HW3

Q1. (a) $s = \frac{x - x_0}{h} = \frac{x - 0.12}{0.12}$

$$P(x) = f_0 + s\Delta f_0 + \frac{s(s-1)}{2!}\Delta^2 f_0$$

$$= 0.79168 - 0.01834s - \tfrac{1}{2}(s^2 - s) \times 0.01129$$

$$= 0.79168 - 0.01270s - 0.00565s^2$$

$$= 0.79168 - 0.01270\left(\frac{x-0.12}{0.12}\right) - 0.00565\left(\frac{x-0.12}{0.12}\right)^2$$

$$P(0.231) = 0.79168 - 0.01270\left(\frac{0.231-0.12}{0.12}\right) - 0.00565\left(\frac{0.231-0.12}{0.12}\right)^2$$

$$= 0.79168 - 0.01270 \times 0.925 - 0.00565 \times 0.925^2$$

$$= 0.77510$$

(b) $s = \frac{x - x_0}{h} = \frac{x - 0.12}{0.12}$

$$P(x) = f_0 + s\Delta f_0 + \frac{s(s-1)}{2!}\Delta^2 f_0 + \frac{s(s-1)(s-2)}{3!}\Delta^3 f_0$$

$$= 0.79168 - 0.01834s - \tfrac{1}{2}(s^2 - s) \times 0.01129 + \tfrac{1}{6}(s^3 - 3s^2 + 2s) \times 0.00134$$

$$= 0.79168 - 0.01225s - 0.00632s^2 + 0.00022s^3$$

$$= 0.79168 - 0.01225\left(\frac{x-0.12}{0.12}\right) - 0.00632\left(\frac{x-0.12}{0.12}\right)^2 + 0.00022\left(\frac{x-0.12}{0.12}\right)^3$$

$$P(0.231) = 0.79168 - 0.01225\left(\frac{0.231-0.12}{0.12}\right) - 0.00632\left(\frac{0.231-0.12}{0.12}\right)^2 + 0.00022\left(\frac{0.231-0.12}{0.12}\right)^3$$

$$= 0.79168 - 0.01225 \times 0.925 - 0.00632 \times 0.925^2 + 0.00022 \times 0.925^3$$

$$= 0.77512$$

(c) Error of (a) $= \dfrac{0.00134 \times s \times (s-1) \times (s-2)}{3!} = 0.0000167$

Error of (b) $= \dfrac{0.00038 \times s \times (s-1) \times (s-2) \times (s-3)}{4!} = -0.00000245$

(d) Since we are using x_0 as a base to estimate other values, we should use the closest x_0 to our to-be-estimated value to have the best result. Thus, we should use $x_0 = 0.36$ instead of $x_0 = 0.24$ to estimate $f(0.42)$.

Q2.

$$f(x) = \begin{cases} 0, & -1 < x < -0.5 \\ 1-|2x|, & -0.5 < x < 0.5 \\ 0, & 0.5 < x < 1 \end{cases}$$

$\rightarrow g_0(x)\ (-1, -0.5)\quad h_0 = 0.5$

$\rightarrow g_1(x)\ (-0.5, 0)\quad h_1 = 0.5$

$\rightarrow g_2(x)\ (0, 0.5)\quad h_2 = 0.5$

$\rightarrow g_3(x)\ (0.5, 1)\quad h_3 = 0.5$

Let $S_0 = S_4 = 0$, $x_0 = -1$, $x_1 = -0.5$, $x_2 = 0$, $x_3 = 0.5$, $x_4 = 1$

$$\begin{bmatrix} 2(h_0+h_1) & h_1 & 0 \\ h_1 & 2(h_1+h_2) & h_2 \\ 0 & h_2 & 2(h_2+h_3) \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \\ S_3 \end{bmatrix} = 6 \begin{bmatrix} f[x_1,x_2] - f[x_0,x_1] \\ f[x_2,x_3] - f[x_1,x_2] \\ f[x_3,x_4] - f[x_2,x_3] \end{bmatrix}$$

