A Comparison of the Black-Scholes Model and Monte-Carlo Model for Options Pricing

Lisa Everest

November 5, 2017

Abstract

Options are contracts that give the holder of the option the right, but not the obligation, to buy or sell an underlying asset at a specified strike price on a specified future date and are useful in the financial industry; they allow investors to trade not only stock movements, but also the passage of time and movements in volatility. Determining the true value of an option is therefore challenging, as its price is highly dependent on a variety of factors, including the underlying asset and market conditions. This paper develops two methods for pricing options given assumptions on the underlying asset's behavior as geometric Brownian motion and the market as efficient. The first method is the Black-Scholes Formula, solved and derived from the Black-Scholes equation, and the second is the Monte-Carlo model. Both methods are applied to price example options and are compared for their accuracy and efficiency.

Contents

1 Motivation for Options Pricing

Options are contracts that give the buyer (the holder of the option) the right, but not the obligation, to buy or sell an underlying asset at a specified strike price on a specified future date. They are useful in the financial industry, as they allow investors to trade not only stock movements, but also the passage of time and movements in volatility. Additionally, options pricing is essential in corporate finance decision making since many corporate liabilities can be expressed in terms of options or combinations of options and such variations help to leverage different methods. Determining the true value of an option, however, is very difficult; its price is highly dependent on a large number of factors, including the underlying asset's return and volatility, as well as the option's time to expiration and strike price. Because options are dependent on so many unpredictable variables, it is difficult to precisely determine the option's price. Furthermore, there are different situations in which one would want to price an option; for example, an investor may be interested in the true price of an option, given that he or she has a lot of information available about that option and its underlying asset. On the other hand, however, another investor may be pricing a different option, for which he or she has little or no information. Ideally, a model for options pricing would provide an accurate solution for many different situations.

Currently, there are two solutions for this problem of options pricing: the Black-Scholes Model, developed in 1973, and the Monte-Carlo Model, introduced in 1964. These two methods, however, are quite different, as the Black-Scholes Model relies on solving a mathematical equation for options pricing, while the Monte-Carlo Model is a simulation-based method. Each method is best suited to different economic environments and investors' needs. This report, therefore, will discuss these two solutions to the problem of options pricing. More specifically, the Black-Scholes Model will be presented in Section 2 and the Monte-Carlo Model will be presented in Section 3. Both sections will cover a derivation of the model, the model's assumptions on the market, and practical applications of the model. Then, in Section 4, this report will delve deeper into comparing the two models, especially with regards what kinds of situations each model is better suited to.

2 Black-Scholes Model

In 1973, Fischer Black and Myron Scholes published "The Pricing of Options and Corporate Liabilities" in *The Journal of Political Economy*[1]. In this paper, they developed the Black-Scholes Model in order to specifically price a European call option. The model revolves around the Black-Scholes Equation, a partial differential equation (PDE) that estimates the price of such an option over time. The solution of this PDE is known as the Black-Scholes Formula, which produces the price of the option based on the Black-Scholes Equation. This section discusses first the idea behind the Black-Scholes Equation, its assumptions about the market, the derivation of both the Black-Scholes Equation and Black-Scholes Formula, and finally, applications of the model.

2.1 Summary

The goal of the Black-Scholes Model is to relate the price of a European call option to the price of the underlying asset. The key idea behind the Black-Scholes Model is to "delta-hedge" the option whose price we are interested in. Delta-hedging is done by buying and selling the underlying asset in such a way that eliminates the risk of the option. From this idea, Black and Scholes developed the Black-Scholes Equation, which can be solved for the Black-Scholes Formula, which gives an explicit expression for the price of the option.

2.2 Assumptions

The Black-Scholes Model makes many assumptions about the underlying asset and the market of that asset. To begin with, it requires that the option being priced is a European call option. In addition, it is assumed that the price S ofthe option's underlying asset/stock follows a geometric Brownian motion[2]. Furthermore, there is an assumption that the payoff of the option at expiration $V(S,T)$, where T is the option's time of expiration, is known. Finally, like most financial models, the Black-Scholes Model assumes that the market is efficient; in other words, investors agree on the likely performance and risk of securities, based on a common time frame, and there exists no arbitrage in the market.

2.3 Derivation of the Black-Scholes Equation

Given that the assumptions in Section 2.2 hold, the goal of the Black-Scholes Model is to relate the price of the option to the price of the underlying asset. In otherwords, the goal is to describe the payoff $V(S,t)$ of the option as a function of S, the price of the underlying asset, and t, the time. With this goal in mind, the Black-Schole Equation may be stated and then proved.

