Fast Computations of Monomial Ideal Invariants Using Constraint Integer Programming

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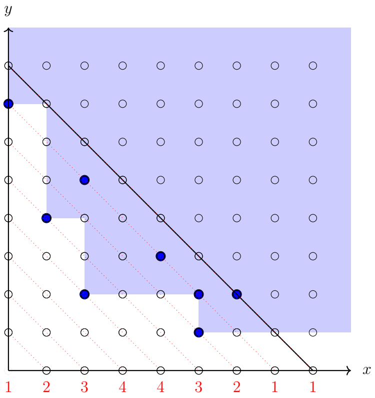
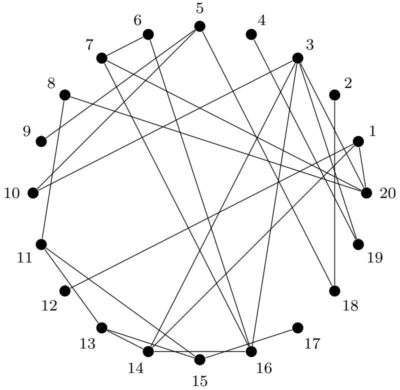


Erdős-Rényi random graphs

$\mathcal{G}(n, p)$ model: n = number of vertices

$p \in [0, 1]$

Include each edge of K_n with probability p (independently).



$n = 20, \quad p = 0.1$

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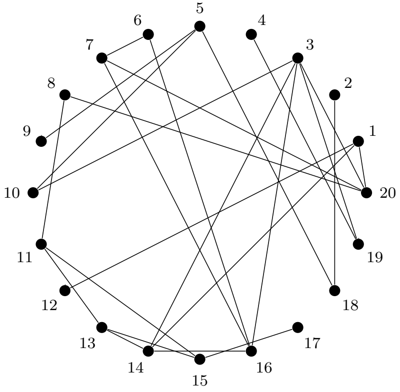
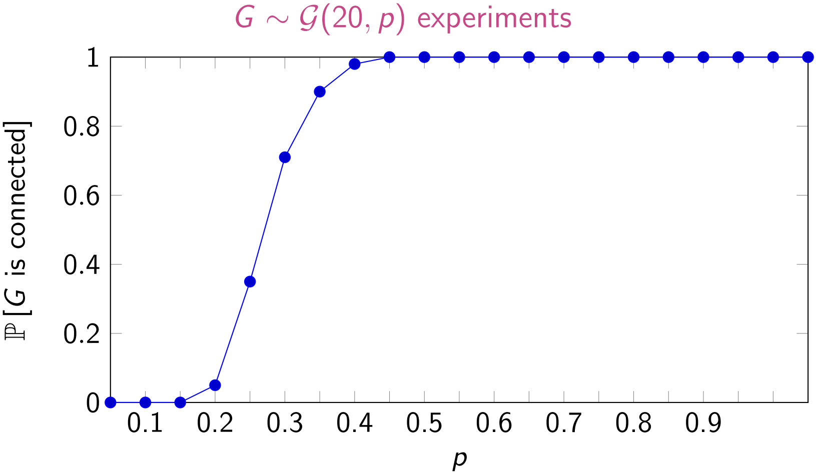


$n = 20, \quad p = 0.1$

If H is a fixed graph with e edges and $G \sim \mathcal{G}(n,p)$, then:

$$\mathbb{P}[G = H] = p^e(1-p)^{\binom{n}{2}-e}.$$

Random graphs exhibit phase transitions & thresholds



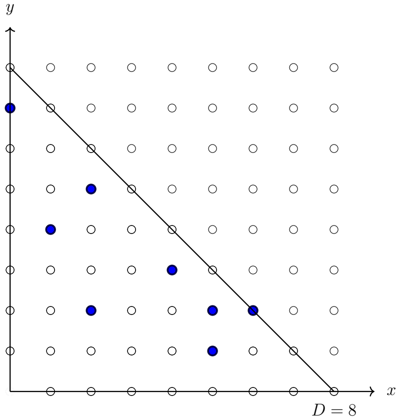
Erdős–Rényi, 1960:

The threshold for connectedness is $p(n) = \dfrac{\ln n}{n}$.

