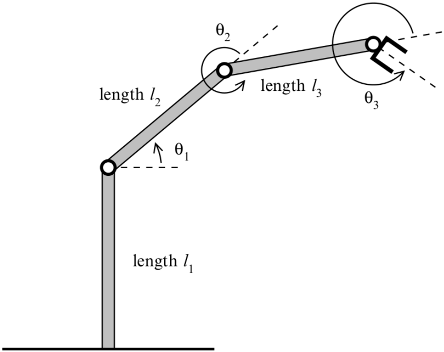
Probabilistic and statistical methods in computational algebra

Lily Silverstein
UC Davis

Cal Poly Pomona, February ♡ 14 ♡ 2019



Robot hand coordinates:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} l_3(c_1c_2 - s_1s_2) + l_2c_1 \\ l_3(s_1c_2 + s_2c_1) + l_2s_1 \end{pmatrix}$$

where

$$c_i = \cos(\Theta_i), \quad s_i = \sin(\Theta_i).$$

Fix bar lengths l_2, l_3, then the points that the hand can reach are all the (x, y) solutions to the polynomial system:

$$\begin{aligned} l_3(c_1c_2 - s_1s_2) + l_2c_1 - x &= 0 \\ l_3(s_1c_2 + s_2c_1) + l_2s_1 - y &= 0 \\ c_1^2 + s_1^2 - 1 &= 0 \\ c_2^2 + s_2^2 - 1 &= 0 \end{aligned}$$

Ideals and varieties

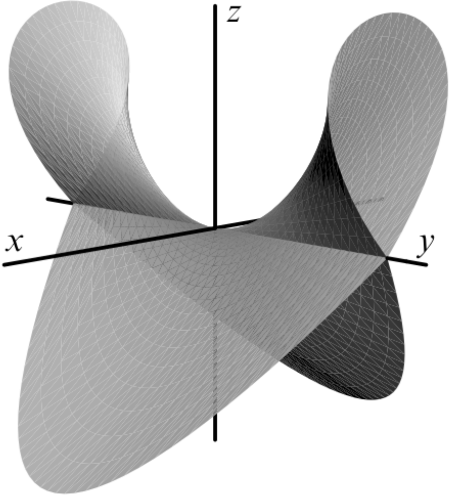
- $\mathbb{C}[x_1, \ldots, x_n]$ = polynomials in the variables $x_1, \ldots, x_n$, with complex coefficients

  Given a system of polynomials $\{f_1, f_2, \ldots, f_m\}$,

- $I = \langle f_1, f_2, \ldots, f_m \rangle$ = ideal generated by $f_1, \ldots, f_m$
  - $I = \{p_1 f_1 + p_2 f_2 + \cdots + p_m f_m : p_i \in \mathbb{C}[x_1, \ldots, x_n]\}$
  - "polynomial linear combinations"

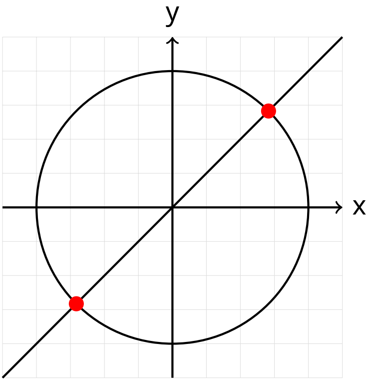
- $\mathcal{V}(I)$ =variety of I
  - Set of points $\vec{v} \in k^n$ where $f(\vec{v}) = 0$ for every $f \in I$.
  - Solutions to the system $\{f_1 = 0, f_2 = 0, \ldots, f_m = 0\}$

- $\dim \mathcal{V}(I)$ or $\dim I$
  - Intuitively: if $\mathcal{V}(I)$ is a finite number of points, $\dim I = 0$, if $\mathcal{V}(I)$ is a line or curve, $\dim I = 1$, etc.