$$f[x_0,x_1] = \frac{f_1-f_0}{x_1-x_0} = \frac{0-0}{-0.5+1} = 0$$

$$f[x_1,x_2] = \frac{f_2-f_1}{x_2-x_1} = \frac{1-0}{0+0.5} = 2$$

$$f[x_2,x_3] = \frac{f_3-f_2}{x_3-x_2} = \frac{0-1}{0.5-0} = -2$$

$$f[x_3,x_4] = \frac{f_4-f_3}{x_4-x_3} = \frac{0-0}{1-0.5} = 0$$

$$\Rightarrow \begin{bmatrix} 2 & 0.5 & 0 \\ 0.5 & 2 & 0.5 \\ 0 & 0.5 & 2 \end{bmatrix} \begin{bmatrix} S_1 \\ S_2 \\ S_3 \end{bmatrix} = 6 \begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} S_1 \\ S_2 \\ S_3 \end{bmatrix} = \begin{bmatrix} 10.2857 \\ -17.1429 \\ 10.2857 \end{bmatrix}$$

$$a_i = \frac{S_{i+1} - S_i}{6h_i} \qquad b_i = \frac{S_i}{2} \qquad c_i = \frac{y_{i+1} - y_i}{h_i} - \frac{2h_i S_i + h_i S_{i+1}}{6} \qquad d_i = y_i$$

$$a_0 = \frac{10.2857 - 0}{6(0.5)} = 3.4286 \qquad b_0 = \frac{0}{2} = 0$$

$$c_0 = \frac{0-0}{0.5} - \frac{2(0.5)(0) + (0.5)(10.2857)}{6} = -0.8571 \qquad d_0 = 0$$

$$a_1 = \frac{-17.1429 - 10.2857}{6(0.5)} = -9.1429 \qquad b_1 = \frac{10.2857}{2} = 5.1429$$

$$c_1 = \frac{1-0}{0.5} - \frac{2(0.5)(10.2857) + (0.5)(-17.1429)}{6} = 1.7143 \qquad d_1 = 0$$

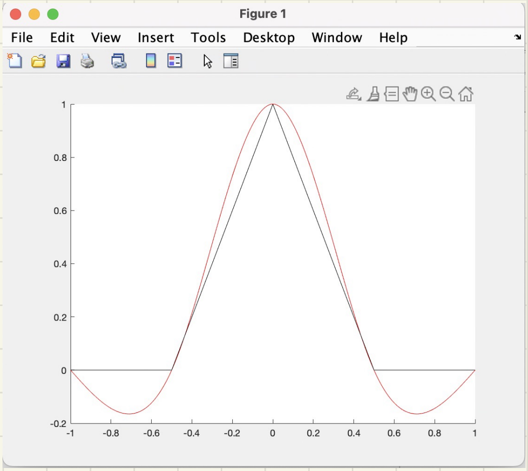
$$a_2 = \frac{10.2857 + 17.1429}{6(0.5)} = 9.1429 \qquad b_2 = \frac{-17.1429}{2} = -8.5715$$

$$c_2 = \frac{0-1}{0.5} - \frac{2(0.5)(-17.1429) + (0.5)(10.2857)}{6} = 0 \qquad d_2 = 1$$

$$a_3 = \frac{0 - 10.2857}{6(0.5)} = -3.4286 \qquad b_3 = \frac{10.2857}{2} = 5.1429$$

$$c_3 = \frac{0-0}{0.5} - \frac{2(0.5)(10.2857) + (0.5)(0)}{6} = -1.7143 \qquad d_3 = 0$$

$$g_0(x) = 3.4286(x+1)^3 - 0.8571(x+1)$$
$$g_1(x) = -9.1429(x+0.5)^3 + 5.1429(x+0.5)^2 + 1.7143(x+0.5)$$
$$g_2(x) = 9.1429x^3 - 8.5714x^2 + 1$$
$$g_3(x) = -3.4286(x-0.5)^3 + 5.1429(x-0.5)^2 - 1.7143(x-0.5)$$

