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# Logistic Map: Stability and Entrance to Chaos

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# Logistic Map: Stability and Entrance to Chaos

**Shaoqiu Chen[1,*], Shiya Feng[2], Wenjun Fu[3], Yingying Zhang[4]**

[1]Shenzhen Middle School, Shenzhen, 518001, China
[2] St.Margaret's School, Victoria, V8X 3P7, Canada
[3]Chengdu Shude High School, Chengdu, 610000, China
[4] Math faculty, University College London, London WC1E 6BT, UK, zcahhack@ucl.ac.uk
*Corresponding Author Email: stevenqiu_8@qq.com

**Abstract:** Chaos and nonlinear dynamics have taken a crucial place in the mathematics, physics, and engineering worlds. The main focus of this paper is about one famous map in the dynamical system that has an extreme sensitivity to the initial conditions, the logistic map.We first discuss the behaviours of the logistic map under different μ: convergence to 0 when $\sqrt{\mu} \in (0,1)$, convergence to $1-1/\mu$ when $\mu \in (1,3)$, 2-cycle when $\mu \in (3,1+6)$, further period doubling and eventual chaos, which is in good accordance with our simulation. In the end, we proved three relevant results: the criteria for stability of cycle, the Coppel Theorem, and the famous slogan "period three implies chaos."

## 1. Introduction

### *1.1 Difference Equation and Chaos*

The study of difference equations leads us to an advanced understanding of differential equations. Through analyzing difference equations, the behaviours of differential equations gradually become clear. The difference equations are the discretization of the differential equations. The Poincar´e Map, which is one of the famous maps in dynamical systems, implies that the pendulum's behaviours are proved by the difference equations [1]. The Poincar´e Map is not the only one using the concept of difference equations; many other mathematical problems are also solved by the difference equations [2]. Thus, the study of difference equations would prove to be a powerful tool in understanding differential equations and the operation of the natural world.

The concept of nonlinear dynamics and chaos has taken a crucial place in the mathematics, engineering and physics worlds. It is true that linear systems have already built a complete theorem in nowadays, and they have simple, understandable notion that are easy to study. However, unlike linear system that have closed-form solutions, nonlinear phenomena are more complicated and abstract [3]. A small difference or a slight change in the initial condition in stable linear systems will only result in small differences in input; however, a phenomenon called chaos will eventually appear in a nonlinear system by that small change. The sensitivity of initial conditions and the unpredictable outputs are the essential characteristics of a nonlinear system. The dropping water from a pipe, or the spread of an infectious disease: nonlinearity is closely related to nature. More fundamentally, our world is nonlinear. And in this world, even a simple nonlinear map could have complicated dynamics. Robert May made a plea in his paper, "an evangelical plea for the introduction of these difference equations into elementary mathematics courses, so that students intuition may be enriched by seeing the wild things that simple nonlinear equations could do" [4]. If we could introduce some difference equations for students to learn, this might change the way they see the world.

### *1.2 Properties of Logistic Map*

One of the examples of a simple nonlinear map that leads to complex behavior and have complicated dynamics is logistic map. It is written as:

$$u_{n+1} = \mu u_n(1 - u_n) \tag{1}$$

This equation defines the rules and dynamics in a population system: where u represents the population at any given time, $\mu$ represents the growth rate, and $1-u_n$ represents the limit given by the environment. The quantity u in $1 - u_n$ also represents the percentage of the theoretical maximum, so the range of the value of $u_n$ is 0 to 1. And this term tends to 0 when the quantity tends to the maximum, so that the quantity can be suppressed. With such an equation, people can observe the fluctuating changes of boom and bust of the population in a certain area at a certain time, in order to better use resources for sustainable development.

In 1976, the biologist Robert May wrote an engrossing and powerful paper in Nature about the Logistic Map [4]. And it sparked a revolution of the analysis of dynamic and gradually developed into the vast dynamics mathematical system it is today. The logistic map becomes one of the most famous maps in dynamical system theorem and chaos. It was originally used to describe the population growth of the world as time passes under a limitation based on a very common S-shaped curve function. And now Logistic Map can be used to simulate many natural processes. The logistic function uses a differential equation that treats time as continuous. The logistic map instead uses a nonlinear difference equation to look at discrete time steps. It's called the logistic map because it maps the population value at any time step to its value at the next time step. And this kind of simple equation shows a complex nature as the value of $\mu$ changes. It eventually leads to chaos once $\mu$ exceeds a specific value.

The exploration of logistic map is far from stopping. People try to graph the logistic map, that is, with the growth rate $\mu$ as the x-axis and the equilibrium number (the number of many generations of population) as the y-axis. At the very beginning when one substitutes a smaller $\mu$, the population becomes extinct, and the equilibrium value is 0. When $\mu$ passes through 1, the population reaches an equilibrium value of 1, and this equilibrium value increases as $\mu$ increases. So far, all of this is quite consistent with common sense. But the results of the next exploration set off a shocking wave in the mathematical world. After $\mu$ has gone through 3, the image splits into two. No matter how many times people iterated it, the resulting number would not be based on a constant; instead, the resulting number would take values back and forth between the two numbers. One year's population will be higher, and another year's population will be slightly lower, thus creating a cycle. Also, such population numbers were once verified in nature. What is even more bizarre is that as the growth rate $\mu$ increases, the bifurcation reappears again and again with new bifurcations, and more multiplicative cycle bifurcations occur and happen faster and faster. The cycles become 8, 16, 32, 64.... It was not until after $\mu$ surpassed 3.57 that the fascinating chaos emerged and the population no longer tended to any number, it moved as if at random. And when people look closely at this bifurcation diagram, they find that it is actually a fractal, a part of the Mandelbrot set.

In 1975, Feigenbaum calculated with an HP-65 calculator that the rate of difference between the parameters when such period-doubling bifurcations occur is a constant, for which he provided a mathematical proof [5]. He further revealed that the same phenomenon and the same constant apply to a wide range of mathematical functions, and this general conclusion enabled mathematicians to take the first step on the road to deciphering chaotic systems. This "rate of convergence" is now commonly referred as Feigenbaum's constant, and in 1978 he published his important paper on the study of mappings, "Quantitative Universality for a Class of Quantitative Universality for a Class of Nonlinear Transformations" [5], in which he discussed, in particular, logistic map.

