5

SEPARATION OF VARIABLES IN THE SPHERICAL COORDINATE SYSTEM

In this chapter, we examine the separation of the heat conduction equation in our last coordinate system of interest, namely, the spherical coordinate system. We determine the elementary solutions, the norms, and the eigenvalues of the separated problems for different combinations of boundary conditions; discuss the solution of the one- and multidimensional homogeneous problems by the method of separation of variables for spheres; examine the solutions of steady-state and transient multidimensional problems with and without the heat generation in the medium; and illustrate the splitting up of nonhomogeneous problems into a set of simpler problems. The reader should consult references 1–7 for additional applications on the solution of heat conduction problems in the spherical coordinate system.

5-1 SEPARATION OF HEAT CONDUCTION EQUATION IN THE SPHERICAL COORDINATE SYSTEM

We will first review the spherical coordinate system, as shown in Figure 5-1, along with the corresponding components of heat flux in the r, ϕ, and θ directions (i.e., Fourier's law), which are given, respectively, by

$$q_r'' = -k\frac{\partial T}{\partial r}, \qquad q_\phi'' = -\frac{k}{r\sin\theta}\frac{\partial T}{\partial \phi} \qquad \text{and} \qquad q_\theta'' = -\frac{k}{r}\frac{\partial T}{\partial \theta} \tag{5-1}$$

Now we consider the three-dimensional, differential equation of heat conduction in the spherical coordinate system, given as

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial\phi^2} + \frac{g_0}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{5-2}$$

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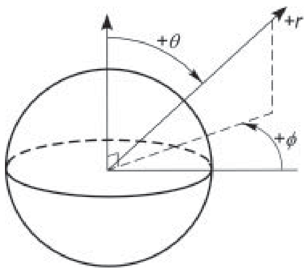

Figure 5-1 Spherical coordinate system.

To aid our solution of equation (5-2), it is convenient to define a new independent variable μ as

$$\boxed{\mu = \cos\theta} \tag{5-3}$$

where the domain of μ is given by $1 \geq \mu \geq -1$, which maps to the variable θ over the corresponding domain $0 \leq \theta \leq \pi$. With this change, equation (5-2) becomes

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \mu}\left[\left(1-\mu^2\right)\frac{\partial T}{\partial \mu}\right] + \frac{1}{r^2\left(1-\mu^2\right)}\frac{\partial^2 T}{\partial \phi^2} + \frac{g_0}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{5-4}$$

where $T \equiv T(r, \mu, \phi, t)$. Finally, we may define a new dependent variable V,

$$\boxed{V(r, \mu, \phi, t) = r^{1/2} T(r, \mu, \phi, t)} \tag{5-5}$$

which upon substitution into equation (5-4) yields

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} - \frac{1}{4}\frac{V}{r^2} + \frac{1}{r^2}\frac{\partial}{\partial \mu}\left[\left(1-\mu^2\right)\frac{\partial V}{\partial \mu}\right] + \frac{1}{r^2\left(1-\mu^2\right)}\frac{\partial^2 V}{\partial \phi^2} + \frac{g_0 r^{1/2}}{k} = \frac{1}{\alpha}\frac{\partial V}{\partial t} \tag{5-6}$$

Equations (5-4) and (5-6) are both alternative forms of the spherical coordinate heat equation that will be considered in this chapter. Equation (5-6) will be used only when temperature depends on the (r, μ, t) or (r, μ, ϕ, t) variables. The reason for this is that when equation (5-6) is used in such situations, the elementary solutions of the separated differential equation for $R(r)$ become the Bessel functions, which have already been developed in Chapter 2. However, if equation (5-4) is used for such a problem, the elementary solutions are the spherical Bessel functions. For all other cases, including the problems involving (r), (r, t), (r, μ), and (r, μ, ϕ) variables, the governing equation will be obtained

TABLE 5-1 Solution Schemes for the Spherical Heat Equation

Variables	Equation to Solve	Transforms	Expected Equations
$T(r)$	(5-2) or (5-4)	None	Cauchy equation
$T(r, \mu)$	(5-4)	None	Cauchy equation in r Legendre equation in μ
$T(r, \mu, \phi)$	(5-4)	None	Cauchy equation in r Associated Legendre in μ
$T(r, t)$	(5-2) or (5-4)	$U(r, t) = rT(r, t)$	$\frac{\partial^2 U}{\partial r^2} + \frac{rg}{k} = \frac{1}{\alpha}\frac{\partial U}{\partial t}$
$T(r, \mu, t)$	(5-6)	$V = r^{1/2}T$	Bessel equation in r Legendre equation in μ
$T(r, \mu, \phi, t)$	(5-6)	$V = r^{1/2}T$	Bessel equation in r Associated Legendre in μ

from equation (5-2) or (5-4). In Table 5-1, we summarize the various solution schemes for the spherical heat equation along with the expected ODE equations.

We now begin with the general separation of the spherical heat equation for the case of $T \equiv T(r, \mu, \phi, t)$, for which we make the transformation to $V(r, \mu, \phi, t)$ and subsequently solve equation (5-6). Here we will consider the homogeneous version of equation (5-6), that is, no internal energy generation, recalling that the generation term is always handled in a steady-state solution via superposition as we developed in Chapter 3. For the homogeneous version of equation (5-6), we can assume a separation of variables in the form

$$V(r, \mu, \phi, t) = R(r)M(\mu)\Phi(\phi)\Gamma(t) \tag{5-7}$$

Substitution of equation (5-7) into the heat equation of (5-6) yields

$$\begin{aligned} \frac{1}{R}\left(\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - \frac{1}{4}\frac{R}{r^2}\right) + \frac{1}{r^2 M}\frac{d}{d\mu}\left[\left(1-\mu^2\right)\frac{dM}{d\mu}\right] \\ + \frac{1}{r^2\left(1-\mu^2\right)}\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = \frac{1}{\alpha\Gamma}\frac{d\Gamma}{dt} = -\lambda^2 \end{aligned} \tag{5-8}$$

where we have introduced the separation constant $-\lambda^2$ to produce a decaying exponential function in time, as we discussed in Chapter 3. Equation (5-8) leads to the separated ordinary differential equation in the t dimension:

$$\frac{d\Gamma}{dt} + \alpha\lambda^2\Gamma(t) = 0 \tag{5-9}$$

with corresponding solution given by

$$\Gamma(t) = C_1 e^{-\alpha\lambda^2 t} \tag{5-10}$$

We now consider the remaining spatial variables of equation (5-8), namely,

$$\frac{1}{R}\left(\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}-\frac{1}{4}\frac{R}{r^2}\right)+\frac{1}{r^2M}\frac{d}{d\mu}\left[\left(1-\mu^2\right)\frac{dM}{d\mu}\right]$$
$$+\frac{1}{r^2\left(1-\mu^2\right)}\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}+\lambda^2=0 \tag{5-11}$$

which is a separated form of the Helmholtz equation for spherical coordinates. Because the initial condition will always provide the nonhomogeneity for transient problems, here we expect all three spatial dimensions to have corresponding homogeneous boundary conditions. Therefore, we seek to force a characteristic value problem in each spatial dimension. Because none of the three spatial dimensions are currently separated, we first multiply by r^2 and then isolate the r terms, yielding

$$\frac{-1}{M}\frac{d}{d\mu}\left[\left(1-\mu^2\right)\frac{dM}{d\mu}\right]-\frac{1}{\left(1-\mu^2\right)}\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2}$$
$$=\frac{r^2}{R}\left(\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}-\frac{1}{4}\frac{R}{r^2}\right)+\lambda^2r^2=n(n+1) \tag{5-12}$$

where we have introduced a new separation constant $n(n+1)$, where n will ultimately be limited to integer values as discussed below, to force the following ODE in the r dimension:

$$\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}-\frac{1}{4}\frac{R}{r^2}+\lambda^2R-\frac{n(n+1)}{r^2}R=0 \tag{5-13}$$

Equation (5-13) is readily rearranged to yield

$$\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}+\left[\lambda^2-\frac{\left(n+\frac{1}{2}\right)^2}{r^2}\right]R=0 \tag{5-14}$$

which is recognized as Bessel's equation of order $n+\frac{1}{2}$. The solution of equation (5-14) for integer n yields half-integer order Bessel functions

$$R(r)=C_2J_{n+1/2}(\lambda r)+C_3Y_{n+1/2}(\lambda r) \tag{5-15}$$

We now consider the left-hand side of equation (5-12), which after multiplication by $1-\mu^2$, yields the following:

$$\frac{1-\mu^2}{M}\frac{d}{d\mu}\left[\left(1-\mu^2\right)\frac{dM}{d\mu}\right]+\left(1-\mu^2\right)n(n+1)=\frac{-1}{\Phi}\frac{d^2\Phi}{d\phi^2}=m^2 \tag{5-16}$$

We have introduced a new separation constant m^2, which yields the desired ODE and corresponding solution for the ϕ dimension, as given by

$$\Phi(\phi) = C_4 \cos m\phi + C_5 \sin m\phi \tag{5-17}$$

The requirement for 2π periodicity, as discussed in detail in the previous chapter, see equation (4-26), is satisfied for integer m, namely,

$$m = 0, 1, 2, 3, \ldots \tag{5-18}$$

and where both constants C_4 and C_5 are retained. Finally, we consider the remaining μ terms of equation (5-16), which yield

$$\frac{d}{d\mu}\left[\left(1-\mu^2\right)\frac{dM}{d\mu}\right] + \left[n(n+1) - \frac{m^2}{1-\mu^2}\right]M = 0 \tag{5-19}$$

We recognize equation (5-19) as the associated Legendre equation, as discussed in Chapter 2 in reference to equation (2-80). As noted above, we now limit n to *integer values*, which along with integer values of m as defined by equation (5-18), yields the orthogonal associated Legendre polynomials as the solution of equation (5-19),

$$M(\mu) = C_6 P_n^m(\mu) + C_7 Q_n^m(\mu) \tag{5-20}$$

As defined for the general characteristic value problem (i.e., with corresponding homogeneous boundary conditions), the functions given by equations (5-15), (5-17), and (5-20) are all orthogonal functions. The homogeneous boundary conditions in the respective spatial dimensions would define the eigenvalues and eigenfunctions, and the general solution of V would then be formed by summation over all eigenfunctions.

An additional comment is offered with regard to the general case of ϕ dependency. As noted in Table 5-1, the presence of ϕ dependency will always yield the associated Legendre functions in the μ dimension as coupled to the integer-order eigenvalues that arise from the conditions of 2π periodicity. When no ϕ dependency is present, this coupling does not exist and Legendre's equation, rather than the associated Legendre equation, is then generated during separation. The integer eigenvalues n originating from the μ dimension will always couple to the order of Bessel's equation, specifically half-integer order, whenever there is μ dependency for a transient problem. Finally, we note that ϕ dependency in the absence of μ dependency is not covered in Table 5-1, as such a scenario is difficult to envision from a physical point of view. We first explore various steady-state problems and then extend our treatment to transient problems of various spatial dimensions.

5-2 SOLUTION OF STEADY-STATE PROBLEMS

In this section we will solve a range of steady-state problems in the spherical coordinate system. Our general guidelines established for separation of variables in the rectangular coordinate system, equally apply to the spherical coordinate problems. Therefore, steady-state problems with a homogeneous PDE must always be solved with only a single nonhomogeneous boundary condition. If more than one boundary condition is nonhomogeneous, the principle of superposition is used to treat each nonhomogeneity individually. For a nonhomogeneous PDE (i.e., when heat generation is present), superposition is used to treat the heat generation within a 1-D ODE, which is coupled to a homogeneous PDE at one or more boundary conditions. Prior to solving a range of problems to illustrate the above procedures, a few additional guidelines are offered with regard to the solution of steady-state problems in spherical coordinates.

For steady-state 1-D problems, $T = T(r)$ only and the solution scheme reduces to the solution of an ODE, which turns out to be the Cauchy equation. For steady-state 2-D and 3-D problems, namely, (r, μ) or (r, μ, ϕ) dependency, respectively, the r dimension must always be the nonhomogeneous dimension and again the ODE in the r dimension will be the Cauchy equation. The orthogonal functions from the μ and ϕ dimensions will then take the form of the Legendre or associated Legendre polynomials and the trigonometric sin and cos functions, respectively.

Example 5-1 $T = T(r, \mu)$ for Solid Hemisphere
A solid hemisphere of radius b is maintained at steady-state conditions with a prescribed surface temperature $T(r = b) = f(\mu)$, and with a perfectly insulated base at $\mu = 0$ (Fig. 5-2). We now calculate the temperature distribution.

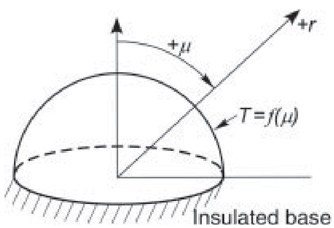

Figure 5-2 Problem description for Example 5-1.

The mathematical formulation of the problem is given as

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \mu}\left[\left(1-\mu^2\right)\frac{\partial T}{\partial \mu}\right] = 0 \quad \text{in} \quad 0 \le r < b \quad 0 < \mu \le 1 \tag{5-21}$$

$$\text{BC1:} \quad T(r \to 0) \Rightarrow \text{finite} \qquad \text{BC2:} \quad T(r = b) = f(\mu) \tag{5-22a}$$

$$\text{BC3:} \quad T(\mu \to +1) \Rightarrow \text{finite} \qquad \text{BC4:} \quad \left.\frac{\partial T}{\partial \mu}\right|_{\mu=0} = 0 \tag{5-22b}$$

The problem formulation contains only a single nonhomogenous boundary condition, namely, BC2; hence the problem is ready for solution using separation of variables. We note that the condition of finiteness imposed as $r \to 0$ may be treated as a homogeneous boundary condition in the context of separation of variables in that it has the effect of eliminating one of the solution constants in the nonhomogeneous r dimension. Similarly so for the condition of finiteness as $\mu \to +1$. We now assume separation of the form

$$T(r, \mu) = R(r)M(\mu) \tag{5-23}$$

Substituting equation (5-23) into the PDE of equation (5-21), multiplying both sides by r^2, and separating yields

$$\frac{r^2}{R}\left[\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr}\right] = \frac{-1}{M}\frac{d}{d\mu}\left[\left(1-\mu^2\right)\frac{dM}{d\mu}\right] = n(n+1) \tag{5-24}$$

where we have introduced the separation constant $n(n+1)$ to force the characteristic value problem in the homogeneous μ dimension. Considering the μ terms, the following ODE, recognized as the Legendre equation, is realized:

$$\frac{d}{d\mu}\left[\left(1-\mu^2\right)\frac{dM}{d\mu}\right] + n(n+1)M = 0 \tag{5-25}$$

The orthogonal Legendre polynomials are generated for integer values of n; hence we define $n = 0, 1, 2, 3, \ldots$ and the solution of equation (5-25) becomes

$$M(\mu) = C_1 P_n(\mu) + C_2 Q_n(\mu) \tag{5-26}$$

We may eliminate the Legendre polynomials of the second kind ($C_2 = 0$) because $Q_n(\mu)$ is infinite at $\mu = \pm 1$, noting that $\mu = 1$(i.e., $\theta = 0$) is in the domain of the problem. For the hemisphere, we have an actual boundary condition (BC4) to consider corresponding to the insulated base of the sphere at $\mu = 0$ (i.e., $\theta = \pi/2$), namely,

$$\left.\frac{dP_n(\mu)}{d\mu}\right|_{\mu=0} = 0 \qquad \text{yielding} \qquad n = 0, 2, 4, \ldots \text{ (even)} \tag{5-27}$$

Examination of the first few Legendre polynomials, see equation (2-70), reveals that the derivatives are all zero at $\mu = 0$ only for the even Legendre polynomials, and we therefore limit n to the even integer values. The remaining r terms in equation (5-24) yield

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - n(n+1)R = 0 \tag{5-28}$$

which is recognized as Cauchy's equation, as developed in equation (4-29). Here we have the specific values of $a_0 = 2$ and $b_0 = -n(n+1)$, which yield the auxiliary equation

$$\gamma^2 + \gamma - n(n+1) = 0 \tag{5-29}$$

Equation (5-29) is readily factored to yields roots $\gamma_1 = n$ and $\gamma_2 = -(n+1)$, giving

$$R(r) = C_3 r^n + C_4 r^{-(n+1)} \tag{5-30}$$

We now apply what we consider to be the homogeneous boundary condition (BC1), namely, the requirement of finiteness at the origin, which eliminates C_4 given that this term goes to infinity as $r \to 0$ for n equal to even positive integers. Having considered all but the final, nonhomogeneous boundary condition, we now recombine our separated solutions as products and sum over all possible solutions:

$$T(r, \mu) = \sum_{\substack{n=0 \\ \text{even}}}^{\infty} C_n r^n P_n(\mu) \tag{5-31}$$

where we have introduced the new constant $C_n = C_1 C_3$. We lastly apply the nonhomogeneous BC2, equation (5-22a), which yields

$$T(r = b) = f(\mu) = \sum_{\substack{n=0 \\ \text{even}}}^{\infty} C_n b^n P_n(\mu) \tag{5-32}$$

