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# Bessel Functions: Theory and Applications

LILY NGUYEN[1] AND ANDRE SAE[1]

[1] *Department of Physics, The University of Texas at Austin*
*Austin, TX 78712, USA*

## ABSTRACT

In this project, we numerically computed and visualized the first five positive roots of the Bessel functions of the first kind, $J_0(x)$, $J_1(x)$, and $J_2(x)$, using Python. These functions solve a second-order differential equation that occur in problems with radial symmetry. We used the `scipy` library to evaluate $J_n(x)$ and locate its roots via the `fsolve` method. We interpreted the computed roots in the context of three physical materials: the radial wavefunction in a quantum infinite square well, heat conduction in cylindrical geometries, and vibrational modes of a circular drumhead. Our results show how Bessel function roots reflect physical boundary conditions and demonstrate the usefulness of numerical methods in modeling such systems.

*Keywords:* Bessel functions — root-finding — radial symmetry — boundary value problems — numerical analysis

## 1. INTRODUCTION

The Bessel functions of the first kind, $J_n(x)$, are solutions to the second-order linear differential equation:

$$z^2 \frac{d^2w}{dz^2} + z\frac{dw}{dz} + (z^2 - n^2)w = 0, \tag{1}$$

where integer $n = 0, 1, 2\ldots$ represents the order of the function. These functions appear naturally in the separation of variables when solving partial differential equations in cylindrical or spherical coordinates, particularly in systems with radial symmetry. Bessel functions also have an integral

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} cos(x\sin\theta - n\theta)d\theta \tag{2}$$

for integers $n = 0, 1, 2\ldots$, which is particularly useful for understanding their oscillatory behavior.

Bessel functions were first introduced by German astoronomer and mathematician Friedrich Wilhelm Bessel in the early 1800s during his study of planetary orbits. However, the equation itself had been investigated earlier by Bernoulli and Euler in problems involving vibrational membranes Abramowitz & Stegun (1972). Today, Bessel functions are foundational tools in mathematical physics and engineering. In quantum mechanics, they appear in radial solutions to the Schrödinger equation for quantum wells CITE, in the vibration modes of circular membranes like drumheads CITE, and in heat conduction problems with cylindrical geometries CITE.

In this project, we numerically compute the first five positive roots of the Bessel functions $J_0(x)$, $J_1(x)$, and $J_2(x)$. These roots correspond to physically meaningful quantities such as resonance modes, cutoff frequencies, or quantized boundary values in systems with radial symmetry. We use Python to visualize each function, estimate the approximation locations of their roots, and apply the `fsolve` method from `scipy.optimize` to compute each root with high precision. Our goal is to obtain accurate roots values for each order and verify them using numerical methods

## 2. DATA AND OBSERVATIONS

This project focuses on computing and visualizing the first five roots of the Bessel functions of the first kind, $J_n(x)$, for orders $n = 0, 1, 2$. These functions solve the second-order linear differential equation:

$$z^2 \frac{d^2w}{dz^2} + z\frac{dw}{dz} + (z^2 - n^2)w = 0, \tag{3}$$

which arise in physical systems with spherical or cylindrical symmetry. The positive roots of $J_n(x)$ represent physically meaningful quantities such as resonant frequencies or quantized energy levels in such systems.

We computed the Bessel functions using Python's `scipy.special.jv` method, which evaluates $J_n(x)$ for an arbitrary order and argument. To find the roots numerically, we used the `fsolve` method from `scipy.optimize`, which refines an initial guess until it finds a point where the function crosses zero. We chose a set of initial guesses based on where the function appeared to cross the x-axis in the plot and passed those values into `fsolve` to calculate each root more precisely. We used the following initial guesses:

- $J_0(x)$: [2, 6.1, 8.6, 11.7, 15]
- $J_1(x)$: [3.9, 7, 10.15, 13.1, 16.4]
- $J_2(x)$: [5.1, 8.3, 11.8, 14.9, 18]

This gave us exactly five positive roots for each Bessel function. The resulting plots of $J_0(x)$, $J_1(x)$, and $J_2(x)$ over the domain $x = 0$ to $x = 20$ are shown in Figure 1, with the first five roots of each function marked as scatter points.