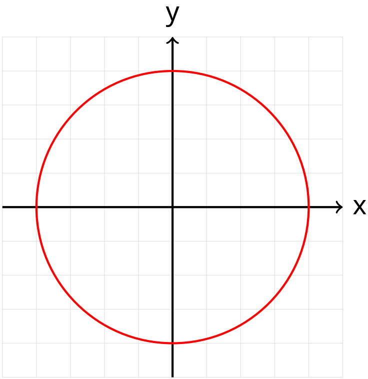
variety of dimension 2

$$I = \langle x^2 + y^2 - 1, x - y \rangle$$



$$\dim I = 0$$

$$I = \langle x^2 + y^2 - 1 \rangle$$



$$\dim I = 1$$

Term orders and initial ideals

- Leading term of $f(x) = x^5 + 3x^4 - x^2 + 1$ is x^5
- For $f(x,y) = x^5 + x^2y^2 + y^6$, we need to decide on a **term order**
  Lexicographic order on monomials in $\mathbb{C}[x,y]$:
  $1 < y < y^2 < y^3 < \cdots < x < xy < xy^2 < \cdots < x^2 < \cdots$
  Degree lexicographic order on monomials in $\mathbb{C}[x,y]$:
  $1 < y < x < y^2 < xy < x^2 < y^3 < xy^2 < x^2y < x^3 < \cdots$

- After fixing a term order, every multivariate polynomial f has a well-defined leading term, $\mathrm{LT}_>(f)$.
  $\mathrm{LT}_{>_{lex}}(x^5 + x^2y^2 + y^6) = x^5$.
  $\mathrm{LT}_{>_{deglex}}(x^5 + x^2y^2 + y^6) = y^6$.

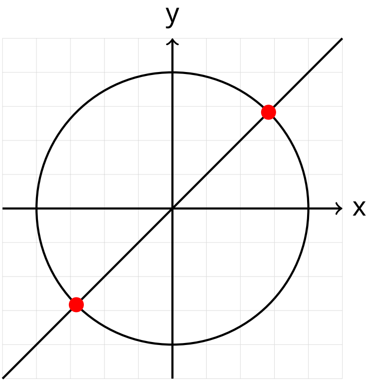
- Every ideal I has an **initial ideal**, $\mathrm{in}_>(I) = \langle \mathrm{LT}_>(f) : f \in I \rangle$, which is a **monomial ideal**.

$$\begin{tikzcd} \text{any polynomial ideal} & & \\ + \ \text{any term order} \arrow[rr, "\text{Buchberger's algorithm}"] & & \text{Gröbner basis} \arrow[d] \\ & & \text{initial ideal} \end{tikzcd}$$

Information about the polynomial ideal can be read from its initial ideal!

- $\dim I = \dim \mathrm{in}_{>}(I)$
- $\#\mathcal{V}(I) = \#$ monomials not in $\mathrm{in}_{>}(I)$
- and much more...

$$I = \langle x^2 + y^2 - 1, x - y \rangle$$

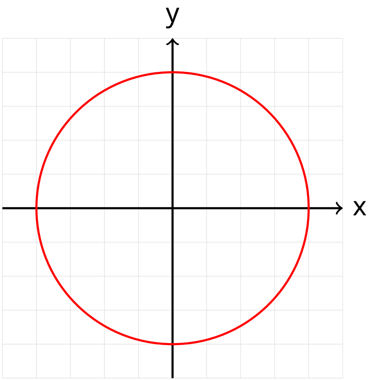


$$\mathrm{in}_{>lex}(I) = \langle x, y^2 \rangle$$

$$\mathcal{V}(\mathrm{in}_{>lex}(I)) = \{(0,0)\}$$

$$\implies \dim I = 0$$

$$I = \langle x^2 + y^2 - 1 \rangle$$



$$\mathrm{in}_{>lex}(I) = \langle x^2 \rangle$$

$$\mathcal{V}(\mathrm{in}_{>lex}(I)) = \{(0,y) : y \in \mathbb{C}\}$$

$$\implies \dim I = 1$$

$$J = \left\langle wy^3z + y^5 - y^3z^2 + 2wz^3 + 2y^2z^2 - 2z^4, \right.$$
$$\left. x^2y^2 - xyz + y^3 + 2z^2, wxz + xy^2 - xz^2 \right\rangle \in k[w, x, y, z]$$

