HW9

27/06/2024

Q1 Let $\mathbf{X}^{(A)} = (X_1, \cdots, X_\theta)^T,\ \mathbf{X}^{(B)} = (X_{\theta+1}, \cdots, X_{112})^T$.

$$\begin{aligned}
f_{\lambda_1|\lambda_2,\alpha,\theta,\mathbf{X}}(\lambda_1 \mid \lambda_2, \alpha, \theta, \mathbf{x}) &= \frac{f_{\lambda_1,\lambda_2,\alpha,\theta,\mathbf{X}}(\lambda_1, \lambda_2, \alpha, \theta, \mathbf{x})}{f_{\lambda_2,\alpha,\theta,\mathbf{X}}(\lambda_2, \alpha, \theta, \mathbf{x})} \\
&\propto f_{\lambda_1,\lambda_2,\alpha,\theta,\mathbf{X}}(\lambda_1, \lambda_2, \alpha, \theta, \mathbf{x}) \\
&= f_{\mathbf{X}|\lambda_1,\lambda_2,\alpha,\theta}(\mathbf{x} \mid \lambda_1, \lambda_2, \alpha, \theta) f_{\lambda_1,\lambda_2|\alpha,\theta}(\lambda_1, \lambda_2 \mid \alpha, \theta) f_{\alpha,\theta}(\alpha, \theta) \\
&\propto f_{\mathbf{X}|\lambda_1,\lambda_2,\alpha,\theta}(\mathbf{x} \mid \lambda_1, \lambda_2, \alpha, \theta) f_{\lambda_1|\alpha,\theta}(\lambda_1 \mid \alpha, \theta) f_{\lambda_2|\alpha,\theta}(\lambda_2 \mid \alpha, \theta) \qquad \text{(independent priors)} \\
&\propto f_{\mathbf{X}|\lambda_1,\lambda_2,\alpha,\theta}(\mathbf{x} \mid \lambda_1, \lambda_2, \alpha, \theta) f_{\lambda_1|\alpha,\theta}(\lambda_1 \mid \alpha, \theta) \\
&\propto f_{\mathbf{X}^{(A)}|\lambda_1,\alpha,\theta}(\mathbf{x}^{(A)} \mid \lambda_1, \alpha, \theta) f_{\lambda_1|\alpha,\theta}(\lambda_1 \mid \alpha, \theta) \qquad (X_i \text{ iid and } \mathbf{X}^{(B)} \text{ not depend on } \lambda_1) \\
&\propto \left(\prod_{i=1}^{\theta} \lambda_1^{x_i} exp(-\lambda_1) \right) \lambda_1^2 exp(-\alpha\lambda_1) \\
&= \left(\lambda_1^{\sum_{i=1}^{\theta} x_i} exp(-\theta\lambda_1) \right) \lambda_1^2 exp(-\alpha\lambda_1) \\
&= \lambda_1^{2+\sum_{i=1}^{\theta} x_i} exp(-\lambda_1(\theta + \alpha))
\end{aligned}$$

$\implies$ kernel of $Ga\left(3 + \sum_{i=1}^{\theta} x_i, \alpha + \theta\right)$

