HW6

30/05/2024

Q1(a)
$P(R_{max} \leq r) = P(R_1 \leq r, ..., R_n \leq r) = \prod_{i=1}^{n} P(R_i \leq r) = (F_R(r))^n$, given $R_1, ..., R_n$ $i.i.d.$
Rewrite $f(r) = \frac{2r}{\rho^2}$ $for\ r \in [0, \rho]$, then integrate wrt. r,

$$F_R(r) = \begin{cases} 0 & for\ r < 0 \\ \frac{r^2}{\rho^2} & for\ r \in [0, \rho] \\ 1 & for\ r > \rho \end{cases}$$

Hence, we have $F_{R_{max}}(r) = P(R_{max} \leq r) = \frac{r^{2n}}{\rho^{2n}}$ $for\ r \in [0, \rho]$

$$\begin{aligned} f_{R_{max}}(r \mid \rho) &= \frac{d}{dr} F_{R_{max}}(r \mid \rho) \\ &= \frac{2nr^{2n-1}}{\rho^{2n}} \qquad for\ r \in [0, \rho] \\ &= 2nr^{2n-1} \cdot \frac{1}{\rho^{2n}} \mathbb{I}_{[0,\rho]}(r) \end{aligned}$$

By factorization theorem, R_{max} is sufficient for ρ.

(b) Given the setting that random distances cannot exceed radius, so $P(R_{max} \leq \rho) = 1$. As the distance measure is continuous, then $P(R_{max} = \rho) = 0 \Rightarrow P(R_{max} < \rho) = 1$. Therefore, the bias of R_{max} cannot be zero.

(c)

$$\begin{aligned}
\mathbb{E}(R_1) &= \int_0^\rho rf(r)\ dr \\
&= \int_0^\rho r\frac{2r}{\rho^2}\ dr = \int_0^\rho \frac{2r^2}{\rho^2}\ dr \\
&= \left[\frac{2}{\rho^2}\frac{r^3}{3}\right]_0^\rho \\
&= \frac{2\rho}{3} \\
Bias &= \mathbb{E}(\hat{\rho}) - \rho \\
&= \mathbb{E}(\frac{3}{2n}\sum_{i=1}^n R_i) - \rho \\
&= \frac{3}{2n}\sum_{i=1}^n \mathbb{E}(R_i) - \rho \\
&= \frac{3}{2}\mathbb{E}(R_1) - \rho \qquad (since\ R_1, ..., R_n\ i.i.d.) \\
&= \frac{3}{2}\frac{2\rho}{3} - \rho = 0
\end{aligned}$$

(d) R_{max} and $\hat{\rho}$ satisfied assumptions to apply RB. As proved in part (a), R_{max} is sufficient for ρ. As proved in part (c), $\hat{\rho}$ is unbiased for ρ.
First, derive conditional expectation for fixed $R_{max} = t \in [0, \rho]$:

$$\begin{aligned}
\mathbb{E}_\rho[\hat{\rho} \mid R_{max} = t] &= \mathbb{E}_\rho[\frac{3}{2n}\sum_{i=1}^n R_i \mid R_{max} = t] \\
&= \frac{3}{2n}\sum_{i=1}^n \mathbb{E}_\rho[R_i \mid R_{max} = t] \\
&= \frac{3}{2n}[t + \sum_{i=2}^n \mathbb{E}_\rho[R_i \mid R_1 = t]] \qquad (W.L.O.G.\ take\ R_1\ as\ R_{max}) \\
&= \frac{3}{2n}[t + (n-1)\frac{2t}{3}] \qquad (from\ part(c)\ \mathbb{E}(R_i)\ with\ upper\ bound\ replaced\ by\ t) \\
&= \frac{3t}{2n} + \frac{(n-1)t}{n} \\
&= \frac{2n+1}{2n}t
\end{aligned}$$

$\Rightarrow \hat{\rho}^* = \frac{2n+1}{2n}R_{max}$

(e) Since RB estimator $\hat{\rho}^*$ is unbiased for ρ,

$$\rho = \mathbb{E}_\rho[\frac{2n+1}{2n} R_{max}] = \frac{2n+1}{2n} \mathbb{E}_\rho[R_{max}]$$
$$\Rightarrow \mathbb{E}_\rho[R_{max}] = \frac{2n}{2n+1}\rho$$
$$Bias_\rho[R_{max}] = \mathbb{E}_\rho[R_{max}] - \rho = \frac{-1}{2n+1}\rho$$

(f)

$$\begin{aligned}
MSE_\rho[R_{max}] &= (Bias_\rho[R_{max}])^2 + Var_\rho[R_{max}] \\
&= (\frac{-1}{2n+1}\rho)^2 + \mathbb{E}_\rho[R_{max}^2] - (\mathbb{E}_\rho[R_{max}])^2 \\
&= \frac{\rho^2}{(2n+1)^2} + \frac{n}{n+1}\rho^2 - (\frac{2n}{2n+1}\rho)^2 \qquad (Given\ \mathbb{E}_\rho[R_{max}^2] = \frac{n}{n+1}\rho^2) \\
&= \rho^2[\frac{(n+1)(1-4n^2) + n(2n+1)^2}{(n+1)(2n+1)^2}] \\
&= \rho^2[\frac{(2n+1)}{(n+1)(2n+1)^2}] \\
&= \frac{\rho^2}{(n+1)(2n+1)}
\end{aligned}$$