Measurable functions

Definition. Let $(X, \mathcal{M})$ be a measurable space. A function $f : X \to \mathbb{R}$ is $\mathcal{M}$-measurable if

$$f^{-1}((a, \infty)) = \{x : f(x) > a\} \in \mathcal{M} \text{ for all } a \in \mathbb{R}.$$

We could have replaced the $>$ in the definition with $<, \geq$, or $\leq$.

Proposition. If $(X, \mathcal{M})$ is a measurable space and $f : X \to \mathbb{R}$, then the following are equivalent:

1. $f^{-1}((a, \infty)) = \{x : f(x) > a\} \in \mathcal{M} \ \forall a \in \mathbb{R}$.
2. $f^{-1}([a, \infty)) = \{x : f(x) \geq a\} \in \mathcal{M} \ \forall a \in \mathbb{R}$.
3. $f^{-1}((-\infty, a)) = \{x : f(x) < a\} \in \mathcal{M} \ \forall a \in \mathbb{R}$.
4. $f^{-1}((-\infty, a]) = \{x : f(x) \leq a\} \in \mathcal{M} \ \forall a \in \mathbb{R}$.

Measurable functions

For example, **1** $\Longleftrightarrow$ **4** because

$$\{x : f(x) \leq a\} = X \setminus \{x : f(x) > a\}\,.$$

A few of the other parts of this proposition are on your homework. The following identities may be helpful.

$$\{x : f(x) \geq a\} = \bigcap_{n=1}^{\infty} \left\{x : f(x) > a - \frac{1}{n}\right\}$$

$$\{x : f(x) > a\} = \bigcup_{n=1}^{\infty} \left\{x : f(x) \geq a + \frac{1}{n}\right\}$$

Examples of measurable functions

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3. Let $(X, \mathcal{M})$ be any measurable space, and let $A \in \mathcal{M}$. Then the characteristic function (or indicator function) $\mathbf{1}_A : X \to \mathbb{R}$ is $\mathcal{M}$-measurable.

Measurable functions

Proposition. If $f, g : (X, \mathcal{M}) \to \mathbb{R}$ are measurable functions, and $c \in \mathbb{R}$, then the following are also $\mathcal{M}$-measurable:

$$-f,\ f+c,\ cf,\ f+g,\ g-f,\ \text{ and }\ fg.$$

For example, let's show that $f + c$ is measurable. Since f is measurable, then for any $a \in \mathbb{R}$,

$$\{x : f(x) > a\} \in \mathcal{M}.$$

Therefore

$$\{x : f(x) + c > a\} = \{x : f(x) > a - c\} \in \mathcal{M}$$

for all $a \in \mathbb{R}$, so $f + c$ is measurable.

Pre-images of Borel sets

Proposition. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $f : X \to \mathbb{R}$ be μ-measurable.[a] If $A \in \mathcal{B}(\mathbb{R})$, then $f^{-1}(A) \in \mathcal{M}$.

[a] Note: this is the same as saying that f is $\mathcal{M}$-measurable

The strategy of this proof may be a bit unexpected. Start by defining $\mathcal{C} = \{B \in \mathcal{B}(\mathbb{R}) : f^{-1}(B) \in \mathcal{M}\}$. By definition $\mathcal{C} \subseteq \mathcal{B}(\mathbb{R})$. To show the reverse inclusion, show that $\mathcal{C}$ is a sigma algebra containing all the open intervals (a, ∞).

μ-almost everywhere

Definition. Let $f, g : (X, \mathcal{M}, \mu) \to \mathbb{R}$. We say $f = g$ **almost everywhere** (or μ-almost everywhere) if

$$\mu(\{x : f(x) \neq g(x)\}) = 0.$$

That is, $f(x) = g(x)$ for all $x \in X$ except perhaps on a set of measure zero.

For example, the Dirichlet function equals the constant zero function m-almost everywhere.

Simple functions

Definition. Let $(X, \mathcal{M})$ be a measurable space. A **simple function** is a finite linear combination, with non-negative coefficients, of characteristic functions on sets in $\mathcal{M}$. That is,

$$f(x) = \sum_{j=1}^{n} a_j \chi_{E_j}(x),$$

where $E_j \in \mathcal{M}$ for all j, and $a_j \in \mathbb{R}$.

Example. $f = 2\chi_{(-5,0)} + 1\chi_{[0,3)}$ is a simple function on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

Definition. (For $X = \mathbb{R}$) A **step function** is a simple function where each E_i is an interval.

Simple functions

Theorem. Let $f : X \to \mathbb{R}$ be non-negative and measurable. Then there exists a sequence S_n of simple functions such that:

1. $S_n(x) \leq S_{n+1}(x) \leq f(x)$ for all $n \geq 1$, and almost every $x \in X$
2. $\lim_{n\to\infty} S_n(x) = f(x)$ for almost every $x \in X$

Outline of proof. Let $E_{n,k} = \{x : \frac{k-1}{2^n} \leq f(x) < \frac{k}{2^n}\}$, $F_n = \{x : f(x) \geq n\}$ for $n \geq 1$, $k = 1, 2, \dots, n2^n$.

Note $E_{n,k}$, F_n all measurable sets.

$$S_n = \sum_{k=1}^{n2^n} \frac{k-1}{2^n} \chi_{E_{n,k}} + n\chi_{F_n} = \begin{cases} n & \text{if } f(x) \geq n \\ \frac{k-1}{2^n} & \text{if } \frac{k-1}{2^n} \leq f(x) < \frac{k}{2^n} \leq n \end{cases}$$

So $S_n \to f$ as desired.