$$J = \langle wy^3z + y^5 - y^3z^2 + 2wz^3 + 2y^2z^2 - 2z^4,$$
$$x^2y^2 - xyz + y^3 + 2z^2, wxz + xy^2 - xz^2\rangle \in k[w,x,y,z]$$

$$I = \mathrm{in}_{>deglex}(J) = \langle wy^3z, x^2y^2, wxz\rangle$$

$$\mathcal{V}(I) = \{(0,0,y,z) : y,z \in k\} \cup \{(0,x,0,z) : x,z \in k\} \cup$$
$$\{(w,x,0,0) : w,x \in k\} \cup \{(w,0,y,0) : w,y \in k\}$$

$$\Longrightarrow \dim I = \dim J = 2$$

$$J = \langle wy^3z + y^5 - y^3z^2 + 2wz^3 + 2y^2z^2 - 2z^4,$$
$$x^2y^2 - xyz + y^3 + 2z^2, wxz + xy^2 - xz^2 \rangle \in k[w,x,y,z]$$

$$I = \mathrm{in}_{>deglex}(J) = \langle wy^3z, x^2y^2, wxz \rangle$$

$$\mathcal{V}(I) = \{(0,0,y,z) : y,z \in k\} \cup \{(0,x,0,z) : x,z \in k\} \cup$$
$$\{(w,x,0,0) : w,x \in k\} \cup \{(w,0,y,0) : w,y \in k\}$$

$$\implies \dim I = \dim J = 2$$

"Easy" combinatorial way to find the dimension gets harder as the number of variables increases. In fact...

Proposition (Bayer–Stillman 1992). The following problem

> Given a monomial ideal $I \subset k[x_1, \ldots, x_n]$, and an integer t, is the dimension of $I \leq t$?

is **NP-complete**.

Table of contents for the rest of this talk:

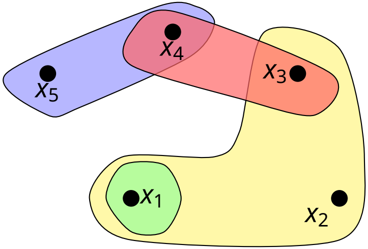
1. How to compute the dimension of a monomial ideal
2. How to compute the dimension of a monomial ideal
3. How to compute the dimension of a monomial ideal
4. How to compute the dimension of a monomial ideal
5. How to compute the dimension of a monomial ideal

Method 1: Graph Theory

From a set of monomials M we define a **hypergraph** H by:

- the vertices of H are $\{x_1, \dots, x_n\}$
- for each $f \in M$ we include the edge $\{x_i : x_i \text{ divides } f\}$

$M = \{x_1^3 x_2^2 x_3, x_1^4, x_3^2 x_4^2, x_4 x_5^3\}$.



A **vertex cover** of H is a set S of vertices, such that every edge of H contains at least one member of S. A vertex cover is **minimal** if no subset of it is a vertex cover.

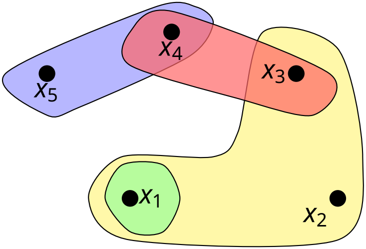
Minimal vertex covers of H: $\{x_1, x_4\}$, $\{x_1, x_3, x_5\}$

$\mathcal{V}(M) = \{(0, x_2, x_3, 0, x_5)\}$
$\cup \{(0, x_2, 0, x_4, 0)\}$

$\dim \mathcal{V}(M) = 3$

$\dim\, (V(M)) =$
$n -$ **minimum size of a vertex cover**

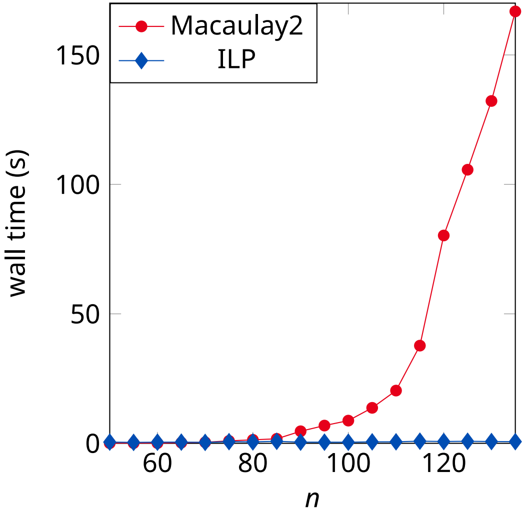
Method 2: Integer Programming



Finding a minimum vertex cover is equivalent to:

$$\text{minimize: } \sum_{i=1}^{5} x_i$$

$$\begin{aligned} \text{subject to: } & x_i \in \{0,1\}\ \forall i \\ & x_1 && \geq 1 \\ & x_1 + x_2 + x_3 && \geq 1 \\ & x_3 + x_4 && \geq 1 \\ & x_4 + x_5 && \geq 1 \end{aligned}$$



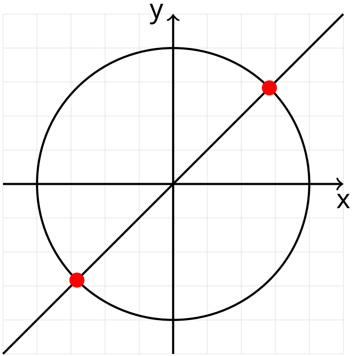
n	M2	IP
50	0.05	0.47
55	0.07	0.35
60	0.08	0.44
65	0.25	0.46
70	0.40	0.36
75	0.94	0.64
80	1.35	0.55
85	1.74	0.78
90	4.65	0.45
95	6.85	0.44
100	8.76	0.43
105	13.70	0.57
110	20.36	0.58
115	37.73	0.83
120	80.27	0.70
125	105.66	0.79
130	132.21	0.68
135	166.81	0.65

time to compute dimension of monomial ideals
with 50 generators of degree 20 in n variables

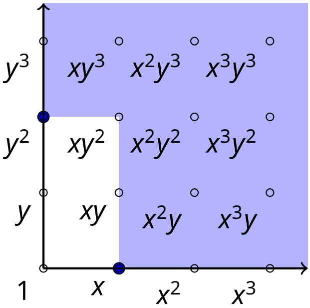
TONS OF OPEN PROBLEMS!!!!!!

Method 3: Staircase diagrams

$$I = \langle x^2 + y^2 - 1, x - y \rangle$$



$$\mathrm{in}_{>_{lex}}(I) = \langle x, y^2 \rangle$$



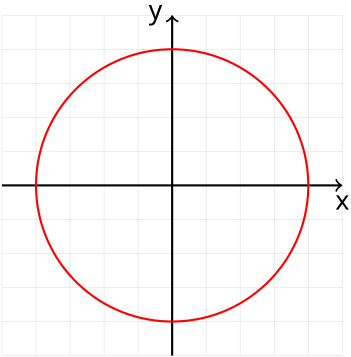
Monomials not in $\mathrm{in}_>(I)$

$=$ monomials "under the staircase"

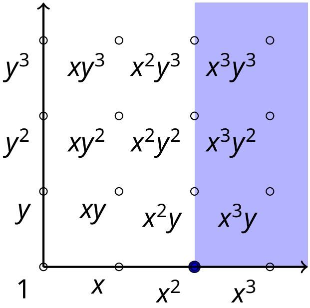
$= \{1, y\} < \infty$

$\implies \dim I = 0$

$$I = \langle x^2 + y^2 - 1 \rangle \subset \mathbb{C}[x, y]$$



$$\mathrm{in}_{>lex}(I) = \langle x^2 \rangle$$



Monomials not in $\mathrm{in}_>(I)$

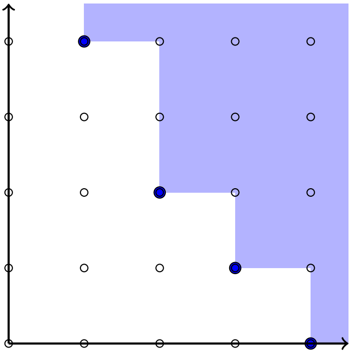
$=$

monomials “under the staircase”

