Generalizing Real-rooted Polynomials to Real Stable Polynomials

Lisa Everest

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Abstract

This paper is an exposition on two classes of polynomials, real-rooted polynomials and real stable polynomials, and how real stable polynomials are a generalization of real-rooted polynomials. We also explore combinatorial applications of both real-rooted polynomials and real stable polynomials, first using real-rooted polynomials to prove the well-known Newton's inequalities. This paper culminates in a break-through proof of the existence of an infinite sequence of regular bipartite Ramanujan graphs that utilizes real stable polynomials in an innovative way.

1 Introduction

Polynomials can be surprisingly useful in unexpected ways. In this paper, we explore just how unusual some applications of polynomials can be. More specifically, we explore two special classes of polynomials: real-rooted polynomials and real stable polynomials. We see how real stable polynomials generalize real stable polynomials and how both classes of polynomials can be utilized to prove interesting combinatorial results that, at first, seem unrelated to polynomials.

Intuitively, we can define a real-rooted polynomial as a single variable polynomial with all real roots. Moreover, this class of polynomials has particularly useful characteristics that are called closure properties, which are operations that can be applied to real-rooted polynomials to produce other real-rooted polynomials. In other words, these properties preserve real-rootedness! These closure properties are especially useful in applications, particularly in proving the well-known Newton's inequalities (Theorem 3.1, Section 3), a claim on the relationship between certain coefficients in real-rooted polynomials. More specifically, the proof of Newton's inequalities hinges on the use of these closure properties on a known real-rooted polynomial to produce a polynomial that is also real-rooted and somehow useful for deriving Newton's inequalities.

Real-rooted polynomials, however, are univariate. The natural question, therefore, is to ask how to generalize real-rooted polynomials into multiple variables. As it turns out, these generalized real-rooted polynomials are called real stable polynomials, and a polynomial $p(z) = p(z_1, ..., z_n)$ is considered real stable if $p(z_1, ..., z_n) \neq 0$ whenever $\operatorname{Im} z_i > 0$ for all $i = 1, ..., n$.

Similar to how real-rooted polynomials have closure properties that preserve real-rootedness, real stable polynomials also have corresponding closure properties that preserve real stability. The closure properties of real stable polynomials can be utilized in various applications and proofs, and in particular, this paper showcases an innovative application of real stable polynomials and their closure properties in Section 5, where we sketch Marcus, Spielman, and Srivastava's [MSS18] break-through proof on Ramanujan graphs, stated as Theorem 1.1. We will define Ramanujan graphs more formally in Section 5.

Theorem 1.1 (Marcus, Spielman, Srivastava 2014 [MSS18]). *For every $d \geq 3$, there is an infinite sequence of d-regular bipartite Ramanujan graphs.*

A key aspect of the proof of this theorem, Theorem 1.1, is the use of real stable polynomials. More specifically, the construction of this infinite sequence of Ramanujan graphs in Theorem 1.1 begins with a complete d-regular bipartite graph that is trivially Ramanujan, and the sequence is produced by a consecutive application of 2-lifts to each preceding graph to achieve the next graph in the sequence. To prove that the resulting graphs are indeed Ramanujan, we utilize real stable polynomials to bound the eigenvalues of the 2-lifts and show that the graphs in the sequence are Ramanujan.

2 Real-Rooted Polynomials

In this section, we explore real-rooted polynomials and interesting properties of real-rooted polynomials called closure properties, which will later be essential in the application of real-rooted polynomials for proving Newton's inequalities in Section 3.

Recall the intuitive definition of a real-rooted polynomial from Section 1. We can more formally define real-rooted polynomials as follows.

Definition 2.1. A univariate polynomial $p(t)$ is *real-rooted* if $\forall$ t s.t. $p(t) = 0, t \in \mathbb{R}$.

Throughout this section, we will use the the following polynomial as a working example of a real-rooted polynomial with 4 roots:

$$p(t) = t^4 + 6t^3 + 12t^2 + 10t + 3.$$

2.1 Closure Properties

Here, we state the three main closure properties of real-rooted polynomials as propositions, as they will be the main mechanism used in the combinatorial application of real-rooted polynomials to prove Newton's inequalities.

Proposition 2.1. [Constant Multiplication Closure Property] If $p(t)$ is real-rooted, then so is $p(ct)$ for all $c \in \mathbb{R}$.

