Transition and Duration Models

The Poisson Process: Applications in Labour Market Models of Search

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The Bellman equations in search models

Unemployed workers search for jobs. Such search takes time, and generates search frictions: Unemployed searchers are without a job, and vacancies are without workers .

The meeting between an unemployed worker and a firm is described by a Poisson process with (arrival rate) $a > 0$, and job separations arrive at rate δ.

Define the random time in unemployment by T_a, the random time in employment by T_δ. These durations or inter-arrival times are therefore *exponentially* distributed with parameters a and δ.

A key modelling object is the life-time utility of unemployed and employed workers. Since the mathematical problem might be difficult to solve, modern methods look at the problem from a recursive perspective. These are the so-called Bellman equations.

We will show that these equations have a very simple structure, thanks to the use of the Poisson process. We have:

$$\begin{aligned} rV_u &= W_u + a(V_e - V_u) \\ rV_e &= W_e - \delta(V_e - V_u). \end{aligned}$$

where W_u (W_e) is the instantaneous pay-off or utility of being unemployed (employed). Often these are specified as wages and unemployment benefit.

Proof. The expected life-time utility of an unemployed worker is

$$\begin{aligned} V_u &= E_{T_a}\left\{\int_0^{T_a} W_u e^{-rt}dt + e^{-rT_a}V_e\right\} \\ &= \frac{W_U}{r+a} + \frac{a}{r+a}V_e \end{aligned}$$

because $E_{T_a}\left\{e^{-rT_a}\right\} = a\int_0^\infty e^{-(a+r)t}dt = a/(a+r)$. Hence

$$rV_u = W_u + a(V_e - V_u)$$

Similarly, $V_e = E_{T_\delta}(\int_0^{T_\delta} W_e e^{-rt}dt + e^{-rT_\delta}V_u)$, which yields

$$rV_e = W_e - \delta(V_e - V_u).$$

The McCall model

Imagine a partial equilibrium setup with a risk neutral individual in discrete time. He starts life as unemployed. When unemployed, he has access to consumption equal to b (from home production, value of leisure or unemployment benefit).

At each time period, he samples a job. All jobs are identical except for their wages, and wages are given by an exogenous stationary distribution of F (w).

At time t = 0, this individual has preferences given by

$$\sum_{t=0}^{\infty} \beta^t c_t$$

Let us assume for now that there is no recall, so that the only thing the individual can do is to take the job offered within that date.

If he accepts a job, he will be employed at that job forever, so the net present value of accepting a job of wage w_t is

$$\frac{w_t}{1-\beta}$$

We define the **value** of the agent when he has sampled a job of $w \in W$. This is

$$v(w) = \max\left\{\frac{w_t}{1-\beta}, \beta v + b\right\}$$

where $v = \int_W v(w)dF(w) = E(v(w))$ is the *continuation* value of not accepting a job. This follows from the fact that from tomorrow on, the individual faces the same distribution of job offers, so v is simply the expected value of $v(w)$ over the stationary distribution of wages.

We are interested in finding both the value function v (w) and the optimal policy of the individual.

The form of the value function definition implies that v (w) is non-decreasing and is piecewise linear with first a flat portion.

This immediately tells us that the optimal policy will take a **reservation wage form**, which is a key result of the sequential search model: there will exist some reservation wage R such that all wages above R will be accepted and those $w < R$ will be turned down.

R is defined by

$$\frac{R}{1-\beta} = b + \beta E(v(w))$$

so that the individual is just indifferent between taking w $=$ R and waiting for one more period.

Since for all $w \geq R$ we have $v(w) = w/(1-\beta)$, and for $w < R$ $v(w) = R/(1-\beta)$, one can split the integral $E(v(w))$, and eventually arrive at

$$R - b = \frac{\beta}{1-\beta} \int_{w \geq R} (w - R) dF(w)$$

which is an important way of characterizing the reservation wage.

The LHS is best understood as the cost of foregoing the wage of R, while the RHS is the expected benefit of one more search.

Clearly, at the reservation wage, these two are equal.

Define the RHS as

$$g(R) = \frac{\beta}{1-\beta} \int_{w \geq R} (w - R) dF(w)$$

Then it follows that

$$g'(R) < 0$$

so the reservation wage equation has a unique solution.

Also, by the implicit function theorem, we have

$$\frac{dR}{db} = \frac{1}{1 - g'(R)} > 0$$

so that as expected, higher benefits when unemployed increase the reservation wage, making workers more picky.

One implication of the reservation wage policy is that the assumption of no recall, made above, was of no consequence.

In a stationary environment, the worker will have a constant reservation wage, and therefore has no desire to go back and take a job that he had previously rejected.

Suppose that there is now a continuum 1 of identical individuals sampling jobs from the same stationary distribution F.

Once a job is created, it lasts until the worker dies, which happens with probability s.

There is a mass of s workers born every period, so that population is constant, and these workers start out as unemployed. The death probability means that the effective discount factor of workers is equal to $\beta(1-s)$.

The reservation wage equation becomes

$$R - b = \frac{\beta(1-s)}{1-\beta(1-s)} \int_{w \geq R} (w - R) dF(w)$$

At time t we have U_t unemployed workers. There will be s new workers born into the unemployment pool. Out of the U_t unemployed workers, those who survive and do not find a job will remain unemployed. Therefore

$$U_{t+1} = s + (1-s)F(R)U_t$$

(1-s)F(R) is the *joint* probability of not finding a job and surviving (i.e., of remaining unemployed).

