What is measure theory and why should I care?

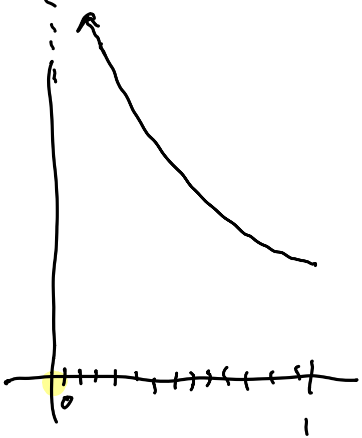
Biggest motivation: the **Lebesgue integral**.

Problem #1 with Riemann integration

It can't handle unbounded functions.

Example

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx$$



Technically, this isn't Riemann integrable. What we really mean when we write this is $\lim_{t\to 0+} \int_t^1 \frac{1}{\sqrt{x}}\,dx$.

However, Lebesgue integration can handle the original form of the integral.

Problem #2 with Riemann integration

It can't handle many discontinuities.

Example

Consider the **Dirichlet function**

$$\mathbf{1}_{\mathbb{Q}}(x) := \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

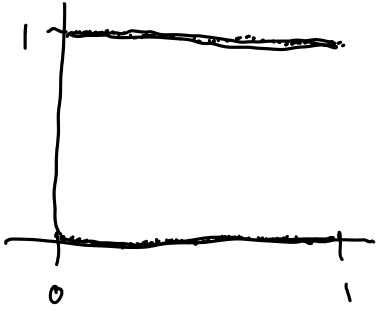
and the integral

$$\int_0^1 \mathbf{1}_{\mathbb{Q}}(x)\, dx$$

Since this function is zero everywhere except on a countable set, the integral "should be" zero. But it's not Riemann integrable.

Indicator function of a set A, written $\mathbb{1}_A$, is defined by

$$\mathbb{1}_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$



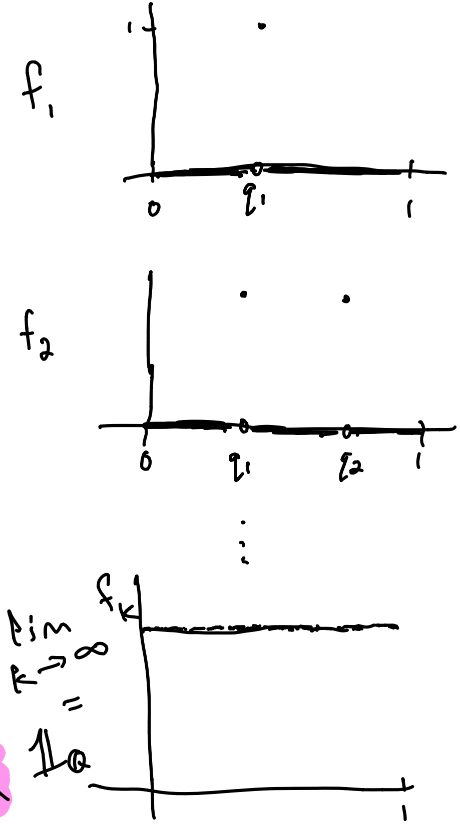
Problem #3 with Riemann integration

It doesn't work well with limits.

Example

Let $q_1, q_2, \ldots$ be an enumeration of the rational numbers in $[0, 1]$, and let

$$f_k = \mathbf{1}_{\{q_1, q_2, \ldots, q_k\}}$$

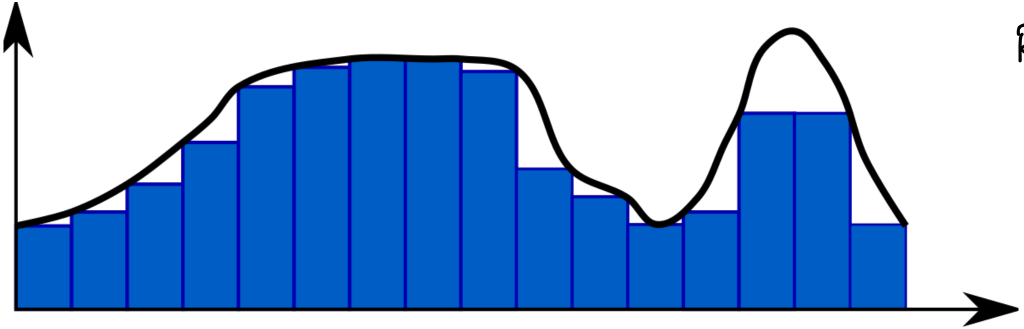


Then $\displaystyle\lim_{k\to\infty} \int_0^1 f_k(x)\,dx = 0$, but $\displaystyle\int_0^1 \lim_{k\to\infty} f_k(x)\,dx$ doesn't exist.