$=$

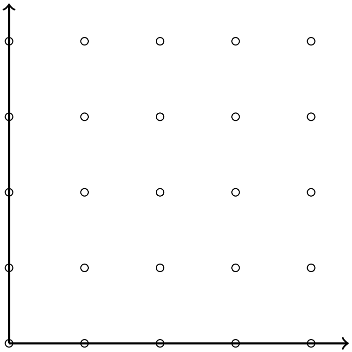
$$\{1, y, y^2, y^3, \ldots, x, xy, xy^2, \ldots\}$$

$$\Longrightarrow \dim I = 1$$



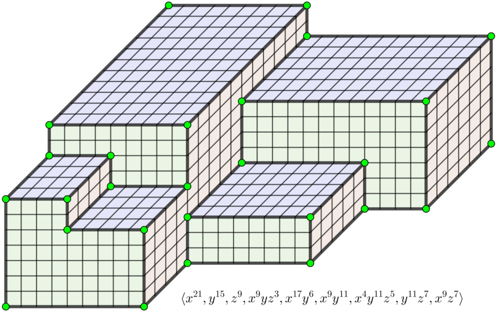
Another 1-dimensional ideal in $\mathbb{C}[x,y]$

$$I = \left\langle x^4, x^3y, x^2y^2, xy^4 \right\rangle$$



Only 2-dimensional ideal in $\mathbb{C}[x,y]$

$$I = \langle 0 \rangle$$



$\langle x^{21}, y^{15}, z^{9}, x^{9}yz^{3}, x^{17}y^{6}, x^{9}y^{11}, x^{4}y^{11}z^{5}, y^{11}z^{7}, x^{9}z^{7}\rangle$

This building in Salt Lake City looks like a staircase diagram of a monomial ideal, so I recreated it in Geogebra and determined what the ideal was.

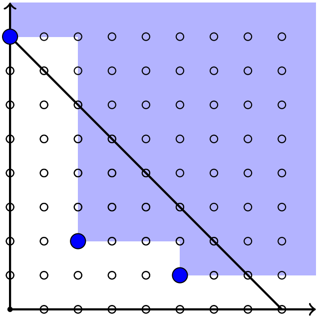
—reddit user **onzie9**

In practice, we don’t try to draw staircases in high dimensions. Instead, we find the **Hilbert function** which enumerates the monomials “under the staircase” in each degree.

Current senior thesis project (Zekai Zhao, UC Davis): finding the fastest way to “slice up” the staircase to find the Hilbert function.

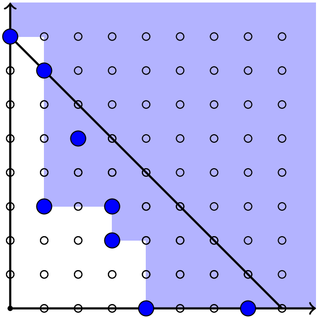
Random monomial ideals

ER-type random monomial ideal $\mathcal{I}(n, D, p)$:, n variables, consider all monomials of total degree at most D, each appears as a generator with independent probability p.



$\mathcal{I}(2, 8, 0.1)$

$\langle x^5y, x^2y^2, y^8 \rangle$



$\mathcal{I}(2, 8, 0.2)$

$\langle x^4, x^3y^2, xy^3, y^8 \rangle$

Theorem (De Loera–Petrović–Stasi–Silverstein–Wilburne, 2019).

Let $I \sim \mathcal{I}(n, D, p(D))$, with $D \to \infty$. The threshold for $\dim I \leq t$ is $p(D) = D^{-(t+1)}$ for every $0 \leq t < n$.

In other words,

$$\text{as } D \to \infty, \begin{cases} \text{if } p \ll D^{-(t+1)}, & \mathbb{P}\left[\dim I > t\right] \to 1 \\ \text{if } p \gg D^{-(t+1)}, & \mathbb{P}\left[\dim I \leq t\right] \to 1. \end{cases}$$

Theorem (De Loera–Petrović–Stasi–Silverstein–Wilburne, 2019).

Let $I \sim \mathcal{I}(n, D, p(D))$, with $D \to \infty$. The threshold for $\dim I \leq t$ is $p(D) = D^{-(t+1)}$ for every $0 \leq t < n$.