Proof. We can write $p(t)$ in its factored form: $p(t) = a_d \prod_{k=1}^{d}(t - r_k)$, where each $r_k \in \mathbb{R}$, $k = 1, ..., d$ are the roots of p and a_d is the coefficient of the highest-degree term in p. Then, fixing any $c \in \mathbb{R}$:

$$p(ct) = a_d \prod_{k=1}^{d}(ct - r_k) = a_d c^t \prod_{k=1}^{d}(t - \frac{r_k}{c}).$$

Observe that each $\frac{r_k}{c}$, $k = 1, ..., d$ are the roots of $p(ct)$, and furthermore, because $r_k, c \in \mathbb{R}$, each $\frac{r_k}{c}$ must also be real. Thus $p(ct)$ is real-rooted. $\square$

We can apply this multiplication property to our working example of a real-rooted polynomial, $p(t) = t^4 + 6t^3 + 12t^2 + 10t + 3$. If $c = 2$, then

$$p(ct) = 16(t + \frac{1}{2})^3(t + \frac{3}{2})$$

is real-rooted, with roots $-\frac{1}{3}$ and -1.

Proposition 2.2. [Inversion Closure Property] If $p(t)$ is a degree d real-rooted polynomial, then so is $t^d p(1/t)$.

Intuitively, this property implies that if the coefficients of a polynomial were reversed (i.e. the coefficient associated with highest-order term were swapped with the coefficient associated with the lowest-order term in the polynomial), then the new polynomial with reversed coefficients is still real-rooted.

Proof. Again write $p(t)$ in its factored form: $p(t) = a_d \prod_{k=1}^{d}(t - r_k)$ as previously discussed. Then,

$$t^d p(\frac{1}{t}) = a_d t^d \prod_{k=1}^{d}(\frac{1}{t} - r_k) = a_d \prod_{k=1}^{d}(1 - \frac{r_k}{t}) = a_d \prod_{k=1}^{d}(\frac{1}{r_k} - t).$$

Observe that each $\frac{1}{r_k}$, $k = 1, ..., d$ are the roots of $p(ct)$, and furthermore, because $r_k \in \mathbb{R}$, each $\frac{1}{r_K}$ must also be real. Thus $t^d p(\frac{1}{t})$ is real-rooted. $\square$

We can apply this inversion property to our working example of a real-rooted polynomial, $p(t) = t^4 + 6t^3 + 12t^2 + 10t + 3$. Here, the degree is $d = 4$, and

$$t^d p(\frac{1}{t}) = 3(t + \frac{1}{3})(t + 1)^3$$

is real-rooted, with roots $-\frac{1}{3}$ and -1.

Proposition 2.3. [Derivative Closure Property] If $p(t)$ is real-rooted, then so is $p'(t)$.

Before proving the Derivative Closure Property, we need the Gauss-Lucas Theorem, which we state here for convenience.

Theorem 2.4 (Gauss-Lucas Theorem)**.** *For any polynomial p, the roots of its derivative p' can be written as a convex combination of the roots of p, where a convex combination of points $x_1, ..., x_n$ is the linear combination $\sum_{i=1}^{n} a_i x_i$, where the a_i are coefficients such that $a_1, ..., a_n \geq 0$ and $\sum_{i=1}^{n} = 1$.*

Proof of Proposition 9. By the Gauss-Lucas Theorem, it is clear that since the roots of $p'(t)$ are convex combinations of the roots of $p(t)$, which are real by assumption, then the roots of $p'(t)$ must also be real. $\square$

We can apply this derivative property to our working example of a real-rooted polynomial, $p(t) = t^4 + 6t^3 + 12t^2 + 10t + 3$, and we have that

$$p'(t) = 2(t+1)^2(2t+5)$$

is real-rooted, with roots -1 and $-\frac{5}{2}$.

3 Application of Real-Rooted Polynomials: Newton's Inequalities

This section explores Theorem 3.1, Newton's inequalities, which is an interesting combinatorial application that showcases the usefulness of the closure properties of real-rooted polynomials proven in the previous section. Using these closure properties, we will prove Newton's inequalities.

To understand Newton's inequalities, we first define ultra log concavity. We then state the theorem of Newton's inequalities.

Definition 3.1. A sequence $\{a_0, a_1, ..., a_x\}$ of nonnegative numbers is said to be *ultra log concave* if for all $0 < k < d$,

$$\frac{a_{k-1}}{\binom{d}{k-1}} \cdot \frac{a_{k+1}}{\binom{d}{k+1}} \leq \left(\frac{a_k}{\binom{d}{k}}\right)^2. \tag{1}$$

Furthermore, we call a polynomial *ultra log concave* if the sequence of its coefficients is ultra log concave.