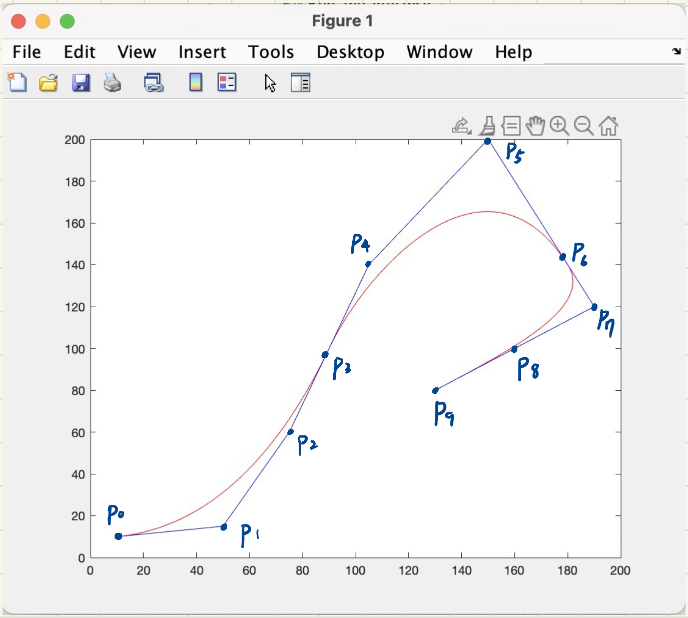


Q3. (a) $P_1(u)$: $p_0=(10,10)$ $p_1=(50,15)$ $p_2=(75,60)$ $p_3=(90,100)$

$P_2(u)$: $p_3=(90,100)$ $p_4=(105,140)$ $p_5=(150,200)$ $p_6=(180,140)$

$P_3(u)$: $p_6=(180,140)$ $p_7=(190,120)$ $p_8=(160,100)$ $p_9=(130,80)$

$$P(u)=p_0(1-u)^3+3p_1u(1-u)^2+3p_2u^2(1-u)+p_3u^3$$
$$P_1(u)=[10,10](1-u)^3+3[50,15]u(1-u)^2+3[75,60]u^2(1-u)+[90,100]u^3$$
$$P_2(u)=[90,100](1-u)^3+3[105,140]u(1-u)^2+3[150,200]u^2(1-u)+[180,140]u^3$$
$$P_3(u)=[180,140](1-u)^3+3[190,120]u(1-u)^2+3[160,100]u^2(1-u)+[130,80]u^3$$



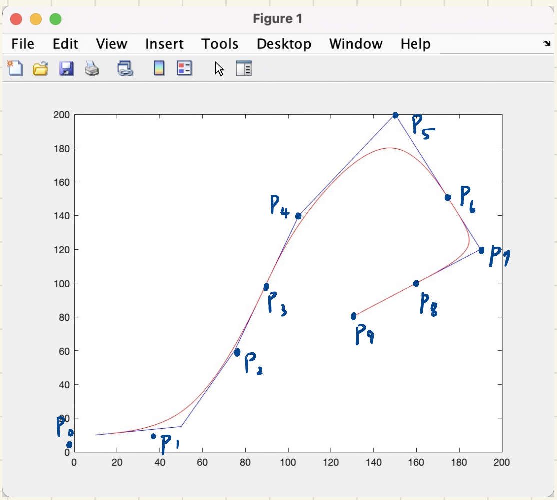
(b) The graph is smoothly connected at point 3 and point 6 because the slope between p_2p_3 = slope between p_3p_4 and the slope between p_5p_6 = slope between p_6p_7. This means that the points $[p_2\ p_3\ p_4]$ and $[p_5\ p_6\ p_7]$ are colinear.

(c)
$$f(u)=\begin{cases} p_0(1-u)^3+p_1(1-u)^2u+p_2(1-u)u^2+p_3u^3 & u\in[0,1] \\ p_3(2-u)^3+p_4(2-u)^2(u-1)+p_5(2-u)(u-1)^2+p_6(u-1)^3 & u\in[1,2] \\ p_6(3-u)^3+p_7(3-u)^2(u-2)+p_8(3-u)(u-2)^2+p_9(u-2)^3 & u\in[2,3] \end{cases}$$

Q4. (a)