Equation (5-32) is recognized as a Fourier–Legendre series expansion of the function $f(\mu)$. We apply the following operator to both sides:

$$* \int_{\mu=0}^{1} P_q(\mu)\, d\mu$$

noting that the arbitrary constant q must be an *even integer*, recalling that

$$\int_{\mu=0}^{1} P_q(\mu) P_n(\mu)\, d\mu = 0 \qquad \text{for} \qquad n \neq q \tag{5-33}$$

provided that n and q are both even or both odd. This yields the coefficients C_n as

$$C_n = \frac{\int_{\mu=0}^{1} f(\mu) P_n(\mu)\, d\mu}{b^n \int_{\mu=0}^{1} \left[P_n(\mu)\right]^2\, d\mu} \tag{5-34}$$

The integral in the denominator of equation (5-34) is recognized as the norm and is equal to $1/(2n+1)$. The solution is now complete with equations (5-31) and

(5-34). We may combine both expressions and simplify, yielding the following expression:

$$T(r,\mu) = \sum_{\substack{n=0\\ \text{even}}}^{\infty} (2n+1)\left(\frac{r}{b}\right)^n P_n(\mu) \int_{\mu'=0}^{1} f\left(\mu'\right) P_n\left(\mu'\right) d\mu' \tag{5-35}$$

Example 5-2 $T = T(r, \mu, \phi)$ for Solid Sphere

A solid sphere of radius b is maintained at steady-state conditions with a prescribed surface temperature $T(r = b) = f(\mu, \phi)$. The mathematical formulation of the problem is given as

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \mu}\left[\left(1-\mu^2\right)\frac{\partial T}{\partial \mu}\right] + \frac{1}{r^2\left(1-\mu^2\right)}\frac{\partial^2 T}{\partial \phi^2} = 0 \tag{5-36}$$

$$\text{in} \qquad 0 \le r < b \qquad -1 \le \mu \le 1 \qquad 0 \le \phi \le 2\pi$$

$$\text{BC1:} \quad T(r \to 0) \Rightarrow \text{finite} \qquad \text{BC2:} \quad T(r = b) = f(\mu, \phi) \tag{5-37a}$$

$$\text{BC3:} \quad T(\mu \to \pm 1) \Rightarrow \text{finite} \tag{5-37b}$$

$$\text{BC4:} \quad T(\phi) = T(\phi + 2\pi) \quad \to \quad 2\pi - \text{periodicity} \tag{5-37c}$$

$$\text{BC5:} \quad \left.\frac{\partial T(\phi)}{\partial \phi}\right|_{\phi} = \left.\frac{\partial T(\phi)}{\partial \phi}\right|_{\phi+2\pi} \quad \to \quad 2\pi - \text{periodicity} \tag{5-37d}$$

where we have imposed finiteness as $r \to 0$ and as $\mu \to \pm 1$. Boundary conditions BC4 and BC5 are recognized as the requirement of 2π periodicity in the ϕ dimension. With only a single nonhomogeneous boundary condition, we proceed with separation of variables in the form

$$T(r, \mu, \phi) = R(r)M(\mu)\Phi(\phi) \tag{5-38}$$

Substituting equation (5-38) into the PDE of equation (5-36) yields

$$\frac{-1}{M}\frac{d}{d\mu}\left[\left(1-\mu^2\right)\frac{dM}{d\mu}\right] - \frac{1}{1-\mu^2}\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = \frac{r^2}{R}\left(\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr}\right) = n(n+1) \tag{5-39}$$

after multiplying both sides by $r^2/R(r)M(\mu)\Phi(\phi)$. The separation constant is selected here to ultimately yield the associated Legendre equation in the μ dimension. The r dimension is now separated to yield the Cauchy equation as the ODE, as detailed in the previous problem, giving the solution

$$R(r) = C_1 r^n + C_2 r^{-(n+1)} \tag{5-40}$$

The requirement of finiteness at the origin eliminates C_2 given that this term goes to infinity as $r \to 0$ for $n > -1$. The term associated with C_1 is finite for

$n \geq 0$. We now consider the left-hand side of equation (5-39), multiplying first by $1 - \mu^2$ and separating, to yield

$$\frac{1-\mu^2}{M}\frac{d}{d\mu}\left[\left(1-\mu^2\right)\frac{dM}{d\mu}\right] + n(n+1)\left(1-\mu^2\right) = \frac{-1}{\Phi}\frac{d^2\Phi}{d\phi^2} = m^2 \tag{5-41}$$

We have forced a characteristic value problem in the homogeneous ϕ dimension with the introduction of the separation constant m^2, which yields the desired solution

$$\Phi(\phi) = C_3 \cos m\phi + C_4 \sin m\phi \tag{5-42}$$

The requirement of 2π periodicity is satisfied for positive integers, namely,

$$m = 0, 1, 2, 3, \ldots$$

and both constants C_3 and C_4 are retained. We finally consider the remaining μ terms in equation (5-41), rearranging as

$$\frac{d}{d\mu}\left[\left(1-\mu^2\right)\frac{dM}{d\mu}\right] + \left[n(n+1) - \frac{m^2}{1-\mu^2}\right]M = 0 \tag{5-43}$$

Equation (5-43) is recognized as the associated Legendre equation. With m set to positive integers to satisfy 2π periodicity in the ϕ dimension, we now limit n to the positive integers,

$$n = 0, 1, 2, 3, \ldots$$

to generate the orthogonal associated Legendre polynomials as the solution of equation (5-43). This provides the solution

$$M(\mu) = C_5 P_n^m(\mu) + C_6 Q_n^m(\mu) \tag{5-44}$$

We may now eliminate the associated Legendre polynomials of the second kind ($C_6 = 0$) in keeping with BC3 because $Q_n^m(\mu)$ is infinite at $\mu = \pm 1$. The complete solution may now be formed by summing over all possible product solutions,

$$T\,(r, \mu, \phi) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} r^n P_n^m\,(\mu)\left(a_{nm}\cos m\phi + b_{nm}\sin m\phi\right) \tag{5-45}$$

where we have defined new constants $a_{nm} = C_1C_3C_5$ and $b_{nm} = C_1C_4C_5$. The inner summation is terminated at $m = n$, noting that the associated Legendre polynomials are zero for $m > n$, as seen by equation (2-83). The nonhomogeneous boundary condition, equation (5-37a), may now be applied:

$$T\,(r = b) = f\,(\mu, \phi) = \sum_{n=0}^{\infty}\sum_{m=0}^{n} b^n P_n^m\,(\mu)\left(a_{nm}\cos m\phi + b_{nm}\sin m\phi\right) \tag{5-46}$$

Equation (5-46) is a double Fourier series expansion in terms of the orthogonal associated Legendre polynomials and trigonometric functions. We first find the Fourier coefficients a_{nm} by applying the following operators to both sides:

$$* \int_{\mu=-1}^{1} P_k^m(\mu)\, d\mu \qquad \text{and} \qquad * \int_{\phi=0}^{2\pi} \cos q\phi\, d\phi$$

which yields the expression

$$a_{nm} = \frac{\displaystyle\int_{\phi=0}^{2\pi} \int_{\mu=-1}^{1} f(\mu, \phi) P_n^m(\mu) \cos m\phi\, d\mu\, d\phi}{\displaystyle b^n \int_{\mu=-1}^{1} [P_n^m(\mu)]^2\, d\mu \int_{\phi=0}^{2\pi} \cos^2 m\phi\, d\phi} \tag{5-47}$$

In a similar manner, the Fourier coefficients b_{nm} are found by applying the following operators to both sides:

$$* \int_{\mu=-1}^{1} P_k^m(\mu)\, d\mu \qquad \text{and} \qquad * \int_{\phi=0}^{2\pi} \sin q\phi\, d\phi$$

to yield the expression

$$b_{nm} = \frac{\displaystyle\int_{\phi=0}^{2\pi} \int_{\mu=-1}^{1} f(\mu, \phi) P_n^m(\mu) \sin m\phi\, d\mu\, d\phi}{\displaystyle b^n \int_{\mu=-1}^{1} [P_n^m(\mu)]^2\, d\mu \int_{\phi=0}^{2\pi} \sin^2 m\phi\, d\phi} \tag{5-48}$$

The norms $N(\mathrm{m})$ of the two trigonometric functions in the denominators of equations (5-47) and (5-48) are equal to π except for the special case of $m = 0$ in equation (5-47), for which the norm is equal to 2π. The norm $N(n,m)$ of the associated Legendre polynomials was previously defined, see equation (2-88), as

$$N(n, m) = \int_{\mu=-1}^{1} \left[P_n^m(\mu)\right]^2\, d\mu = \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!} \tag{5-49}$$

We may introduce the Fourier coefficients into equation (5-45) to yield

$$T(r, \mu, \phi) = \frac{1}{\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left[\frac{2n+1}{2} \frac{(n-m)!}{(n+m)!} \left(\frac{r}{b}\right)^n P_n^m(\mu) \cdot \int_{\phi'=0}^{2\pi} \int_{\mu'=-1}^{1} \right.$$
$$\left. \times\ f\left(\mu', \phi'\right) P_n^m\left(\mu'\right) \cos m\left(\phi - \phi'\right)\, d\mu'\, d\phi' \right] \tag{5-50}$$

where π should be replaced by 2π for the case of $m = 0$, and noting that we have introduced a trigonometric substitution to combine the sin and cos terms into a single expression.

5-3 SOLUTION OF TRANSIENT PROBLEMS

In this section we will solve a range of transient problems in the spherical coordinate system. Our general guidelines established for separation of variables in the rectangular coordinate system equally apply to the spherical coordinate problems. Therefore, transient problems with a homogeneous PDE must always be solved with all homogeneous boundary conditions and with the single nonhomogeneity being the initial condition. If one or more boundary conditions are nonhomogeneous, removal of the nonhomogeneities via a temperature shift may be used, or the principle of superposition must be used to treat the nonhomogeneous boundary conditions separately in a steady-state problem. For a nonhomogeneous PDE (i.e., when heat generation is present), superposition is used to treat the heat generation within a steady-state problem, which is then coupled to a homogeneous transient problem at the initial condition. Prior to solving a range of problems to illustrate the above procedures, a few additional guidelines are offered for the transient heat equation in spherical coordinates.

As introduced in Section 5-1, the last three entries of Table 5-1 concern the transient spherical problems. For the 1-D transient problem, namely, $T = T(r, t)$, the transformation $U = rT$ is used to remove the nonconstant coefficients from the PDE, effectively reducing the spherical heat equation to the form of the Cartesian heat equation, for which we have the necessary tools to solve. The 2-D and 3-D transient problems involve $T = T(r, \mu, t)$ and $T = T(r, \mu, \phi, t)$, respectively. For both instances, we utilize the $V = r^{1/2}T$ transformation, and expect orthogonal functions in all spatial dimensions. For the r dimension, half-integer-order Bessel functions will always arise due to the separation constant introduced to force the Legendre polynomials (or associated Legendre polynomials) in the μ dimension. For the 3-D problem, as with cylindrical systems, the ϕ dimension must always be homogeneous, generating the trigonometric functions $\cos m\phi$ and $\sin m\phi$ for the condition of 2π periodicity for the full sphere $(0 \le \phi \le 2\pi)$. The necessity for 2π periodicity will limit the separation constants m to integer values, which will then couple to the order of the associated Legendre polynomials with ϕ dependency. For the spherical coordinate system, the ϕ dimension will always correspond to the full sphere $(0 \le \phi \le 2\pi)$, with a hemisphere defined by limiting the domain of the μ dimension, namely, to $0 \le \mu \le 1$. Finally, as described above, ϕ dependency in the absence of μ dependency is not covered from a physical point of view. Similarly, any combination of ϕ or μ dependency in the absence of r dependency is not considered (i.e., physically unrealistic).

Example 5-3 $T = T(r, t)$ for Solid Sphere with Convective Boundary
A solid sphere of radius b is initially at temperature $F(r)$. For $t > 0$, the boundary condition at $r = b$ dissipates heat by convection, with convection coefficient h into a fluid at zero temperature. The mathematical formulation of the problem is given as

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad \text{in} \qquad 0 \le r < b \qquad t > 0 \tag{5-51}$$

$$\text{BC1:} \qquad T(r \to 0) \Rightarrow \text{finite} \qquad \text{BC2:} \qquad -k\left.\frac{\partial T}{\partial r}\right|_{r=b} = h\, T|_{r=b} \tag{5-52a}$$

$$\text{IC:} \qquad T(t = 0) = F(r) \tag{5-52b}$$

Equation (5-51) may be recast in the equivalent form

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(rT) = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{5-53}$$

We now introduce a new dependent variable $U(r,t)$, using the transform

$$\boxed{U(r, t) = rT(r, t)} \tag{5-54}$$

With this substitution, the PDE of equations (5-51) and (5-53) becomes

$$\frac{\partial^2 U}{\partial r^2} = \frac{1}{\alpha}\frac{\partial U}{\partial t} \tag{5-55}$$

which is equivalent to the 1-D Cartesian formulation. It is also necessary to transform the boundary conditions. We first consider a change of dependent variable for the spatial derivative of T, namely,

$$\frac{\partial T}{\partial r} = \frac{\partial}{\partial r}\left(\frac{U}{r}\right) = \frac{1}{r}\frac{\partial U}{\partial r} - \frac{1}{r^2}U \tag{5-56}$$

Equation (5-56) has the result of adding an additional term, $1/r^2 U$, to each derivative term in a boundary condition; hence both insulated and convective boundary conditions each obtain an additional term. One must also consider the condition of finiteness as $r \to 0$, which now takes the form

$$T(r \to 0) \quad \Rightarrow \quad \lim_{r \to 0} \frac{U(r)}{r} \tag{5-57}$$

which is only finite for the expected trigonometric functions if the numerator is zero in the limit, which then enables the application of L'Hôpital's rule. Thus finiteness at the origin requires $U(r = 0) = 0$. We now summarize the complete transformation of the general $U(r, t) = rT(r, t)$ formulation in Table 5-2.

In view of the above discussion and Table 5-2, we now reformulate the problem, including both the boundary conditions and initial condition:

$$\frac{\partial^2 U}{\partial r^2} = \frac{1}{\alpha}\frac{\partial U}{\partial t} \qquad \text{in} \qquad 0 \le r < b \qquad t > 0 \tag{5-58}$$

TABLE 5-2 PDE and Boundary Condition Transformations for $U = rT$

Equation	$T\ (r, t)$	$U\ (r, t)$
PDE	$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} + \frac{g_0}{k} = \frac{1}{\alpha}\frac{\partial T(r,t)}{\partial t}$	$\frac{\partial^2 U}{\partial r^2} + \frac{g_0 r}{k} = \frac{1}{\alpha}\frac{\partial U(r,t)}{\partial t}$
BC1	$T(r = b) = 0$	$U(r = b) = 0$
BC2	$\left.\frac{\partial T}{\partial r}\right\|_{r=b} = 0$	$\left.\frac{\partial U}{\partial r}\right\|_{r=b} - \frac{1}{b}\ U\|_{r=b} = 0$
BC3	$\left.\frac{\partial T}{\partial r}\right\|_{r=b} + \frac{h}{k}\ T\|_{r=b} = 0$	$\left.\frac{\partial U}{\partial r}\right\|_{r=b} + \left(\frac{h}{k} - \frac{1}{b}\right) U\|_{r=b} = 0$
BC4	$T(r \to 0) \Rightarrow$ finite	$U(r = 0) = 0$

$$\text{BC1:} \quad U(r = 0) = 0 \qquad \text{BC2:} \quad \left.\frac{\partial U}{\partial r}\right|_{r=b} + \left(\frac{h}{k} - \frac{1}{b}\right) U|_{r=b} = 0 \tag{5-59a}$$

$$\text{IC:} \quad U(t = 0) = rF(r) \tag{5-59b}$$

For convenience, we will introduce the constant $K = (h/k - 1/b)$ into equation (5-59a). With the successful transformation and with all homogeneous boundary conditions, we are now ready to proceed with separation of the form

$$U(r, t) = R(r)\Gamma(t) \tag{5-60}$$

which after substitution into equation (5-58) yields

$$\frac{1}{R}\frac{d^2 R}{dr^2} = \frac{1}{\alpha\Gamma}\frac{d\Gamma}{dt} = -\lambda^2 \tag{5-61}$$

Solution of separated ODE in the t dimension yields

$$\Gamma(t) = C_1 e^{-\alpha\lambda^2 t} \tag{5-62}$$

while solution of the separated ODE in the r dimension gives

$$R(r) = C_2 \cos\lambda r + C_3 \sin\lambda r \tag{5-63}$$

Boundary condition BC1 yields $C_2 = 0$, while BC2 yields the following transcendental equation:

$$\lambda_n \cot\lambda_n b = -K \quad \rightarrow \quad \lambda_n \quad \text{for} \quad n = 1, 2, 3, \ldots \tag{5-64}$$

The transcendental equation (5-64) may be recast in the form $b\lambda_n \cot \lambda_n b + bK = 0$, which has all real roots if $bK > -1$, and noting that $\lambda_0 = 0$ is an eigenvalue only for the special case of $bK = -1$. When the value of K as defined above is introduced into the inequality $bK > -1$, we find the necessary condition for all real roots given as $bh/k > 0$ (i.e., Bi >0). This result is satisfied with the physical requirements of a positive radius, convection coefficient, and thermal conductivity. We now sum over all possible solutions, yielding the general solution

$$U(r, t) = \sum_{n=1}^{\infty} C_n \sin(\lambda_n r) e^{-\alpha\lambda_n^2 t} \tag{5-65}$$

with $C_n = C_1 C_3$. The initial condition, equation (5-29b), is now applied, yielding

$$U(t = 0) = rF(r) = \sum_{n=1}^{\infty} C_n \sin(\lambda_n r) \tag{5-66}$$

We solve for the Fourier coefficients by applying the operator

$$* \int_{r=0}^{b} \sin(\lambda_q r)\, dr$$

which yields the result

$$C_n = \frac{\int_{r=0}^{b} rF(r) \sin(\lambda_n r)\, dr}{\int_{r=0}^{b} \sin^2(\lambda_n r)\, dr} \tag{5-67}$$

The denominator is the norm $N(\lambda_n)$ and may be simplified by case 7 in Table 2-1, with the replacement of H_2 with K. We finally transform the problem back to $T(r,t)$, yielding the result

$$T(r, t) = \frac{U(r, t)}{r} = \sum_{n=1}^{\infty} C_n \frac{\sin(\lambda_n r)}{r} e^{-\alpha\lambda_n^2 t} \tag{5-68}$$

We know that the limit as $r \to 0$ exists and is finite per L'Hôpital's rule. In fact, in this limit, the temperature at the center of the sphere becomes

$$T(r = 0, t) = \sum_{n=1}^{\infty} C_n \lambda_n e^{-\alpha\lambda_n^2 t} \tag{5-69}$$

The steady-state solution for the entire sphere is simply $T = 0$, which follows directly from equations (5-68) and (5-69) as $t \to \infty$, and is in agreement with the physics of the problem, namely, equilibration at the fluid temperature of zero.

