Random Monomial Ideals

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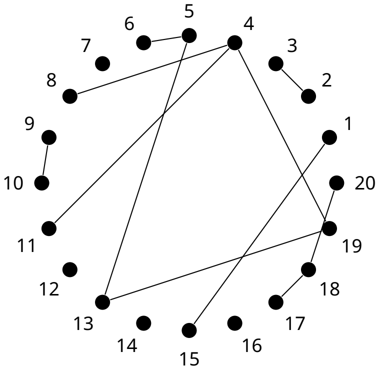
Joint Mathematics Meetings

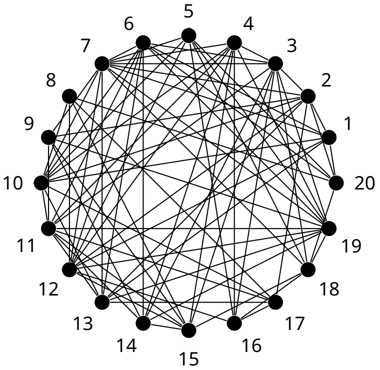
18 January 2019

Techniques from probabilistic combinatorics

The **Erdős-Rényi random graph**, $\mathcal{G}(n, p)$, is a graph with n vertices, where each possible edge appears with probability p.



$G \sim \mathcal{G}(20, 0.05)$



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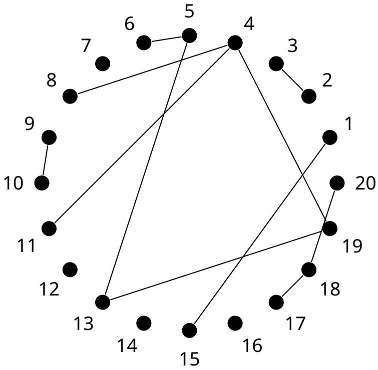

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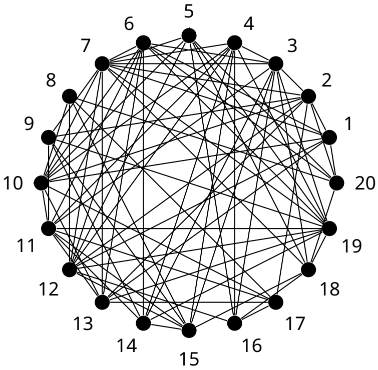
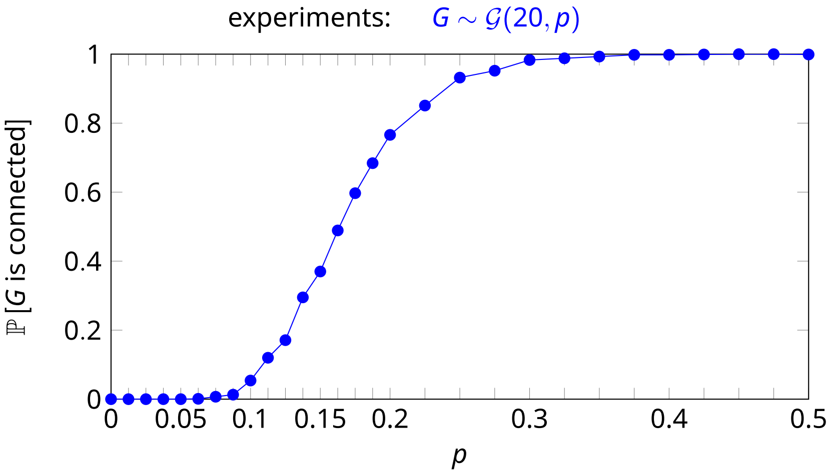