Theorem 3.1 (Newton's inequalities)**.** *For any real-rooted polynomial* $p(t) = \sum_{i=0}^{d} a_i t^i$*, if* $a_0, ..., a_k \geq 0$ *(i.e. all coefficients are nonnegative), then* $p(t)$ *is ultra log concave.*

Proof. Intuitively, the idea of this proof is to begin with a polynomial that is assumed to be real-rooted and apply the closure properties of real-rooted polynomials to this polynomial. Because we want to show that this polynomial is ultra-log concave, we must show that any set of three coefficients a_{i-1}, a_i, a_{i+1}, for $0 < i < d$, satisfies Equation 1. To do so, we apply the closure properties of real-rooted polynomials to the original polynomial to eliminate all terms except for those associated with the coefficients a_{i-1}, a_i, a_{i+1}, for any $0 < k < d$. The resulting polynomial will be real-rooted, and we can utilize its real-rootedness and its discriminant to derive Equation 1 and prove ultra log concavity.

More precisely, pick any real-rooted polynomial $p(t) = \sum_{i=k}^{d} a_k t^k$. Also, pick any $i = 1, ..., d-1$. Then eliminate the first $i-1$ coefficients $a_0, ..., a_{i-2}$ of $p(t)$ by applying Proposition 2.3, the Multiplication Closure Property of real-rooted polynomials, $i-1$ times to $p(t)$. Call the resulting polynomial $p_1(t) = p^{(i-1)}(t)$:

$$\begin{aligned} p_1(t) &= p^{(i-1)}(t) \\ &= \sum_{k=i-1}^{d} a_k k(k-1)...(k-i+2)t^{k-i+1}. \end{aligned}$$

Next, reverse the coefficients of $p_1(t)$ by applying Proposition 2.2, the Inversion Closure Property for real-rooted polynomials, on $p_1(t)$, which has degree $d-i+1$, to derive

$$\begin{aligned} p_2(t) &= t^{(d-i+1)}p_1(\frac{1}{t}) \\ &= k(k-1)...(k-i+2)t^{d-k}. \end{aligned}$$

Finally, we want to remove the last $d-i+1$ coefficients $a_{i+1},...,a_n$ of $p(t)$, so we apply Proposition 2.3 $(d-i-1)$ times to p_2 to obtain $p_3(t) = p_2^{(d-i+1)}(t)$:

$$\begin{aligned} p_3(t) &= p_2^{(d-i+1)}(t) \\ &= \frac{d!}{2}\left(\frac{a_{i-1}}{\binom{d}{i-1}}t^2 + \frac{2a_i}{\binom{d}{i}}t + \frac{a_{i+1}}{\binom{d}{i+1}}\right). \end{aligned}$$

Note that $p_3(t)$ must be real-rooted since the closure properties of Proposition 2.2 and 2.3 preserve real-rootedness and the original polynomial $p(t)$ was assumed to be real-rooted. Because $p_3(t)$ has real roots, therefore, its discriminant must be nonnegative:

$$\begin{aligned} &\left(\frac{d!}{2}\frac{2a_i}{\binom{d}{i}}\right)^2 - 4\left(\frac{d!}{2}\frac{a_{i-1}}{\binom{d}{i-1}}\right)\left(\frac{d!}{2}\frac{a_{i+1}}{\binom{d}{i+1}}\right) \geq 0 \\ &\qquad\qquad \implies \frac{a_{i-1}}{\binom{d}{i-1}} \cdot \frac{a_{i+1}}{\binom{d}{i+1}} \leq \left(\frac{a_i}{\binom{d}{i}}\right)^2. \end{aligned}$$

But this is exactly Equation 1. So we have shown that for any chosen $0 < i < d$, Equation 1 holds and that therefore $p(t)$ is extra log concave, as desired. So Newton's inequalities hold.

□

4 Real Stable Polynomials

This section explores the next natural step in real-rooted polynomials - their generalization into multiple variables. Here, we prove exactly how a class of polynomials called real stable polynomials generalizes real-rooted polynomials. Furthermore, we present closure properties of real stable polynomials that are analogous to those of real-rooted polynomials from Section 2.1. We also see how to easily generate real stable polynomials from positive semidefinite matrices. These closure properties – along with the relationship between positive semidefinite matrices and real stable polynomials – will prove useful in Marcus, Spielman, and Srivastava's [MSS18] breakthrough application of real stable polynomials for Ramanujan graphs in Section 5.