Example 5-4 $T = T(r, t)$ for Solid Sphere with Insulated Boundary
A solid sphere of radius b is initially at temperature $F(r)$. For $t > 0$, the boundary condition at $r = b$ is perfectly insulated. The mathematical formulation of the problem is given as

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad \text{in} \qquad 0 \le r < b \qquad t > 0 \tag{5-70}$$

$$\text{BC1:} \qquad T(r \to 0) \Rightarrow \text{finite} \qquad \text{BC2:} \qquad \left.\frac{\partial T}{\partial r}\right|_{r=b} = 0 \tag{5-71a}$$

$$\text{IC:} \qquad T(t = 0) = F(r) \tag{5-71b}$$

As in the previous problem, we introduce the $U(r, t) = rT(r, t)$ transformation of the PDE, boundary conditions, and initial condition. Using Table 5-2, this yields

$$\frac{\partial^2 U}{\partial r^2} = \frac{1}{\alpha}\frac{\partial U}{\partial t} \qquad \text{in} \qquad 0 \le r < b \qquad t > 0 \tag{5-72}$$

$$\text{BC1:} \qquad U(r = 0) = 0 \qquad \text{BC2:} \qquad \left.\frac{\partial U}{\partial r}\right|_{r=b} - \frac{1}{b}\,U|_{r=b} = 0 \tag{5-73a}$$

$$\text{IC:} \qquad U(t = 0) = rF(r) \tag{5-73b}$$

With the successful transformation, and with all homogeneous boundary conditions, we are now ready to proceed with separation of the form

$$U(r, t) = R(r)\Gamma(t) \tag{5-74}$$

which after substitution into equation (5-72) yields

$$\frac{1}{R}\frac{d^2 R}{dr^2} = \frac{1}{\alpha\Gamma}\frac{d\Gamma}{dt} = -\lambda^2 \tag{5-75}$$

Solution of separated ODE in the t dimension yields the desired solution

$$\Gamma(t) = C_1 e^{-\alpha\lambda^2 t} \tag{5-76}$$

while solution of the separated ODE in the r dimension gives

$$R(r) = C_2 \cos\lambda r + C_3 \sin\lambda r \tag{5-77}$$

Boundary condition BC1 yields $C_2 = 0$, while BC2 yields the following transcendental equation:

$$b\lambda_n \cot\lambda_n b = 1 \qquad \to \qquad \lambda_n \quad \text{for} \quad n = 0, 1, 2, 3, \ldots \tag{5-78}$$

noting that $\lambda_0 = 0$ is an eigenvalue for this special case of the right-hand side equal to unity (see Appendix II). The general solution is now formed by summing over all possible solutions, yielding

$$U(r,t) = \sum_{n=0}^{\infty} C_n \sin(\lambda_n r) e^{-\alpha\lambda_n^2 t} \tag{5-79}$$

with $C_n = C_1 C_3$. Given the physics of the problem with regard to perfect insulation, as discussed below, it is useful to transform the problem back to $T(r,t)$ prior to considering the initial condition, which gives

$$T(r,t) = \sum_{n=0}^{\infty} C_n \frac{\sin(\lambda_n r)}{r} e^{-\alpha\lambda_n^2 t} \tag{5-80}$$

Examination of equation (5-80) as $t \to \infty$ yields a steady-state temperature equal to zero, which is inconsistent with the physics given that finite energy is initially contained within the sphere, as given by the initial temperature $F(r)$. Unlike with the case of the perfectly insulated cylinder, see Example 4-8, the necessary steady-state temperature does not arise directly from the zero eigenvalue term. Nevertheless, as noted by Carslaw and Jaeger [1, p. 203], for the case of the perfectly insulated sphere, $\lambda_0 = 0$ is an eigenvalue, and an additional term must be added to the solution (see the note at the end of this chapter). With this in mind, we express our general solution in the form

$$T(r,t) = C_0 + \sum_{n=1}^{\infty} C_n \frac{\sin(\lambda_n r)}{r} e^{-\alpha\lambda_n^2 t} \tag{5-81}$$

Ultimately, we expect C_0 to be the steady-state solution, although we continue here using our standard approach to evaluate the constants. We now apply the initial condition as given by equation (5-71b), which yields

$$T(t=0) = F(r) = C_0 + \sum_{n=1}^{\infty} C_n \frac{\sin(\lambda_n r)}{r} \tag{5-82}$$

The orthogonality of the sin function is with respect to a weighting function of unity; hence we first multiply both sides by r, giving

$$rF(r) = C_0 r + \sum_{n=1}^{\infty} C_n \sin(\lambda_n r) \tag{5-83}$$

We now evaluate the Fourier coefficients C_n by applying the operator to both sides

$$* \int_{r=0}^{b} \sin(\lambda_m r)\, dr$$

which yields the following expression:

$$\int_{r=0}^{b} rF(r)\sin(\lambda_m r)\,dr = C_0\int_{r=0}^{b} r\sin(\lambda_m r)\,dr + \sum_{n=1}^{\infty} C_n \int_{r=0}^{b} \sin(\lambda_n r)\sin(\lambda_m r)\,dr \tag{5-84}$$

We note that the integral $\int_{r=0}^{b} r\sin(\lambda_m r)\,dr$ is identically zero for the eigenvalues as defined by equation (5-78). Using the property of orthogonality for the remaining integral on the right-hand side yields the Fourier coefficients

$$C_n = \frac{\int_{r=0}^{b} rF(r)\sin(\lambda_n r)\,dr}{\int_{r=0}^{b} \sin^2(\lambda_n r)\,dr} \tag{5-85}$$

The norm $N(\lambda_n)$ corresponds to case 7 in Table 2-1 with the replacement of H_2 with $-1/b$; hence the norm becomes

$$\int_{r=0}^{b} \sin^2(\lambda_n r)\,dr = \frac{b\left(\lambda_n^2 + 1/b^2\right) - 1/b}{2\left(\lambda_n^2 + 1/b^2\right)} \tag{5-86}$$

To now evaluate the remaining coefficient C_0, we apply the following operator to both sides of equation (5-83)

$$* \int_{r=0}^{b} r\,dr$$

which yields the following expression:

$$\int_{r=0}^{b} r^2 F(r)\,dr = C_0\int_{r=0}^{b} r^2\,dr + \sum_{n=1}^{\infty} C_n \int_{r=0}^{b} r\sin(\lambda_n r)\,dr \tag{5-87}$$

As described above, the rightmost integral is zero for all terms in the summation, yielding the expression for C_0 as

$$C_0 = \frac{\int_{r=0}^{b} r^2 F(r)\,dr}{\int_{r=0}^{b} r^2\,dr} = \frac{3}{b^3}\int_{r=0}^{b} r^2 F(r)\,dr \tag{5-88}$$

Equations (5-81), (5-85), and (5-88) complete the problem, where C_0 is now recognized as the steady-state temperature of the sphere. This is readily checked by considering conservation of energy between the initial and final states. For a perfectly insulated sphere, we must conserve total energy, namely,

$$E_{\text{initial}} \equiv E_{\text{final}} \qquad \text{(J)} \tag{5-89}$$

We may evaluate the left-hand term by integrating the initial energy over the full sphere, while the right-hand term follows from $E_{\text{final}} = mcT_{\text{ss}}$, yielding

$$\int_{r=0}^{b} (\rho c) 4\pi r^2 F(r)\, dr = (\rho c)\left(\frac{4}{3}\pi b^3\right) T_{\text{ss}} \tag{5-90}$$

Simplification of equation (5-90) directly yields equation (5-88) with $C_0 \equiv T_{\text{ss}}$; hence our solution of the heat equation, including the steady-state solution, correctly reflects conservation of energy.

Example 5-5 $\boldsymbol{T = T(r, t)}$ for Hollow Sphere

A hollow sphere of inner radius a and outer radius b is initially at temperature $F(r)$ (Fig. 5-3). For $t > 0$, the inner and outer surfaces are maintained at zero temperature.

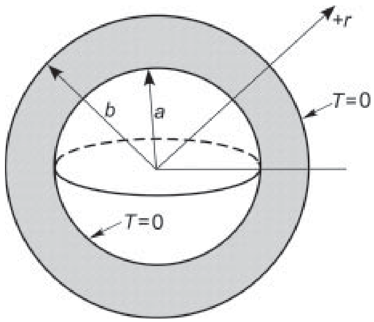

Figure 5-3 Problem description for Example 5-5.

The mathematical formulation of the problem is given as

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad \text{in} \qquad a < r < b \qquad t > 0 \tag{5-91}$$

$$\text{BC1:} \qquad T(r = a) = 0 \qquad \text{BC2:} \qquad T(r = b) = 0 \tag{5-92a}$$

$$\text{IC:} \qquad T(t = 0) = F(r) \tag{5-92b}$$

We introduce the $U(r, t) = rT(r, t)$ transformation of the PDE, boundary conditions, and initial condition. Using Table 5-2, this yields

$$\frac{\partial^2 U}{\partial r^2} = \frac{1}{\alpha}\frac{\partial U}{\partial t} \qquad \text{in} \qquad a < r < b \qquad t > 0 \tag{5-93}$$

$$\text{BC1:} \qquad U(r = a) = 0 \qquad \text{BC2:} \qquad U(r = b) = 0 \tag{5-94a}$$

$$\text{IC:} \qquad U(t = 0) = rF(r) \tag{5-94b}$$

With all homogeneous boundary conditions, we proceed with separation of the form

$$U(r,t) = R(r)\Gamma(t) \tag{5-95}$$

which after substitution into equation (5-93) yields

$$\frac{1}{R}\frac{d^2R}{dr^2} = \frac{1}{\alpha\Gamma}\frac{d\Gamma}{dt} = -\lambda^2 \tag{5-96}$$

Solution of separated ODE in the t dimension yields

$$\Gamma(t) = C_1 e^{-\alpha\lambda^2 t} \tag{5-97}$$

while the solution of the separated ODE in the r dimension yields

$$R(r) = C_2 \cos\lambda r + C_3 \sin\lambda r \tag{5-98}$$

Boundary condition BC1 gives

$$C_3 = -C_2 \frac{\cos\lambda a}{\sin\lambda a} \tag{5-99}$$

which upon substitution into equation (5-98) gives

$$R(r) = C_2 \cos\lambda r - C_2 \frac{\cos\lambda a}{\sin\lambda a}\sin\lambda r \tag{5-100}$$

We may now factor out the $1/\sin(\lambda a)$ term, yielding

$$R(r) = \frac{C_2}{\sin\lambda a}\left[\sin\lambda a\cos\lambda r - \cos\lambda a\sin\lambda r\right] \tag{5-101}$$

As we have done before, the term in front of the bracket is now defined as a new constant, say C_4, and a trigonometric substitution is applied to the bracket, giving

$$R(r) = C_4 \sin\left[\lambda(r-a)\right] \tag{5-102}$$

We now apply the boundary condition BC2,

$$0 = C_4 \sin\lambda\,(b-a) \quad \to \quad \lambda_n = \frac{n\pi}{b-a} \qquad \text{for} \qquad n = 0, 1, 2, \ldots \tag{5-103}$$

to yield the eigenvalues λ_n, noting that $\lambda_0 = 0$ is a trivial eigenvalue. We note that equation (5-103) is equivalent to the roots generated by the equation $\cot\lambda_n a = \cot\lambda_n b$. The general solution now becomes

$$U(r,t) = \sum_{n=1}^{\infty} C_n \sin\left[\lambda_n(r-a)\right] e^{-\alpha\lambda_n^2 t} \tag{5-104}$$

We now apply the initial condition as given by equation (5-94b), which yields

$$U(t=0) = rF(r) = \sum_{n=1}^{\infty} C_n \sin\left[\lambda_n (r-a)\right] \tag{5-105}$$

We evaluate the Fourier coefficients C_n by applying the operator to both sides

$$* \int_{r=a}^{b} \sin\left[\lambda_m (r-a)\right] dr$$

which yields the following expression:

$$C_n = \frac{\int_{r=a}^{b} rF(r) \sin\left[\lambda_n (r-a)\right] dr}{\int_{r=a}^{b} \sin^2\left[\lambda_n (r-a)\right] dr} \tag{5-106}$$

Transforming back to $T(r,t)$, the solution takes the form

$$T(r,t) = \sum_{n=1}^{\infty} C_n \frac{\sin\left[\lambda_n (r-a)\right]}{r} e^{-\alpha\lambda_n^2 t} \tag{5-107}$$

The norm of equation (5-106) is $(b-a)/2$ for the eigenvalues of equation (5-103). We may finally substitute the Fourier coefficients and the norm into equation (5-107), yielding the solution of the form

$$T\,(r,t) = \frac{2}{b-a} \sum_{n=1}^{\infty} \frac{\sin\left[\lambda_n (r-a)\right]}{r} e^{-\alpha\lambda_n^2 t} \int_{r'=a}^{b} r' F\left(r'\right) \sin\left[\lambda_n \left(r'-a\right)\right] dr' \tag{5-108}$$

Example 5-6 $T = T(r,t)$ for Solid Sphere with Internal Energy Generation

A solid sphere of radius b is initially at temperature $F(r)$. For $t > 0$, the boundary condition at $r = b$ is subjected to convection heat transfer with convection coefficient h and zero fluid temperature. In addition, the sphere is subjected to uniform internal energy generation g_0 (W/m^3). The mathematical formulation of the problem is given as

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} + \frac{g_0}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad \text{in} \qquad 0 \le r < b \qquad t > 0 \tag{5-109}$$

$$\text{BC1:} \qquad T(r \to 0) \Rightarrow \text{finite} \qquad \text{BC2:} \qquad -k\left.\frac{\partial T}{\partial r}\right|_{r=b} = h\left. T\right|_{r=b} \tag{5-110a}$$

$$\text{IC:} \qquad T(t=0) = F(r) \tag{5-110b}$$

We introduce the $U(r, t) = rT(r, t)$ transformation of the PDE, boundary conditions, and initial condition. Using Table 5-2, this yields

$$\frac{\partial^2 U}{\partial r^2} + \frac{g_o r}{k} = \frac{1}{\alpha}\frac{\partial U}{\partial t} \qquad \text{in} \qquad 0 \le r < b \quad t > 0 \tag{5-111}$$

$$\text{BC1:} \qquad U(r = 0) = 0 \qquad \text{BC2:} \qquad \left.\frac{\partial U}{\partial r}\right|_{r=b} + \left(\frac{h}{k} - \frac{1}{b}\right) U|_{r=b} = 0 \tag{5-112a}$$

$$\text{IC:} \qquad U(t = 0) = rF(r) \tag{5-112b}$$

The PDE of equation (5-111) is nonhomogeneous; hence we must seek superposition of the form $U(r, t) = \Psi(r, t) + \Phi(r)$, where $\Psi(r, t)$ takes the homogeneous form of the PDE and boundary conditions, while $\Phi(r)$ is a nonhomogeneous ODE. We first consider the 1-D problem of $\Phi(r)$:

$$\frac{d^2\Phi}{dr^2} + \frac{g_0 r}{k} = 0 \qquad \text{in} \qquad 0 \le r < b \tag{5-113}$$

$$\text{BC1:} \qquad \Phi(r = 0) = 0 \qquad \text{BC2:} \qquad \left.\frac{d\Phi}{dr}\right|_{r=b} + K\ \Phi|_{r=b} = 0 \tag{5-114}$$

where we have introduced the constant $K = (h/k - 1/b)$. The solution of equation (5-113) takes the form of a homogeneous solution plus a particular solution, giving

$$\Phi(r) = C_1 + C_2 r - \frac{g_0}{6k}r^3 \tag{5-115}$$

Boundary condition BC1 eliminates constant C_1, while BC2 yields C_2,

$$C_2 = \left(\frac{bg_0}{6h}\right)\left(2 + \frac{hb}{k}\right) \tag{5-116}$$

which gives the final expression for $\Phi(r)$ as

$$\Phi(r) = r\left(C_2 - \frac{g_0}{6k}r^2\right) \tag{5-117}$$

We now define the homogeneous $\Psi(r, t)$ problem, noting that $\Phi(r)$ is coupled to the initial condition, as detailed in Chapter 3. The formulation becomes

$$\frac{\partial^2\Psi}{\partial r^2} = \frac{1}{\alpha}\frac{\partial\Psi}{\partial t} \qquad \text{in} \qquad 0 \le r < b \quad t > 0 \tag{5-118}$$

$$\text{BC1:} \qquad \Psi(r = 0) = 0 \qquad \text{BC2:} \qquad \left.\frac{\partial\Psi}{\partial r}\right|_{r=b} + K\ \Psi|_{r=b} = 0 \tag{5-119a}$$

$$\text{IC:} \qquad \Psi(t = 0) = rF(r) - \Phi(r) \tag{5-119b}$$

With homogeneous PDE and boundary conditions, we are ready to separate using

$$\Psi(r, t) = R(r)\Gamma(t) \tag{5-120}$$

which after substitution into equation (5-118) yields

$$\frac{1}{R}\frac{d^2R}{dr^2} = \frac{1}{\alpha\Gamma}\frac{d\Gamma}{dt} = -\lambda^2 \tag{5-121}$$

