Accounting and Finance

Class 4
Time Value of Money

Class Outline

- **Topics**
  - 4.1) Compounding and Future Value
  - 4.2) Discounting and Present Value
  - 4.3) Useful Present Value Formulas
  - 4.4) Net Present Value (NPV)
  - 4.5) More Frequent Compounding
- **Readings**
  - Berk and DeMarzo, sections 3.1-3.3, chapter 4, and sections 5.1-5.2
  - Case: Natasha Kingery
- **Practice Problems**
  - Canvas: Graded Problem Set #1 (due at 11:59pm on March 11)
  - MyLab: Practice Problem Set #1 (not graded)
  - Download: Practice Problem Set #1 (not graded)

Section 4.1
Compounding and Future Value

Motivation

- At the most general level, an *investment* is a claim to a stream of cash flows.
  - Real investments: new plant, new project, acquisition, etc.
  - Financial investments: bonds, stocks, options, mortgage-backed securities, credit default swaps, etc.
- These cash flows differ along three dimensions:
  - size;
  - timing;
  - risk.

Motivation

- At the most general level, an *investment* is a claim to a stream of cash flows.
  - Real investments: new plant, new project, acquisition, etc.
  - Financial investments: bonds, stocks, options, mortgage-backed securities, credit default swaps, etc.
- These cash flows differ along three dimensions:
  - size;
  - timing;
  - risk.
- Thus, in order to choose between alternative investments, we must find a way to compare cash flows differing in these three dimensions.
  - Let us postpone our treatment of risk until later and focus for now on comparing *certain* (or *risk-free*) cash flows.
  - The techniques of compounding and discounting allow us to compare cash flows differing in size and timing.

Compounding and Future Value: Numerical Example

- Suppose that you invest $1,000 in a bank account paying an interest rate of 10%, and that interest is credited to your account once a year.
  - Money in the account after one year:

Investment:	$1,000.00	
Interest (1,000 × 10%):	$100.00	
Total:	$1,100.00	= $1,000(1.10)

Compounding and Future Value: Numerical Example

- Suppose that you invest $1,000 in a bank account paying an interest rate of 10%, and that interest is credited to your account once a year.
  - Money in the account after one year:

Investment:	\$1,000.00	
Interest (1,000 × 10%):	\$100.00	
Total:	\$1,100.00	= \$1,000(1.10)

  - Money in the account after two years:

Re-Investment:	\$1,100.00	
Interest (1,100 × 10%):	\$110.00	[>\$100.00]
Total:	\$1,210.00	$= \$1,000(1.10)^2$

Compounding and Future Value: Numerical Example

- Suppose that you invest $1,000 in a bank account paying an interest rate of 10%, and that interest is credited to your account once a year.
  - Money in the account after one year:

Investment:	\$1,000.00	
Interest (1,000 × 10%):	\$100.00	
Total:	\$1,100.00	$= \$1{,}000(1.10)$

  - Money in the account after two years:

Re-Investment:	\$1,100.00	
Interest (1,100 × 10%):	\$110.00	[>\$100.00]
Total:	\$1,210.00	$= \$1{,}000(1.10)^2$

  - Money in the account after three years:

Re-Investment:	\$1,210.00	
Interest (1,210 × 10%):	\$121.00	
Total:	\$1,331.00	$= \$1{,}000(1.10)^3$

- Continuing this reasoning, you will have $\$1{,}000(1.10)^T$ after T years.

Compounding and Future Value: General Case

- Suppose now that you invest C dollars in a bank account paying an interest rate r, and that interest is credited to your account once a year.
  - Money in the account after one year:

Investment:	$\$C$
Interest:	$\$Cr$
Total:	$\$C(1+r)$

Compounding and Future Value: General Case

- Suppose now that you invest C dollars in a bank account paying an interest rate r, and that interest is credited to your account once a year.
  - Money in the account after one year:

Investment:	$\$C$
Interest:	$\$Cr$
Total:	$\$C(1+r)$

  - Money in the account after two years:

Re-Investment:	$\$C(1+r)$	
Interest:	$\$C(1+r)r$	
Total:	$\$C(1+r)(1+r)$	$= \$C(1+r)^2$

Compounding and Future Value: General Case

- Suppose now that you invest C dollars in a bank account paying an interest rate r, and that interest is credited to your account once a year.
  - Money in the account after one year:

Investment:	$\$C$
Interest:	$\$Cr$
Total:	$\$C(1+r)$

  - Money in the account after two years:

Re-Investment:	$\$C(1+r)$	
Interest:	$\$C(1+r)r$	
Total:	$\$C(1+r)(1+r)$	$= \$C(1+r)^2$

- Continuing this reasoning, you will have $\$C(1+r)^T$ after T years.
- We say that this quantity is the *future value* in T years of C dollars invested at a rate of r *compounded annually*:

$$\boxed{FV_T = C(1+r)^T}$$

Compounding: Example

- **Background**
  - Mutual funds charge annual fees to investors.
  - Fees are usually a percentage of the amount in the account.
  - Fees can go from 0.02%/yr all the way up to 2.00%/yr.
- **Question 1**
  - Kira is 30 years old and plans to retire at 65 (in 35 years).
  - So far, she has saved $80,000 toward retirement.
  - How much will Kira have when she retires if the money is invested at 12% per year?

Compounding: Example (cont'd)

- **Question 1**. We can use the formula from page 4.6 to find the future value of Kira's investment:

$$FV_{35} = \$80{,}000(1.12)^{35} = \$4.224 \text{ million.}$$

Compounding: Example

- **Background**
  - Mutual funds charge annual fees to investors.
  - Fees are usually a percentage of the amount in the account.
  - Fees can go from 0.02%/yr all the way up to 2.00%/yr.
- **Question 1**
  - Kira is 30 years old and plans to retire at 65 (in 35 years).
  - So far, she has saved $80,000 toward retirement.
  - How much will Kira have when she retires if the money is invested at 12% per year?
- **Question 2**
  - Kira just discovered that her fund will charge her 1% per year in fees.
  - This means that her investment will compound at a rate of only 11% per year.
  - How much will the $80,000 that she saves now be worth when she retires in 35 years?

Compounding: Example (cont'd)

- **Question 1**. We can use the formula from page 4.6 to find the future value of Kira's investment:

$$FV_{35} = \$80{,}000(1.12)^{35} = \$4.224 \text{ million.}$$

- **Question 2**. If her investment compounds at only 11% per year, then its future value is

$$FV_{35} = \$80{,}000(1.11)^{35} = \$3.086 \text{ million.}$$

Compounding: Example

- **Background**
  - Mutual funds charge annual fees to investors.
  - Fees are usually a percentage of the amount in the account.
  - Fees can go from 0.02%/yr all the way up to 2.00%/yr.
- **Question 1**
  - Kira is 30 years old and plans to retire at 65 (in 35 years).
  - So far, she has saved $80,000 toward retirement.
  - How much will Kira have when she retires if the money is invested at 12% per year?
- **Question 2**
  - Kira just discovered that her fund will charge her 1% per year in fees.
  - This means that her investment will compound at a rate of only 11% per year.
  - How much will the $80,000 that she saves now be worth when she retires in 35 years?
- **Question 3**: What fraction of her money will have gone to fees?

Compounding: Example (cont'd)

- **Question 1**. We can use the formula from page 4.6 to find the future value of Kira's investment:

$$FV_{35} = \$80{,}000(1.12)^{35} = \$4.224 \text{ million.}$$

- **Question 2**. If her investment compounds at only 11% per year, then its future value is

$$FV_{35} = \$80{,}000(1.11)^{35} = \$3.086 \text{ million.}$$

- **Question 3**. Kira will have lost \$4.224 million – \$3.086 million = \$1.138 million in fees, which represents

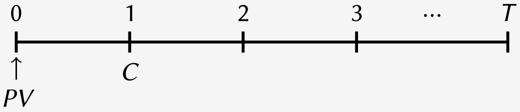
$$\frac{\$1.138 \text{ million}}{\$4.224 \text{ million}} = 26.9\%$$

of her money.

Section 4.2
Discounting and Present Value

Discounting Over One Period

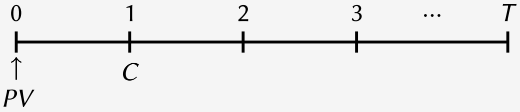
- Suppose you want to have C dollars in your account in one year, and that the current annual interest rate is r. How much do you have to invest today?
- Let PV denote this amount.



