Calculus 3

11/29/23 HW 11 Josiah Schmitz

283. 16π

$r(u,v) = \langle u\cos v, u\sin v, 0\rangle \quad 0<u<4,\ 0<v<2\pi$

$t_u \times t_v = \begin{vmatrix} i & j & k \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 0 \end{vmatrix} = \langle 0, 0, u\rangle \quad \|t_u \times t_v\| = u$

$\int_0^4\int_0^{2\pi} u\,dv\,du \quad \int_0^{2\pi} u\,dv = uv\Big]_0^{2\pi} = 2\pi u \quad 2\pi\int_0^4 u\,du = \pi u^2\Big]_0^4 = 16\pi$

286. π

$r(u,v) = \langle u\cos v, u\sin v, u\sin v + 1\rangle \quad 0\le u\le 1,\ 0\le v\le 2\pi$

$t_u \times t_v = \begin{vmatrix} i & j & k \\ \cos v & \sin v & \sin v \\ -u\sin v & u\cos v & u\cos v \end{vmatrix} = \langle 0, -u, u\rangle$

$\int_0^{2\pi}\int_0^1 \langle u^2\cos^2 v, u^2\sin^2 v, u^2\sin^2 v + 2u\sin v + 1\rangle \cdot \langle 0, -u, u\rangle\,du\,dv$

$\int_0^{2\pi}\int_0^1 (2u^2\sin v + u)\,du\,dv \quad \left(\frac{2}{3}u^3\sin v + \frac{1}{2}u^2\right)\Big]_0^1 = \frac{2}{3}\sin v + \frac{1}{2}$

$\int_0^{2\pi}\left(\frac{2}{3}\sin v + \frac{1}{2}\right)dv = \left(-\frac{2}{3}\cos v + \frac{1}{2}v\right)\Big]_0^{2\pi} = \pi$

297. $\frac{1023\sqrt{2}\pi}{5}$

$r(u,v) = \langle u\cos v, u\sin v, u\rangle \quad 1\le u\le 4,\ 0\le v\le 2\pi$

$t_u \times t_v = \begin{vmatrix} i & j & k \\ \cos v & \sin v & 1 \\ -u\sin v & u\cos v & 0 \end{vmatrix} = \langle -u\cos v, -u\sin v, u\rangle$

$\|t_u \times t_v\| = \sqrt{2u^2} = \sqrt{2}\,u$

$\int_0^{2\pi}\int_1^4 \sqrt{2}\,u^4\cos^2 v\,du\,dv \quad \int_1^4 \sqrt{2}\,u^4\cos^2 v\,du = \left(\frac{\sqrt{2}}{5}u^5\cos^2 v\right)\Big]_1^4 = \frac{1023\sqrt{2}}{5}\cos^2 v$

$\int_0^{2\pi}\frac{1023\sqrt{2}}{5}\cos^2 v\,dv = \frac{1023\sqrt{2}}{10}(\cos v\sin v + v)\Big]_0^{2\pi} = \frac{1023\sqrt{2}\pi}{5}$

301. πa^3

$r(u,v) = \langle u\cos v, u\sin v, \sqrt{a^2-u^2} \rangle \quad 0 \le u \le a, \ 0 \le v \le 2\pi$

$t_u \times t_v = \begin{vmatrix} i & j & k \\ \cos v & \sin v & \frac{-u}{\sqrt{a^2-u^2}} \\ -u\sin v & u\cos v & 0 \end{vmatrix} = \left\langle \frac{-u^2\cos v}{\sqrt{a^2-u^2}}, \frac{-u^2\sin v}{\sqrt{a^2-u^2}}, u \right\rangle$

$\|t_u \times t_v\| = \sqrt{\frac{u^4}{a^2-u^2} + u^2} = \sqrt{\frac{u^2a^2}{a^2-u^2}}$

$= \frac{ua}{\sqrt{a^2-u^2}} \quad \int_0^{2\pi}\int_0^a ua\,du\,dv \quad \int_0^a ua\,du = \frac{1}{2}u^2a\Big]_0^a = \frac{1}{2}a^3$

$\int_0^{2\pi} \frac{1}{2}a^3\,dv = \frac{1}{2}a^3v\Big]_0^{2\pi} = a^3\pi$

305. $\frac{3}{4}$

$\iint_S F \cdot N\,ds = 3\int_0^1\int_0^1 xy\,dx\,dy = 3\int_0^1 \frac{1}{2}y\,dy = 3\left(\frac{1}{4}\right) = \frac{3}{4}$

