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Bessel Functions: Theory and Applications

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ABSTRACT

We go over three physical scenarios where Bessel Functions are used. The first scenario is the infinite square well quantum mechanical wave function in spherical coordinates. The second example scenario involves solving the temperature equation for thermal diffusion through a material. The final example models the vibration along the drum head right after it is struck in the middle.

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INTRODUCTION

For our applications we go over three different physical scenarios where the conversion of coordinate from cartesian to the coordinate with a radial dependency has solutions that contain a Bessel function. The key point between each scenario is that the radial dependency in the new coordinate systems transforms the second order differential equation whose solution is sinusoidal with a damping term or critical damping into a Bessel function along the radial basis. The Bessel function decays to 0 as the input parameter for each physical situation radial basis moves toward infinity. However, the physical model does resemble the exponential decay pattern found in the physical problems such as a spring in fluid simulation.

DATA AND OBSERVATIONS

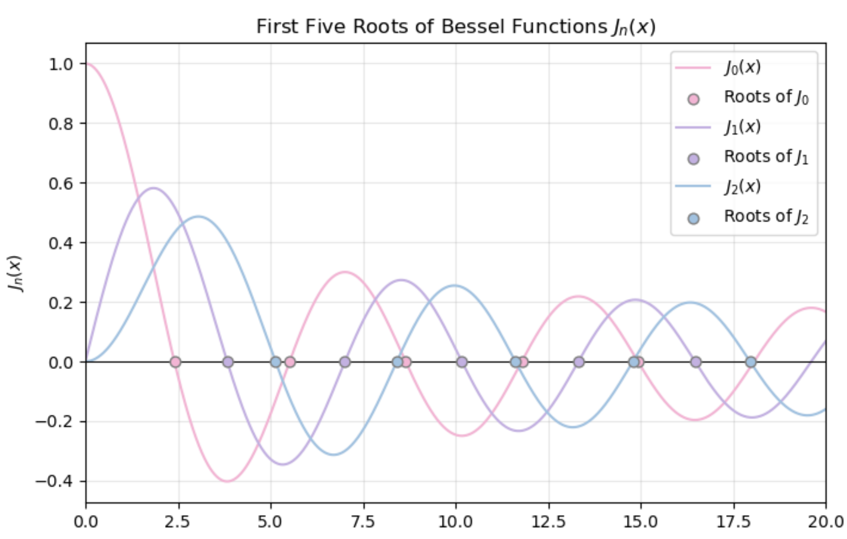

Figure 1. some caption here

APPLICATIONS

Quantum Mechanics: The Infinite Square Well

The Bessel function appears in the solution to the wave function of an infinite square well in spherical coordinates. To solve the quantum mechanics problem, we define a potential well:

$$V(x) = \begin{cases} 0, & \text{if } r < a \\ \infty, & \text{if } r \geq a \end{cases} \tag{1}$$

The Hamiltonian operator $\hat{H}$ and energy operator $\hat{E}$ must also be defined to compute the wavefunction. The n and i are quantum numbers where n is the principle quantum number and I is the angular momentum quantum number where both span the integer range from zero to infinity. $\hbar$ is Planck's constant, m is the mass of the particle, r is the radius, and $R_{n,l}$ is the wavefunction in the radial basis.

$$\hat{H} = \frac{-\hbar^2}{2m}\left(\frac{\partial}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{l(l+1)}{r^2}\right) + V(r) \tag{2}$$

$$\hat{H}\psi - E\psi = \hat{H}R_{n,l} - ER_{n,l} = 0 \tag{3}$$

Applying the Hamiltonial operator on the wavefunction gives us the spherical Bessel function in differential form, which looks similar to the origianl Bessel function in differential form:

$$r^2 \frac{\partial R_{n,l}}{partial r^2} + 2r\frac{\partial R_{n,l}}{\partial r} + (k^2r^2 - l(l+1))R_{n,l} = 0 \tag{4}$$

To obtain a solution, we impose a boundary condition $j_l(k_{n,l}a) = 0$ at the bounds of the well. This quantizes our solution by setting a discrete energy level $E_{n,l} = \frac{\hbar^2 k_{n,l}^2}{2ma^2}$ where the solution exists. Hence, we are left with a radial solution to the wave equation containing the original Bessel Function:

$$R_{n,l} = Aj_l(k_{n,l}r) \tag{5}$$

where $j_l(k_{n,l}, r) = \sqrt{\frac{\pi}{2kr}} J_{l+\frac{1}{2}}(k_{n,l}R)$.

Thermal Diffusion

Drum Wave Propagation

Drum wave propagation is similar to thermal diffusion regarding classical wave mechanics and quantum mechanics through the quantization of boundary conditions. We define a wavefunction for the propagation across a drum surface where σ is the surface mass density of the membrane and S is the surface tension across the membrane:

$$\frac{\partial^2 z}{\partial t^2} = c^2 \nabla^2 z \tag{6}$$

where $c^2 = \frac{\sigma^2}{S}$.

The first boundary condition is $z(R_f, t) = 0$. The displacement ...

RESULTS

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REFERENCES