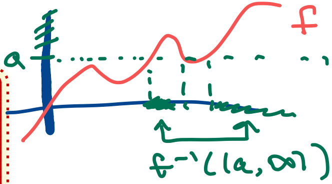
Measurable functions



Definition. Let $(X, \mathcal{M})$ be a measurable space. A function $f : X \to \mathbb{R}$ is $\mathcal{M}$-measurable if

$$f^{-1}((a, \infty)) = \{x : f(x) > a\} \in \mathcal{M} \text{ for all } a \in \mathbb{R}.$$

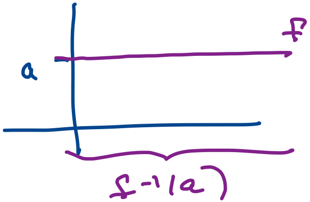
Note: if f is cntns then $f^{-1}((a,\infty))$ is open.

We could have replaced the $>$ in the definition with $<, \geq$, or $\leq$.

Proposition. If $(X, \mathcal{M})$ is a measurable space and $f : X \to \mathbb{R}$, then the following are equivalent: T.F.A.E.

1. $f^{-1}((a, \infty)) = \{x : f(x) > a\} \in \mathcal{M}\ \forall a \in \mathbb{R}$.
2. $f^{-1}([a, \infty)) = \{x : f(x) \geq a\} \in \mathcal{M}\ \forall a \in \mathbb{R}$. write $f^{-1}([a,\infty))$ as...
3. $f^{-1}((-\infty, a)) = \{x : f(x) < a\} \in \mathcal{M}\ \forall a \in \mathbb{R}$.
4. $f^{-1}((-\infty, a]) = \{x : f(x) \leq a\} \in \mathcal{M}\ \forall a \in \mathbb{R}$.

$1 \Rightarrow 2$



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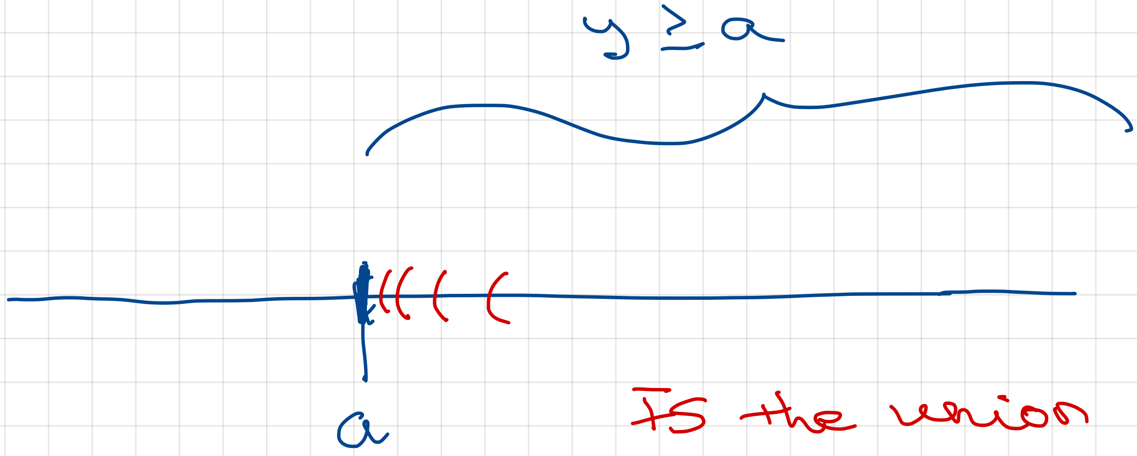
$1 \Rightarrow 2$ $f^{-1}([a,\infty))$