Random monomial ideals

Erdős-Rényi-type model $\mathcal{I}(n, D, p)$:

- n = variables in $S = k[x_1, \ldots, x_n]$
- D, maximum total degree
- $p \in [0, 1]$
- Sample generating set $\mathfrak{B}$: include each $x^\alpha \in S$, $1 \leq |\alpha| \leq D$, with (independent) probability p.



$$\mathfrak{B} = \{x^6y^2, x^5y, x^5y^2, x^4y^3, x^2y^2, x^2y^5, xy^4, y^7\}$$

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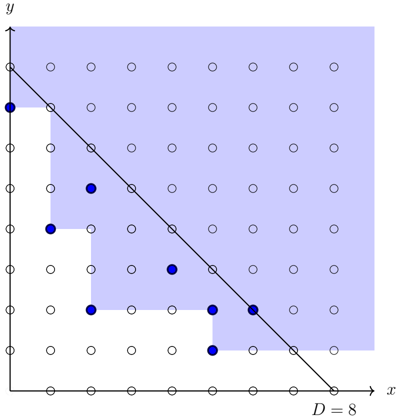
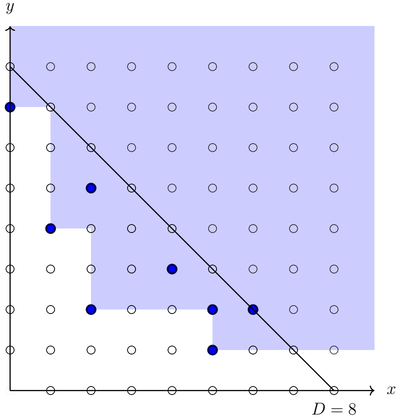
- Define random ideal $I = (\mathfrak{B})$.



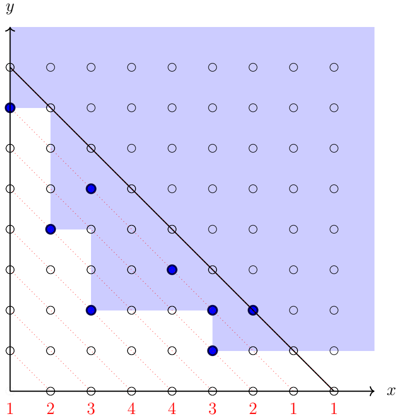
$$I = (x^5y, x^2y^2, xy^4, y^7)$$



Theorem (De Loera, Petrović, Stasi, S, Wilburne 2017)

Let $J \subseteq S$ be a monomial ideal with Hilbert function $H_J(d)$ and b minimal generators of degree at most D, and let $I \sim \mathcal{I}(n, D, p)$. Then

$$\mathbb{P}[I = J] = p^b(1-p)^{\sum_{d=1}^{D} H_J(d)}.$$



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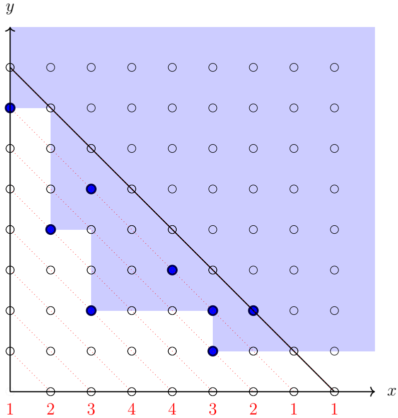
For $I \sim \mathcal{I}(2, 8, p)$,