[1]Black, Fischer and Myron Scholes. "The Pricing of Options and Corporate Liabilities," *The Journal of Political Economy*¡https://www.cs.princeton.edu/courses/archive/fall09/cos323/papers/black_scholes73.pdf¿

[2]Lawler, Gregory F. "Stochastic Calculus: An Introduction with Applications." ¡http://www.math.uchicago.edu/~lawler/finbook2.pdf¿

Theorem 1. *Let $V(S,t)$ be the value of the option, dependent on S, the underlying asset's price, and T, the time to expiration of the option. Then the partial differential equation describing the value of the option is*

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \tag{1}$$

Proof. From the assumptions in Section 2.2, it is assumed that the price SS of the option's underlying asset follows a Brownian motion: $\frac{dS}{S} = \mu dt + \sigma dW$. Rearranging the equation, we have

$$dS = S\mu dt + S\sigma dW. \tag{2}$$

Equation (2), however, satsifies Ito's Lemma, which we state now[3].

Lemma 1. *Ito's Lemma. Let X_t be an Ito process satisfying the stochastic differential equation $dX_t = \mu_t dt + \sigma_t dW_t$, where W_t is a Wiener process (standard Brownian motion process[4]). Then, for a function $f(x,t)$, Ito's Lemma states that*

$$df = \left(\frac{\partial f}{\partial t} + \mu_t\frac{\partial f}{\partial x} + \frac{\sigma_t^2}{2}\frac{\partial^2 f}{\partial x^2}\right)dt + \sigma_t\frac{\partial f}{\partial x}\,dW_t. \tag{3}$$

Because Equation (2) satisfies the conditions for Ito's Lemma stated above, we can apply the lemma by replacing $f(x,t)$ in Ito's Lemma with $V(S,T)$ to determine how V changes over time:

$$dV = (\frac{\partial V}{\partial T} + S\mu\frac{\partial V}{\partial S} + \frac{(S\sigma)^2}{2}\frac{\partial^2 V}{\partial S^2})dt + S\sigma\frac{\partial V}{\partial S}dW. \tag{4}$$

Then, since the Black-Scholes Model uses a delta-hedge portfolio, the price Π of this portfolio is

$$\Pi = -V + \frac{\partial V}{\partial S}S^5. \tag{5}$$

The change $\Delta\Pi$ in the value of this portfolio over the time period $[t, t+\Delta t]$ is

$$\Delta\Pi = -\Delta V + \frac{\partial V}{\partial S}\Delta S. \tag{6}$$

We can now make Equation (4) discrete in order to determine an expression for ΔS:

$$\Delta S = S\mu\Delta t + S\sigma\Delta W. \tag{7}$$

Similarly, we discretize Equation (5) to find an expression for ΔV:

$$\Delta V = (\frac{\partial V}{\partial t} + S\mu\frac{\partial V}{\partial S} + \frac{(S\sigma)^2}{2}\frac{\partial^2 V}{\partial S^2})\Delta t + S\sigma\frac{\partial V}{\partial S}\Delta W. \tag{8}$$

---

[3]For a full proof of Ito's Lemma, please see ¡http://lsc.fie.umich.mx/~juan/Materias/Posgrado/Finance/fin/node10.html¿

[4]See Appendix A for details on a Brownian motion process.

[5]For more information, see ¡https://en.wikipedia.org/wiki/Delta_neutral¿

Plugging Equation (8) and Equation (9) into Equation (7), we have

$$\Delta\Pi = -(\frac{\partial V}{\partial t} + S\mu\frac{\partial V}{\partial S} + \frac{(S\sigma)^2}{2}\frac{\partial^2 V}{\partial S^2})\Delta t + S\sigma\frac{\partial V}{\partial S}\Delta W) + \frac{\partial V}{\partial S}(S\mu\Delta t + S\sigma\Delta W).$$

Simplifying, we have

$$\Delta\Pi = -(\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2})\Delta t. \tag{9}$$

Notice that Equation (10) no longer has any terms involving W. This makes since, as the Black-Scholes Model utilizes a delta-hedged portfolio that must be riskless. The uncertainty associated with W is therefore gone. Thus, the rate of return on the portfolio must be equivalent to the rate of return on any other riskless instrument. Therefore, let r be the risk-free rate of return. Then by definition, we have

$$\frac{\Delta\Pi}{\Delta t} = r\Pi,$$

which simplifies to

$$\Delta\Pi = r\Pi\Delta t.$$

Substituting Equation (6) for Π and Equation (10) for $\Delta\Pi$, we have

$$-(\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2})\Delta t = r(-V + \frac{\partial V}{\partial S}S)\Delta t.$$

This then simplifies to

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \tag{10}$$

which is the Black-Scholes Equation stated in Section 2.3.3. ■

2.4 Black-Scholes Formula

Now that the Black-Scholes Equation has been stated and derived in Section 2.3, it may be solved to derive the Black-Scholes Formula.

