Calculus 3

11/1/23 — HW 8 — Josiah Schmitz

134. 272π

$\int_0^{2\pi}\int_3^5 (r^2\cos^2\theta + r^2\sin^2\theta) r\,dr\,d\theta \quad \int_3^5 r^3\,dr = \frac{1}{4}r^4\Big]_3^5 = \frac{1}{4}(625-81) = 136$

$\int_0^{2\pi} 136\,d\theta = 136(2\pi) = 272\pi$

138. $\frac{3\pi}{16}$

$\int_{\frac{\pi}{2}}^{\pi}\int_0^1 \sqrt[3]{(r^2\cos^2\theta + r^2\sin^2\theta)}\, r\,dr\,d\theta \quad \int_0^1 \sqrt[3]{r^2}\, r\,dr = \int_0^1 r^{\frac{5}{3}}\,dr = \frac{3}{8}r^{\frac{8}{3}}\Big]_0^1 = \frac{3}{8}$

$\int_{\frac{\pi}{2}}^{\pi} \frac{3}{8}\,d\theta = \frac{3}{8}\left(\frac{\pi}{2}\right) = \frac{3\pi}{16}$

140. $\frac{\pi\sqrt{2}}{12}$

$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\int_1^2 \sin\left(\tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta}\right)\right)dr\,d\theta \quad \int_1^2 \sin(\tan^{-1}(1))\,dr = \frac{\sqrt{2}}{2}r\Big]_1^2 = \frac{\sqrt{2}}{2}$

$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{2}}{2}\,d\theta = \frac{\sqrt{2}}{2}\left(\frac{\pi}{6}\right) = \frac{\sqrt{2}\pi}{12}$

148. $\frac{81\pi}{8}$

$0 \le r \le 3 \quad 0 \le \theta \le \frac{\pi}{2} \quad \int_0^{\frac{\pi}{2}}\int_0^3 (r^2\cos^2\theta + r^2\sin^2\theta) r\,dr\,d\theta$

$\int_0^3 r^3\,dr = \frac{1}{4}r^4\Big]_0^3 = \frac{81}{4} \quad \int_0^{\frac{\pi}{2}} \frac{81}{4}\,d\theta = \frac{81}{4}\left(\frac{\pi}{2}\right) = \frac{81\pi}{8}$

149. $\frac{32\pi}{3}$

$\int_0^{\pi}\int_0^2 (r^2\cos^2\theta + r^2\sin^2\theta)^2 r\,dr\,d\theta \quad \int_0^2 r^5\,dr = \frac{1}{6}r^6\Big]_0^2 = \frac{32}{3}$

$\int_0^{\pi} \frac{32}{3}\,d\theta = \frac{32\pi}{3}$

150. $\frac{2}{3}$

$\int_0^{\frac{\pi}{2}}\int_0^1 (r\cos\theta + r\sin\theta) r\,dr\,d\theta \quad \int_0^1 r^2(\cos\theta + \sin\theta)\,dr = \frac{1}{3}r^3(\cos\theta + \sin\theta)\Big]_0^1 = \frac{1}{3}(\cos\theta + \sin\theta)$

$\int_0^{\frac{\pi}{2}} \frac{1}{3}(\cos\theta + \sin\theta)\,d\theta = \frac{1}{3}(\sin\theta - \cos\theta)\Big]_0^{\frac{\pi}{2}} = \frac{1}{3}\left(\sin\frac{\pi}{2} - \cos\frac{\pi}{2} - \sin 0 + \cos 0\right)$

$= \frac{1}{3}(1 - 0 - 0 + 1) = \frac{2}{3}$

151. $\frac{-\pi}{2}(\cos 16 - 1)$

$\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \int_0^4 \sin(r^2\cos^2\theta + r^2\sin^2\theta) r\, dr\, d\theta \quad \int_0^4 \sin(r^2) r\, dr = \frac{-1}{2}\cos(r^2)\Big]_0^4$
$= \frac{-1}{2}(\cos 16 - \cos 0) = \frac{-1}{2}(\cos 16 - 1) \quad \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{-1}{2}(\cos 16 - 1)\, d\theta = \frac{-1}{2}(\cos 16 - 1)(\pi)$

5.4

182. 26

$\int_1^3 \int_0^2 \int_1^2 (xy + yz + xz)\, dx\, dy\, dz \quad \int_1^2 (xy + yz + xz)\, dx = \left(\frac{1}{2}x^2y + xyz + \frac{1}{2}x^2z\right)\Big]_1^2$
$= 2y + 2yz + 2z - \frac{1}{2}y - yz - \frac{1}{2}z = \frac{3}{2}y + \frac{3}{2}z + yz$
$\int_0^2 \left(\frac{3}{2}y + \frac{3}{2}z + yz\right) dy = \left[\frac{3}{4}y^2 + \frac{3}{2}yz + \frac{1}{2}y^2z\right]_0^2 = 3 + 3z + 2z = 5z + 3$
$\int_1^3 5z + 3\, dz = \left(\frac{5}{2}z^2 + 3z\right)\Big]_1^3 = \frac{45}{2} + 9 - \frac{5}{2} - 3 = 26$