$$\mathbb{P}\left[I = (x^5y, x^2y^2, xy^4, y^7)\right]$$
$$= p^4(1-p)^{20}$$



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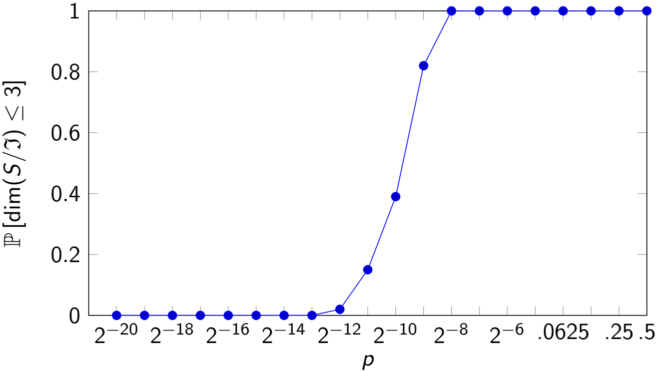
Random mon. ideals have phase transitions & thresholds!!

$\mathfrak{I} \sim \mathcal{I}(5, 25, p)$ experiments



Theorem (DPSSW 2017)

Let n be fixed, and $p = p(D)$. For each integer $1 \leq t \leq n$,

$$D^{-t-1} \text{ is a threshold for } \dim(S/I) \leq t.$$

- ⋆ Jesús A. De Loera, Sonja Petrović, LS, Despina Stasi, and Dane Wilburne. **Random monomial ideals**. To appear in *Journal of Algebra*. (arXiv 1701.07130)

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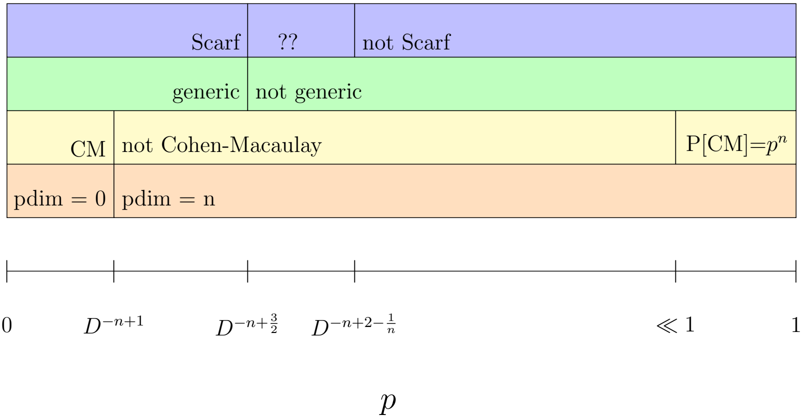

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$$\mathbb{P}[H_I(\cdot) = (1, 2, 3, 4, 4, 3, 2, 1, 1, \ldots)] = ???$$

$$\mathbb{P}[H_I(\cdot) = (1, 2, 3, 4, 4, 3, 2, 1, 1, \ldots) \text{ and } \beta_1 = 4] = ???$$

$$\mathbb{P}[H_I(\cdot) = (1, 2, 3, 4, 4, 3, 2, 1, 1, \ldots) \text{ and } \beta_{1,\cdot} = (0, 0, 0, 0, 1, 1, 1, 1, 0)] = ???$$

the "monopolytope"

Lemma 2.3, DPSSW 2017

Denote by $NMon(n, D, h)$ the number of possible monomial ideals in n variables, with generating monomials of total degree no more than D and given Hilbert function h. Then $NMon(n, D, h)$ is equal to the number of vertices of the $0 - 1$ convex polytope defined by

$$\sum_{|\alpha|=d} x_\alpha = \binom{n+d-1}{d} - h(d), \quad \forall d = 1, \dots, D$$

$$x_\alpha \leq x_\gamma, \quad \forall\, \alpha \leq \gamma,\, |\alpha| + 1 = |\gamma|,$$

where α, γ denote exponent vectors of monomials with n variables and total degree no more than D, thus the system has $\binom{n+D}{D} - 1$ variables.

another monomial ideal IP

Let $I \subseteq k[x_1, \ldots, x_n]$ be a squarefree monomial ideal with minimal generating set $G = \{g_1, \ldots, g_m\}$.
The codimension of I is the minimum height of a prime ideal containing I.

