Non-Local Regularization for Image Reconstruction

Research Case Studies

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1 Abstract

This paper focuses on the use of non-local regularization for image reconstruction. Non-local regularization is a technique that takes into account the similarity of the patches in the image in order to better preserve the structure and details of the image during the reconstruction process. Here, a non-local regularization algorithm is presented that takes non-local information in order to improve the accuracy of image reconstruction. Two approaches with non-local setting have been performed, one with non-neighbouring pixels and another with non-neighbouring patches. The results from the later approach has then been compared to results from the local regularization technique. Also, the effects from the varying values of the multiple hyper-parameters have been studied and discussed at the end of the paper. Overall, the review has highlighted the importance of incorporating non-local information in image processing to achieve better results and has identified areas for further research.

2 Background

To present the image reconstruction as an ill-posed problem, considering a simple ill-posed problem as below:

$$u = \phi f + \epsilon \in \mathbb{R}^p \tag{1}$$

where, ϕ is the masking operator, u is the masked image or the input, ϵ is the additive noise, and $f \in \mathbb{R}^n$ of n pixels from a set of $p \leq n$ noisy linear measurements, is the image that needs to be recovered.

To solve this ill-posed problem in equation (1), there is a need to introduce additional information or constraints to stabilize the solution as:

$$\min_f \|\phi f - u\|_2^2 + \lambda J(f) \tag{2}$$

where λ is the weight ,and the $J(f)$ is a regularization function. Here, the weight λ needs to be adapted to match the amplitude of the noise ϵ, which might be a non-trivial task in practical situations.

Here, the masking operator ϕ is not of full rank. Hence, the regular solution does not exist. So instead, the solution will be selected from regularized least squares problem by solving the linear equation, which has a form of normal equation (3) for the optimization of the image. This equation will give a closed-form solution which is used to find the optimal f.

$$(\Phi^T\Phi + 2\lambda P^T G_\omega^T G_\omega P) f(\omega) = \Phi^T u \tag{3}$$

2.1 Related works

2.1.1 Classical smoothness priors

Smoothness priors are well-known and most commonly used methods in the analysis of stochastic processes. They are particularly suited for smoothing the noisy data and use for instance, a discretized Sobolev norm.

$$J^{sob}(f) = \sum_x \|\nabla f(x)\|^2 \tag{4}$$

For recovery of sharp features, a total variation norm is proposed [8], which can be used for inpainting smaller holes [14] and can solve for super resolution [9]. However, inpainting larger holes or patches require higher-order regularization [1, 2, 10].

$$J^{TV}(f) = \sum_x \|\nabla f(x)\| \tag{5}$$

Sparsity prior [5] has also been used to solve general inverse problems [5, 6].

$$J^{spars}(f) = \sum_m |\langle f, \psi_{\mathrm{m}} \rangle| \tag{6}$$

2.1.2 Non-local diffusion

In order to better respect the edges when compared to classical smoothness priors, the non-local means filter [3] goes one step further by averaging pixels that can be arbitrarily far away, using a similarity measure based on the distance between patches. These edge adaptive filters are related to the minimization of equation (2) using a graph based regularization over the image.

$$J_{\mathrm{w}}^{graph}(f) = \sum_{x,y} w_{\mathrm{x,y}} |f(x) - f(y)|^\alpha \tag{7}$$

where α = 2. The weights $w_{\mathrm{x,y}}$ are computed from the input noisy image u using either the distance $|u(x) - u(y)|$ between the noisy pixel values [15, 17, 18] or the distance $\|p_x(f) - p_y(f)\|$ between the patches [3, 4, 16].

3 Literature Review

3.1 Image Inpainting

Image inpainting is the process of adding missing pixels to an image in order to make it realistic-looking and consistent with its original (actual) context. Given a distorted or masked input image, it is typically defined as i) invalid/missing/hole pixels as the pixels situated in the region(s) to be filled; and ii) valid/remaining/ground truth pixels as the pixels can be used to assist in filling in the missing pixels.

