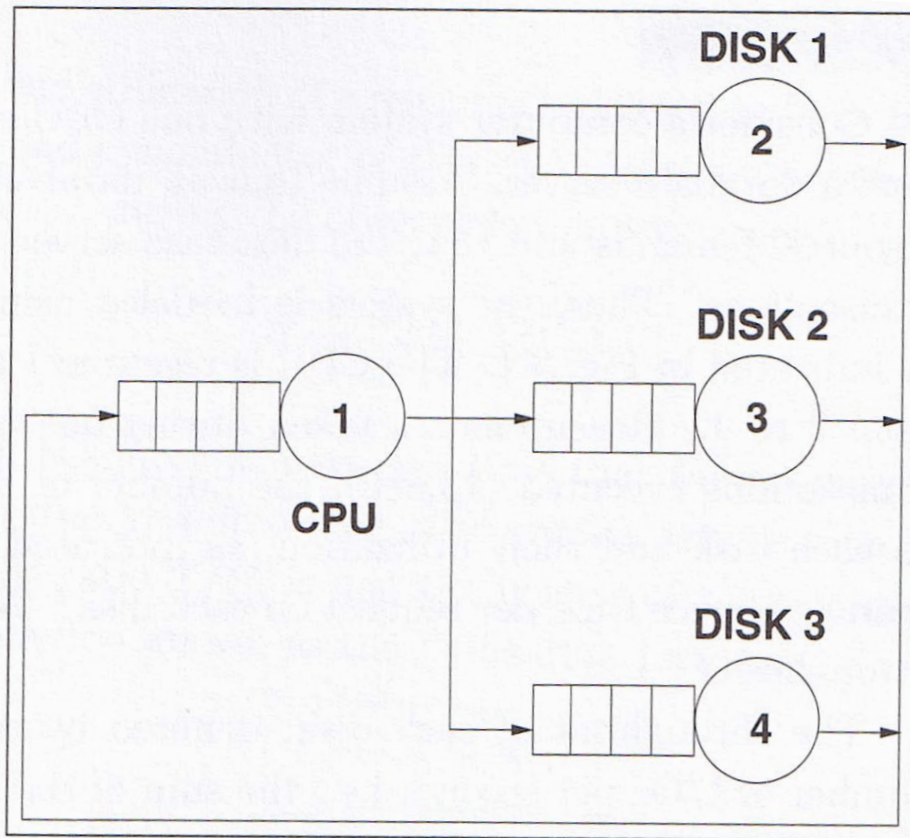
$$X(0) \leq \frac{1}{D(j)} \quad \text{Should hold for all } K \text{ devices in the system}$$

$$i.e. X(0) \leq \frac{1}{D(1)}, \dots, X(0) \leq \frac{1}{D(K)}$$

$$\Rightarrow X(0) \leq \min \frac{1}{D(j)}$$

$$\Rightarrow X(0) \leq \frac{1}{\max D(j)} \quad \text{Bottleneck throughput is limited by the maximum service demand}$$

## Bottleneck exercise



	D(j)	Utilisation
Disk 1	79ms	0.30
Disk 2	108ms	0.41
Disk 3	142ms	0.54
CPU	92ms	0.35

The system throughput is upper bounded by  $\frac{1}{0.142} = 7.04$  jobs/s

If we upgrade Disk 3 by a new disk which is 2 times faster, which device will be the bottleneck after the upgrade? You can assume that service time is inversely proportional to disk speed.