Ex.

$$I = (\mathbf{x_1} x_2, \mathbf{x_3} x_4, x_2 \mathbf{x_3} x_5, \mathbf{x_1} x_4 x_6) \subset \mathbb{C}[x_1, \ldots, x_6]$$

$$\text{minimal primes: } \{(\mathbf{x_1}, \mathbf{x_3}), (x_2, x_4), (x_1, x_4, x_5), (x_2, x_3, x_6)\}$$

$$\operatorname{codim}(I) = 2$$

$$\dim(I) = 4$$

I.e., $\operatorname{codim}(I)$ is the minimum size of a set $A \subseteq \{x_1, \ldots, x_n\}$ s.t. $\forall g \in G(I)$, $\exists x_i \in A$ with $x_i \in \operatorname{supp}(g)$.

codim(I) is equivalent to:

$$
\begin{array}{rll}
\text{minimize:} & \displaystyle\sum_{i=1}^{n} x_i & \\
\text{subject to:} & x_i \in \{0,1\} & 1 \le i \le n \\
& \displaystyle\sum_{x_i \in \mathrm{supp}(g_j)} x_i \ge 1 & 1 \le j \le m
\end{array}
$$

deg(I) is the number of **optimal** solutions

Constraint programming:

- constraint propagation
- nonlinear constraints possible
- fast detection of (un)satisfiability

vs.

Integer linear programming:

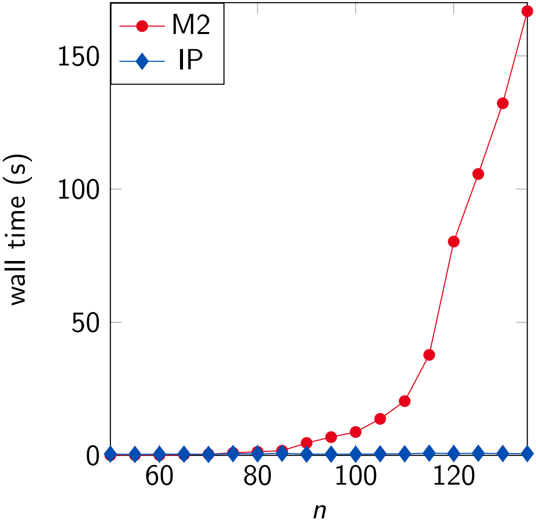
- LP relaxation
- primal-dual bounds, cutting planes
- optimal as well as feasible solutions

SCIP: Solving Constraint Integer Programs

- `https://scip.zib.de/`
- Free for academic, non-commercial use (*ZIB Academic License*)

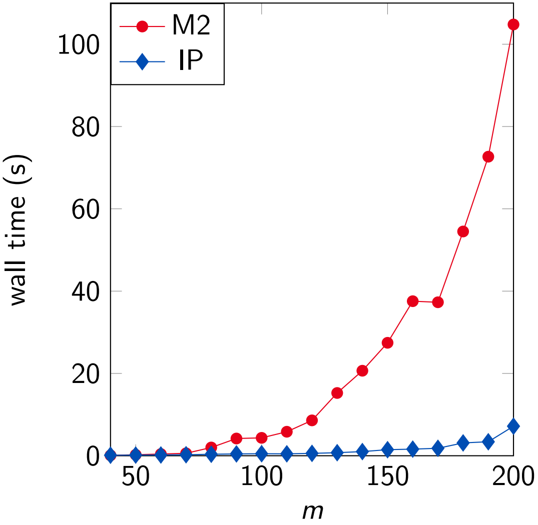
ZIMPL: Zuse Institute Mathematical Programming Language

- User guide: `https://zimpl.zib.de/download/zimpl.pdf`
- Thorsten Koch. *Rapid Mathematical Programming*. PhD Thesis, TU Berlin, 2004.