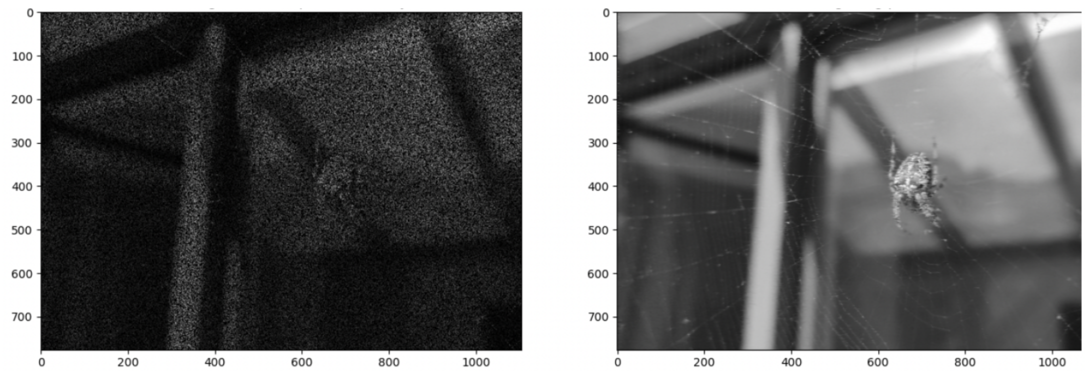

Figure 1: Image inpainting example.

Copying and pasting is the easiest approach to adding the missing pieces to an image. The basic concept is to first look through the remaining pixels in the image to find the image patches that are most comparable, then paste those patches where they are needed. However, the search process takes a lot of time and uses custom distance measurements.

3.2 Non-Local Regularization of inverse problems

The function $J(f)$ defined above as a regularization function has a huge part in solving the ill-posed problem described in equation (1). Moreover, there are a lot of regularization functions that are available. In this particular task, only the non-local regularize-rs are in consideration. The objective is to recover the underlying parameters of a system from indirect or incomplete measurements using non-local regularization technique. This helps to adapt the solution and improve the accuracy by incorporating additional prior information or constraints.

In traditional regularization techniques, the regularization operator is applied locally, meaning that the regularization term at each point in the solution depends only on the neighboring points. However, in many real-world problems, the underlying structure is

non-local, and the prior information or constraints may depend on the relationship between distant points.
Non-local regularization addresses this issue by incorporating non-local prior information or constraints into the regularization term. This is achieved by defining a kernel function that measures the similarity between different points in the solution and weighting the regularization term accordingly.
One approach to non-local regularization is to use the total variation (TV) regularization framework [13]. It can be represented as seen in equation (5). In this framework, the regularization term penalizes the total variation of the solution, which measures the amount of variation in the intensity values of adjacent pixels. Non-local TV regularization extends this framework by introducing a non-local term that takes into account the similarity between distant pixels [13]. It introduces a technique used in inverse problems to regularize the solution by incorporating information from different parts of the image. Traditional regularization methods such as Tikhonov regularization or total variation regularization, are only based on local information around the pixel. However, non-local regularization includes the similarities between the pixels, even if they are far away in the image plane.

3.3 Patches

Patches are the pixels grouped together. The idea is that, patches of the image with damaged or missing pixels can be replaced with patches from a nearby undamaged region of the image.

3.3.1 Patch Based Signatures

The aim is to obtain the non-local regularization by comparing small patches that can be far away in the image plane. In this case, a group of the neighbouring pixels for a certain pixel. Then, instead of simply comparing the pixels, a patch of $\tau \times \tau$ pixels at location x $\in \{0, ..., \sqrt{n} - 1\}^2$ in the image is defined as:

$$\forall t \in \{-(\tau - 1)/2 + 1...., (\tau - 1)/2\}^2, \pi_x f(t) = f(x + t) \tag{8}$$

where τ is assumed to be an odd integer. A patch $\pi_{(x)} f$ is a vector of size τ^2. A local signature of dimension q $\leq \tau^2$ is obtained from $p_{\mathrm{x}}(f)$ using an orthogonal projector U $\in \mathbb{R}^{q * \tau^2}$ that reduces the dimensionality of the patch.

$$p_x(f) = U\pi_x(f) \in \mathbb{R}^q \tag{9}$$

where U satisfies UU^* = Id. The parameters q and τ in consideration are used as hyper parameters during implementation, to tune the results and check different cases with varying values of them.