Discounting Over One Period

- Suppose you want to have C dollars in your account in one year, and that the current annual interest rate is r. How much do you have to invest today?

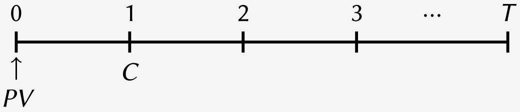
- Let PV denote this amount.



So PV must satisfy: $PV(1 + r) = C$

Discounting Over One Period

- Suppose you want to have C dollars in your account in one year, and that the current annual interest rate is r. How much do you have to invest today?
- Let PV denote this amount.



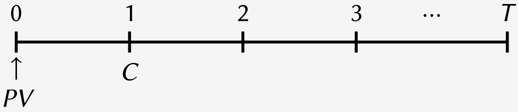
So PV must satisfy: $PV(1+r) = C$ or, equivalently,

$$PV = \frac{C}{1+r}.$$

Discounting Over One Period

- Suppose you want to have C dollars in your account in one year, and that the current annual interest rate is r. How much do you have to invest today?

- Let PV denote this amount.



So PV must satisfy: $PV(1 + r) = C$ or, equivalently,

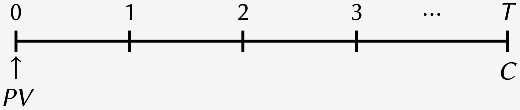
$$PV = \frac{C}{1 + r}.$$

- We say that PV is the *present value* of C dollars delivered in one year from now.

- As long as you can borrow and lend at the rate r, you would be indifferent between receiving PV dollars now, or C dollars in one year.

Discounting Over Multiple Periods

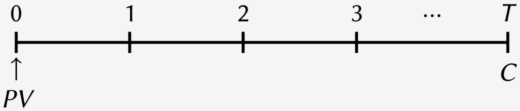
- Now, suppose you want to have C dollars in your account in T years, and that the current annual interest rate is r. How much do you have to invest today?
- Again, let PV denote this amount.



Discounting Over Multiple Periods

- Now, suppose you want to have C dollars in your account in T years, and that the current annual interest rate is r. How much do you have to invest today?

- Again, let PV denote this amount.

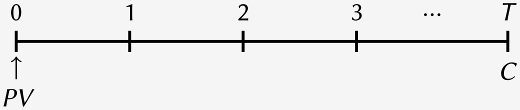


So PV must satisfy: $PV(1+r)^T = C$

Discounting Over Multiple Periods

- Now, suppose you want to have C dollars in your account in T years, and that the current annual interest rate is r. How much do you have to invest today?

- Again, let PV denote this amount.



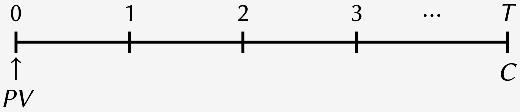
So PV must satisfy: $PV(1+r)^T = C$ or, equivalently,

$$\boxed{PV = \frac{C}{(1+r)^T}}$$

Discounting Over Multiple Periods

- Now, suppose you want to have C dollars in your account in T years, and that the current annual interest rate is r. How much do you have to invest today?

- Again, let PV denote this amount.



So PV must satisfy: $PV(1+r)^T = C$ or, equivalently,

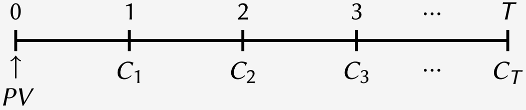
$$\boxed{PV = \frac{C}{(1+r)^T}}$$

- We say that PV is the *present value* of C dollars delivered in T years from now.

- Again, as long as you can borrow and lend at the rate r, you would be indifferent between receiving PV dollars now, or C dollars in T years.

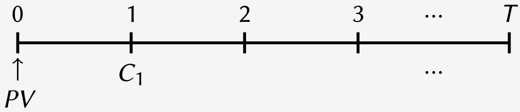
Discounting Multiple Cash Flows

- More generally, suppose you want to receive C_1 dollars from your account in one year, C_2 in two years, ..., C_T in T years. Suppose also that the current annual interest rate is r. How much do you have to invest today?
- That is, we seek to find the present value PV of an investment paying C_1 dollars in one year, C_2 dollars in two years, ..., C_T dollars in T years.



Discounting Multiple Cash Flows

- More generally, suppose you want to receive C_1 dollars from your account in one year, C_2 in two years, ..., C_T in T years. Suppose also that the current annual interest rate is r. How much do you have to invest today?
- That is, we seek to find the present value PV of an investment paying C_1 dollars in one year, C_2 dollars in two years, ..., C_T dollars in T years.

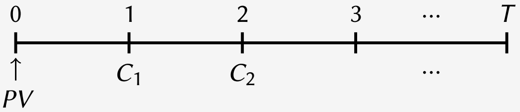


- Using arguments similar to those on pages 4.10 and 4.11, we should find

$$\boxed{PV = \frac{C_1}{1+r}}$$

Discounting Multiple Cash Flows

- More generally, suppose you want to receive C_1 dollars from your account in one year, C_2 in two years, …, C_T in T years. Suppose also that the current annual interest rate is r. How much do you have to invest today?
- That is, we seek to find the present value PV of an investment paying C_1 dollars in one year, C_2 dollars in two years, …, C_T dollars in T years.

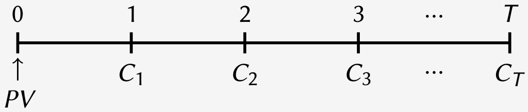


- Using arguments similar to those on pages 4.10 and 4.11, we should find

$$\boxed{PV = \frac{C_1}{1+r} + \frac{C_2}{(1+r)^2}}$$

Discounting Multiple Cash Flows

- More generally, suppose you want to receive C_1 dollars from your account in one year, C_2 in two years, …, C_T in T years. Suppose also that the current annual interest rate is r. How much do you have to invest today?
- That is, we seek to find the present value PV of an investment paying C_1 dollars in one year, C_2 dollars in two years, …, C_T dollars in T years.

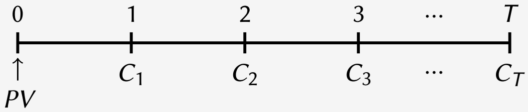


- Using arguments similar to those on pages 4.10 and 4.11, we should find

$$\boxed{PV = \frac{C_1}{1+r} + \frac{C_2}{(1+r)^2} + \cdots + \frac{C_T}{(1+r)^T}}$$

Discounting Multiple Cash Flows

- More generally, suppose you want to receive C_1 dollars from your account in one year, C_2 in two years, …, C_T in T years. Suppose also that the current annual interest rate is r. How much do you have to invest today?
- That is, we seek to find the present value PV of an investment paying C_1 dollars in one year, C_2 dollars in two years, …, C_T dollars in T years.



- Using arguments similar to those on pages 4.10 and 4.11, we should find

$$\boxed{PV = \frac{C_1}{1+r} + \frac{C_2}{(1+r)^2} + \cdots + \frac{C_T}{(1+r)^T} = \sum_{t=1}^{T} \frac{C_t}{(1+r)^t}}$$

  - The present value of a sequence of cash flows is the sum of the present values of each individual cash flow (value additivity).
  - We can add the present values of cash flows, but we cannot simply add cash flows that occur at different points in time.

Discounting: Example

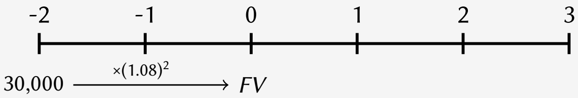
- Two years ago, you put $30,000 in a savings account earning an annual interest rate of 8%.
- At the time, you thought that these savings would grow enough for you to buy a new car five years later (i.e., in three years from now).
- However, you just re-estimated the price that you will have to pay for the new car in three years at $54,000.

Discounting: Example

- Two years ago, you put \$30,000 in a savings account earning an annual interest rate of 8%.
- At the time, you thought that these savings would grow enough for you to buy a new car five years later (i.e., in three years from now).
- However, you just re-estimated the price that you will have to pay for the new car in three years at \$54,000.
- Questions.
  (i) How much more money do you need to put in your savings account now for it to grow to this new estimate in three years?