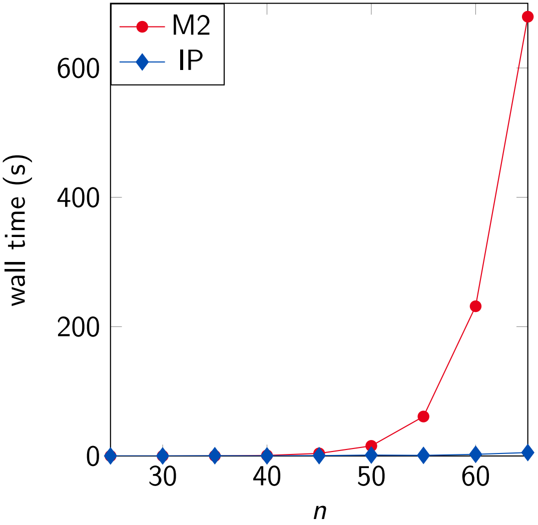
n	M2	IP
50	0.05	0.47
55	0.07	0.35
60	0.08	0.44
65	0.25	0.46
70	0.40	0.36
75	0.94	0.64
80	1.35	0.55
85	1.74	0.78
90	4.65	0.45
95	6.85	0.44
100	8.76	0.43
105	13.70	0.57
110	20.36	0.58
115	37.73	0.83
120	80.27	0.70
125	105.66	0.79
130	132.21	0.68
135	166.81	0.65

time to compute **dimension** of squarefree monomial ideals
with 50 generators of degree 20 in n variables



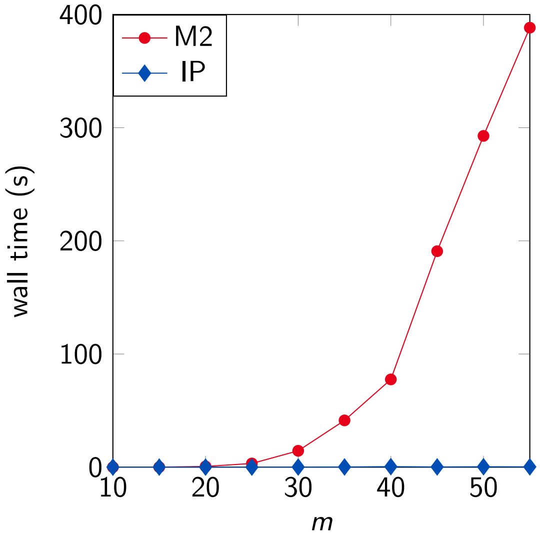
m	M2	IP
40	0.09	0.12
50	0.24	0.16
60	0.42	0.18
70	0.59	0.24
80	2.00	0.36
90	4.19	0.45
100	4.34	0.49
110	5.83	0.47
120	8.59	0.58
130	15.25	0.74
140	20.66	1.01
150	27.44	1.47
160	37.55	1.61
170	37.29	1.81
180	54.49	3.12
190	72.67	3.40
200	104.79	7.18

time to compute **dimension** of squarefree monomial ideals with m generators of degree 5 in 50 variables



n	M2	IP
25	0.01	0.08
30	0.03	0.08
35	0.17	0.42
40	0.86	0.19
45	3.99	0.50
50	15.62	1.41
55	61.04	0.94
60	231.53	2.57
65	679.28	5.31

time to compute **degree** of squarefree monomial ideals with 50 generators of degree 20 in n variables



m	M2	IP
10	0.01	0.06
15	0.09	0.08
20	0.67	0.09
25	3.29	0.08
30	14.48	0.15
35	41.35	0.21
40	77.56	0.50
45	190.85	0.23
50	292.73	0.40
55	388.35	0.31

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Thanks for your attention!

★ Jesús A. De Loera, Sonja Petrović, LS, Despina Stasi, and Dane Wilburne. **Random monomial ideals**. To appear in *Journal of Algebra*. (arXiv 1701.07130, 2017)

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★ Preliminary version of Monomial Integer Programs `Macaulay2` package available at

`github.com/lilysilverstein/MonIP`

★ See also the Random Monomial Ideals `Macaulay2` package, distributed with `M2` and described in:

★ Sonja Petrović, Despina Stasi, and Dane Wilburne. **Random monomial ideals `Macaulay2` package**. (arXiv 1711.10075, 2017)