## Another throughput bound

- Little's law

$$N = R \times X(0) \geq \left( \sum_{i=1}^K D_i \right) \times X(0)$$

$$\Rightarrow X(0) \leq \frac{N}{\sum_{i=1}^K D_i}$$

Previously, we have

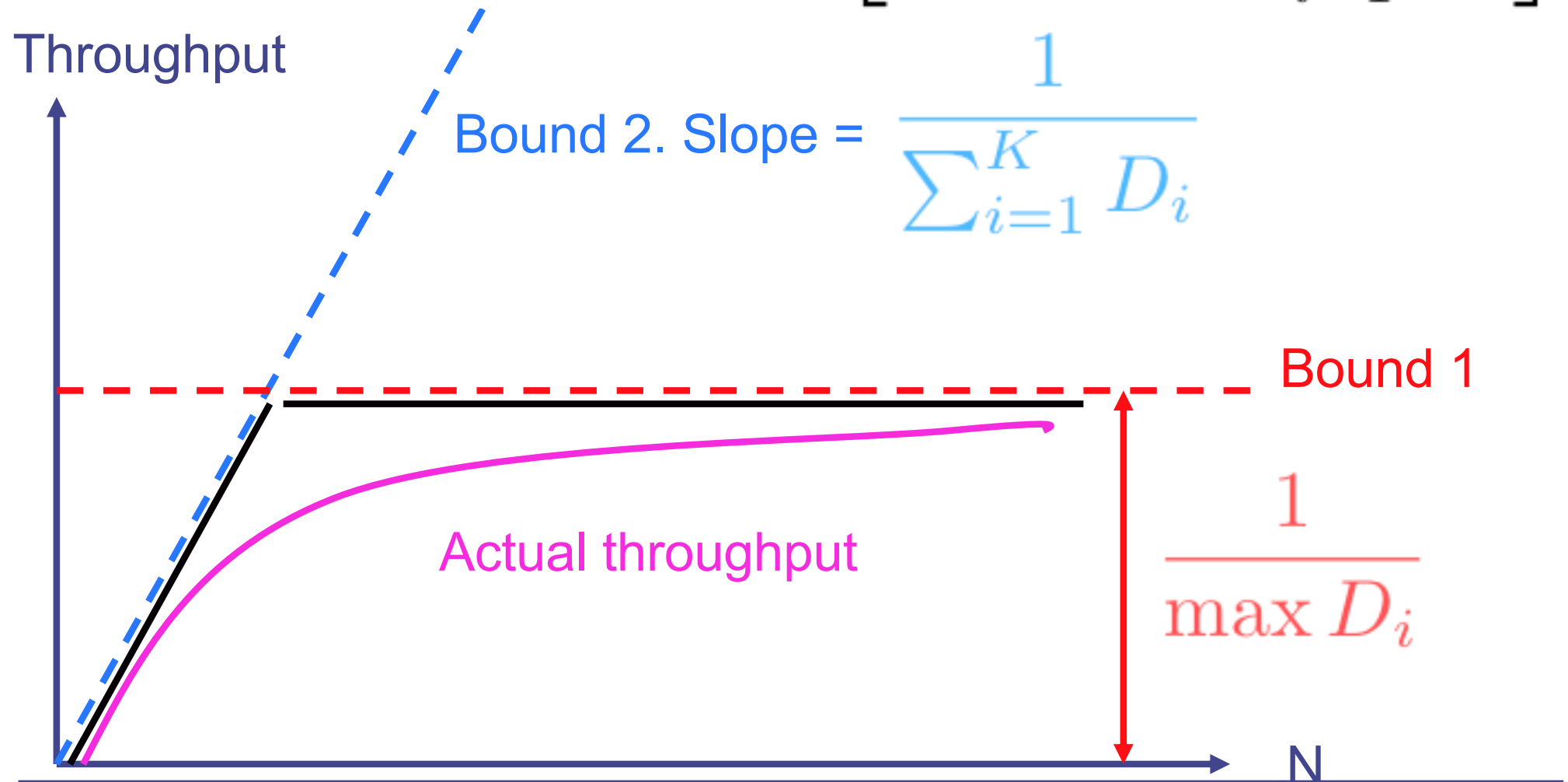
$$X(0) \leq \frac{1}{\max D(j)}$$

Therefore:

$$X(0) \leq \min \left[ \frac{1}{\max D_i}, \frac{N}{\sum_{i=1}^K D_i} \right]$$

# Throughput bounds

$$X(0) \leq \min \left[ \frac{1}{\max D_i}, \frac{N}{\sum_{i=1}^K D_i} \right]$$



# Bottleneck analysis

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- Simple to use
  - Needs only utilisation of various components
- Assumes service demand is load independent

# Modification analysis (1)

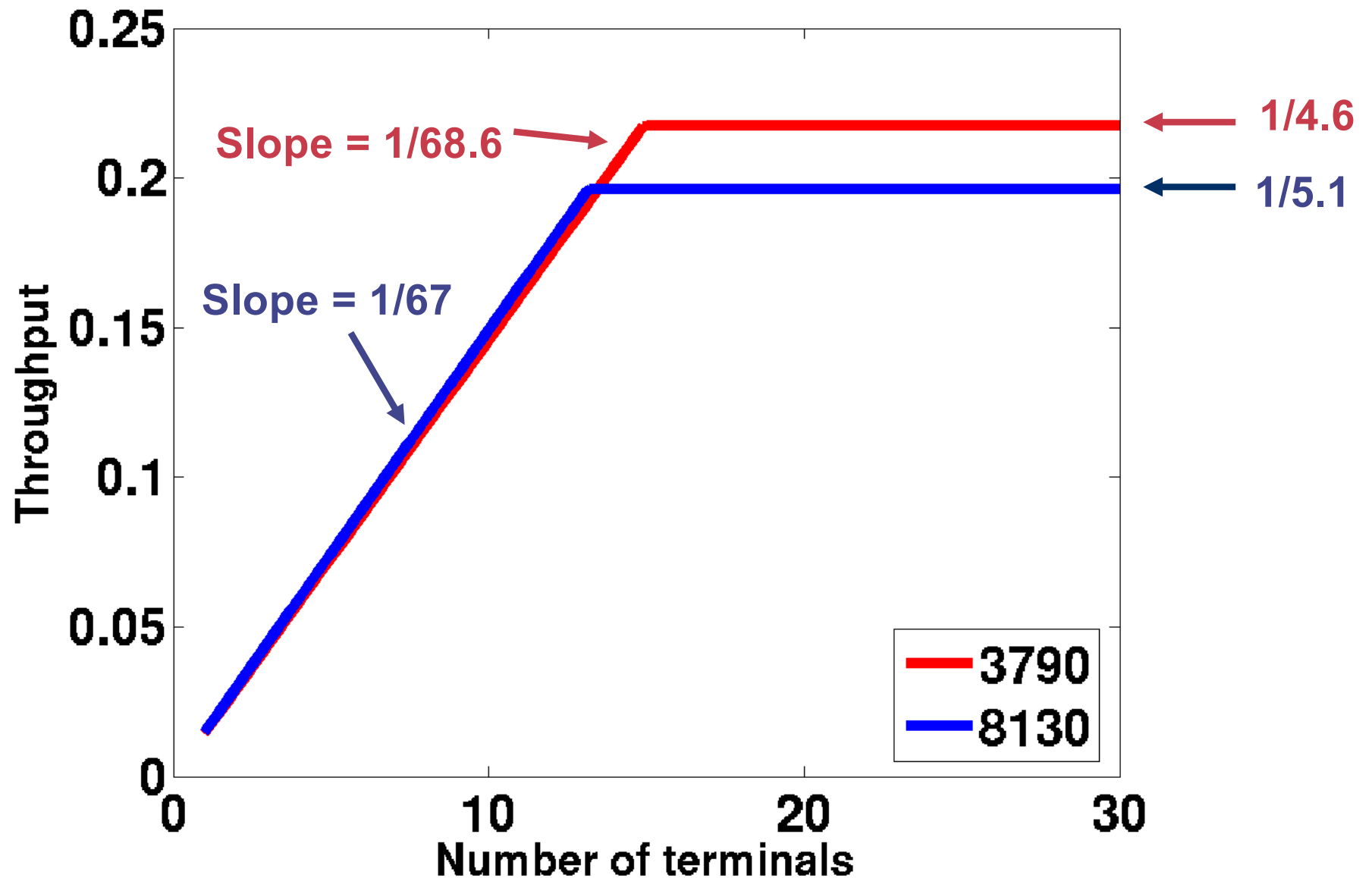
- (Reference: Lazowska Section 5.3.1)
- A company currently has a system (3790) and is considering switching to a new system (8130). The service demands for these two systems are given below:

System	Service demand (seconds)	
	CPU	Disk
3790	4.6	4.0
8130	5.1	1.9

- The company uses the system for interactive application with a think time of 60s.
- Given the same workload, should the company switch to the new system?
- Exercise: Answer this question by using bottleneck analysis. For each system, plot the upper bound of throughput as a function of the number of interactive users.



## Modification analysis (2)



# Operational analysis

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- Operational analysis allows you to bound the system performance but it does NOT allow you to find the throughput and response time of a system
- To order to find the throughput and response time, we need to use queueing analysis
- To order to use queueing analysis, we need to specify the workload in a probabilistic manner

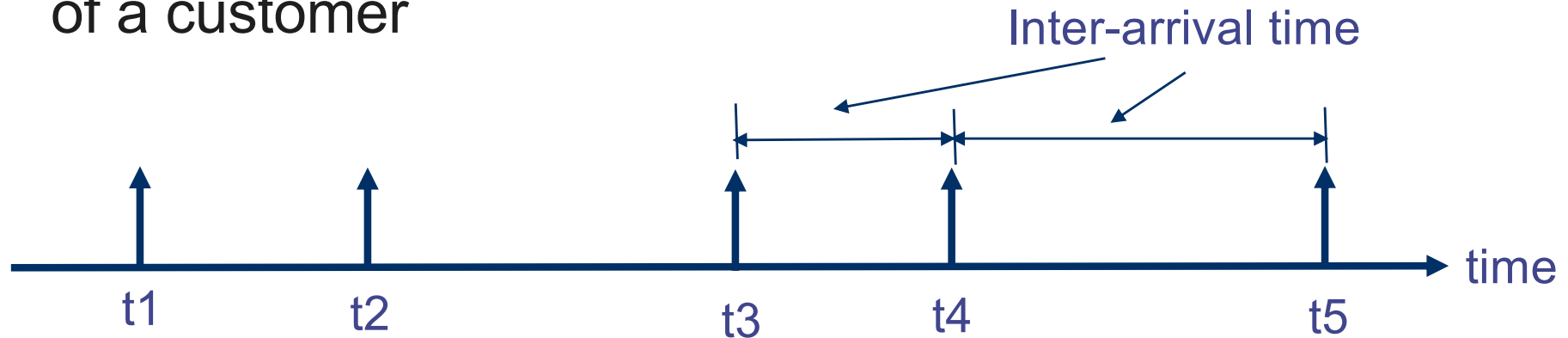
# Workload analysis

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- Performance depends on workload
  - When we look at the performance bound earlier, the bounds depend on **number of users** and **service demand**
  - Queue response time depends on the **job arrival probability distribution** and **job service time distribution**
    - Recall from Lecture 1A:
      - Uniform arrival times and uniform processing times result in zero waiting time
      - But non-uniform distributions give non-zero waiting time
- Need to specify workload by using probability distribution.
- We will look at a well-known arrival process called Poisson process today.

# Arrival process

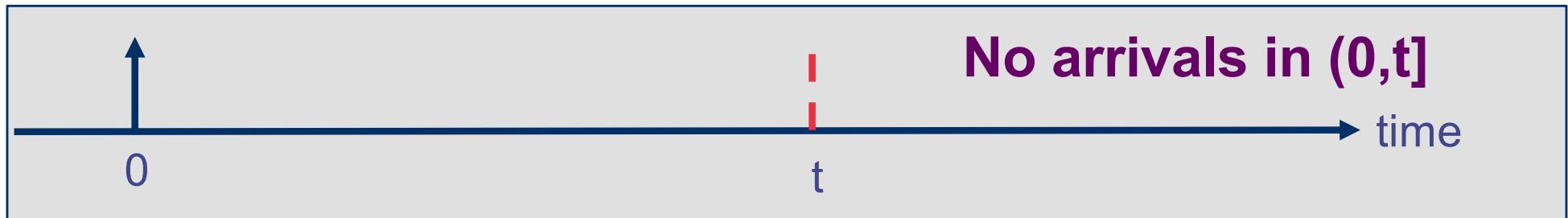
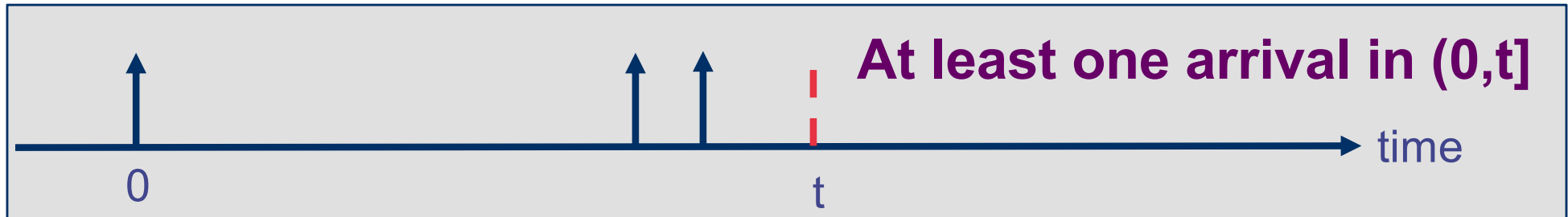
- Each vertical arrow in the time line below depicts the arrival of a customer



