PHY338K Electronic Techniques

Fall 2025

Homework 2 Solutions

1. (30 points) <u>Oscilloscope probe</u>

a) C_{cable} is in parallel with C_{scope} to make a total capacitance

$C_T = C_{cable} + C_{scope} = 81 + 14 = 95$ pF.

C_T and R_{scope} are in parallel, which gives $Z_L = \dfrac{R_{scope}\dfrac{1}{j\omega C_T}}{R_{scope} + \dfrac{1}{j\omega C_T}} = \dfrac{R_{scope}}{1 + j\omega C_T R_{scope}}$

b) $|Z_L| = \dfrac{R_{scope}}{|1 + j\omega C_T R_{scope}|} = \dfrac{R_{scope}}{\sqrt{1 + \left(\omega C_T R_{scope}\right)^2}}$

For $f = 0.50$ kHz, $|Z_L| = \dfrac{1\ \text{M}\Omega}{\sqrt{1 + \left(2\pi \times 500 \times 95 \times 10^{-12} \times 10^6\right)^2}} = \dfrac{1\ \text{M}\Omega}{\sqrt{1 + (0.299)^2}} = 958\ \text{k}\Omega$

For $f = 3.0$ kHz, $|Z_L| = \dfrac{1\ \text{M}\Omega}{\sqrt{1 + \left(2\pi \times 3.0 \times 10^3 \times 95 \times 10^{-12} \times 10^6\right)^2}} = \dfrac{1\ \text{M}\Omega}{\sqrt{1 + (1.79)^2}} = 488\ \text{k}\Omega$

For $f = 20$ kHz, $|Z_L| = \dfrac{1\ \text{M}\Omega}{\sqrt{1 + \left(2\pi \times 20 \times 10^3 \times 94 \times 10^{-12} \times 10^6\right)^2}} = \dfrac{1\ \text{M}\Omega}{\sqrt{1 + (11.9)^2}} = 83.5\ \text{k}\Omega$

For the frequency $f = \dfrac{1}{2\pi R_{scope} C_T} = 1.68\ \text{kHz},$ the resistor impedance R_{scope} is equal to the magnitude of the capacitor impedance $\dfrac{1}{\omega C_T}$. The capacitor impedance falls with increasing frequency. At frequencies above 1.68 kHz the capacitor impedance will be smaller than the resistor impedance. Since the components are in parallel, the falling capacitor impedance causes $|Z_L|$ to fall with increasing frequency, with the effect being more prominent at frequencies above 1.68 kHz.

c) The circuit is an AC voltage divider consisting of Z_L and *R*, with the oscilloscope measuring the voltage across Z_L. The complex impedance Z_L is equal to

$$Z_L = \frac{R_{scope}}{1 + j\omega C_T R_{scope}} = \frac{R_{scope}}{1 + j\omega C_T R_{scope}} \frac{1 - j\omega C_T R_{scope}}{1 - j\omega C_T R_{scope}} = \frac{R_{scope} - j\omega C_T R_{scope}{}^2}{1 + \left(\omega C_T R_{scope}\right)^2}$$

At $\underline{f = 0.50\text{ kHz}}$, $\omega C_T R_{scope} = 2\pi \times 500 \times 95 \times 10^{-12} \times 10^6 = 0.298$ and

$$Z_L = \frac{R_{scope} - j\omega C_T R_{scope}{}^2}{1 + \left(\omega C_T R_{scope}\right)^2} = \frac{10^6 - j(0.298)10^6}{1 + 0.298^2} = 918\text{ k}\Omega - j274\text{ k}\Omega$$

In polar form this is $Z_L = \left(\sqrt{918^2 + 274^2},\ \tan^{-1}\left(\frac{-274}{918}\right)\right) = \left(958\text{ k}\Omega,\ \angle -16.6°\right)$

The total impedance seen by V_{trans} is $Z_T = R_{trans} + Z_L = 500 + 918 - j274 = 1418\text{ k}\Omega - j274\text{ k}\Omega$.