But this was not the end of the inquiry, it was just the beginning. Mathematicians started to investigate further with respect to chaos, and they gradually discovered that chaotic systems are a subtype of simple nonlinear dynamical systems. They may contain few interacting parts, and these parts may follow very simple rules, but these systems are very sensitive to their initial conditions. Despite their decisive simplicity, over time these systems may produce completely unpredictable and very different (aka chaotic) behavior, like the previously mentioned dripping faucets, fluid experiments, etc. It is destined that logistic map will become one of the extraordinary and important topics in mathematics today.

## 2. Definition

**Fixed Point** *Let f be a function, then a point l in the domain of f is said to be a fixed point of f is we have f(l) = l.*

**Convergence** *Let $u_n$ be a sequence in* R*, then we say that $u_n$ converge to l where l is a real number such that* $\lim_{n\to\infty} u_n = l$

**Proposition 1** *let f be a continuous function from [0,1] to [0,1]. let ($u_n$) be a sequence defined as $u_0 \in$* [0,1] *and $u_{n+1} = f(u_n)$ for $n \in N$. It has convergence if the sequence converges to the fixed point l.*

**Proof of Proposition 1**

Since function $f$ is continuous and we know that sequence ($u_n$) converges to $l$, we have $\lim_{n\to\infty} f(u_n) = f\left(\lim_{n\to\infty} u_n\right) = f(l)$. but $f(u_n) = u_{n+1}$, and the sequence $\{u_{n+1}\}$ also converges to $l$. Therefore, we get $\lim_{n\to\infty} f(u_n) = \lim_{n\to\infty} u_{n+1}$, and thus $l = f(l)$, making $l$ a fixed point.

**Theorem 1 – Monotone Convergence Theorem** *Let $u_n$ be a monotone sequence in* R*. Then sequence $u_n$ converges if and only if it is bounded.*

The sequence is monotonous if it satisfies either $u_n \leq u_{n+1}$ or $u_n \geq u_{n+1}$ for all $n \in$ N. A sequence is bounded is there exist a boundary value $D$ such that $|u_n| \leq D$ for all $n$. If a sequence is monotonously increasing and bounded above, the minimum boundary is called supremum. In contrast, a sequence that is monotonously decreasing and bounded below, the maximum boundary is called infimum. The increasing sequence will converge to the supremum, and the decreasing sequence will converge to the infimum.

**Divergence** *let, $u_n$ denotes a sequence for $n \in$* N*, we say the sequence diverge if and only if* $\lim_{n\to\infty} un = \pm\infty$

The phenomenon of divergence is when a sequence or function doesn't converge to a specific value and wonder "off the chart".

**Attractive Fixed Point** *Let ($u_n$) be a sequence in* R*, if ($u_n$) converges to a fixed point l for $u_0$ sufficiently close to l, we say the fixed point l is attractive.*

In another word, the orbits of all points in (a, b) approach $l$ if there's an interval (a, b) containing $l$, it called an attractive basin.

$$
\begin{gathered}
l \in (a, b)\ f^n(l) \in (a, b)\ \forall n > 0 \\
s.t.\ \forall l \in (a, b)\ f^n(l) \to l \\
\text{where } n \to \infty
\end{gathered}
\tag{2}
$$

One criteria for such fixed point is $f(l) = l$ with its derivative $|f\,'(l)| < 1$

**Repulsive Fixed Point** *Let ($u_n$) be a sequence in* R*, if, for $u_0$ starting sufficiently close to the fixed point l, $|u_n - l|$ increases with n, we say the fixed point is repulsive*

*Remark* When inputting a parameter into a function, and iterate it leads to *diverge* the fixed point $l$, we called repulsive fixed point.

**k-Cycle** *let k be a positive integer and let f be a function on* R*. Then we say that function f has a k-cycle if there exists values $u_1, u_2, \ldots, u_k$ in the domain of f, such that they are distinct and we have*

$$
\begin{gathered}
f(u_1) = u_2 \\
f(u_2) = u_3 \\
\ldots
\end{gathered}
\tag{3}
$$

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$$f(u_k) = u_1$$

*For a k-cycle of function f, we say that the period of this cycle is k. It is very important for the readers to note that the values must be distinct from each other because otherwise we will have a smaller cycle.*
**Period** *A period is the number of* distinct *points in the cycle. All the multiples of the period is called multiple periods.*

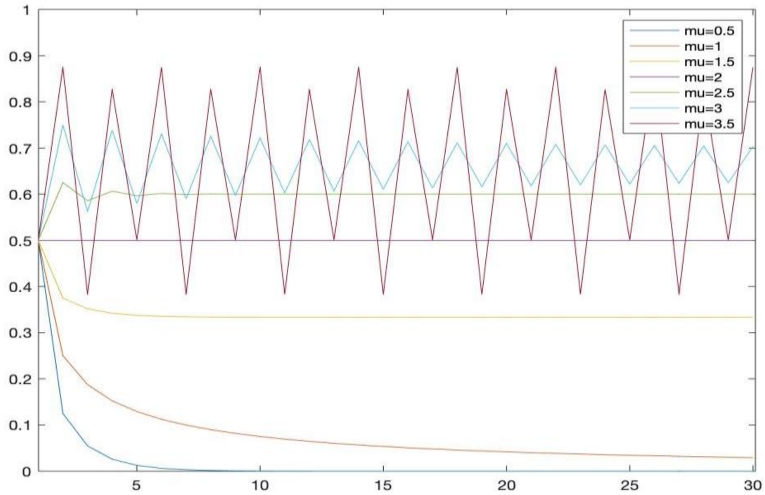


**Figure 1.** evolution of $u_n$ with different $\mu$'s

## 3. Behaviors of Logistic Model

### *3.1 Overview*

For logistic map with $\mu \in (0,4)$, we have $f([0, 1]) \subset [0, 1]$ where $f$ is continuously differentiable on (0,1) and $(u_n)$ is a sequence in [0,1] with

$$u_0 \in [0,1]$$
$$u_{n+1} = f(u_n) = \mu u_n(1 - u_n) \tag{4}$$

The behaviours of the fixed points and cycles for $\mu \in [0, 4]$ will be analyzed in this paper. The following Figure.1 shows the values of generation of $u_n$ versus $n$. It follows from the plot that for $\mu$'s smaller than 3, the x quickly stabilizes to a stable solution; for $\mu$ = 3, the x oscillates between two values; and for $\mu$ = 3.5, a stable oscillation between 4 values occurs.