- An arrival can mean
  - A telephone call arriving at a call centre
  - A transaction arriving at a computer system
  - A customer arriving at a checkout counter
  - An HTTP request arriving at a web server
- The **inter-arrival time** distribution will impact on the response time.
- We will study an inter-arrival time distribution that results from a large number of **independent** customers.

# Describing arrivals probabilistically

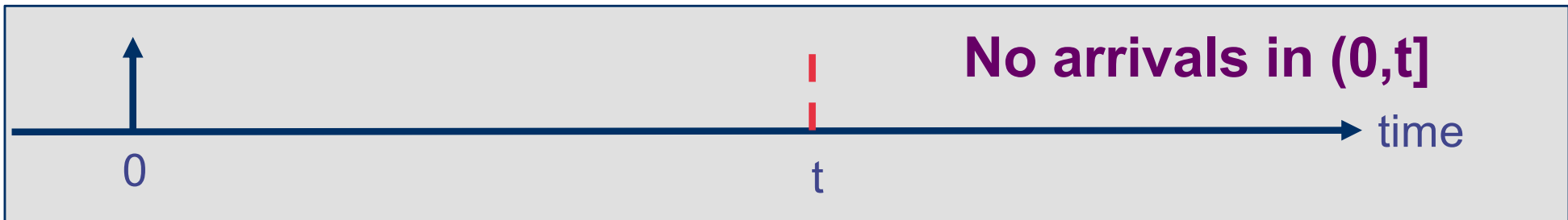
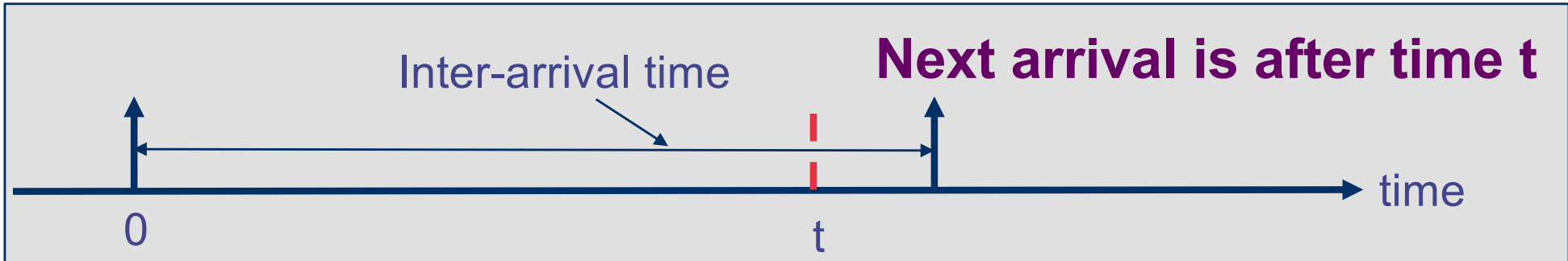
- Assume a customer arrives at time 0



- Quiz: What is the relation between the following two probabilities?
  - Prob[at least one arrival in  $(0,t]$  ]
  - Prob[no arrivals in  $(0,t]$  ]
- Answer: They add up to 1
- Moral: "No arrivals" is not boring, it tells you something