We can write this in polar form as $Z_T = \left(\sqrt{1418^2 + 274^2},\ \tan^{-1}\left(\frac{-274}{1418}\right)\right) = \left(1444\text{ k}\Omega,\ \angle -10.9°\right)$

The voltage displayed by the oscilloscope is

$$\tilde{V}_{scope} = \tilde{V}_{trans}\frac{Z_L}{Z_T} = \frac{\left(100\text{ mV},\ \angle 0°\right)\left(958\text{ k}\Omega,\ \angle -16.6°\right)}{\left(1444\text{ k}\Omega,\ \angle -10.9°\right)} = \boxed{66.3\text{ mV},\ \angle -5.7°}.$$

Repeating this for $\underline{f = 3.0\text{ kHz}}$ gives $\omega C_T R_{scope} = 2\pi \times 3 \times 10^3 \times 95 \times 10^{-12} \times 10^6 = 1.791$

$$Z_L = \frac{R_{scope} - j\omega C_T R_{scope}{}^2}{1 + \left(\omega C_T R_{scope}\right)^2} = \frac{10^6 - j(1.791)10^6}{1 + 1.791^2} = 238\text{ k}\Omega - j426\text{ k}\Omega$$

$$= \left(\sqrt{238^2 + 426^2},\ \tan^{-1}\left(\frac{-426}{238}\right)\right) = \left(488\text{ k}\Omega,\ \angle -60.8°\right)$$

$$Z_T = R_{trans} + Z_L = 500 + 488 - j426 = 988\text{ k}\Omega - j426\text{ k}\Omega$$

$$= \left(\sqrt{988^2 + 426^2},\ \tan^{-1}\left(\frac{-426}{988}\right)\right) = \left(1076\text{ k}\Omega,\ \angle -23.3°\right)$$

$$\tilde{V}_{scope} = \tilde{V}_{trans}\frac{Z_L}{Z_T} = \frac{\left(100\text{ mV},\ \angle 0°\right)\left(488\text{ k}\Omega,\ \angle -60.8°\right)}{\left(1076\text{ k}\Omega,\ \angle -23.3°\right)} = \boxed{45.4\text{ mV},\ \angle -37.5°}.$$

Repeating this for $\underline{f = 20\text{ kHz}}$ gives $\omega C_T R_{scope} = 2\pi \times 20 \times 10^3 \times 95 \times 10^{-12} \times 10^6 = 11.94$

$$Z_L = \frac{R_{scope} - j\omega C_T R_{scope}{}^2}{1 + \left(\omega C_T R_{scope}\right)^2} = \frac{10^6 - j(11.94)10^6}{1 + 11.94^2} = 6.97\text{ k}\Omega - j83.2\text{ k}\Omega$$

$$= \left(\sqrt{6.97^2 + 83.2^2},\ \tan^{-1}\left(\frac{-83.2}{6.97}\right)\right) = \left(83.5\text{ k}\Omega,\ \angle -85.2°\right)$$

$$Z_T = R_{trans} + Z_L = 500 + 6.97 - j83.2 = 507\text{ k}\Omega - j83.2\text{ k}\Omega$$

$$= \left(\sqrt{507^2 + 83.2^2},\ \tan^{-1}\left(\frac{-83.2}{507}\right)\right) = \left(514\text{ k}\Omega,\ \angle -9.3°\right)$$

$$\tilde{V}_{scope} = \tilde{V}_{trans}\frac{Z_L}{Z_T} = \frac{(100\text{ mV}, \angle 0°)(83.5\text{ k}\Omega, \angle -85.2°)}{(514\text{ k}\Omega, \angle -9.3°)} = \boxed{16.2\text{ mV}, \angle -75.9°}.$$

The loading of the signal by the impedance Z_L causes the displayed voltage to be smaller than and phase-shifted from $\tilde{V}_{trans}$. The effects become especially large at higher frequencies. So no, the oscilloscope does not display a true and accurate measurement of $V_{trans}(t)$. The discrepancy is worse at higher frequencies because the capacitive loading increases with frequency.

d) The loading has a significant effect if $|Z_L|$ is comparable to our less than $|Z_S|$, where Z_S is the impedance of the signal source. In the above example, $|Z_S| = 500\text{ k}\Omega$ was quite high, so that the loading effect was significant even for $|Z_L| = 1\text{ M}\Omega$. However in Lab 2 we measured the voltage on *RC* circuits. We mostly used a value of $R = 20\text{ k}\Omega$ and $C = 5\text{ nF}$. The impedance of the resistor was small compared to the $1\text{ M}\Omega$ input resistance of the oscilloscope. Also, at any given frequency the magnitude of the capacitor impedance $\dfrac{1}{\omega C} = \dfrac{1}{\omega(5\text{ nF})}$ was small compared to the magnitude of the capacitive part of the load impedance $\dfrac{1}{\omega C_T} = \dfrac{1}{\omega(95\text{ pF})}$. Therefore the lab 2 measurements satisfied the condition $|Z_L| >> |Z_S|$, and this kept the loading effects on the measurements small.