Another way to visualize the behavior of logistic map is through a technique called cob-webbing. In the cobweb diagram, we plot a diagonal line $y = x$ and and quadratic curve $y = \mu x(1 - x)$, and we connect each $u_n$ on diagonal and $u_{n+1}$ on quadratic curve by vertical line and each $u_n$ in quadratic curve to diagonal line by horizontal line. In this way, each y-value of points on the quadratic curve $y = \mu x(1-x)$ indicates the value of the next iteration of $u_{n+1}$. In the following cobweb graphs (Figures.2,3,4), the value of $\mu$ progressively gets larger, as can be seen in the "humps" of the quadratic curves rising higher.

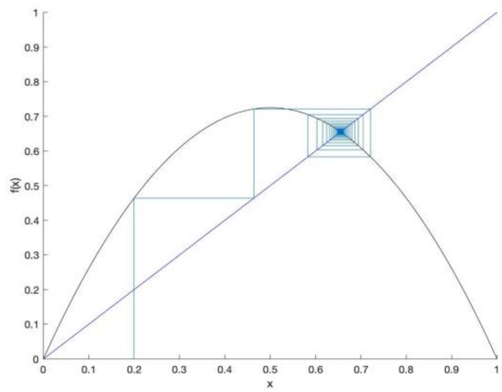

**Figure 2.** $u_0$ starts near the divergent fixed point 0 but got repelled away to the other convergent fixed point $1-\frac{1}{\mu}$

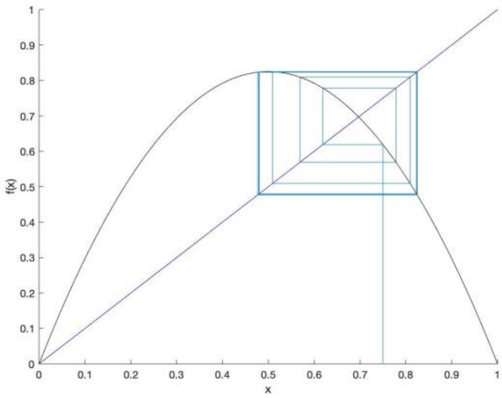


**Figure 3.** $u_0$ starts near the divergent fixed point $1-\frac{1}{\mu}$ and is attracted to the convergent 2-cycle.

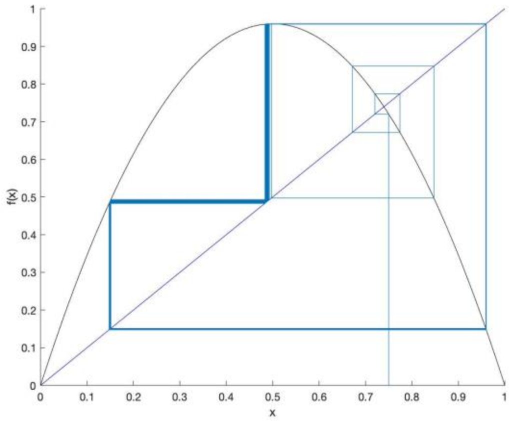


**Figure 4.** $u_0$ starts near the divergent fixed point $1-\frac{1}{\mu}$ and is attracted to the convergent 3-cycle.

### *3.2 $\mu \in (0,3)$: Convergent Fixed Point(s) Only*

The fixed points of $f$ are given by:

$$\mu x(1-x)=x \tag{5}$$
$$x=0 \text{ and } x=1-\frac{1}{\mu}$$

Next we compute the derivative of $f_\mu$ at the fixed points:

$$f(x)=\mu(1-2x) \text{ so we get } f'(0)=\mu \text{ and } f'\left(1-\frac{1}{\mu}\right)=2-\mu \tag{6}$$

Thus x=0 is a fixed point for all $\mu$ and $x=1-\frac{1}{\mu}$ is a fixed point only if $\mu \geq 1$. Stability is depending on $f'(x)=\mu-2\mu x$. So we get $x=0$ is stable for $0 \leq \mu < 1$ and unstable for $\mu > 1$. And $x=1-\frac{1}{\mu}$ is stable for $1<\mu<3$ and unstable for $\mu>3$. For $1 \leq \mu < 3$ it can be shown that all initial points $x_0 \in (0, 1)$ lie in the basin of attraction on $1-\frac{1}{\mu}$, so there can be no 2-cycle existing for $0<\mu<3$. Moreover, we don't need to check for cycle of period higher than 2, due to the first statement of the Coppel Theorem, which says that if 2 cycle doesn't exist, neither will any higher cycles. Also, the second statement of the Coppel Theorem, that a sequence without a 2-cycle will converge, confirms our conclusion about that there must be at least one stable fix point for $\mu \in (0, 3)$, which is in accordance to figure.2.

*3.3 $\mu \in (3, 1+\sqrt{\mathbf{6}})$: Divergent Fixed Points and Convergent 2-Cycle*

At $\mu = 3$ we have:

$$f'\left(1-\frac{1}{\mu}\right) = -1 \tag{7}$$

$$(f^2)'\left(1-\frac{1}{\mu}\right) = (-1)(-1) = 1 \tag{8}$$

When $\mu$ just exceeds 3, the slope of the graph of $f^2$ at the fixed point becomes greater than 1, and this graph intersects two new points on either side of the fixed point of the line $y = x$. The new fixed points of $f^2$ are not fixed points of $f_\mu$, so they must form a new cycle of $f$ with period 2. Furthermore, the slope of $f_\mu^2 < 1$at each new point, so this 2-cycle cycle is an attractor, in accordance with the behavior shown in figure.3. It can be shown that when this 2-cycle remains an attractor, every point of (0, 1), except the repulsive fixed point and its pre-image, is in the 2-cycle basin of attraction. Therefore, there are no cycles other than the 2-cycle and the two fixed points $(0, 1-\frac{1}{\mu})$. And a 2-cycle exists if and only if there are two points p and q such that $f(p) = q$ and $f(q) = p$. Equivalently, such a p must satisfy $f^2(p) = p$ where $f(x) = \mu x(1-x)$. Hence, p is a fixed point of the second iterate map $f^2(u_n) = u_{n+1}$. Since $f(x)$ is a quadratic map, $f^2(x)$ is a quartic polynomial.

