HW2

09/05/2024

Q1(a)

$$\begin{aligned} R(T, a, b) &= MSE_{a,b}(T(X)) \\ &= Var_{a,b}(T(X)) + (Bias_{a,b}(T(X)))^2 \end{aligned}$$

$$\begin{aligned} Var_{a,b}(T(X)) &= Var_{a,b}(\frac{1}{n}\sum_{i=1}^{n} X_i) \\ &= \frac{1}{n^2}\sum_{i=1}^{n} Var_{a,b}(X_i) \\ &= \frac{b^2\pi^2}{3n} \end{aligned}$$

$$\begin{aligned} (Bias_{a,b}(T(X)))^2 &= (\mathbb{E}_{a,b}(T(X)) - a)^2 \\ &= (\mathbb{E}_{a,b}(\frac{1}{n}\sum_{i=1}^{n} X_i) - a)^2 \\ &= (\frac{1}{n}\sum_{i=1}^{n} \mathbb{E}_{a,b}(X_i) - a)^2 \\ &= (\frac{1}{n}\sum_{i=1}^{n} a - a)^2 \\ &= 0 \end{aligned}$$

$$\begin{aligned} R(T, a, b) &= Var_{a,b}(T(X)) + (Bias_{a,b}(T(X)))^2 \\ &= \frac{b^2\pi^2}{3n} \end{aligned}$$

(b)

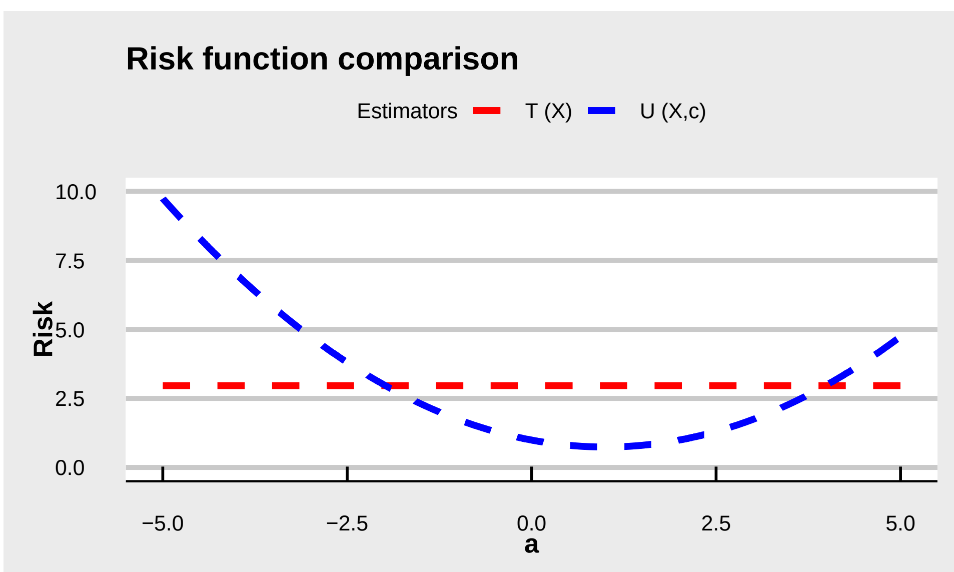
$$\begin{aligned} R(U, a, b) &= MSE_{a,b}(U(X, c)) \\ &= Var_{a,b}(U(X, c)) + (Bias_{a,b}(U(X, c)))^2 \end{aligned}$$

$$\begin{aligned}
Var_{a,b}(U(X,c)) &= Var_{a,b}(\frac{c}{2} + \frac{1}{2n}\sum_{i=1}^{n} X_i) \\
&= \frac{1}{4n^2}\sum_{i=1}^{n} Var_{a,b}(X_i) \\
&= \frac{b^2\pi^2}{12n}
\end{aligned}$$

$$\begin{aligned}
(Bias_{a,b}(U(X,c)))^2 &= (\mathbb{E}_{a,b}(U(X,c)) - a)^2 \\
&= (\mathbb{E}_{a,b}(\frac{c}{2} + \frac{1}{2n}\sum_{i=1}^{n} X_i) - a)^2 \\
&= (\frac{c}{2} + \frac{1}{2n}\sum_{i=1}^{n} \mathbb{E}_{a,b}(X_i) - a)^2 \\
&= (\frac{c}{2} + \frac{1}{2n}\sum_{i=1}^{n} a - a)^2 \\
&= (\frac{c}{2} + \frac{a}{2} - a)^2 \\
&= \frac{(c-a)^2}{4}
\end{aligned}$$

$$\begin{aligned}
R(U,a,b) &= Var_{a,b}(U(X,c)) + (Bias_{a,b}(U(X,c)))^2 \\
&= \frac{b^2\pi^2}{12n} + \frac{(c-a)^2}{4}
\end{aligned}$$

(c) Given $a \in [-5, 5], b = 3, c = 1$ and $n = 10$.



(d) Observed that T(X) is constant with respect to a, the minimax estimator is $T(X) = \text{argmin}_{\{T,U\}} \max_a R$.