Discounting: Example (cont'd)

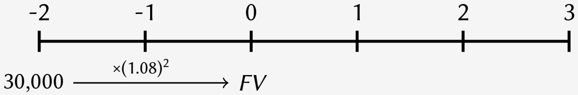
(i) Let us first figure out how much money FV is now in the account.



$$\hookrightarrow \quad FV = 30{,}000(1.08)^2 = 34{,}992.00.$$

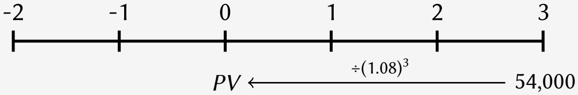
Discounting: Example (cont'd)

(i) Let us first figure out how much money FV is now in the account.



$$\hookrightarrow \quad FV = 30{,}000(1.08)^2 = 34{,}992.00.$$

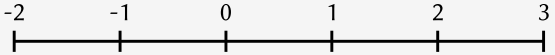
Now, the account should have an amount PV in it for it to grow to \$54,000 in three years.



$$\hookrightarrow \quad PV = \frac{54{,}000}{(1.08)^3} = 42{,}866.94.$$

Discounting: Example (cont'd)

(i) Let us first figure out how much money *FV* is now in the account.



$$30{,}000 \xrightarrow{\times(1.08)^2} FV$$

$$\hookrightarrow \quad FV = 30{,}000(1.08)^2 = 34{,}992.00.$$

Now, the account should have an amount *PV* in it for it to grow to \$54,000 in three years.

$$PV \xleftarrow{\div(1.08)^3} 54{,}000$$

$$\hookrightarrow \quad PV = \frac{54{,}000}{(1.08)^3} = 42{,}866.94.$$

So, you need to put \$42,866.94 – \$34,992.00 = \$7,874.94 in the account.

Discounting: Example

- Two years ago, you put \$30,000 in a savings account earning an annual interest rate of 8%.
- At the time, you thought that these savings would grow enough for you to buy a new car five years later (i.e., in three years from now).
- However, you just re-estimated the price that you will have to pay for the new car in three years at \$54,000.
- Questions.
  (i) How much more money do you need to put in your savings account now for it to grow to this new estimate in three years?
  (ii) Now suppose that you know that the car company will offer you to pay for the car over some time. In particular, you will have the opportunity
    - to make a down-payment of \$4,000 at the time you get the car (three years from now), and
    - to make additional payments of \$27,000 at the end of each of the following two years.

  With this offer, how much money do you need to add to your account now?

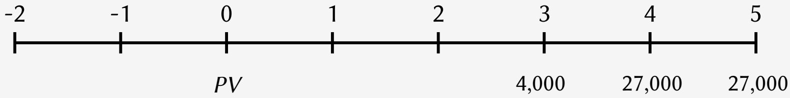
Discounting: Example (cont'd)

(ii) The time line for the payments to be made later is as follows:

-2	-1	0	1	2	3	4	5
		PV			4,000	27,000	27,000

Discounting: Example (cont'd)

(ii) The time line for the payments to be made later is as follows:



The present value (at time 0) of these three payments is

$$PV = \frac{4{,}000}{(1.08)^3} + \frac{27{,}000}{(1.08)^4} + \frac{27{,}000}{(1.08)^5} = 41{,}396.88.$$

So, you need to add \$41,396.88 – \$34,992.00 = \$6,404.88 to the account.

from part (i) on page 4.14

Section 4.3
Useful Present Value Formulas

Shortcuts to Calculating PVs: Perpetuities

- A *perpetuity* is an investment paying a fixed sum C at the end of every year forever.

Shortcuts to Calculating PVs: Perpetuities

- A *perpetuity* is an investment paying a fixed sum C at the end of every year forever.

- From our general formula (on page 4.12), we know that the present value of the perpetuity is given by

$$PV = \frac{C}{1+r} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \cdots$$

Shortcuts to Calculating PVs: Perpetuities

- A *perpetuity* is an investment paying a fixed sum C at the end of every year forever.

0	1	2	3	4	$\cdots$
$\uparrow$ PV	C	C	C	C	$\cdots$

- From our general formula (on page 4.12), we know that the present value of the perpetuity is given by

$$PV = \frac{C}{1+r} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \cdots$$

- As shown on pages 4.18-4.19, this infinite sum simplifies to

$$\boxed{PV = \frac{C}{r}}$$

Shortcuts to Calculating PVs: Perpetuities

- A *perpetuity* is an investment paying a fixed sum C at the end of every year forever.

0	1	2	3	4	…
↑ PV	C	C	C	C	…

- From our general formula (on page 4.12), we know that the present value of the perpetuity is given by

$$PV = \frac{C}{1+r} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \cdots$$

- As shown on pages 4.18-4.19, this infinite sum simplifies to

$$\boxed{PV = \frac{C}{r}}$$

- Note: The perpetuity formula gives the PV one period *before* the first cash flow is paid (as do the other PV formulas we discuss below).

[Optional] The Perpetuity Formula

- We seek to simplify the following expression:

$$PV = \frac{C}{1+r} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \cdots \tag{1}$$

- Dividing both sides of the above equation by $1+r$ gives

$$\frac{PV}{1+r} = \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \frac{C}{(1+r)^4} + \cdots \tag{2}$$

[Optional] The Perpetuity Formula

- We seek to simplify the following expression:

$$PV = \frac{C}{1+r} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \cdots \tag{1}$$

- Dividing both sides of the above equation by $1 + r$ gives

$$\frac{PV}{1+r} = \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \frac{C}{(1+r)^4} + \cdots \tag{2}$$

- We can now subtract (2) from (1) to obtain

$$\begin{aligned} PV - \frac{PV}{1+r} &= \frac{C}{1+r} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \cdots \\ &\quad - \left[\frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \cdots \right]. \end{aligned}$$

[Optional] The Perpetuity Formula

- We seek to simplify the following expression:

$$PV = \frac{C}{1+r} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \cdots \tag{1}$$

- Dividing both sides of the above equation by $1+r$ gives

$$\frac{PV}{1+r} = \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \frac{C}{(1+r)^4} + \cdots \tag{2}$$

- We can now subtract (2) from (1) to obtain

$$\begin{aligned} PV - \frac{PV}{1+r} &= \frac{C}{1+r} + \cancel{\frac{C}{(1+r)^2}} + \cancel{\frac{C}{(1+r)^3}} + \cdots \\ &\quad - \left[\cancel{\frac{C}{(1+r)^2}} + \cancel{\frac{C}{(1+r)^3}} + \cdots \right]. \end{aligned}$$

- Notice that all but one term on the right-hand side of the above equation can be canceled out.

[Optional] The Perpetuity Formula (cont'd)

- We can now solve for PV as follows:

$$PV - \frac{PV}{1+r} = \frac{C}{1+r} \iff PV(1+r) - PV = C \iff PV \times r = C$$

- The present value of the perpetuity of C paid at the end of every year forever is therefore

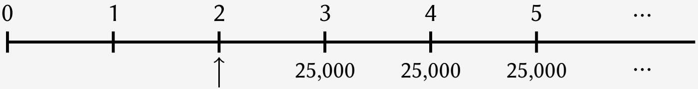
$$PV = \frac{C}{r}.$$

Example: Deferred Perpetuity

- A rich entrepreneur would like to set up a foundation that, every year, will pay $25,000 in the form of a scholarship to one deserving student.
  - The first such scholarship is to be awarded in 3 years, and
  - a scholarship will be awarded in perpetuity every year after that (even after the entrepreneur's death).
- How much money should the entrepreneur put in the foundation's account today, if the account earns 8% compounded annually?

Example: Deferred Perpetuity (cont'd)

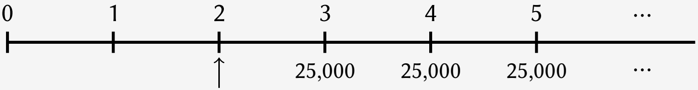
- First, let us calculate how much money will need to be in the account at the end of the second year; let us denote that amount by PV_2.



$$PV_2 = \frac{25{,}000}{0.08} = 312{,}500$$

Example: Deferred Perpetuity (cont'd)

- First, let us calculate how much money will need to be in the account at the end of the second year; let us denote that amount by PV_2.



$$PV_2 = \frac{25{,}000}{0.08} = 312{,}500$$

- For the account to be worth this much in two years, the amount that the entrepreneur needs to contribute initially is

$$PV = \frac{PV_2}{(1.08)^2} = \frac{312{,}500}{(1.08)^2} = 267{,}918.$$

Example: Deferred Perpetuity (cont'd)

- First, let us calculate how much money will need to be in the account at the end of the second year; let us denote that amount by PV_2.