Solution of the separated ODE in the t dimension yields the solution

$$\Gamma(t) = C_1 e^{-\alpha\lambda^2 t} \tag{5-122}$$

while solution of the separated ODE in the r dimension gives

$$R(r) = C_2 \cos\lambda r + C_3 \sin\lambda r \tag{5-123}$$

Boundary condition BC1 yields $C_2 = 0$, while BC2 yields the following transcendental equation:

$$\lambda_n \cot\lambda_n b = -K \quad \rightarrow \quad \lambda_n \quad \text{for} \quad n = 1, 2, 3, \ldots \tag{5-124}$$

We now sum over all possible solutions, yielding the general solution

$$\Psi(r, t) = \sum_{n=1}^{\infty} C_n \sin(\lambda_n r) e^{-\alpha\lambda_n^2 t} \tag{5-125}$$

with $C_n = C_1 C_3$. The initial condition, equation (5-119b), is now applied, yielding

$$\Psi(t = 0) = rF(r) - \Phi(r) = \sum_{n=1}^{\infty} C_n \sin(\lambda_n r) \tag{5-126}$$

We solve for the Fourier coefficients by applying the operator

$$* \int_{r=0}^{b} \sin(\lambda_q r)\, dr$$

which yields the result

$$C_n = \frac{\int_{r=0}^{b} [rF(r) - \Phi(r)] \sin(\lambda_n r)\, dr}{\int_{r=0}^{b} \sin^2(\lambda_n r)\, dr} \tag{5-127}$$

The denominator is the norm $N(\lambda_n)$ and may be simplified by case 7 in Table 2-1, with the replacement of H_2 with K. We finally transform the problem back to $T(r,t)$, yielding the result

$$T(r,t) = \frac{\Psi(r,t)}{r} + \frac{\Phi(r)}{r} = \sum_{n=1}^{\infty} C_n \frac{\sin(\lambda_n r)}{r} e^{-\alpha\lambda_n^2 t} + \left(C_2 - \frac{g_0}{6k} r^2\right) \tag{5-128}$$

where the constant C_2 is defined by equation (5-116). We note that the rightmost term in equation (5-128) is the steady-state solution, giving the parabolic temperature distribution expected for a steady-state, 1-D sphere with generation.

Example 5-7 $T = T(r, \mu, t)$ for Solid Sphere

A solid sphere of radius b is initially at temperature $F(r,\mu)$. For $t > 0$, the boundary condition at $r = b$ is maintained at temperature T_1. Because all boundary conditions must be homogeneous for a transient problem, we first shift the temperature, defining $\Theta(r, \mu, t) = T(r, \mu, t) - T_1$. With this shift, the mathematical formulation of the problem is given as

$$\frac{\partial^2\Theta}{\partial r^2} + \frac{2}{r}\frac{\partial\Theta}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial\mu}\left[\left(1-\mu^2\right)\frac{\partial\Theta}{\partial\mu}\right] = \frac{1}{\alpha}\frac{\partial\Theta}{\partial t} \quad \text{in} \quad 0 \le r < b,\ -1 \le \mu \le 1,\ t > 0 \tag{5-129}$$

BC1: $\Theta(r \to 0) \Rightarrow$ finite BC2: $\Theta(r = b) = 0$ (5-130a)

BC3: $\Theta(\mu \to \pm 1) \Rightarrow$ finite (5-130b)

IC: $\Theta(t = 0) = F(r, \mu) - T_1 = G(r, \mu)$ (5-130c)

As discussed above and detailed in Table 5-1, transient problems with both r and μ dependency are solved using the $V(r, \mu, t) = r^{1/2}\Theta(r, \mu, t)$ transformation. Using such a transformation, the mathematical formulation becomes

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} - \frac{1}{4}\frac{V}{r^2} + \frac{1}{r^2}\frac{\partial}{\partial\mu}\left[\left(1-\mu^2\right)\frac{\partial V}{\partial\mu}\right] = \frac{1}{\alpha}\frac{\partial V}{\partial t} \quad \text{in} \quad 0 \le r < b,\ -1 \le \mu \le 1,\ t > 0 \tag{5-131}$$

BC1: $V(r \to 0) \Rightarrow$ finite BC2: $V(r = b) = 0$ (5-132a)

BC3: $V(\mu \to \pm 1) \Rightarrow$ finite (5-132b)

IC: $V(t = 0) = r^{1/2} G(r, \mu)$ (5-132c)

In addition, it will be necessary to check that $V/r^{1/2}$ is finite as $r \to 0$ after transforming back to $\Theta(r, \mu, t)$. We now assume separation of the form

$$V(r, \mu, t) = R(r)M(\mu)\Gamma(t) \tag{5-133}$$

Substituting equation (5-133) into equation (5-131) yields

$$\frac{1}{R}\left[\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}-\frac{1}{4}\frac{R}{r^2}\right]+\frac{1}{r^2M}\frac{d}{d\mu}\left[\left(1-\mu^2\right)\frac{dM}{d\mu}\right]=\frac{1}{\alpha\Gamma}\frac{d\Gamma}{dt}=-\lambda^2 \tag{5-134}$$

Solution of separated ODE in the t dimension yields the solution

$$\Gamma(t)=C_1e^{-\alpha\lambda^2t} \tag{5-135}$$

with the remaining r terms and μ term of equation (5-134) yielding

$$\frac{r^2}{R}\left[\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}-\frac{1}{4}\frac{R}{r^2}\right]+\lambda^2r^2=-\frac{1}{M}\frac{d}{d\mu}\left[\left(1-\mu^2\right)\frac{dM}{d\mu}\right]=n(n+1) \tag{5-136}$$

after introduction of a new separation constant $n(n+1)$. Solution of the μ dimension ODE yields the orthogonal Legendre polynomials for integer values of n; hence we define $n = 0, 1, 2, 3, \ldots$, and the solution becomes

$$M(\mu)=C_2P_n(\mu)+C_3Q_n(\mu) \tag{5-137}$$

We may eliminate the Legendre polynomials of the second kind ($C_3 = 0$) because $Q_n(\mu)$ is infinite at $\mu = \pm 1$, with both values within the domain of the problem. We next consider the remaining r terms of equation (5-136), which we reorganize to the form

$$\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}+\left[\lambda^2-\frac{(n+1/2)^2}{r^2}\right]R=0 \tag{5-138}$$

Equation (5-138) is recognized as Bessel's equation of half-integer order, with the solution

$$R(r)=C_4J_{n+1/2}(\lambda r)+C_5Y_{n+1/2}(\lambda r) \tag{5-139}$$

The requirement for finiteness as $r \to 0$ stated by BC1 eliminates the $Y_{n+1/2}(\lambda r)$ term from further consideration, while BC2 yields the equation

$$J_{n+1/2}(\lambda_{nm}b)=0 \quad \to \quad \lambda_{nm} \quad \text{for} \quad m=1,2,3,\ldots \quad \text{for each } n \tag{5-140}$$

which generates eigenvalues λ_{nm}. As discussed previously, we use a double subscript for the eigenvalues to denote that for each value of n, there is a *full set* of eigenvalues λ_m. We now form the general solution by summing the product solutions over all eigenvalues n and λ_{nm}, yielding

$$V(r,\mu,t)=\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}C_{nm}J_{n+1/2}(\lambda_{nm}r)P_n(\mu)e^{-\alpha\lambda_{nm}^2t} \tag{5-141}$$

with constant $C_{nm} = C_1 C_2 C_4$. Application of the initial condition, equation (5-132c), yields

$$r^{1/2} G(r, \mu) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} C_{nm} J_{n+1/2}(\lambda_{nm} r) P_n(\mu) \tag{5-142}$$

to which we apply successively the two operators

$$* \int_{r=0}^{b} r J_{n+1/2}(\lambda_{nk} r)\, dr \qquad \text{and} \qquad * \int_{\mu=-1}^{1} P_q(\mu)\, d\mu$$

to yield the Fourier–Bessel–Legendre coefficients C_{nm} as

$$C_{nm} = \frac{\int_{\mu=-1}^{1} \int_{r=0}^{b} r^{3/2} G(r, \mu) J_{n+1/2}(\lambda_{nm} r) P_n(\mu)\, dr\, d\mu}{\int_{r=0}^{b} r J_{n+1/2}^2(\lambda_{nm} r)\, dr \int_{\mu=-1}^{1} \left[P_n(\mu)\right]^2 d\mu} \tag{5-143}$$

The norms are defined by equation (2-79) for the Legendre polynomials and by case 3 with $\nu = n + 1/2$ in Table 2-2 for the Bessel functions. Finally, we shift back to $\Theta\,(r, \mu, t)$, and then $T\,(r, \mu, t)$, giving

$$T(r, \mu, t) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} C_{nm} \frac{J_{n+1/2}(\lambda_{nm} r)}{r^{1/2}} P_n(\mu) e^{-\alpha \lambda_{nm}^2 t} + T_1 \tag{5-144}$$

As a final step, it is necessary to explore the requirement for finiteness as $r \to 0$. Considering only the r dependence of the problem, we have

$$\lim_{r \to 0} \frac{J_{n+1/2}(\lambda r)}{r^{1/2}} = \lim_{r \to 0} \frac{(\text{constant}) r^{n+1/2}}{r^{1/2}} \sum_k \frac{(\lambda r)^{2k}}{f(n, k)} \tag{5-145}$$

where we have considered the series form of the Bessel functions, see Appendix IV. As seen in equation (5-145), the $r^{1/2}$ term cancels from the denominator, and the overall solution is finite as $r \to 0$ for all values of $n \geq 0$.

Example 5-8 $T = T(r, \mu, t)$ for a Hemisphere
A hemisphere of radius b is initially at temperature $F(r,\mu)$ (Fig. 5-4). For $t > 0$, the boundary condition at $r = b$ is maintained at temperature T_1, while the boundary condition at $\mu = 0$ (the base) is maintained at temperature T_2.

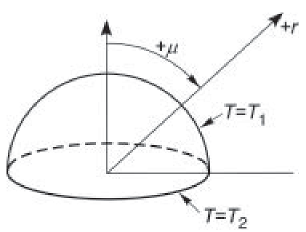

Figure 5-4 Problem description for Example 5-8.

The problem formulation becomes

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \mu}\left[\left(1-\mu^2\right)\frac{\partial T}{\partial \mu}\right] = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad \text{in} \qquad 0 \le r < b,\ 0 < \mu \le 1,\ t > 0 \tag{5-146}$$

BC1: $T(r \to 0) \Rightarrow$ finite BC2: $T(r = b) = T_1$ (5-147a)

BC3: $T(\mu \to +1) \Rightarrow$ finite BC4: $T(\mu = 0) = T_2$ (5-147b)

IC: $T(t = 0) = F(r, \mu)$ (5-147c)

Because all boundary conditions must be homogeneous for a transient problem, and we are unable to remove both nonhomogeneous conditions by a simple temperature shift, we use superposition of the form

$$T(r, \mu, t) = \Psi(r, \mu, t) + \Phi(r, \mu) \tag{5-148}$$

The formulation for the steady-state problem becomes

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{2}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \mu}\left[\left(1-\mu^2\right)\frac{\partial \Phi}{\partial \mu}\right] = 0 \qquad \text{in} \qquad 0 \le r < b \quad 0 < \mu \le 1 \tag{5-149}$$

BC1: $\Phi(r \to 0) \Rightarrow$ finite BC2: $\Phi(r = b) = T_1$ (5-150a)

BC3: $\Phi(\mu \to +1) \Rightarrow$ finite BC4: $\Phi(\mu = 0) = T_2$ (5-150b)

where the steady-state solution takes both nonhomogeneous boundary conditions. However, the steady-state formulation must contain only a single nonhomogeneity for the solution, and since the μ dimension *must be homogeneous*, we introduce a temperature shift $\Theta(r, \mu) = \Phi(r, \mu) - T_2$, which leads to the new formulation:

$$\frac{\partial^2 \Theta}{\partial r^2} + \frac{2}{r}\frac{\partial \Theta}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \mu}\left[\left(1-\mu^2\right)\frac{\partial \Theta}{\partial \mu}\right] = 0 \qquad \text{in} \qquad 0 \le r < b \quad 0 < \mu \le 1 \tag{5-151}$$

BC1: $\Theta(r \to 0) \Rightarrow$ finite BC2: $\Theta(r = b) = T_1 - T_2 = T_3$ (5-152a)

BC3: $\Theta(\mu \to +1) \Rightarrow$ finite BC4: $\Theta(\mu = 0) = 0$ (5-152b)

The problem is now correctly set for separation of the form

$$\Theta(r, \mu) = R(r)M(\mu) \tag{5-153}$$

Substituting equation (5-153) into the PDE of equation (5-151), multiplying both sides by r^2, and separating yields

$$\frac{r^2}{R}\left[\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}\right]=\frac{-1}{M}\frac{d}{d\mu}\left[\left(1-\mu^2\right)\frac{dM}{d\mu}\right]=n(n+1) \tag{5-154}$$

where we have introduced the separation constant $n(n+1)$ to force the characteristic value problem in the homogeneous μ dimension. As with Example 5-1, the μ terms yield the orthogonal Legendre polynomials for integer values of n ($n = 0, 1, 2, 3, \ldots$), giving the solution

$$M(\mu) = C_1 P_n(\mu) + C_2 Q_n(\mu) \tag{5-155}$$

The Legendre polynomials of the second kind are eliminated ($C_2 = 0$) because $Q_n(\mu)$ is infinite at $\mu = +1$. The boundary condition at $\mu = 0$ is given by equation (5-152b), namely,

$$P_n(\mu = 0) = 0 \qquad \text{yielding} \qquad n = 1, 3, 5, \ldots \quad \text{(odd integers)} \tag{5-156}$$

Examination of the first few Legendre polynomials, see equation (2-70), reveals that these functions are zero at $\mu = 0$ only for the odd Legendre polynomials, and we therefore limit n to the odd integer values. The remaining r terms in equation (5-154) yield Cauchy's equation as described above, with the solution

$$R(r) = C_3 r^n + C_4 r^{-(n+1)} \tag{5-157}$$

where we eliminate the constant C_4 for the condition of finiteness as $r \to 0$ for n equal to odd positive integers. We now recombine our separated solutions as products, and sum over all possible solutions:

$$\Theta(r,\mu) = \sum_{\substack{n=1\\ \text{odd}}}^{\infty} C_n r^n P_n(\mu) \tag{5-158}$$

where we have introduced the new constant $C_n = C_1 C_3$. We lastly apply the nonhomogeneous BC2, equation (5-152a), which yields

$$\Theta(r=b) = T_3 = \sum_{\substack{n=1\\ \text{odd}}}^{\infty} C_n b^n P_n(\mu) \tag{5-159}$$

Equation (5-159) is recognized as a Fourier–Legendre series expansion of a constant. We apply the following operator to both sides:

$$*\int_{\mu=0}^{1} P_q(\mu)\, d\mu$$

noting that the arbitrary constant q must be an odd integer, recalling that

$$\int_{\mu=0}^{1} P_q(\mu) P_n(\mu)\, d\mu = 0 \qquad \text{for} \qquad n \neq q \tag{5-160}$$

provided that n and q are both even or both odd. This yields the coefficients C_n as

$$C_n = \frac{T_3 \int_{\mu=0}^{1} P_n(\mu)\, d\mu}{b^n \int_{\mu=0}^{1} \left[P_n(\mu)\right]^2\, d\mu} \tag{5-161}$$

The integral in the denominator of equation (5-161) is recognized as the norm and is equal to $1/(2n+1)$. The solution is now complete and may be combined and the temperature shifted back, yielding the following expression:

$$\Phi(r, \mu) = \left(T_1 - T_2\right) \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} (2n+1) \left(\frac{r}{b}\right)^n P_n(\mu) \int_{\mu'=0}^{1} P_n\left(\mu'\right) d\mu' + T_2 \tag{5-162}$$

With the steady-state solution now fully solved, we formulate the homogeneous transient problem as

$$\frac{\partial^2 \Psi}{\partial r^2} + \frac{2}{r}\frac{\partial \Psi}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \mu}\left[\left(1-\mu^2\right)\frac{\partial \Psi}{\partial \mu}\right] = \frac{1}{\alpha}\frac{\partial \Psi}{\partial t} \quad \text{in} \quad 0 \leq r < b,\ 0 < \mu \leq 1,\ t > 0 \tag{5-163}$$