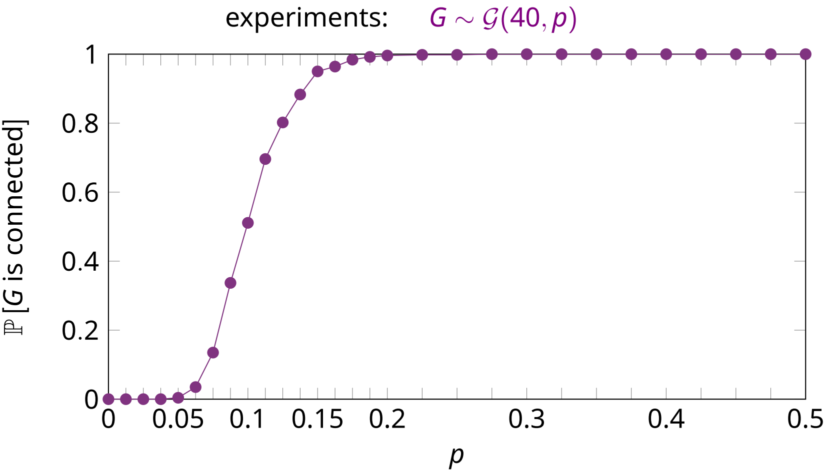
$G \sim \mathcal{G}(20, 0.5)$

$$\mathbb{P}\left[G \text{ is connected}\right] = ??$$

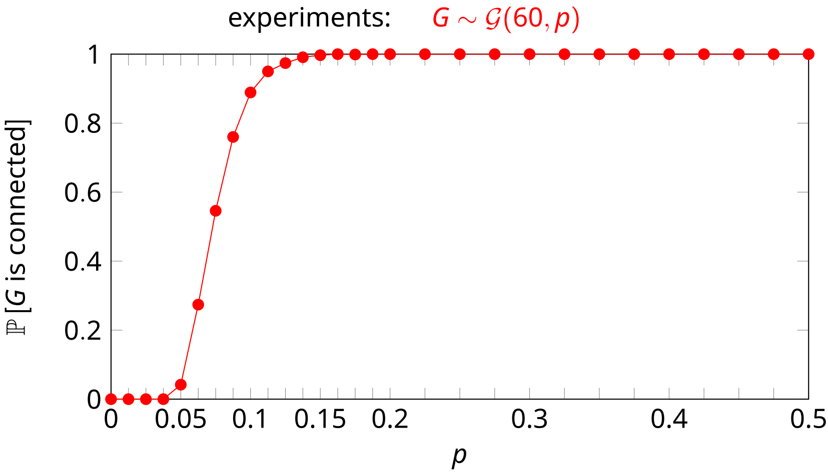
Techniques from probabilistic combinatorics



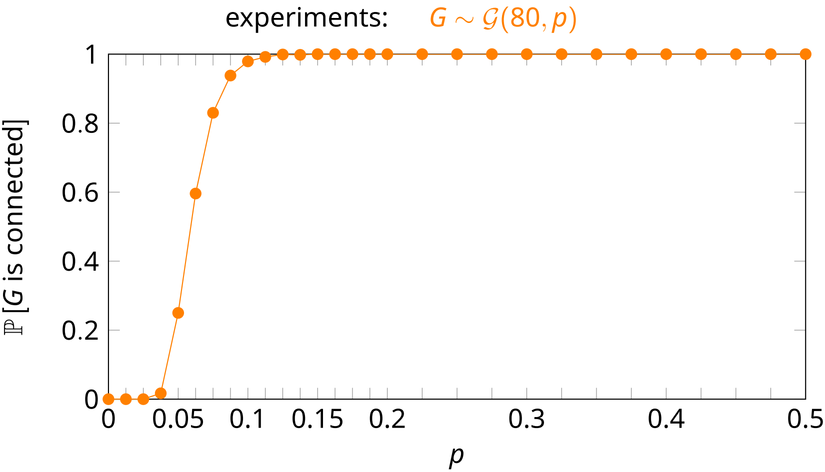
Techniques from probabilistic combinatorics



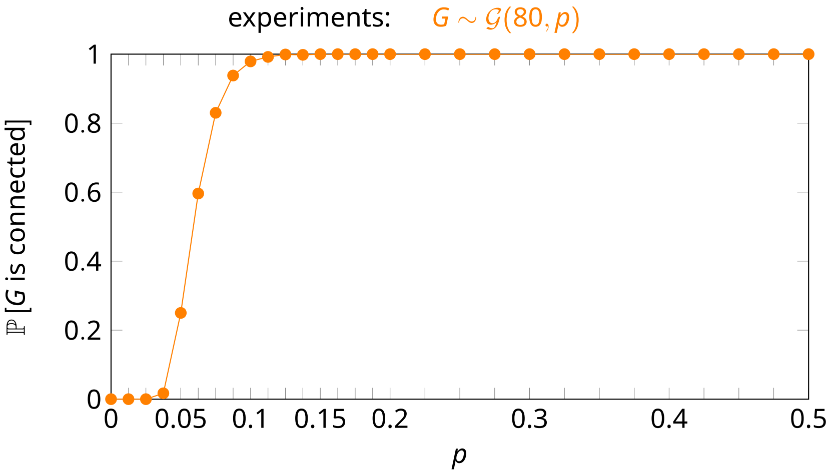
Techniques from probabilistic combinatorics