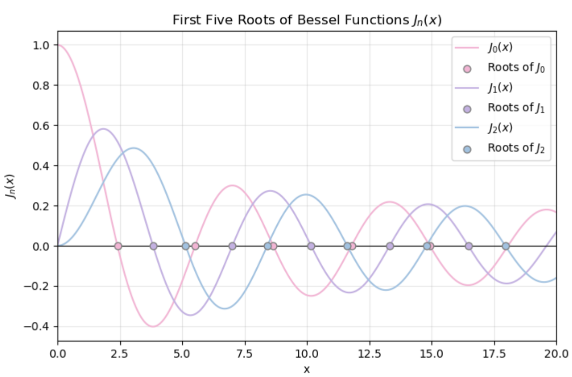


**Figure 1.** Bessel functions $J_0(x)$, $J_1(x)$, and $J_2(x)$ plotted from $x = 0$ to $x = 20$, with their first five positive roots represented as circular markers.

## 3. RESULTS

The plots of $J_0(x)$, $J_1(x)$, and $J_2(x)$ from $x = 0$ to $x = 20$ show that Bessel functions of the first kind exhibit oscillatory behavior with gradually decaying amplitude. As expected, $J_0(x)$ begins at 1, while higher-order functions satisfy $J_n(x) = 0$. The zero crossings become slightly less frequent as $x$ increases.

The computed roots correspond to the first five positive values of $x$ for which $J_n(x) = 0$ and are marked as circular points in Figure 1. These roots are important in radial problems where boundary conditions require the function to vanish at a specific radius. For example, they determine the allowed energy levels in circular quantum wells, resonance frequencies in drumhead vibrations, and decay rates of thermal modes in cylindrical heat conduction.

Our numerical results followed the expected trend that root values increase with both the order $n$ and the root index. This quantized structure highlights the physical significance of Bessel function solutions in bounded radial domains.

## 4. APPLICATIONS

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Bessel functions arise naturally in the solutions to a wide variety of physical problems that exhibit radial symmetry. In this section, we highlight three examples: the infinite square well in quantum mechanics, radial thermal diffusion, and wave propagation on a circular membrane. In each case, Bessel functions emerge from imposing boundary conditions on the radial part of a separable partial differential equation.

### 4.1. *Quantum Mechanics: The Infinite Square Well*

Bessel functions appear in the solution to the Schrödinger equation for a particle confined in a three-dimensinoal infinite spherical potential well. The potential is defined as:

$$V(x) = \begin{cases} 0, & \text{if } r < a \\ \infty, & \text{if } r \geq a \end{cases} \tag{4}$$

The time-independent radial Schrödinger equation in spherical coordinates becomes:

$$\hat{H}\psi - E\psi = \hat{H}R_{n,l} - ER_{n,l} = 0 \tag{5}$$

$$\hat{H} = \frac{-\hbar^2}{2m}\left(\frac{\partial}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{l(l+1)}{r^2}\right) + V(r) \tag{6}$$

where $\hat{H}$ is the Hamiltonian operator and $E$ is the energy eigenvalue.

Inside the well ($r < a$), the radial equation reduces to:

$$r^2\frac{\partial R_{n,l}}{\partial r^2} + 2r\frac{\partial R_{n,l}}{\partial r} + (k^2r^2 - l(l+1))R_{n,l} = 0 \tag{7}$$

This is recognizeed as the spherical Bessel differential equation. Its solutions are the spherical Bessel functions $j_l(k_{n,l}, r)$, which are related to the ordinary Bessel functions by:

$$j_l(k_{n,l}, r) = \sqrt{\frac{\pi}{2kr}}J_{l+\frac{1}{2}}(k_{n,l}r) \tag{8}$$

Imposing the boundary condition $j_l(ka) = 0$ leads to discrete values $k_{n,l}$, which quantize the energy levels as:

$$E_{n,l} = \frac{\hbar^2 k_{n,l}^2}{2ma^2} \tag{9}$$

Thus, the Bessel function roots determine the allowed energy states of the system.

