Draft version October 21, 2025
Typeset using LaTeX **RNAAS** style in AASTeX631

Bessel Functions: Theory and Applications

Lily Nguyen[1] and Andre Sae[1]

[1] *Department of Physics, The University of Texas at Austin*
Austin, TX 78712, USA

ABSTRACT

We go over three physical scenarios where Bessel Functions are used. The first scenario is the infinite square well quantum mechanical wave function in spherical coordinates. The second example scenario involves solving the temperature equation for thermal diffusion through a material. The final example models the vibration along the drum head right after it is struck in the middle.

Keywords: word 1 (1) — word 2 (2) — word 3 (3) — word 4 (4)

INTRODUCTION

For our applications we go over three different physical scenarios where the conversion of coordinate from cartesian to the coordinate with a radial dependency has solutions that contain a Bessel function. The key point between each scenario is that the radial dependency in the new coordinate systems transforms the second order differential equation whose solution is sinusoidal with a damping term or critical damping into a Bessel function along the radial basis. The Bessel function decays to 0 as the input parameter for each physical situation radial basis moves toward infinity. However, the physical model does resemble the exponential decay pattern found in the physical problems such as a spring in fluid simulation.

DATA AND OBSERVATIONS

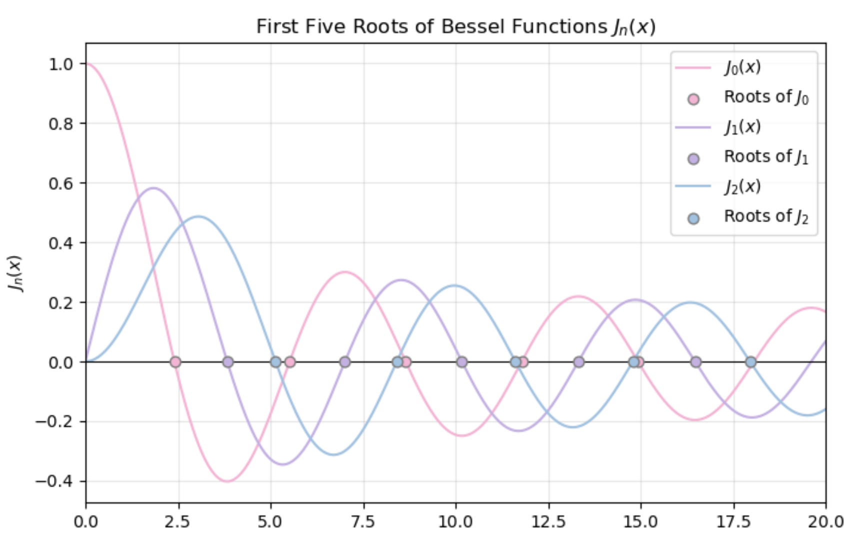

Figure 1. some caption here

RESULTS

APPLICATIONS

Quantum Mechanics: The Infinite Square Well

The Bessel function appears in the solution to the wave function of an infinite square well in spherical coordinates. To solve the quantum mechanics problem, we define a potential well:

$$V(x) = \begin{cases} 0, & \text{if } r < a \\ \infty, & \text{if } r \geq a \end{cases} \tag{1}$$

The Hamiltonian operator $\hat{H}$ and energy operator $\hat{E}$ must also be defined to compute the wavefunction. The n and i are quantum numbers where n is the principle quantum number and I is the angular momentum quantum number where both span the integer range from zero to infinity. $\hbar$ is Planck's constant, m is the mass of the particle, r is the radius, and $R_{n,l}$ is the wavefunction in the radial basis.

$$\hat{H} = \frac{-\hbar^2}{2m}\left(\frac{\partial}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{l(l+1)}{r^2}\right) + V(r) \tag{2}$$

$$\hat{H}\psi - E\psi = \hat{H}R_{n,l} - ER_{n,l} = 0 \tag{3}$$

Applying the Hamiltonial operator on the wavefunction gives us the spherical Bessel function in differential form, which looks similar to the origianl Bessel function in differential form:

$$r^2\frac{\partial R_{n,l}}{\partial r^2} + 2r\frac{\partial R_{n,l}}{\partial r} + (k^2r^2 - l(l+1))R_{n,l} = 0 \tag{4}$$

To obtain a solution, we impose a boundary condition $j_l(k_{n,l}a) = 0$ at the bounds of the well. This quantizes our solution by setting a discrete energy level $E_{n,l} = \frac{\hbar^2 k_{n,l}^2}{2ma^2}$ where the solution exists. Hence, we are left with a radial solution to the wave equation containing the original Bessel Function:

$$R_{n,l} = Aj_l(k_{n,l}r) \tag{5}$$

where $j_l(k_{n,l}, r) = \sqrt{\frac{\pi}{2kr}}J_{l+\frac{1}{2}}(k_{n,l}R)$.

