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Numerical Analysis of Bessel Function Roots and Applications in Physical Systems

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ABSTRACT

In this project, we numerically computed and visualized the first five positive roots of the Bessel functions of the first kind, $J_0(x)$, $J_1(x)$, and $J_2(x)$, using Python. These functions solve a second-order differential equation that occur in problems with radial symmetry. We used the `scipy` library to evaluate $J_n(x)$ and locate its roots via the `fsolve` method. We interpreted the computed roots in the context of three physical materials: the radial wavefunction in a quantum infinite square well, heat conduction in cylindrical geometries, and vibrational modes of a circular drumhead. Our results show how Bessel function roots reflect physical boundary conditions and demonstrate the usefulness of numerical methods in modeling such systems.

Keywords: Bessel functions — root-finding — radial symmetry — boundary conditions — numerical analysis

1. INTRODUCTION

The Bessel functions of the first kind, $J_n(x)$, are solutions to the second-order linear differential equation:

$$z^2\frac{d^2w}{dz^2} + z\frac{dw}{dz} + (z^2 - n^2)w = 0, \tag{1}$$

where integer $n = 0, 1, 2\ldots$ represents the order of the function. These functions appear naturally in the separation of variables when solving partial differential equations in cylindrical or spherical coordinates, particularly in systems with radial symmetry Abramowitz & Stegun (1972). Bessel functions also have an integral

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} cos(x\sin\theta - n\theta)d\theta \tag{2}$$

for integers $n = 0, 1, 2\ldots$, which is particularly useful for understanding their oscillatory behavior.

Bessel functions were first introduced by German astoronomer and mathematician Friedrich Wilhelm Bessel in the early 1800s during his study of planetary orbits. However, the equation itself had been investigated earlier by Bernoulli and Euler in problems involving vibrational membranes Abramowitz & Stegun (1972). Today, Bessel functions are foundational tools in mathematical physics and engineering. In quantum mechanics, they appear in radial solutions to the Schrödinger equation for quantum wells CITE, in the vibration modes of circular membranes like drumheads CITE, and in heat conduction problems with cylindrical geometries CITE.

In this project, we numerically compute the first five positive roots of the Bessel functions $J_0(x)$, $J_1(x)$, and $J_2(x)$. These roots correspond to physically meaningful quantities such as resonance modes, cutoff frequencies, or quantized boundary values in systems with radial symmetry. We use Python to visualize each function, estimate the approximation locations of their roots, and apply the `fsolve` method from `scipy.optimize` to compute each root with high precision. Our goal is to obtain accurate roots values for each order and verify them using numerical methods

2. DATA AND OBSERVATIONS

This project focuses on computing and visualizing the first five roots of the Bessel functions of the first kind, $J_n(x)$, for orders $n = 0, 1, 2$. These functions solve the second-order linear differential equation:

$$z^2\frac{d^2w}{dz^2} + z\frac{dw}{dz} + (z^2 - n^2)w = 0, \tag{3}$$

which arise in physical systems with spherical or cylindrical symmetry Abramowitz & Stegun (1972). The positive roots of $J_n(x)$ represent physically meaningful quantities such as resonant frequencies or quantized energy levels in such systems.

We computed the Bessel functions using Python's `scipy.special.jv` method, which evaluates $J_n(x)$ for an arbitrary order and argument. To find the roots numerically, we used the `fsolve` method from `scipy.optimize`, which refines an initial guess until it finds a point where the function crosses zero. We chose a set of initial guesses based on where the function appeared to cross the x-axis in the plot and passed those values into `fsolve` to calculate each root more precisely. We used the following initial guesses:

- $J_0(x)$: [2, 6.1, 8.6, 11.7, 15]
- $J_1(x)$: [3.9, 7, 10.15, 13.1, 16.4]
- $J_2(x)$: [5.1, 8.3, 11.8, 14.9, 18]

This gave us exactly five positive roots for each Bessel function. The resulting plots of $J_0(x)$, $J_1(x)$, and $J_2(x)$ over the domain $x = 0$ to $x = 20$ are shown in Figure 1, with the first five roots of each function marked as scatter points.

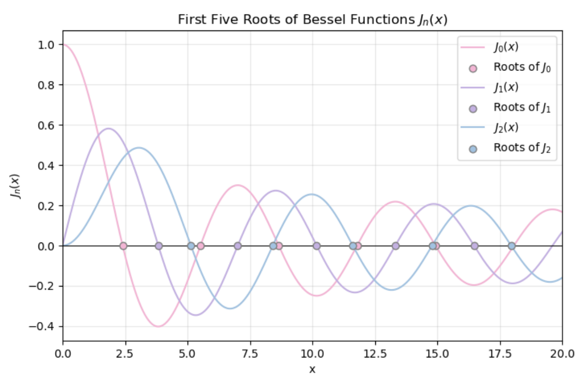

Figure 1. Bessel functions $J_0(x)$, $J_1(x)$, and $J_2(x)$ plotted from $x = 0$ to $x = 20$, with their first five positive roots represented as circular markers.

