Example.
For any set X, $\mathcal{A} = \{E \subseteq X : E \text{ is countable or } \underbrace{X \backslash E \text{ is countable}}_{E \text{ is cocountable}}\}$ is a σ-algebra.

1. $\emptyset \in \mathcal{A}$ because $\emptyset$ is countable.
2. Let $E \in \mathcal{A}$. Then either E is countable or $X \backslash E$ is countable. In the first case, $X \backslash E$ is the complement of a countable set, so $X \backslash E \in \mathcal{A}$. In the second case, $X \backslash E \in \mathcal{A}$ since it's countable. So $\mathcal{A}$ is closed under complements.

Example.
For any set X, $\mathcal{A} = \{E \subseteq X : E \text{ is countable or } X \backslash E \text{ is countable}\}$ is a σ-algebra.

3. Let $\{E_j\}_1^\infty \subseteq \mathcal{A}$. We need to show that either $\cup_1^\infty E_j$ is countable or that $X \backslash (\cup_1^\infty E_j)$ is countable.
First suppose that E_j is countable for all j. Then $\cup_1^\infty E_j$ is countable, since the countable union of countable sets is countable.
On the other hand, suppose at least one of the E_j, WLOG let's say E_1, is uncountable. Since $E_1 \in \mathcal{A}$ it must be that $X \backslash E_1$ is countable. Then

$$X \backslash \bigcup_{j=1}^{\infty} E_j = \bigcap_{j=1}^{\infty} (X \backslash E_j) \subseteq X \backslash E_1,$$

so $X \backslash (\cup_1^\infty E_j)$ is contained in a countable set which means it's countable and in $\mathcal{A}$. Thus $\mathcal{A}$ is closed under countable unions. $\square$

Definition. Let X be a set and $\mathcal{A}$ a set of subsets of X. Then the intersection of all σ-algebras on X that contain $\mathcal{A}$ is a σ-algebra on X. We call this the **σ-algebra generated by** $\mathcal{A}$ and write $\sigma(\mathcal{A})$.

Example. Suppose X is a set and S is the set of subsets of X that consist of exactly one element:

$$S = \{\{x\} : x \in X\}.$$

Then $\sigma(S) = \{E \subseteq X : E \text{ is countable or } X \setminus E \text{ is countable}\}$.

Ex. $X = \mathbb{R}$

$S = \{ \{x\} : x \in \mathbb{R} \}$

$\{1\} \in S$

$\{1\} \in \sigma(S)$

Argue why <u>any</u> σ-algebra on X that contains S needs to contain $\sigma(S)$.

$S \subseteq \sigma(S)$

Borel sets

Definition. The *Borel σ-algebra* on $\mathbb{R}$, written $\mathcal{B}(\mathbb{R})$ (or $\mathcal{B}_{\mathbb{R}}$ or sometimes just $\mathcal{B}$) is the σ-algebra generated by the open sets of $\mathbb{R}$. Elements of $\mathcal{B}(\mathbb{R})$ are called *Borel sets*.

$\mathcal{B}$ is generated by each of the following collections:

1. $C_1 = \{(a, b) : a, b \in \mathbb{R}\}$
2. $C_2 = \{[a, b] : a, b \in \mathbb{R}\}$
3. $C_3 = \{(a, b] : a, b \in \mathbb{R}\}$
4. $C_4 = \{(a, \infty) : a \in \mathbb{R}\}$
5. $C_5 = \{(-\infty, a] : a \in \mathbb{R}\}$

(work in groups)

1. Claim: $\sigma(C_1) = \mathcal{B}(\mathbb{R})$

First we need the fact that every open set in $\mathbb{R}$ can be written as a countable union of open intervals.

So all open sets belong to $\sigma(C_1)$

Next we'll use the following important fact:

if $A \subseteq B$ then $\sigma(A) \subseteq \sigma(B)$

so $\sigma(\{\text{open sets}\}) \subseteq \sigma(\sigma(C_1))$

$= \sigma(C_1)$

OTOH every open interval is an open set,

so

$C_1 \subseteq \{\text{open sets}\}$

$\Rightarrow \sigma(C_1) \subseteq \sigma(\{\text{open sets}\})$

Therefore $\sigma(C_1) = \mathcal{B}(\mathbb{R})$

Claim: $\mathcal{B}(\mathbb{R}) = \sigma(C_5)$, where $C_5 = \{(-\infty, a] : a \in \mathbb{R}\}$

First let's show

$(a,b) \in \sigma(C_5)$ for all $a<b$.

$$(a,b) = \bigcup_{n=1}^{\infty} (a, b_n] \qquad \text{where } b_n = b - \frac{1}{n}$$

$$= \bigcup_{n=1}^{\infty} \left((-\infty, b_n] \cap (a, \infty) \right)$$

$$= \bigcup_{n=1}^{\infty} \left((-\infty, b_n] \cap (-\infty, a]^c \right)$$

$\in C_5$ $\in \sigma(C_5)$

$C_5 \subseteq \sigma(C_5)$ $\qquad$ $\sigma(C_5)$

So $(a,b) \in \sigma(C_5)$ for all (a,b)

$$C_1 \subseteq \sigma(C_5)$$

$$\Rightarrow \underset{\mathcal{B}(\mathbb{R})}{\underset{\shortparallel}{\sigma(C_1)}} \subseteq \sigma(\sigma(C_5)) = \sigma(C_5)$$