Theorem 2. *The Black-Scholes Formula (Equation (11))describes the relationship between the price $V(S,T)$ of a European call option to S and T, where S is the underlying asset's price and T is the time:*

$$V(S, t = T) = S\Phi(D_1) - Ke^{-r(T-t)}\Phi(D_2), \tag{11}$$

where

$$D_1 = \frac{\log\frac{S}{K} + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

and

$$D_2 = \frac{\log\frac{S}{K} + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{(T-t)}}.$$

Proof. In order to prove Theorem 2, it is first necessary to state and prove a lemma.

Lemma 2. *Suppose V(x,τ) satisfies* $\frac{\partial V}{\partial \tau} = A\frac{\partial^2 V}{\partial x^2}\frac{\partial V}{\partial x}+cV$, *with initial conditions* $V(x,0) = f(x) and A, B, C$ *are constants where A ¿ 0. Then,*

$$V(x,\tau) = \frac{e^{c\tau}}{\sqrt{4\pi A\tau}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(\frac{y-x-B\tau}{\sqrt{2A\tau}})^2} f(y)dy. \tag{12}$$

Proof. To prove this lemma, we must do a change of variables. Let $u = e^{\alpha x+\beta\tau}V$. Then $V = e^{-(\alpha x+\beta\tau)}u$. Then we may differentiate V with respect to τ and x:

$$\frac{\partial V}{\partial \tau} = e^{-(\alpha x+\beta\tau)}(\frac{\partial u}{\partial \tau} - \beta u),$$

$$\frac{\partial V}{\partial x} = e^{-(\alpha x+\beta\tau)}(\frac{\partial u}{\partial x} - \alpha u),$$

and

$$\frac{\partial^2 V}{\partial x^2} = e^{-(\alpha x+\beta\tau)}(\frac{\partial^2 u}{\partial x^2} - 2\alpha\frac{\partial u}{\partial x} + \alpha^2 u).$$

Since V(x,τ) satisfies $\frac{\partial V}{\partial \tau} = A\frac{\partial^2 V}{\partial x^2}\frac{\partial V}{\partial x} + cV$, we can plug in the above three derivatives, as well as $V = e^{-(\alpha x+\beta\tau)}u$ to get

$$e^{-(\alpha x+\beta\tau)}(\frac{\partial u}{\partial \tau}-\beta u) = Ae^{-(\alpha x+\beta\tau)}(\frac{\partial^2 u}{\partial x^2}-2\alpha\frac{\partial u}{\partial x}+\alpha^2 u)+Be^{-(\alpha x+\beta\tau)}(\frac{\partial u}{\partial x}-\alpha u)+e^{-(\alpha x+\beta\tau)}u.$$

This simplifies to

$$\frac{\partial u}{\partial \tau} - \beta u = A(\frac{\partial^2 u}{\partial x^2} - 2\alpha\frac{\partial u}{\partial x} + \alpha^2 u) + B(\frac{\partial u}{\partial x} - \alpha u) + Cu,$$

which may be rearranged to get

$$\frac{\partial u}{\partial \tau} = A\frac{\partial^2 u}{\partial x^2} + (B - 2A\alpha)\frac{\partial u}{\partial x} + (C + \beta - \alpha B + A\alpha^2)u.$$

Since in the beginning of this proof, we set $u = e^{\alpha x+\beta\tau}V$ for any α, β, we may now let $\alpha = \frac{B}{2A}$ in order to remove the $\frac{\partial u}{\partial x}$ term:

$$\frac{\partial u}{\partial \tau} = A\frac{\partial^2 u}{\partial x^2} + (C + \beta - \frac{B^2}{4A})u.$$

Similarly, since we set $u = e^{\alpha x+\beta\tau}V$ for any α, β in the beginning of this proof, we may now let $\beta = \frac{B^2}{4A} - C$ in order to eliminate the $\frac{\partial^2 u}{\partial x^2}$ term:

$$\frac{\partial u}{\partial \tau} = A\frac{\partial^2 u}{\partial x^2}.$$

But this is the heat equation. Since the initial conditions for $V(x,\tau)$ are $V(x,0) = f(x)$ and $u = e^{\alpha x+\beta\tau}V$, we have that the initial condition for the above heat equation is $u(x,0) = e^{\alpha x}V(x,0) = e^{\frac{B}{2A}}f(x)$. The solution of this heat equation with this initial value is

$$u(x,\tau) = \frac{1}{\sqrt{4\pi A\tau}} \int_{\infty}^{\infty} e^{-\frac{(x-y)^2}{4A\tau}} e^{\frac{B}{2A}y} f(y)dy.^6.$$

Completing the square in the exponentials, this simplifies to

$$u(x,\tau) = \frac{e^{\alpha x+\beta\tau}e^{C\tau}}{\sqrt{4\pi A\tau}} \int_{-\infty}^{\infty} e^{-(\frac{y-x-\beta\tau}{2\sqrt{A\tau}})^2} f(y)dy.$$

Since $V = e^{-(\alpha x+\beta\tau)}u$, we can multiply both sides by $e^{-\alpha x+\beta\tau}$ and simplify to get

$$V(x,\tau) = \frac{e^{C\tau}}{\sqrt{4\pi A\tau}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(\frac{y-x-\beta\tau}{\sqrt{2A\tau}})^2} f(y)dy. \tag{13}$$