Techniques from probabilistic combinatorics



Techniques from probabilistic combinatorics

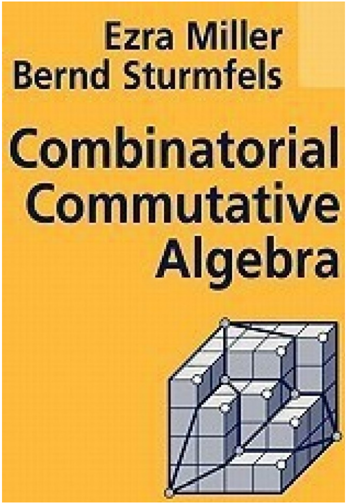


Erdős–Rényi, 1960: **threshold** for connectedness is $\mathbf{p}(\mathbf{n}) = \dfrac{\ln \mathbf{n}}{\mathbf{n}}$.

What can we learn about commutative algebra using this approach?

Start with **monomial ideals** for several reasons:

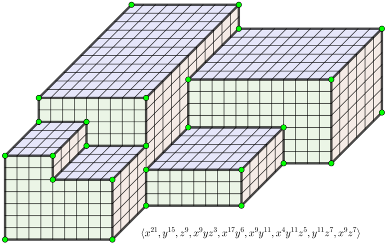
- Past work on randomness in algebra has focused on random polynomial **coefficients**.
- Monomial ideals are a **combinatorial** way to study random polynomial systems.
- Crucial in computations because of **Gröbner basis theory**. Invariants of a polynomial ideal, like **dimension**, **degree**, and **Hilbert function**, can be read from an initial ideal.



- Combinatorial constructions like **cellular resolutions** are key to developing faster algorithms in computer algebra.

Posted Wednesday night on reddit by **u/onzie9**:

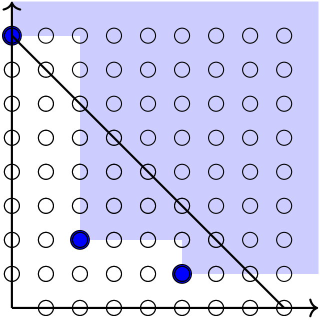
This building in Salt Lake City looks like a staircase diagram of a monomial ideal, so I recreated it in Geogebra and determined what the ideal was.



$$\langle x^{21}, y^{15}, z^{9}, x^{9}yz^{3}, x^{17}y^{6}, x^{9}y^{11}, x^{4}y^{11}z^{5}, y^{11}z^{7}, x^{9}z^{7} \rangle$$

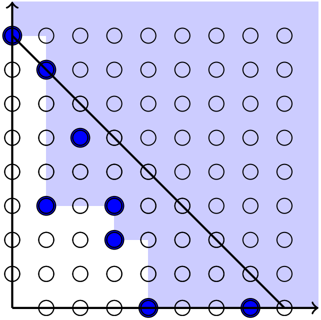
Random monomial ideals

The **ER-type random monomial ideal**, $\mathcal{I}(n, D, p)$, is a monomial ideal in n variables, where each monomial of total degree at most D appears as a generator with probability p.