Thermal Diffusion

We define the heat equation propagating through a two-dimensional surface in the radial direction. T is the temperature, t is the time, r is the radius, k is the material conductivity, ρ is the density of the material, and c_p is the specific heat capacity.

$$\frac{\partial T}{\partial t} = \alpha\nabla^2 T \tag{6}$$

$$= \alpha\left(\frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial r^2}\right) \tag{7}$$

where $\alpha = \frac{k}{\rho c_p}$.

To solve this equation, we use separation of variables to split the differential equation into two separate differential equations and assume that one takes the form $T(r,t) = X(r)\theta(t)$. The first equation takes the form of exponential decay over time:

$$\frac{d\theta}{dt} + \lambda^2 \alpha \theta = 0 \tag{8}$$

whereas the second equation is the Bessel function of the zeroth-order since $\alpha = 0$:

$$\frac{d^2X}{dr^2} + \frac{1}{r}\frac{X}{r} + \lambda^2 X = 0. \tag{9}$$

We set a boundary condition of $T(r,t) = T_1$, where R is the radius of the circle, t is time with $t > 0$, and T_1 is a constant real value. The initial condition for the radial space is defined as $T(r,0) = T_2$, where T_2 is a constant real value.

$$T^* = \frac{T(r,t) - T(R,t)}{T(r,0) - T(R,t)} = 2\sum_{n=0}^{\infty} e^{-\beta_n^2 \frac{\alpha t}{R^2}} \frac{J_0(\beta_n \frac{r}{R})}{\beta_n J_1(\beta_n)} \tag{10}$$

$$T(r,t) = T^*(T(r,0) - T(R,t)) + T(R,t) \tag{11}$$

We define temperature as a unitless quantity T^*. β_n is the nth-root solution where $J_0(r) = 0$. The temperature function uses both the zeroth and first order Bessel function in the solution. Hence, Equation 10 and Equation 11 is the final solution for that the temperature takes.

Drum Wave Propagation

Drum wave propagation is similar to thermal diffusion regarding classical wave mechanics and quantum mechanics through the quantization of boundary conditions. We define a wavefunction for the propagation across a drum surface where σ is the surface mass density of the membrane and S is the surface tension across the membrane:

$$\frac{\partial^2 z}{\partial t^2} = c^2 \nabla^2 z \tag{12}$$

where $c^2 = \frac{\sigma^2}{S}$.

The first boundary condition is $z(R_f, t) = 0$. The displacement from the origin along the z-axis must be zero at the edge of the drum since those points are fixed. This condition quantizes the solution, yielding discrete Bessel functions that satisfy the boundary conditions. To simplify the example, we impose the boundary condition $z(r,\theta,0) = f(r)$ for $0 \leq r \leq a$, representing the initial pertubation caused by striking the surface at its center. This approach preserves the θ-dependence in the solution, allowing the Bessel function to appear explicitly in the radial component.

$$z(r,\theta,t) = R(r)T(t)\Theta(\theta) = R(r)T(t) \tag{13}$$

$$r^2\frac{\partial^2 R}{\partial r^2} + r\frac{\partial R}{\partial r} + (\lambda^2 r^2 - n^2)R = 0 \tag{14}$$

$$\frac{dT}{dt} + \lambda^2 c T = 0 \tag{15}$$

$$n = \lambda_{m,k} \tag{16}$$

$$z(r,t) = \sum_{m=0}^{\infty}\sum_{k=1}^{\infty} J_m(\lambda_{m,k} r) e^{-c\lambda_{m,n} t} \tag{17}$$

Finally, impose a boundary condition enforcing zero displacement at the drum's boundary:

$$J_m(\lambda_{m,k} R_f) = 0, \tag{18}$$

where m is the order of the Bessel function and k is the wavenumber. For $m = 0$, the Bessel function itself vanishes at the boundary. There exist infinitely many integer values of m and k for which this condition is satisfied, corresponding to the distinct vibration modes of the drumhead. The resulting function $z(r, t)$ describes the final waveform of the drum's vibration in time.

SUMMARY AND CONCLUSION

Acknowledgements

We thank Professor Mitra for his guidance and instruction throughout the course, as well as the teaching assistants for their helpful feedback. We also acknowledge the Department of Physics for providing an excellent academic foundation to support this work. This work was completed as part of the requirements for C S 323E: Elements of Scientific Computing at the University of Texas at Austin.

REFERENCES