Outer measure on $\mathbb{R}$

Definition. The length of an open interval $I \subseteq \mathbb{R}$ is defined by

$$\ell(I) = \begin{cases} 0 & I = \emptyset \\ b - a & I = (a, b) \text{ with } a < b \\ \infty & I = (-\infty, a) \text{ or } (a, \infty) \text{ or } (-\infty, \infty) \end{cases}$$

Definition. Define the **Lebesgue outer measure** of $E \subseteq \mathbb{R}$, written $|E|$ (Axler) or $m^*(E)$ (Bass) by

$$m^*(E) = \inf\left\{ \sum_{i=1}^{\infty} \ell(A_i) : A_i \text{ is an open interval } \forall i \text{ and } E \subseteq \bigcup_{i=1}^{\infty} A_i \right\}.$$

Outer measure is **not** a measure on $(\mathbb{R}, \mathcal{P}(\mathbb{R}))$. We will soon see that it **is** a measure on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$, and in fact there is a somewhat larger σ-algebra $\mathcal{L}$ such that outer measure is a measure on $(\mathbb{R}, \mathcal{L})$.

Outer measure on $\mathbb{R}$

Example. Each of the following subsets of $\mathbb{R}$ has Lebesgue outer measure zero.

1. $\emptyset$
2. $\{x\}$ for any real number x
3. Any countable subset of $\mathbb{R}$, for instance $\mathbb{Q}$ (homework)
4. The Cantor set (homework)

Special kinds of measures

Definition.

- μ is a *finite* measure if $\mu(X) < \infty$
- μ is a *probability measure* if $\mu(X) = 1$
- μ is called σ-*finite* if $X = \bigcup_{n=1}^{\infty} X_n$ where $\mu(X_n) < \infty$ for all n.

Null sets

Definition. If $(X, \mathcal{M}, \mu)$ is a measure space, a set $N \in \mathcal{M}$ such that $\mu(N) = 0$ is called a **null set**.

Trick question: If N is a null set of $(X, \mathcal{M}, \mu)$, and $E \subseteq N$, is E necessarily a null set?

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Definition. $(X, \mathcal{M}, \mu)$ is called a **complete measure space** if every subset of a μ-null set belongs to $\mathcal{M}$.

Theorem. If $(X, \mathcal{M}, \mu)$ is a measure space, there exists a complete measure space $(X, \widehat{\mathcal{M}}, \widehat{\mu})$ such that $\mathcal{M} \subseteq \widehat{\mathcal{M}}$ and $\mu(A) = \widehat{\mu}(A)$ for all $A \in \mathcal{M}$.

Proof: homework

Constructing Lebesgue measure

Definition. An **outer measure** on a set X is a function $\mu^* : \mathcal{P}(X) \to [0, \infty]$ such that:

1. $\mu^*(\emptyset) = 0$
2. If $A \subseteq B$, then $\mu^*(A) \leq \mu^*(B)$
3. If $A_1, A_2, \ldots \subseteq X$, then $\mu^*(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} \mu^*(A_i)$

Proposition. Suppose $\{\emptyset, X\} \subseteq \mathcal{C} \subseteq \mathcal{P}(X)$. If $\ell : \mathcal{C} \to [0, \infty]$ and $\ell(\emptyset) = 0$, then

$$\mu^*(E) = \inf \left\{ \sum_{i=1}^{\infty} \ell(A_i) : A_i \in \mathcal{C} \text{ and } E \subseteq \bigcup_{i=1}^{\infty} A_i \right\}$$

is an outer measure.

Proof: homework

Constructing Lebesgue measure

Definition. Let μ^* be an outer measure on X. Then $E \subseteq X$ is μ^*-measurable if

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c) \text{ for all } A \subseteq X.$$

Theorem. (Carathéodory) If μ^* is an outer measure on X, then the collection $\mathcal{M}$ of all μ^*-measurable sets forms a σ-algebra. If μ is the restriction of μ^* to $\mathcal{M}$, then μ is a complete measure.

Constructing Lebesgue measure

Theorem. Suppose $\{\emptyset, X\} \subseteq \mathcal{C} \subseteq \mathcal{P}(X)$, $\ell : \mathcal{C} \to [0, \infty]$, $\ell(\emptyset) = 0$. Then there exists a complete measure space $(X, \mathcal{M}, \mu)$ such that $\sigma(\mathcal{C}) \subseteq \mathcal{M}$ and

$$\mu(E) = \inf \left\{ \sum_{i=1}^{\infty} l(A_i) : A_i \in \mathcal{C} \text{ and } E \subseteq \bigcup_{i=1}^{\infty} A_i \right\}.$$

Now consider:
$X = \mathbb{R}$
$\mathcal{C} = \{(a, b) : a < b\} \cup \{\emptyset\}$
ℓ is usual length of intervals, i.e. $\ell((a, b)) = b - a$, $\ell(\emptyset) = 0$.

Then the measure guaranteed by this theorem is what we call **Lebesgue measure** on $\mathbb{R}$. (But what is the associated σ-algebra?)

Lebesgue measurable sets

That is, the only difference between m^*, outer measure on $\mathbb{R}$, and m, Lebesgue measure on $\mathbb{R}$, is their domains.

$$m^* : \mathcal{P}(\mathbb{R}) \to [0, \infty]$$
$$m : \mathcal{L} \to [0, \infty]$$
$$m = m^*|_{\mathcal{L}} \text{ (that is, } m(E) = m^*(E) \text{ for all } E \in \mathcal{L})$$

Where $\mathcal{L}$ is the σ-algebra, as described by Carathéodory's Theorem, of all sets that are Lebesgue measurable.

Definition. $E \subseteq \mathbb{R}$ is Lebesgue measurable if for all $A \subseteq \mathbb{R}$,

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^C).$$

Approximation of measurable sets

Definition.

1. A countable intersection of open sets is called a G_δ set.
2. A countable union of closed sets is called an F_σ set.

Even though the class of Lebesgue measurable sets is very large, any measurable set can be approximated arbitrarily well from above by an open set, and from below by a closed one. Moreover, any measurable set differs by at most a set of measure 0 from both a G_δ set containing it and an F_σ set contained in it.

Proposition. For $E \subseteq \mathbb{R}$, the following are equivalent:

1. E is Lebesgue measurable.
2. Given $\epsilon > 0, \exists$ open set $G \supset E$ such that $m^*(G \backslash E) < \epsilon$.
3. Given $\epsilon > 0, \exists$ closed set, $F \subseteq E$ such that $m^*(E \backslash F) < \epsilon$.
4. There is a G_δ set G with $E \subseteq G, m^*(G \backslash E) = 0$.
5. There is an F_σ set F with $F \subseteq E, m^*(E \backslash F) = 0$.