$\mathcal{I}(2, 8, 0.05)$

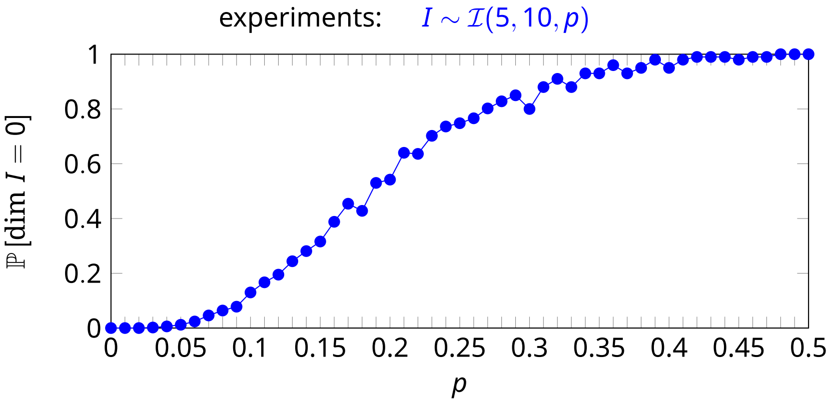
(x^5y, x^2y^2, y^8)



$\mathcal{I}(2, 8, 0.1)$

(x^4, x^3y^2, xy^3, y^8)

$\dim I \sim \mathcal{I}(n, D, p)$



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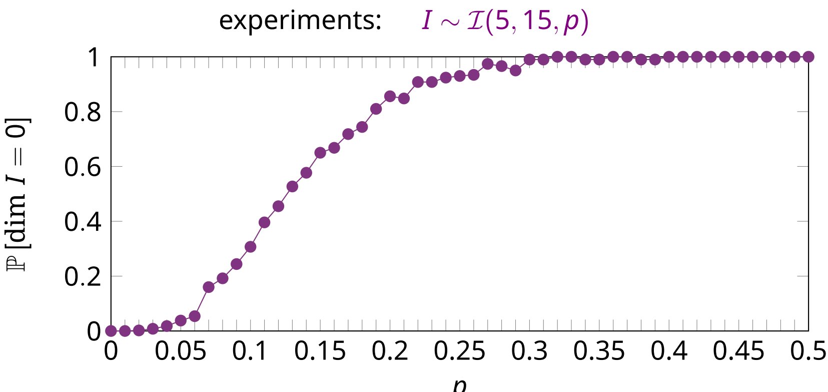

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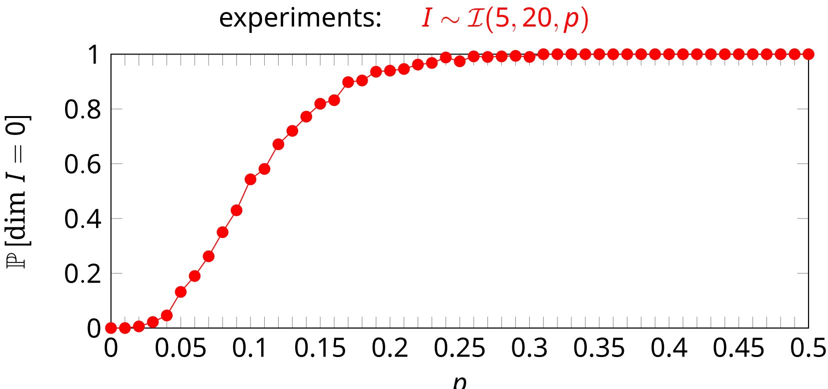

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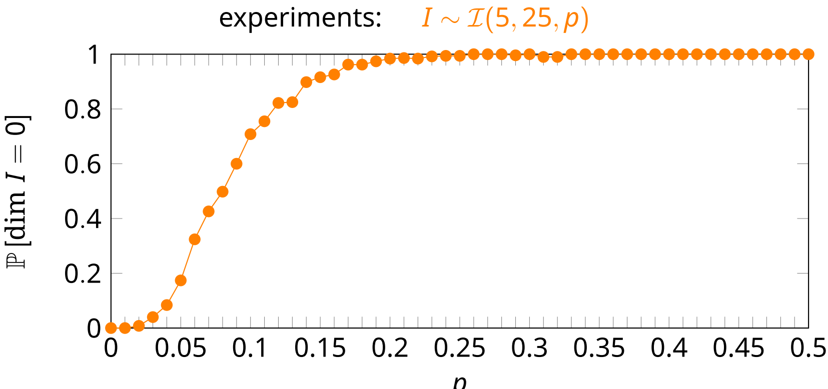