192. $\frac{1}{2}e$

$\int_0^1 \int_0^{\ln x} \int_1^e (y\ln x + z)\, dx\, dy\, dz \quad 1 \le x \le e \quad 0 \le y \le \ln x \quad 0 \le z \le 1$
$\int_0^1 \int_1^e \int_0^1 (y\ln x + z)\, dy\, dx\, dz \quad \int_0^1 (y\ln x + z)\, dy = \left(\frac{1}{2}y^2\ln x + yz\right)\Big]_0^1 = \frac{1}{2}\ln x + z$
$\int_1^e \left(\frac{1}{2}\ln x + z\right) dx = \left(\frac{1}{2}x(\ln x - 1) + xz\right)\Big]_1^e = ez - z + \frac{1}{2}$
$\int_0^1 \left(ez - z + \frac{1}{2}\right) dz = \left(\frac{1}{2}ez^2 - \frac{1}{2}z^2 + \frac{1}{2}z\right)\Big]_0^1 = \frac{1}{2}e - 1 + 1 = \frac{1}{2}e$

197. 0

$\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^{1-x^2-y^2} y\, dz\, dy\, dz \quad \int_0^{1-x^2-y^2} y\, dz \quad yz\Big]_0^{1-x^2-y^2} = y - x^2y - y^3$
$\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (y - x^2y - y^3)\, dy = \left(\frac{1}{2}y^2 - \frac{1}{2}x^2y^2 - \frac{1}{4}y^4\right)\Big]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} = 0$

200. $\frac{256}{3}$

$\int_0^4 \int_0^2 \int_{-y^4}^{y^4} (\sin x + y)\, dx\, dy\, dz \quad \int_{-y^4}^{y^4} (\sin x + y)\, dx = (-\cos x + xy)\Big]_{-y^4}^{y^4}$
$= -\cos y^4 + y^5 + \cos -y^4 + y^5 = 2y^5 \quad \int_0^2 2y^5\, dy = \frac{1}{3}y^6\Big]_0^2 = \frac{64}{3}$
$\int_0^4 \frac{64}{3}\, dz = \frac{64}{3}z\Big]_0^4 = \frac{256}{3}$

210. $21\sqrt{3}$

$x^2+1\le 4 \quad x\le\sqrt{3} \quad y\le 2 \quad \int_0^{4x^2+4y^2} yz\,dz = yz\Big]_0^{4x^2+4y^2} = 4x^2y+4y^3$

$\int_1^2 (4x^2y+4y^3)\,dy = (2x^2y^2+y^4)\Big]_1^2 = 8x^2+16-2x^2+1 = 6x^2+15$

$\int_0^{\sqrt{3}} (6x^2+15)\,dx = (2x^3+15x)\Big]_0^{\sqrt{3}} = 6\sqrt{3}+15\sqrt{3} = 21\sqrt{3}$

235. $\frac{35}{2}$

$0\le x\le 5 \quad 0\le y\le 5-x \quad 0\le z\le 5-x-y \quad \int_0^5\int_0^{5-x}\int_0^{5-x-y}(xz-5z+10)\,dz\,dy\,dx = \frac{4375}{12}$

$\frac{1}{V(E)} = \int_0^5\int_0^{5-x}\int_0^{5-x-y} dz\,dy\,dx = \frac{125}{6} \quad \frac{4375}{12}\cdot\frac{6}{125} = \frac{35}{2}$

5.5

242. $\frac{-256}{9}$

$f(r,\theta,z) = rz^2\cos\theta \quad r^2\cos^2\theta + r^2\sin^2\theta \le 16 \quad -4\le r\le 4 \quad r\sin\theta\le 0\le r\cos\theta \quad \frac{-\pi}{2}\le\theta\le 0$

$-1\le z\le 1 \quad \int_{-\frac{\pi}{2}}^0\int_{-4}^4\int_{-1}^1 r^2z^2\cos\theta\,dz\,dr\,d\theta \quad \int_{-1}^1 r^2z^2\cos\theta\,dz = \frac{1}{3}r^2z^3\cos\theta\Big]_{-1}^1$

$= \frac{2}{3}r^2\cos\theta \quad \int_{-4}^4 \frac{2}{3}r^2\cos\theta\,dr = \frac{2}{9}r^3\cos\theta\Big]_{-4}^4 = \frac{256}{9}\cos\theta$

