Simple functions

Definition. Let $(X, \mathcal{M})$ be a measurable space. A **simple function** is a finite linear combination, with non-negative coefficients, of characteristic functions on sets in $\mathcal{M}$. That is,

$$f(x) = \sum_{j=1}^{n} a_j \chi_{E_j}(x),$$

where $E_j \in \mathcal{M}$ for all j, and $a_j \in \mathbb{R}$.

Example. $f = 2\chi_{(-5,0)} + 1\chi_{[0,3)}$ is a simple function on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

Definition. (For $X = \mathbb{R}$) A **step function** is a simple function where each E_i is an interval.

Simple functions

Theorem. Let $f : X \to \mathbb{R}$ be non-negative and measurable. Then there exists a sequence S_n of simple functions such that:

1. $S_n(x) \leq S_{n+1}(x) \leq f(x)$ for all $n \geq 1$, and almost every $x \in X$
2. $\lim_{n\to\infty} S_n(x) = f(x)$ for almost every $x \in X$

Outline of proof. Let $E_{n,k} = \{x : \frac{k-1}{2^n} \leq f(x) < \frac{k}{2^n}\}$, $F_n = \{x : f(x) \geq n\}$ for $n \geq 1$, $k = 1, 2, \ldots, n2^n$.

Note $E_{n,k}$, F_n all measurable sets.

$$S_n = \sum_{k=1}^{n2^n} \frac{k-1}{2^n} \chi_{E_{n,k}} + n\chi_{F_n} = \begin{cases} n & \text{if } f(x) \geq n \\ \frac{k-1}{2^n} & \text{if } \frac{k-1}{2^n} \leq f(x) < \frac{k}{2^n} \leq n \end{cases}$$

So $S_n \to f$ as desired.

Simple functions

Corollary. Let $f : X \to \mathbb{R}$ be measurable. (difference: not assuming non-negative.) Then there exists a sequence S_n of simple functions so

1. $|S_n(x)| \leq |S_{n+1}(x)| \leq |f(x)|$ for all n and a.e. x
2. $\lim_{n\to\infty} S_n(x) = f(x)$ for a.e. $x \in X$.

Outline of proof.

- Define $f^+(x) = \max\{f(x), 0\}$, and $f^-(x) = \max\{-f(x), 0\}$.
- Claim: $f = f^+ - f^-$ and $|f| = f^+ + f^-$.
- Claim: both functions are measurable.
- (draw picture)

Corollary. If $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable, then there exists a sequence f_n of *step* functions so that $f_n \to f$ a.e.

Integral of simple function

Going back to the simple function $f = 2\chi_{(-5,0)} + 1\chi_{[0,3)}$ on $\mathbb{R}$.

$$f(x) = \begin{cases} 2, & \text{if } -5 < x < 0, \\ 1, & \text{if } 0 \leq x < 3, \\ 0, & \text{otherwise} \end{cases}$$

What should the integral of f be?

Integrating simple functions

Definition. Fix a measure space $(X, \mathcal{M}, \mu)$. If $S = \sum_{j=1}^{n} a_j \chi_{E_j}$, we define the integral of S with respect to μ by

$$\int S \, d\mu = \int_X S \, d\mu = \sum_{j=1}^{n} a_j \mu(E_j).$$

Important observations.

- We allow $\int S \, d\mu = \infty$.
- We must use the convention that $0 \cdot \infty = 0$.
- Wait: is this even well-defined? Writing a simple function as a sum of characteristic functions is not unique. However, the integral has a unique value no matter how the simple function is represented.
- To simplify things, we can assume the a_j's are distinct and the $\{E_j\}$ are disjoint, with $\cup E_j = X$ (why?). This is called the **standard representation** of a simple function.

Integrating simple functions

Proposition. Let f, g be simple functions. Then

a. If $c \geq 0$, $\int cf\, d\mu = c \int f\, d\mu$.

b. $\int (f+g)\, d\mu = \int f\, d\mu + \int g\, d\mu$.

c. If $f \leq g$, then $\int f\, d\mu \leq \int g\, d\mu$. ($\leftarrow$ homework!)

Integrating simple functions

Proposition. Let f, g be simple functions. Then

a. If $c \geq 0$, $\int cf\, d\mu = c \int f\, d\mu$.

b. $\int (f+g)\, d\mu = \int f\, d\mu + \int g\, d\mu$.

c. If $f \leq g$, then $\int f\, d\mu \leq \int g\, d\mu$. ($\leftarrow$ homework!)

Proof of a. Write $f = \sum_{j=1}^{n} a_j \chi_{E_j}$.

Then $cf = c \sum_{j=1}^{n} a_j \chi_{E_j} = \sum_{j=1}^{n} c a_j \chi_{E_j}$

So $\int cf\, d\mu = \sum_{j=1}^{n} c a_j \mu(E_j) = c \sum_{j=1}^{n} a_j \mu(E_j) = c \int f\, d\mu$.