2. a) I'm sorry that there was a typo in this problem. The goal should have been to make the complex impedance Z_2 to be exactly 1/9 of the complex impedance Z_1, not 1/10. I'll proceed with this correction. Let $C_T = C_{scope} + C_{comp} + C_{cable}$ be the capacitance of the parallel combination of C_{scope}, C_{comp}, and C_{cable}.

The impedances are $Z_1 = \dfrac{R_1 \dfrac{1}{j\omega C_1}}{R_1 + \dfrac{1}{j\omega C_1}} = \dfrac{R_1}{1 + j\omega C_1 R_1}$ and $Z_2 = \dfrac{R_{scope}\dfrac{1}{j\omega C_T}}{R_{scope} + \dfrac{1}{j\omega C_T}} = \dfrac{R_{scope}}{1 + j\omega C_T R_{scope}}$.

The ratio is $\dfrac{Z_1}{Z_2} = \dfrac{R_1}{R_{scope}}\left(\dfrac{1 + j\omega C_T R_{scope}}{1 + j\omega C_1 R_1}\right)$. It will be exactly equal to 9 if $R_1 = 9R_{scope} = 9\text{ M}\Omega$, and if also $C_T R_{scope} = C_1 R_1$. The latter condition is satisfied for $C_T = C_1\dfrac{R_1}{R_{scope}} = 9C_1 = 9(12\text{ pF}) = 108\text{ pF}$.

We can obtain this value with $C_{comp} = C_T - C_{scope} - C_{cable} = 108 - 14 - 80.5 = \boxed{13.5\text{ pF}}$.

b) The circuit of Figure 5 with Z_1 and Z_2 in series has impedance

$$Z_L = Z_1 + Z_2 = \frac{R_1}{1 + j\omega C_1 R_1} + \frac{R_1/9}{1 + j\omega C_1 R_1} = \frac{\frac{10}{9}R_1}{1 + j\omega C_1 R_1}$$

The parallel equivalent circuit in Figure 6 has impedance

$$Z_L = \frac{R_L \frac{1}{j\omega C_L}}{R_L + \frac{1}{j\omega C_L}} = \frac{R_L}{1 + j\omega C_L R_L}$$

These are equal for $\boxed{R_L = \frac{10}{9}R_1 = 10\ \text{M}\Omega}$ and $C_L R_L = C_1 R_1$, which requires that

$$C_L = C_1 \frac{R_1}{R_L} = 14\frac{9}{10} = 12.6\ \text{pF}.$$

The oscilloscope with the probe attached has an impedance consisting of the parallel combination of a resistor that is 10 times as high as the oscilloscope resistive impedance and a capacitor that is about 10 times smaller than the oscilloscope capacitive impedance including a connecting cable.

c) For 0.50 kHz, the new values are

$$\omega C_L R_L = 2\pi \times 500 \times 12.6 \times 10^{-12} \times 10^{7} = 0.396$$

$$Z_L = \frac{R_L - j\omega C_L R_L{}^2}{1 + \left(\omega C_L R_L\right)^2} = \frac{10^7 - j(0.396)10^7}{1 + 0.396^2} = 8.64\ \text{M}\Omega - j3.42\ \text{M}\Omega$$

$$= \left(\sqrt{8.64^2 + 3.42^2},\ \tan^{-1}\left(\frac{-3.42}{8.64}\right)\right) = \left(9.29\ \text{M}\Omega,\ \angle -21.6^\circ\right)$$

$$Z_T = R_{trans} + Z_L = 0.5 + 8.64 - j3.42 = 9.14\ \text{M}\Omega - j3.42\ \text{M}\Omega$$