4.1 Real Stable Polynomials as a Generalization of Real-Rooted Polynomials

Recall from Section 1 that a multivariate polynomial is $p(z) = p(z_1, ..., z_n)$ is real stable if $p(z_1, ..., z_n) \neq 0$ whenever $\operatorname{Im} z_i > 0$ for all $i = 1, ..., n$. We can intuitively see how real stable variables may generalize real-rooted polynomials, since real-rooted polynomials require that all roots have zero imaginary part, while real stable polynomials require that at least one root have nonnegative imaginary part. Lemma 4.1, however, more formally articulates the relationship between real-rooted and real stable polynomials.

Lemma 4.1. *Suppose $p(z_1, ..., z_d)$ is a multivariate real stable polynomial; if we let $z_i = t$ for all i and call this new univariate polynomial $q(t)$, then $q(t)$ is real-rooted.*

To prove Lemma 4.1, we use the following lemma.

Lemma 4.2. *For any univariate polynomial $p(t)$ with all real coefficients, we have that for $a, b \in \mathbb{R}$, $p(a + ib) = 0 \iff p(a - ib) = 0$. In other words, if $p(t)$ has all real coefficients, then its imaginary roots come in conjugate pairs.*

Proof of Lemma 4.1. We prove by contradiction; assume that instead $q(t)$ is not real-rooted. Then $q(t)$ must have some root with nonzero imaginary value; call this root $a + bi$, where $b \neq 0$. But note that $p(z_1, .., z_d)$ is real stable and therefore has all real coefficients. Then $q(t)$ must also have all real coefficients, since q was generated from p without affecting the coefficients. Then by Lemma 4.2, since $q(t)$ has all real coefficients, its imaginary roots come in conjugate pairs and therefore both $a + bi, a - bi$, $b \neq 0$ are roots of q. But either b or $-b$ has positive value, and we can pick $b > 0$ without loss of generality. Then $\operatorname{Im}(a + bi) > 0$.

Now observe that $a + bi$ being a root of q means $(a + bi, ..., a + bi)$ is a root of p, such that $p(a+bi, ..., a+bi) = 0$. This is a contradiction however, since p is real stable such that $p(z_1, ..., z_n) \neq 0$ whenever $\operatorname{Im} z_k > 0$ for all $k = 1, ..., d$, but here, we have found a root of p where $p(z_1 = a+bi, ..., z_n = a + bi) = 0$ and $z_k = a + bi > 0$ for all $k = 1, ..., d$. □

4.2 Closure Properties

Similar to the closure properties of real-rooted polynomials in Section 2.1, real stable polynomials also have closure properties that, when applied to real stable polynomials, produce other polynomials that are also real stable. In this section, we state three such closure properties that are analogous to those mentioned Section 2.1 for real-rooted polynomials. In particular, Proposition 4.5 will prove to be an essential part of the proof of Theorem 5.1, in Section 5, that discusses the application of real stable polynomials in Ramanujan graphs.

Proposition 4.3. [Product Closure Property] If $p_1, p_2, ..., p_m$ are real stable polynomials, then $\prod_{i=1}^{m} p_i$ is also real stable. In particular, if $p(z_1, ..., z_d)$ is a real stable polynomial, for any constant $c \in \mathbb{R}$, $cp(z_1, ..., z_d)$ is real stable.

Proposition 4.4. [Inversion Closure Property] Let $p(z_1, ..., z_n)$ be a real stable polynomial, where d_i is the degree of z_i, for all $i = 1, ..., n$. Then

$$p(\frac{1}{z_1}, ..., \frac{1}{z_n}) \prod_{i=1}^{n} z_i^{d_i}$$

is a real stable polynomial.

Proposition 4.5. [Differentiation Closure Property] Let $p(z_1, ..., z_n)$ be a real stable polynomial. Then for all $i = 1, ..., n$, $\frac{\partial p}{\partial z_i}$ is a real stable polynomial.

4.3 Generating Real Stable Polynomials from Positive Semidefinite Matrices

Though we have explored some properties of real stable polynomials, it is unclear if there is some relatively easy way to generate such polynomials. It turns out that real stable polynomials can be created quite easily from positive semidefinite matrices, using the procedure in Lemma 4.6. Eventually in Section 5, this method of generating real stable polynomials will be a key step in proving Theorem 5.1.