$P_{-1}(u)$:	$p_{-2}=(10,10)$	$p_{-1}=(10,10)$	$p_0=(10,10)$	$p_1=(50,15)$
$P_0(u)$:	$p_{-1}=(10,10)$	$p_0=(10,10)$	$p_1=(50,15)$	$p_2=(75,60)$
$P_1(u)$:	$p_0=(10,10)$	$p_1=(50,15)$	$p_2=(75,60)$	$p_3=(90,100)$
$P_2(u)$:	$p_1=(50,15)$	$p_2=(75,60)$	$p_3=(90,100)$	$p_4=(105,140)$
$P_3(u)$:	$p_2=(75,60)$	$p_3=(90,100)$	$p_4=(105,140)$	$p_5=(150,200)$
$P_4(u)$:	$p_3=(90,100)$	$p_4=(105,140)$	$p_5=(150,200)$	$p_6=(180,140)$
$P_5(u)$:	$p_4=(105,140)$	$p_5=(150,200)$	$p_6=(180,140)$	$p_7=(190,120)$
$P_6(u)$:	$p_5=(150,200)$	$p_6=(180,140)$	$p_7=(190,120)$	$p_8=(160,100)$
$P_7(u)$:	$p_6=(180,140)$	$p_7=(190,120)$	$p_8=(160,100)$	$p_9=(130,80)$
$P_8(u)$:	$p_7=(190,120)$	$p_8=(160,100)$	$p_9=(130,80)$	$p_{10}=(130,80)$
$P_9(u)$:	$p_8=(160,100)$	$p_9=(130,80)$	$p_{10}=(130,80)$	$p_{11}=(130,80)$

$$P(u) = p_{i-1}\frac{(1-u)^3}{6} + p_i\left(\frac{u^3}{2} - u^2 + \frac{2}{3}\right) + p_{i+1}\left(-\frac{u^3}{2} + \frac{u^2}{2} + \frac{u}{2} + \frac{1}{6}\right) + p_{i+2}\frac{u^3}{6}$$



(b) B-spline curve has C^1 continuity everywhere.

(c) Define $g(u) = u\cdot\frac{(i+1)}{3}$ for $u\in\left[\frac{i}{3},\frac{i+1}{3}\right]$ and $i\in[0,8]$

Then $F(u) = G(3g(u))$ for $u\in\left[\frac{i}{3},\frac{i+1}{3}\right]$ and $i\in[0,8]$ and $0\le u\le 3$

Q5. (a) $AX = b$, $z = ax + by + c$

$$A = \begin{bmatrix} \Sigma x_i x_i & \Sigma x_i y_i & \Sigma x_i \\ \Sigma x_i y_i & \Sigma y_i y_i & \Sigma y_i \\ \Sigma x_i & \Sigma y_i & \Sigma 1 \end{bmatrix} \quad X = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad b = \begin{bmatrix} \Sigma x_i z_i \\ \Sigma y_i z_i \\ \Sigma z_i \end{bmatrix}$$

$$A = \begin{bmatrix} 200.79 & 213.49 & 31.3 \\ 213.49 & 229.42 & 35 \\ 31.3 & 35 & 7 \end{bmatrix} \quad b = \begin{bmatrix} 177.43 \\ 187.33 \\ 26.92 \end{bmatrix}$$

$$X = \begin{bmatrix} 1.5961 \\ -0.7024 \\ 0.2207 \end{bmatrix} \Rightarrow z = 1.5961x - 0.7024y + 0.2207$$

(c) $\text{Deviation} = \Sigma |1.5961 x_i - 0.7024 y_i + 0.2207 - z_i|^2$

$= 0.1132 + 0.0740 + 0.0485 + 0.0007 + 0.0005 + 0.0709 + 0.0116$

$= 0.3194$

Q6. $\cos^2(x) = 1 - x^2 + \frac{1}{3}x^4 - \frac{2}{45}x^6$

$$1 - x^2 + \frac{1}{3}x^4 - \frac{2}{45}x^6 = \frac{a_0 + a_1 x + a_2 x^2 + a_3 x^3}{1 + b_1 x + b_2 x^2 + b_3 x^3}$$