This concludes the proof of the lemma. ■

Now that we have proved this lemma, we may now solve the Black-Scholes Equation to derive the Black-Scholes Formula for a European call option. Let $\tau = T - t$, where T is the maturity of the option. Then $\frac{\partial V}{\partial \tau} = -\frac{partialV}{\partial t}$ and Equation (11) becomes

$$\frac{\partial V}{\partial \tau} = \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2} + rs\frac{\partial V}{\partial S} - rV. \tag{14}$$

Then, let $x = \log S$, so $s = e^x$. Then

$$\frac{\partial V}{\partial S} = \frac{\partial V}{\partial x}\frac{\partial x}{\partial S} = \frac{1}{S}\frac{\partial V}{\partial x}$$

and

$$\frac{\partial^2 V}{\partial S^2} = \frac{\partial}{\partial S}(\frac{1}{S}\frac{\partial V}{\partial x}) = \frac{1}{S^2}\frac{\partial^2 V}{\partial x^2} - \frac{1}{S^2}\frac{\partial V}{\partial x}.$$

Plugging these derivatives into Equation (14) and simplifying, we arrive at

$$\frac{\partial V}{\partial \tau} = \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial x^2} + (r - \frac{\sigma^2}{2})\frac{\partial V}{\partial x} - rV. \tag{15}$$

This equation is of the form $\frac{\partial V}{\partial \tau} = A\frac{\partial^2 V}{\partial x^2} + B\frac{\partial V}{\partial x} + CV$. We can then apply Lemma 1 to Equation (15), where $A = \frac{\sigma^2}{2}$, $B = r - \frac{\sigma^2}{2}$, and $C = -r$ to conclude that

$$V(x,\tau) = \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(\frac{y-x-(r-\frac{\sigma^2}{2}\tau)}{\sigma\sqrt{\tau}})^2} f(y)dy, \tag{16}$$

where $x = \log S$.

[6] For the full solution of the heat equation, see: https://web.stanford.edu/class/math220b/handouts/heateqn.pdf

Now, the value $V(S, t = T)$ of a call option is

$$V(S, t = T) = \begin{cases} S - K & \text{if } S > K \\ 0 & \text{if } S \le K, \end{cases}$$

where K is the strike price of the option.

Since $x = \log S$, we have that $S = e^x$ and so

$$V(S, t = T) = V(e^x, t = T) = f(x)$$

and

$$f(x) = \begin{cases} e^x - K & \text{if } e^x > K \\ 0 & \text{if } e^x \le K. \end{cases}$$

Since $f(x) = 0$ when $e^x \le K$, we may then rewrite Equation (16):

$$V(x, \tau) = \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} \int_{\log K}^{\infty} e^{-\frac{1}{2}(\frac{y - x - (r - \frac{\sigma^2}{2}\tau)}{\sigma\sqrt{\tau}})^2} (e^y - K) dy. \tag{17}$$

Let us now make a change of variables. Let $z = \frac{y - x - (r - \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}$. Then $dz = \frac{dy}{\sigma\sqrt{\tau}}$ and $y = x + (r - \frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}z$. We can apply this change of variables to rewrite and simplify Equation (17):

$$V(x, \tau) = \frac{e^{-r\tau}}{\sqrt{2\pi}} \int_{\frac{\log K - x - (r - \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}}^{\infty} e^{-\frac{1}{2}z^2} (e^{x + (r - \frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}z} - K) dz. \tag{18}$$

Equation (17) is the sum of two integrals:

$$V(x, \tau) = \frac{e^{-r\tau}}{\sqrt{2\pi}} \int_{\frac{\log K - x - (r - \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}}^{\infty} e^{-\frac{1}{2}z^2} e^{x + (r - \frac{\sigma^2}{2})\tau + \sigma\sqrt{\tau}z} dz - K \frac{e^{-r\tau}}{\sqrt{2\pi}} \int_{\frac{\log K - x - (r - \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}}^{\infty} e^{-\frac{1}{2}z^2} dz.$$