Lemma 4.6 (Lemma 6.2 in [MSS18]). *Let $A_1, ..., A_m$ be positive semidefinite (PSD) matrices. Then*

$$p(z_1, ..., z_n) = \det\left(\sum_{i=1}^{n} z_i A_i\right)$$

is real stable.

5 Application of Real Stable Polynomials: Ramanujan Graphs

Using the closure properties of real stable polynomials from the previous section, this sections illustrates an interesting application of real stable polynomials for a special class of graphs. More specifically, we sketch the break-through proof of the existence of an infinite sequence of d-regular bipartite Ramanujan graphs by following the proof published in 2014 by Marcus, Spielman, and Srivastava [MSS18], where they utilize real stable polynomials to prove their claim.

For the reader's convenience, we give the formal definition of a Ramanujan graph, followed by the a restatement of Theorem 1.1, henceforth known as Theorem 5.1.

Definition 5.1. A d-regular bipartite graph is said to be *Ramanujan* if its non-trivial eigenvalues are between $-2\sqrt{d-1}$ and $2\sqrt{d-1}$. An eigenvalue is considered trivial for a d-regular bipartite graph if it is equivalent to d or 0, as these are fixed eigevnalues for this class of graphs.

Theorem 5.1 (Marcus, Spielman, Srivastava 2014 [MSS18]). *For every $d \geq 3$, there is an infinite sequence of d-regular bipartite Ramanujan graphs.*

Intuitively, the proof of Theorem 5.1 relies on the construction of an infinite sequence of graphs that begins with the complete d-regular bipartite graph. From this graph, the infinite sequence is produced by repeatedly performing random 2-lifts on the previous graph in the sequence. Though this idea of performing 2-lifts on graphs is not a new notion, the innovation in the proof here is the invocation of real stable polynomials to prove that this infinite sequence consists of graphs that are all indeed Ramanujan.

5.1 2-Lifts and Signings

Because the construction of the infinite sequence of regular bipartite graphs described in Theorem 5.1 relies on the idea of random 2-lifts, we give the following of a 2-lift.

Definition 5.2. Given a graph $G = (V, E)$, a *2-lift* of G is a graph that has two vertices for each vertex in V and two edges for every edge in E. If (u, v) is an edge in E, then the 2-lift either contains the pair of edges $\{(u_0, v_0), (u_1, v_1)\}$ (called a Type 1 edge) or $\{(u_0, v_1), (u_1, v_0\}$ (called a Type 2 edge), where the vertices u_0, u_1 and v_0, v_1 in the 2-lift correspond to the vertices u and v in G, respectively, as shown in Figure 1.

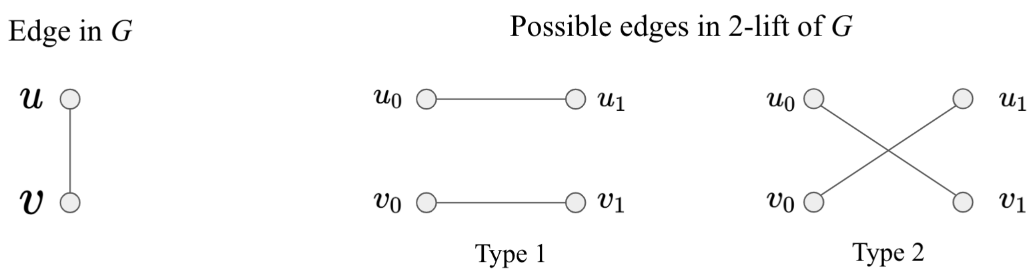

Figure 1: Possible 2-lift Edges

The question now is how to prove that at least one of these possible random 2-lifts at each point in the construction of the sequence of graphs is Ramanujan. To prove if a 2-lift is Ramanujan or not, we must analyze the eigenvalues of the new 2-lift, and for this, we need the following concepts: signings, signed adjacency matrices, and characteristic polynomials.

Definition 5.3. A *signing* $s : E \to \{\pm 1\}$ of the edges of G is the function

$$s(u, v) = \begin{cases} 1 & (u, v) \text{ is a Type 1 edge} \\ -1 & (u, v) \text{ is a Type 2 edge} \end{cases}.$$

Definition 5.4. A *signed adjacency matrix* of a 2-lift of a graph G is the same as the adjacency matrix of G, except that the entries corresponding to an edge (u,v) are now $s(u, v)$.