3.3.2 Non-Local Patch Operator

The signature extraction process in equation (8) defines a mapping from the pixel domain $\mathbb{R}^n$ to the signature domain P.

$$P : \begin{cases} \mathbb{R}^n \to P, \\ f \mapsto \{p_x(f)\}_x \end{cases} \tag{10}$$

A set of signatures $\{p_x(f)\}_x \in P$ is stored as a matrix of $q * n$ elements.
The non-local energy defined in equation (15) is a vectorial l^1 norm.

$$J_w(f) = ||G_w P f||_1 \tag{11}$$

where the signature-valued gradient maps signatures in P to signature differentials in D

$$G_w : \begin{cases} P \to D, \\ \{p_x\}_x \mapsto \{d_{x,y}\}_{||x-y|| \leq \rho} \end{cases} \tag{12}$$

where $d_{x,y} = w_{x,y}(p_x - p_y)$
The ad-joint of this signature-valued gradient is a signature-valued divergence:

$$G_w^* : \begin{cases} D \to P, \\ \{d_{x,y}\}_{||x-y|| \leq \rho} \mapsto \{p_x\}_x \end{cases} \tag{13}$$

$p_x = \sum_{||x-y|| \leq \rho} w_{x,y} d_{x,y} - w_{y,x} d_{y,x}$

3.4 Optimization on the Graph

3.4.1 Graph based priors on Images

A non-local graph is a set of weights w = $\{w_{x,y}\}_{x,y}$, which assigns to each pair of pixels, a weight $w_{x,y} \geq 0$. These weights correspond to a probability distribution and the graph only connects pixels that are not too far away.

$$C = \{w / w_{x,y} \geq 0, \sum_y w_{x,y} = 1 \text{and} ||x - y|| > \rho \to w_{x,y} = 0\} \tag{14}$$

The parameter ρ controls the degree of non-locality of the graph. This is also a hyper-parameter for the implementation.

This weighted graph is used to indicate, which local signatures should be compared in the image, and leads to the following non-local regularization function.

$$J_w(f) = \sum(||x-y|| \leq \rho) w_{x,y} ||p_x(f) - p_y(f)|| \tag{15}$$

where $||p_x(f) - p_y(f)||$ denotes the patch variation.

3.4.2 Optimization of Weights

With f fixed, the weights are optimized, which is the solution to the following strictly convex optimization problem:

$$w(f) = \arg\min_{w \in C} J_w(f) + \gamma E(w) = \sum_{||x-y|| \leq \rho} w_{x,y} ||p_x(f) - p_y(f)|| + \gamma w_{x,y} log(w_{x,y}) \tag{16}$$

Then, the optimal graph corresponds to the exponential weights:

$$w(f)_{x,y} = \frac{\tilde{w_{x,y}}}{Z_x} \tag{17}$$

where, $\tilde{w_{x,y}} = \begin{cases} e^{\frac{-(||p_x - p_y||)}{\gamma}}, & \text{if} ||x-y|| \leq \rho \\ 0, & \text{Otherwise} \end{cases}$

where the normalizing constant is:

$$Z_x = \sum_y \tilde{w_{x,y}} \tag{18}$$

3.5 Optimization on the Image

3.5.1 Difference Matrix

The difference matrix gives the pairwise differences between the pixel in consideration with other pixels within its neighbourhood (neighbourhood $\leq \rho$). The idea of the difference matrix is to use the non-local information from the direct neighbours. It is computed as:

$$D = \tilde{D} + Id \tag{19}$$

where, $\tilde{D}$ = -w and w computed using equation (17).

3.5.2 Update on the Image

The image update is then performed using a variation of the normal equation (3).

$$\phi^T \phi + \lambda D f = \phi^T u \tag{20}$$

where D is the difference matrix in equation (19)

4 Implementation

4.1 Algorithm

In this study, a slightly different variation of the algorithm suggested in [7] have been implemented . The algorithm can be seen below.

Algorithm 1 The non-local regularization algorithm

Initialize: $f_0 = \phi * u$, maximum-iteration
  for $i = 0$ to $maximum - iteration$ **do**
    — **Weight Update:** Compute w using equation (17)
    — **Create Difference Matrix** : Using equation (19)
    — **Image Update:** Compute f_i using equation (20)
    — **Set:** f = f_i
  end for
Output: f

4.2 Implementation of the Algorithm

The algorithm 1 was implemented on Python 3.8 using NumPy, SciPy and Pillow libraries, for loading and computation of the different parts of the algorithm.