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The Hamiltonian operator $\hat{H}$ and energy operator $\hat{E}$ must also be defined to compute the wavefunction. The $n$ and $i$ are quantum numbers where $n$ is the principle quantum number and $I$ is the angular momentum quantum number where both span the integer range from zero to infinity. $\hbar$ is Planck's constant, $m$ is the mass of the particle, $r$ is the radius, and $R_{n,l}$ is the wavefunction in the radial basis.

$$\hat{H} = \frac{-\hbar^2}{2m}\left(\frac{\partial}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{l(l+1)}{r^2}\right) + V(r) \tag{10}$$

$$\hat{H}\psi - E\psi = \hat{H}R_{n,l} - ER_{n,l} = 0 \tag{11}$$

Applying the Hamiltonial operator on the wavefunction gives us the spherical Bessel function in differential form, which looks similar to the origianl Bessel function in differential form:

$$r^2\frac{\partial R_{n,l}}{\partial r^2} + 2r\frac{\partial R_{n,l}}{\partial r} + (k^2r^2 - l(l+1))R_{n,l} = 0 \tag{12}$$

To obtain a solution, we impose a boundary condition $j_l(k_{n,l}a) = 0$ at the bounds of the well. This quantizes our solution by setting a discrete energy level $E_{n,l} = \frac{\hbar^2 k_{n,l}^2}{2ma^2}$ where the solution exists. Hence, we are left with a radial solution to the wave equation containing the original Bessel Function:

$$R_{n,l} = Aj_l(k_{n,l}r) \tag{13}$$

where $j_l(k_{n,l}, r) = \sqrt{\frac{\pi}{2kr}}J_{l+\frac{1}{2}}(k_{n,l}r)$.

### 4.2. *Thermal Diffusion*

Bessel functions also arise in radial heat conduction problems. The temperature distribution $T(r,t)$ in a two-dimensional circular region is governed by the heat equation:

$$\frac{\partial T}{\partial t} = \alpha\nabla^2 T \tag{14}$$

$$= \alpha\left(\frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial r^2}\right) \tag{15}$$

where $\alpha = \frac{k}{\rho c_p}$.

Using separation of variables with $T(r,t) = X(r)\theta(t)$, the spatial equation becomes:

$$\frac{d^2X}{dr^2} + \frac{1}{r}\frac{X}{r} + \lambda^2 X = 0. \tag{16}$$

THis is the Bessel equation of order zero. Solutions involve the Bessel function $J_0(\lambda r)$, and applying the boundary condition $T(r,t) = T_1$ leads to quantization in terms of the roots $\beta_n$ such that $J_0(\beta_0) = 0$.

The full solution is:

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The decay of each mode over time depends on the root $\beta_n$, which emphasizes how Bessel zeroes influence the thermal dynamics of circular geometries.

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We define the heat equation propagating through a two-dimensional surface in the radial direction. $T$ is the temperature, $t$ is the time, $r$ is the radius, $k$ is the material conductivity, $\rho$ is the density of the material, and $c_p$ is the specific heat capacity.

$$\frac{\partial T}{\partial t} = \alpha\nabla^2 T \tag{17}$$

$$= \alpha\left(\frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial r^2}\right) \tag{18}$$

where $\alpha = \frac{k}{\rho c_p}$.

To solve this equation, we use separation of variables to split the differential equation into two separate differential equations and assume that one takes the form $T(r,t) = X(r)\theta(t)$. The first equation takes the form of exponential decay over time:

$$\frac{d\theta}{dt} + \lambda^2\alpha\theta = 0 \tag{19}$$

whereas the second equation is the Bessel function of the zeroth-order since $\alpha = 0$:

$$\frac{d^2X}{dr^2} + \frac{1}{r}\frac{X}{r} + \lambda^2 X = 0. \tag{20}$$

We set a boundary condition of $T(r,t) = T_1$, where $R$ is the radius of the circle, $t$ is time with $t > 0$, and $T_1$ is a constant real value. The initial condition for the radial space is defined as $T(r,0) = T_2$, where $T_2$ is a constant real value.

$$T^* = \frac{T(r,t) - T(R,t)}{T(r,0) - T(R,t)} = 2\sum_{n=0}^{\infty} e^{-\beta_n^2 \frac{\alpha t}{R^2}} \frac{J_0(\beta_n \frac{r}{R})}{\beta_n J_1(\beta_n)} \tag{21}$$

$$T(r,t) = T^*(T(r,0) - T(R,t)) + T(R,t) \tag{22}$$

We define temperature as a unitless quantity $T^*$. $\beta_n$ is the nth-root solution where $J_0(r) = 0$. The temperature function uses both the zeroth and first order Bessel function in the solution. Hence, Equation 21 and Equation 22 is the final solution for that the temperature takes.