BC1: $\Psi(r \to 0) \Rightarrow$ finite BC2: $\Psi(r = b) = 0$ (5-164a)

BC3: $\Psi(\mu \to +1) \Rightarrow$ finite BC4: $\Psi(\mu = 0) = 0$ (5-164b)

IC: $\Psi(t = 0) = F(r, \mu) - \Phi(r, \mu) = G(r, \mu)$ (5-164c)

where we have coupled the steady-state solution to the initial condition, as always required for superposition of the form given by equation (5-148). As detailed in Table 5-1, we now use the $V(r, \mu, t) = r^{1/2}\Psi(r, \mu, t)$ transformation, which gives

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} - \frac{1}{4}\frac{V}{r^2} + \frac{1}{r^2}\frac{\partial}{\partial \mu}\left[\left(1-\mu^2\right)\frac{\partial V}{\partial \mu}\right] = \frac{1}{\alpha}\frac{\partial V}{\partial t} \quad \text{in} \quad 0 \leq r < b,\ 0 < \mu \leq 1,\ t > 0 \tag{5-165}$$

BC1: $V(r \to 0) \Rightarrow$ finite BC2: $V(r = b) = 0$ (5-166a)

BC3: $V(\mu \to +1) \Rightarrow$ finite BC4: $V(\mu = 0) = 0$ (5-166b)

IC: $V(t = 0) = r^{1/2}\, G(r, \mu)$ (5-166c)

We now assume separation of the form

$$V(r,\mu,t) = R(r)M(\mu)\Gamma(t) \tag{5-167}$$

which upon substitution into equation (5-165) yields

$$\frac{1}{R}\left[\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - \frac{1}{4}\frac{R}{r^2}\right] + \frac{1}{r^2 M}\frac{d}{d\mu}\left[\left(1-\mu^2\right)\frac{dM}{d\mu}\right] = \frac{1}{\alpha\Gamma}\frac{d\Gamma}{dt} = -\lambda^2 \tag{5-168}$$

Solution of separated ODE in the t dimension yields the solution

$$\Gamma(t) = C_1 e^{-\alpha\lambda^2 t} \tag{5-169}$$

with the r terms and μ term of equation (5-168) yielding

$$\frac{r^2}{R}\left[\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - \frac{1}{4}\frac{R}{r^2}\right] + \lambda^2 r^2 = -\frac{1}{M}\frac{d}{d\mu}\left[\left(1-\mu^2\right)\frac{dM}{d\mu}\right] = n(n+1) \tag{5-170}$$

after introduction of a new separation constant $n(n+1)$. Solution of the μ-dimension ODE yields the orthogonal Legendre polynomials for integer values $n = 0, 1, 2, 3, \ldots$, and the solution becomes

$$M(\mu) = C_2 P_n(\mu) + C_3 Q_n(\mu) \tag{5-171}$$

We eliminate the Legendre polynomials of the second kind ($C_3 = 0$) for the requirement of finiteness at $\mu = +1$. The boundary condition at $\mu = 0$ as given by equation (5-166b) further limits the values of n, namely,

$$P_n(\mu = 0) = 0 \qquad \text{yielding} \qquad n = 1, 3, 5, \ldots \text{(odd integers)} \tag{5-172}$$

The remaining r terms of equation (5-170) yield Bessel's equation of half-integer order, with the solution

$$R(r) = C_4 J_{n+1/2}(\lambda r) + C_5 Y_{n+1/2}(\lambda r) \tag{5-173}$$

The requirement for finiteness as $r \to 0$ stated by BC1 eliminates the $Y_{n+1/2}(\lambda r)$ term, while BC2 yields the transcendental equation

$$J_{n+1/2}(\lambda_{nm} b) = 0 \quad \to \quad \lambda_{nm} \quad \text{for} \quad m = 1, 2, 3 \ldots \text{for each } n \tag{5-174}$$

which produces eigenvalues λ_{nm}, noting that there is a full set of eigenvalues λ_m for each value of n. We now form the general solution by summing the product solutions over all eigenvalues n and λ_{nm}, yielding

$$V(r,\mu,t) = \sum_{\substack{n=1\\ \text{odd}}}^{\infty} \sum_{m=1}^{\infty} C_{nm} J_{n+1/2}(\lambda_{nm} r) P_n(\mu) e^{-\alpha\lambda_{nm}^2 t} \tag{5-175}$$

with constant $C_{nm} = C_1C_2C_4$. Application of the initial condition, equation (5-166c), yields

$$r^{1/2}G(r,\mu) = \sum_{\substack{n=1\\ \text{odd}}}^{\infty}\sum_{m=1}^{\infty} C_{nm}J_{n+1/2}(\lambda_{nm}r)P_n(\mu) \tag{5-176}$$

to which we apply successively the two operators

$$*\int_{r=0}^{b} r\,J_{n+1/2}(\lambda_{nk}r)\,dr \qquad \text{and} \qquad *\int_{\mu=0}^{1} P_q(\mu)\,d\mu$$

to yield the Fourier–Bessel–Legendre coefficients C_{nm} as

$$C_{nm} = \frac{\int_{\mu=0}^{1}\int_{r=0}^{b} r^{3/2}G(r,\mu)J_{n+1/2}(\lambda_{nm}r)P_n(\mu)\,dr\,d\mu}{\int_{r=0}^{b} r\,J_{n+1/2}^2(\lambda_{nm}r)\,dr\int_{\mu=0}^{1}\left[P_n(\mu)\right]^2\,d\mu} \tag{5-177}$$

The norms are defined by equation (2-79) for the Legendre polynomials, and by case 3 with $\nu = n + \frac{1}{2}$ in Table 2-2 for the Bessel functions. Finally, we shift back to $\Psi\,(r,\mu,t)$, giving

$$\Psi(r,\mu,t) = \sum_{\substack{n=1\\ \text{odd}}}^{\infty}\sum_{m=1}^{\infty} C_{nm}\frac{J_{n+1/2}(\lambda_{nm}r)}{r^{1/2}}P_n(\mu)e^{-\alpha\lambda_{nm}^2 t} \tag{5-178}$$

The total solution is then formed by superposition of the transient and steady-state solutions,

$$\begin{aligned} T(r,\mu,t) &= \sum_{\substack{n=1\\ \text{odd}}}^{\infty}\sum_{m=1}^{\infty} C_{nm}\frac{J_{n+1/2}(\lambda_{nm}r)}{r^{1/2}}P_n(\mu)e^{-\alpha\lambda_{nm}^2 t} \\ &\quad + \left(T_1 - T_2\right)\sum_{\substack{n=1\\ \text{odd}}}^{\infty}(2n+1)\left(\frac{r}{b}\right)^n P_n\,(\mu)\int_{\mu'=0}^{1} P_n\left(\mu'\right)\,d\mu' + T_2 \end{aligned} \tag{5-179}$$

where C_{nm} is defined by equation (5-177). As explored in the previous example, the overall solution is indeed finite as $r \to 0$ for all values of $n \geq 0$.

Example 5-9 $\quad T = T(r,\mu,t)$ for a Hemisphere

A hemisphere of radius b is initially at temperature $F(r,\mu)$. For $t > 0$, the boundary condition at $r = b$ dissipates heat by convection with convection coefficient

h into a fluid of zero temperature, while the boundary condition at $\mu = 0$ (the base) is perfectly insulated. The problem formulation becomes

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \mu}\left[\left(1-\mu^2\right)\frac{\partial T}{\partial \mu}\right] = \frac{1}{\alpha}\frac{\partial T}{\partial t} \quad \text{in} \quad 0 \le r < b,\ 0 < \mu \le 1,\ t > 0 \tag{5-180}$$

BC1: $T(r \to 0) \Rightarrow \text{finite}$ BC2: $-k\left.\frac{\partial T}{\partial r}\right|_{r=b} = h\, T|_{r=b}$ (5-181a)

BC3: $T(\mu \to +1) \Rightarrow \text{finite}$ BC4: $\left.\frac{\partial T}{\partial \mu}\right|_{\mu=0} = 0$ (5-181b)

IC: $T(t = 0) = F(r, \mu)$ (5-181c)

As detailed in Table 5-1, we now use the $V(r, \mu, t) = r^{1/2}T(r, \mu, t)$ transformation, which gives

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} - \frac{1}{4}\frac{V}{r^2} + \frac{1}{r^2}\frac{\partial}{\partial \mu}\left[\left(1-\mu^2\right)\frac{\partial V}{\partial \mu}\right] = \frac{1}{\alpha}\frac{\partial V}{\partial t} \quad \text{in} \quad 0 \le r < b,\ 0 < \mu \le 1,\ t > 0 \tag{5-182}$$

BC1: $V(r \to 0) \Rightarrow \text{finite}$ BC2: $\left.\frac{\partial V}{\partial r}\right|_{r=b} + \left(\frac{h}{k} - \frac{1}{2b}\right) V|_{r=b} = 0$ (5-183a)

BC3: $V(\mu \to +1) \Rightarrow \text{finite}$ BC4: $\left.\frac{\partial V}{\partial \mu}\right|_{\mu=0} = 0$ (5-183b)

IC: $V(t = 0) = r^{1/2}F(r, \mu)$ (5-183c)

The derivative boundary condition BC2 of equation (5-181a) was modified using

$$\boxed{\frac{\partial T}{\partial r} = \frac{\partial}{\partial r}\left(\frac{V}{r^{1/2}}\right) = \frac{1}{r^{1/2}}\frac{\partial V}{\partial r} - \frac{1}{2r^{3/2}}V} \tag{5-184}$$

which has the effect of introducing an additional term to BC2, as seen in equation (5-183a). We list all such boundary condition transformations in Table 5-3 corresponding to the $V = r^{1/2}T$ transformation.

We now assume separation of the form

$$V(r, \mu, t) = R(r)M(\mu)\Gamma(t) \tag{5-185}$$

TABLE 5-3 Boundary Condition Transformations for $V = r^{1/2}T$

Boundary Equation	$T(r,\mu,t),\ T(r,\mu,\phi,t)$	$V(r,\mu,t),\ V(r,\mu,\phi,t)$
BC1	$T(r=b)=0$	$V(r=b)=0$
BC2	$\left.\frac{\partial T}{\partial r}\right\|_{r=b}=0$	$\left.\frac{\partial V}{\partial r}\right\|_{r=b}-\frac{1}{2b}\,V\|_{r=b}=0$
BC3	$\left.\frac{\partial T}{\partial r}\right\|_{r=b}+\frac{h}{k}\,T\|_{r=b}=0$	$\left.\frac{\partial V}{\partial r}\right\|_{r=b}+\left(\frac{h}{k}-\frac{1}{2b}\right)V\|_{r=b}=0$
BC4	$T(r\to 0)\Rightarrow$ finite	$V(r\to 0)\Rightarrow$ finite

which upon substitution into equation (5-182) yields

$$\frac{1}{R}\left(\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}-\frac{1}{4}\frac{R}{r^2}\right)+\frac{1}{r^2M}\frac{d}{d\mu}\left[\left(1-\mu^2\right)\frac{dM}{d\mu}\right]=\frac{1}{\alpha\Gamma}\frac{d\Gamma}{dt}=-\lambda^2 \tag{5-186}$$

Solution of the separated ODE in the t dimension yields the solution

$$\Gamma(t)=C_1e^{-\alpha\lambda^2t} \tag{5-187}$$

with the remaining r terms and μ term of equation (5-186) yielding

$$\frac{r^2}{R}\left(\frac{d^2R}{dr^2}+\frac{1}{r}\frac{dR}{dr}-\frac{1}{4}\frac{R}{r^2}\right)+\lambda^2r^2=-\frac{1}{M}\frac{d}{d\mu}\left[\left(1-\mu^2\right)\frac{dM}{d\mu}\right]=n(n+1) \tag{5-188}$$

after introduction of a new separation constant $n(n+1)$. Solution of the μ dimension ODE yields the orthogonal Legendre polynomials for integer values $n=0,1,2,3,\ldots$, with the solution becoming

$$M(\mu)=C_2P_n(\mu)+C_3Q_n(\mu) \tag{5-189}$$

We eliminate the Legendre polynomials of the second kind ($C_3=0$) for the requirement of finiteness at $\mu=+1$. The boundary condition BC4 at $\mu=0$, as given by equation (5-183b), further limits the values of n, namely,

$$\left.\frac{dP_n(\mu)}{d\mu}\right|_{\mu=0}=0 \qquad \text{yielding} \qquad n=0,2,4,\ldots\text{(even integers)} \tag{5-190}$$

The remaining r terms of equation (5-188) yield Bessel's equation of half-integer order, with the solution

$$R(r)=C_4J_{n+1/2}(\lambda r)+C_5Y_{n+1/2}(\lambda r) \tag{5-191}$$

The requirement for finiteness as $r \to 0$ stated by BC1 eliminates the $Y_{n+1/2}(\lambda r)$ term, while BC2 yields the transcendental equation

$$\frac{d}{dr} J_{n+1/2}(\lambda_{nm} r)\bigg|_{r=b} + K J_{n+1/2}(\lambda_{nm} b) = 0 \quad \to \quad \lambda_{nm} \quad \text{for} \quad m = 1, 2, 3, \ldots \tag{5-192}$$

where we have introduced the constant $K = (h/k - 1/2b)$. Equation (5-192) produces eigenvalues λ_{nm}, noting that there is a full set of eigenvalues λ_m for each value of n. We now form the general solution by summing the product solutions over all eigenvalues n and λ_{nm}, yielding

$$V(r, \mu, t) = \sum_{\substack{n=0 \\ \text{even}}}^{\infty} \sum_{m=1}^{\infty} C_{nm} J_{n+1/2}(\lambda_{nm} r) P_n(\mu) e^{-\alpha \lambda_{nm}^2 t} \tag{5-193}$$

with constant $C_{nm} = C_1 C_2 C_4$. Application of the initial condition, equation (5-183c), yields

$$V(t = 0) = r^{1/2} F(r, \mu) = \sum_{\substack{n=0 \\ \text{even}}}^{\infty} \sum_{m=1}^{\infty} C_{nm} J_{n+1/2}(\lambda_{nm} r) P_n(\mu) \tag{5-194}$$

to which we apply successively the two operators

$$* \int_{r=0}^{b} r J_{n+1/2}(\lambda_{nk} r)\, dr \qquad \text{and} \qquad * \int_{\mu=0}^{1} P_q(\mu)\, d\mu$$

to yield the Fourier–Bessel–Legendre coefficients C_{nm} as

$$C_{nm} = \frac{\int_{\mu=0}^{1} \int_{r=0}^{b} r^{3/2} F(r, \mu) J_{n+1/2}(\lambda_{nm} r) P_n(\mu)\, dr\, d\mu}{\int_{r=0}^{b} r J_{n+1/2}^2(\lambda_{nm} r)\, dr \int_{\mu=0}^{1} \left[P_n(\mu)\right]^2\, d\mu} \tag{5-195}$$

The norms are defined by equation (2-79) for the Legendre polynomials and by case 1 with $\nu = n + 1/2$ and $H = K$ in Table 2-2 for the Bessel functions. Finally, we shift back to $T(r, \mu, t)$, giving

$$T(r, \mu, t) = \sum_{\substack{n=0 \\ \text{even}}}^{\infty} \sum_{m=1}^{\infty} C_{nm} \frac{J_{n+1/2}(\lambda_{nm} r)}{r^{1/2}} P_n(\mu) e^{-\alpha \lambda_{nm}^2 t} \tag{5-196}$$

As explored in the previous example, the overall solution is indeed finite as $r \to 0$ for all values of $n \geq 0$.

For the special case of the surface at $r = b$ now being perfectly insulated, several modifications to the above solution are made. The eigenvalues of equation (5-192) are adjusted by replacing K with $-1/2b$ (i.e., letting $h = 0$), as seen in Table 5-5. For the case of $n = 0$ only, there is a new eigenvalue $\lambda_{00} = 0$. This can be derived by expanding the derivative of the Bessel function using

$$\frac{d}{dr} J_{1/2}(\lambda r)\bigg|_{r=b} = \lambda J_{-1/2}(\lambda b) - \frac{1}{2b} J_{1/2}(\lambda b) \tag{5-197}$$

in addition to the following substitutions for Bessel functions of order $\frac{1}{2}$, namely,

$$J_{1/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2} \sin z \tag{5-198}$$

$$J_{-1/2}(z) = \left(\frac{2}{\pi z}\right)^{1/2} \cos z \tag{5-199}$$

The term that results from the zero eigenvalue solution should then be added to the solution, giving

$$T(r, \mu, t) = \frac{\int_{\mu=0}^{1} \int_{r=0}^{b} F(r, \mu) r^2 \, dr \, d\mu}{\int_{\mu=0}^{1} \int_{r=0}^{b} r^2 \, dr \, d\mu} + \sum_{\substack{n=0 \\ \text{even}}}^{\infty} \sum_{m=1}^{\infty} C_{nm} \frac{J_{n+1/2}(\lambda_{nm} r)}{r^{1/2}} P_n(\mu) e^{-\alpha \lambda_{nm}^2 t} \tag{5-200}$$

where the integral-containing expression is the steady-state temperature and is reflective of conservation of energy. Conservation of energy may be independently checked by considering the following:

$$E_{\text{initial}} \equiv E_{\text{final}} \qquad \text{(J)} \tag{5-201}$$

We may evaluate the left-hand term by integrating the initial energy over the hemisphere, while the right-hand term follows from $E_{\text{final}} = mcT_{\text{ss}}$, yielding

$$\int_{\mu=0}^{1} \int_{r=0}^{b} (\rho c) F(r, \mu)(2\pi r) r \, dr \, d\mu = (\rho c) \frac{1}{2} \left(\frac{4}{3} \pi b^3\right) T_{\text{ss}} \tag{5-202}$$

Simplification of equation (5-202) is in exact agreement with the steady-state solution of equation (5-200); hence the solution correctly reflects conservation of energy.