To simplify Equation (18), let's handle each term of V separately. To do so, we will use the cumulative distribution function of a Gaussian distribution in the next steps. Let us denote the Gaussian cumulative distribution function as $\Phi(x)$, such that

$$\Phi(x) = \int_{-\infty} x e^{-\frac{z^2}{2}} dz.$$

Since $e^{-\frac{z^2}{2}}$ is an even function and $\int_{-\infty} \infty e^{-\frac{z^2}{2}} dz = 1$, then

$$\Phi(-x) = \int_{x} \infty e^{-\frac{z^2}{2}} dz.$$

Let us now consider the first term of V (the first integral):

$$\frac{e^{-r\tau}}{\sqrt{2\pi}}\int_{\frac{\log K - x-(r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}}^{\infty} e^{-\frac{1}{2}z^2}e^{x+(r-\frac{\sigma^2}{2})\tau+\sigma\sqrt{\tau}z}dz = \frac{e^{x-r\tau}}{\sqrt{2\pi}}\int_{\frac{\log K - x-(r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}}^{\infty} e^{-\frac{1}{2}(z^2-2r\tau+\sigma^2\tau-2\sigma\sqrt{\tau}z)}dz$$
$$= \frac{e^{x-r\tau}}{\sqrt{2\pi}}\int_{\frac{\log K - x-(r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}}^{\infty} e^{-\frac{1}{2}(z-\sigma\sqrt{\tau})^2+r\tau}dz$$
$$= \frac{e^{x-}}{\sqrt{2\pi}}\int_{\frac{\log K - x-(r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}}^{\infty} e^{-\frac{1}{2}((z-\sigma\sqrt{\tau})^2}dz$$

Remember that $x = \log S$, so $S = e^x$. Also, let us do another change of variables and let $w = z - \sigma\sqrt{\tau}$. Then $dw = dz$, $z = w + \sigma\sqrt{\tau}$ and the lower limit of the integral becomes $\frac{\log K - x-(r+\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}$. It is now convient to also use the fact that $\tau = T - t$. Then the first term of V is

$$\frac{S}{\sqrt{2\pi}}\int_{\frac{\log K - \log S-(r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}}^{\infty} e^{-\frac{1}{2}w^2}dw = S\Phi(-(\frac{\log K - \log S - (r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}))$$
$$= S\Phi(\frac{\log\frac{S}{K} + (r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}})$$

We now consider the second term (the second integral) of V, where we may also use the fact that $\tau = T - t$ and $x = \log S$:

$$-K\frac{e^{-r\tau}}{\sqrt{2\pi}}\int_{\frac{\log K - x-(r-\frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}}^{\infty} e^{-\frac{1}{2}z^2}dz = -Ke^{-r(T-t)}F(-(\frac{\log K - \log S - (r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{(T-t)}}))$$
$$= -Ke^{-r(T-t)}\Phi(\frac{\log\frac{S}{K} + (r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{(T-t)}})$$

Putting the two terms together, we have an equation for $V(S, t = T)$:

$$V(S,t=T) = S\Phi(\frac{\log\frac{S}{K} + (r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}) - Ke^{-r(T-t)}\Phi(\frac{\log\frac{S}{K} + (r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{(T-t)}}) \quad (19)$$

We may simplify this a bit to get

$$V(S,t=T) = S\Phi(D_1) - Ke^{-r(T-t)}\Phi(D_2), \quad (20)$$

where

$$D_1 = \frac{\log\frac{S}{K} + (r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

and

$$D_2 = \frac{\log\frac{S}{K} + (r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{(T-t)}}.$$

Thus, we have just derived Equation (19) (or its alternate version, Equation (20)), the Black-Scholes Formula for the price of a European call option. ■

2.5 Application/Example

Let us now discuss how to apply the Black-Scholes Formula in practice. We will calculate the price of a European call option with Apple stock as its underlying asset. From NASDAQ, the prices of Apple stock from January 3, 2017 to November 10, 2017 are obtained[7]. Using this data, the annualized volatility of Apple stock is calculated to be 17.86%. The current price Apple stock is $174.67 per share. The risk-free interest rate is estimated using the Treasury 3-month rate, which is 1.18%[8]. Say the strike price of a European call option on Apple's stock, expiring in 5 days, is $170. Using this information, we can calculate the value or price of this option using Black-Scholes Formula.

First, we calculate D_1 and D_2:

$$\begin{aligned} D_1 &= \frac{\log \frac{S}{K} + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \\ &= \frac{\log \frac{174.67}{170} + (1.18\% + \frac{17.68\%^2}{2})(5)}{17.68\%\sqrt{5}} \\ &\approx 0.4155 \end{aligned}$$

and

$$\begin{aligned} D_2 &= \frac{\log \frac{S}{K} + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{(T-t)}} \\ &= \frac{\log \frac{174.67}{170} + (1.18\% - \frac{17.68\%^2}{2})(5)}{17.68\%\sqrt{(5)}} \\ &\approx 0.02012 \end{aligned}$$

We can now calculate $V(S, t = T)$:

$$\begin{aligned} V(S, t = T) &= S\Phi(D_1) - Ke^{-r(T-t)}\Phi(D_2) \\ &= 174.67\Phi(0.4155) - 170e^{-1.18\%(5)}\Phi(0.02012) \\ &\approx \$34.06 \end{aligned}$$

Thus, the European call option with Apple stock as its underlying asset is worth roughly $34.06.

3 Monte-Carlo Model

While the Black-Scholes Model provides a mathematical equation to solve for options pricing, the Monte-Carlo Model is a simulation-based method that calculates an option's average payoff.