0	1	2	3	4	5	⋯
		↑	25,000	25,000	25,000	⋯

$$PV_2 = \frac{25{,}000}{0.08} = 312{,}500$$

- For the account to be worth this much in two years, the amount that the entrepreneur needs to contribute initially is

$$PV = \frac{PV_2}{(1.08)^2} = \frac{312{,}500}{(1.08)^2} = 267{,}918.$$

- The perpetuity starts in 3 years, but the exponent on the discount factor is a 2.
  - This is because the perpetuity formula used in the first step calculates the value of a stream of cash flows *starting a year later*, i.e., the perpetuity formula from page 4.17 gives us the value of the stream at time 2.
  - Only two more years of discounting are needed after that.

Shortcuts to Calculating PVs: Growing Perpetuities

- A *growing perpetuity* is an investment paying a growing sum (at a rate g) every year forever, i.e., the amount paid in year t is $C(1+g)^{t-1}$.

0	1	2	3	4	…
$\uparrow$ PV	C	$C(1+g)$	$C(1+g)^2$	$C(1+g)^3$	…

Shortcuts to Calculating PVs: Growing Perpetuities

- A ***growing perpetuity*** is an investment paying a growing sum (at a rate g) every year forever, i.e., the amount paid in year t is $C(1+g)^{t-1}$.

0	1	2	3	4	$\cdots$
$\uparrow$ PV	C	$C(1+g)$	$C(1+g)^2$	$C(1+g)^3$	$\cdots$

- From our general formula (on page 4.12), we know that the present value of the growing perpetuity is given by

$$PV = \frac{C}{1+r} + \frac{C(1+g)}{(1+r)^2} + \frac{C(1+g)^2}{(1+r)^3} + \cdots$$

Shortcuts to Calculating PVs: Growing Perpetuities

- A *growing perpetuity* is an investment paying a growing sum (at a rate g) every year forever, i.e., the amount paid in year t is $C(1+g)^{t-1}$.

0	1	2	3	4	…
$\uparrow$ PV	C	$C(1+g)$	$C(1+g)^2$	$C(1+g)^3$	…

- From our general formula (on page 4.12), we know that the present value of the growing perpetuity is given by

$$PV = \frac{C}{1+r} + \frac{C(1+g)}{(1+r)^2} + \frac{C(1+g)^2}{(1+r)^3} + \cdots$$

- As shown on pages 4.23-4.24, as long as $r > g$, this infinite sum simplifies to

$$\boxed{PV = \frac{C}{r-g}}$$

[Optional] The Growing Perpetuity Formula

- We seek to simplify the following expression:

$$PV = \frac{C}{1+r} + \frac{C(1+g)}{(1+r)^2} + \frac{C(1+g)^2}{(1+r)^3} + \cdots \tag{1}$$

- Let us multiply both sides of equation (1) by $\frac{1+g}{1+r}$:

$$PV\left(\frac{1+g}{1+r}\right) = \frac{C(1+g)}{(1+r)^2} + \frac{C(1+g)^2}{(1+r)^3} + \cdots \tag{2}$$

- Now we subtract (2) from (1):

$$\begin{aligned} PV - PV\left(\frac{1+g}{1+r}\right) &= \frac{C}{1+r} + \frac{C(1+g)}{(1+r)^2} + \frac{C(1+g)^2}{(1+r)^3} + \cdots \\ &\quad - \left[\frac{C(1+g)}{(1+r)^2} + \frac{C(1+g)^2}{(1+r)^3} + \cdots\right]. \end{aligned}$$

- Notice that all but one term on the right-hand side of this last equation can be cancelled out.

[Optional] The Growing Perpetuity Formula (cont'd)

- We can now solve for PV as follows (provided that $g < r$):

$$PV - PV\left(\frac{1+g}{1+r}\right) = \frac{C}{1+r} \iff PV(1+r) - PV(1+g) = C \iff PV(r-g) = C$$

- The present value of the growing perpetuity is therefore

$$PV = \frac{C}{r-g}.$$

Example: An Endowed Chair

- A benefactor wishes to endow a chair in finance at the Fuqua School of Business.
- The aim is to provide an amount equaling $150,000 in the first year and growing at a rate of 5% each subsequent year in order to adjust for the expected growth in salaries.
- Suppose that the interest rate is 10%. How much should the benefactor donate?
- Solution.
  - We have $r = 10\%$, $g = 5\%$, $C = 150{,}000$.
  - Therefore,

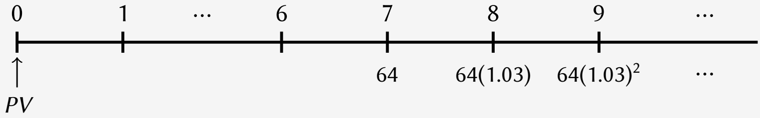
$$PV = \frac{C}{r-g} = \frac{150{,}000}{0.10 - 0.05} = 3{,}000{,}000.$$

Example: Deferred Growing Perpetuity

- SDRR Inc. is developing a new drug.
- The company is interested in figuring out the present value of the cash inflows that they can expect from selling the drug once it is ready for the market.
- They figure that the first sales will occur in 7 years from now, and they estimate them to be $64 million.
- In subsequent years, they expect their sales to grow in perpetuity at 3% per year.
- What is the present value of these sales if SDRR's annual discount rate is 11%?

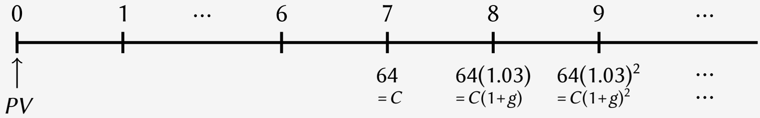
Example: Deferred Growing Perpetuity (cont'd)

- Let us start by putting the cash flows on a time line. We are looking for *PV* in this diagram.



Example: Deferred Growing Perpetuity (cont'd)

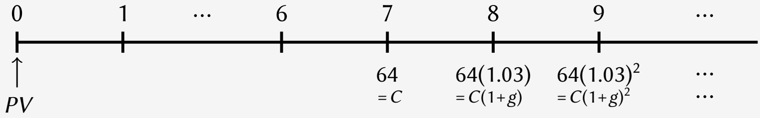
- Let us start by putting the cash flows on a time line. We are looking for PV in this diagram.



- The growing perpetuity formula from page 4.22 gives us the value of a perpetuity *one year before the first payment*.

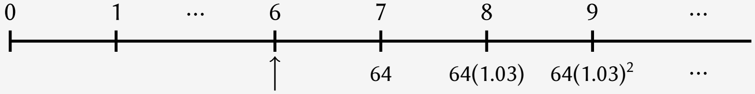
Example: Deferred Growing Perpetuity (cont'd)

- Let us start by putting the cash flows on a time line. We are looking for *PV* in this diagram.



- The growing perpetuity formula from page 4.22 gives us the value of a perpetuity *one year before the first payment*.

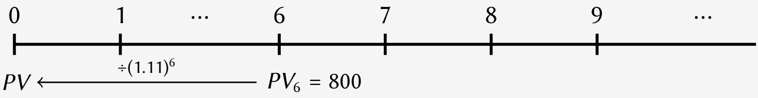
- So, let us first figure out the value of this perpetuity at the end of year 6 (denoted by PV_6), one year before the first cash flow of \$64 million.



$$PV_6 = \frac{C}{r-g} = \frac{64}{0.11-0.03} = 800$$

Example: Deferred Growing Perpetuity (cont'd)

- To find PV, we now only need to discount PV_6 to time zero, that is, we need to discount it for six (not seven) years.

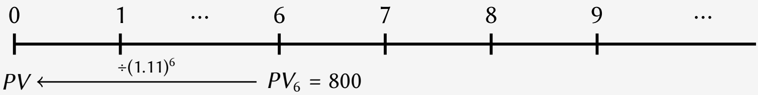


- Thus the present value of cash inflows coming from drug sales is given by

$$PV = \frac{PV_6}{(1.11)^6} = \frac{800}{(1.11)^6} = 427.71.$$

Example: Deferred Growing Perpetuity (cont'd)

- To find PV, we now only need to discount PV_6 to time zero, that is, we need to discount it for six (not seven) years.