Definition 5.5. The *characteristic polynomial* of a signed adjacency matrix A_s corresponding to a signing s is $f_s(x) := \det(xI - A_s)$.

Note that signings and signed adjacency matrices clearly have a one-to-one correspondence with 2-lifts, so one could easily define a possible 2-lift of a graph G by taking the original adjacency matrix of G and randomly assigning nonzero entries with values of $+1$ or -1. Furthermore, the roots of the characteristic polynomial are precisely the eigenvalues of the associated signed adjacency matrix, so from here on, we focus on bounding the roots of a 2-lift's characteristic polynomial in order to bound its eigenvalues.

It turns out that we can utilize the roots of a 2-lift's characteristic polynomial – equivalently, the eigenvalues of the 2-lift's signed adjacency matrix – to bound the eigenvalues of the 2-lift's (unsigned) adjacency matrix. More precisely, the eigenvalues of a 2-lift of a graph G is in fact equivalent to the union of the eigenvalues of the adjacency matrix of original graph G that is being lifted and the signed adjacency matrix of the 2-lift. This is stated more formally as Lemma 5.2.

Lemma 5.2 (Lemma 3.1 in [BL06]). *Let A be the adjacency matrix of a graph G and A_s the signed adjacency matrix of a 2-lift G^* of G. Then every eigenvalue of A and every eigenvalue of A_s is an eigenvalue of G^*. Furthermore, the multiplicity of each eigenvalue of G^* is the sum of its multiplicities in A and A_s.*

Thus, proving a 2-lift is Ramanujan by bounding its eigenvalues requires us to somehow bound the eigenvalues of its signed adjacency matrix or, equivalently, the roots of the 2-lift's characteristic polynomial. To bound the characteristic polynomial roots, we introduce the concepts of an interlacing polynomial and a matching polynomial in Sections 5.2 and 5.3 .

5.2 The Matching Polynomial

Matching polynomials are a class of polynomials that will prove useful in proving the existence of a 2-lift whose characteristic polynomial has the desired bounded roots, since we can relate the matching polynomial of a graph with all the possible 2-lifts of that graph. Though for our purposes the definition of a matching polynomial is unnecessary, we state it here for clarity.

Definition 5.6. For a graph G, let m_i denote the number of matchings in G, where for a graph G, a *matching* (or independent edge set) in G is a set of edges without common vertices. Then the *matching polynomial* of G is the polynomial

$$\mu_G(x) := \sum_{i \geq 0} x^{n-2i}(-1)^i m_i.$$

It turns out that the matching polynomial and 2-lifts are highly related; in fact, we can express the matching polynomial of a graph as the expectation of all possible 2-lifts of that graph. We state this formally in the following theorem.

Theorem 5.3 (Corollary 2.2 in [GG81]). *Let $\{f_s(x)\}_{s \in \{\pm 1\}^m}$ denote the set of characteristic polynomials of all possible 2-lifts of a graph G. Then,*

$$\mathbb{E}_{s \in \{\pm 1\}^m}[f_s(x)] = \mu_G(x).$$

Because of this relationship between 2-lift characteristic polynomials and matching polynomials in Theorem 5.3, we can utilize the following theorems on matching polynomials in order to bound the roots of the 2-lift characteristic polynomials.

Theorem 5.4 (Theorem 4.2 in [HL72])**.** *For every graph $\mu_G(x)$ has only real roots.*

Theorem 5.5 (Theorems 4.2 and 4.3 in [HL72])**.** *For every graph G of maximum degree d, all of the roots of $\mu_G(x)$ are real have absolute value at most $2\sqrt{d-1}$.*

5.3 Interlacing Polynomials and Real Stable Polynomials

This section now introduces the class of interlacing polynomials, which – like the class of matching polynomials from Section 5.2 – is necessary to bound the eigenvalues of a 2-lift's characteristic polynomial. In fact, we will eventually combine the ideas of matching polynomials with interlacing polynomials to prove the existence of some 2-lift with bounded roots, in each step of the construction of infinite sequence when proving Theorem 5.1. We define an interlacing polynomial as follows.

Definition 5.7. We say a polynomial $g(x) = \prod_{i=1}^{n-1}(x - \alpha_i)$ *interlaces* a polynomial $f(x) = \prod_{i=1}^{n}(x - \beta_i)$ if

$$\beta_1 \leq \alpha_1 \leq \beta_2 \leq \alpha_2 \leq ... \leq \alpha_{n-1} \leq \beta_n.$$

Furthermore, if a set of functions interlace the same function, we say that this set of functions has a common interlacing, which we define more formally as follows.