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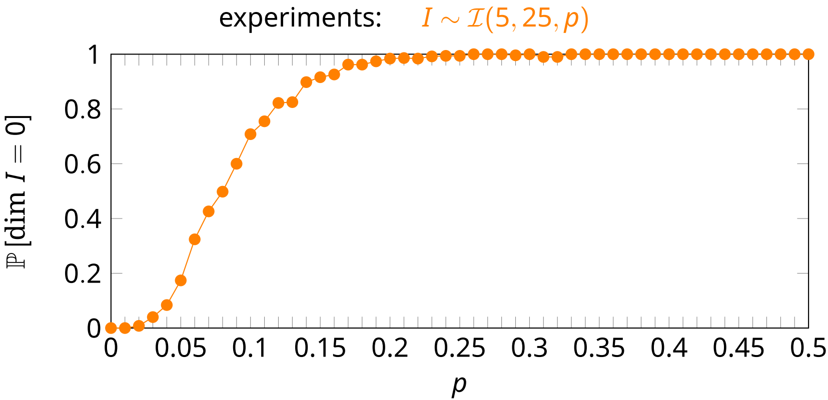


De Loera–Petrović–Stasi–Silverstein–Wilburne, 2019

As $D \to \infty$, the **threshold** for $\dim I \leq t$ is $\mathbf{p}(\mathbf{D}) = \mathbf{D}^{-\mathbf{t}-\mathbf{1}}$ for every $0 \leq t < n$.

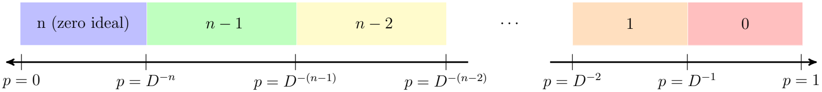
dim $I \sim \mathcal{I}(n, D, p)$

De Loera–Petrović–Stasi–Silverstein–Wilburne, 2019

Let $I \sim \mathcal{I}(n, D, p(D))$. As $D \to \infty$,

$$\text{if } D^{-t-1} \ll p(D) \ll D^{-t}, \text{ then } \dim I = t$$

asymptotically almost surely as $D \to \infty$.



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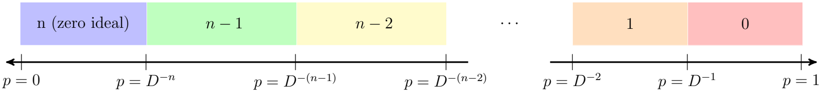
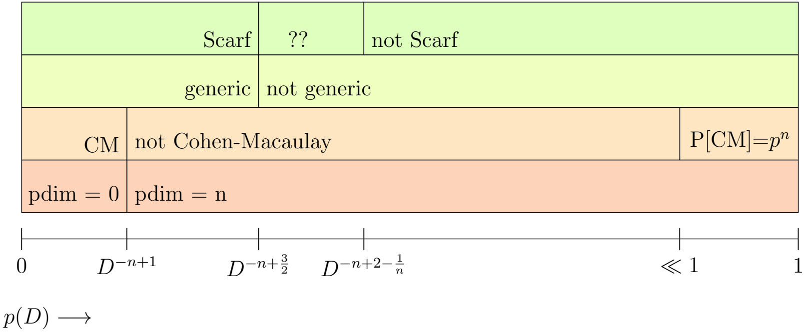


Bayer–Stillman, 1992

The following problem is NP-complete:
Given a monomial ideal $I \subset k[x_1, \ldots, x_n]$, and an integer t, is the dimension of $I \leq t$?

Many other thresholds and phase transitions for algebraic properties!

De Loera–Hoşten–Krone–Silverstein, 2019:



Genericity and “Scarfness”

A monomial ideal is called (strictly) **generic** if no two distinct minimal generators have the same positive degree in the same variable x_i.

By Bayer–Peeva–Sturmfels (1998), the **Scarf complex** of a generic monomial ideal gives its minimal free resolution.

