Random Errors

When the results of an experiment are governed by random errors, the distribution of measurement values will follow a specific probability distribution called the Gaussian distribution:

$$G_{\bar{x},\sigma}(x) = \tfrac{1}{\sigma\sqrt{2\pi}}\ exp[-\tfrac{(x-\bar{x})^2}{2\sigma^2}].$$

where the variable x represents repeated measurements of the same quantity, the constant $\bar{x}$ is the mean and the constant σ is the standard deviation about the mean. Associated statistical indicators can be reasoned out intuitively, although a formal derivation from the Gaussian function produces minor modifications for a few of these, such as replacing N with $N-1$ in the definition of variance and standard deviation. In each definition, x_i are the individual measurements and N is the number of measured values.

mean:

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i = \frac{1}{N}(x_1 + x_2 + \ldots + x_N)$$

deviation:

$$d_i = (x_i - \bar{x})$$

average deviation or dispersion:

$$\bar{d} = \frac{1}{N}\sum_{i=1}^{N} |d_i| = \frac{1}{N}\sum_{i=1}^{N} |(x_i - \bar{x})|$$

variance:

$$(\sigma_x)^2 = \frac{1}{N-1}\sum_{i=1}^{N} (d_i)^2 = \frac{1}{N-1}\sum_{i=1}^{N} (x_i - \bar{x})^2$$

standard deviation:

$$\sigma_x = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N} (d_i)^2} = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N} (x_i - \bar{x})^2}$$

Error Propagation Rules, from Taylor's Introduction to Error Analysis

Sums and Differences

$$R = X + Y - Z$$

$$\delta R \approx \delta X + \delta Y + \delta Z$$

$$\delta R = \sqrt{(\delta X)^2 + (\delta Y)^2 + (\delta Z)^2}$$

Products and Quotients

$$R = \frac{X \cdot Y}{Z}$$

$$\frac{\delta R}{|R|} \approx \frac{\delta X}{|X|} + \frac{\delta Y}{|Y|} + \frac{\delta Z}{|Z|}$$

$$\delta R = |R| \cdot \sqrt{\left(\frac{\delta X}{X}\right)^2 + \left(\frac{\delta Y}{Y}\right)^2 + \left(\frac{\delta Z}{Z}\right)^2}$$

Multiplication by Constant

$$R = c \cdot X$$

$$\delta R = |c| \cdot \delta X$$

Power Law

$$R = X^n$$

$$\delta R = |n| \cdot \frac{\delta X}{|X|} \cdot |R|$$

General Case

$$R = R(X, Y, \ldots)$$

$$\delta R = \sqrt{\left(\frac{\partial R}{\partial X} \cdot \delta X\right)^2 + \left(\frac{\partial R}{\partial Y} \cdot \delta Y\right)^2 + \ldots}$$