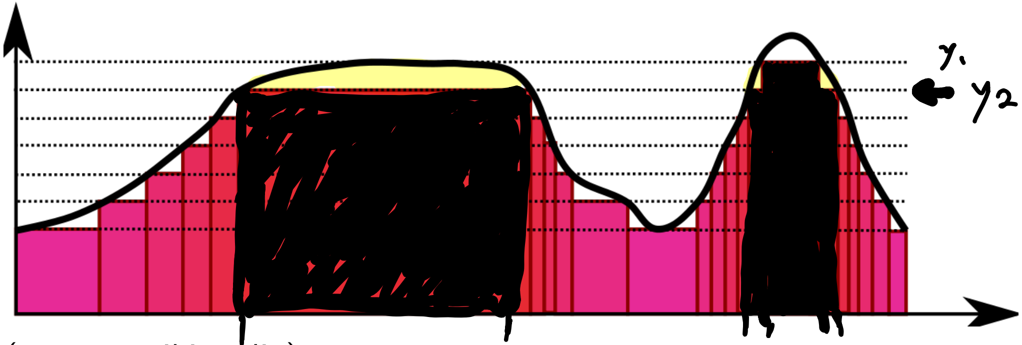
This example shows that the space of Riemann integrable functions on $[0, 1]$ is **not complete**.

Analogy: Riemann integrability IS TO Lebesgue integrability as $\mathbb{Q}$ IS TO $\mathbb{R}$

Intuition for how the Lebesgue integral works



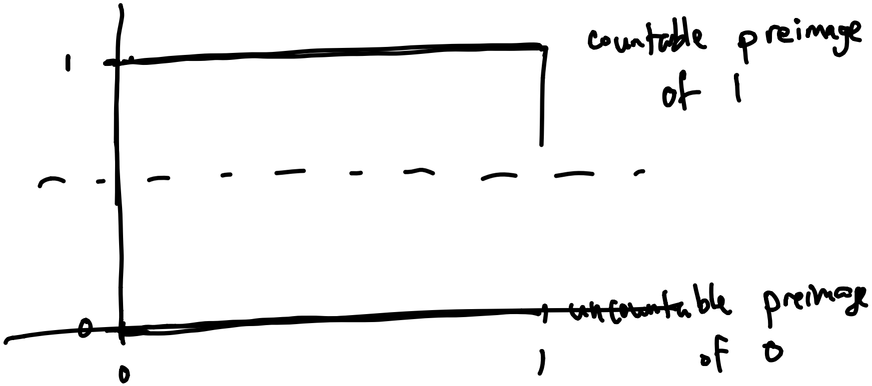
Riemann sums converge to Riemann integral



Approximation to the Lebesgue integral

(source: wikimedia)

$\mathbb{1}_{\mathbb{Q}}$ on $[0,1]$



$$1 \cdot \text{"size"}(\mathbb{Q} \cap [0,1]) + 0 \cdot \text{"size"}([0,1] \setminus \mathbb{Q})$$

what we want

$$1 \cdot 0 + 0 \cdot 1 = 0$$

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Problems with the Riemann integral:

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- It can't handle many discontinuities.
- It doesn't work well with limits / lack of completeness.

These problems come from the way we rely on partitioning the domain into intervals.

For Lebesgue integration, we need to define the "size" or **measure** of sets more complicated than intervals.

"Wish list" for measuring sets in $\mathbb{R}$

Call our wannabe measure μ

- $\mu(A) \geq 0$ for any $A \subseteq \mathbb{R}$
- If A and B are disjoint subsets of $\mathbb{R}$, then
  $\mu(A \cup B) = \mu(A) + \mu(B)$
- $\mu(A)$ is defined for every $A \subseteq \mathbb{R}$ (domain of μ is $\mathcal{P}(\mathbb{R})$)

  power set of $\mathbb{R}$ i.e. set of all subsets of $\mathbb{R}$
- $\mu([a,b]) = b-a$, $\mu((a,b)) = b-a = \mu([a,b)) = \mu((a,b])$
- $\mu(\emptyset) = 0$
- "translation invariance": $\mu(A) = \mu(t+A)$ where
  $t+A = \{t+a : a \in A\}$, $t \in \mathbb{R}$

The bad news...

2.22 ***nonexistence of extension of length to all subsets of* R**

There does not exist a function μ with all the following properties:

(a) μ is a function from the set of subsets of **R** to $[0, \infty]$.

(b) $\mu(I) = \ell(I)$ for every open interval I of **R**.

(c) $\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k)$ for every disjoint sequence $A_1, A_2, \ldots$ of subsets of **R**.

(d) $\mu(t + A) = \mu(A)$ for every $A \subset \mathbf{R}$ and every $t \in \mathbf{R}$. translation invariance

$$\ell(I) = \begin{cases} b-a & \text{if } I=(a,b) \\ \text{etc.} & \\ \text{(see Axler)} & \end{cases}$$

(source: Axler)

proof: Axler proof refers back to section on outer measure on $\mathbb{R}$

Bass section on nonmeasurable sets

google "nonmeasurable set" or "Vitali set"

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For Lebesgue integration, we need to define the "size" or **measure** of sets more complicated than intervals... but we can't measure every set in $\mathbb{R}$.