We assume that the orbit for this is $p_1$,$p_2$. We have the equation:

$$f^2(x) = x \tag{9}$$

$$f^2(x) = \mu(\mu x(1-x))(1-\mu x(1-x)) = 0$$
$$(-\mu^3)x^4 + (2\mu^3)x^3 + (-\mu^3-\mu^2)x^2 + \mu^2 x = 0$$

We divide out the trivial solutions $x_1 = 0$ and$x_2 = 1-\frac{1}{\mu}$. Then we can find:

$$x^2 - \left(\frac{\mu+1}{\mu}\right)x + \frac{\mu+1}{\mu^2} = 0 \tag{10}$$

So, in specific, real $p_1$ and $p_2$, which is of the forms

$$p, q = \frac{\mu + 1 \pm \sqrt{(\mu-3)(\mu+1)}}{2\mu} \tag{11}$$

exist if and only if:

$$\left(\frac{\mu+1}{\mu}\right)^2 \geq \frac{4(\mu+1)}{\mu^2} \tag{12}$$
$$\mu \geq 3$$

Thus the 2 cycle exist for all value of $\mu \geq 3$. But if we want to find an attractor we need to compute the value of $f'(p_1)f'(p_2)$

$$f'(p_1)f'(p_2) = (\mu - 2\mu p_1)(\mu - 2\mu p_2) \tag{13}$$

$$= \mu^2(1 - 2(p_1+p_2) + 4p_1p_2)$$
$$= \mu^2\left(1 - 2\left(\frac{\mu+1}{\mu}\right) + 4\left(\frac{\mu+1}{\mu}\right)\right)$$
$$= -\mu^2 + 2\mu + 4$$

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When $\mu = 3$, $-\mu^2 + 2\mu + 4 = 1$, So for $\mu > 3$, it decreases monotonically. And it reaches to 0 when $\mu = 1 + \sqrt{5}$ and reaches to -1 when $\mu = 1 + \sqrt{\mu}$. Thus, the 2 cycle is attractive for $3 < \mu < 1 + \sqrt{6}$ and becomes a repellent for $\mu > 1 + \sqrt{6}$.

Here, what actually happens is a Flip (Period Doubling) Bifurcation [6], which, if we consider $x_{n+1} = f(x, \mu) = \mu x_n(1 - x_n)$, happens when a stable fixed point of $f$ loses its stability and a stable 2-cycle of $f$ emerge as $\mu$ passes a critical value. That is, a stable 2-cycle solution *bifurcates* from the stable fixed point, depriving it of its stability. Here the critical point is

$$(x_c, \mu_c) = \left(\frac{2}{3}, 3\right) \tag{14}$$

### 3.4 Stable 3-cycle Interval

From the previous calculation, we know that the first bifurcation occurs at the two cycle $\mu = 3$. So in general, the set of n+1 equations which can be solved to give the onset of an arbitrary n-cycle is for example: $x_2 = \mu x_1(1 - x_1)$, $x_3 = \mu x_2(1 - x_2)$, $x_4 = \mu x_3(1 - x_3)$.

...

And now let's look at the set of the 3-cycle, as shown in figure.4. It considers the typical equation for 3-cycle with the trivial solution of fixed point divided out:

$$\frac{f^3(x) - x}{f(x) - x} = 0 \tag{15}$$

Then we have:

$$\begin{gathered} 1 + \mu + \mu^2 - (\mu^4 + 2\mu^3 + 2\mu^2 + \mu)x + (2\mu^5 + 3\mu^4 + 3\mu^3 + \mu^2)x^2 \\ -(\mu^6 + 5\mu^5 + 3\mu^4 + \mu^3)x^3 + (3\mu^6 + 4\mu^5 + \mu^4)x^4 - (3\mu^6 - \mu^5)x^5 + \mu^6 x^6 = 0 \end{gathered} \tag{16}$$

But the root of this equation are all imaginary numbers for$\mu$ less than some specific $\mu_3$. At this point, two of their roots are transformed into real roots. $\mu$ can then find the specific value by the discriminant.

$$\frac{(\mu^2 - 5\mu + 7)^2(\mu^2 - 2\mu - 7)^3(1 + \mu + \mu^2)^2}{\mu^{30}} \tag{17}$$

When this discriminant above is equal to 0, those two real roots coincide. So:

$$\mu = 1 + 2\sqrt{2} = 3.828427.... \tag{18}$$

Of course, this is only one of the most primitive methods of calculating the value of the 3-cycle $\mu$. And in the follow-up, in 1995, mathematicians Saha and Strogatz successively gave a more simplified algebraic treatment of the 3-cycle and found the solution. In 1996, mathematicians Bechhoeffer and Gordon provided a further simplified approach, although such an algorithm cannot be easily generalized to higher cycles.

### 3.5 Cascade of Period Doubling

As the value of $\mu$ goes higher, the previous 4-cycle would lose stability and give rise to a stable 8-cycle, and so on. That is, the $f^{(4)}$, $f^{(8)}$, $f^{(16)}$... successively undergoes flip bifurcations [6]. The readers can refer to figure.5,6 for a visual presentation of the successive period doublings. This doubling periods of 1, 2, 4, 8, 16... to infinity is in consistent with the "tail" of Sharkovsky ordering, whcih is defined as followed:

$$\begin{gathered} 3 \rhd 5 \rhd 7 \ldots \rhd 2 \times 3 \rhd 2 \times 5 \ldots \rhd 2^2 \times 3 \rhd 2^2 \times 5 \ldots \\ \rhd 2^3 \times 3 \rhd 2^3 \times 5 \ldots \rhd 2^3 \rhd 2^2 \rhd 2 \rhd 1 \end{gathered} \tag{19}$$

The Sharkovsky Forcing Theorem tells us that if $m$ is a period of f(x) and $m$ B $l$, then $l$ is also a period of f [7]. Indeed, we see in the logistic map that the periods of cycles occur in the order specified above. The proof of this period doubling, however, requires much more complex numerical calculation as done above, for we would be dealing with polynomial with higher orders. Granted, we can still solve these equation to determine the cycles, with the techniques shown in the above sections, but to determine the range of $\mu$ for which these cycles are stable would be more and more complicated. To prove this, we may use the condition for flip bifurcation and check it for $f^{(4)}$, $f^{(8)}$, $f^{(16)}$...