Example 5-10 $T = T(r, \mu, \phi, t)$ for a Sphere
A solid sphere of radius b is initially at temperature $F(r, \mu, \phi)$. For $t > 0$, the boundary condition at $r = b$ is maintained at zero temperature. The problem formulation becomes

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \mu}\left[\left(1-\mu^2\right)\frac{\partial T}{\partial \mu}\right] + \frac{1}{r^2\left(1-\mu^2\right)}\frac{\partial^2 T}{\partial \phi^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad (5\text{-}203)$$

$$\text{in} \qquad 0 \le r < b \qquad -1 \le \mu \le 1 \qquad 0 \le \phi \le 2\pi \qquad t > 0$$

$$\text{BC1:} \qquad T(r \to 0) \Rightarrow \text{finite} \qquad \text{BC2:} \qquad T(r = b) = 0 \qquad (5\text{-}204a)$$

$$\text{BC3:} \qquad T(\mu \to \pm 1) \Rightarrow \text{finite} \qquad (5\text{-}204b)$$

$$\text{BC4:} \qquad T(\phi) = T(\phi + 2\pi) \quad \to \quad 2\pi\text{-periodicity} \qquad (5\text{-}204c)$$

$$\text{IC:} \qquad T(t = 0) = F(r, \mu, \phi) \qquad (5\text{-}204d)$$

As detailed in Table 5-1, we now use the $V(r, \mu, \phi, t) = r^{1/2} T(r, \mu, \phi, t)$ transformation, which gives the mathematical formulation

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} - \frac{1}{4}\frac{V}{r^2} + \frac{1}{r^2}\frac{\partial}{\partial \mu}\left[\left(1-\mu^2\right)\frac{\partial V}{\partial \mu}\right] + \frac{1}{r^2\left(1-\mu^2\right)}\frac{\partial^2 V}{\partial \phi^2} = \frac{1}{\alpha}\frac{\partial V}{\partial t} \qquad (5\text{-}205)$$

$$\text{in} \qquad 0 \le r < b \qquad -1 \le \mu \le 1 \qquad 0 \le \phi \le 2\pi \qquad t > 0$$

$$\text{BC1:} \qquad V(r \to 0) \Rightarrow \text{finite} \qquad \text{BC2:} \qquad V(r = b) = 0 \qquad (5\text{-}206a)$$

$$\text{BC3:} \qquad V(\mu \to \pm 1) \Rightarrow \text{finite} \qquad (5\text{-}206b)$$

$$\text{BC4:} \qquad V(\phi) = V(\phi + 2\pi) \quad \to \quad 2\pi\text{-periodicity} \qquad (5\text{-}206c)$$

$$\text{IC:} \qquad V(t = 0) = r^{1/2} F(r, \mu, \phi) \qquad (5\text{-}206d)$$

For all homogeneous boundary conditions, we can assume a separation of variables in the form

$$V(r, \mu, \phi, t) = R(r) M(\mu) \Phi(\phi) \Gamma(t) \qquad (5\text{-}207)$$

Substitution of equation (5-207) into the heat equation of (5-205) yields

$$\frac{1}{R}\left(\frac{d^2 R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - \frac{1}{4}\frac{R}{r^2}\right) + \frac{1}{r^2 M}\frac{d}{d\mu}\left[\left(1-\mu^2\right)\frac{dM}{d\mu}\right] + \frac{1}{r^2\left(1-\mu^2\right)}\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = \frac{1}{\alpha\Gamma}\frac{d\Gamma}{dt} = -\lambda^2 \qquad (5\text{-}208)$$

where we have introduced the separation constant $-\lambda^2$. Solution of the separated ODE in the t dimension yields

$$\Gamma(t) = C_1 e^{-\alpha\lambda^2 t} \tag{5-209}$$

We now consider the remaining spatial variables of equation (5-208), first multiplying by r^2, and then isolating the r terms to yield

$$\begin{aligned} &\frac{-1}{M}\frac{d}{d\mu}\left[\left(1-\mu^2\right)\frac{dM}{d\mu}\right] - \frac{1}{1-\mu^2}\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} \\ &\qquad = \frac{r^2}{R}\left(\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - \frac{1}{4}\frac{R}{r^2}\right) + \lambda^2 r^2 = n(n+1) \end{aligned} \tag{5-210}$$

where we have introduced a new separation constant $n(n+1)$. Separation of the r terms produces Bessel's equation of order $\left(n+\frac{1}{2}\right)$,

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left[\lambda^2 - \frac{\left(n+\frac{1}{2}\right)^2}{r^2}\right]R = 0 \tag{5-211}$$

The solution of equation (5-211) for integer n yields half-integer-order Bessel functions, namely,

$$R(r) = C_2 J_{n+1/2}(\lambda r) + C_3 Y_{n+1/2}(\lambda r) \tag{5-212}$$

Applying boundary condition BC1 eliminates the Bessel functions of the second kind ($C_3 = 0$), while boundary condition BC2 yields the transcendental equation

$$J_{n+1/2}(\lambda_{np} b) = 0 \quad \rightarrow \quad \lambda_{np} \quad \text{for} \quad p = 1, 2, 3, \ldots \quad \text{for each } n \tag{5-213}$$

which produces eigenvalues λ_{np}, noting that there is a full set of eigenvalues λ_p for each value of n. We now consider the left-hand side of equation (5-210), which after multiplication by $1-\mu^2$, yields the following separated equation:

$$\frac{1-\mu^2}{M}\frac{d}{d\mu}\left[\left(1-\mu^2\right)\frac{dM}{d\mu}\right] + \left(1-\mu^2\right)n(n+1) = \frac{-1}{\Phi}\frac{d^2\Phi}{d\phi^2} = m^2 \tag{5-214}$$

We have introduced a new separation constant m^2, which yields the desired ODE and corresponding solution for the ϕ dimension, as given by

$$\Phi(\phi) = C_4 \cos m\phi + C_5 \sin m\phi \tag{5-215}$$

The requirement for 2π periodicity is satisfied for integer m, namely,

$$m = 0, 1, 2, 3, \ldots \tag{5-216}$$

and both constants C_4 and C_5 are retained. Finally, we consider the remaining μ terms of equation (5-214), which yield

$$\frac{d}{d\mu}\left[\left(1-\mu^2\right)\frac{dM}{d\mu}\right]+\left[n(n+1)-\frac{m^2}{1-\mu^2}\right]M=0 \tag{5-217}$$

We recognize equation (5-217) as the associated Legendre equation, and now limit n to integer values ($n = 0, 1, 2, 3, \ldots$), which along with integer values of m as defined by equation (5-216), yield the orthogonal associated Legendre polynomials as the solution, giving

$$M(\mu) = C_6 P_n^m(\mu) + C_7 Q_n^m(\mu) \tag{5-218}$$

We eliminate the Legendre polynomials of the second kind ($C_7 = 0$) for the requirement of finiteness at $\mu = \pm 1$. With all three boundary value problems considered for the three spatial dimensions, we now form a solution as the sum of all four separated functions, and sum over all possible solutions, giving

$$\begin{aligned} V(r,\mu,\phi,t) = \sum_{n=0}^{\infty}\sum_{p=1}^{\infty}\sum_{m=0}^{n} J_{n+1/2}(\lambda_{np} r) P_n^m(\mu)\big(a_{nmp}\cos m\phi \\ + b_{nmp}\sin m\phi\big)\, e^{-\alpha\lambda_{np}^2 t} \end{aligned} \tag{5-219}$$

with constants $a_{nmp} = C_1C_2C_4C_6$ and $b_{nmp} = C_1C_2C_5C_6$. Application of the initial condition, equation (5-206d), yields

$$r^{1/2}F(r,\mu,\phi) = \sum_{n=0}^{\infty}\sum_{p=1}^{\infty}\sum_{m=0}^{n} J_{n+1/2}(\lambda_{np} r) P_n^m(\mu)\big(a_{nmp}\cos m\phi + b_{nmp}\sin m\phi\big) \tag{5-220}$$

to which we apply successively the three operators

$$*\int_{r=0}^{b} r J_{n+1/2}(\lambda_{nl} r)\, dr, \qquad *\int_{\mu=-1}^{1} P_q^m(\mu)\, d\mu \qquad \text{and} \qquad *\int_{\phi=0}^{2\pi}\cos k\phi\, d\phi$$

to yield the Fourier–Bessel–Legendre coefficients a_{nmp} as

$$a_{nmp} = \frac{\int_{\phi=0}^{2\pi}\int_{\mu=-1}^{1}\int_{r=0}^{b} r^{3/2}F(r,\mu,\phi) J_{n+1/2}(\lambda_{np} r) P_n^m(\mu)\cos m\phi\, dr\, d\mu\, d\phi}{\int_{r=0}^{b} r J_{n+1/2}^2(\lambda_{np} r)\, dr \int_{\mu=-1}^{1}[P_n^m(\mu)]^2\, d\mu \int_{\phi=0}^{2\pi}\cos^2 m\phi\, d\phi} \tag{5-221}$$

In a similar manner, we apply successively the three operators

$$*\int_{r=0}^{b} r J_{n+1/2}(\lambda_{nl} r)\, dr, \qquad *\int_{\mu=-1}^{1} P_q^m(\mu)\, d\mu \qquad \text{and} \qquad *\int_{\phi=0}^{2\pi}\sin k\phi\, d\phi$$

to yield the Fourier–Bessel–Legendre coefficients b_{nmp} as

$$b_{nmp} = \frac{\int_{\phi=0}^{2\pi}\int_{\mu=-1}^{1}\int_{r=0}^{b} r^{3/2} F(r,\mu,\phi) J_{n+1/2}(\lambda_{np} r) P_n^m(\mu) \sin m\phi \, dr\, d\mu\, d\phi}{\int_{r=0}^{b} r J_{n+1/2}^2(\lambda_{np} r)\, dr \int_{\mu=-1}^{1} [P_n^m(\mu)]^2 \, d\mu \int_{\phi=0}^{2\pi} \sin^2 m\phi \, d\phi} \tag{5-222}$$

The norms for the three orthogonal functions have all been previously defined. The associated Legendre polynomials, see equation (2-88), yield the norm

$$\int_{\mu=-1}^{1} \left[P_n^m(\mu)\right]^2 d\mu = \frac{2}{2n+1}\frac{(n+m)!}{(n-m)!} \tag{5-223}$$

while the Bessel functions, see case 3 of Table 2-2, yield the norm

$$\int_{r=0}^{b} r J_{n+1/2}^2(\lambda_{np} r)\, dr = \frac{b^2 J_{n+3/2}^2(\lambda_{np} b)}{2} \tag{5-224}$$

The trigonometric functions yield the norm π for $m > 0$, with the norm of the $\cos(m\phi)$ becoming 2π for the special case of $m = 0$. The Fourier coefficients and the norms may be substituted into equation (5-219), which yields the following overall solution after making a trigonometric substitution and transforming back to $T(r,\mu,\phi,t)$ from $V(r,\mu,\phi,t)$:

$$T(r,\mu,\phi,t) = \frac{1}{\pi}\sum_{n=0}^{\infty}\sum_{p=1}^{\infty}\sum_{m=0}^{n}$$

$$\left\{ \begin{array}{l} \dfrac{(2n+1)\,(n-m)!}{b^2 J_{n+3/2}^2(\lambda_{np} b)(n+m)!} \cdot \dfrac{J_{n+1/2}(\lambda_{np} r)}{r^{1/2}} P_n^m(\mu) e^{-\alpha\lambda_{np}^2 t} \\ \cdot \displaystyle\int_{\phi'=0}^{2\pi}\int_{\mu'=-1}^{1}\int_{r'=0}^{b} \begin{bmatrix} r'^{3/2} J_{n+1/2}\left(\lambda_{np} r'\right) P_n^m\left(\mu'\right) \\ \cdot \cos m\left(\phi-\phi'\right) F(r',\mu',\phi') \end{bmatrix} dr'\, d\mu'\, d\phi' \end{array} \right\} \tag{5-225}$$

where the π term is replaced by 2π for the case of $m = 0$. As discussed previously, this solution is finite as $r \to 0$ for all values of $n \ge 0$.

5-4 CAPSTONE PROBLEM

In this chapter, we have developed the solution schemes for the solution of the spherical heat equation using both the method of separation of variables and superposition. Our focus has been to develop the analytical solutions, with emphasis on the resulting eigenfunctions and eigenvalues, and the necessary Fourier coefficients, while also considering conservation of energy. The general approach has been to leave the Fourier coefficients in integral form, although

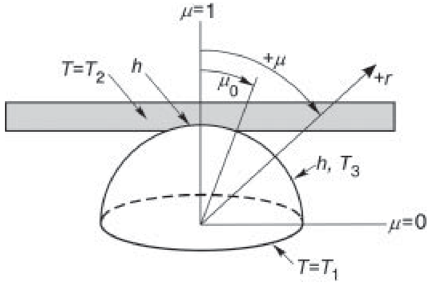

Figure 5-5 Description for capstone problem.

at times we have performed the integrals and substituted back into the series summation as in the above problem. Here we consider a final problem in detail in an attempt to integrate and illustrate the many aspects of our temperature field solutions presented in this chapter, including the temperature field in the context of conservation of energy.

We consider here a hemisphere as shown in Figure 5-5. The hemisphere is of radius b and at steady-state conditions. The base ($\mu = 0$) is maintained at a constant temperature T_1, while the surface at $r = b$ is exposed to convection heat transfer with convection coefficient h and with the reference fluid temperature $T_\infty = f(\mu)$. Specifically, for $0 < \mu \le \mu_0$, the fluid temperature is equal to T_3, while for $\mu_0 < \mu \le 1$, the reference temperature is equal to T_2. By letting the convection coefficient $h = h_c$, with h_c being the contact conductance, see Table 1-3, we approximate the problem of heat transfer through the hemisphere in thermal contact over $\mu_0 < \mu \le 1$ with an isothermal reservoir at T_2, while the base is in perfect thermal contact with an isothermal reservoir at T_1. Ideally, different values of the convection coefficient should be assigned to regions in and out of the contact region, with an actual convection coefficient used over the region of fluid contact, namely, $0 < \mu \le \mu_0$, and a true contact conductance used over the region $\mu_0 < \mu \le 1$. However, formulation of the problem does not allow the convection coefficient to vary with μ; hence adjustment can be made to the value of T_3 as appropriate to best match the expected convective heat transfer through the exposed surface region outside of the zone of contact.