To describe which sets are measurable, and which functions $\mathbb{R} \to \mathbb{R}$ are Lebesgue integrable, we start with studying σ-**algebras**, which are certain well-behaved collections of subsets.

σ-algebras

2.23 Definition ***σ-algebra***

Suppose X is a set and $\mathcal{S}$ is a set of subsets of X. Then $\mathcal{S}$ is called a *σ-algebra* on X if the following three conditions are satisfied:

- $\varnothing \in \mathcal{S}$;
- if $E \in \mathcal{S}$, then $X \setminus E \in \mathcal{S}$; (closed under complements)
- if $E_1, E_2, \ldots$ is a sequence of elements of $\mathcal{S}$, then $\bigcup_{k=1}^{\infty} E_k \in \mathcal{S}$. (closed under countable unions)

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- Let $X = \{a, b, c\}$, $S = \{\emptyset, \{a\}, \{b, c\}, X\}$. Then S is a σ-algebra on X.

  - $\emptyset \in S$ is given
  - $X \setminus \emptyset = X \in S$

    $X \setminus \{a\} = \{b, c\} \in S$

    $X \setminus \{b, c\} = \{a\} \in S$

    $X \setminus X = \emptyset \in S$
  - closed under countable unions

- Let $X = \{a, b, c\}$, $S = \{\emptyset, \{a\}, \{a, b\}, \{b, c\}, X\}$. Then S is not a σ-algebra on X.

Not closed under complements because $\{a, b\} \in S$ but $X \setminus \{a, b\} = \{c\} \notin S$.

σ-algebras

Examples.

- Let $X = \{a, b, c\}$, $S = \{\emptyset, \{a\}, \{b, c\}, X\}$. Then S is a σ-algebra on X.

- Let $X = \{a, b, c\}$, $S = \{\emptyset, \{a\}, \{a, b\}, \{b, c\}, X\}$. Then S is not a σ-algebra on X.

power set of X

- Let X be any set. Then $\mathcal{P}(X)$ is a σ-algebra.

- Let X be any set. Then $\{X, \emptyset\}$ is a σ-algebra. (trivial σ-algebra on X)

- Let X be any set. Then $\mathcal{A} := \{E \subseteq X : E \text{ is countable or } X \backslash E \text{ is countable}\}$ is a σ-algebra.

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For example, if $X = \mathbb{R}$, then $\{1, 2, \pi\} \in \mathcal{A}$, and $(\mathbb{R} \setminus \mathbb{Q}) \in \mathcal{A}$ because its complement $\mathbb{Q}$ is countable, but $[0,1] \notin \mathcal{A}$ since $[0,1]$ is not countable and $\mathbb{R} \setminus [0,1] = (-\infty, 0) \cup (1, \infty)$ is also not countable.

Proof:

σ-algebras

2.27 *smallest σ-algebra containing a collection of subsets*

Suppose X is a set and $\mathcal{A}$ is a set of subsets of X. Then the intersection of all σ-algebras on X that contain $\mathcal{A}$ is a σ-algebra on X. Write $\sigma(\mathcal{A})$.

see proof in Axler or Bass

$\boxed{\text{De Morgan's Laws}}$

Examples.

- Let $X = [0,1]$ and $\mathcal{A} = \{\underbrace{[0,\frac{1}{4}]}_{A_1}, \underbrace{[\frac{1}{2},1]}_{A_2}\}$. Find $\sigma(\mathcal{A})$.

$A_1^c \cap A_2^c$

$=$

$$\sigma(\mathcal{A}) = \{\emptyset, A_1, A_2, \underbrace{A_1^c}_{(\frac{1}{4},1]}, \underbrace{A_2^c}_{[0,\frac{1}{2})}, X, \underbrace{A_1 \cup A_2}_{[0,\frac{1}{4}]\cup[\frac{1}{2},1]}, \underbrace{(A_1 \cup A_2)^c}_{(\frac{1}{4},\frac{1}{2})}\}$$

σ-algebras

The word *measurable* is used in the terminology below because in the next section we introduce a size function, called a measure, defined on measurable sets.

2.26 Definition ***measurable space; measurable set***

- A *measurable space* is an ordered pair $(X, \mathcal{S})$, where X is a set and $\mathcal{S}$ is a σ-algebra on X.
- An element of $\mathcal{S}$ is called an $\mathcal{S}$-*measurable set*, or just a *measurable set* if $\mathcal{S}$ is clear from the context.

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To describe which sets are measurable, and which functions $\mathbb{R} \to \mathbb{R}$ are Lebesgue integrable, we start with studying σ-**algebras**, which are certain well-behaved collections of subsets.

The usefulness of Lebesgue integration goes far beyond addressing the problems with Riemann integration on $\mathbb{R}$ and $\mathbb{R}^n$. Because the Lebesgue integral only uses measurable subsets of the domain (not intervals), it lets us integrate functions on all kinds of weird/interesting/important spaces (where there is no such thing as an interval).

e.g. probability spaces