HW3

15/05/2024

Q1(a)
For $X_i \overset{i.i.d.}{\sim} \mathsf{Bin}(1,\pi),\ \mathbb{E}(X_i)=\pi$,
$\mathbb{E}(U(X)) = \mathbb{E}(\sum_{i=1}^n X_i) = \sum_{i=1}^n \mathbb{E}(X_i) = n\pi$
$\Rightarrow$ U(X) is biased for π.

(b)
Given that U(X) is sufficient statistic for π, U(X) will be minimal sufficient if proved completeness.
Note $U(X) = \sum_{i=1}^n X_i \in \{0,1,...,n\}$
Consider real function $g : \{0,1,...,n\} \to \mathbb{R}$
Let $\mathbb{E}_\pi[g(U(X))] = 0$ hold for $\forall\pi \in (0,1)$. Then for all $\pi \in (0,1)$:

$$
\begin{aligned}
0 = \mathbb{E}_\pi[g(U(X))] &= \sum_{k=0}^n g(k)\mathbb{P}_\pi(U(X)=k) \\
&= \sum_{k=0}^n g(k)\binom{n}{k}\pi^k(1-\pi)^{n-k},\ for\ U(X) = \sum_{i=1}^n X_i \overset{i.i.d.}{\sim} \mathsf{Bin}(n,\pi) \\
&= \underbrace{(1-\pi)^n}_{>0}\sum_{k=0}^n \underbrace{g(k)\binom{n}{k}}_{:=a_k}\underbrace{(\frac{\pi}{1-\pi})^k}_{:=x\in(0,\infty)}
\end{aligned}
$$

(Using the hint of Ex.3 Q3)

$$
\begin{aligned}
&\Leftrightarrow 0 = a_k = g(k)\underbrace{\binom{n}{k}}_{>0},\ \forall k \in \{0,1,...,n\} \\
&\Leftrightarrow \mathbb{P}_\pi(g(U(x))=0) = 1,\ \forall\pi \in (0,1)
\end{aligned}
$$

So, U(X) is complete, and hence minimal sufficient for π as reasoned above.

(c)

For $U(X) = \sum_{i=1}^{n} X_i \overset{i.i.d.}{\sim} \mathsf{Bin}(n, \pi),\ \mathbb{E}(U(X)) = n\pi,\ Var(U(X)) = n\pi(1-\pi),$

$$\begin{aligned}
\mathbb{E}[V(X)] &= \mathbb{E}[\frac{U(X)[U(X)-1]}{n(n-1)}] \\
&= \frac{1}{n(n-1)}(\mathbb{E}[(U(X))^2] - \mathbb{E}[U(X)]) \\
&= \frac{1}{n(n-1)}(\mathbb{E}[(U(X))^2] - (\mathbb{E}[U(X)])^2 + (\mathbb{E}[U(X)])^2 - \mathbb{E}[U(X)]) \\
&= \frac{1}{n(n-1)}(Var[U(X)] + (\mathbb{E}[U(X)])^2 - \mathbb{E}[U(X)]) \\
&= \frac{1}{n(n-1)}(n\pi(1-\pi) + (n\pi)^2 - n\pi) \\
&= \frac{1}{n(n-1)}(-n\pi^2 + (n\pi)^2) = \pi^2
\end{aligned}$$

$\Rightarrow$ V(X) is unbiased for π^2.