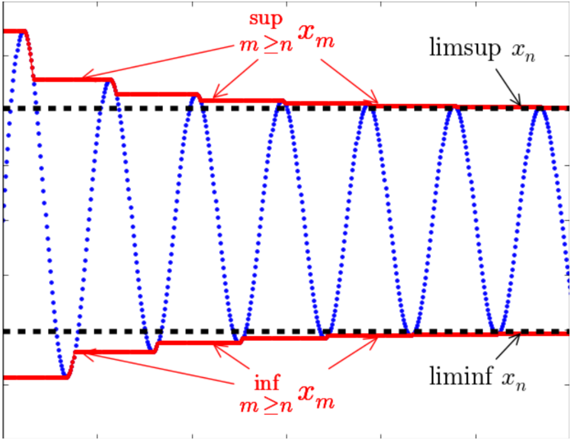
Limits of measurable functions are measurable

Proposition. If $\{f_n\}$ are measurable functions $(X, \mathcal{M}) \to \mathbb{R}$, then

$$\sup_n f_n(x), \ \inf_n f_n(x), \ \limsup_{n\to\infty} f_n(x), \quad \text{and} \quad \liminf_{n\to\infty} f_n(x)$$

are all measurable.

$$\limsup_{n\to\infty} f_n(x) = \inf_{n\geq 1}(\sup_{k\geq n} f_k(x)) \qquad \liminf_{n\to\infty} f_n(x) = \sup_{n\geq 1}(\inf_{k\geq n} f_k(x))$$



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are all measurable. And if $\lim_{n\to\infty} f_n(x)$ exists, it is measurable.

Proof of $\sup_n$: Define $g(x) = \sup_n f_n(x)$. We want to show g is measurable, so we want to show that for each $a \in \mathbb{R}$, $\{x : g > a\} \in \mathcal{M}$.

Claim: $\{x : g(x) > a\} = \bigcup_{n=1}^{\infty} \{x : f_n(x) > a\}$. (Why?)

Since each f_n is a measurable function, each $\{x : f_n(x) > a\} \in \mathcal{M}$. Therefore $\{x : g(x) > a\} \in \mathcal{M}$ Since this works for all a, $\sup_n \ f_n(x)$ is a measurable function.

Trick question: is the limit of integrable functions integrable?

Bounded Convergence Theorem

Theorem. Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$. Let $\{f_n\}$ be a sequence of measurable functions with $f_n \to f$ pointwise a.e. If $\exists M$ such that $|f_n(x)| \leq M$ for all n (and a.e. x), then f is integrable and

$$\int f = \lim_{n \to \infty} \int f_n.$$

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Example of why we need boundedness.

Let $X = [0,1]$, $\mu = m$, and $f_n = n\chi_{(0,1/n)}$.

$$\int \lim_{n\to\infty} f_n = 0 \neq 1 = \lim_{n\to\infty} \int f_n$$

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Example of why we need finiteness.

Let $X = [0, \infty)$, $\mu = m$, and $f_n = \chi_{[n,n+1]}$. Then $f_n \to 0$ pointwise.

$$\int \lim_{n\to\infty} f_n(x)\, dm = 0 \text{ but } \lim_{n\to\infty} \int f_n\, dm = 1.$$

Dominated Convergence Theorem

Theorem. Let $(X, \mathcal{M}, \mu)$ be a measure space, $\{f_n\}$ a sequence of measurable functions with $f_n \to f$ pointwise a.e. If $\exists g \in L^1(X, \mu)$ such that $|f_n(x)| \leq g$ for all n and a.e. x, then $f \in L^1(X, \mu)$ and

$$\int f = \lim_{n\to\infty} \int f_n.$$

Monotone Convergence Theorem

Theorem. Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space, $\{f_n\}$ a sequence of non-negative measurable functions such that $f_1 \leq f_2 \leq \cdots$ a.e. and $f_n \to f$ a.e. Then

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Why we need non-negativity

Let $f_n : [0, \infty) \to \mathbb{R}$ be defined by $f_n = (-1/n)\chi_{[n,2n]}$. Then f_n increases pointwise to $f = 0$, but

$$\int \lim_{n\to\infty} f_n = \int f = 0 \not\leq -1 = \lim_{n\to\infty} \int f_n.$$

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Why we need f_n increasing to f

Let $f_n = n\chi_{(0,1/n)}$. Then f_n converges pointwise to $f = 0$, but

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Monotone convergence does not hold for Riemann integrals

Recall our sequence on $[0,1]$ whose limit was $\mathbf{1}_{\mathbb{Q}}$ on $[0,1]$.

Fatou's Lemma

Theorem. Let $(X, \mathcal{M}, \mu)$ be a measure space, $\{f_n\}$ a sequence of measurable functions with $f_n \to f$ pointwise a.e. If $f_n \geq 0$, then

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Littlewood's Three Principles

- ★ *Every measurable set is nearly a finite union of intervals*.
- ★ *Every measurable function is nearly continuous*.
- ★ *Every convergent sequence of measurable functions is nearly uniform convergent*.

Littlewood's Three Principles

★ *Every measurable set is nearly a finite union of intervals.*

★ *Every measurable function is nearly continuous.*

★ *Every convergent sequence of measurable functions is nearly uniform convergent.*

The first principle refers to the definition of outer measure, based on approximating measurable sets with unions of intervals.

Littlewood's Three Principles

The second principle (*every measurable function is nearly continuous*) is captured by **Lusin's Theorem**:

Theorem. Let μ be a finite Borel measure on $X \subseteq \mathbb{R}^n$, and let $f : X \to \mathbb{R}$ be measurable. For all $\epsilon > 0$, there exists a closed set $F \subseteq X$ such that $f : F \to \mathbb{R}$ is continuous and $\mu(X \backslash F) < \epsilon$.

Lusin's Theorem might not do what you expect

Let $X = [0,1]$, $f = \chi_{\mathbb{Q}\cap[0,1]}$. Let E be an open cover of $\mathbb{Q} \cap [0,1]$ with $m(E) < \epsilon$, and let $F = E^C$. Then $m(F^C) < \epsilon$ and F is closed. Since f is identically 1 on F, $f : F \to \mathbb{R}$ is continuous.
But of course $f : [0,1] \to \mathbb{R}$ isn't continuous *anywhere*, including the points of F. So Lusin's Theorem might trim the domain in a weird way—it doesn't necessarily find a subset on which the original function is continuous.

Littlewood's Three Principles

The third principle (*every convergent sequence of measurable functions is nearly uniform convergent*) is probably a reference to **Egorov's Theorem**:

Theorem. Let μ be a finite measure on a metric space X. Let $f_n : X \to \mathbb{R}$ be μ-measurable functions that converge to f μ-a.e. Then for all $\delta > 0$ there exists a closed set A so $\mu(A^C) < \delta$ and $f_n \to f$ uniformly on A.