3. RESULTS

The plots of $J_0(x)$, $J_1(x)$, and $J_2(x)$ from $x = 0$ to $x = 20$ show that Bessel functions of the first kind exhibit oscillatory behavior with gradually decaying amplitude. As expected, $J_0(x)$ begins at 1, while higher-order functions satisfy $J_n(x) = 0$. The zero crossings become slightly less frequent as x increases.

The computed roots correspond to the first five positive values of x for which $J_n(x) = 0$ and are marked as circular points in Figure 1. These roots are important in radial problems where boundary conditions require the function to vanish at a specific radius. For example, they determine the allowed energy levels in circular quantum wells, resonance frequencies in drumhead vibrations, and decay rates of thermal modes in cylindrical heat conduction.

Our numerical results followed the expected trend that root values increase with both the order n and the root index. This quantized structure highlights the physical significance of Bessel function solutions in bounded radial domains.

4. APPLICATIONS

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Bessel functions arise naturally in the solutions to a wide variety of physical problems that exhibit radial symmetry. In this section, we highlight three examples: the infinite square well in quantum mechanics, radial thermal diffusion, and wave propagation on a circular membrane. In each case, Bessel functions emerge from imposing boundary conditions on the radial part of a separable partial differential equation.

4.1. *Quantum Mechanics: The Infinite Square Well*

Bessel functions appear in the solution to the Schrödinger equation for a particle confined in a three-dimensinoal infinite spherical potential well. The potential is defined as:

$$V(x) = \begin{cases} 0, & \text{if } r < a \\ \infty, & \text{if } r \geq a \end{cases} \tag{4}$$

Inside the well ($r < a$), the time-independent Schrödinger equation in spherical coordinates becomes:

$$\hat{H}\psi - E\psi = \hat{H}R_{n,l} - ER_{n,l} = 0 \tag{5}$$

$$\hat{H} = \frac{-\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{l(l+1)}{r^2}\right) + V(r) \tag{6}$$

where $R_{n,l}(r)$ is the radial wavefunction, n is the principle quantum number, l is the angular momentum quantum number, m is the particle mass, and $\hbar$ is the reduced Planck constant.

With $V(r) = 0$ inside the well, the radial differential equation reduces to:

$$r^2\frac{\partial^2 R_{n,l}}{\partial r^2} + 2r\frac{\partial R_{n,l}}{\partial r} + (k^2r^2 - l(l+1))R_{n,l} = 0 \tag{7}$$

This is the spherical Bessel differential equation, and its solutions are spherical Bessel functions $j_l(k_{n,l}r)$, which relate to the ordinary Bessel functions by:

$$j_l(k_{n,l}r) = \sqrt{\frac{\pi}{2kr}} J_{l+\frac{1}{2}}(k_{n,l}r) \tag{8}$$

To satisfy the boundary condition that the wavefunction vanishes at $r = a$, we require that $j_l(k_{n,l}a) = 0$. This quantization conditions selects discrete values $k_{n,l}$ which in turn determine the allowed energy levels:

$$E_{n,l} = \frac{\hbar^2 k_{n,l}^2}{2ma^2} \tag{9}$$

Thus, the roots of the spherical Bessel functions determine the quantized energy spectrum of the particle in the well. This example illustrates how the mathematical properties of Bessel functions have direct physical consequences in quantum systems with spherical symmetry.

4.2. *Thermal Diffusion*

Bessel functions also appear in heat conduction problems with radial symmetry. In a two-dimensional circular region, the temperature distribution $T(r,t)$ evolves according to the heat equation:

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T \tag{10}$$

$$= \alpha \left(\frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial r^2} \right) \tag{11}$$

where $\alpha = \frac{k}{\rho c_p}$ is the thermal diffusivity, with k as the thermal conductivity, ρ the material density, and c_p the specific heat capacity.

To solve this equation, we assume separable solutions of the form $T(r,t) = X(r)\theta(t)$, which leads us to two ordinary differential equations. The time-dependent part gives exponential decay:

$$\frac{d\theta}{dt} + \lambda^2 \alpha \theta = 0, \tag{12}$$

and the spatial part gives the Bessel differential equation of order zero:

$$\frac{d^2X}{dr^2} + \frac{1}{r}\frac{dX}{dr} + \lambda^2 X = 0. \tag{13}$$

The general solution involves $J_0(\lambda r)$, and applying boundary conditions quantizes the solutions in terms of the roots β_n where $J_0(\beta_n) = 0$.