Definition 5.8. Polynomials $f_1, ..., f_k$ have a *common interlacing* if there is a polynomial g such that g interlaces f_i for each i.

Though initially interlacing polynomials seem unrelated to both real stable polynomials and the 2-lift characteristic polynomials, the idea of a common interlacing is actually related to real stable polynomials in one variable - in other words, real-rooted polynomials. This relationship is formally stated in Lemma 5.6.

Lemma 5.6 (Lemma 4.5 in [MSS18])**.** *Let $f_1, ..., f_k$ be (univariate) polynomials of the same degree with positive leading coefficients. Then $f_1, ..., f_k$ have a common interlacing if and only if $\sum_{i=1}^{k} \lambda_i f_i$ is real stable in one variable and therefore real-rooted for all convex combinations $\lambda_i \geq 0, \sum_{i=1}^{k} \lambda_i = 1$.*

Furthermore, we can then relate the idea of a common interlacing of a set of polynomials with the roots of the these polynomials, which will later become useful in bounding the roots of a 2-lift characteristic polynomials in the proof of 5.1. We state this relationship formally as Lemma 5.7.

Lemma 5.7 (Lemma 4.2 in [MSS18])**.** *Let $f_1, ..., f_k$ be polynomials of the same degree that are real-rooted and have positive leading coefficients. Define*

$$f_\emptyset = \sum_{i=1}^{k} f_i.$$

If $f_1, ..., f_k$ have a common interlacing, then there exists an i such that the largest root of f_i is at most the largest root of $f_\emptyset$.

Intuitively, the usefulness of Lemmas 5.6 and 5.7 relies on proving that all possible convex combinations of all possible 2-lift characteristic polynomials are real-rooted in Theorem 5.8. If this is true, then by Lemma 5.6, these characteristic polynomials must have a common interlacing. Then, since in Section 5.2 we showed that the matching polynomial is the discrete expectation of all possible 2-lift characteristic polynomials, we essentially equated the matching polynomial to the sum of all 2-lift characteristic polynomials of a graph, multiplied by some constant. Lemma 5.7, therefore, allows us to bound the roots of at least one 2-lift characteristic polynomial with the roots of the matching polynomial of the original graph, which we already bounded in Theorem 5.1. This would then get us to the correct upper bound for the eigenvalues of a Ramanujan graph.

5.4 Proof of Theorem 5.1

The crux of the construction of the infinite sequence of regular bipartite Ramanujan graphs relies on the existence of some 2-lift being Ramanujan, which relies on the real-rootedness of the convex combinations of all possible 2-lift characteristic polynomials, as explained at the end of the previous section. It turns out, however, that the convex combinations of all possible 2-lift characteristic polynomials are indeed always real-rooted, by Theorem 5.8 stated as follows.

Theorem 5.8 (Theorem 5.1 in [MSS18])**.** *Let $p_1, ..., p_m$ be numbers in [0,1]. Also let $\{f_s(x)\}_{s\in\{\pm 1^m\}}$ be the set of characteristic polynomials of all possible 2-lifts of a graph G. Then, the polynomial*

$$\sum_{s\in\{\pm 1\}^m} \left(\prod_{i:s_i=1} p_i\right)\left(\prod_{i:s_i=-1} (1-p_i)\right) f_s(x) \tag{2}$$

is real-rooted.

We now provide a brief sketch of the proof of Theorem 5.8. This proof relies on an analogous procedure by which Theorem 3.1 was proved earlier. We begin with a multivariate polynomial that is generated from the determinant of the sum of positive semidefinite matrices, which is real stable by Theorem 4.6. Applying certain closure properties to this polynomial – specifically, the Derivative Closure Property in Proposition 4.5 – eliminates all but one variable and results in another polynomial that is univariate and also real stable. This polynomial is therefore real-rooted by Lemma 4.1. This polynomial can then be manipulated to be equivalent to Equation 2.

Now, in the previous section, Section 5.3, we argued that if we could show that the 2-lift characteristic polynomials have a common interlacing, then we could obtain the desired bound for the roots of the charactristic polynomial of at least on of these 2-lifts. Indeed, we can prove that this is true, which we state as a theorem as follows.

