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Bessel Functions: Theory and Applications

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ABSTRACT

We go over three physical scenarios where Bessel Functions are used. The first scenario is the infinite square well quantum mechanical wave function in spherical coordinates. The second example scenario involves solving the temperature equation for thermal diffusion through a material. The final example models the vibration along the drum head right after it is struck in the middle.

Keywords: word 1 (1) — word 2 (2) — word 3 (3) — word 4 (4)

INTRODUCTION

The Bessel functions of the first kind, $J_n(x)$, are solutions to the second-order linear differential equation:

$$z^2 \frac{d^2w}{dz^2} + z\frac{dw}{dz} + (z^2 - n^2)w = 0, \tag{1}$$

where integer $n = 0, 1, 2 \ldots$ represents the order of the function. These functions appear naturally in the separation of variables when solving partial differential equations in cylindrical or spherical coordinates, particularly in systems with radial symmetry. Bessel functions also have an integral

$$J_n(x) = \frac{1}{\pi}\int_0^\pi cos(x\sin\theta - n\theta)d\theta \tag{2}$$

for integers $n = 0, 1, 2 \ldots$, which is particularly useful for understanding their oscillatory behavior.

Bessel functions were first introduced by German astoronomer and mathematician Friedrich Wilhelm Bessel in the early 1800s during his study of planetary orbits. However, the equation itself had been investigated earlier by Bernoulli and Euler in problems involving vibrational membranes Abramowitz & Stegun (1972). Today, Bessel functions are foundational tools in mathematical physics and engineering. In quantum mechanics, they appear in radial solutions to the Schrödinger equation for quantum wells CITE, in the vibration modes of circular membranes like drumheads CITE, and in heat conduction problems with cylindrical geometries CITE.

In this project, we numerically compute the first five positive roots of the Bessel functions $J_0(x)$, $J_1(x)$, and $J_2(x)$. These roots correspond to physically meaningful quantities such as resonance modes, cutoff frequencies, or quantized boundary values in radial systems. We use Python to visualize each function, estimate locations of roots from their plots, and apply the `fsolve` method from `scipy.optimize` to define each root with high precision.

DATA AND OBSERVATIONS

This project focuses on computing and visualizing the first five roots of the Bessel functions of the first kind, $J_n(x)$, for orders $n = 0, 1, 2$. These functions solve the second-order linear differential equation:

$$J_n(x) = \frac{1}{\pi}\int_0^\pi cos(x\sin\theta - n\theta)d\theta, \tag{3}$$

which arise in physical systems with spherical or cylindrical symmetry. The positive roots of $J_n(x)$ represent physically meaningful quantities such as resonant frequencies or quantized energy levels in such systems.

We computed the Bessel fucntions using Python's `scipy.special.jv` method, which evaluates $J_n(x)$ for an arbitrary order and argument. To find the roots numerically, we used the `fsolve` method from `scipy.optimize`, which refines an initial guess until it finds a point where the function crosses zero. We chose a set of initial guesses based on where the function appeared to cross the x-axis in the plot and passed those values into `fsolve` to calculate each root more precisely. We use the following initial guesses:

- $J_0(x)$: [2, 6.1, 8.6, 11.7, 15]
- $J_1(x)$: [3.9, 7, 10.15, 13.1, 16.4]
- $J_2(x)$: [5.1, 8.3, 11.8, 14.9, 18]

This gave us exactly five positive roots for each Bessel function. The resulting plots of $J_0(x)$, $J_1(x)$, and $J_2(x)$ over the domain $x = 0$ to $x = 20$ are shown in Figure 1, with the first five roots of each function marked as scatter points.

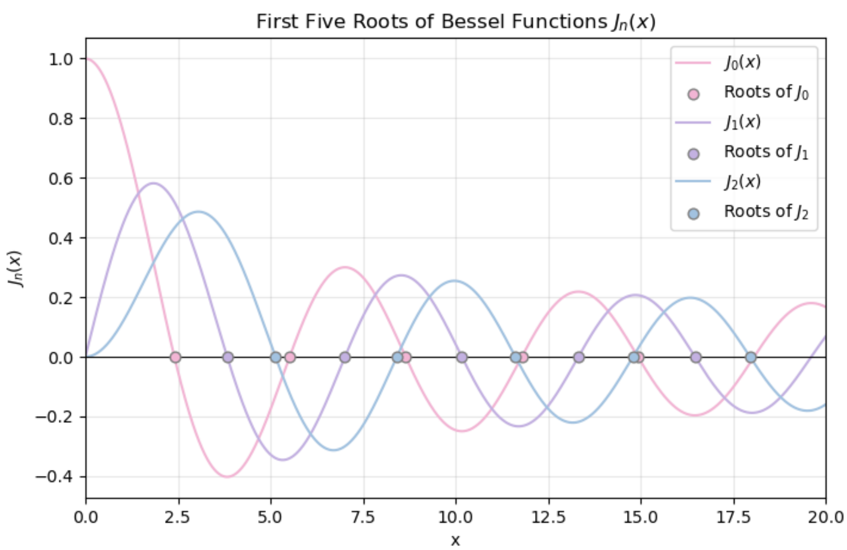

Figure 1. Bessel functions $J_0(x)$, $J_1(x)$, and $J_2(x)$ plotted from $x = 0$ to $x = 20$, with their first five positive roots represented as circular markers.