$= \{x : f(x) \geq a\}$

$= \bigcap_{n=1}^{\infty} \{x : f(x) > a - \frac{1}{n}\}$

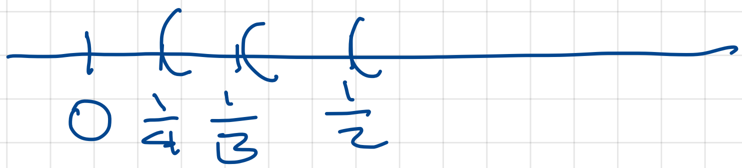
Since $\mathcal{M}$ is closed under intersections, and $\{x : f(x) > a - \frac{1}{n}\} \in \mathcal{M}$ (by 1), $\{x : f(x) \geq a\} \in \mathcal{M}$

Ex: f is "Borel Measurable" if $\mathcal{M} = \{\text{Borel } \sigma\text{-algebra}\}$ = smallest σ-alg. containing open sets



Is the union of open intervals ever closed?

<u>Ex</u>:



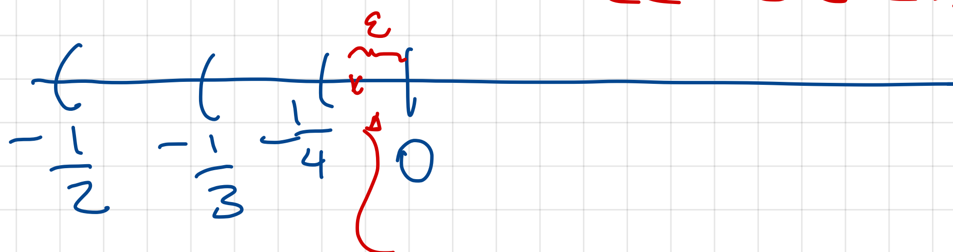
$$\bigcup_{n=1}^{\infty} \left(\frac{1}{n}, \infty\right) = (0, \infty)$$

Does this contain 0? No.

Since $0 \notin (\frac{1}{n}, \infty)$ for any n.

$$\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \infty\right) = [0, \infty)$$

This intersection contains 0 since $0 \in (-\frac{1}{n}, \infty)\ \forall n$.



$\exists\, n$ s.t. $\frac{1}{n} < \varepsilon$

Measurable functions

For example, **1** $\iff$ **4** because

$$\{x : f(x) \leq a\} = X \setminus \{x : f(x) > a\}.$$

A few of the other parts of this proposition are on your homework. The following identities may be helpful.

$$\{x : f(x) \geq a\} = \bigcap_{n=1}^{\infty} \left\{x : f(x) > a - \frac{1}{n}\right\}$$

$$\{x : f(x) > a\} = \bigcup_{n=1}^{\infty} \left\{x : f(x) \geq a + \frac{1}{n}\right\}$$

Examples of measurable functions

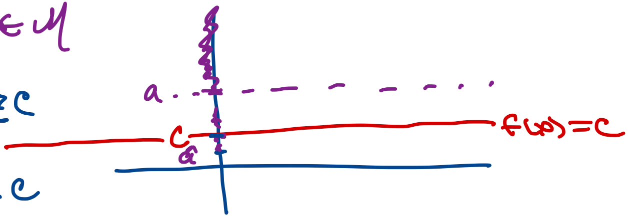
1. Let $(X, \mathcal{M})$ be any measurable space. Then every constant function $f : X \to \mathbb{R}$ is $\mathcal{M}$-measurable.

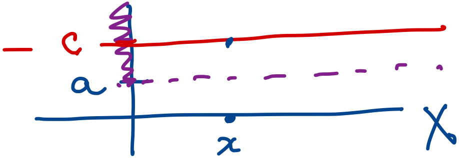
Let $f(x) = c$, all $x \in X$, c some real number.