Theorem 5.9 (Theorem 5.2 in [MSS18])**.** *The polynomials $\{f_s\}_{s\in\{\pm 1\}^m}$, which are the characteristic polynomials of all possible 2-lifts of a graph G, have a common interlacing.*

Proof. Pick any k such that $0 \leq k \leq m-1$, any partial assignment $s_1 \in \pm 1, ..., s_k \in \pm 1$, and any $\lambda \in [0,1]$. Then let $p_i = \frac{1+s_i}{2}$ for $1 \leq i \leq k$ and $p_{k+1} = \lambda$, $p_j = \frac{1}{2}$ for $k+2 \leq j \leq m$. Since the $p_i \in [01]$

for all $1 \leq i \leq m$, we apply Theorem 5.8 and get that $\lambda f_{s_1,\ldots,s_k,1}(x)+(1-\lambda)f_{s_1,\ldots,s_k,-1}(x)$ is real-rooted for all possible convex combinations. By Lemma 5.6, we therefore have that the set of characteristic polynomials of all possible 2-lifts of a graph G, $\{f_s\}_{s\in\{\pm1\}^m}$, has a common interlacing. $\square$

The last piece before proving Theorem 5.1 is to take the relationship between characteristic polynomials and matching polynomials and prove the existence of a 2-lift – which is equivalent to a signing – of a graph G such that its eigenvalues can successfully be bounded to at most the threshold value in a Ramanujan graph, $2\sqrt{d-1}$. The following theorem, Theorem 5.10, gives this formal claim, and we prove it is true.

Theorem 5.10 (Theorem 5.3 in [MSS18])**.** *Let G be a d-regular graph with adjacency matrix A. Then there is a signing s of A such that all of the eigenvalues of A_s are at most $2\sqrt{d-1}$.*

Proof. Note that the conditions for Theorems 5.5 and 5.3 and Lemma 5.7 are satisfied by the characteristic polynomials of some signing A_s. Also observe that the roots of the characteristic polynomials are the eigenvalues of A_s. Then combining Theorems 5.5 and 5.3 and Lemma 5.7 and noting that the expectation in Theorem 5.3 is discrete and is therefore a sum multiplied by some constant, we can apply Lemma 5.7 to this expectation to get that there is a characteristic polynomial whose largest root is no greater than the largest root of the matching polynomial of the graph G. Since $\square$

With all the machinery fully developed, we are finally ready to prove Theorem 5.1.

Proof of Theorem 5.1. It is clear that the nontrivial eigenvalues of a complete d-regular bipartite graph are 0. Thus, the complete bipartite graph of degree d is Ramanujan. By Theorem 5.10, for every d-regular bipartite Ramanujan graph G, there is a 2-lift in which every non-trivial eigenvalue is at most $2\sqrt{d-1}$. As the 2-lift of a bipartite graph is bipartite, and the eigenvalues of a bipartite graph are symmetric about 0, this 2-lift's signed adjacency matrix has non-trivial eigenvalues between $-2\sqrt{d-1}$ and $d\sqrt{d-1}$. Finally, by Theorem 5.2, the eigenvalues of a 2-lift of a Ramanujan graph G is the union of the eigenvalues of the 2-lift's adjacency matrix and the eigenvalues of G itself (its adjacency matrix). Thus, this 2-lift is also a regular bipartite Ramanujan graph.

Thus, from any d-regular bipartite Ramanujan graph, we can construct another d-regular bipartite Ramanujan graph with twice aof ts many vertices, so we have an infinite sequence of d-regular bipartite Ramanujan graphs. $\square$

6 Conclusion

We have shown that real stable polynomials are a generalization of real-rooted polynomials. Furthermore, we found that both real-rooted and real stable polynomials are very useful in applications. We presented proofs using real stable and real stable polynomials to Newton's inequalities and the existence of an infinite sequence of d-regular bipartite Ramanujan graphs, respectively.

Though these two examples were presented, they certainly not the only applications of real-rooted and real stable polynomials. More specifically, polynomials can be used to prove results in a variety of fields within mathematics. In fact, one can utilize real stable polynomials in an optimization application; Straszak and Vishnoi [SV17] uses this class of polynomials to obtain a guarantee on the approximation for counting and optimization problems on matroids, which are structures that generalizes the idea of linear independence in vector spaces. Thus, it is clear polynomials are useful in many applications – sometimes in very surprising and seemingly unrelated ways, as demonstrated in this paper.

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