ASTRONOMY 224

Probability Functions and the Standard Deviation

Random Errors: The Gaussian (Normal) Distribution (Taylor Ch. 5)

When the results of an experiment are governed by random errors, the distribution of measurement values will follow a specific probability distribution, named after the great mathematician and astronomer Karl Friedrich Gauss (1777-1855). This "bell curve" or normal distribution will be centered on the mean value of the measurement. *The width of the distribution reflects the inherent accuracy in the measurement process and is usually characterized by the standard deviation.* The standard deviation (σ) is the "error per point", where the probability that any individual measurement with lie within one σ of the experimental mean is 68%, within two σ is 95.4% and within three σ is 99.7%.

Mathematically, one can describe the Gaussian distribution as follows. If x is a random variable with mean X and standard deviation σ, then values of x follow the Gaussian distribution:

$$G_{X,\sigma}(x) = \frac{1}{\sigma\sqrt{2\pi}} exp[-\frac{(x-X)^2}{2\sigma^2}].$$

The coefficient of the exponential factor guarantees that the integral of G over all x is 1. That is, x lies between $-\infty$ and $+\infty$ with perfect certainty. Following the usual definition of a function in the calculus, $G_{X,\sigma}(x_1)dx$ is the probability that x lies in the infinitesimal interval between x_1 and $x_1 + dx$. Then, the probability that x lies in the finite interval between x_1 and x_2 is the definite integral of G between the limits x_1 and x_2. This basic definition leads to the results in Taylor's Section 5.4 (see attached pages).

Counting Errors: The Poisson Distribution(Taylor Ch. 11)

The distribution of measurements will be somewhat different in a counting experiment where the thing that you are counting is itself subject to inherent random fluctuations about a mean rate. For example when acquiring a digital image, the CCD captures photons from each star that arrive at a definite average rate, but there are random fluctuations in the number of detections in successive time intervals. Similar results occur for detections of radioactive decays and for public opinion polls. (In the latter case, however, it takes skill to ensure that the sample really is random.) The probabilty distribution that occurs in a counting experiment is called a Poisson distribution. Like a Gaussian distribution it will have a peak at the mean value, in this case the average rate of arrival of photons from a distant source. *In a counting experiment the width of the distribution is NOT governed by the accuracy of your measuring technique, but insteady by the randomness in the arrival of the quantity being counted.* In this case there is a UNIQUE value for the standard deviation associated with a specific mean value- if the mean number of counts is N, then the standard deviation σ will be $N^{1/2}$!

Mathematically, this can be expressed as follows. If μ is the average number of recorded events, the probability of measuring n is given by the Poisson Distribution:

$$P_\mu(n) = \frac{\mu^n}{n!}\ e^{-\mu}\ .$$

Note that the Poisson Distribution has a only single parameter, μ, in contrast to the Gaussian distribution which has both X and σ. It can be shown that the associated standard deviation is just $\sigma = \mu^{1/2}$, which is what you found in the empirical result from your photon-counting data. Thus, when you are counting photons, a *single* measurement allows you to estimate *both* the average count (what you measured) and its standard deviation! How cool!!

When you glance at the mathematical forms of the two distributions they look quite different. Interestingly when μ is large (in excess of 10 or so) the shape of the two distributions become nearly indistinguishable from each other. The difference is that in an experiment governed by Gaussian statistics you can reduce the standard deviation by improving your measuring technique while for an experiment governed by Poisson statistics the standard deviation is inextricably linked to the mean number of counts.

Thus, for photon-counting experiments, you can readily estimate μ and σ and you may interpret the chances of extreme events using the characteristics of the Gaussian Distribution. For example, suppose you measured 25 counts for one star, which you understand as 25 ± 5. For another star you measure 36 counts, i.e., 36 ± 6. How much confidence would you have that the second star actually is brighter than the first? The difference is 11 counts- but is this statistically significant? A rough estimate of the uncertainty is the sum of the individual uncertainties, 11 counts. (A better estimate is $(5^2 + 6^2)^{1/2} \approx 8$.) You can conclude that the second star *probably* is brighter than the first, but you would not want to bet next semester's tuition on the outcome of repeated measurements. It works out that there is about an 8% chance that the first star is at least as bright as the second. To know for sure you need to get more counts for each of them!