- Thus the present value of cash inflows coming from drug sales is given by

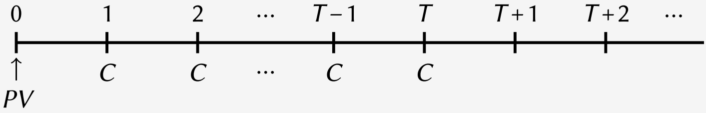
$$PV = \frac{PV_6}{(1.11)^6} = \frac{800}{(1.11)^6} = 427.71.$$

- Note that we could have calculated this as follows (again, note that the exponent is 6, not 7):

$$PV = \frac{64}{0.11 - 0.03} \times \frac{1}{(1.11)^6} = 427.71.$$

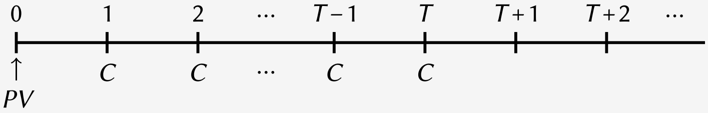
Shortcut to Calculating PVs: Annuities

- An *annuity* is an investment that pays a fixed sum C at the end of each year for T years.



Shortcut to Calculating PVs: Annuities

- An *annuity* is an investment that pays a fixed sum C at the end of each year for T years.

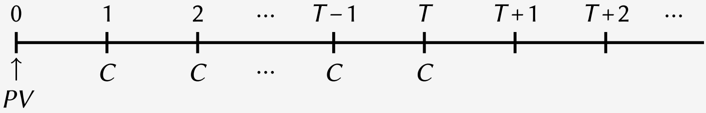


- Using our general formula from page 4.12, we can write the present value of the annuity as

$$PV = \frac{C}{1+r} + \frac{C}{(1+r)^2} + \cdots + \frac{C}{(1+r)^T}.$$

Shortcut to Calculating PVs: Annuities

- An ==*annuity*== is an investment that pays a fixed sum C at the end of each year for T years.



- Using our general formula from page 4.12, we can write the present value of the annuity as

$$PV = \frac{C}{1+r} + \frac{C}{(1+r)^2} + \cdots + \frac{C}{(1+r)^T}.$$

- Although we can use a spreadsheet to calculate this finite sum, pages 4.30-4.32 show that there is a simple formula for the present value of an annuity:

$$\boxed{PV = \frac{C}{r}\left[1 - \frac{1}{(1+r)^T}\right]}$$

[Optional] The Annuity Formula

- To reach the formula on page 4.29, first observe that the cash flows from the annuity equal the difference between the cash flows of two perpetuities:
  - one starting at time 1;

0	1	2	$\cdots$	$T-1$	T	$T+1$	$T+2$	$\cdots$
PV_A	C	C	$\cdots$	C	C	C	C	$\cdots$

[Optional] The Annuity Formula

- To reach the formula on page 4.29, first observe that the cash flows from the annuity equal the difference between the cash flows of two perpetuities:
  - one starting at time 1;
  - the other starting at time $T + 1$.

0	1	2	$\cdots$	$T-1$	T	$T+1$	$T+2$	$\cdots$
PV_A	C	C	$\cdots$	C	C	C	C	$\cdots$
$-PV_B$						$-C$	$-C$	$\cdots$

[Optional] The Annuity Formula

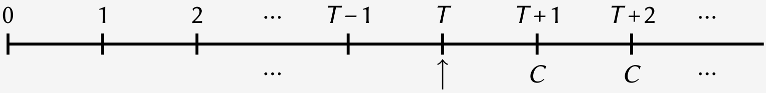
- To reach the formula on page 4.29, first observe that the cash flows from the annuity equal the difference between the cash flows of two perpetuities:
  - one starting at time 1;
  - the other starting at time $T + 1$.

0	1	2	⋯	$T-1$	T	$T+1$	$T+2$	⋯
PV_A	C	C	⋯	C	C	C	C	⋯
$-PV_B$						$-C$	$-C$	⋯

- The present value of the first perpetuity is $PV_A = \frac{C}{r}$, as shown on page 4.17.
- What about the second perpetuity, which is *deferred* for T years?

[Optional] The Annuity Formula (cont'd)

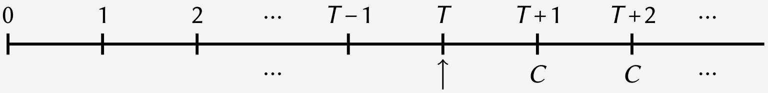
- Let us first calculate the value of that perpetuity at the end of T years. We call this value $PV_B(\text{at } T)$.



$$PV_B(\text{at } T) = \frac{C}{r}$$

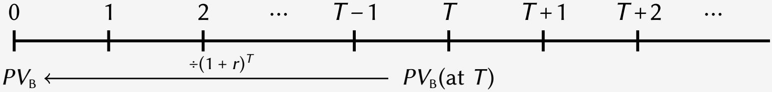
[Optional] The Annuity Formula (cont'd)

- Let us first calculate the value of that perpetuity at the end of T years. We call this value $PV_B(\text{at } T)$.



$$PV_B(\text{at } T) = \frac{C}{r}$$

- Now, since $PV_B(\text{at } T)$ is the value in T years from now, we need to discount this value to time 0 to get the value of the perpetuity:



$$\hookrightarrow \quad PV_B = \frac{PV_B(\text{at } T)}{(1+r)^T} = \frac{C/r}{(1+r)^T}.$$

[Optional] The Annuity Formula (cont'd)

- The calculation of the present value of the annuity then simply involves a difference of two perpetuities:

$$PV = PV_{\text{A}} - PV_{\text{B}} = \frac{C}{r} - \frac{C/r}{(1+r)^T}.$$

- After simplification, the present value of the annuity is therefore given by

$$PV = \frac{C}{r}\left[1 - \frac{1}{(1+r)^T}\right].$$

[Optional] The Annuity Formula (cont'd)

- The calculation of the present value of the annuity then simply involves a difference of two perpetuities:

$$PV = PV_{\text{A}} - PV_{\text{B}} = \frac{C}{r} - \frac{C/r}{(1+r)^T}.$$

- After simplification, the present value of the annuity is therefore given by

$$PV = \frac{C}{r}\left[1 - \frac{1}{(1+r)^T}\right].$$

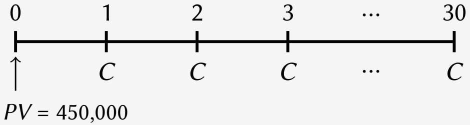
- Intuition.
  - $\dfrac{C}{r}$ would be the present value if the payments went on forever.
  - The term in square brackets, which is always smaller than 1, accounts for the fact that the payments stop (i.e., the PV is not as large as that of a perpetuity).

Example: Mortgage Payment

- You have decided to buy a house for $500,000, with an initial down-payment of $50,000.
- To finance the balance, you have negotiated a 30-year mortgage at an annual rate of 6%.
- Assume that your mortgage calls for equal payments at the end of every year (and that the 6% is compounded annually).
- What is your annual payment?

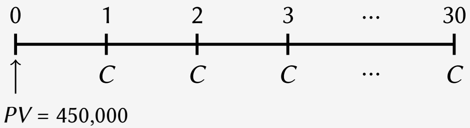
Example: Mortgage Payment (cont'd)

- Let us denote the annual payment by C. Because a down-payment of \$50,000 has been made on the house, the present value of this annuity must be \$450,000.



Example: Mortgage Payment (cont'd)

- Let us denote the annual payment by C. Because a down-payment of \$50,000 has been made on the house, the present value of this annuity must be \$450,000.

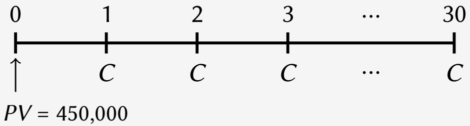


- Mathematically, with an annual interest rate of 6%, we seek to solve

$$450{,}000 = \frac{C}{0.06}\left[1 - \frac{1}{(1.06)^{30}}\right]$$

Example: Mortgage Payment (cont'd)

- Let us denote the annual payment by C. Because a down-payment of \$50,000 has been made on the house, the present value of this annuity must be \$450,000.