### 4.3. *Drum Wave Propagation*

The vibration of a circular membrane, such as a drumhead, is another problem where Bessel functions appear. The vertical displacement $z(r,\theta,t)$ satisfies the two-dimensional wave equation:

$$\frac{\partial^2 z}{\partial t^2} = c^2 \nabla^2 z \tag{23}$$

where $c^2 = \frac{\sigma^2}{S}$.

Assuming axisymmetric vibrations and separating variables, we arrive at the radial equation:

$$r^2 \frac{\partial^2 R}{\partial r^2} + r\frac{\partial R}{\partial r} + (\lambda^2 r^2 - n^2)R = 0 \tag{24}$$

whose solutions are $J_m(\lambda r)$. The boundary condition $(R_f) = 0$ implies that $\lambda$ must be a root $\lambda_{m,k}$ of the Bessel function $J_m(\lambda r)$. Thus, the full solution becomes:

REWRITE/SIMPLLIFY EQN TMRW

These roots define the resonance frequencies of the drumhead and determine its modes of vibration.

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Drum wave propagation is similar to thermal diffusion regarding classical wave mechanics and quantum mechanics through the quantization of boundary conditions. We define a wavefunction for the propagation across a drum surface where $\sigma$ is the surface mass density of the membrane and $S$ is the surface tension across the membrane:

$$\frac{\partial^2 z}{\partial t^2} = c^2 \nabla^2 z \tag{25}$$

where $c^2 = \frac{\sigma^2}{S}$.

The first boundary condition is $z(R_f,t) = 0$. The displacement from the origin along the z-axis must be zero at the edge of the drum since those points are fixed. This condition quantizes the solution, yielding discrete Bessel functions that satisfy the boundary conditions. To simplify the example, we impose the boundary condition $z(r,\theta,0) = f(r)$ for $0 \leq r \leq a$, representing the initial pertubation caused by striking the surface at its center. This approach preserves the $\theta$-dependence in the solution, allowing the Bessel function to appear explicitly in the radial component.

$$z(r,\theta,t) = R(r)T(t)\Theta(\theta) = R(r)T(t) \tag{26}$$

$$r^2 \frac{\partial^2 R}{\partial r^2} + r\frac{\partial R}{\partial r} + (\lambda^2 r^2 - n^2)R = 0 \tag{27}$$

$$\frac{dT}{dt} + \lambda^2 cT = 0 \tag{28}$$

$$n = \lambda_{m,k} \tag{29}$$

$$z(r,t) = \sum_{m=0}^{\infty}\sum_{k=1}^{\infty} J_m(\lambda_{m,k} r) e^{-c\lambda_{m,n} t} \tag{30}$$

Finally, impose a boundary condition enforcing zero displacement at the drum's boundary:

$$J_m(\lambda_{m,k} R_f) = 0, \tag{31}$$

where $m$ is the order of the Bessel function and $k$ is the wavenumber. For $m = 0$, the Bessel function itself vanishes at the boundary. There exist infinitely many integer values of $m$ and $k$ for which this condition is satisfied, corresponding to the distinct vibration modes of the drumhead. The resulting function $z(r,t)$ describes the final waveform of the drum's vibration in time.

## 5. SUMMARY AND CONCLUSION

In this project, we explored the roots of the Bessel functions of the first kind, $J_n(x)$, for orders $n = 0, 1, 2$ using Python's `scipy` library. Our numerical approach combined the built-in evaluation of $J_n(x)$ with the `fsolve` method to accurately identify the first five positive roots for each order. Our results aligned well with the expected theoretical behavior, including the trends in oscillation decay and root spacing.

The numerical methods we used were sufficiently accurate for our goals. Plotting the functions alongside their roots helped us visualize their role in physical boundary value problems with radial symmetry. While the method was effective, it relied on manually chosen initial guesses. This could be improved in future work by using analytical approximations or implementing a more automated root-finding strategy.

Overall, this project demonstrated how Bessel functions can be approached computationally, how their oscillatory behavior varies with order, and why their roots are important physical systems. Further work could involve exploring other families of Bessel functions or extending the root-finding method to more advanced applications involving partial differential equations where these functions naturally arise due to symmetry. These additions would provide deeper insight into the mathematical structure and wide-ranging applications of Bessel functions.

### 5.1. *Acknowledgements*

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## REFERENCES

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