# Inter-arrival time probability

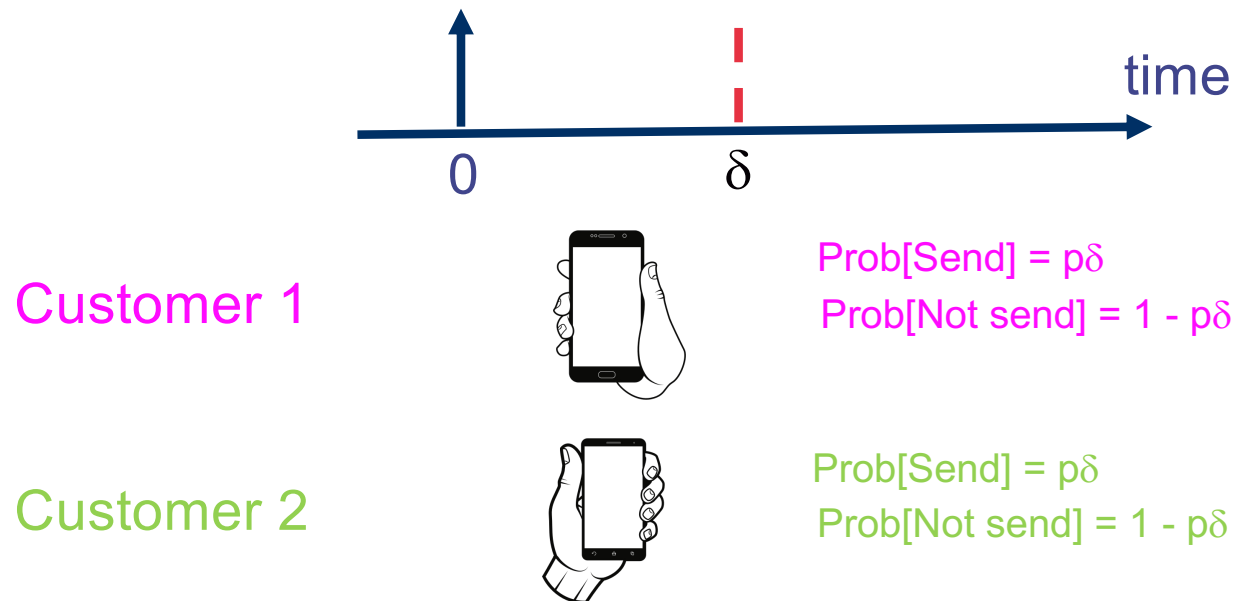
- Assume a customer arrives at time 0



- Quiz: What is the relation between the following two probabilities?
  - $\text{Prob}[\text{Inter-arrival time is } > t]$
  - $\text{Prob}[\text{no arrivals in } (0,t] ]$
- Answer: Equal
- Next step: Find  $\text{Prob}[\text{no arrivals in } (0,t] ]$  for independent customers

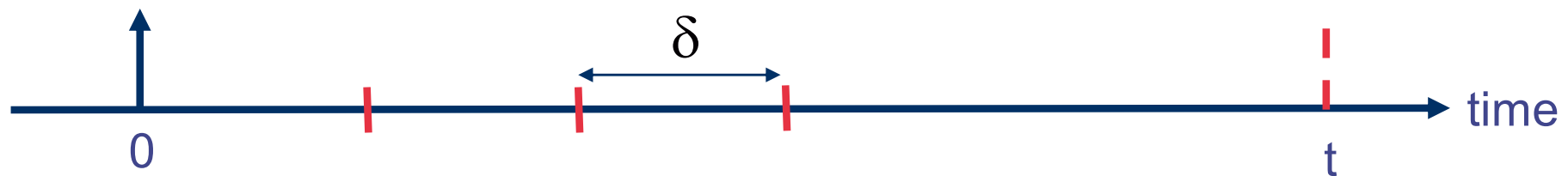
# Many independent arrivals (1)

- Problem set up:
  - An arrival at time 0
  - A large pool of  $N$  **independent** customers
  - Behaviour of each customer: Within a small time interval of  $\delta$ , a customer sends a request (or arrives) with a probability of  $p\delta$ 
    - $p$  is a constant
- Quiz: If there are 2 ( $= N$ ) customers, what is the probability that both of them do not send any request in the time interval  $\delta$ 
  - Answer:  $(1 - p\delta)^2$



## Many independent arrivals (2)

- Aim: Want to find the probability of no arrivals in  $(0,t]$
- Divide the time  $t$  into intervals of width  $\delta$



- No arrivals in  $(0,t]$  = no arrivals in each interval  $\delta$  from  $N$  users
- Probability of no arrivals in  $\delta = (1 - p \delta)^N \approx 1 - Np\delta$
- There are  $t / \delta$  intervals
- Probability of no arrivals in  $(0,t]$  is

$$(1 - Np\delta)^{\frac{t}{\delta}} \rightarrow e^{-Npt} \text{ as } \delta \rightarrow 0$$



