Draft version September 25, 2025
Typeset using LaTeX **RNAAS** style in AASTeX631

Bifurcation and Chaos in the Logistic Map

Lily Nguyen[1] and Raymond Vu[2]

[1] *Department of Physics, The University of Texas at Austin*
Austin, TX 78712, USA
[2] *Department of —, The University of Texas at Austin*
Austin, TX 78712, USA

ABSTRACT

The logistic map is a simple nonlinear difference equation that displays various behaviors ranging from stable fixed points to periodic oscillations and chaotic dynamics. In this project, we generated the bifurcation diagram of the logistic map over the parameter range $2.4 < \mu < 4.0$. To reduce dependence on initial conditions, we followed the standard approach of discarding transient iterations and recording the long-term dynamics of x for each μ. The resulting diagram illustrates the progression from stability to successive period-doubling behavior and the eventual onset of chaos. The transition occurred near $\mu \approx 3.57$, consistent with the universal period-doubling scenario first described by Feigenbaum Feigenbaum (1978) and later confirmed in numerical studies Chen et al. (2021). Overall, our results align with the mathematical analysis of the logistic map Bubolo (2018) and highlight the importance of visualization for understanding nonlinear systems Boeing (2016).

Keywords: logistic map — bifurcation diagram — nonlinear dynamics — chaos theory — period-doubling bifurcation

INTRODUCTION

The concept of logistic growth was first introduced by Pierre-François Verhulst in the 19th century to describe how populations stabilize as they approach a maximum carrying capacity Schtickzelle (1981). Although not widely recognized at the time, his work laid the foundation for later studies of population dynamics and mathematical modeling.

The modern study of chaos developed much later in the 20th century. Edward Lorenz, often described as the father of chaos theory, found that small changes of initial conditions in a simple model of atmospheric convection could lead to completely different outcomes, making long-term weather prediction fundamentally impossible Boeing (2016).

In 1976, Robert May showed that the discrete logistic map, despite its simplicity, could exhibit oscillations and even chaotic behavior May (1976). Mitchell Feigenbaum later demonstrated that this transition to chaos follows a period-doubling route, with bifurcations accumulating near $\mu \approx 3.57$ Feigenbaum (1978). Together, these discoveries established the logistic map as a central model in chaos theory.

Today, the logistic map remains a key tool in nonlinear dynamics. Its mathematical structure has been studied in detail Bubolo (2018), and its bifurcation diagram provides a clear visualization of how order gives way to chaos Boeing (2016). As such, the logistic map continues to demonstrate how simple deterministic rules can generate complex and unpredictable behavior.

DATA AND OBSERVATIONS

The logistic map is a nonlinear recurrence relation defined by

$$x_{n+1} = \mu x_n (1 - x_n) \tag{1}$$

where $0 < x_n < 1$ is the normalized state variable at the iteration n and μ is the control parameter with $0 < \mu < 4$. The goal of this project is to generate the bifurcation diagram of the logistic map for the parameter range $2.4 \leq \mu \leq 4$ to study the transition from stability to chaos. The diagram shows the long-term values of x_n for each μ after discarding transient behavior, revealing fixed points, periodic oscillations, and chaotic dynamics.

To compute the diagram, we varied μ uniformly between 2.4 and 4.0 with a resolution of 1000 distinct values. For each μ, we initialized the map with $x_0 = 0.5$ and iterated 400 times. We discarded the first 200 iterations as transients and recorded the final 200 iterations to represent long-term behavior. We binned the results to produce a clear density plot of the bifurcation diagram. The resulting figure (Figure 1) displays the bifurcation diagram of the logistic map. Stable fixed points appear at low values of μ, followed by successive period-doubling bifurcations, and eventually the onset of chaos near $\mu \approx 3.57$.

