HW7

13/06/2024

Q1(a) Likelihood:

$$\begin{aligned}
L(\lambda, \mu) &= f(\mathbf{x}, \mathbf{y} \mid \lambda, \mu) \\
&= f(x_1, \ldots, x_m \mid \lambda) f(y_1, \ldots, y_n \mid \mu) \qquad (\mathbf{X}, \mathbf{Y} \ \textit{independent}) \\
&= \prod_{i=1}^{m} f(x_i \mid \lambda) \prod_{j=1}^{n} f(y_j \mid \mu) \qquad \left(X_i \overset{i.i.d.}{\sim} Exp(\lambda),\ Y_j \overset{i.i.d.}{\sim} Exp(\mu) \right)
\end{aligned}$$

Log-likelihood:

$$\begin{aligned}
l(\lambda, \mu) &= log \left(\prod_{i=1}^{m} f(x_i \mid \lambda) \prod_{j=1}^{n} f(y_j \mid \mu) \right) \\
&= \sum_{i=1}^{m} log\ f(x_i \mid \lambda) + \sum_{j=1}^{n} log\ f(y_j \mid \mu) \\
&= \sum_{i=1}^{m} log(\lambda\ exp(-\lambda x_i)) + \sum_{j=1}^{n} log(\mu\ exp(-\mu y_j)) \\
&= m\ log\ \lambda - \lambda m \bar{x} + n\ log\ \mu - \mu n \bar{y}
\end{aligned}$$

Score function:

$$\begin{aligned}
s(\lambda, \mu \mid \mathbf{x}, \mathbf{y}) &= \left(\frac{\partial l(\lambda, \mu)}{\partial \lambda},\ \frac{\partial l(\lambda, \mu)}{\partial \mu} \right)^T \\
&= \left(\frac{m}{\lambda} - m\bar{x},\ \frac{n}{\mu} - n\bar{y} \right)^T \overset{!}{=} (0, 0)^T \\
&\Leftrightarrow \begin{cases} \frac{m}{\lambda} = m\bar{x} \\ \frac{n}{\mu} = n\bar{y} \end{cases} \\
\Rightarrow (\hat{\lambda}_{ML},\ \hat{\mu}_{ML})^T &= \left(\frac{1}{\bar{X}},\ \frac{1}{\bar{Y}} \right)^T
\end{aligned}$$

(b)

$$
\begin{aligned}
\mathbb{E}[V(\mathbf{X},\mathbf{Y})] &= \left(\mathbb{E}\left(\frac{1}{\hat{\lambda}_{ML}}\right),\ \mathbb{E}\left(\frac{1}{\hat{\mu}_{ML}}\right)\right)^T \\
&= (\mathbb{E}(\bar{X}),\ \mathbb{E}(\bar{Y}))^T \\
&= \left(\frac{1}{m}\sum_{i=1}^{m}\mathbb{E}(X_i),\ \frac{1}{n}\sum_{j=1}^{n}\mathbb{E}(Y_j)\right)^T \\
&= \left(\frac{1}{\lambda},\frac{1}{\mu}\right)^T \qquad \left(X_i \overset{i.i.d.}{\sim} Exp(\lambda),\ Y_j \overset{i.i.d.}{\sim} Exp(\mu)\right) \\
Cov(V(\mathbf{X},\mathbf{Y})) &= Cov\left(\left(\frac{1}{\hat{\lambda}_{ML}},\ \frac{1}{\hat{\mu}_{ML}}\right)^T\right) \\
&= Cov((\bar{X},\ \bar{Y})^T) \\
&= \begin{pmatrix} Var(\bar{X}) & 0 \\ 0 & Var(\bar{Y}) \end{pmatrix} \qquad (\mathbf{X},\mathbf{Y}\ independent) \\
&= \begin{pmatrix} \frac{1}{m^2}Var(\sum_{i=1}^{m}X_i) & 0 \\ 0 & \frac{1}{n^2}Var(\sum_{j=1}^{n}Y_j) \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{m\lambda^2} & 0 \\ 0 & \frac{1}{n\mu^2} \end{pmatrix} \qquad \left(X_i \overset{i.i.d.}{\sim} Exp(\lambda),\ Y_j \overset{i.i.d.}{\sim} Exp(\mu)\right)
\end{aligned}
$$

(c) Based on the given assumption on $V(\mathbf{X},\mathbf{Y})$, we let $\boldsymbol{\theta} = (\frac{1}{\lambda},\frac{1}{\mu})^T,\ \hat{\boldsymbol{\theta}}_n = V(\mathbf{X},\mathbf{Y}),\ V(\boldsymbol{\theta}) = \mathbf{\Sigma}_V$ for Delta method notation.