W.T.S. $\forall a \in \mathbb{R}$ $f^{-1}((a,\infty)) \in \mathcal{M}$

$$f^{-1}((a,\infty)) = \begin{cases} \emptyset \in \mathcal{M}, & a \geq c \\ X \in \mathcal{M} & a < c \end{cases}$$

$= \{x : f(x) > a\}$





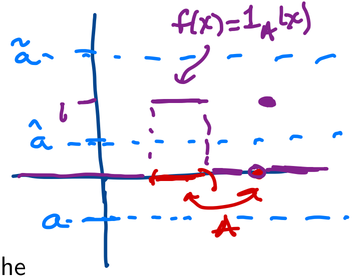
$f(x) = c > a \Rightarrow$,

Examples of measurable functions

1. Let $(X, \mathcal{M})$ be any measurable space. Then every constant function $f : X \to \mathbb{R}$ is $\mathcal{M}$-measurable.
2. Let X be a metric space. Then every continuous function $f : X \to \mathbb{R}$ is $\mathcal{B}(X)$-measurable (where $\mathcal{B}(X)$ is the Borel σ-algebra on X).

If f is continuous, $f^{-1}((a,\infty))$ is open.
Any open set is a countable union of open intervals
$\Rightarrow$ $f^{-1}((a,\infty))$ is a countable union of open intervals
$\Rightarrow$ $f^{-1}((a,\infty)) \in \mathcal{B}(X)$.

Examples of measurable functions



1. Let $(X, \mathcal{M})$ be any measurable space. Then every constant function $f : X \to \mathbb{R}$ is $\mathcal{M}$-measurable.
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3. Let $(X, \mathcal{M})$ be any measurable space, and let $A \in \mathcal{M}$. Then the characteristic function (or indicator function) $\mathbf{1}_A : X \to \mathbb{R}$ is $\mathcal{M}$-measurable.

Note $1_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$

Let $f(x) = 1_A(x)$ some $A \in \mathcal{M}$.

$$f^{-1}(a,\infty) = \begin{cases} X & a<0 \quad \leftarrow \in \mathcal{M} \\ A & 0 \le a < 1 \quad \leftarrow \in \mathcal{M} \text{ (by hypothesis)} \\ \emptyset & a \geq 1 \quad \leftarrow \in \mathcal{M} \end{cases}$$

Measurable functions

Proposition. If $f, g : (X, \mathcal{M}) \to \mathbb{R}$ are measurable functions, and $c \in \mathbb{R}$, then the following are also $\mathcal{M}$-measurable:

$$-f,\ f + c,\ cf,\ f + g,\ g - f,\ \text{ and }\ fg.$$

For example, let's show that $f + c$ is measurable. Since f is measurable, then for any $a \in \mathbb{R}$,

$$\{x : f(x) > a\} \in \mathcal{M}.$$

work through these ....

Therefore

$$\{x : f(x) + c > a\} = \{x : f(x) > a - c\} \in \mathcal{M}$$

for all $a \in \mathbb{R}$, so $f + c$ is measurable.

Pre-images of Borel sets

Proposition. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $f : X \to \mathbb{R}$ be μ-measurable.[a] If $A \in \mathcal{B}(\mathbb{R})$, then $f^{-1}(A) \in \mathcal{M}$.

[a] Note: this is the same as saying that f is $\mathcal{M}$-measurable

The strategy of this proof may be a bit unexpected. Start by defining $\mathcal{C} = \{B \in \mathcal{B}(\mathbb{R}) : f^{-1}(B) \in \mathcal{M}\}$. By definition $\mathcal{C} \subseteq \mathcal{B}(\mathbb{R})$. To show the reverse inclusion, show that $\mathcal{C}$ is a sigma algebra containing all the open intervals (a, ∞).

1. show $(a,\infty) \in \mathcal{C}$ for all a. True, since $f^{-1}(a,\infty) \in$
2. If $C_1, C_2 \in \mathcal{C} \Rightarrow C_1 \cup C_2 \in \mathcal{C}$
3. If $C_1, C_2 \in \mathcal{C} \Rightarrow C_1 \cap C_2 \in \mathcal{C}$.