The mathematical formulation of the problem is given as

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \mu}\left[\left(1-\mu^2\right)\frac{\partial T}{\partial \mu}\right] = 0 \qquad \text{in} \qquad 0 \le r < b \qquad 0 < \mu \le 1 \tag{5-225}$$

$$\text{BC1:} \qquad T(r \to 0) \Rightarrow \text{finite} \qquad \text{BC2:} \qquad -k\left.\frac{\partial T}{\partial r}\right|_{r=b} = h\left(T|_{r=b} - T_\infty\right) \tag{5-226a}$$

$$\text{where} \qquad T_\infty = f(\mu) = \begin{cases} T_2 & \text{for } \mu_0 < \mu \le 1 \\ T_3 & \text{for } 0 < \mu \le \mu_0 \end{cases}$$

$$\text{BC3:} \qquad T(\mu \to +1) \Rightarrow \text{finite} \tag{5-227b}$$

$$\text{BC4:} \qquad T(\mu = 0) = T_1 \tag{5-227c}$$

With two nonhomogeneous boundary conditions, one must be removed prior to separation of variables. Since the μ dimension must be homogeneous, we will introduce a shift of temperature as $\Theta(r, \mu) = T(r, \mu) - T_1$. With this shift, the problem formulation becomes

$$\frac{\partial^2 \Theta}{\partial r^2} + \frac{2}{r}\frac{\partial \Theta}{\partial r} + \frac{1}{r^2}\frac{\partial}{\partial \mu}\left[\left(1-\mu^2\right)\frac{\partial \Theta}{\partial \mu}\right] = 0 \qquad \text{in} \qquad 0 \le r < b \quad 0 < \mu \le 1 \tag{5-228}$$

$$\text{BC1:} \quad \Theta(r \to 0) \Rightarrow \text{finite} \quad \text{BC2:} \quad -k\left.\frac{\partial \Theta}{\partial r}\right|_{r=b} = h\left[\left.\Theta\right|_{r=b} - \left(T_\infty - T_1\right)\right] \tag{5-229a}$$

$$\text{where} \qquad T_\infty = f(\mu) = \begin{cases} T_2 & \text{for } \mu_0 < \mu \le 1 \\ T_3 & \text{for } 0 < \mu \le \mu_0 \end{cases}$$

$$\text{BC3:} \qquad \Theta(\mu \to +1) \Rightarrow \text{finite} \tag{5-229b}$$

$$\text{BC4:} \qquad \Theta(\mu = 0) = 0 \tag{5-229c}$$

Now with only a single nonhomogeneous boundary condition, we proceed with separation of variables in the form

$$\Theta(r, \mu) = R(r)M(\mu) \tag{5-230}$$

which upon substitution into equation (5-228) yields

$$\frac{r^2}{R}\left(\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr}\right) = \frac{-1}{M}\frac{d}{d\mu}\left[\left(1-\mu^2\right)\frac{dM}{d\mu}\right] = n(n+1) \tag{5-231}$$

We have introduced the separation constant $n(n+1)$ to force the characteristic value problem in the homogeneous μ dimension. The resulting ODE is recognized as the Legendre equation, with solution

$$M(\mu) = C_1 P_n(\mu) + C_2 Q_n(\mu) \tag{5-232}$$

We may eliminate the Legendre polynomials of the second kind ($C_2 = 0$) because $Q_n(\mu)$ is infinite at $\mu = \pm 1$, noting that $\mu = 1$ (i.e., $\theta = 0$) is in the domain of the problem. We then consider the boundary condition BC4 at $\mu = 0$ (i.e., $\theta = \pi/2$), namely,

$$P_n(\mu = 0) = 0 \qquad \text{yielding} \qquad n = 1, 3, 5, \ldots \text{(odd integers)} \tag{5-233}$$

and therefore limit n to the odd integer values. The remaining r terms in equation (5-231) yield

$$r^2\frac{d^2R}{dr^2}+2r\frac{dR}{dr}-n(n+1)R=0 \tag{5-234}$$

which is recognized as Cauchy's equation, as developed in equation (4-29). As described in previous examples, the solution is

$$R(r)=C_3r^n+C_4r^{-(n+1)} \tag{5-235}$$

We now consider the requirement of finiteness at the origin, which eliminates C_4 given that this terms goes to infinity as $r \to 0$ for n equal to odd positive integers. Having considered all but the final, nonhomogeneous boundary condition, we now recombine our separated solutions as products, and sum over all possible solutions:

$$\Theta(r,\mu)=\sum_{\substack{n=1\\ \text{odd}}}^{\infty}C_nr^nP_n(\mu) \tag{5-236}$$

We lastly apply the nonhomogeneous BC2, equation (5-229a), which yields

$$-k\frac{d}{dr}\left[\sum_{\substack{n=1\\ \text{odd}}}^{\infty}C_nr^nP_n(\mu)\right]_{r=b}=h\left[\sum_{\substack{n=1\\ \text{odd}}}^{\infty}C_nb^nP_n(\mu)-\left(T_\infty-T_1\right)\right] \tag{5-237}$$

The two summations may be combined into a single summation as follows:

$$h\left(T_\infty-T_1\right)=\sum_{\substack{n=1\\ \text{odd}}}^{\infty}C_n\left(nkb^{n-1}+hb^n\right)P_n(\mu) \tag{5-238}$$

Equation (5-238) is recognized as a Fourier–Legendre series expansion of the function $h\left(T_\infty-T_1\right)\equiv h\left[f(\mu)-T_1\right]$. We apply the following operator to both sides:

$$*\int_{\mu=0}^{1}P_q(\mu)\,d\mu$$

noting that the arbitrary constant q must be an odd integer, recalling that

$$\int_{\mu=0}^{1}P_q(\mu)P_n(\mu)\,d\mu=0 \qquad \text{for} \qquad n\neq q \tag{5-239}$$

provided that n and q are both even or both odd. This yields the coefficients C_n as

$$C_n=\frac{\int_{\mu=0}^{1}h\left[f(\mu)-T_1\right]P_n(\mu)\,d\mu}{\left(nkb^{n-1}+hb^n\right)\int_{\mu=0}^{1}\left[P_n(\mu)\right]^2\,d\mu} \tag{5-240}$$

The integral in the denominator of equation (5-240) is recognized as the norm and is equal to $1/(2n+1)$. The solution is now complete with equations (5-236) and (5-240). We may combine both expressions, simplify, and shift back to $T(r, \mu)$, yielding the following expression:

$$T(r, \mu) = \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} (2n+1) \frac{r^n P_n(\mu)}{nkb^{n-1} + hb^n} \int_{\mu'=0}^{1} h \left[f(\mu') - T_1 \right] P_n(\mu')\, d\mu' + T_1 \tag{5-241}$$

If we further define $f(\mu)$ as given by equation (5-229a), the above expression becomes

$$\boxed{T(r, \mu) = \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} (2n+1) \frac{hr^n P_n(\mu)}{nkb^{n-1} + hb^n} \left[\begin{array}{l} (T_3 - T_1) \displaystyle\int_{\mu'=0}^{\mu_0} P_n(\mu')\, d\mu' \\ + (T_2 - T_1) \displaystyle\int_{\mu'=\mu_0}^{1} P_n(\mu')\, d\mu' \end{array} \right] + T_1} \tag{5-242}$$

The above expressions may now be used to explore the behavior of the 2-D temperature profiles and heat transfer for a set of parameters corresponding to a stainless steel hemisphere in contact with a large solid surface made of stainless steel. For these calculations, let the hemisphere have a radius of 1 cm ($b = 0.01$ m) and have properties for stainless steel: Thermal conductivity $k = 14$ W/m · K, and thermal diffusivity $\alpha = 3.5 \times 10^{-6}$ m^2/s. Let the contact interface be at 10 atm pressure, with air as the interfacial fluid, and with a mean surface roughness of the two surfaces equal to 2.5 μm. Using Table 1-3, a representative contact conductance (i.e., reciprocal of the thermal contact resistance) is taken as $h = h_c = 3400$ W/m^2 · K for this steel-on-steel interface. The upper and lower temperatures are maintained at $T_1 = 325$ K and $T_2 = 300$ K. To minimize actual convective heat transfer to the surrounding air outside of the contact zone, we will let $T_3 = T_1 = 325$ K. Finally, we will assume a contact angle of 5°; hence $\mu_0 = 0.9962$.

The first 150 nonzero terms ($n = 1 - 299$) were used for the series summation, as given by equation (5-242). The Legendre polynomials were generated using the recursion formula given by equation (2-71). Over the range of spatial variables explored below, these parameters were found to yield reasonable convergence for temperature and heat flux, although there was significant spatial dependence. In particular, convergence was always greater near the center ($r \sim 0$) and poorest near the edge ($r \sim b$). For example, convergence along the radial line corresponding to $\mu = 0.9659$ ($\theta = 15°$) for various values of r is explored in Figure 5-6, specifically at $r = 0.25b$, $r = 0.5b$, $r = 0.75b$, and at $r = b$. As observed in the figure, convergence is very rapid for the two innermost positions explored, with the first 5 terms of the series providing accuracy to within 0.001 K. At $r = 0.75b$, such accuracy is achieved with the first 9 terms. However, at the surface of the hemisphere, the first 19 terms achieve a convergence to less

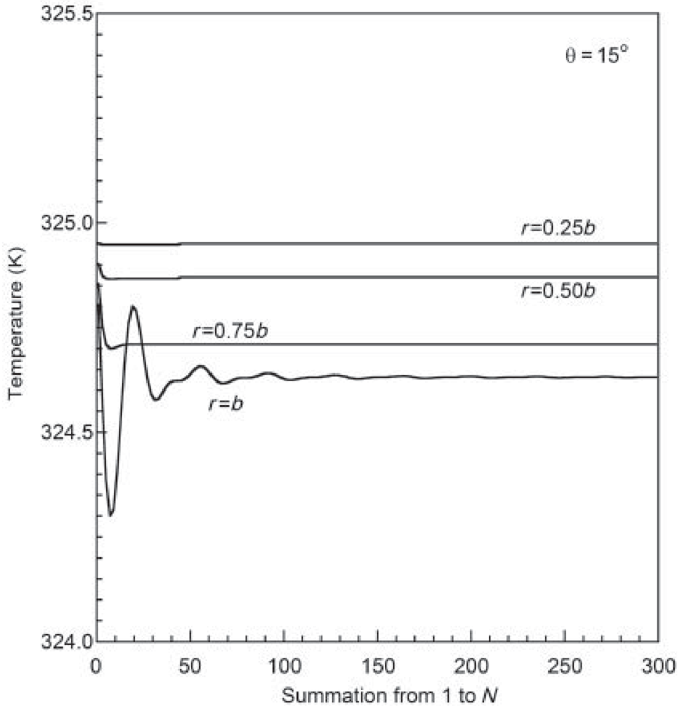

Figure 5-6 Convergence of the temperature solution.

than 0.1 K, while the first 150 terms are required to reach a convergence to less than 0.001 K. This behavior was consistent along the surface for all values of μ.

It is noted, however, that the coefficients of the Legendre polynomials grow very large with increasing degree, and for Legendre polynomials greater than a degree of about 37, round-off error becomes significant even with double-precision computation. Therefore, rather than calculate and store the Legendre polynomial coefficients for subsequent evaluation of the Legendre polynomials at each value of μ, the first two Legendre polynomials were evaluated at each value of μ, and the recurrence relations where then used to evaluate the Legendre polynomial value for each higher degree. This avoids the precision issues associated with very large coefficients, as the evaluated Legendre polynomials themselves over the range of $0 \leq \mu \leq 1$ present no computational challenge. For the Fourier coefficients, the integrals of the Legendre polynomials were done numerically using the trapezoid rule to once again avoid the calculation and storage of the Legendre coefficients for higher-order polynomials. This procedure was checked against the exact analytical integration for lower-order polynomials and found to be satisfactory within the precision of 0.001 K.

Convergence also tended to be worse as μ was varied from 1 to 0, although the boundary condition ($T_1 = 325$ K) is recovered exactly at $\mu = 0$ (i.e., along the base) for the odd Legendre polynomials. For calculation of the heat flux using Fourier's law (i.e., using the appropriate derivatives), convergence was again poor toward the outer surface. Therefore, for calculating the heat flux along the surface at $r = b$, the convective condition ($h\Delta T$) was used rather than Fourier's law ($-k\partial T/\partial r$), which provided sufficient convergence for the first 150 nonzero terms. In particular, with the heat flux calculations using Fourier's law, the terms in the series summation alternate signs, making convergence very slow, as the terms tend to cancel out in pairs. This was the case for the heat flux along the base of the sphere near the outer edge ($r \sim b$). For calculation of the derivatives, the same approach outlined above was used, namely, the recurrence relationships were used to calculate the exact values rather than the coefficients themselves. With these comments in mind, we now explore the behavior of our solution.

Figure 5-7 presents the radial temperature profile along the central axis of the hemisphere, namely, $\mu = 1$ (i.e., $\theta = 0$). As observed in the profile, the temperature decreases from the boundary condition of $T_1 = 325$ K at $r = 0$ to a value of 320.8 K at the point of contact $r = b$. This corresponds to a temperature drop of 20.8 K across the thermal contact resistance, with a corresponding heat flux of 70.9 kW/m^2. The influence of T_2 within the zone of contact is apparent in

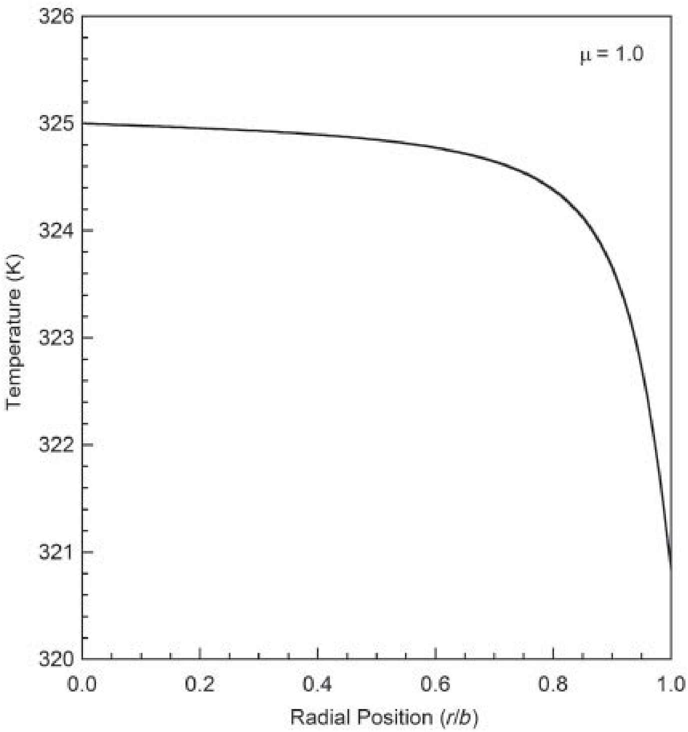

Figure 5-7 Radial temperature profile within the hemisphere at $\mu = 1$.

the temperature profile, with the pronounced increase in the temperature gradient when approaching the contact surface.

Figure 5-8 presents the radial temperature profiles for various values of μ, namely, for $\mu = 0.9848$, 0.9659, and 0.9397, corresponding to values of $\theta = 10^\circ$, 15° and 20°. As observed in the profile for $\theta = 10^o$, the temperature decreases monotonically to a minimum value just below the surface and then increases to a value of 324.3 K at the surface $r = b$. This is in contrast to the radial profiles that end within the zone of contact, as shown in Figure 5-4. The profiles for $\theta = 15^o$ and 20^o reveal similar trends. Specifically, there is a slight increase in temperature nearing the very edge, reflecting the influence of $T_3 = 325$ K. In other words, while the central region of the hemisphere is suppressed in the temperature reflecting the temperature of the region in contact (i.e., a gradient toward the reservoir T_2), the outer region is actually being warmed by the convective fluid outside of the contact region; hence heat is flowing into the hemisphere from the surrounding convective fluid.

Another way to explore the temperature profile behavior, notably the decrease in temperature about the central axis region, is to make a horizontal cut through

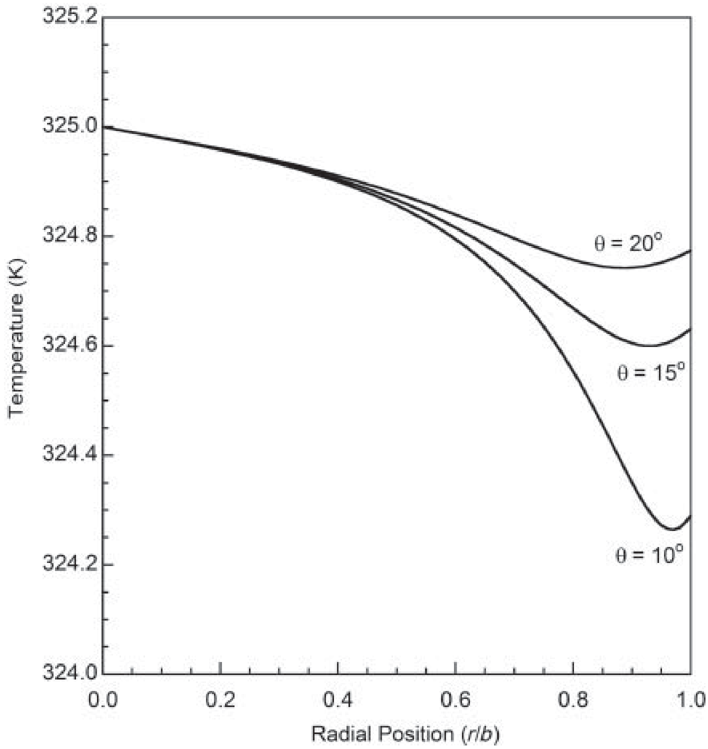

Figure 5-8 Radial temperature profiles within the hemisphere for $\theta = 10^\circ$, 15° and 20°, corresponding to $\mu = 0.9848$, 0.9659, and 0.9397.

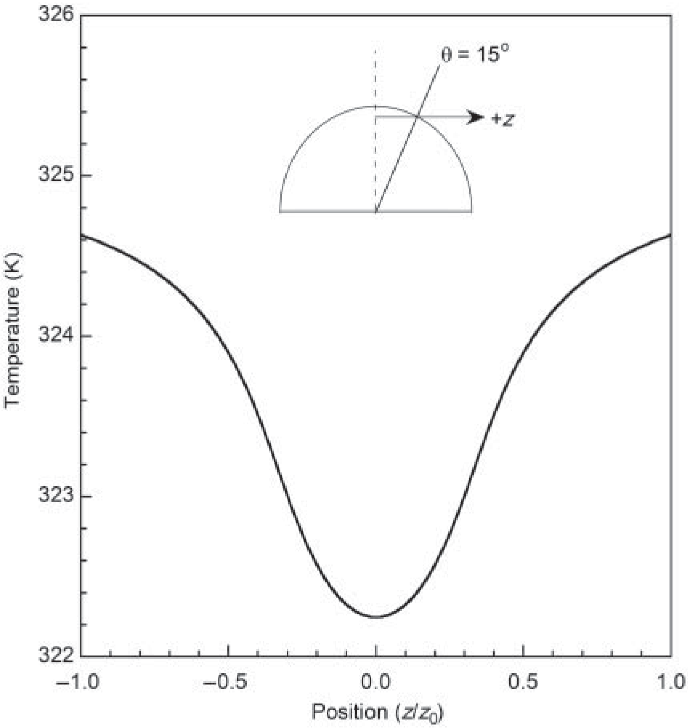

Figure 5-9 Temperature profile across the horizontal cross section intersecting the outer surface at $\theta = 15^{o}$. The z direction is defined as the radial distance perpendicular to the central axis of $\mu = 1$, and z_0 is the radius of this horizontal slice, as shown in the inset.

the hemisphere. This was done by taking a horizontal line from the center axis outward such that it intersects the outer surface at the same location as a radial line from the origin for $\theta = 15^{\circ}$. We show the temperature profile along such a slice in Figure 5-9, where we have passed fully from one side to the other to emphasize the symmetry across the centerline, as observed, for no ϕ dependence. We have normalized the spatial coordinate along this cut with respect to the outer radius of this circle, therefore the temperature at $z/z_0 = 1$ in Figure 5-9 is identical to the temperature at $r/b = 1$ in Figure 5-8 for the $\theta = 15^{o}$ trace. Clearly heat is flowing inward from the outer surface toward the central axis region and ultimately into the region of contact. The slope of this profile is identically zero across the center line for this horizontal slice, properly reflecting the physics of axial symmetry for this problem formulation. It is interesting to note that this behavior does not arise from any boundary conditions in this region. In fact, the spherical coordinate system has no orthogonal boundary for such a horizontal slice. Rather, such behavior arises from conservation of energy at each location inherent in the heat equation.