$$
\begin{pmatrix} \sqrt{m} & 0 \\ 0 & \sqrt{n} \end{pmatrix} (\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}) \xrightarrow{d} N_2(\mathbf{0}, V(\boldsymbol{\theta}))
$$

Let $h : \mathbb{R}^2 \to \mathbb{R}^2, \qquad h(\boldsymbol{\theta}) = h((\frac{1}{\lambda},\frac{1}{\mu})^T) = (\lambda,\mu)^T$
Now check conditions to apply the Delta method are fulfilled:

$$
\begin{aligned}
H(\boldsymbol{\theta}) &= \left(\frac{\partial h(\boldsymbol{\theta})}{\partial \lambda},\ \frac{\partial h(\boldsymbol{\theta})}{\partial \mu}\right) \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}
$$

so, H is of full rank $r = 2$ while every row not equal to zero vector. h is component-wise continuously partially differentiable.

Apply Delta method,

$$
\begin{aligned}
h(\hat{\boldsymbol{\theta}}_n) &= T(\mathbf{X}, \mathbf{Y}) \\
\begin{pmatrix} \sqrt{m} & 0 \\ 0 & \sqrt{n} \end{pmatrix} (h(\hat{\boldsymbol{\theta}}_n) - h(\boldsymbol{\theta})) &\xrightarrow{d} N_2(0, H(\boldsymbol{\theta}) V(\boldsymbol{\theta}) H(\boldsymbol{\theta})^T) \\
\begin{pmatrix} \sqrt{m} & 0 \\ 0 & \sqrt{n} \end{pmatrix} (T(\mathbf{X}, \mathbf{Y}) - (\lambda, \mu)^T) &\xrightarrow{d} N_2 \left(0, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \boldsymbol{\Sigma}_V \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \\
\implies T(\mathbf{X}, \mathbf{Y}) &\overset{a}{\sim} N_2 \left((\lambda, \mu)^T, \begin{pmatrix} \sqrt{m} & 0 \\ 0 & \sqrt{n} \end{pmatrix}^{-1} \boldsymbol{\Sigma}_V \begin{pmatrix} \sqrt{m} & 0 \\ 0 & \sqrt{n} \end{pmatrix}^{-T} \right)
\end{aligned}
$$

(d) Since Cramer-Rao bound for $T(\mathbf{X}, \mathbf{Y})$ is $I^{-1}((\lambda, \mu)^T)$,

$$
\begin{aligned}
I((\lambda, \mu)^T) &= \mathbb{E}_{\lambda, \mu} \left[-\frac{\partial s(\lambda, \mu \mid \mathbf{X}, \mathbf{Y})}{\partial(\lambda, \mu)} \right] \\
&= \mathbb{E}_{\lambda, \mu} \left[- \begin{pmatrix} \frac{-m}{\lambda^2} & 0 \\ 0 & \frac{-n}{\mu^2} \end{pmatrix} \right] \\
&= \begin{pmatrix} \frac{m}{\lambda^2} & 0 \\ 0 & \frac{n}{\mu^2} \end{pmatrix} \\
I^{-1}((\lambda, \mu)^T) &= \frac{\lambda^2 \mu^2}{mn} \begin{pmatrix} \frac{n}{\mu^2} & 0 \\ 0 & \frac{m}{\lambda^2} \end{pmatrix} \\
&= \begin{pmatrix} \frac{\lambda^2}{m} & 0 \\ 0 & \frac{\mu^2}{n} \end{pmatrix}
\end{aligned}
$$

For single observation $(X_i,\ Y_j)$,

$$
i^{-1}((\lambda, \mu)^T) = \begin{pmatrix} \lambda^2 & 0 \\ 0 & \mu^2 \end{pmatrix}
$$

Cramer-Rao bound is achieved if

$$
\begin{aligned}
&\boldsymbol{\Sigma}_V = i^{-1}((\lambda, \mu)^T) \\
&\Leftrightarrow \begin{cases} \frac{1}{\lambda^2} = \lambda^2 \\ \frac{1}{\mu^2} = \mu^2 \end{cases} \\
&\Leftrightarrow \begin{cases} \lambda = 1 \\ \mu = 1 \end{cases} \qquad \textit{(consider real solution only and given } \lambda,\ \mu > 0)
\end{aligned}
$$