$\int_{-\frac{\pi}{2}}^0 \frac{256}{9}\cos\theta\,d\theta = \frac{256}{9}\sin\theta\Big]_{-\frac{\pi}{2}}^0 = \frac{-256}{9}$

245. $\frac{\pi e^2}{6}$

$f(r,\theta,z) = e^r \quad 1\le r^2\le 4 \quad 1\le r\le 2 \quad r\sin\theta\le 0 \quad r\cos\theta\le\sqrt{3}\,r\sin\theta \quad \frac{1}{\sqrt{3}}\le\tan\theta \quad \frac{7\pi}{6}\le\theta\le\frac{3\pi}{2}$

$\int_{\frac{7\pi}{6}}^{\frac{3\pi}{2}}\int_1^2\int_2^3 re^r\,dz\,dr\,d\theta \quad \int_2^3 re^r\,dz = zre^r\Big]_2^3 = re^r \quad \int_1^2 re^r\,dr = (re^r-e^r)\Big]_1^2 = 2e^2-e^2-e+e$

$= e^2 \quad \int_{\frac{7\pi}{6}}^{\frac{3\pi}{2}} e^2\,d\theta = \theta e^2\Big]_{\frac{7\pi}{6}}^{\frac{3\pi}{2}} = \frac{\pi e^2}{6}$

270. $\frac{-45\pi}{2}$

$f(\rho,\theta,\varphi) = 1-\rho \quad \rho\le 3 \quad \rho\cos(\varphi)\ge 0 \quad 0\le\varphi\le\pi \quad \rho\sin\varphi\sin\theta\ge 0 \quad 0\le\theta\le\pi$

$\int_0^\pi\int_0^\pi\int_0^3 (\rho^2\sin\varphi-\rho^3\sin\varphi)\,d\rho\,d\varphi\,d\theta \quad \int_0^3(\rho^2\sin\varphi-\rho^3\sin\varphi)\,d\rho = 9\sin\varphi-\frac{81}{4}\sin\varphi$

$\int_0^\pi \left(9\sin\varphi-\frac{81}{4}\sin\varphi\right)d\varphi = \frac{-45}{2} \quad \int_0^\pi \frac{-45}{2}\,d\theta = \frac{-45\pi}{2}$

286. $\frac{(64-3\sqrt{2})\pi}{6}$

$-2 \le x \le 2 \quad -\sqrt{4-x^2} \le y \le \sqrt{4-x^2} \quad \sqrt{x^2+y^2} \le z \le \sqrt{16-x^2-y^2}$

$0 \le \theta \le \frac{\pi}{2} \quad 0 \le \rho \le 4 \quad \sqrt{x^2+y^2}^2 + z^2 = 16 \quad 2z^2 = 16 \quad z = 2\sqrt{2}$

$4\cos\varphi = 2\sqrt{2} \quad \cos\varphi = \frac{\sqrt{2}}{2} \quad \varphi = \frac{\pi}{4} \quad \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{4}}\int_0^4 \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$

$\int_0^4 \rho^2 \sin\varphi \, d\rho = \frac{1}{3}\rho^3 \sin\varphi \Big]_0^4 = \frac{64}{3}\sin\varphi \quad \int_0^{\frac{\pi}{4}} \frac{64}{3}\sin\varphi \, d\varphi = -\frac{64}{3}\cos\varphi \Big]_0^{\frac{\pi}{4}} = \frac{-32\sqrt{2}}{3} + \frac{64}{3}$

$= \frac{64-3\sqrt{2}}{3} \quad \int_0^{\frac{\pi}{2}} \frac{64-3\sqrt{2}}{3} \, d\theta = \frac{(64-3\sqrt{2})\pi}{6}$

295. $\pi k r^4 \ \mu C$

$q(\rho, \theta, \varphi) = k\rho \quad \int_0^r \int_0^{2\pi} \int_0^{\pi} k\rho^3 \sin\varphi \, d\varphi \, d\theta \, d\rho$

$\int_0^{\pi} k\rho^3 \sin\varphi \, d\varphi = -k\rho^3 \cos\varphi \Big]_0^{\pi} = 2k\rho^3 \quad \int_0^{2\pi} 2k\rho^3 \, d\theta = 2k\rho^3\theta \Big]_0^{2\pi}$

$= 4\pi k\rho^3 \quad \int_0^r 4\pi k\rho^3 \, d\rho = \pi k\rho^4 \Big]_0^r = \pi k r^4$

296. $20\pi r^4 \ \mu C$

$Q = \pi k r^4 = 20\pi r^4 \ \mu C$