As shown experimentally by Feigenbaum in $19^{th}$ century, distance between two successive $\mu$'s in which period doubling occurs has a limit. That is:

$$\lim_{k\to\infty} \frac{\mu_k - \mu_{k-1}}{\mu_{k+1} - \mu_k} = 4.669202\ldots \tag{20}$$

in which $\mu_k$ is the critical value in which a k-cycle emerges.
which is often called Feigenbaum constant [5]. This constant has two amazing property: First, it is a transcendental like $\pi$, which cannot be expressed as the root of a non-zero polynomial of finite degree with rational coefficients; Second, it is universal, and appears in all families of

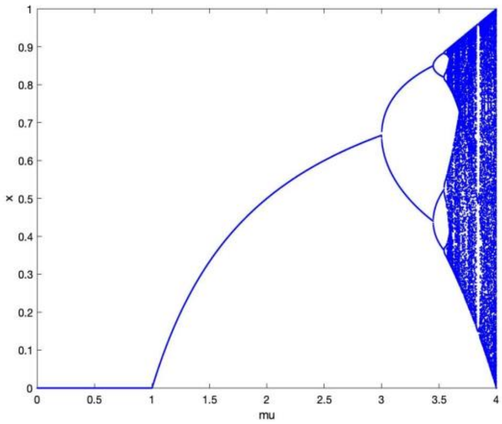


**Figure 5.** bifurcation diagram for $x_{n+1} = f(x_n) = \mu x(1 - x)$

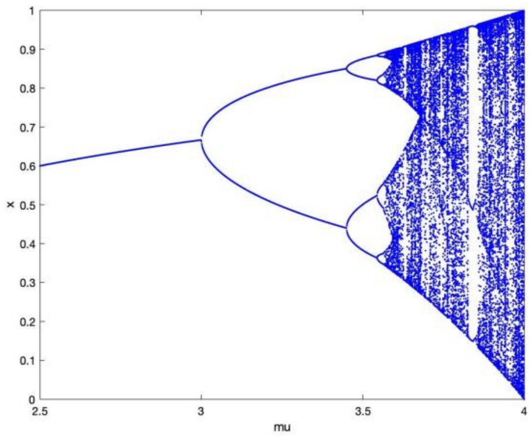


**Figure 6.** This is a closer view of the above bifurcation diagram

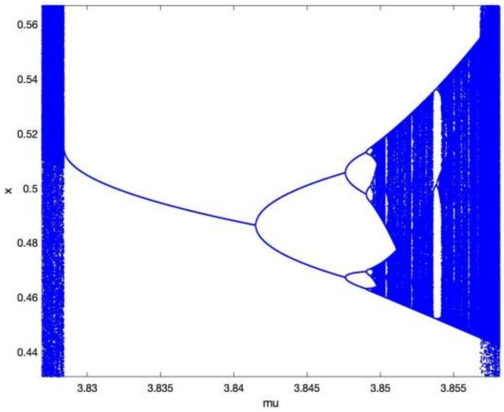


**Figure 7.** close-up view of the "middle branch" of the stable 3-cycle window.

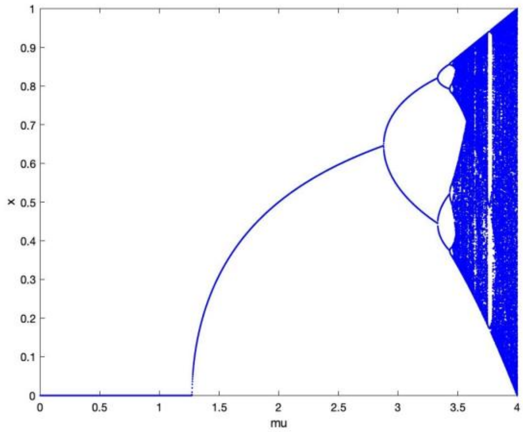


**Figure 8.** Bifurcation diagram for $x_{n+1} = f(x_n) = \mu\sin(\pi x_n)/4$

"quadratic-like-maps" (one humped maps with non-zero second derivative at the peak). The analytic proof of the universality such constant uses the idea of "re-normalization theory" and was given by Dennis Sullivan in 1980's [8].

To many's surprise, The phenomenon of period doubling is not a mere highly-idealized and approximated (discrete time) mathematical trickery, but has been observed in many experiments, like in the convection roll in water or mercury, the nonlinear electronic circuits, and even population of some insects in controlled laboratory [5, 8]. However, as the doubling period get larger, the sensitivity required to detect such change of cycle increase exponentially, for the change in parameter would mingle with the background noise in the observation.

### *3.6 Entrance to Chaos*

As $\mu$ increase pass a threshold $\mu_\infty = 3.61547...$, the bifurcation map exhibits advances further into the Sharkovsky ordering and result in a fascinatingly complex bifurcation diagram 5 (value of $\mu$ versus the stable cycle).

Here other parts of the Sharkovsky ordering occur, most conspicuously in the "periodic windows" dotted among the chaotic background. For example, at about $\mu = 3.629$, the diagram exhibits a 6-cycle; at about $\mu = 3.739$, the diagram exhibits a 5-cycle; and at about $\mu = 3.84$, the diagram exhibits a 3-cycle, which, according to Sharkovsky Forcing Theorem, implies that existence of all other (unstable) natural number cycles, hence the slogan: "period three implies chaos." The widths of these periodic windows differ,

with the largest being the window of 3cycle. More amazingly, if we zoom in to one branch of the 3-cycle, we discover that, locally, this branch undergoes period-doubling just like what happened for $\mu \in (2.5, \mu_\infty)$, as can be seen in figure.7. Due to this property of local similarity to the whole, this bifurcation diagram, or what is commonly called logistic map, has become a famous example of fractal structure. This behaviors of period-doubling followed by chaos has been shown to exist for all of "quadraticlike-maps", thus making the logistic map truly a general phenomenon worthy of investigation. For example, we display in figure.8 the bifurcation diagram of another quadratic-like-map of [0, 1] → [0, 1] defined by $x_{n+1} = f(x_n) = \mu/4\sin(\pi x_n)$:
The bifurcation diagram for $x_{n+1} = f(x_n) = \mu\sin\pi x_n/4$ is shown in figure 8, which is very similar to that of the logistic map with similar period doubling pattern and eventual chaos. However, the specific values of $\mu$ in which the period changing occurs is slightly different between the two.