[7] For the full data, see: http://www.nasdaq.com/symbol/aapl/historical

[8] For the full data, see: https://www.treasury.gov/resource-center/data-chart-center/interest-rates/Pages/TextView.aspx?data=yield

3.1 Summary

With one of the earliest variants in 1930, Monte-Carlo methods are actually a large class of algorithms that rely on repeated random sampling to obtain numerical results. They use randomness to solve problems that might be deterministic in principle. Thus, these methods are very applicable for finance and specifically for options pricing. In fact, the Monte-Carlo Model for options pricing was first introduced to finance in 1964 by David B. Hertz. This model simulates the various sources of uncertainty affecting the value of an option in order to create random sample for the price of its underlying asset and use this random sample to determine the price of the option from the discounted expected value of the underlying.

3.2 Assumptions

To use the Monte-Carlo Model to price options, one must make some assumptions. First and foremost, the model assumes that the risk neutrality assumption for pricing holds. This is the assumption that the current value of financial assets is equal to their expected payoffs in the future discounted at the risk-free rate. Furthermore, it assumes that the price of the underlying asset is a Brownian motion. Finally, like the Black-Scholes Model, the Monte-Carlo Model assumes that the market is efficient; in other words, investors agree on the likely performance and risk of securities, based on a common time frame, and there exists no arbitrage in the market.

3.3 Derivation

Because Monte-Carlo methods are such a large class of algorithms, it is important to first discuss general Monte-Carlo methods in mathematics and the problem they solve.

The Monte-Carlo methods are a solution to the generic problem of evaluating the expected value of some function $h(X)$ where X takes on values in the set χ:

$$E(h(X)) = \int_{\chi} h(x)f(x)dx.$$

These methods evaluate the above integral by first generating a sample of values for X, $\{X_1, X_2, ..., X_n\}$ from the density function f. Then, the integral is approximated using the sample mean $\overline{h}_n$, which is a discrete average:

$$\overline{h}_n = \frac{1}{n}\sum_{i=1}^{n} h(X_i).$$

This discrete average approximation is valid since, as n increases, $\overline{h}_n$ approaches $E(h(X))$. This is made more precise with with the Strong Law of Large Numbers.

Theorem 3. *The Strong Law of Large Numbers*[9] *states that the sample average surely convergest to the expected value. In other words,*

$$\overline{h}_n = \frac{1}{n}\sum_{i=1}^{n} h(X_i) \to \int_{\chi} h(x)f(x)dx = E(h(X)). \tag{21}$$

Furthermore, $\overline{h}_n$ may by normalized to a standard normal random variable using the sample variance $v_n = \frac{1}{n^2}\sum_{i=1}^{n}(h(X_i) - \overline{h}_n)^2$, which follows from the Central Limit Theorem.

Theorem 4. *The Central Limit Theorem*[10] *states that when $h^2(X)$ has a finite expectation under f,*

$$\frac{\overline{h}_n - E(h(X))}{\sqrt{v_n}} \sim N(0,1). \tag{22}$$

Now that the general Monte-Carlo methods and how they solve the problem of evaluating $E(h(X)) = \int_{\chi} h(x)f(x)dx$ have been discussed, the derivation of the Monte-Carlo Model for options pricing may be presented. Similar to the general Monte-Carlo methods, the Monte-Carlo Model relies on generating a sample set using random sampling methods to specifically evaluate a simple expected value for the option being priced: $E(h(X)) = \int_{\chi} h(x)f(x)dx$, where $h(X) = X$ and $f(X)$ is the distribution of the payoff for the option.

The Monte-Carlo method for options pricing assumes that the price of the underlying aset $S(T)$ is a simple Brownian motion:

$$S(T) = S_t e^{(r-\frac{\sigma^2}{2})(T-t)+\epsilon\sigma\sqrt{T-t}}. \tag{23}$$

In Equation (16), S_T is the underlying asset's price at expiration, S_t is the current asset price, T is the time to expiration, t is the current time, r is the risk-free rate, σ is the expected volatility of the asset, and ϵ is a number sampled from a standard normal distribution. The random sampling using Equation (16), therefore, is derived from the random sampling of ϵ.