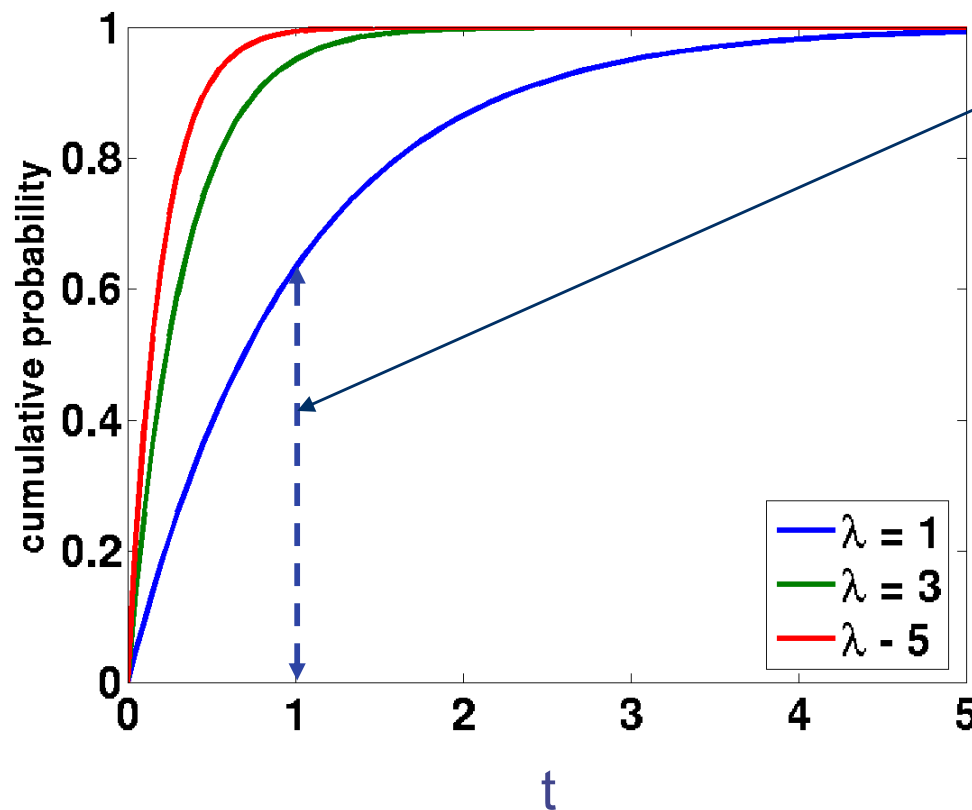
# Exponential inter-arrival time

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- We have showed  
Probability(no arrivals in  $(0,t]$ ) =  $\exp(-Npt)$
- Probability(inter-arrival time  $> t$ ) =  $\exp(-Npt)$
- This means  
Probability(inter-arrival time  $\leq t$ ) =  $1 - \exp(-Npt)$
- What this shows is the inter-arrival time distribution for independent arrival is exponentially distributed
- Define:  $\lambda = Np$ 
  - $\lambda$  is the mean arrival rate of customers

# Exponential distribution - cumulative distribution

- Cumulative distribution of inter-arrival time with customer arrival rate  $\lambda$ 
  - $\text{Prob}(\text{inter-arrival time} \leq t) = 1 - \exp(-\lambda t)$



Prob that at least a customer will arrive before  $t = 1$  for  $\lambda = 1$

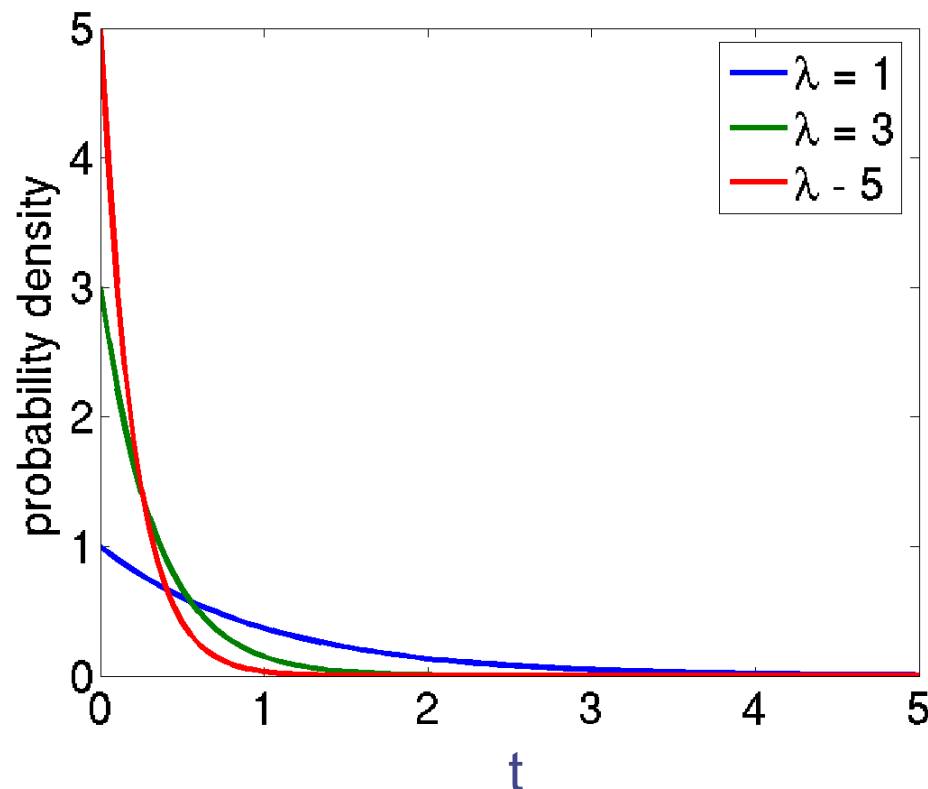
$\wedge$

Prob that at least a customer will arrive before  $t = 1$  for  $\lambda = 3$

# Exponential distribution

- A continuous random variable is exponentially distributed with rate  $\lambda$  if it has probability density function

$$f(t) = \begin{cases} \lambda \exp(-\lambda t) & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$



Reminder:

Probability  
density  
function

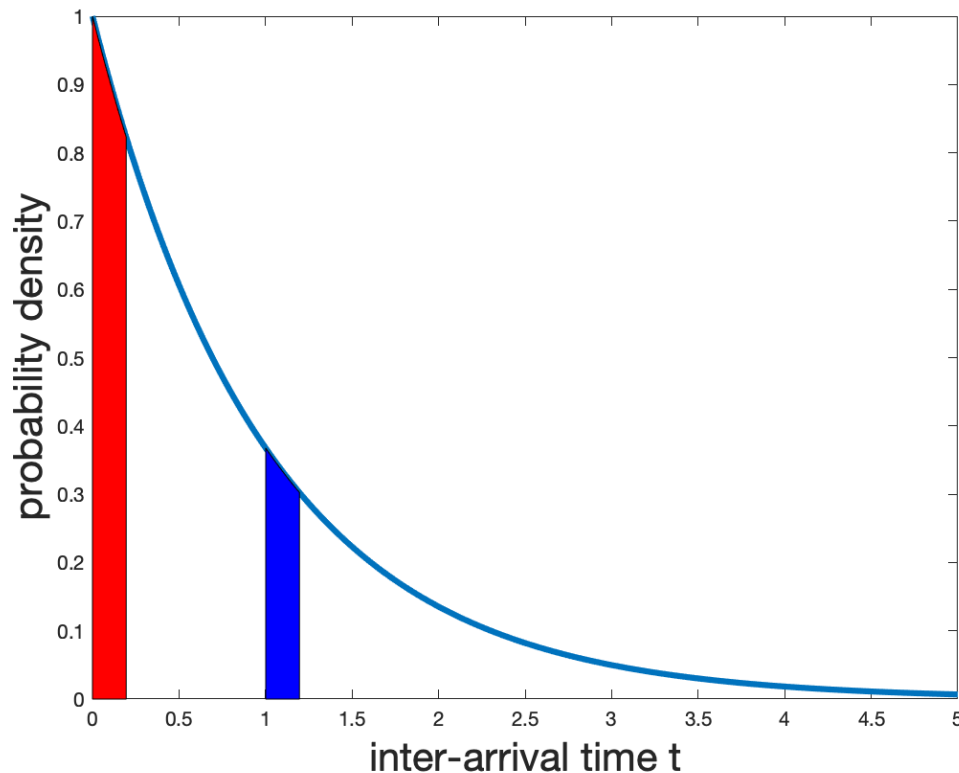
integration

differentiation

Cumulative  
probability

$$1 - \exp(-\lambda t)$$

# Probability density function (PDF)



Reminder: PDF  $f(t)$

Probability( $t \leq T \leq t + \delta t$ )

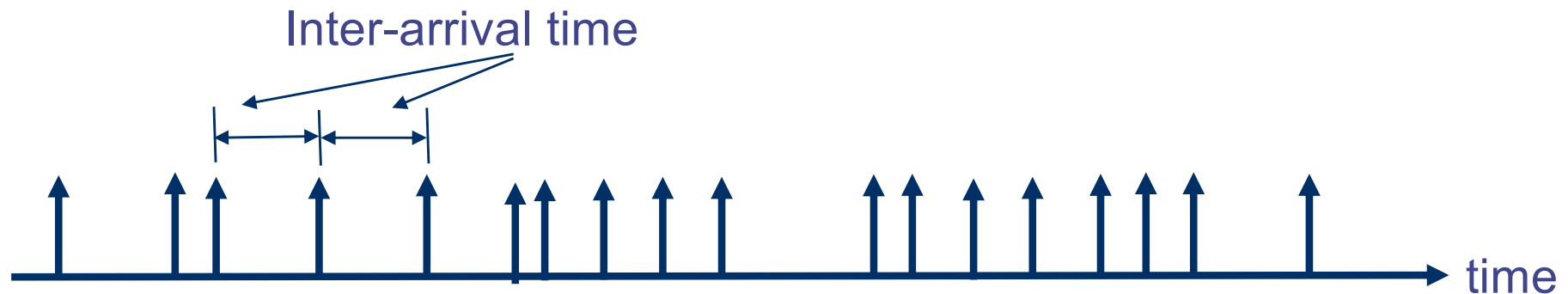
$$= f(t) \delta t$$

Red area = probability  
that inter-arrival time is in  
the interval  $[0, 0.2]$

Blue area = probability  
that the inter-arrival time  
is in the interval  $[1, 1.2]$

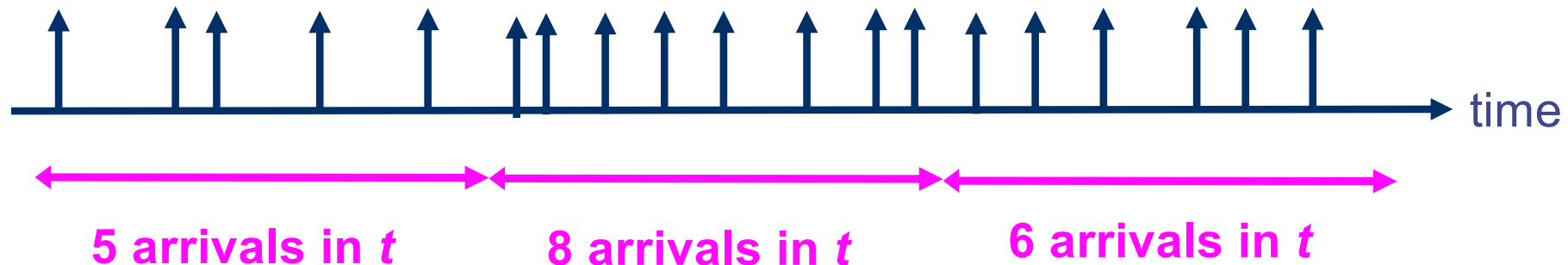
# Two different methods to describe arrivals

Method 1: Continuous probability distribution of inter-arrival time



## Two different methods to describe arrivals

Method 2: Use a fixed time interval (say  $t$ ), and count the number of arrivals within  $t$ .