317. 16π

$x^2+y^2=4,\ 1\le z\le 3$ is a cylinder with base 4π, circumference 4π, and height 2. S.A. of cylinder is $\pi dh + 2\pi r^2$.
S.A. $= \pi\cdot 4\cdot 2 + 2\cdot\pi\cdot 2^2 = 8\pi + 8\pi = 16\pi$

318. 9π

$x^2+y^2\le 9,\ z=4$ is a circle with radius 3. Area of a circle is πr^2. Area $= \pi\cdot 3^2 = 9\pi$

322. 617.2

$F = -\nabla T = \langle 100e^{-x+y}, 100e^{-x-y}, 0\rangle$
Flux $= \sum_{i=1}^{6}(F\cdot n) = 100e^{-1} + 100e^{1} + 100e^{-1} + 100e^{1} + 0 + 0 = 200e + \frac{200}{e} = 617.2$

323. $8\pi a$

$F = -\nabla T = \left\langle \frac{2x}{x^2+y^2+z^2}, \frac{2y}{x^2+y^2+z^2}, \frac{2z}{x^2+y^2+z^2} \right\rangle$ $r(\varphi,\theta) = \langle a\cos\theta\sin\varphi, a\sin\theta\sin\varphi, a\cos\varphi$

$$t_\varphi \times t_\theta = \begin{vmatrix} i & j & k \\ a\cos\theta\cos\varphi & a\sin\theta\cos\varphi & -a\sin\varphi \\ -a\sin\theta\sin\varphi & a\cos\theta\sin\varphi & 0 \end{vmatrix} = \langle a^2\sin^2\varphi\cos\theta, a^2\sin^2\varphi\sin\theta, a^2\cos\varphi$$

$F(r(\varphi,\theta)) = \left\langle \frac{2\cos\theta\sin\varphi}{a}, \frac{2\sin\theta\sin\varphi}{a}, \frac{2\cos\varphi}{a} \right\rangle$

$F\cdot N = 2a\sin^3\varphi\cos^2\theta + 2a\sin^3\varphi\sin^2\theta + 2a\cos^2\varphi\sin\varphi = 2a(\sin^3\varphi + \cos^2\varphi\sin\varphi$

$= 2a\sin\varphi(\sin^2\varphi + \cos^2\varphi) = 2a\sin\varphi$ $\int_0^{2\pi}\int_0^{\pi} 2a\sin\varphi\, d\varphi\, d\theta$

$\int_0^{\pi} 2a\sin\varphi\, d\varphi = -2a\cos\varphi\Big]_0^{\pi} = 4a$ $\int_0^{2\pi} 4a\, d\theta = 4a\theta\Big]_0^{2\pi} = 8\pi a$

6.7

335. $\int_C F \cdot dS = 0$

$$\nabla F = \begin{vmatrix} i & j & k \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ 2xy^2z & 2x^2yz & x^2y^2-2z \end{vmatrix} = \langle 0, 0, 0 \rangle$$

339. $\iint_S \nabla F \cdot dS = 0$

$$\nabla F = \begin{vmatrix} i & j & k \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ e^{xy}\cos z & x^2z & xy \end{vmatrix} = \langle x-x^2, y+e^{xy}\sin z, 2xz-xe^{xy}\cos z \rangle$$

$r(\varphi,\theta) = \langle \sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi \rangle$

$$\frac{dr}{d\varphi} \times \frac{dr}{d\theta} = \begin{vmatrix} i & j & k \\ \cos\varphi\cos\theta & \cos\varphi\sin\theta & -\sin\varphi \\ -\sin\varphi\sin\theta & \sin\varphi\cos\theta & 0 \end{vmatrix} = \langle \sin^2\varphi\cos\theta, \sin^2\varphi\cos\theta, \sin\varphi\cos\varphi \rangle$$

$\nabla F \cdot \frac{dr}{d\varphi} \times \frac{dr}{d\theta} = 0$

344. 12

$12x-36+6y+18z=0 \quad z = \frac{-2}{3}x - \frac{1}{3}y + \frac{1}{2} \quad \langle x, y, \frac{-2}{3}x - \frac{2}{3}y + \frac{1}{2} \rangle$