## 4. Other Relevant Results

*4.1 Criteria for Stability of k-Cycle*

**Statement 1** *If the sequence $\{u_n\}$ defined by $u_{n+1} = f(u_n)$ have a k-cycle $l_1, l_2...l_k$. We define $g(x) = f^k(x)$, then the cycle is attractive (for a starting point $x_0$ close to $l_i$ ($i$ = 1,2...k), then the sequence $\{u_m\}$ defined by $u_{m+1} = g(u_m)$ will converge to $l_i$) if $|g^0(l_1)| < 1$.*
*Similarly, the cycle is repulsive (for a starting point $x_0$ close to $l_i$ ($i$ = 1,2...k), then the sequence $\{u_m\}$ defined by $u_{m+1} = g(u_m)$ will diverge from $l_i$) if $|g^0(l_1)| > 1$.*
*Remark* By chain rule: $g'(l_1) = \prod_{i=1}^{k} f'(l_i)$

**Proof of Statement 1**

We prove the part when $|g'(l_1)| > 1$, the proof for when $|g'(l_1)| < 1$ is similar.
Assuming $g'(l_1) = k$, since $g'$ is continuous, there is $\delta > 0$ such that $\forall x \in (l_1 - \delta, l_1 + \delta)$, we have

$$|g'(x)| \geq \frac{1+k}{2} > 1$$

Then for $u_n \in (l_1, l_1 + \delta)$ (the case when $u_n = l_1$ is trivial and the case when $u_n \in (l_1 - \delta, l_1)$ is similar), then using the first order Taylor expansion

$$\begin{aligned} u_{n+1} &= g(u_n) \\ &= g(l_1) + g'^{(c)(u_n - l_1)} \\ u_{n+1} - l_1 &= g'^{(c)(u_n - l1)} since g(l_1) = l_1 \end{aligned} \tag{21}$$

or some $c \in (l_1, x_n)$. Then $s \in (l_1 - \delta, l_1 + \delta), so\ |g'(s)| \geq \frac{1+k}{2}$

$$|u_{n+1} - l_1| \geq \left(\frac{k+1}{2}\right)|u_n - l_1| > |u_n - l_1| \tag{22}$$

Thus, unless $u_n = l_1$ for some index n and then stay at $l_1$, the sequence $\{u_n\}$ cannot converge to $l_1$.

*4.2 Coppel Theorem*

*4.2.1 Part 1.* ***Statement 2.1*** *Let I be a closed interval on R and $f(u_n)$ a continuous [0, 1] → [0, 1] function. If the sequence $(u_n)$ in that region has no 2-cycle, then n-cycle will not occur.*

*Lemma 1* If there is a c, d$\in$ [0, 1] with $f(d) \leq c < d \leq f(c)$, then $f$ has a 2-cycle.
*Proof of Lemma 1*
To find a 2-cycle of $f$ ($f^2(p) = p$), we equivalently find $f^{-1}(p) = f(p)$ such that $f(p)$ 6= $p$. To do this, we plot the graphs of $y = f(x)$ and $x = f(y)$ in the plane $[a, b] \times [a, b]$. The two will be two symmetrical curves across the diagonal line $x = y$. The points $(c, f(c))$ and $(d, f(d))$ would be reflected to $(f(c), c)$ and $(f(d), d)$, respectively. Our goal is finding the intersection of curve $C_1$ ($y = f(x)$) and curve $C_2$ ($x = f(y)$) besides that on the diagonal.

In the first case shown in figure.9, $f(f(c)) < d$, thus ($f(d)$, $d$), which is part of $C_2$, would be trapped below by the part of $C_1$ in the plane [0, $c$] × [0, $d$], denoted by $C'_1$.
However, $C_2$ also contains ($f(c)$, $c$), which is on the opposite side of the plane [0, 1] ∗ [$c$, $d$], separated from ($f(d)$, $d$) by $C'_1$. Since the movement of $C_2$ between ($f(d)$, $d$) and ($f(c)$, $c$) is limited to the plane [0, 1] × [$c$, $d$], there must be a intersection of $C_2$ and $C'_1$ in the plane [0, $c$] × [$c$, $d$]. The intersection could not happen at (c, c) for $f(c) \neq c$. Thus there is an intersection of $C_1$ and $C_2$ not located at the diagonal line $y = x$.
In the second case shown in figure.9, $f(f(c)) > d$, thus ($f(d)$, $d$), which is part of $C_2$ would be trapped above by the part of $C_1$ in the plane [0, $d$] × [0, 1], denoted by $C'_1$.
Since $C_2$ must end at the line $y = 1$, and the movement of $C_2$ between ($f(d)$, $d$) and ($f^{-1}(1)$, 1) is limited to the plane [0, 1] × [$d$, 1], there must be an intersection of $C_2$ from ($f(d)$, $d$) to

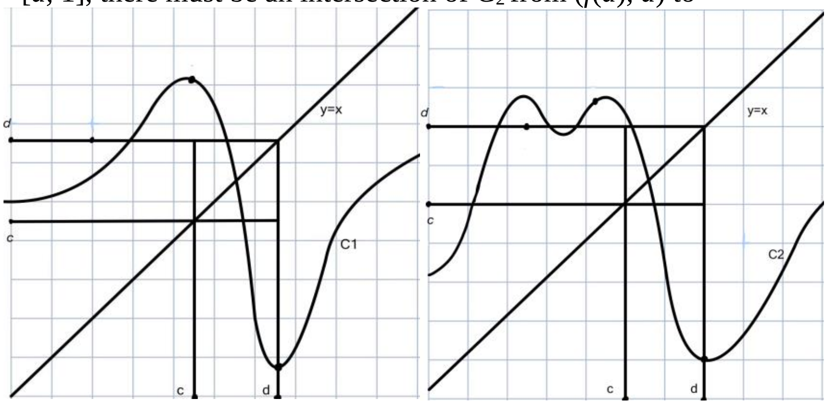


**Figure 9.** (Left)The first case, (Right) The second case

($f^{-1}(1)$ ,1) and $C'_1$. Furthermore, this intersection can only occur at [0, $d$] ∗ [$d$, 1], and not at ($d$, $d$) because $f(d) \neq d$. Thus, there is an intersection of $C_1$ and $C_2$ not located at diagonal line $y = x$.

