Example.
For any set X, $\mathcal{A} = \{E \subseteq X : E \text{ is countable or } X \backslash E \text{ is countable}\}$ is a σ-algebra.

1. $\emptyset \in \mathcal{A}$ because $\emptyset$ is countable.
2. Let $E \in \mathcal{A}$. Then either E is countable or $X \backslash E$ is countable. In the first case, $X \backslash E$ is the complement of a countable set, so $X \backslash E \in \mathcal{A}$. In the second case, $X \backslash E \in \mathcal{A}$ since it's countable. So $\mathcal{A}$ is closed under complements.

Example.
For any set X, $\mathcal{A} = \{E \subseteq X : E \text{ is countable or } X \backslash E \text{ is countable}\}$ is a σ-algebra.

3. Let $\{E_j\}_1^\infty \subseteq \mathcal{A}$. We need to show that either $\cup_1^\infty E_j$ is countable or that $X \backslash (\cup_1^\infty E_j)$ is countable.
First suppose that E_j is countable for all j. Then $\cup_1^\infty E_j$ is countable, since the countable union of countable sets is countable.
On the other hand, suppose at least one of the E_j, WLOG let's say E_1, is uncountable. Since $E_1 \in \mathcal{A}$ it must be that $X \backslash E_1$ is countable. Then

$$X \backslash \bigcup_{j=1}^{\infty} E_j = \bigcap_{j=1}^{\infty} (X \backslash E_j) \subseteq X \backslash E_1,$$

so $X \backslash (\cup_1^\infty E_j)$ is contained in a countable set which means it's countable and in $\mathcal{A}$. Thus $\mathcal{A}$ is closed under countable unions. $\square$

Definition. Let X be a set and $\mathcal{A}$ a set of subsets of X. Then the intersection of all σ-algebras on X that contain $\mathcal{A}$ is a σ-algebra on X. We call this the σ-**algebra generated by** $\mathcal{A}$ and write $\sigma(\mathcal{A})$.

Example. Suppose X is a set and S is the set of subsets of X that consist of exactly one element:

$$S = \{\{x\} : x \in X\}.$$

Then $\sigma(S) = \{E \subseteq X : E \text{ is countable or } X \backslash E \text{ is countable}\}$.

Borel sets

Definition. The *Borel σ-algebra* on $\mathbb{R}$, written $\mathcal{B}(\mathbb{R})$ (or $\mathcal{B}_{\mathbb{R}}$ or sometimes just $\mathcal{B}$) is the σ-algebra generated by the open sets of $\mathbb{R}$. Elements of $\mathcal{B}(\mathbb{R})$ are called *Borel sets*.

$\mathcal{B}$ is generated by each of the following collections:

1. $C_1 = \{(a, b) : a, b \in \mathbb{R}\}$
2. $C_2 = \{[a, b] : a, b \in \mathbb{R}\}$
3. $C_3 = \{(a, b] : a, b \in \mathbb{R}\}$
4. $C_4 = \{(a, \infty) : a \in \mathbb{R}\}$
5. $C_5 = \{(-\infty, a] : a \in \mathbb{R}\}$

(work in groups)

Measures

Definition. Let $(X, \mathcal{M})$ be a measurable space. A **measure** on $(X, \mathcal{M})$ is a function $\mu : \mathcal{M} \to [0, \infty]$ such that

1. $\mu(\emptyset) = 0$
2. (countable additivity) If $\{E_j\}$ is a collection of pairwise disjoint sets in $\mathcal{M}$, then $\mu(\bigcup_{j=1}^{\infty} E_j) = \sum_{j=1}^{\infty} \mu(E_j)$.

Definition

If μ is a measure on $(X, \mathcal{M})$, then $(X, \mathcal{M}, \mu)$ is called a **measure space**.

Measures

Examples and non-examples of measures (work in groups)

For the following let $(X, \mathcal{M}) = (\mathbb{R}, \mathcal{P}(\mathbb{R}))$.

1. $\alpha : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ defined by $\alpha(E) = \begin{cases} 0 & \text{if } E \text{ is countable,} \\ \infty & \text{if } E \text{ is uncountable.} \end{cases}$

2. $\beta : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ defined by $\beta(E) = \begin{cases} 0 & \text{if } E \text{ is finite,} \\ \infty & \text{if } E \text{ is infinite.} \end{cases}$

3. $\gamma : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ defined by $\gamma(E) = $ cardinality of E.

4. Fix $x_0 \in \mathbb{R}$, and define $\delta_{x_0} : \mathcal{P}(\mathbb{R}) \to [0, \infty]$ by
$$\delta_{x_0}(E) = \begin{cases} 1 & x_0 \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Outer measure on $\mathbb{R}$

Definition. The length of an open interval $I \subseteq \mathbb{R}$ is defined by

$$\ell(I) = \begin{cases} 0 & I = \emptyset \\ b - a & I = (a, b) \text{ with } a < b \\ \infty & I = (-\infty, a) \text{ or } (a, \infty) \text{ or } (-\infty, \infty) \end{cases}$$

Definition. Define the **Lebesgue outer measure** of $E \subseteq \mathbb{R}$, written $|E|$ (Axler) or $m^*(E)$ (Bass) by

$$\inf \left\{ \sum_{i=1}^{\infty} \ell(A_i) : \text{ each } A_i \text{ is an open interval of } \mathbb{R} \text{ and } E \subseteq \bigcup_{i=1}^{\infty} A_i \right\}.$$

Outer measure is **not** a measure on $(\mathbb{R}, \mathcal{P}(\mathbb{R}))$. We will soon see that it **is** a measure on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$, and in fact there is a somewhat larger σ-algebra $\mathcal{L}$ such that outer measure is a measure on $(\mathbb{R}, \mathcal{L})$.

Special kinds of measures

Definition.

- μ is a *finite* measure if $\mu(X) < \infty$
- μ is a *probability measure* if $\mu(X) = 1$
- μ is called σ-*finite* if $X = \bigcup_{n=1}^{\infty} X_n$ where $\mu(X_n) < \infty$ for all n.

Properties of measures

Theorem. Let $(X, \mathcal{M}, \mu)$ be a measure space. Then for all $E, F \in \mathcal{M}$ and for all $\{E_j\}_1^\infty \subseteq \mathcal{M}$:

1. (monotonicity) If $E \subseteq F$ then $\mu(E) \leq \mu(F)$.

2. (countable subadditivity) $\mu(\bigcup_{j=1}^\infty E_j) \leq \sum_{j=1}^\infty \mu(E_j)$.

3. (continuity from below) If $E_j \subseteq E_{j+1}$ for all j, then
$$\mu\left(\bigcup_{j=1}^\infty E_j\right) = \lim_{j\to\infty} \mu(E_j).$$

4. (continuity from above) If $\mu(E_1) < \infty$ and $E_j \supseteq E_{j+1}$ for all j, then $\mu\left(\bigcap_{j=1}^\infty E_j\right) = \lim_{j\to\infty} \mu(E_j)$.