$$\begin{aligned}
f_{\lambda_2|\lambda_1,\alpha,\theta,\mathbf{X}}(\lambda_2 \mid \lambda_1, \alpha, \theta, \mathbf{x}) &= \frac{f_{\lambda_1,\lambda_2,\alpha,\theta,\mathbf{X}}(\lambda_1, \lambda_2, \alpha, \theta, \mathbf{x})}{f_{\lambda_1,\alpha,\theta,\mathbf{X}}(\lambda_1, \alpha, \theta, \mathbf{x})} \\
&\propto f_{\lambda_1,\lambda_2,\alpha,\theta,\mathbf{X}}(\lambda_1, \lambda_2, \alpha, \theta, \mathbf{x}) \\
&= f_{\mathbf{X}|\lambda_1,\lambda_2,\alpha,\theta}(\mathbf{x} \mid \lambda_1, \lambda_2 \alpha, \theta) f_{\lambda_1,\lambda_2|\alpha,\theta}(\lambda_1, \lambda_2 \mid \alpha, \theta) f_{\alpha,\theta}(\alpha, \theta) \\
&\propto f_{\mathbf{X}|\lambda_1,\lambda_2,\alpha,\theta}(\mathbf{x} \mid \lambda_1, \lambda_2 \mid \alpha, \theta) f_{\lambda_1|\alpha,\theta}(\lambda_1 \mid \alpha, \theta) f_{\lambda_2|\alpha,\theta}(\lambda_2 \mid \alpha, \theta) \qquad \text{(independent priors)} \\
&\propto f_{\mathbf{X}|\lambda_1,\lambda_2,\alpha,\theta}(\mathbf{x} \mid \lambda_1, \lambda_2, \alpha, \theta) f_{\lambda_2|\alpha,\theta}(\lambda_2 \mid \alpha, \theta) \\
&\propto f_{\mathbf{X}^{(B)}|\lambda_2,\alpha,\theta}(\mathbf{x}^{(B)} \mid \lambda_2, \alpha, \theta) f_{\lambda_2|\alpha,\theta}(\lambda_2 \mid \alpha, \theta) \qquad (X_i \text{ iid and } \mathbf{X}^{(A)} \text{ not depend on } \lambda_2) \\
&\propto \left(\prod_{i=\theta+1}^{112} \lambda_2^{x_i} exp(-\lambda_2) \right) \lambda_2^2 exp(-\alpha\lambda_2) \\
&= \left(\lambda_2^{\sum_{i=\theta+1}^{112} x_i} exp(-(112-\theta)\lambda_2) \right) \lambda_2^2 exp(-\alpha\lambda_2) \\
&= \lambda_2^{2+\sum_{i=\theta+1}^{112} x_i} exp(-\lambda_2(112 - \theta + \alpha))
\end{aligned}$$

$\implies$ kernel of $Ga\left(3 + \sum_{i=\theta+1}^{112} x_i, \alpha + (112 - \theta)\right)$

$$
\begin{aligned}
f_{\alpha\mid\lambda_1,\lambda_2,\theta,\mathbf{X}}(\alpha \mid \lambda_1,\lambda_2,\theta,\mathbf{x}) &= \frac{f_{\lambda_1,\lambda_2,\alpha,\theta,\mathbf{X}}(\lambda_1,\lambda_2,\alpha,\theta,\mathbf{x})}{f_{\lambda_1,\lambda_2,\theta,\mathbf{X}}(\lambda_1,\lambda_2,\theta,\mathbf{x})} \\
&\propto f_{\lambda_1,\lambda_2,\alpha,\theta,\mathbf{X}}(\lambda_1,\lambda_2,\alpha,\theta,\mathbf{x}) \\
&= f_{\mathbf{X}\mid\lambda_1,\lambda_2,\theta}(\mathbf{x} \mid \lambda_1,\lambda_2,\theta) f_{\lambda_1,\lambda_2,\alpha,\theta}(\lambda_1,\lambda_2,\alpha,\theta) \\
&\propto f_{\lambda_1,\lambda_2\mid\alpha,\theta}(\lambda_1,\lambda_2 \mid \alpha,\theta) f_{\alpha,\theta}(\alpha,\theta) \\
&= f_{\lambda_1\mid\alpha,\theta}(\lambda_1 \mid \alpha,\theta) f_{\lambda_2\mid\alpha,\theta}(\lambda_2 \mid \alpha,\theta) f_{\alpha,\theta}(\alpha,\theta) \qquad \text{(independent priors)} \\
&= f_{\lambda_1\mid\alpha,\theta}(\lambda_1 \mid \alpha,\theta) f_{\lambda_2\mid\alpha,\theta}(\lambda_2 \mid \alpha,\theta) f_{\alpha}(\alpha) \qquad \text{(assumed fixed } \theta\text{)} \\
&\propto \alpha^3 exp(-\alpha\lambda_1)\alpha^3 exp(-\alpha\lambda_2)\alpha^9 exp(-10\alpha) \\
&= \alpha^{15} exp(-\alpha(\lambda_1+\lambda_2+10))
\end{aligned}
$$

$\implies$ kernel of $Ga\,(16, 10+\lambda_1+\lambda_2)$