In other words,

$$\text{as } D \to \infty, \begin{cases} \text{if } p \ll D^{-(t+1)}, & \mathbb{P}\left[\dim I > t\right] \to 1 \\ \text{if } p \gg D^{-(t+1)}, & \mathbb{P}\left[\dim I \leq t\right] \to 1. \end{cases}$$

Now plug in $t - 1$:

$$\text{as } D \to \infty, \begin{cases} \text{if } p \ll D^{-t}, & \mathbb{P}\left[\dim I > t - 1\right] \to 1 \\ \text{if } p \gg D^{-t}, & \mathbb{P}\left[\dim I \leq t - 1\right] \to 1. \end{cases}$$

Corollary (DPSSW).

Let $I \sim \mathcal{I}(n, D, p(D))$. As $D \to \infty$,

$$\text{if } D^{-t-1} \ll p(D) \ll D^{-t}, \text{ then } \mathbb{P}\left[\dim I = t\right] \to 1.$$

Complete understanding of dimension in the random case!

Thresholds for many more algebraic properties.

★ Projective dimension and other homological properties (with Jesús De Loera, Robert Krone, Serkan Hoşten, 2019)

★ Expected degree, Hilbert polynomial (current project with Serkan Hoşten, Dane Wilburne, Jay Yang)

TONS OF OPEN PROBLEMS!!!!!!

★ Distributions of initial ideals coming from random polynomial ideals

★ Distributions of initial ideals coming from families of polynomial systems that arise in applications

★ Characterizing the affect of term order on expected initial ideals

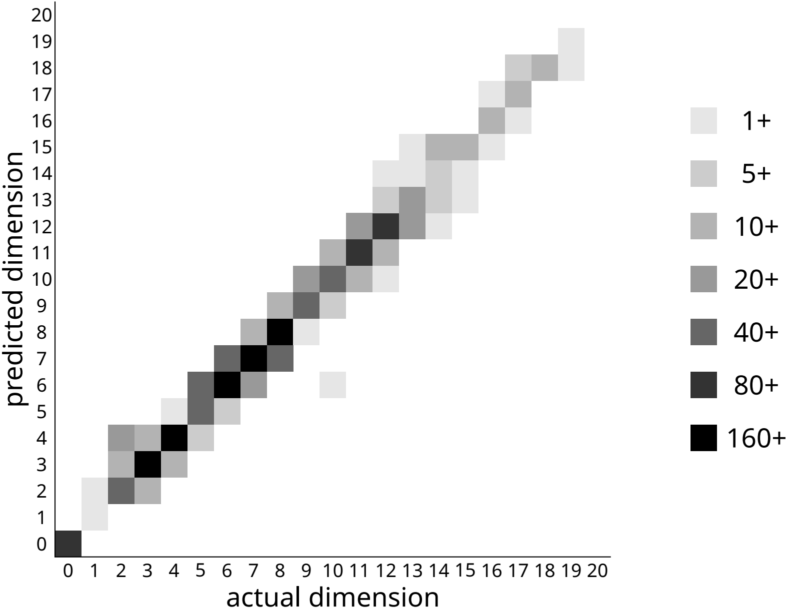
★ etc.

Method 5: Machine learning

Monomial ideals observed from an unknown distribution.

Monomial ideals observed from an unknown distribution.

$|$**prediction** $-$ **actual dimension**$| \leq 1$ **for 99% of test set**



Potential applications:

- polynomial-time *confirmation* of prediction in some cases (not guaranteed)
- numerical solving methods
- real-time updating in algebraic vision and other applications (?)
- next: hard problem of computing Gröbner bases

Potential applications:

- polynomial-time *confirmation* of prediction in some cases (not guaranteed)
- numerical solving methods
- real-time updating in algebraic vision and other applications (?)
- next: hard problem of computing Gröbner bases

Not enough time for a part 6:

6. How to compute the dimension of a monomial ideal

Random monomial ideals are a strict generalization of random simplicial complexes

TONS of connections to computational topology, especially **multidimensional persistence** (multiparameter persistent homology)

Thanks for your attention!