Exercise Set 1
(Introduction to Poisson processes)
Sketch Answers

1. (a) $E(X) = \lambda \int_0^\infty x e^{-\lambda x} dx$. Integrate by parts.

   (b)
$$E(X) = \sum_{k=0}^{\infty} k \frac{\lambda^k e^{-\lambda}}{k!} = \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} = \lambda e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{(k)!}$$

   To see that the last two terms cancel, consider (i) $1 = \sum_{k=0}^{\infty} \Pr(X = k) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}$; or (ii) Taylor expand e^{λ}.

2. The easiest way to show this is to use the definition of the Poisson process, ie need to check that $N(t)$ verifies the 3 properties.

   (i) N(0)=0; (ii) Each process $N_i(t)$, i = 1, 2 has independent increment, hence the sum has independent increment. (iii) The sum of 2 Poisson random variables is another Poisson random variable, hence $N(t) \sim Poisson((\lambda_1 + \lambda_2)t)$. Thus, $N(t)$ is a Poisson process. re. (iii) explicitly, let $S = N_1 + N_2$ and consider $\Pr\{S = s\}$:

$$\begin{aligned}
\Pr\{S = s\} &= \sum_{n=0}^{s} \Pr\{N_2 = s - n | N_1 = n\} \Pr\{N_1 = n\} \\
&= \sum_{n=0}^{s} \Pr\{N_2 = s - n\} \Pr\{N_1 = n\} \\
&= \sum_{n=0}^{s} \frac{\lambda_2^{s-n} e^{-\lambda_2}}{(s-n)!} \frac{\lambda_1^n e^{-\lambda_1}}{(n)!} \\
&= e^{-(\lambda_2 + \lambda_1)} \sum_{n=0}^{s} \frac{\lambda_2^{s-n}}{(s-n)!} \frac{\lambda_1^n}{(n)!}
\end{aligned}$$

The last term can be simplified using the binomial formula

$$(\lambda_1 + \lambda_2)^s = \sum_{n=0}^{s} \frac{s!}{i!(s-i)!} \lambda_1^i \lambda_1^{s-i}$$

giving

$$\Pr\{S = s\} = e^{-(\lambda_2+\lambda_1)} \frac{(\lambda_1 + \lambda_2)^s}{s!}$$

which is the probability mass function of a Poisson with parameter $\lambda = \lambda_1 + \lambda_2$.

1. *Let $\{X(t),\ t \geq 0$ be a Poisson process with rate parameter $\lambda = 0.7$. Let T_k be the time of the k'th event. Compute*

   (a) $\Pr(X(4) = 5 \,|\, X(3.5) = 4)$.

   (b) $\Pr(X(4) = 4 \,|\, X(3.5) = 5)$.

   (c) $\Pr(T_1 < 5)$.

   (d) $\Pr(T_3 < 5 \,|\, T_2 = 3.5)$.

   (e) $\Pr(X(7) = 8 \,|\, T_3 = 6)$.

   (f) $\Pr(X(7) = 8 \,|\, T_9 = 6)$.

   (a) $\Pr(X(4) = 5 \,|\, X(3.5) = 4) = \Pr(1 \text{ event in } [3.5; 4) \,|\, 4 \text{ events in } [0, 3.5))$. Since process increments on disjoint time intervals are independent this equals

   $$\begin{aligned}\Pr(1 \text{ event in } [3.5; 4)) &= \Pr(1 \text{ event in } [0, 0.5)) = P_1(0.5) \\ &= 0.7 * 0.5\, \mathrm{e}^{-0.7*0.5} = 0.35\, \mathrm{e}^{-.35}.\end{aligned}$$

   (b) As $X(t)$ is non-decreasing $\Pr(X(4) = 4 \,|\, X(3.5) = 5) = 0$.

   (c) $\Pr(T_1 < 5) = 1 - \Pr(T_1 \geq 5) = 1 - \Pr(\text{no events in } [0, 5)) = 1 - \Pr(X(5) = 0) = 1 - \mathrm{e}^{-\lambda t} = 1 - \mathrm{e}^{-3.5}$ with $\lambda = 0.7$ and $t = 5$.