$$= \left(\sqrt{9.14^2 + 3.42^2},\ \tan^{-1}\left(\frac{-3.42}{9.14}\right)\right) = \left(9.76\ \text{M}\Omega,\ \angle -20.5^\circ\right)$$

$$\tilde{V}_{scope} = \tilde{V}_{trans}\frac{Z_L}{Z_T} = \frac{\left(100\ \text{mV},\ \angle 0^\circ\right)\left(8.64\ \text{M}\Omega,\ \angle -21.6^\circ\right)}{\left(9.76\ \text{M}\Omega,\ \angle -20.5^\circ\right)} = \boxed{88.5\ \text{mV},\ \angle -1.1^\circ}.$$

For 3.0 kHz, the new values are

$$\omega C_L R_L = 2\pi \times 3000 \times 12.6 \times 10^{-12} \times 10^7 = 2.375$$

$$Z_L = \frac{R_L - j\omega C_L R_L{}^2}{1+\left(\omega C_L R_L\right)^2} = \frac{10^7 - j(2.375)10^7}{1+2.375^2} = 1.506\ \text{M}\Omega - j3.58\ \text{M}\Omega$$

$$= \left(\sqrt{1.506^2+3.58^2},\ \tan^{-1}\left(\frac{-3.58}{1.506}\right)\right) = \left(3.88\ \text{M}\Omega,\ \angle -67.2^\circ\right)$$

$$Z_T = R_{trans} + Z_L = 0.5 + 1.506 - j3.58 = 2.01\ \text{M}\Omega - j3.58\ \text{M}\Omega$$

$$= \left(\sqrt{2.01^2+3.58^2},\ \tan^{-1}\left(\frac{-3.58}{2.01}\right)\right) = \left(4.11\ \text{M}\Omega,\ \angle -60.7^\circ\right)$$

$$\tilde{V}_{scope} = \tilde{V}_{trans}\frac{Z_L}{Z_T} = \frac{\left(100\ \text{mV},\ \angle 0^\circ\right)\left(3.88\ \text{M}\Omega,\ \angle -67.2^\circ\right)}{\left(4.11\ \text{M}\Omega,\ \angle -60.7^\circ\right)} = \boxed{94.4\ \text{mV},\ \angle -6.5^\circ}.$$

For 20 kHz, the new values are

$$\omega C_L R_L = 2\pi \times 20 \times 10^3 \times 12.6 \times 10^{-12} \times 10^7 = 15.83$$

$$Z_L = \frac{R_L - j\omega C_L R_L{}^2}{1+\left(\omega C_L R_L\right)^2} = \frac{10^7 - j(15.83)10^7}{1+15.83^2} = 0.0397\ \text{M}\Omega - j0.629\ \text{M}\Omega$$

$$= \left(\sqrt{0.0397^2+0.629^2},\ \tan^{-1}\left(\frac{-0.629}{0.0397}\right)\right) = \left(0.630\ \text{M}\Omega,\ \angle -86.4^\circ\right)$$

$$Z_T = R_{trans} + Z_L = 0.5 + 0.0397 - j0.629 = 0.540\ \text{M}\Omega - j0.629\ \text{M}\Omega$$

$$= \left(\sqrt{0.540^2+0.629^2},\ \tan^{-1}\left(\frac{-0.629}{0.540}\right)\right) = \left(0.829\ \text{M}\Omega,\ \angle -49.4^\circ\right)$$

$$\tilde{V}_{scope} = \tilde{V}_{trans}\frac{Z_L}{Z_T} = \frac{\left(100\ \text{mV},\ \angle 0^\circ\right)\left(0.630\ \text{M}\Omega,\ \angle -86.4^\circ\right)}{\left(0.829\ \text{M}\Omega,\ \angle -49.4^\circ\right)} = \boxed{76.0\ \text{mV},\ \angle -37.0^\circ}.$$

The loading effect on the signal is substantially less than without the scope probe. For instance at 2.0 kHz without the scope probe, the displayed amplitude was only 46.4% of the true amplitude and the phase shift was -37.5°. With the scope probe the displayed amplitude was 94.4% of the true amplitude and the phase shift was -6.5°.