RESULTS

APPLICATIONS

Quantum Mechanics: The Infinite Square Well

The Bessel function appears in the solution to the wave function of an infinite square well in spherical coordinates. To solve the quantum mechanics problem, we define a potential well:

$$V(x) = \begin{cases} 0, & \text{if } r < a \\ \infty, & \text{if } r \geq a \end{cases} \tag{4}$$

The Hamiltonian operator $\hat{H}$ and energy operator $\hat{E}$ must also be defined to compute the wavefunction. The n and i are quantum numbers where n is the principle quantum number and I is the angular momentum quantum number where both span the integer range from zero to infinity. $\hbar$ is Planck's constant, m is the mass of the particle, r is the radius, and $R_{n,l}$ is the wavefunction in the radial basis.

$$\hat{H} = \frac{-\hbar^2}{2m} \left(\frac{\partial}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{l(l+1)}{r^2} \right) + V(r) \tag{5}$$

$$\hat{H}\psi - E\psi = \hat{H}R_{n,l} - ER_{n,l} = 0 \tag{6}$$

Applying the Hamiltonial operator on the wavefunction gives us the spherical Bessel function in differential form, which looks similar to the origianl Bessel function in differential form:

$$r^2\frac{\partial R_{n,l}}{\partial r^2} + 2r\frac{\partial R_{n,l}}{\partial r} + (k^2r^2 - l(l+1))R_{n,l} = 0 \tag{7}$$

To obtain a solution, we impose a boundary condition $j_l(k_{n,l}a) = 0$ at the bounds of the well. This quantizes our solution by setting a discrete energy level $E_{n,l} = \frac{\hbar^2 k_{n,l}^2}{2ma^2}$ where the solution exists. Hence, we are left with a radial solution to the wave equation containing the original Bessel Function:

$$R_{n,l} = Aj_l(k_{n,l}r) \tag{8}$$

where $j_l(k_{n,l}, r) = \sqrt{\frac{\pi}{2kr}}J_{l+\frac{1}{2}}(k_{n,l}R)$.

Thermal Diffusion

We define the heat equation propagating through a two-dimensional surface in the radial direction. T is the temperature, t is the time, r is the radius, k is the material conductivity, ρ is the density of the material, and c_p is the specific heat capacity.

$$\frac{\partial T}{\partial t} = \alpha\nabla^2 T \tag{9}$$

$$= \alpha\left(\frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial r^2}\right) \tag{10}$$

where $\alpha = \frac{k}{\rho c_p}$.

To solve this equation, we use separation of variables to split the differential equation into two separate differential equations and assume that one takes the form $T(r,t) = X(r)\theta(t)$. The first equation takes the form of exponential decay over time:

$$\frac{d\theta}{dt} + \lambda^2\alpha\theta = 0 \tag{11}$$

whereas the second equation is the Bessel function of the zeroth-order since $\alpha = 0$:

$$\frac{d^2X}{dr^2} + \frac{1}{r}\frac{X}{r} + \lambda^2 X = 0. \tag{12}$$

We set a boundary condition of $T(r,t) = T_1$, where R is the radius of the circle, t is time with $t > 0$, and T_1 is a constant real value. The initial condition for the radial space is defined as $T(r,0) = T_2$, where T_2 is a constant real value.

$$T^* = \frac{T(r,t) - T(R,t)}{T(r,0) - T(R,t)} = 2\sum_{n=0}^{\infty} e^{-\beta_n^2\frac{\alpha t}{R^2}}\frac{J_0(\beta_n\frac{r}{R})}{\beta_n J_1(\beta_n)} \tag{13}$$

$$T(r,t) = T^*(T(r,0) - T(R,t)) + T(R,t) \tag{14}$$

We define temperature as a unitless quantity T^*. β_n is the nth-root solution where $J_0(r) = 0$. The temperature function uses both the zeroth and first order Bessel function in the solution. Hence, Equation 13 and Equation 14 is the final solution for that the temperature takes.

Drum Wave Propagation

Drum wave propagation is similar to thermal diffusion regarding classical wave mechanics and quantum mechanics through the quantization of boundary conditions. We define a wavefunction for the propagation across a drum surface where σ is the surface mass density of the membrane and S is the surface tension across the membrane:

$$\frac{\partial^2 z}{\partial t^2} = c^2 \nabla^2 z \tag{15}$$

where $c^2 = \frac{\sigma^2}{S}$.

The first boundary condition is $z(R_f, t) = 0$. The displacement from the origin along the z-axis must be zero at the edge of the drum since those points are fixed. This condition quantizes the solution, yielding discrete Bessel functions that satisfy the boundary conditions. To simplify the example, we impose the boundary condition $z(r, \theta, 0) = f(r)$ for $0 \leq r \leq a$, representing the initial pertubation caused by striking the surface at its center. This approach preserves the θ-dependence in the solution, allowing the Bessel function to appear explicitly in the radial component.

$$z(r, \theta, t) = R(r)T(t)\Theta(\theta) = R(r)T(t) \tag{16}$$

$$r^2 \frac{\partial^2 R}{\partial r^2} + r \frac{\partial R}{\partial r} + (\lambda^2 r^2 - n^2)R = 0 \tag{17}$$

$$\frac{dT}{dt} + \lambda^2 c T = 0 \tag{18}$$

$$n = \lambda_{m,k} \tag{19}$$

$$z(r, t) = \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} J_m(\lambda_{m,k} r) e^{-c\lambda_{m,n} t} \tag{20}$$

Finally, impose a boundary condition enforcing zero displacement at the drum's boundary:

$$J_m(\lambda_{m,k} R_f) = 0, \tag{21}$$

where m is the order of the Bessel function and k is the wavenumber. For $m = 0$, the Bessel function itself vanishes at the boundary. There exist infinitely many integer values of m and k for which this condition is satisfied, corresponding to the distinct vibration modes of the drumhead. The resulting function $z(r, t)$ describes the final waveform of the drum's vibration in time.

SUMMARY AND CONCLUSION

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