To show 2 we know $f^{-1}(C_1) \in \mathcal{M}$, $f^{-1}(C_2) \in \mathcal{M}$

$f^{-1}(C_1 \cup C_2) \stackrel{?}{=} f^{-1}(C_1) \cup f^{-1}(C_2)$

$$f^{-1}(C_1 \cup C_2) \subset f^{-1}(C_1) \cup f^{-1}(C_2)$$

Let $x \in f^{-1}(C_1 \cup C_2) \implies f(x) \in (C_1 \cup C_2)$

$\implies$ either $f(x) \in C_1$

or $f(x) \in C_2$

$\implies x \in f^{-1}(C_1)$ OR $x \in f^{-1}(C_2)$

$\implies x \in f^{-1}(C_1) \cup f^{-1}(C_2)$

μ-almost everywhere

Definition. Let $f, g : (X, \mathcal{M}, \mu) \to \mathbb{R}$. We say $f = g$ **almost everywhere** (or μ-almost everywhere) if

$$\mu(\{x : f(x) \neq g(x)\}) = 0.$$

That is, $f(x) = g(x)$ for all $x \in X$ except perhaps on a set of measure zero.

$f = g$ a.e. μ

For example, the Dirichlet function equals the constant zero function m-almost everywhere.

Simple functions

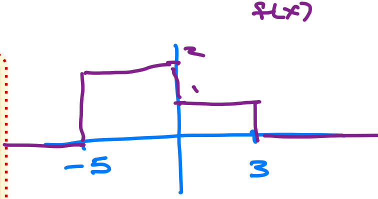
Definition. Let $(X, \mathcal{M})$ be a measurable space. A **simple function** is a finite linear combination, with non-negative coefficients, of characteristic functions on sets in $\mathcal{M}$. That is,

$$f(x) = \sum_{j=1}^{n} a_j \chi_{E_j}(x),$$

where $E_j \in \mathcal{M}$ for all j, and $a_j \in \mathbb{R}$.

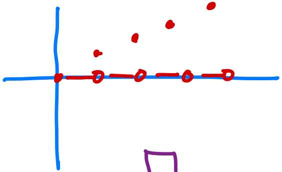
Example. $f = 2\chi_{(-5,0)} + 1\chi_{[0,3)}$ is a simple function on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

Definition. (For $X = \mathbb{R}$) A **step function** is a simple function where each E_i is an interval.



$f(x) = \sum j \mathbb{1}_{E_j}$

$E_j = \{j\}$

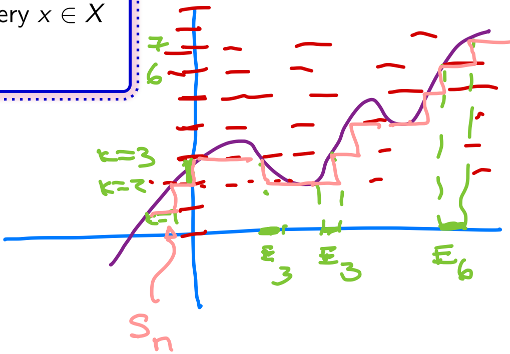


Simple functions

Theorem. Let $f : X \to \mathbb{R}$ be non-negative and measurable. Then there exists a sequence S_n of simple functions such that:

1. $S_n(x) \leq S_{n+1}(x) \leq f(x)$ for all $n \geq 1$, and almost every $x \in X$
2. $\lim_{n\to\infty} S_n(x) = f(x)$ for almost every $x \in X$

divide vert. axis into pieces of size $\frac{1}{2^n}$



Outline of proof. Let $E_{n,k} = \{x : \frac{k-1}{2^n} \leq f(x) < \frac{k}{2^n}\}$, $F_n = \{x : f(x) \geq n\}$ for $n \geq 1$, $k = 1, 2, \ldots, n2^n$.

Note $E_{n,k}$, F_n all measurable sets.

$$S_n = \sum_{k=1}^{n2^n} \frac{k-1}{2^n} \chi_{E_{n,k}} + n\chi_{F_n} = \begin{cases} n & \text{if } f(x) \geq n \\ \frac{k-1}{2^n} & \text{if } \frac{k-1}{2^n} \leq f(x) < \frac{k}{2^n} \leq n \end{cases}$$

$|S_n - f(x)| < \frac{1}{2^n}$ for any x

So $S_n \to f$ as desired.

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