Repeated sampling using Equation (16), through repeated sampling of ϵ, provides a sample set $\{S_{T_1}, S_{T_2}, ..., S_{T_n}\}$ of potential future prices for the underlying asset. From this sample set, the corresponding set of payoffs for the option being priced may be determined. With a European call option, the expected present value C_t is

$$C_t = e^{-r(T-t)}E(max(0, S_T - X)). \tag{24}$$

Calculating C_t for each potential asset price $\{S_1, S_2, ..., S_n\}$ produces a sample set for the payoffs of the option:

$$C = \{C_{T_1}, C_{T_2}, ..., C_{T_n}\}$$

[9] For more information, see ¡`http://mathworld.wolfram.com/StrongLawofLargeNumbers.html`¿

[10] For more information, see ¡`http://mathworld.wolfram.com/CentralLimitTheorem.html`¿

, where
$$C_{T_i} = e^{-r(T-t)}E(\max(0, S_{T_i} - X)).$$

Using this sample set C, the sample mean $\overline{C}_n$ may be calculated:

$$\overline{C}_n = \frac{1}{n}\sum_{i=1}^{n} C_{T_i}. \tag{25}$$

Thus, Equation (17) and the sample mean $\overline{C}_n$ is the estimation of the option's price. Note that similar to in the general Monte-Carlo methods,

$$\lim_{n\to\infty} \overline{C_n} = \int_C x f_C(x)dx,$$

by the Strong Law of Large Numbers. The more random samples that are taken for the option's underlying asset, therefore, the more accurate the sample mean $\overline{C}_n$ should be to the true price of the option.

3.4 Application

Now that the Monte-Carlo Model has been proved and derived, it can be used in practice. We will use the model to calculate the price of the same European call option as in Section 2.5, with Apple stock as its underlying asset. Using the Section 2.5's data, the annualized volatility of Apple stock is calculated to be 17.86%. Again, the current price Apple stock is \$174.67 per share and the risk-free rate estimated using the Treasury 3-month rate, which is 1.18%. Say the strike price of a European call option on Apple's stock, expiring in 5 days, is \$170. Then, using Python's `numpy` data analysis library, we may use the Monte-Carlo Model to estimate the price of this call option.

The first thing needed to is to sample ϵ from Equation (16) in Section 3.2. We will create a random sample by sampling ϵ from a standard normal distribution 100 times and store them in the variable `sampleEpsilonSet`:

```
import numpy as np
sampleEpsilonSet = np.random.normal(0, 1, 100)
```

From this set of 100 random samples of ϵ, we may calculate the associated stock prices using Equation (16) from Section 3.2, storing them in the variable `assetPrices`:

```
import math
assetPrices = []
for epsilon in sampleSet:
    assetPrice = 174.67 * math.e ** ((.0118 - (.1786**2)/2)*5
        + epsilon*.1786*math.sqrt(5))
    assetPrices.append(assetPrice)
```

Then, we calculated the corresponding option payoffs for each potential asset price in `assetPrices` using Equation (17), storing these payoffs in the variable `optionPrices`:

```
optionPrices = []
for assetPrice in assetPrices:
    optionPrice = (math.e ** (-.0118*5)) *
        max(0, assetPrice - 170)
    optionPrices.append(optionPrice)
```

Finally, from `optionPrices`, a sample mean may be calculated and used as the option's price:

```
sampleMean = sum(optionPrices)/len(optionPrices)
```

After executing the above code, the result is `sampleMean` ≈ `$32.2571`. Therefore, the price of the European call option with Apple stock as its underlying asset is estimated to be $32.2571 using the Monte-Carlo Model.

4 Comparison of Black-Scholes and Monte-Carlo

After discussing the assumptions, derivation, and application of the Black-Scholes Model and Monte-Carlo Model, this section will explore which model is better suited to what kind of situations. This is important to examine because in the financial industry, there are various different cases in which investors would be interested in pricing an option. For example, the investor may have a lot or very little information about the option and its underlying asset. This then becomes a question regarding whether the option in question satisfies each model's requirements, which is discussed in Section 4.1. Or, the investor may have a lot or very little computing power or time, and as seen in Section 4.2, this becomes a matter of efficiency. Furthermore, the market conditions for the option in question may or may not satisfy the assumptions of each model, and this is explored more in Section 4.3. Finally, in consideration of the three previous factors, it may be possible that one model is better than another, in certain situation. This is discussed in Section 4.4.

4.1 Satsifaction of Requirements

Depending on how strict the requirements are for options pricing models, it may be very difficult or very easy to apply them to pricing a specific option. In the case of the Black-Scholes Model, the requirements are quite strict. As discussed in Section 2, the Black-Scholes Model, as presented in this paper, is applicable only for European call options who's strike price and time until expiration are known. The current underlying asset's current price, expected return, and volatility must also be known, as well as the risk-free rate. Though the Black-Scholes Model may be extended to types of options other than European call options, it requires another set of mathematical equations and solutions. For the purpose of this paper, therefore, it can be considered that the Black-Scholes Model is only easily applied for European call options.

The Monte-Carlo Model, however, only requires the strike price and time of expiration of the option, in addition to the risk-free rate and the underlying asset's current price and daily volatility. More importantly, there are no requirements on the type of option being priced. Though this paper only discussed the Monte-Carlo Model with respect to European call options, it is very simple to apply the Monte-Carlo Model to other options, including European put options, American call and put options, or more exotic options like Asian options. Pricing these other types of options would only require a slight modification to the model - that is, changing Equation (17) in Section 3.2 from the present value of a European call option to the present value of the other type of option being priced.