- Mathematically, with an annual interest rate of 6%, we seek to solve

$$450{,}000 = \frac{C}{0.06}\left[1 - \frac{1}{(1.06)^{30}}\right] = C \times 13.765.$$

- The annual payment is $C = \dfrac{450{,}000}{13.765} = 32{,}692.$

Shortcut to Calculating PVs: Growing Annuities

- A ***growing annuity*** is an investment that pays a growing sum (at a rate g) at the end of every year, and stops at the end of year T.

0	1	2	$\cdots$	$T-1$	T
$\uparrow$ PV	C	$C(1+g)$	$\cdots$	$C(1+g)^{T-2}$	$C(1+g)^{T-1}$

Shortcut to Calculating PVs: Growing Annuities

- A *growing annuity* is an investment that pays a growing sum (at a rate g) at the end of every year, and stops at the end of year T.

$$
\begin{array}{cccccc}
0 & 1 & 2 & \cdots & T-1 & T \\
\uparrow & C & C(1+g) & \cdots & C(1+g)^{T-2} & C(1+g)^{T-1} \\
PV & & & & &
\end{array}
$$

- There is also a shortcut to compute the present value of a growing annuity:

$$PV = \frac{C}{1+r} + \frac{C(1+g)}{(1+r)^2} + \frac{C(1+g)^2}{(1+r)^3} + \cdots + \frac{C(1+g)^{T-1}}{(1+r)^T}$$

Shortcut to Calculating PVs: Growing Annuities

- A *growing annuity* is an investment that pays a growing sum (at a rate g) at the end of every year, and stops at the end of year T.

0	1	2	$\cdots$	$T-1$	T
$\uparrow$ PV	C	$C(1+g)$	$\cdots$	$C(1+g)^{T-2}$	$C(1+g)^{T-1}$

- There is also a shortcut to compute the present value of a growing annuity:

$$PV = \frac{C}{1+r} + \frac{C(1+g)}{(1+r)^2} + \frac{C(1+g)^2}{(1+r)^3} + \cdots + \frac{C(1+g)^{T-1}}{(1+r)^T}$$

leads to

$$\boxed{PV = \frac{C}{r-g}\left[1 - \left(\frac{1+g}{1+r}\right)^T\right]}$$

- We will use this shortcut in an example in the next section.

Excel's Annuity Functions

- Excel has several functions for annuities.
  - There are no such functions for perpetuities, growing perpetuities, or growing annuities. For these cash flow streams, just use the relevant formula.
- Excel's annuity functions.
  - `=PV(r,T,C)`: present value (at time 0)
  - `=FV(r,T,C)`: future value (at time T)
  - `=PMT(r,T,PV0)`: constant periodic payment amount (C)
  - `=NPER(r,C,PV0)`: number of periods (T)
  - `=RATE(T,C,PV0)`: rate (r)
- Excel also has a net present value formula, `=NPV(r,C1,[C2],...)`, which is designed for any general cash flow stream. We will use it later.

Excel's Annuity Functions: Example

- Let us solve the mortgage payment example from page 4.33 using Excel's `PMT(r,T,PV0)` function.
  - The interest rate is 6% $\longrightarrow$ `r` = 0.06
  - The mortgage is for 30 years $\longrightarrow$ `T` = 30
  - We need to borrow \$450,000 $\longrightarrow$ PV_0 = 450000

Excel's Annuity Functions: Example

- Let us solve the mortgage payment example from page 4.33 using Excel's $\texttt{PMT(r,T,PV}_0\texttt{)}$ function.
  - The interest rate is 6% $\longrightarrow$ $\texttt{r} = 0.06$
  - The mortgage is for 30 years $\longrightarrow$ $\texttt{T} = 30$
  - We need to borrow \$450,000 $\longrightarrow$ $\texttt{PV}_0 = 450000$
- The answer is the same as that on page 4.34:

	A	B	C
1	Interest Rate:	0.06	
2	Number of Years:	30	
3	Amount Borrowed:	450000	
4	Annual Payment:	(32,692)	=PMT(B1,B2,B3)

- Notice that Excel returns the answer as a negative number: this is what you have to pay, i.e., it is a negative cash flow.

Section 4.4
Net Present Value (NPV)

The Concept of Net Present Value

- Now that we have seen the rules of “time travel” (compounding and discounting), we can use them to make financial decisions.
- The idea is compare the costs and benefits in present-value terms.
- For this purpose, let us define the *net present value* (NPV) of an investment as

$$NPV = PV(\text{benefits}) - PV(\text{costs}).$$

- The investment should be made when $NPV > 0$, as the benefits then exceed the costs (again, in PV terms).

Example: Using Net Present Value to Make a Decision

- We make financial decisions using the concept of NPV on a regular basis.
- Consider the decision to enter an MBA program with the following estimates.
    - You are 30 years old. You plan to work until the age of 65.
    - You have a job paying $60,000 a year and you expect your salary to increase by 2% per year.
    - The MBA program will cost you $100,000 a year for 2 years (payable at the beginning of each year).
    - After you are done with the program, you expect your salary to start at $120,000 (in 3 years from now) and to increase at 3% a year thereafter.
    - Assuming an annual discount rate of 5%, should you enter the program?
- To solve this problem, we will use the growing annuity formula from page 4.35.

Example: Using Net Present Value to Make a Decision

- We make financial decisions using the concept of NPV on a regular basis.
- Consider the decision to enter an MBA program with the following estimates.
  - You are 30 years old. You plan to work until the age of 65.
  - You have a job paying \$60,000 a year and you expect your salary to increase by 2% per year.
  - The MBA program will cost you \$100,000 a year for 2 years (payable at the beginning of each year).
  - After you are done with the program, you expect your salary to start at \$120,000 (in 3 years from now) and to increase at 3% a year thereafter.
  - Assuming an annual discount rate of 5%, should you enter the program?
- To solve this problem, we will use the growing annuity formula from page 4.35.

Example: Using Net Present Value to Make a Decision

- We make financial decisions using the concept of NPV on a regular basis.
- Consider the decision to enter an MBA program with the following estimates.
  - You are 30 years old. You plan to work until the age of 65.
  - You have a job paying $60,000 a year and you expect your salary to increase by 2% per year.
  - The MBA program will cost you $100,000 a year for 2 years (payable at the beginning of each year).
  - After you are done with the program, you expect your salary to start at $120,000 (in 3 years from now) and to increase at 3% a year thereafter.
  - Assuming an annual discount rate of 5%, should you enter the program?
- To solve this problem, we will use the growing annuity formula from page 4.35.

Example: Using Net Present Value to Make a Decision

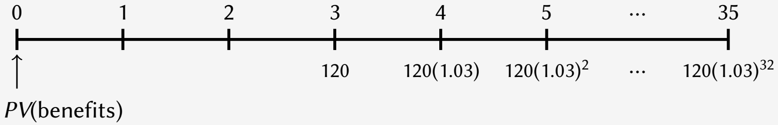
- We make financial decisions using the concept of NPV on a regular basis.
- Consider the decision to enter an MBA program with the following estimates.
  - You are 30 years old. You plan to work until the age of 65.
  - You have a job paying \$60,000 a year and you expect your salary to increase by 2% per year.
  - The MBA program will cost you \$100,000 a year for 2 years (payable at the beginning of each year).
  - After you are done with the program, you expect your salary to start at \$120,000 (in 3 years from now) and to increase at 3% a year thereafter.
  - Assuming an annual discount rate of 5%, should you enter the program?
- To solve this problem, we will use the growing annuity formula from page 4.35.

Example: Using Net Present Value to Make a Decision

- We make financial decisions using the concept of NPV on a regular basis.
- Consider the decision to enter an MBA program with the following estimates.
  - You are 30 years old. You plan to work until the age of 65.
  - You have a job paying \$60,000 a year and you expect your salary to increase by 2% per year.
  - The MBA program will cost you \$100,000 a year for 2 years (payable at the beginning of each year).
  - After you are done with the program, you expect your salary to start at \$120,000 (in 3 years from now) and to increase at 3% a year thereafter.
  - Assuming an annual discount rate of 5%, should you enter the program?
- To solve this problem, we will use the growing annuity formula from page 4.35.