$0 \le y \le 6, \; 0 \le x \le 3$

$$\nabla F = \begin{vmatrix} i & j & k \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ z & x & y \end{vmatrix} = \langle 1, 1, 1 \rangle$$

$$t_x \times t_y = \begin{vmatrix} i & j & k \\ 1 & 0 & \frac{-2}{3} \\ 0 & 1 & \frac{-1}{3} \end{vmatrix} = \langle \frac{-2}{3}, \frac{1}{3}, 1 \rangle \quad \int_0^3\int_0^6 \langle 1,1,1 \rangle \cdot \langle \frac{-2}{3}, \frac{1}{3}, 1 \rangle \, dy\,dx$$

$= \int_0^3\int_0^6 \frac{2}{3} \, dy\,dx$

$\int_0^6 \frac{2}{3}dy = \frac{2}{3}y\Big]_0^6 = 4 \quad \int_0^3 4\,dx = 4x\Big]_0^3 = 12$

348.

$$\nabla F = \begin{vmatrix} i & j & k \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ x & y^2 & ze^{xy} \end{vmatrix} = \langle xze^{xy}, yze^{xy}, 0 \rangle$$

$r = \langle x, y, 1-x^2-2y^2 \rangle$

$$\frac{dr}{dx} \times \frac{dr}{dy} = \begin{vmatrix} i & j & k \\ 1 & 0 & -2x \\ 0 & 1 & -4y \end{vmatrix} = \langle -2x, 4y, 1 \rangle$$

$\iint_S (\nabla F \cdot N) dS = \int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} \int_{-1}^{1} \langle xze^{xy}, yze^{xy}, 0 \rangle \cdot \langle -2x, 4y, 1 \rangle \, dx\, dy$

$\int_{-1}^{1} (-2x^2 z e^{xy} + 4y^2 z e^{xy}) dx$

<u>6.8</u>

385. $\frac{\pi}{3}$

$\text{div} F = 0+1+1 = 2 \quad \iiint_E 2 dV = \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^1 2r^2 \sin\theta \, dr\, d\theta\, d\varphi$

$\int_0^1 2r^2 \sin\theta \, dr = \frac{2}{3}\sin\theta \quad \int_0^{\frac{\pi}{2}} \frac{2}{3}\sin\theta \, d\theta = \frac{2}{3} \quad \int_0^{\frac{\pi}{2}} \frac{2}{3} d\varphi = \frac{\pi}{3}$

387. 0

$\text{div} F = 2z^3 + 2xz^3 + 4xz^3 = 2z^3(3x+1) \quad \iiint_E 2z^3(3x+1) dV$

$\int_{-3}^{3}\int_{-2}^{2}\int_{-1}^{1} 2z^3(3x+1) dx\,dy\,dz \quad \int_{-1}^{1} 2z^3(3x+1)dx = 2z^3(\frac{3}{2}x^2 + x)\Big]_{-1}^{1} = 6z^3$

$\int_{-2}^{2} 6z^3 dy = 6yz^3\Big]_{-2}^{2} = 12z^3 - 12z^3 = 0$

392. $\frac{5\pi}{3}$

$\text{div} F = 4z^3 + 2 \quad \int_0^{2\pi}\int_0^1\int_0^r (4z^3+2) r\, dz\, dr\, d\varphi$

$\int_0^r (4z^3+2) r\, dz = (z^4 + 2z)r\Big]_0^r = r^5 + 2r^2 \quad \int_0^1 (r^5 + 2r^2) dr = (\frac{1}{6}r^6 + \frac{2}{3}r^3)\Big]_0^1 = \frac{5}{6}$

$\int_0^{2\pi} \frac{5}{6} d\theta = \frac{5\pi}{3}$

410. 48π

$\operatorname{div} F = 3x^2 + z\sin xyz - z\sin xyz + 3y^2 = 3x^2 + 3y^2$

$\int_0^{2\pi}\int_0^2\int_0^4 3r^3\,dz\,dr\,d\theta \quad \int_0^4 3r^3\,dz = 3zr^3\Big]_0^4 = 12r^3 \quad \int_0^2 12r^3\,dr = 3r^4\Big]_0^2 = 24$

$\int_0^{2\pi} 24\,d\theta = 48\pi$