HW10

04/07/2024

Q1

(a) The kernel of posterior density for $\boldsymbol{\beta}|\boldsymbol{y}, \boldsymbol{X}$ is not easily recognisable as any know distribution, so analytic solution may not be easily obtained and MH algorithm is then suitable for sampling from the posterior.
Let $q(\boldsymbol{\beta_{new}} \mid \boldsymbol{\beta_{old}})$ be the density of given proposal distribution $N_p(\boldsymbol{\beta_{old}}, \widehat{Cov}(\hat{\boldsymbol{\beta}}_{\boldsymbol{ML}}))$. The algorithm in pseudo code is as follows:

(i) Set the staring value of $\boldsymbol{\beta^{(0)}}$, e.g. using prior mean $\boldsymbol{b_0}$.

(ii) For $t = 1, \ldots, 1000$:

1. Draw $\boldsymbol{\beta^*}$ from $q(\boldsymbol{\beta^*} \mid \boldsymbol{\beta^{(t-1)}})$.
2. Compute the acceptance probability: $\alpha(\boldsymbol{\beta^*} \mid \boldsymbol{\beta^{(t-1)}}) = min\left(1, \frac{p(\boldsymbol{\beta}^*|\boldsymbol{y},\boldsymbol{X})q(\boldsymbol{\beta^{(t-1)}}|\boldsymbol{\beta^*})}{p(\boldsymbol{\beta}^{(t-1)}|\boldsymbol{y},\boldsymbol{X})q(\boldsymbol{\beta^*}|\boldsymbol{\beta^{(t-1)}})}\right)$
3. With probability $\alpha(\boldsymbol{\beta^*} \mid \boldsymbol{\beta^{(t-1)}})$ set $\boldsymbol{\beta^{(t)}} = \boldsymbol{\beta^*}$. Else, set $\boldsymbol{\beta^{(t)}} = \boldsymbol{\beta^{(t-1)}}$.

(iii) Optionally, discard first B iterations as burn-in.

(iv) Output samples $\boldsymbol{\beta^{(B+1)}}, \ldots, \boldsymbol{\beta^{(1000)}}$.

(b) Point estimates of individual parameters can be approximated from the generated random draws. The posterior mean is approximated by the sample average:

$$\mathbb{E}(\boldsymbol{\beta} \mid \boldsymbol{y}, \boldsymbol{X}) \approx \frac{1}{1000 - B} \sum_{t=B+1}^{1000} \boldsymbol{\beta}^{(t)}$$

(c) The credible interval can be defined via empirical quantiles of the respective random draws, e.g. 90% credible interval:

$$\hat{Q}_{0.05}(\boldsymbol{\beta} \mid \boldsymbol{y}, \boldsymbol{X}) \approx q_{0.05}(\boldsymbol{\beta^{(B+1)}}, \ldots, \boldsymbol{\beta^{(1000)}})$$
$$\hat{Q}_{0.95}(\boldsymbol{\beta} \mid \boldsymbol{y}, \boldsymbol{X}) \approx q_{0.95}(\boldsymbol{\beta^{(B+1)}}, \ldots, \boldsymbol{\beta^{(1000)}})$$