In terms of flows

$$U_{t+1} - U_t = s(1 - U_t) + (1-s)(1 - F(R))U_t$$

The unique steady-state unemployment rate where $U_{t+1} = U_t$ is given by

$$U = \frac{s}{s + (1-s)(1-F(R))}$$

This is the canonical formula of the flow approach: The steady-state unemployment rate is equal to the job destruction rate (here the rate at which workers die, s) divided by the job destruction rate plus the job creation rate.

An increases in s will raise steady-state unemployment.

An increase in R, will also depress job creation and increase unemployment.

Diamond's Paradox

Assume that all workers are identical, and have the same reservation wage. Firms post wages (there is no bargaining). We will add labour market and firm dynamics, and examine the model in steady state (when inflows equal outflows).

The worker's search strategy will be simple: It is optimal to accept a new job if the offered wage is above the individual's reservation wage w_r (reservation wage rule).

What is the reservation wage, which wages will be offered by firms, and do we get non-generate realised wage distribution ?

The Bellman equation for the employed is a function of the current wage, and gives the instantaneous benefit (the wage) and the expected change in status

$$rV_e(w) = w + \delta\left(V_u - V_e(w)\right)$$

The reservation wage is given by the condition

$$V_e(w) - V_u = 0,$$

which yields

$$w_r = rV_e = rV_u.$$

The Bellman equation. for the unemployed gives the instantaneous benefit (unemployment benefit minus search cost) and the expected change in status

$$rV_u = (b - c) + a \int_{w_r}^{\infty} (V_e(w) - V_u) dF(w)$$

where b is the unemployment benefit and c is the cost of search as it takes effort to look for a new job, a the job offer arrival rate, and F is the wage offer curve.

Note that only wage offers above w_r would be accepted since firms know that lower offers will be rejected.

Using the two preceding results, the reservation wage can we re-written implicitly as

$$w_r = (b - c) + \frac{a}{r + \delta} \int_{w_r}^{\infty} (w - w_r) dF(w).$$

Consider now the labour market dynamics in steady state (SS), so that inflows are equal to outflows. The measure of unemployed is u, and of employed is $(1-u)$ who lose their jobs with probability δ. The unemployed only accept jobs, which arrive with probability a, if the offered wage at least equals the reservation rate w_r. Therefore:

$$0 = (1-u) \times \delta - u \times a \times (1 - F(w_r)).$$

Next, the employment dynamics of an individual firm. Denote the employment size in a firm by $l(w)$.

The firm's SS dynamics are, for $w \geq w_r$,

$$0 = au - \delta l(w),$$

Then using the SS equation for the entire labour market yields

$$l(w) = \frac{a}{\delta + a(1 - F(w_r))} \quad \text{if } w \geq w_r$$

and zero otherwise. Therefore $l(w)$ does not depend on w: all firms have the same size!

Now consider explicitly the hiring strategy of firms, wage offers being their only instrument. Firms post wages that maximise their profits

$$\pi = \max_{w}(y - w)I(w).$$

Since $I(w)$ does not depend on w, this is achieved by offering the lowest wage feasible, which is the reservation wage w_r. If $w = w_r$, we know from a previous equation that $w_r = b - c$:

$$w^* = w_r = b - c.$$

This is the Nash equilibrium: there is no wage dispersion!

But ...

Assume that the unemployed who do not search get b forever.

Then then the participation constraint is $w_r \geq b$.

The equilibrium condition is $w^* = w_r = b - c$.

Now if $c = 0$, then everyone searches and accepts any job.

But if $c > 0$, then the participation constraint is violated, $w_r = b - c \ngeq b$ and no one searches. This is Diamond's paradox.

Resolutions

Albrecht and Axel seek to escape the Diamond paradox by considering two types of workers: the prop. of 'high' types is n_0 (with $z_0 = b_0 - c_0 > z_1$). Otherwise, the model is the same as Diamond's. We have then two reservation wages such that $w_{r_0} > w_{r_1}$.

Unemployment dynamics are considered separately for each group (u_i).

Firms size dynamics now have to distinguish between pooling ($w \geq w_{r_0}$) and only employing low types ($w \in [w_{r_1}, w_{r_0})$).

Profit maximisation by firms shows that they will only pay reservation wages (so $l(w)$ does not depend on w).

If both types are employed, this implies that profits in pooling case equals profits of firms only employing low types.

The labour market equilibrium. We have three candidates. (i) If firms post only wages $w^* = w_{r_1}$, then high (0) types do not participate; since low (1) types have search costs $c_1 > 0$, we are back to the Diamond no participation case. (ii) All firms post $w^* = w_{r_0}$, so $p_0 = 1$. Then all workers participate. (iii) A fraction of firms p_0 posts w_{r_0}, a fraction $1 - p_0$ posts w_{r_1}. For this to happen, firms must be indifferent between posting the two wage levels; this equal profit condition then allows determining p_0.

Burdett and Mortenson (1998, IER) take an alternative approach to escape Diamond's paradox.

Instead of considering different reservation levels b, they consider the effects of on-the-job search: Ex ante identical workers will move up the wage ladder as they get poached by higher paying firms.