It is also useful to consider the heat flux entering and leaving the hemisphere. We first explore the heat flux crossing the outer surface at $r = b$, which as noted above, may be calculated by two different means, namely,

$$q_r''(r = b) = -k\frac{\partial T}{\partial r}\bigg|_{r=b} = h\left(T|_{r=b} - T_\infty\right) \tag{5-243}$$

which simply reflects the boundary condition given by equation (2-226a). However, as discussed above, convergence was significantly better using the convective boundary condition; hence the surface temperature was used to calculate the surface heat flux. Figure 5-10 presents the surface heat flux at $r = b$ as a function of μ, over the entire surface of the hemisphere, namely, over the range $0 < \mu \leq 1$. As consistent with the physics of the problem, heat is flowing out of the hemisphere within the zone of contact with T_2, while heat is flowing into the sphere outside of the contact zone via convection heat transfer with the fluid at temperature T_3. There is a discontinuity in heat flux at $\theta = 5^{\mathrm{o}}$ due to the step change in T_∞. Within the region of the convection heat transfer ($\theta > 5^{\mathrm{o}}$), the heat flux is negative (i.e., into the hemisphere) as consistent with Newton's law of cooling,

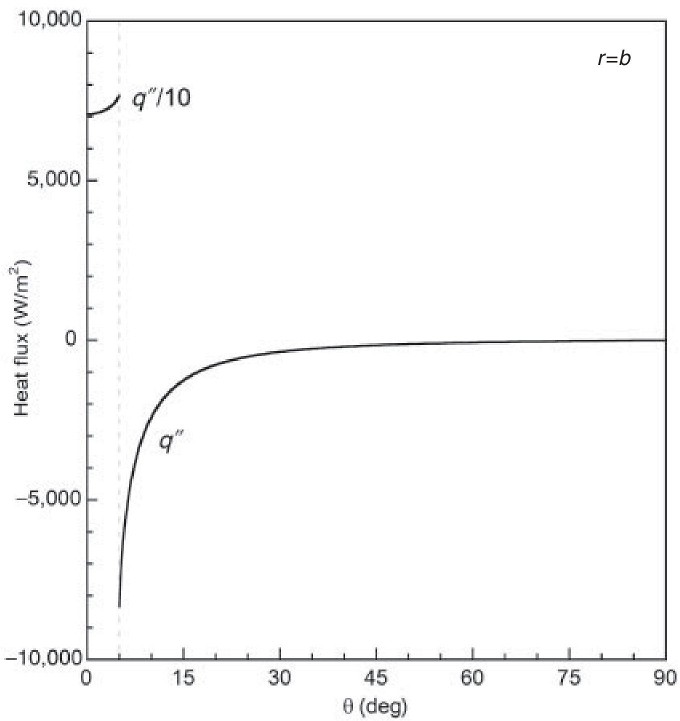

Figure 5-10 Surface heat flux ($r = b$) as a function of μ. Values within the zone of contact are scaled by a factor of 10.

where it decays monotonically from a value of -8335 W/m^2 at the boundary of the contact zone, to zero at the base of the hemisphere ($\theta = 90^{\circ}$, $\mu = 0$).

The heat flux entering the base is readily calculated using Fourier's law in the θ direction, given as

$$q''_{\theta} = -\frac{k}{r}\frac{\partial T}{\partial \theta} \tag{5-244}$$

With the change of variables to the μ coordinate, Fourier's law becomes

$$q''_{\mu} = \frac{k\left(1-\mu^2\right)^{1/2}}{r}\frac{\partial T}{\partial \mu} \tag{5-245}$$

Equation (5-245) is readily evaluated from equation (5-242) by simply differentiating the Legendre polynomials, noting that the integral terms (i.e., the Fourier coefficient terms) are unchanged. Evaluating equation (5-245) along the base of the hemisphere yields

$$q''_{\mu}(\mu=0) = \sum_{\substack{n=1\\ \text{odd}}}^{\infty} \frac{(2n+1)\,khr^{n-1}}{nkb^{n-1}+hb^n}\left.\frac{dP_n}{d\mu}\right|_{\mu=0}\left[\begin{array}{l}\left(T_3-T_1\right)\displaystyle\int_{\mu'=0}^{\mu_o} P_n\left(\mu'\right)\,d\mu' \\ +\left(T_2-T_1\right)\displaystyle\int_{\mu'=\mu_o}^{1} P_n\left(\mu'\right)\,d\mu'\end{array}\right] \tag{5-246}$$

The surface heat flux along the base of the hemisphere is plotted in Figure 5-11. As discussed above, convergence of the series summation in equation (5-246) becomes an issue near the outer surface ($r \sim b$). In fact, convergence was sufficient for $r < 0.99b$ using the first 150 terms, although convergence directly on the surface ($r = b$) was insufficient for 150 nonzero terms; hence we have extrapolated the final value in Figure 5-11 at $r = b$. We note that the negative values in Figure 5-11 reflect heat entering the sphere (i.e., in the negative μ direction). Overall, the heat flux entering the base of the hemisphere is observed to decay monotonically from a maximum flux of 282.5 W/m^2 to a flux of 44 W/m^2 at the edge of the base. The heat flux at $r = 0$ as calculated with equation (2-246) is in exact agreement with the heat flux calculated using

$$q''_r(r=0, \mu=1) = -k\left.\frac{\partial T}{\partial r}\right|_{\substack{r=0\\ \mu=1}} \tag{5-247}$$

noting that only the first term of the series contributes at the origin (r=0).

Finally, it is useful to consider conservation of energy under steady-state conditions, which may be expressed as

$$\dot{Q}_{\text{in}} = \dot{Q}_{\text{out}} \tag{5-248}$$

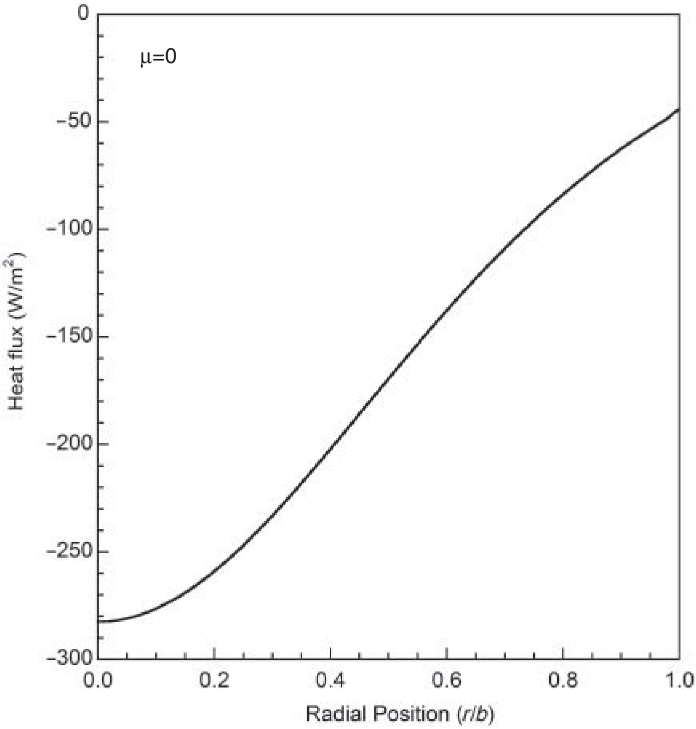

Figure 5-11 Heat flux entering the base of the hemisphere.

where the rate of energy entering the hemisphere must be balanced by the rate of energy leaving the sphere through the contact area with the solid at T_2. In view of the problem formulation, equation (5-248) may be expressed as

$$\int_{r=0}^{b} 2\pi r\, q''_{\mu}\big|_{\mu=0}\, dr + \int_{\theta=\theta_o}^{\pi/2} 2\pi b^2\, q''_r\big|_{r=b} \sin\theta d\theta = \int_{\theta=0}^{\theta_o} 2\pi b^2\, q''_r\big|_{r=b} \sin\theta d\theta \tag{5-249}$$

where we have set up the differential surface area over the surface $r = b$ in terms of the spherical coordinate θ. Rather than integrating the exact series summations analytically, the flux profiles of Figures 5-5 to 5-10 and 5-11 were integrated using the trapezoid rule, which yields

$$39.14 + 135.84 = 174.98 \quad \text{(mW)} \tag{5-250}$$

As observed with equation (5-250), conservation of energy is indeed realized, and it is observed that with respect to the $\sim$175 mW of heat leaving the zone of contact, about 23.4% enters the sphere through the base, with the remaining 77.6% being transferred via convection heat transfer from the surrounding fluid.

REFERENCES

1. H. S. Carslaw and J. C. Jaeger, *Conduction of Heat in Solids*, Clarendon Press, London, 1959.
2. M. N. Özişik, *Boundary Value Problems of Heat Conduction*, International Textbook, Scranton, PA, 1968.
3. A. V. Luikov, *Analytical Heat Diffusion Theory*, Academic, New York, 1968.
4. J. Crank, *The Mathematics of Diffusion*, Clarendon Press, London, 1957.
5. T. M. MacRobert, *Spherical Harmonics*, 3rd ed., Pergamon, New York, 1967.
6. E. T. Whitaker and G. N. Watson, *A Course of Modern Analysis*, Cambridge University Press, London, 1965.
7. W. E. Byerly, *Fourier's Series and Spherical, Cylindrical and Ellipsoidal Harmonics*, Dover, New York, 1959.

PROBLEMS

5-1 A hollow sphere $a \le r \le b$ is initially at temperature $T = F(r)$. For times $t > 0$, the boundary surface at $r = a$ is kept insulated, and the boundary at $r = b$ dissipates heat by convection with convection coefficient h into a medium at zero temperature. Obtain an expression for the temperature distribution $T(r, t)$ in the sphere for times $t > 0$.

5-2 A hollow sphere $a \le r \le b$ is initially at temperature $T = F(r)$. For times $t > 0$, the boundary surface at $r = a$ is maintained at a constant temperature of zero, and the boundary at $r = b$ dissipates heat by convection with convection coefficient h into a medium at zero temperature. Obtain an expression for the temperature distribution $T(r, t)$ in the sphere for times $t > 0$.

5-3 A hollow sphere $a \le r \le b$ is initially at temperature $T = F(r)$. For times $t > 0$, the boundary surface at $r = a$ dissipates heat by convection with convection coefficient h_1 into a medium at temperature T_∞, and the boundary at $r = b$ dissipates heat by convection with convection coefficient h_2 into a medium at temperature T_∞. Obtain an expression for the temperature distribution $T(r, t)$ in the sphere for times $t > 0$.

5-4 A solid sphere $0 \le r \le b$ is initially at temperature $T = F(r)$. For times $t > 0$, the boundary surface at $r = b$ is maintained at a constant temperature of T_1. Obtain an expression for the temperature distribution $T(r, t)$ in the sphere for times $t > 0$.

5-5 A solid sphere $0 \le r \le b$ is initially at temperature $T = F(r)$. For times $t > 0$, the boundary surface at $r = b$ is maintained at a constant temperature of T_1. In addition, the sphere is subjected to uniform internal energy generation g_0 (W/m^3). Obtain an expression for the temperature distribution $T(r, t)$ in the sphere for times $t > 0$.

5-6 A hemisphere $0 \le r \le b$, $0 \le \mu \le 1$ is maintained at steady-state conditions with a prescribed surface temperature $T(r = b) = f(\mu)$, and with the base ($\mu = 0$) maintained at a constant temperature of T_1. Obtain an expression for the steady-state temperature distribution $T(r, \mu)$.

5-7 A solid sphere $0 \le r \le b$ is maintained at steady-state conditions with a prescribed surface temperature $T(r = b) = f(\mu)$. Obtain an expression for the steady-state temperature distribution $T(r, \mu)$.

5-8 A hollow sphere $a \le r \le b$ is initially at temperature $T = F(r)$. For times $t > 0$, the boundary surface at $r = a$ is maintained at temperature T_1, and the boundary surface at $r = b$ is maintained at temperature T_2. In addition, the sphere is subjected to uniform internal energy generation g_0 (W/m^3). Obtain an expression for the temperature distribution $T(r, t)$ in the sphere for times $t > 0$.

5-9 A solid sphere $0 \le r \le b$ is maintained at steady-state conditions. The surface at $r = b$ is subjected to convection heat transfer with convection coefficient h and with fluid temperature $T_\infty = f(\mu, \phi)$. Obtain an expression for the steady-state temperature distribution $T(r, \mu, \phi)$.

5-10 A hemisphere $0 \le r \le b$, $0 \le \mu \le 1$ is maintained at steady-state conditions. The surface at $r = b$ is maintained at a prescribed temperature $T(r = b) = f(\mu, \phi)$, while the base is maintained at zero temperature. Obtain an expression for the steady-state temperature distribution $T(r,\mu,\phi)$. See note 2.

5-11 A solid sphere $0 \le r \le b$, $-1 \le \mu \le 1$ is maintained at steady-state conditions. The surface at $r = b$ is subjected to a prescribed heat flux $f(\mu, \phi)$. Obtain an expression for the steady-state temperature distribution $T(r,\mu,\phi)$.

5-12 A hemisphere $0 \le r \le b$, $0 \le \mu \le 1$ is maintained at steady-state conditions. The surface at $r = b$ is subjected to convection heat transfer with convection coefficient h and with fluid temperature $T_\infty = f(\mu, \phi)$, while the base is maintained at zero temperature. Obtain an expression for the steady-state temperature distribution $T(r,\mu,\phi)$. See note 2.

5-13 A hemisphere $0 \le r \le b$, $0 \le \mu \le 1$ is maintained at steady-state conditions. The surface at $r = b$ is subjected to a prescribed heat flux $f(\mu, \phi)$, while the base is maintained at zero temperature. Obtain an expression for the steady-state temperature distribution $T(r,\mu,\phi)$. See note 2.

5-14 A hemisphere $0 \le r \le b$, $0 \le \mu \le 1$ is initially at temperature $T = F(r, \mu)$. For times $t > 0$, the boundary at the spherical surface $r = b$ is maintained at zero temperature, and at the base $\mu = 0$ is perfectly insulated. Obtain an expression for the temperature distribution $T(r,\mu,t)$ for times $t > 0$.

5-15 A solid sphere $0 \le r \le b$, $-1 \le \mu \le 1$ is initially at temperature $T = F(r, \mu)$. For times $t > 0$, the boundary at the surface $r = b$ dissipates heat by convection with convection coefficient h into a medium at temperature T_∞. Obtain an expression for the temperature distribution $T(r,\mu,t)$ for times $t > 0$.

5-16 A solid sphere $0 \le r \le b$, $-1 \le \mu \le 1$, $0 \le \phi \le 2\pi$ is initially at temperature $T = F(r, \mu, \phi)$. For times $t > 0$, the boundary at the surface $r = b$ dissipates heat by convection with convection coefficient h into a medium at temperature T_∞. Obtain an expression for the temperature distribution $T(r,\mu,\phi,t)$ for times $t > 0$.

5-17 A solid sphere $0 \le r \le b$, $-1 \le \mu \le 1$, $0 \le \phi \le 2\pi$ is initially at temperature $T = F(r, \mu, \phi)$. For times $t > 0$, the boundary at the surface $r = b$ is maintained at temperature T_1. In addition, the sphere is subjected to uniform internal energy generation g_0 (W/m^3). Obtain an expression for the temperature distribution $T(r, \mu, \phi, t)$ for times $t > 0$.

NOTES

1. For the fully insulated sphere in Example 5-4, we consider equation (5-75) for the special case of $\lambda = 0$, which yields in the r dimension

$$\frac{1}{R(r)} \frac{d^2 R(r)}{dr^2} = 0 \tag{1}$$

This ODE yields a solution of the form

$$R(r) = C_0 r + C_1 \tag{2}$$

which in view of BC1, namely,

$$\text{BC1:} \qquad U(r = 0) = 0 \quad \rightarrow \quad R(r = 0) = 0 \tag{3}$$

yields $C_1 = 0$. We then have $R_0(r) = C_0 r$ for the $U(r, t)$ problem, or simply $R_0(r) = C_0$ after transforming back to $T(r, t)$, which is consistent with our addition of C_0 to equation (5-81).

2. The associated Legendre polynomials, $P_n^m(\mu)$ and $P_k^m(\mu)$, are orthogonal over the interval $0 \le \mu \le 1$ for both n and k even integers, or for both n and k odd integers, for a given integer value of m. For integer values of m and n, $P_n^m(0) = 0$ for $m = n + 1$.