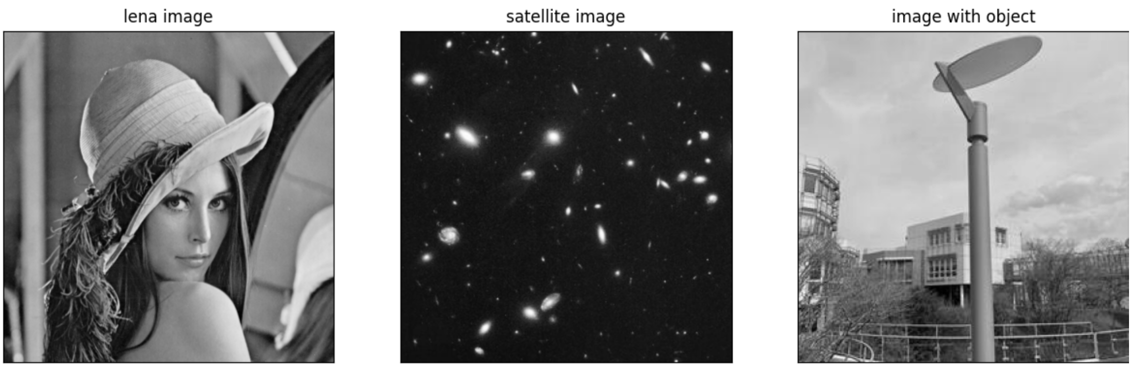

Figure 2: Original Images for Implementation

Three images were considered for testing, ”lena” image [12], which is commonly used for image processing problems, a ”satellite” image [11], to test if the scattered visualizations in the image, can be inpainted in a better manner, using a non-local setting, lastly a natural image, in which an object was masked, and tested of how well the algorithm will fill the damaged pixels, as in this special case a local technique, will not affect the change, because of the large area with damaged pixels. The images can be seen in Figure 2.
The processing was performed iteratively, tuning on different parameters, specifically the non-local distance ρ, the regularization parameter λ and also patch/pixel variation, to test if the calculation of the weights based on the pixel value will differ than the calculation in the patch around the pixel.

4.3 Implementation Issues

Due to the computational complexity arising from complex matrix multiplication in the case of higher non-locality that is considered in the algorithm, the testing was performed with different types of images in case of different values of ρ and λ. Setting the optimal values for such parameters seem to be highly challenging.

4.3.1 Patch Size

The smaller patch size leads to better results in case of fine details, but larger patches can provide better regularization for large structures. Defining the optimal size of the patch depends on the specific problem to be solved.

4.3.2 Computational Complexity

Non-local regularization based on patches is computationally expensive, majorly because of comparisons among each patch in the image to all other patches. Also, the complexity increases concerning the non-locality. To reduce the complexity, vectorization has been used by performing operations on large matrices rather than individual elements.

4.3.3 Memory Requirements

A significant amount of memory is required to store all pairwise patch comparisons during the computation, which may not be feasible for large or high-dimensional images. To tackle this problem, sparse matrices with the SciPy library was used. Furthermore, the images has been rescaled to a smaller size, 512 x 512 for the lena[12] and satellite images[11], and 256 x 256 for the image missing an object. The rescaling of the images, might have caused loss of important information and therefore affected the inpainting process.

4.3.4 Parameter Tuning

Several parameters need to be tuned, to get the optimal results, such as regularization parameter λ, number of iterations and locality parameter ρ. These parameters can affect the quality of the output and may need to be optimized for each problem. Choosing the appropriate parameters is highly challenging and requires extensive experimentation.

4.3.5 Boundary effects

Patches near the boundary of the image may not have enough neighbours to compute the similarity measure and might produce artifacts near the boundaries, if the locality parameter ρ, extends the image boundaries. Symmetric padding has been applied to solve this issue. It extends the borders of the image by adding values to the edges in a symmetrical way.

5 Results and Discussion

The results obtained are as follows:

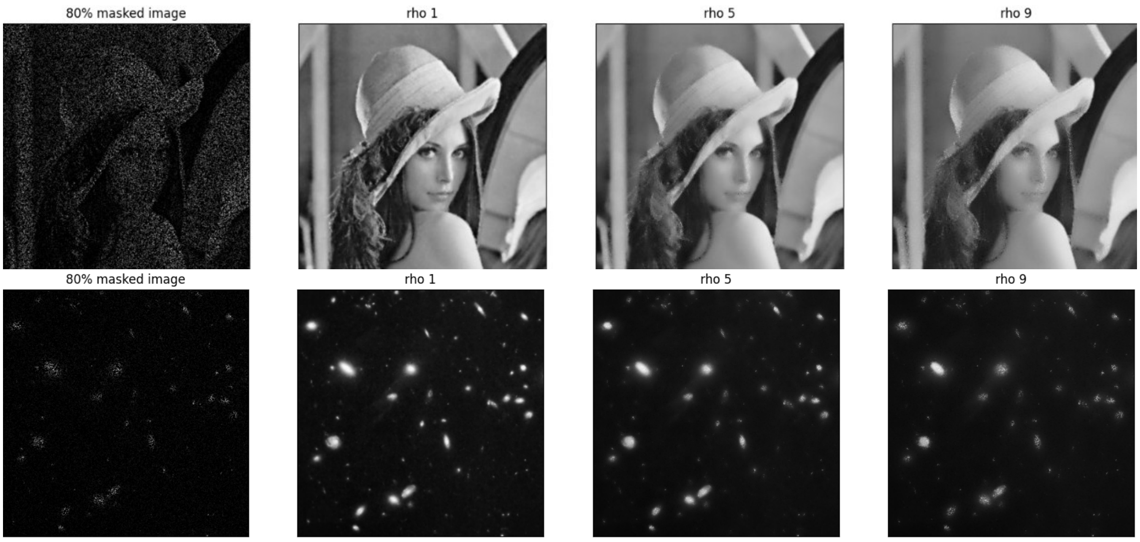

Figure 3: Results for multiple images with non-local regularization

To measure the quality of the reconstructed images, compared to the original images,the PSNR (Peak Signal-to-Noise Ratio) value has been used. A higher PSNR value indicates better quality, as the error between the original and reconstructed image is smaller. The PSNR is calculated as follows:

$$PSNR = 10 \log_{10} \left(\frac{MAX^2}{MSE} \right) \tag{21}$$

Table 1: PSNR Values with varying ρ and λ for Lena Image[12]

ρ values							
λ	1	3	5	7	9	11	13
0.01	28.34	26.09	24.78	23.84	23.07	22.43	21.88
0.06	28.27	26.00	24.70	23.76	22.99	22.35	21.81
0.11	28.20	25.91	24.61	23.67	22.91	22.27	

Table 2: PSNR Values with varying ρ and λ for Satellite Image [11]

ρ values							
λ	1	3	5	7	9	11	13
0.01	33.92	29.91	27.46	25.53	24.29	23.55	22.91
0.06	33.83	29.81	27.34	25.44	24.23	23.50	22.87
0.11	33.73	29.69	27.23	25.34	24.17	23.44	22.82

Table 1 and 2, presents the results of the non-local regularization algorithm applied to a masked image, using different regularization parameters. Specifically, the locality parameter ρ varies from 1 to 13 pixels, and the regularization parameter λ from 0.01 to 0.11. For each combination of ρ and λ, the PSNR values of the resulting images have been measured. From the results, it can be observed that a decrease in ρ generally led to an improvement in PSNR up to a certain point. For example, in table 2, for λ = 0.01 , the PSNR decreases from 33.92 dB with a $= rho$ = 1 to 22.91 dB with ρ = 13.
Overall, these results demonstrate the importance of carefully tuning the locality parameter and regularization parameter in non-local regularization algorithms to achieve the best possible trade-off between noise reduction and preservation of image details.

Figure 4: Comparing local and non-local approaches for image reconstruction

Figure 4 shows the significant progress from non-local regularization, in situations where a big chunk of an image is expected to be removed. In the case of local regularization, after a certain number of iterations, it starts adding more noise and blur to the results or stopped progressing. The results were captured by implementing both local and non-local regularization for 400 iterations.

6 Conclusion

The process of finding the solution of an ill-posed problem like image reconstruction is always a challenging task. Inverse problems are characterized by their sensitivity to small changes in the input data, which can result in large variations in the output. This makes it difficult to obtain a unique and stable solution to the problem. To overcome this challenge, various techniques have been developed in the field of computational imaging, one of which is non-local regularization.

The image that contains a large structure that extends throughout the entire image, a local regularization seems to be unable to capture a significant amount of global information to produce expected outcomes. It seems to perform well only up to a certain point. After that point, it starts to introduce additional noise or blur. In contrast, non-local approach shows impressive progress. The progress of the non-local approach highly depends on the setting of the hyper-parameters in consideration, which include locality parameter ρ, regularization parameter λ, patch size τ, and many more. They might add more complexity to the computations but they assist in tuning the results better.

In conclusion, the choice between local and non-local regularization techniques for image reconstruction depends on the specific characteristics of the image and the goal of the reconstruction. The choice between these two approaches depends on the specific requirements of the reconstruction problem, and it is important to carefully evaluate and compare different regularization methods to achieve optimal results.

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