What is measure theory and why should I care?

Biggest motivation: the **Lebesgue integral**.

Problem #1 with Riemann integration

It can't handle unbounded functions.

Example

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx$$

Technically, this isn't Riemann integrable. What we really mean when we write this is $\lim_{t\to 0+} \int_t^1 \frac{1}{\sqrt{x}}\,dx$.

However, Lebesgue integration can handle the original form of the integral.

Problem #2 with Riemann integration

It can't handle many discontinuities.

Example

Consider the **Dirichlet function**

$$\mathbf{1}_{\mathbb{Q}}(x) := \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

and the integral

$$\int_0^1 \mathbf{1}_{\mathbb{Q}}(x)\, dx$$

Since this function is zero everywhere except on a countable set, the integral "should be" zero. But it's not Riemann integrable.

Problem #3 with Riemann integration

It doesn't work well with limits.

Example

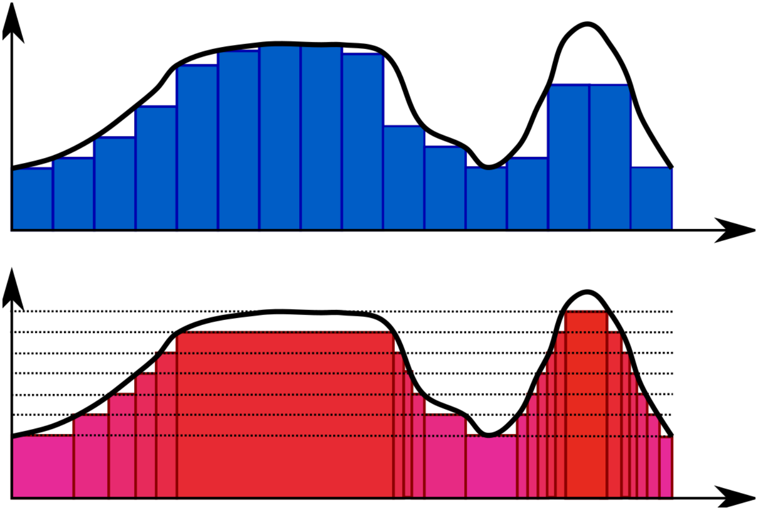
Let $q_1, q_2, \ldots$ be an enumeration of the rational numbers in $[0, 1]$, and let

$$f_k = \mathbf{1}_{\{q_1, q_2, \ldots, q_k\}}$$

Then $\lim_{k\to\infty} \int_0^1 f_k(x)\,dx = 0$, but $\int_0^1 \lim_{k\to\infty} f_k(x)\,dx$ doesn't exist.

This example shows that the space of Riemann integrable functions on $[0, 1]$ is **not complete**.

Intuition for how the Lebesgue integral works



(source: wikimedia)

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Problems with the Riemann integral:

- It can't handle unbounded functions.
- It can't handle many discontinuities.
- It doesn't work well with limits / lack of completeness.

These problems come from the way we rely on partitioning the domain into intervals.

For Lebesgue integration, we need to define the "size" or **measure** of sets more complicated than intervals.

"Wish list" for measuring sets in $\mathbb{R}$

The bad news...

2.22 ***nonexistence of extension of length to all subsets of* $\mathbf{R}$**

There does not exist a function μ with all the following properties:

(a) μ is a function from the set of subsets of $\mathbf{R}$ to $[0, \infty]$.

(b) $\mu(I) = \ell(I)$ for every open interval I of $\mathbf{R}$.

(c) $\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k)$ for every disjoint sequence $A_1, A_2, \ldots$ of subsets of $\mathbf{R}$.

(d) $\mu(t + A) = \mu(A)$ for every $A \subset \mathbf{R}$ and every $t \in \mathbf{R}$.

(source: Axler)

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These problems come from reliance on partitioning the domain into intervals.

For Lebesgue integration, we need to define the "size" or **measure** of sets more complicated than intervals... but we can't measure every set in $\mathbb{R}$.

To describe which sets are measurable, and which functions $\mathbb{R} \to \mathbb{R}$ are Lebesgue integrable, we start with studying σ-**algebras**, which are certain well-behaved collections of subsets.

σ-algebras

2.23 Definition ***σ-algebra***

Suppose X is a set and $\mathcal{S}$ is a set of subsets of X. Then $\mathcal{S}$ is called a *σ-algebra* on X if the following three conditions are satisfied:

- $\varnothing \in \mathcal{S}$;
- if $E \in \mathcal{S}$, then $X \setminus E \in \mathcal{S}$;
- if $E_1, E_2, \ldots$ is a sequence of elements of $\mathcal{S}$, then $\displaystyle\bigcup_{k=1}^{\infty} E_k \in \mathcal{S}$.

σ-algebras

Examples.

- Let $X = \{a, b, c\}$, $S = \{\emptyset, \{a\}, \{b, c\}, X\}$. Then S is a σ-algebra on X.

- Let $X = \{a, b, c\}$, $S = \{\emptyset, \{a\}, \{a, b\}, \{b, c\}, X\}$. Then S is not a σ-algebra on X.

- Let X be any set. Then $\mathcal{P}(X)$ is a σ-algebra.

- Let X be any set. Then $\{X, \emptyset\}$ is a σ-algebra.

- Let X be any set. Then $\mathcal{A} := \{E \subseteq X : E$ is countable or $X \backslash E$ is countable$\}$ is a σ-algebra.

σ-algebras

2.27 *smallest σ-algebra containing a collection of subsets*

Suppose X is a set and $\mathcal{A}$ is a set of subsets of X. Then the intersection of all σ-algebras on X that contain $\mathcal{A}$ is a σ-algebra on X.

Examples.

- Let $X = [0, 1]$ and $\mathcal{A} = \{[0, \frac{1}{4}],\ [\frac{1}{2}, 1]\}$. Find $\sigma(\mathcal{A})$.

σ-algebras

The word *measurable* is used in the terminology below because in the next section we introduce a size function, called a measure, defined on measurable sets.

2.26 Definition ***measurable space; measurable set***

- A *measurable space* is an ordered pair $(X, \mathcal{S})$, where X is a set and $\mathcal{S}$ is a σ-algebra on X.
- An element of $\mathcal{S}$ is called an $\mathcal{S}$*-measurable set*, or just a *measurable set* if $\mathcal{S}$ is clear from the context.

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The usefulness of Lebesgue integration goes far beyond addressing the problems with Riemann integration on $\mathbb{R}$ and $\mathbb{R}^n$. Because the Lebesgue integral only uses measurable subsets of the domain (not intervals), it lets us integrate functions on all kinds of weird/interesting/important spaces (where there is no such thing as an interval).