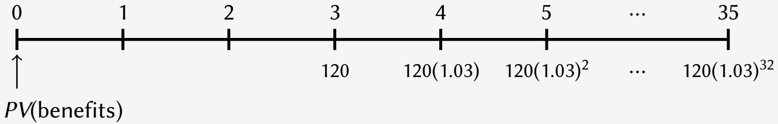
Example: Using Net Present Value to Make a Decision (cont'd)

- The decision-maker is 30 years old at time 0, and 65 years old at time 35.
- Let us first calculate the present value of the benefits of entering the program.
  - The cash flows (in \$000s) are as follows:

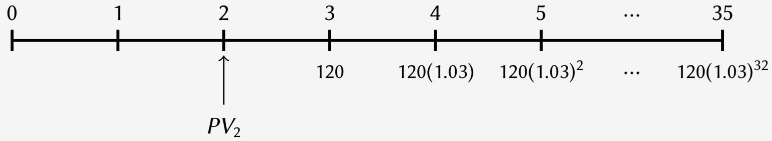


Example: Using Net Present Value to Make a Decision (cont'd)

- The decision-maker is 30 years old at time 0, and 65 years old at time 35.
- Let us first calculate the present value of the benefits of entering the program.
  - The cash flows (in \$000s) are as follows:

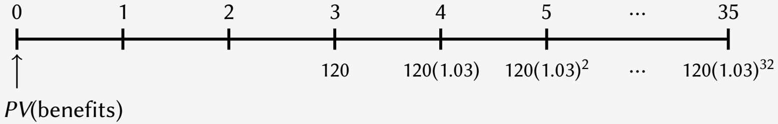


  - The growing annuity formula from page 4.35 gives the value of the (33-year) annuity one year before the first payment.

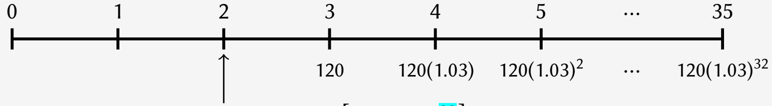


Example: Using Net Present Value to Make a Decision (cont'd)

- The decision-maker is 30 years old at time 0, and 65 years old at time 35.
- Let us first calculate the present value of the benefits of entering the program.
  - The cash flows (in \$000s) are as follows:



  - The growing annuity formula from page 4.35 gives the value of the (33-year) annuity one year before the first payment.



$$PV_2 = \frac{120}{0.05-0.03}\left[1-\left(\frac{1.03}{1.05}\right)^{33}\right] = 2{,}819$$

Example: Using Net Present Value to Make a Decision (cont'd)

- The decision-maker is 30 years old at time 0, and 65 years old at time 35.
- Let us first calculate the present value of the benefits of entering the program.
  - The cash flows (in $000s) are as follows:

0	1	2	3	4	5	⋯	35
↑ PV(benefits)			120	120(1.03)	$120(1.03)^2$	⋯	$120(1.03)^{32}$

  - The growing annuity formula from page 4.35 gives the value of the (33-year) annuity one year before the first payment.

0	1	2	3	4	5	⋯	35
		↑ PV_2	120	120(1.03)	$120(1.03)^2$	⋯	$120(1.03)^{32}$

$$PV_2 = \frac{120}{0.05-0.03}\left[1-\left(\frac{1.03}{1.05}\right)^{33}\right] = 2{,}819$$

  - Thus, we have $PV(\text{benefits}) = \dfrac{PV_2}{(1.05)^2} = \dfrac{2{,}819}{(1.05)^2} = 2{,}557$.

Example: Using Net Present Value to Make a Decision (cont'd)

- The present value of the costs (in $000s) of attending the program is simply

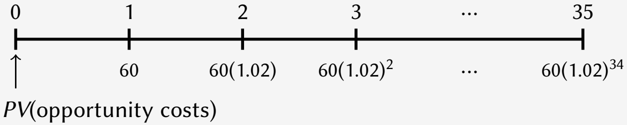
$$PV(\text{cost of program}) = 100 + \frac{100}{1.05} = 195.$$

Example: Using Net Present Value to Make a Decision (cont'd)

- The present value of the costs (in \$000s) of attending the program is simply

$$PV(\text{cost of program}) = 100 + \frac{100}{1.05} = 195.$$

- Also, by entering the program, you will stop collecting the salary from your current job. This is an opportunity cost of entering the program.
  - The cash flows (in \$000s) are as follows:

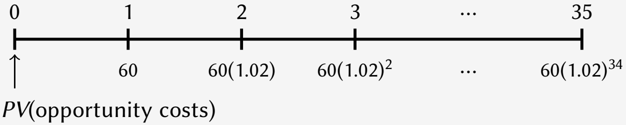


Example: Using Net Present Value to Make a Decision (cont'd)

- The present value of the costs (in \$000s) of attending the program is simply

$$PV(\text{cost of program}) = 100 + \frac{100}{1.05} = 195.$$

- Also, by entering the program, you will stop collecting the salary from your current job. This is an opportunity cost of entering the program.
    - The cash flows (in \$000s) are as follows:



    - The growing annuity formula from page 4.35 gives the value of the (35-year) annuity one year before the first payment (i.e., at time 0).

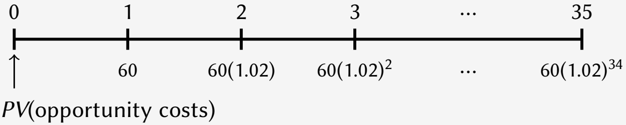
$$PV(\text{opp. costs}) = \frac{C}{r-g}\left[1-\left(\frac{1+g}{1+r}\right)^T\right] = \frac{60}{0.05-0.02}\left[1-\left(\frac{1.02}{1.05}\right)^{35}\right] = 1{,}275.$$

Example: Using Net Present Value to Make a Decision (cont'd)

- The present value of the costs (in \$000s) of attending the program is simply

$$PV(\text{cost of program}) = 100 + \frac{100}{1.05} = 195.$$

- Also, by entering the program, you will stop collecting the salary from your current job. This is an opportunity cost of entering the program.
  - The cash flows (in \$000s) are as follows:



  - The growing annuity formula from page 4.35 gives the value of the (35-year) annuity one year before the first payment (i.e., at time 0).

$$PV(\text{opp. costs}) = \frac{C}{r-g}\left[1-\left(\frac{1+g}{1+r}\right)^T\right] = \frac{60}{0.05-0.02}\left[1-\left(\frac{1.02}{1.05}\right)^{35}\right] = 1{,}275.$$

- Thus, we have $NPV = 2{,}557 - 195 - 1{,}275 = 1{,}087 > 0$, so it is a good idea to enter the program.

Section 4.5

More Frequent Compounding

More Frequent Compounding

- Sometimes, an annual rate is compounded more than once a year and, as a result, interest gets credited more rapidly.
- *Semiannual compounding*: Suppose that you invest $1,000 in a bank account paying an interest rate of 10%, and interest is credited to your account *twice* a year.
  - Every six months, your account will generate $\frac{10\%}{2} = 5\%$ interest.

More Frequent Compounding

- Sometimes, an annual rate is compounded more than once a year and, as a result, interest gets credited more rapidly.
- *Semiannual compounding*: Suppose that you invest $1,000 in a bank account paying an interest rate of 10%, and interest is credited to your account *twice* a year.
  - Every six months, your account will generate $\frac{10\%}{2} = 5\%$ interest.
  - Money in the account after 6 months:

Investment:	\$1,000.00	
Interest ($1{,}000 \times \frac{10\%}{2}$):	\$50.00	
Total:	\$1,050.00	= \$1,000(1.05)

More Frequent Compounding

- Sometimes, an annual rate is compounded more than once a year and, as a result, interest gets credited more rapidly.

- *Semiannual compounding*: Suppose that you invest $1,000 in a bank account paying an interest rate of 10%, and interest is credited to your account *twice* a year.
  - Every six months, your account will generate $\frac{10\%}{2} = 5\%$ interest.
  - Money in the account after 6 months:

Investment:	\$1,000.00	
Interest $(1{,}000 \times \frac{10\%}{2})$:	\$50.00	
Total:	\$1,050.00	$= \$1{,}000(1.05)$

  - Money in the account after one year (12 months):

Re-Investment:	\$1,050.00	
Interest $(1{,}050 \times \frac{10\%}{2})$:	\$52.50	[>\$50.00]
Total:	\$1,102.50	$= \$1{,}000(1.05)^2$

More Frequent Compounding

- Sometimes, an annual rate is compounded more than once a year and, as a result, interest gets credited more rapidly.
- *Semiannual compounding*: Suppose that you invest $1,000 in a bank account paying an interest rate of 10%, and interest is credited to your account *twice* a year.
  - Every six months, your account will generate $\frac{10\%}{2} = 5\%$ interest.
  - Money in the account after 6 months:

Investment:	\$1,000.00	
Interest ($1{,}000 \times \frac{10\%}{2}$):	\$50.00	
Total:	\$1,050.00	$= \$1{,}000(1.05)$

  - Money in the account after one year (12 months):

Re-Investment:	\$1,050.00	
Interest ($1{,}050 \times \frac{10\%}{2}$):	\$52.50	[>\$50.00]
Total:	\$1,102.50	$= \$1{,}000(1.05)^2$

- When the interest rate *r is compounded m times a year*, it is often referred to as an *annual percentage rate* (APR) *r* compounded *m* times a year.