Aside: It is possible to do an even better job of reducing oscilloscope loading effects with *active* oscilloscope probes.

3. a) The impedance of L and C in parallel is

$$Z_{\parallel} = \frac{Z_C Z_L}{Z_C + Z_L} = \frac{\frac{1}{j\omega C} j\omega L}{\frac{1}{j\omega C} + j\omega L} = \frac{j\omega L}{1-\omega^2 LC} = \frac{j\omega L}{1-\frac{\omega^2}{\omega_0{}^2}}$$

The total impedance attached to the signal source is

$$Z_T = R + Z_{\parallel} = R + \frac{j\omega L}{1-\frac{\omega^2}{\omega_0{}^2}} = \frac{R\left(1-\frac{\omega^2}{\omega_0{}^2}\right) + j\omega L}{1-\frac{\omega^2}{\omega_0{}^2}}$$

The output voltage amplitude is $\tilde{V}_{out} = \tilde{V}_{out}\frac{Z_{\parallel}}{Z_T} = \tilde{V}_{in}\frac{j\omega L}{R\left(1-\frac{\omega^2}{\omega_0{}^2}\right) + j\omega L} = \frac{1}{1-j\frac{R}{\omega L}\left(1-\frac{\omega^2}{\omega_0{}^2}\right)}$

The magnitude of the voltage ratio is therefore $\frac{\left|\tilde{V}_{out}\right|}{\left|\tilde{V}_{in}\right|} = \frac{1}{\sqrt{1+\left(\frac{R}{\omega L}\right)^2\left(1-\frac{\omega^2}{\omega_0{}^2}\right)^2}}$

b) At exact resonance the voltage ratio is $\frac{\left|\tilde{V}_{out}\right|}{\left|\tilde{V}_{in}\right|} = 1$

c) The voltage ratio is 0.707 for $\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{1+\left(\frac{R}{\omega L}\right)^2\left(1-\frac{\omega^2}{\omega_0{}^2}\right)^2}}$

$$\Rightarrow \quad 1+\left(\frac{R}{\omega L}\right)^2\left(1-\frac{\omega^2}{\omega_0{}^2}\right)^2 = 2 \qquad\qquad \Rightarrow \quad \left(1-\frac{\omega^2}{\omega_0{}^2}\right) = \pm\frac{\omega L}{R}$$

Here, we note that $\omega_0 = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{2.5\times10^{-3}\times10\times10^{-9}}} = 2.00\times10^5\ \text{s}^{-1}$, $\frac{R}{L} = \frac{25\times10^3}{2.5\times10^{-3}} = 1.00\times10^7\ \text{s}^{-1}$.

We expect the filter to transmit frequencies in some range centered at $\omega = \omega_0$. Thus $\frac{\omega L}{R} \sim \frac{\omega_0 L}{R} = 0.02$. From the above equation, this implies that $\left|1-\frac{\omega_{\pm}{}^2}{\omega_0{}^2}\right| << 1$, *i.e.* that

$|\omega_\pm - \omega_0| \ll \omega_0$. In other words, the resonance response of the filter has a high Q. It follows that we can take $\frac{\omega L}{R} \approx \frac{\omega_0 L}{R}$ and that $\left(1-\frac{\omega^2}{\omega_0{}^2}\right) \approx \pm\frac{\omega_0 L}{R} \quad \Rightarrow \quad \left(1-\frac{\omega_\pm{}^2}{\omega_0{}^2}\right) \approx \pm\frac{\omega_0 L}{R}$

$$\Rightarrow \quad \omega_\pm \approx \omega_0\sqrt{1\pm\frac{\omega_0 L}{R}} = 2\times10^5\sqrt{1\pm0.020} = \begin{cases} 2.020\times10^5 \text{ s}^{-1} & (\omega_+) \\ 1.980\times10^5 \text{ s}^{-1} & (\omega_-) \end{cases}$$

d) The FWHM is $\Delta\omega_{FWHM} = \omega_+ - \omega_- = 4\times10^3 \text{ s}^{-1}$.

e) $Q = \frac{\omega_0}{\Delta\omega_{FWHM}} = \frac{2\times10^5}{4\times10^3} = 50.$

f) I drew this with Matplotlib, but a similar hand-drawn sketch is fine.