**Proof of Statement 2.1**

The purpose is to prove that there doesn't have a 2-cycle and n-cycle in the sequence ($u_n$) in [0, 1]. We first assume that the function $f$ has a n-cycle, and $n \geq 3$. Because it is in [0,1], we could get that

$$0 \leq x_1 < x_2 < x_3 < \ldots < x_n \leq 1 \tag{23}$$

When we input $x_1$, we will get $x_2$. Inputting $x_2$ in it, we will get $x_3$. By that analogy, input $x_n$ into it we will get $x_1$ again, which formed a n-cycle system.

$$x_1 < f(x_1) < f^2(x_1) < f^3(x_1) < \ldots f^n(x_1) < x_n \tag{24}$$

By the iteration, we get that $x_1 < f(x_1)$, and $f(x_n) < x_n$ due to iterate $x_n$ we will eventually get $x_1$, and $x_1 < x_n$. We could define it as

$$xk = \max\{x1, x2, x3 \ldots xn\} \tag{25}$$
$$\text{and the sequence (x) } \forall f(xn) < xn$$

If the maximum number in the sequence (x) that satisfy $f(x_n) < x_n$ is $x_k$, then it means that $f(x_k) > x_k$ and $f(x_{k+1}) < x_k + 1$. This is because $f(x_{k+1})$ will go back to $x_1$, and $x_1 < x_{k+1}$.
[$c$, $d$] is in the interval [0, 1], and [$x_k$, $x_{k+1}$] is also in the interval [0, 1]. Therefor, we could get that [$c$, $d$]⊂[$x_k$, $x_{k+1}$] and this satisfies the assumption of Lemma. A 2-cycle occurs in that sequence.
We know that the condition of that statement is no 2-cycle, so no n-cycle. We could prove this statement by contraposition.
If the sequence doesn't have 2-cycle, then there's no n-cycle.

If the sequence has a 2-cycle, then it means a n-cycle will occur.
If the sequence has a n-cycle, then there must be a 2-cycle.
If the sequence doesn't have n-cycle, then there's no 2-cycle.
We proved that and defined that there is 2-cycle in the sequence. According to the idea of contraposition, the sequence also has n-cycle. However, if 2-cycle doesn't occur in the sequence, then it means there's no n-cycle.

*4.2.2 Part 2.* ***Stattement 2.2*** *Let I be a closed interval on R and f(x) a continuous* $R \to R$ *function, such that* $f(I) \subseteq I$*. Then for the sequence* $\{u_n\}$ *for* $k \in N$ *defined as* $u_o \in I$ *and* $u_{n+1} = u_n$*, if there is no 2-cycle, then all sequence* $\{u_n\}$ *will converge.*

*Remark* Due to Statement 1 of Coppel Theorem, the condition that 2-cycle doesn't existence implies that all cycles don't exist, except that of period one, which is a fixed point.
*Lemma 2* If there is no k-cycle, for every $2 \le k \le n$, then

$$f(c) > c \Leftrightarrow f^n(c) > c \tag{26}$$
$$f(c) = c \Leftrightarrow f^n(c) = c \qquad \text{trivial case}$$
$$f(c) < c \Leftrightarrow f^n(c) < c$$

*Proof of Lemma 2*
We prove the lemma by contradiction.
Assume $f(c) > c$ and $f^n(c) < c$ (The equality cannot occur because there is no n-cycle), let $I$ be $[a_1, a_2]$, and let $b$ be the last fixed point of $f^n$ on $[a_1, c]$. $c \neq a_1$ because $f^n(c) < c$ and $f(I) \subseteq I$. Since $f(I) \subseteq I$, we must have $f^n(a_1) \geq a_1$. Thus $f^n(a_1) - a_1 > 0$ and $f^n(c) - c < 0$, which means the fixed point $b$ for which $f^n(b) = b$ must exist. Since there is no cycles of period beyond 1, $b$ must also the last fixed point of $f$ in $[a_1, c]$. Therefore, $f(x) > x$ and $f^n(x) < x$ for $x \in (b,c]$ Also, since $f(b) = b$ and $f(c) > c$, there exists a first $d_1$ in $(b,c)$ such that $f(d_1) = c$. Therefore, $f([b, d_1]) = [b, c]$ By the same argument, there must exist a first $d_2$ in $(b,d_1)$ such that $f(d_2) = d_1$, and so on by induction. In this way, with $d_0 = c$, we get a sequence $d_0, d_1, d_2 \ldots d_{n-1}$ such that $f([b, d_k]) = [b, d_{k-1}]$ for $k = n-1, n-2 \ldots 1$. Notice that this sequence is strict decreasing therefore $[b, d_k] \subseteq [b, c]$.
We choose $e$ in $(b, d_{n-1}]$. Thus $f^m(e) \in (b, c]$ for $m = 0,1,2 \ldots n$, which means

$$e < f(e) < f^2(e) \ldots f^{n-1}(e) < f^n(e) \tag{27}$$

but $f^n(x) < x$ for $x \in (b, c]$. Therefore we have a contradiction. The case when $f(c) > c$ is similar except we find the first fixed point of $f^n$ in $[c, a_2]$.

**Proof of Statement 2.2**

We split the sequence $\{u_n\}$ into two "subsequences": $\{u_{\phi 1(n)}\}$ and $\{u_{\phi 2(n)}\}$, in which $\phi : N \to N$ is an increasing function that selects some indices as follows:

$n$ is part of $\phi_1(n)$ if $f(u_n) > u_n$.
$n$ is part of $\phi_2(n)$ if $f(u_n) < u_n$.

The lemma 1 tells us that $\{u_{\phi 1(n)}\}$ is increasing and $\{u_{\phi 2(n)}\}$ is decreasing. Since $\{u_n\}$ is bounded in $[a_1, a_2]$, by monotonous convergence theorem, $\{u_{\phi 1(n)}\}$ converges to some $l_1$ and $\{u_{\phi 2(n)}\}$ converges to some $l_2$. We assume here that both sequences are infinite and add up to the whole sequence $\{u_n\}$, otherwise $\{u_n\}$ would eventually be monotonous and so converge. Since $\{u_{\phi 1(n)}\}$ and $\{u_{\phi 2(n)}\}$ are infinite, there is an infinite sequence of "flipping" indices $N_k$ such that $f(u_{N_k}) > u_{N_k}$ (part of $\{u_{\phi 1(n)}\}$) but $f(u_{N_{k+1}}) < u_{N_{k+1}}$ (part of $\{u_{\phi 2(n)}\}$). Note that $N_k \to \infty$ as $k \to \infty$. Therefore

$$u_{N_{k+1}} = f(u_{N_k}) \tag{28}$$
$$\lim_{k\to\infty} u_{N_{k+1}} = l_2$$
$$\lim_{k\to\infty} u_{N_k} = l_1$$
$$l_2 = f(l_1)$$

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Similarly, we have $l_1$ = f($l_2$), so $l_1$ and $l_2$ is a 2-cycle, but there is no 2-cycle, so $\boldsymbol{l_1 = l_2}$ is a fixed point and the sequence converge.