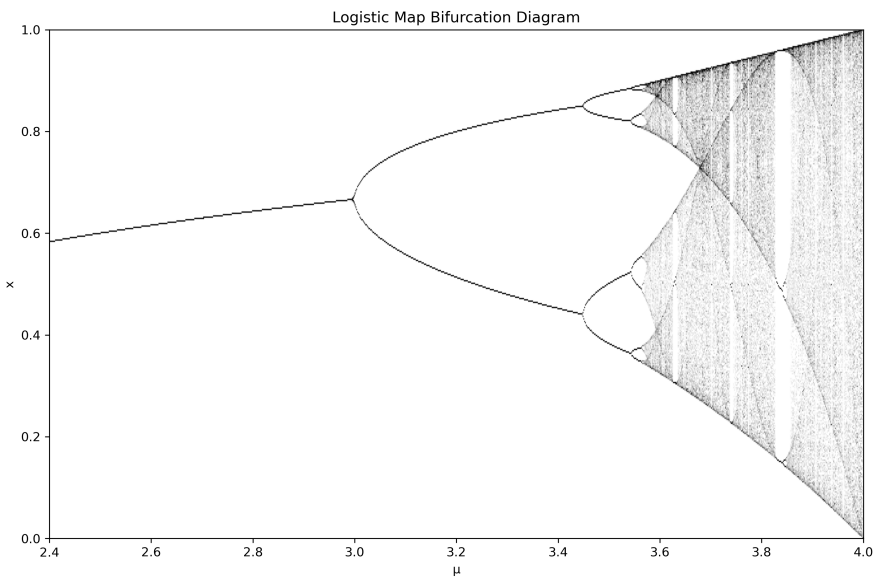

Figure 1. The logistic map bifurcation diagram for $2.4 \leq \mu \leq 4.0$. Stable fixed points exist for low values of μ, followed by successive period-doubling bifurcations that lead to chaotic behavior.

RESULTS

The bifurcation diagram generated for $2.4 \leq \mu \leq 4.0$ illustrates the transition from stability to chaos in the logistic map. For small values of μ, the system converges to a stable fixed point. As μ increases beyond $\mu \approx 3.0$, the fixed point becomes unstable through a period-doubling bifurcation, leading to oscillations of period two. Further increases in μ cause successive period doublings, with period four, eight, and higher powers of two.

These bifurcations accumulate near $\mu \approx 3.57$, the onset of chaos Feigenbaum (1978); Chen et al. (2021). Within the chaotic region, the diagram also reveals windows of periodicity, demonstrating that ordered behavior can re-emerge even within chaos.

Potential sources of error include the finite resolution in μ, which limits how precisely the bifurcation points can be located, as well as the limited number of iterations used, which constrains the model's accuracy in chaotic regions. Despite these limitations, the bifurcation diagram reproduced the expected features with enough clarity to capture the route from stability to chaos.

SUMMARY AND CONCLUSION

This project demonstrated the transition from stability to chaos in the logistic map by generating its bifurcation diagram. Our results align with theoretical predictions, showing successive period doublings and the onset of chaos near $\mu \approx 3.57$. Our findings confirm the logistic map as an accepted model for illustrating how a deterministic equation can generate highly complex and unpredictable behavior.

Beyond reproducing known features, this project highlights the value of visualization in understanding nonlinear dynamics. Future work may include calculating the Lyapunov exponent to quantify chaos or comparing the logistic map with continuous systems such as the Lorenz attractor. Investigating the fractal geometry could also deepen our understanding of the the scaling behavior found in chaotic systems.

Acknowledgements

We thank Professor Mitra for his guidance and instruction throughout the course, as well as the teaching assistants for their helpful feedback. We also acknowledge the Department of Physics for providing the broader academic foundation to support this work. This paper was completed as part of the requirements for C S 323E: Elements of Scientific Computing at the University of Texas at Austin.

REFERENCES

Boeing, G. 2016, Systems, 4, 37, doi: 10.3390/systems4040037

Bubolo, M. 2018, The Dynamics of the Logistic Map and Difference Equations

Chen, S., Feng, S., Fu, W., & Zhang, Y. 2021, Journal of Physics: Conference Series, 2014, 012009, doi: 10.1088/1742-6596/2014/1/012009

Feigenbaum, M. 1978, J. Stat. Phys., 19, 25, doi: 10.1007/BF01020332

May, R. 1976, Nature, 26, 457, doi: 10.1038/261459a0

Schtickzelle, M. 1981, Population (French Edition), 36, 541, doi: 10.2307/1532327