Example: Mortgage Payment Revisited

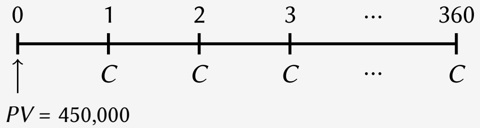
- In consumer finance (mortgages, car loans, credit cards, student loans), monthly compounding is typical.
- Let us revisit the example from page 4.33. The following information is as before.
  - You have decided to buy a house for $500,000, with an initial down-payment of $50,000.
  - To finance the balance, you have negotiated a 30-year mortgage at an annual rate of 6%.

Example: Mortgage Payment Revisited

- In consumer finance (mortgages, car loans, credit cards, student loans), monthly compounding is typical.
- Let us revisit the example from page 4.33. The following information is as before.
  - You have decided to buy a house for $500,000, with an initial down-payment of $50,000.
  - To finance the balance, you have negotiated a 30-year mortgage at an annual rate of 6%.
- However, let us now assume the following.
  - Your mortgage calls for equal payments at the end of every month.
  - The annual rate (APR) of 6% is compounded monthly.
- What is your monthly payment?

Example: Mortgage Payment Revisited (cont'd)

- Let us now denote the monthly payment by C.
  - Because a down-payment of \$50,000 has been made on the house, the present value of this annuity must be \$450,000.
  - Also, there are $30 \times 12 = 360$ months in 30 years.



Example: Mortgage Payment Revisited (cont'd)

- Let us now denote the monthly payment by C.
  - Because a down-payment of \$50,000 has been made on the house, the present value of this annuity must be \$450,000.
  - Also, there are $30 \times 12 = 360$ months in 30 years.

0	1	2	3	$\cdots$	360
$\uparrow$	C	C	C	$\cdots$	C

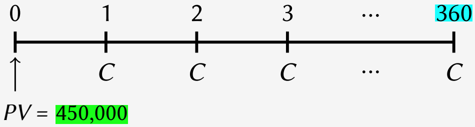
$PV = 450{,}000$

- Because an annual rate of 6% compounded monthly is really an effective monthly rate of $\frac{6\%}{12} = 0.5\%$, we seek to solve

$$450{,}000 = \frac{C}{0.005}\left[1 - \frac{1}{(1.005)^{360}}\right]$$

Example: Mortgage Payment Revisited (cont'd)

- Let us now denote the monthly payment by C.
  - Because a down-payment of \$50,000 has been made on the house, the present value of this annuity must be \$450,000.
  - Also, there are $30 \times 12 = 360$ months in 30 years.



- Because an annual rate of 6% compounded monthly is really an effective monthly rate of $\frac{6\%}{12} = 0.5\%$, we seek to solve

$$450{,}000 = \frac{C}{0.005}\left[1 - \frac{1}{(1.005)^{360}}\right] = C \times 166.79.$$

- The monthly payment is $C = \dfrac{450{,}000}{166.79} = 2{,}698.$

Effective Annual Rate

- In the mortgage example, the annual percentage rate (APR) $r = 6\%$ is compounded monthly.
  - The compounding interval is one month and there are 12 month in a year, so $m = 12$.
  - The interest rate you pay per month is $\frac{r}{m} = \frac{6\%}{12} = 0.5\%$.

Effective Annual Rate

- In the mortgage example, the annual percentage rate (APR) $r = 6\%$ is compounded monthly.
  - The compounding interval is one month and there are 12 month in a year, so $m = 12$.
  - The interest rate you pay per month is $\frac{r}{m} = \frac{6\%}{12} = 0.5\%$.
- Since compounding is monthly, this is equivalent to paying a higher interest rate per year:

$$(1 + 0.5\%)^{12} = 1.0617 = 1 + 6.17\%$$

- This rate (of 6.17%) is called the *effective annual rate* (denoted r_{EAR}) or the *equivalent annual rate.*

Effective Annual Rate

- In the mortgage example, the annual percentage rate (APR) $r = 6\%$ is compounded monthly.
  - The compounding interval is one month and there are 12 month in a year, so $m = 12$.
  - The interest rate you pay per month is $\frac{r}{m} = \frac{6\%}{12} = 0.5\%$.
- Since compounding is monthly, this is equivalent to paying a higher interest rate per year:

$$(1 + 0.5\%)^{12} = 1.0617 = 1 + 6.17\%$$

- This rate (of 6.17%) is called the *effective annual rate* (denoted r_{EAR}) or the *equivalent annual rate.*
- In general, for an APR of r compounded m times a year, it can be computed by solving

$$\boxed{1 + r_{\text{EAR}} = \left(1 + \frac{r}{m}\right)^m}$$

Effective Annual Rate: Example

Two Ways to Get Great Rates

Certificates of Deposit

MINIMUM DEPOSIT $2,500	ANNUAL PERCENTAGE YIELD %	INTEREST RATE %
5 Year CD	**6.72**	**6.50**

Minimum Balance for IRA/KEOGH CDs is $500

Personal Money Market Accounts

Balances of $75,000 or more	**4.86**	**4.75**
Balances of $1,000-$74,999	**3.05**	**3.00**

For more information on all our accounts, stop by your local branch or call our Customer Information Center today at

1-800-REPUBLIC

Republic National Bank™

Interest is compounded daily. Rates are subject to change. Early withdrawals from CDs are subject to consent of the Bank and substantial penalties. In addition there is a substantial IRS penalty for IRA/Keogh funds withdrawn before age 59 1/2. Rates and yields apply to accounts opened by 08/13/96. Personal, domestic accounts only.

Member FDIC

Effective Annual Rate: Example (cont'd)

- Why do you think these certificates of deposits (CD's) seem to offer two different interest rates?
- The small print says that the interest rate is a *daily compounded* rate. This means that a dollar invested in these CD's will grow to

$$FV_1 = \left(1 + \frac{0.065}{365}\right)^{365} = 1.0672$$

at the end of one year.

Effective Annual Rate: Example (cont'd)

- Why do you think these certificates of deposits (CD's) seem to offer two different interest rates?
- The small print says that the interest rate is a *daily compounded* rate. This means that a dollar invested in these CD's will grow to

$$FV_1 = \left(1 + \frac{0.065}{365}\right)^{365} = 1.0672$$

  at the end of one year.
- The "annual percentage yield" is how some financial institutions refer to the effective annual rate r_{EAR}, which can be found as follows:

$$1 + r_{\text{EAR}} = 1.0672 \quad \Longrightarrow \quad r_{\text{EAR}} = 6.72\%.$$

Time Value of Money: Main Takeaways

- Decision making requires comparing the cash flows of different courses of action.
- Only values at the same point in time are comparable.
- Valuation and NPV decision making thus require discounting or compounding cash flows that occur at different points in time.
- Simple formulas for valuing perpetuities and annuities are useful in practice.

Time Value of Money: Formulas

- Compounding and future value: $FV = C(1+r)^T$
- Discounting and present value: $PV = \dfrac{C_1}{1+r} + \dfrac{C_2}{(1+r)^2} + \cdots + \dfrac{C_T}{(1+r)^T}$
- Perpetuity.
  - Constant: $PV = \frac{C}{r}$
  - Growing: $PV = \frac{C}{r-g}$
- Annuity.
  - Constant: $PV = \frac{C}{r}\left[1 - \frac{1}{(1+r)^T}\right]$
  - Growing: $PV = \frac{C}{r-g}\left[1 - \left(\frac{1+g}{1+r}\right)^T\right]$
- Net present value: $NPV = PV(\text{benefits}) - PV(\text{costs})$
- Interest rate per compounding interval if APR r compounded m times a year: $\frac{r}{m}$
- Effective (or equivalent) annual rate: $1 + r_{\text{EAR}} = \left(1 + \frac{r}{m}\right)^m$