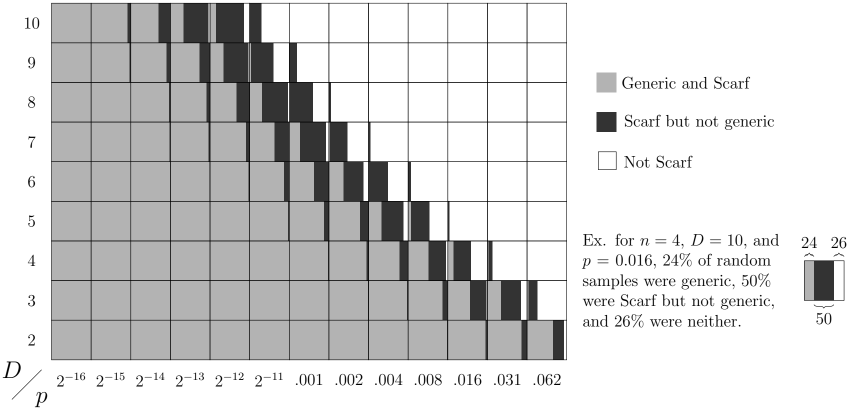
For the graded model, we proved that $p(D) = D^{-n+3/2}$ is the threshold for genericity of a monomial ideal, i.e.

$$\lim_{D\to\infty} \mathbb{P}\left[\mathcal{M}(n,D,p) \text{ is generic}\right] = \begin{cases} 1, & p \ll D^{-n+3/2} \\ 0, & p \gg D^{-n+3/2}. \end{cases}$$

On the other hand, we showed:

for $p \gg D^{-n+2-1/n}$, the Scarf complex will not resolve $\mathcal{M}(n,D,p)$.

The twilight zone



`Macaulay2` experiments, $n = 8$

Projective dimension

De Loera–Hoşten–Krone–Silverstein, 2019+

Let $S = k[x_1, \ldots, x_n]$, and let $M \sim \mathcal{M}(n, D, p)$ be a random monomial ideal generated in degree D.
Then $p(D) = D^{-n+1}$ is a threshold for the projective dimension of S/M. In particular, as $D \to \infty$:

- If $p \ll D^{-n+1}$, then $\mathbb{P}\left[\mathrm{pdim}(S/M) = 0\right] \to 1$.
- If $p \gg D^{-n+1}$, then $\mathbb{P}\left[\mathrm{pdim}(S/M) = n\right] \to 1$.

Projective dimension

Krull dimension is a **monotone** property of ideals.

$$I \subseteq J \implies \dim\,(S/I) \geq \dim\,(S/J).$$

For random graphs, it's known that *every* monotone graph property (e.g., connectedness) has a threshold.

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Unfortunately, projective dimension isn't monotone...

$$\begin{aligned} S &= k[v, w, x, y, z] \\ \mathrm{pdim}(S/(wz, y^3, xy^2, vwx^3)) &= & 3 \\ \mathrm{pdim}(S/(wz, y^3, xy^2, vwx^3, v^2wy)) &= & 4 \\ \mathrm{pdim}(S/(w, y^3, xy^2)) &= & 3 \end{aligned}$$

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\operatorname{pdim}(S/(w,y^3,xy^2)) &= 3
\end{aligned}$$

...nor is it monotone w/r/t the number of minimal generators.

$$\begin{aligned}
\operatorname{pdim}(S/(v^2xyz,vw^2yz,vwx^2z,vwxy^2,wxyz^2)) &= 5 \\
\operatorname{pdim}(S/(v^2xyz,vw^2yz,vwx^2z,vwxy^2,wxyz^2,vwxyz)) &= 2
\end{aligned}$$

Alesandroni, 2017

Let M be a monomial ideal of $S = k[x_1, \ldots, x_n]$ with minimal generating set G. Then $\mathrm{pdim}(S/M) = n$ if and only if there exists a subset $L \subseteq G$ satisfying:

1. $\#L = n$
2. For each $m \in L$, there is a variable x_i such that the x_i exponent of m is strictly larger than the x_i exponent of any other $l \in L$. (L is a **dominant** set.)
3. No $g \in G$ **strongly divides** $\mathrm{lcm}(L)$. (Strongly divides: each nonzero exponent is *strictly* smaller.)

Ex. $S = k[x, y, z]$, $M = (x^9y^2z^4, x^7y^6z^2, x^6y^6z^3, x^6y^4z^5, x^5y^7z^3)$.

The subset of generators $L = \{x^9y^2z^4, x^7y^6z^2, x^6y^4z^5\}$ satisfies the theorem, so $\mathrm{pdim}(S/M) = 3$.

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