<u>OTOH</u> for all $a \in \mathbb{R}$,

$$[-\infty, a] = \mathbb{R} \setminus ((a, \infty))$$

$$= \mathbb{R} \setminus \left(\bigcup_{n=1}^{\infty} (a,n) \right) \in \sigma(C_1)$$

$$\Rightarrow \dots \Rightarrow \sigma(C_5) \subseteq \sigma(C_1) = \mathcal{B}(\mathbb{R})$$

Measures

set $\quad$ σ-algebra on X

Definition. Let $(X, \mathcal{M})$ be a measurable space. A **measure** on $(X, \mathcal{M})$ is a function $\mu : \mathcal{M} \to [0, \infty]$ such that $\leftarrow$ ∞ is allowed

1. $\mu(\emptyset) = 0$
2. (countable additivity) If $\{E_j\}$ is a collection of pairwise disjoint sets in $\mathcal{M}$, then $\mu(\bigcup_{j=1}^{\infty} E_j) = \sum_{j=1}^{\infty} \mu(E_j)$.

for all E_i and E_j
$E_i \cap E_j = \emptyset$

Definition

If μ is a measure on $(X, \mathcal{M})$, then $(X, \mathcal{M}, \mu)$ is called a **measure space**.

Measures

Examples and non-examples of measures (work in groups)

For the following let $(X, \mathcal{M}) = (\mathbb{R}, \mathcal{P}(\mathbb{R}))$.

1. $\alpha : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ defined by $\alpha(E) = \begin{cases} 0 & \text{if } E \text{ is countable,} \\ \infty & \text{if } E \text{ is uncountable.} \end{cases}$ Yes measure

2. $\beta : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ defined by $\beta(E) = \begin{cases} 0 & \text{if } E \text{ is finite,} \\ \infty & \text{if } E \text{ is infinite.} \end{cases}$ NOT a measure

3. $\gamma : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ defined by $\gamma(E) =$ cardinality of E. Yes, this is called <u>counting measure</u>

4. Fix $x_0 \in \mathbb{R}$, and define $\delta_{x_0} : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ by
$$\delta_{x_0}(E) = \begin{cases} 1 & x_0 \in E, \\ 0 & \text{otherwise.} \end{cases}$$

4) • $x_0 \notin \emptyset$ so $\delta_{x_0}(\emptyset) = 0$

• consider $\bigcup_{j=1}^{\infty} E_j$ where each $E_j \in \mathcal{P}(\mathbb{R})$

and the E_j are pairwise disjoint.

Case 1: suppose $x_0 \in E_j$ for some j, then x_0 is in exactly one of the E_j. So

$$\sum_{j=1}^{\infty} \delta_{x_0}(E_j) = 1 = \delta_{x_0}\left(\bigcup_{j=1}^{\infty} E_j\right) \checkmark$$

Case 2: x_0 is not in any of the E_j. Then x_0 does not belong to their union either. So

$$\sum_{j=1}^{\infty} \delta_{x_0}(E_j) = 0 = \delta_{x_0}\left(\bigcup_{j=1}^{\infty} E_j\right) \checkmark$$

So δ_{x_0} is a measure on $(\mathbb{R}, \mathcal{P}(\mathbb{R}))$

called the point mass at x_0 or Dirac delta measure

2) • $\beta(\emptyset) = 0$ because $\emptyset$ is finite

• countable additivity condition fails

ex. take $\bigcup_{i=1}^{\infty} \{i\}$

for each i, $\beta(\{i\}) = 0$ since $\{i\}$ is finite

and $\beta\left(\bigcup_{i=1}^{\infty} \{i\}\right) = \infty$

so $$\sum_{i=1}^{\infty} \beta(\{i\}) = \sum_{i=1}^{\infty} 0 = 0 \neq \infty = \beta\left(\bigcup_{i=1}^{\infty} \{i\}\right)$$

Outer measure on $\mathbb{R}$ — come back to this tomorrow

Definition. The length of an open interval $I \subseteq \mathbb{R}$ is defined by

$$\ell(I) = \begin{cases} 0 & I = \emptyset \\ b - a & I = (a, b) \text{ with } a < b \\ \infty & I = (-\infty, a) \text{ or } (a, \infty) \text{ or } (-\infty, \infty) \end{cases}$$

Definition. Define the **Lebesgue outer measure** of $E \subseteq \mathbb{R}$, written $|E|$ (Axler) or $m^*(E)$ (Bass) by

$$\inf \left\{ \sum_{i=1}^{\infty} \ell(A_i) : \text{ each } A_i \text{ is an open interval of } \mathbb{R} \text{ and } E \subseteq \bigcup_{i=1}^{\infty} A_i \right\}.$$

Outer measure is **not** a measure on $(\mathbb{R}, \mathcal{P}(\mathbb{R}))$. We will soon see that it **is** a measure on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$, and in fact there is a somewhat larger σ-algebra $\mathcal{L}$ such that outer measure is a measure on $(\mathbb{R}, \mathcal{L})$.