   (d) $\Pr(T_3 < 5 \,|\, T_2 = 3.5) = \Pr(T_3 - T_2 < 1.5 \,|\, T_2 = 3.5) = \Pr(T_3 - T_2 < 1.5) = \Pr(T_1 < 1.5) = 1 - \Pr(X(1.5) = 0) = 1 - \mathrm{e}^{0.7*1.5} = 1 - \mathrm{e}^{-1.05}$. In the second equation we have used independence of increments on disjoint time intervals and the fourth equation we have used c) with 5 replaced by 1.5;

   (e) $\Pr(X(7) = 8 \,|\, T_3 = 6) = \Pr(X(7) = 8 \,|\, X(6) = 3) = \Pr(X(7) - X(6) = 8 - 3 \,|\, X(6) - X(0) = 3 - 0) = \Pr(5 \text{ events in } [6, 7)) = \Pr(5 \text{ events in } [0, 1)) = P_5(1) = \frac{1}{5!}(0.7)^5\, \mathrm{e}^{-0.7}$. In the third equation we have used independence of increments on disjoint time intervals

   (f) As $X(t)$ is non-decreasing, $\Pr(X(7) = 8 \,|\, T_9 = 6) = 0$.

Queestion 4:

(a) exponential with parameter λ_w.

(b) Let t_R denote the interarrival between two telephone consultations. Since the arrival process is Poisson, t_R is exponentially distributed with parameter λ_R: $\Pr\{t_R > t\} = e^{-\lambda_r t}$.

(c) Let t_W denote the interarrival between two walk-in consultations. Note that t_W is exponentially distributed with parameter λ_W . Thus Pr (read consultation first) = $\Pr\{t_R < t_W\} = \frac{\lambda_R}{\lambda_R+\lambda_W}$. Explicitly:

$$\begin{aligned}
\Pr\{t_R < t_W\} &= \int_0^\infty \Pr\{t_R < t | t_W = t\} f_{t_W}(t) \\
&= \int_0^\infty \Pr\{t_R < t\} f_{t_W}(t) \\
&= \int_0^\infty (1 - e^{-\lambda_r t}) \lambda_w e^{-\lambda_w t} \\
&= 1 - \lambda_w \int_0^\infty \exp(-(\lambda_w + \lambda_r)t) \\
&= \frac{\lambda_r}{\lambda_r + \lambda_w}
\end{aligned}$$

(d) Let $N_W(t)$ denote the number walk-in consultations arriving in the interval $[0, t)$. Then look at $\Pr\{N_W(t) = i\}$. P (At most 3 consultations) equals $\sum_{i=0}^3 \Pr\{N_W(t) = i\}$.

(e) Because of the closure property of independent Poisson processes, the joint arrival process is Poisson with rate $\lambda_R + \lambda_W$. Let $N(t) = N_R(t) + N_W(t)$, and compute $\Pr\{N(t) = i\}$. P(At least 2 consultations) = $1 - \Pr\{N(t) = 0\} - \Pr\{N(t) = 1\}$.

(f) Let T be the time of arrival of the first walk-in customer, and write N for the number of telephone calls received in $[0, T)$. We have

$$\Pr\{N = i\} = \int_0^\infty \Pr\{N = i | T = t\} f_T(t) dt$$

where of course T has an exponential distribution. Then

$$\Pr\{N = i\} = \int_0^\infty \frac{e^{-\lambda_r t} (\lambda_r t)^i}{i!} \lambda_w e^{-\lambda_w t} dt$$

Set $x = (\lambda_w + \lambda_r)t$ and use this change of variable to get

$$\begin{aligned}
\Pr\{N = i\} &= \lambda_w \frac{\lambda_r^i}{i!} \int_0^\infty (\frac{x}{\lambda_w + \lambda_r})^i e^{-x} \frac{dx}{\lambda_w + \lambda_w} \\
&= \frac{\lambda_w}{\lambda_r + \lambda_w} (\frac{\lambda_r}{\lambda_r + \lambda_w})^i \times \frac{1}{i!} \int_0^\infty x^i e^{-x} dx \\
&= \frac{\lambda_w}{\lambda_r + \lambda_w} (\frac{\lambda_r}{\lambda_r + \lambda_w})^i
\end{aligned}$$

To get a simpler intuition, define

$$p = \frac{\lambda_w}{\lambda_r + \lambda_w}$$

so

$$\Pr\{N = i\} = p(1-p)^i$$

which is the law of a geometric distribution (tossing a coin: the distribution of the number of tails tossed before the first head).