### *4.3 Period 3 implies all Periods*

We adapt proof of this special case of Sharkovsky Theorem

**Statement 3** *For a sequence* $(u_n)$ *for* $n \in \mathrm{N}$ *defined by* $u_{n+1} = f(u_n)$ *for a* $\mathrm{R} \rightarrow \mathrm{R}$ *function f, if there exist a 3-cycle, then all k-cycle's exist for* $k \in \mathrm{N}$.

**Coverage** *Interval with endpoints in the cycle O of f is called an O-*interval*. An interval I* covers *an interval J, denoted by* $I \rightarrow J$*, if* $J \subset f(I)$.

*Remark* By intermediate value theorem, $I \rightarrow J$ if and only if the end point of $J = (j_1, j_2)$ is contained in $f(I)$. That is, there exists some $x_l \in I$ such that $f(x_l) = j_1$ and some $x_r \in I$ such that $f(x_r) = j_2$

*Lemma 3.1* If $[a, b] \rightarrow [a, b]$, there is a fixed point of $f$ in $[a, b]$.

*Proof of Lemma 3.1*

There is $c, d \in [a, b]$ such that $f(c) = a$ and $f(d) = b$. Then $f(c) - c \leq 0 \leq f(d) - d$, and, by intermediate value theorem, we have some $e \in [c, d]$ such that $f^n(e) = e$, a fixed point of multiple period n.

*Lemma 3.2* If $J_0, J_1, J_2...J_{n-1}$ are closed bounded intervals and $J_0 \rightarrow J_1 \rightarrow J_2 \rightarrow ... \rightarrow J_{n-1} \rightarrow J_0$ (denoted as a loop or a n-loop of intervals), then there is a fixed point of $f^n$ such that $f^k(p) \in J_k$ for $0 < n$. We say that p follows the loop.

*Proof of Lemma 3.2*

We denotes $f(I) = J$ by $I \mapsto J$. If $I \rightarrow J$, then there is an interval $K \subset I$ such that $K \mapsto J$. We can select the interval by choosing an *adjacent* pair of $x'_l$ and $x'_r$ such that there is no other $x_r$'s or $x_l$'s in the open interval bounded by the two.

Therefore, there is an interval $K_{n-1} \subset J_{n-1}$ such that $K_{n-1} \mapsto J_0$. Then $J_{n-2} \rightarrow K_{n-1}$, and there is an interval $K_{n-2} \subset J_{n-2}$ such that $K_{n-2} \mapsto K_{n-1}$. Due to induction, we have intervals $K_i \subset J_i$, $0 \leq i < n$, such that $K_0 \mapsto K_1 \mapsto ... \mapsto K_{n-1} \mapsto J_0$. All $x \in K_0$ satisfies $f^i(x) \in K_i$ for $0 \leq i < n$ and $f^n(x) \in J_0$. Since $K_0 \subset J_0 = f^n(K_0)$, then, by lemma 3.1, there is a fixed-point p of $f^n$.

However, to ensure that the point following the loop indeed has period n rather than some multiple divisors of n, we need another criterion.

*Lemma 3.3* A loop $J_0 \rightarrow J_1 \rightarrow ... \rightarrow J_{n-1} \rightarrow J_0$ of *O-intervals* is followed by a point of period n, denoted by saying the loop if *authentic*, if the loop is not followed by a point of O and the *interior* of $J_0$, denoted as $Int(J_0)$ is *disjoint* from each of $J_1...j_{n-1}$, that is, $Int(J_0) \cap J_k = \varnothing$ for $0 < k \leq n - 1$.

*Proof of Lemma 3.3*

If $p$ follows the loop, then $p \notin Int(J_0)$. If $0 < i < n$ then $f^i(p) \notin Int(J_0)$, so $q \neq f^i(p)$. Thus, $p$ has period $n$.


**Figure 10.** Two kinds of 3-cycle

**Proof of Statement 3**

There are two symmetrical versions of 3-cycle, as shown in figure.10: For $x_1 < x_2 < x_3$, we have $f(x_1) = x_2$, $f(x_2) = x_3$, $f(x_3) = x_1$ or $f(x_3) = x_2$, $f(x_2) = x_1$, $f(x_1) = x_3$. We discuss the first version, for that of the second version is similar except that we choose the intervals oppositely. We define $I_1$ as $[x_2, x_3]$ and $I_2$ as $[x_1, x_2]$. Then $I_1 \rightarrow I_1$ and $I_1 \rightarrow I_2$. From the first relation, there is a fixed point (period 1) of $f$ by lemma 3.1. We also have the relation $I_1 \rightarrow I_2 \rightarrow I_1$, which requires the point following it to have period 1 or 2 and therefore cannot be followed by points in O with the least period being 3. It follows from lemma 3.2 that $f$ has a 2-cycle. For our last relation $I_2 \rightarrow I_1 \rightarrow I_1 \rightarrow ... \rightarrow I_1 \rightarrow I_2$, with l-1 copies of $I_1$ in the middle for $l > 3$. The loop cannot be followed by points of O for they cannot stay in $I_1$ for more than 2 consecutive iterations of $f$. Therefore, it follows from lemma 3.2 that $f$ has a period l for every $l > 3$. Therefore, the existence of period 3 implies the existence of all periods.

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## 5. Conclusion

In this paper, we obtained the classical results of the behaviors of logistic equation under different values of $\mu$, from stable cycles of increasing periods to eventual chaos. Then, we discussed and showcased many of the famous results from the logistic map like Feigenbaum number and fractal structure. In the end, we presents proofs for three relevant results important to the study of logistic map: the criteria for stability of cycle, the Coppel Theorem, and the famous slogan "period three implies chaos." One complex behaviour derived from a simple model-The logistic map shows how the overall map behaves as $\mu$ continuously having slight changes. The simple equation $u_{n+1} = \mu u_n(1 - u_n)$ eventually goes into an intricate, and unpredictable map. Complex systems can be built on simple rules and applied to the real world: weather, stars, celestial bodies, human population, and even bird flocks [9, 10].

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