We impose the initial condition $T(r,0) = T_2$ and the boundary condition $T(R,t) = 0$, where R is the radius of the domain and T_1, T_2 are constant values. The resulting dimensionsionless solution can be written as:

$$T^*(r,t) = \frac{T(r,t) - T(R,t)}{T(r,0) - T(R,t)} = 2\sum_{n=0}^{\infty} e^{-\beta_n^2 \frac{\alpha t}{R^2}} \frac{J_0(\beta_n \frac{r}{R})}{\beta_n J_1(\beta_n)} \tag{14}$$

thus, making the full temperature solution:

$$T(r,t) = T^*(T(r,0) - T(R,t)) + T(R,t). \tag{15}$$

The temperature function uses both the zeroth and first order Bessel function in the solution. Thus, we see how the Bessel function determine the the dynamics of thermal modes, with each mode decaying exponentially at a rate set by β_n.

4.3. *Drum Wave Propagation*

The vibration of a circular membrane, such as a drumhead, is another problem where Bessel functions appear. The vertical displacement $z(r,\theta,t)$ satisfies the two-dimensional wave equation:

$$\frac{\partial^2 z}{\partial t^2} = c^2 \nabla^2 z \tag{16}$$

where $c^2 = \frac{\sigma^2}{S}$.

Assuming axisymmetric vibrations and separating variables, we arrive at the radial equation:

$$r^2 \frac{\partial^2 R}{\partial r^2} + r\frac{\partial R}{\partial r} + (\lambda^2 r^2 - n^2)R = 0 \tag{17}$$

whose solutions are $J_m(\lambda r)$. The boundary condition $(R_f) = 0$ implies that λ must be a root $\lambda_{m,k}$ of the Bessel function $J_m(\lambda r)$. Thus, the full solution becomes:

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These roots define the resonance frequencies of the drumhead and determine its modes of vibration.

Drum wave propagation is similar to thermal diffusion regarding classical wave mechanics and quantum mechanics through the quantization of boundary conditions. We define a wavefunction for the propagation across a drum surface where σ is the surface mass density of the membrane and S is the surface tension across the membrane:

$$\frac{\partial^2 z}{\partial t^2} = c^2 \nabla^2 z \tag{18}$$

where $c^2 = \frac{\sigma^2}{S}$.

The first boundary condition is $z(R_f, t) = 0$. The displacement from the origin along the z-axis must be zero at the edge of the drum since those points are fixed. This condition quantizes the solution, yielding discrete Bessel functions that satisfy the boundary conditions. To simplify the example, we impose the boundary condition $z(r, \theta, 0) = f(r)$ for $0 \leq r \leq a$, representing the initial pertubation caused by striking the surface at its center. This approach preserves the θ-dependence in the solution, allowing the Bessel function to appear explicitly in the radial component.

$$z(r, \theta, t) = R(r)T(t)\Theta(\theta) = R(r)T(t) \tag{19}$$

$$r^2 \frac{\partial^2 R}{\partial r^2} + r \frac{\partial R}{\partial r} + (\lambda^2 r^2 - n^2)R = 0 \tag{20}$$

$$\frac{dT}{dt} + \lambda^2 c T = 0 \tag{21}$$

$$n = \lambda_{m,k} \tag{22}$$

$$z(r, t) = \sum_{m=0}^{\infty} \sum_{k=1}^{\infty} J_m(\lambda_{m,k} r) e^{-c\lambda_{m,n} t} \tag{23}$$

Finally, impose a boundary condition enforcing zero displacement at the drum's boundary:

$$J_m(\lambda_{m,k} R_f) = 0, \tag{24}$$

where m is the order of the Bessel function and k is the wavenumber. For $m = 0$, the Bessel function itself vanishes at the boundary. There exist infinitely many integer values of m and k for which this condition is satisfied, corresponding to the distinct vibration modes of the drumhead. The resulting function $z(r, t)$ describes the final waveform of the drum's vibration in time.

5. SUMMARY AND CONCLUSION

In this project, we explored the roots of the Bessel functions of the first kind, $J_n(x)$, for orders $n = 0, 1, 2$ using Python's `scipy` library. Our numerical approach combined the built-in evaluation of $J_n(x)$ with the `fsolve` method to accurately identify the first five positive roots for each order. Our results aligned well with the expected theoretical behavior, including the trends in oscillation decay and root spacing.

The numerical methods we used were sufficiently accurate for our goals. Plotting the functions alongside their roots helped us visualize their role in physical boundary value problems with radial symmetry. While the method was effective, it relied on manually chosen initial guesses. This could be improved in future work by using analytical approximations or implementing a more automated root-finding strategy.

Overall, this project demonstrated how Bessel functions can be approached computationally, how their oscillatory behavior varies with order, and why their roots are important physical systems. Further work could involve exploring other families of Bessel functions or extending the root-finding method to more advanced applications involving partial differential equations where these functions naturally arise due to symmetry. These additions would provide deeper insight into the mathematical structure and wide-ranging applications of Bessel functions.

5.1. *Acknowledgements*

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