Special kinds of measures – come back tomorrow

Definition.

- μ is a *finite* measure if $\mu(X) < \infty$
- μ is a *probability measure* if $\mu(X) = 1$
- μ is called σ-*finite* if $X = \bigcup_{n=1}^{\infty} X_n$ where $\mu(X_n) < \infty$ for all n.

$(X, \mathcal{M}, \mu)$

Properties of measures

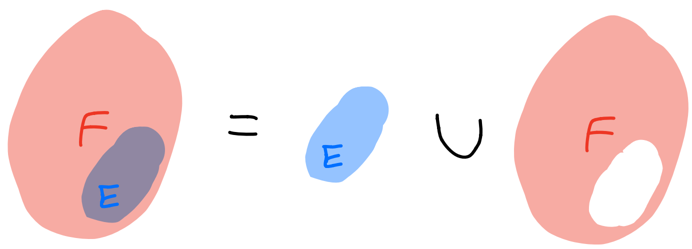
Theorem. Let $(X, \mathcal{M}, \mu)$ be a measure space. Then for all $E, F \in \mathcal{M}$ and for all $\{E_j\}_1^\infty \subseteq \mathcal{M}$:

1. (monotonicity) If $E \subseteq F$ then $\mu(E) \leq \mu(F)$.
2. (countable subadditivity) $\mu(\bigcup_{j=1}^\infty E_j) \leq \sum_{j=1}^\infty \mu(E_j)$.
3. (continuity from below) If $E_j \subseteq E_{j+1}$ for all j, then
$$\mu\left(\bigcup_{j=1}^\infty E_j\right) = \lim_{j\to\infty} \mu(E_j).$$
4. (continuity from above) If $\mu(E_1) < \infty$ and $E_j \supseteq E_{j+1}$ for all j, then $$\mu\left(\bigcap_{j=1}^\infty E_j\right) = \lim_{j\to\infty} \mu(E_j).$$

E_j are not necessarily pw disjoint

Proof of $\boxed{1}$

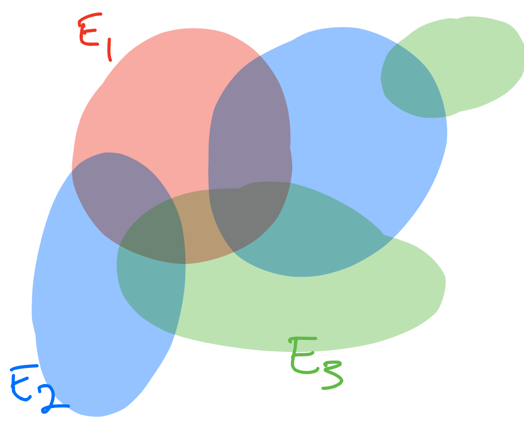
Since $E \subseteq F$, $F = E \cup (F \setminus E)$ where E and $F \setminus E$ are disjoint. $E \in \mathcal{M}$ by hypothesis and $F \setminus E =$ $F \cap E^c = (F^c \cup E)^c \in \mathcal{M}$ by closure under complements and unions.



Then by countable additivity of measure,

$$\mu(F) = \mu(E) + \mu(F \setminus E) \geq \mu(E) + 0 \geq \mu(E).$$

Proof of $\boxed{2}$



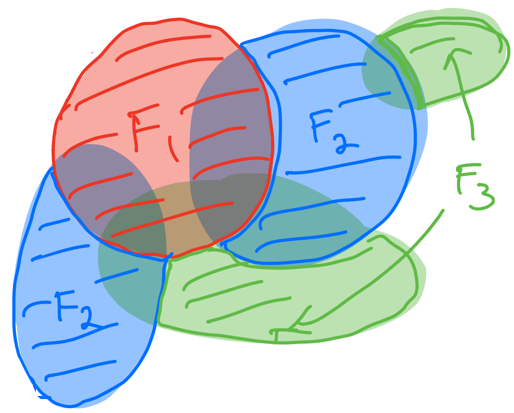
Let $F_1 = E_1$

$F_2 = E_2 \setminus E_1$

$F_3 = E_3 \setminus (E_1 \cup E_2)$

$\vdots$

$F_k = E_k \setminus \left(\bigcup_{j=1}^{k} E_j \right)$



Then the $\{F_k\}$ are pairwise disjoint sets in $\mathcal{M}$ and

$$\bigcup_{j=1}^{n} F_j = \bigcup_{j=1}^{n-1} E_j \quad \text{for all } n.$$

Now we can use countable additivity of measure on p.w. disjoint sets to compute

$$\mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \mu\left(\bigcup_{j=1}^{\infty} F_j\right) = \sum_{j=1}^{\infty} \mu(F_j)$$

Since $F_j \subseteq E_j$ for each j, by monotonicity:

$$\leq \sum_{j=1}^{\infty} \mu(E_j). \qquad \square$$