Integrating simple functions

Proposition. Let f, g be simple functions. Then

a. If $c \geq 0$, $\int cf\, d\mu = c \int f\, d\mu$.

b. $\int (f+g)\, d\mu = \int f\, d\mu + \int g\, d\mu$.

c. If $f \leq g$, then $\int f\, d\mu \leq \int g\, d\mu$. ($\leftarrow$ homework!)

Proof of b. Write $f = \sum_{j=1}^{n} a_j \chi_{E_j}$, $g = \sum_{k=1}^{m} b_k \chi_{F_k}$, and assume these are standard representations. Then,

$$E_j = E_j \cap X = E_j \cap (\cup_1^m F_k) = \cup_{k=1}^m (E_j \cap F_k),$$

$$F_k = F_k \cap X = F_k \cap (\cup_1^m E_j) = \cup_{j=1}^n (F_k \cap E_j),$$

and these unions are disjoint.

So $f + g = \sum_{j=1}^{n} \sum_{k=1}^{m} (a_j + b_k) \chi_{E_j \cap F_k}$.

Then, by definition,

$$
\begin{aligned}
\int (f+g)\ d\mu &= \sum_{j=1}^{n}\sum_{k=1}^{m}(a_j+b_k)\mu(E_j\cap F_k) \\
&= \sum_{j=1}^{n}\sum_{k=1}^{m} a_j\mu(E_j\cap F_k) + \sum_{k=1}^{m}\sum_{j=1}^{n} b_k\mu(E_j\cap F_k) \\
&= \sum_{j=1}^{n} a_j\mu(E_j) + \sum_{j=1}^{n} b_k\mu(F_k) \\
&= \int f\, d\mu + \int g\, d\mu.
\end{aligned}
$$

The Lebesgue integral

Definition. For $f \geq 0$ measurable, define

$$\int_X f \, d\mu = \sup \left\{ \int_X s \, d\mu : 0 \leq s \leq f, s \text{ simple} \right\}.$$

What if f is measurable, but not necessarily nonnegative?

Definition. Suppose f is measurable. As we've seen before, let $f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$. Then

$$\int_X f \, d\mu = \int_X f^+ \, d\mu - \int_X f^- \, d\mu,$$

...as long as $\int_X f^+ \, d\mu$ and $\int_X f^- \, d\mu$ are *not both infinite.*

The Lebesgue integral

Definition.
If f is measurable and $\int |f| \, d\mu < \infty$, we say f is **integrable**.

Basic properties of the Lebesgue integral

Bass: "The proof of the next proposition follows from the definitions."
(This is Prop 6.3.)

Proposition.

- **a.** If f is a real-valued measurable function with $a \leq f(x) \leq b$ for all x and $\mu(X) < \infty$, then $a\mu(X) \leq \int f\,d\mu \leq b\mu(X)$;
- **b.** If f and g are measurable, real-valued, and integrable and $f(x) \leq g(x)$ for all x, then $\int f\,d\mu \leq \int g\,d\mu$;
- **c.** If f is integrable, then $\int cf\,d\mu = c\int f\,d\mu$ for all complex c;
- **d.** If $\mu(A) = 0$ and f is integrable, then $\int f\chi_A\,d\mu = 0$.

Let's prove part **d.**.

If $\mu(A) = 0$ and f is integrable, then $\int f \chi_A \, d\mu = 0$.

I.e., the integral of any function over a nullset is zero.

Proof. First suppose f is a simple function, $f = \sum_{j=1}^{n} a_j \chi_{E_j}$. Then

$$\int_X f \, d\mu = \sum_{j=1}^{n} a_j \mu(E_j).$$

So for a nullset E, we have

$$\int_E f \, d\mu = \sum_{j=1}^{n} a_j \mu(E_j \cap E) \leq \sum_{j=1}^{n} a_j \mu(E) = 0.$$

If $\mu(A) = 0$ and f is integrable, then $\int f \chi_A \, d\mu = 0$.

I.e., the integral of any function over a nullset is zero.

Next, suppose $f \geq 0$ is measurable, so its integral is by definition

$$\int_X f \, d\mu = \sup \left\{ \int_X s \, d\mu : 0 \leq s \leq f, s \text{ simple} \right\}.$$

Then for the nullset E, for any simple function s we already proved $\int_E s \, d\mu = 0$. Thus

$$\int_E f \, d\mu = \sup \left\{ \int_E s \, d\mu : 0 \leq s \leq f, s \text{ simple} \right\} = 0.$$

Finally, since every measurable function can be written as $f = f^+ - f^-$ and $f^+, f^- \geq 0$, for any nullset E,

$$\int_E f \, d\mu = \int_E f^+ \, d\mu - \int_E f^- \, d\mu = 0.$$