Thus, if an investor lacks information about the option in question, or if the investor is trying to price an option that is something other than a European call option, it is much more beneficial to use the Monte-Carlo Model, as it has less information requirements and is much more applicable to a large variety of options.

4.2 Efficiency

While the Monte-Carlo Model is better than the Black-Scholes Model in terms of information requirements and flexibility for applications in pricing different types of options, the Black-Scholes Model is much more efficeint than the Monte-Carlo Model. In many cases in the financial industry, investors must make quick decisions based on the price of an option. In other cases, investors may not have the appropriate computing power to fully utilize the Monte-Carlo Model. The Black-Scholes Model, therefore, is much quicker to use for options pricing than the Monte-Carlo Model.

The Black-Scholes Model for options pricing simply involves plugging in known numbers into Equation (11)from Theorem 2 in Section 2.4 to calculate the price of the option in question. There is no time nor computational power involved.

The Monte-Carlo Model, however, requires much more time and computational power, in comparison to the Black-Scholes Model. Since the Monte-Carlo is based on random sampling of the underlying asset's potential price, the time used to compute the option's price increases linearly with the number of random samples generated. In order to maintain some accuracy, however, one could not just take a single random sample and use it to approximate the option's price. As shown in Section 3.2 using the Strong Law of Large Numbers, the more random samples, the closer the Monte-Carlo approximation for the option's price is to the true price of the option. Thus, using a single random sample for the Monte-Carlo Model, in order to match the efficiency of the Black-Scholes Model, is most likely insufficient to price the option in question.

Thus, if an investor needed to compute the price of an option efficiently, it is clear that the single computation in the Black-Scholes Model is much faster than the Monte-Carlo Model. In fact, there is less computing power and memory needed for the Black-Scholes Model than the Monte-Carlo Model.

4.3 Assumptions

Although the Black-Scholes Model beats the Monte-Carlo Model in efficiency, neither of these models is usable if their assumptions are not satisfied. Both assume that the underlying asset's price follows standard Brownian motion and that the market is efficient, with a lack of arbitrage and with investors agreeing on the likely perofrmance and risk of securities.

While the two models share some assumptions, however, they also differ in other assumptions. The Black-Scholes Model relies on the assumption that the return on the underlying asset follows a normal distribution. The Monte-Carlo Model, however, assumes that the risk neutrality assumption for pricing holds. This is the assumption that the current value of financial assets is equal to their expected payoffs in the future discounted at the risk-free rate.

Because the two models share and differ in certain assumptions regarding the market and option in question, therefore, it is difficult to say definitively which model performs better in certain situations. However, if the assumptions of one model are clearly satisfied by a specific environment that an investor is interested in, then for that investor, that model could potentially be better than the other.

4.4 Accuracy

Similarly to how it is difficult to determine which of the two models for options pricing are better than the other with regards to their economic assumptions, it is also difficult to deduce whether the Black-Scholes Model or Monte-Carlo Model is more accurate when pricing an option.

For pricing a European call option, the Black-Scholes Model is definitively more accurate, as it was derived in order to price this kind of option. The Monte-Carlo Model, however, is just an approximation calculated from simulation. Theoretically, however, by the Strong Law of Large Numbers in Section 3.2, the Monte-Carlo Model's approximation for the option's price will converge to the price produced by the Black-Scholes Model, if the number of random samples is high enough. Of course, this convergence for the Monte-Carlo Model costs time and efficiency, as discussed in Section 4.2. Because the Monte-Carlo Model is a simulation, therefore, the theory-based Black-Scholes Model is more accurate when pricing a European call option.

For other types of call options like put options, American options, and even more exotic options, however, the Monte-Carlo Model is much more suited. Because the Black-Scholes Model as discussed in this paper is designed to price European call options, it is essentially unusable for pricing other types of options. The Monte-Carlo Model, however, which may be easily adjusted to price other options, would therefore be able to price such options more accurately.

Thus, since the Black-Scholes Model was derived for the purpose of pricing European call options, it is much more accurate when pricing such a specific type of option. The Monte-Carlo Model, however, is more accurate when pricing options that are not European call options.

5 Conclusion

Both the Black-Scholes Model and Monte-Carlo Model solve the same recurring problem in the financial industry. That is, how can investors accurately and quickly price an option? This paper explored this question in-depth through an examination of the two models individually, discussing each model's assumptions, derivation, and application in the financial industry. Then, a comparison between the two models was done, with respect to four categories: requirements, efficiency, assumptions, and accuracy. Each model, however, had strengths and weaknesses in each of these categories. While the Black-Scholes Model involves simply plugging in numbers to an equation and is very efficient in pricing European call options, it cannot price other types of options. The Monte-Carlo Model of simulation, however, is much more flexible and can reasonably approximate the prices of all types of options. Thus, it is difficult to determine exactly which model better solves the question of accurately and quickly pricing an option. Both models, however, solve this question reasonably well.