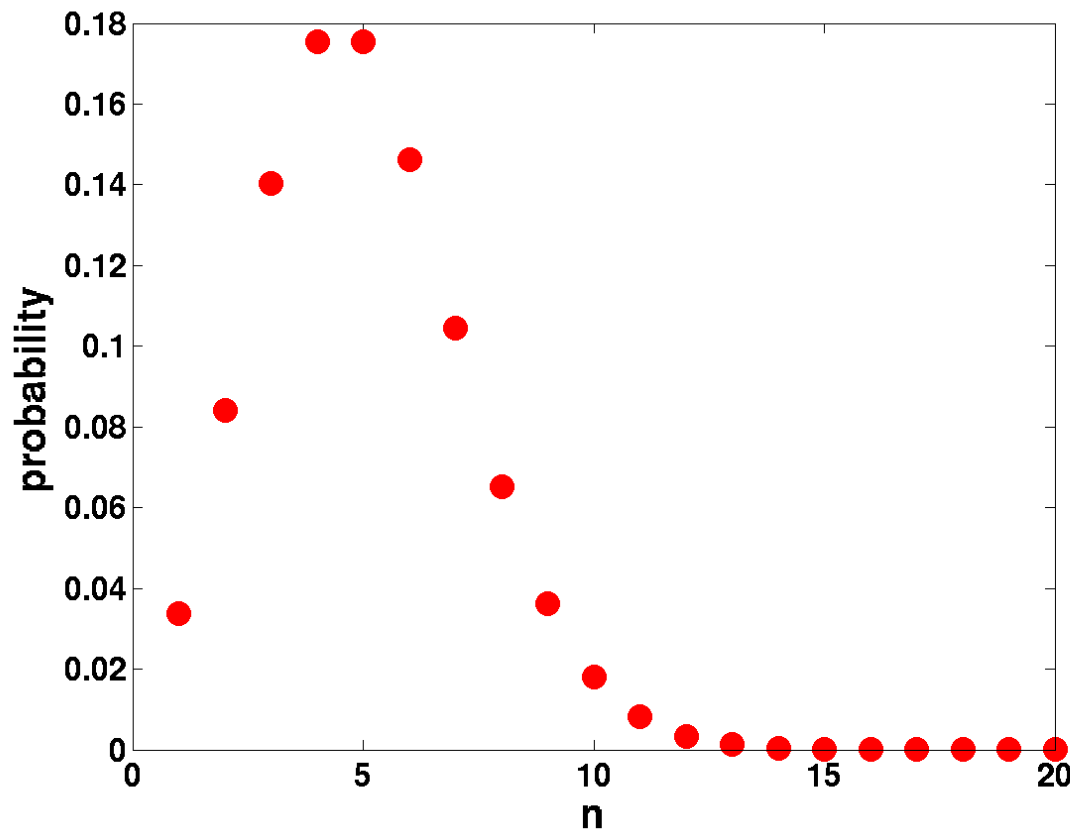


- The number of arrivals in  $t$  is random
- The number of arrivals must be a non-negative integer
- We need a discrete probability distribution:
  - $\text{Prob}[\text{\#arrivals in } t = 0]$
  - $\text{Prob}[\text{\#arrivals in } t = 1]$
  - etc.

# Poisson process (1)

- Definition: An arrival process is Poisson with parameter  $\lambda$  if the probability that  $n$  customers arrive in any time interval  $t$  is

$$\frac{(\lambda t)^n e^{-\lambda t}}{n!}$$



Example:

Example:

$\lambda = 5$  and  $t = 1$

Note: Poisson is a discrete probability distribution.

## Poisson process (2)

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- Theorem: An exponential inter-arrival time distribution with parameter  $\lambda$  gives rise to a Poisson arrival process with parameter  $\lambda$
- How can you prove this theorem?
  - A possible method is to divide an interval  $t$  into small time intervals of width  $\delta$ . A finite  $\delta$  will give a binomial distribution and with  $\delta \rightarrow 0$ , we get a Poisson distribution.



## Customer arriving rate

- Given a Poisson process with parameter  $\lambda$ , we know that the probability of  $n$  customers arriving in a time interval of  $t$  is given by:

$$\frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

- What is the mean number of customers arriving in a time interval of  $t$ ?

$$\sum_{n=0}^{\infty} n \frac{(\lambda t)^n e^{-\lambda t}}{n!} = \lambda t$$

- That's why  $\lambda$  is called the arrival rate.

## Customer inter-arrival time

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- You can also show that if the inter-arrival time distribution is exponential with parameter  $\lambda$ , then the mean inter-arrival time is  $1/\lambda$
- Quite nicely, we have  
Mean arrival rate =  $1 / \text{mean inter-arrival time}$

# Application of Poisson process

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- Poisson process has been used to model the arrival of telephone calls to a telephone exchange successfully
- Queueing networks with Poisson arrival is tractable
  - We will see that in the next few weeks.
- Beware that not all arrival processes are Poisson! Many arrival processes we see in the Internet today are not Poisson. We will see that later.

# References

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- Operational analysis
  - Lazowska et al, Quantitative System Performance, Prentice Hall, 1984. (Classic text on performance analysis. Now out of print but can be downloaded from <http://www.cs.washington.edu/homes/lazowska/qsp/>
    - Chapters 3 and 5 (For Chapter 5, up to Section 5.3 only)
  - Alternative 1: You can read Menasce et al, “Performance by design”, Chapter 3. Note that Menasce doesn’t cover certain aspects of performance bounds. So, you will also need to read Sections 5.1-5.3 of Lazowska.
  - Alternative 2: You can read Harcol-Balter, Chapters 6 and 7. The treatment is more rigorous. You can gross over the discussion on ergodicity.
- Poisson process: Harcol-Balter Chapter 11