$\Rightarrow$
$a_0 = 1$
$a_1 = b_1$
$a_2 = -1 + b_2$
$a_3 = -b_1 + b_3$
$0 = \frac{1}{3} - b_2$
$0 = \frac{1}{3}b_1 - b_3$
$0 = -\frac{2}{45} + \frac{1}{3}b_2$

$\Rightarrow$
$a_0 = 1$
$a_1 = 0$
$a_2 = -\frac{2}{3}$
$a_3 = 0$
$b_1 = 0$
$b_2 = \frac{1}{3}$
$b_3 = 0$

$$\Rightarrow \cos^2(x) = \frac{1 - \frac{2}{3}x^2}{1 + \frac{1}{3}x^2}$$

$$\sin(x^4-x) = -x+\frac{1}{6}x^3+x^4-\frac{1}{120}x^5$$

$$-x+\frac{1}{6}x^3+x^4-\frac{1}{120}x^5 = \frac{a_0+a_1x+a_2x^2+a_3x^3}{1+b_1x+b_2x^2+b_3x^3}$$

$$\Rightarrow \begin{aligned} a_0 &= 0 \\ a_1 &= -1 \\ a_2 &= -b_1 \\ a_3 &= -b_2+\frac{1}{6} \\ 0 &= -b_3+\frac{1}{6}b_1+1 \\ 0 &= \frac{1}{6}b_2+b_1-\frac{1}{120} \\ 0 &= \frac{1}{6}b_3+b_2+\frac{1}{120}b_1 \end{aligned} \qquad \Rightarrow \begin{aligned} a_0 &= 0 \\ a_1 &= -1 \\ a_2 &= -78/2147 \\ a_3 &= 43109/128820 \\ b_1 &= 78/2147 \\ b_2 &= -7213/42940 \\ b_3 &= 2160/2147 \end{aligned}$$

$$\Rightarrow \sin(x^4-x) = \frac{-x-\frac{78}{2147}x^2+\frac{43109}{128820}x^3}{1+\frac{78}{2147}x-\frac{7213}{42940}x^2+\frac{2160}{2147}x^3}$$

$$xe^x = x+x^2+\frac{1}{2}x^3+\frac{1}{6}x^4+\frac{1}{24}x^5+\frac{1}{120}x^6$$

$$x+x^2+\frac{1}{2}x^3+\frac{1}{6}x^4+\frac{1}{24}x^5+\frac{1}{120}x^6 = \frac{a_0+a_1x+a_2x^2+a_3x^3}{1+b_1x+b_2x^2+b_3x^3}$$

$$\Rightarrow \begin{aligned} a_0 &= 0 \\ a_1 &= 1 \\ a_2 &= 1+b_1 \\ a_3 &= \frac{1}{2}+b_1+b_2 \\ 0 &= \frac{1}{6}+\frac{1}{2}b_1+b_2+b_3 \\ 0 &= \frac{1}{24}+\frac{1}{6}b_1+\frac{1}{2}b_2+b_3 \\ 0 &= \frac{1}{120}+\frac{1}{24}b_1+\frac{1}{6}b_2+\frac{1}{2}b_3 \end{aligned} \qquad \Rightarrow \begin{aligned} a_0 &= 0 \\ a_1 &= 1 \\ a_2 &= 2/5 \\ a_3 &= 1/20 \\ b_1 &= -3/5 \\ b_2 &= 3/20 \\ b_3 &= -1/60 \end{aligned}$$

$$\Rightarrow xe^x = \frac{x+\frac{2}{5}x^2+\frac{1}{20}x^3}{1-\frac{3}{5}x+\frac{3}{20}x^2-\frac{1}{60}x^3}$$

Q7. $f(x) = xe^{-x}$ $\quad f'(x) = (1-x)e^{-x}$

x	$f(x)$	$f'(x)$
1	e^{-1}	0
2	$2e^{-2}$	$-e^{-2}$
3	$3e^{-3}$	$-2e^{-3}$

$$P_1(u) = [u^3\ u^2\ u\ 1] \cdot \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} e^{-1} \\ 2e^{-2} \\ 0 \\ -e^{-2} \end{bmatrix}$$

$$= [u^3\ u^2\ u\ 1] \begin{bmatrix} 2e^{-1} - 5e^{-2} \\ -3e^{-1} + 7e^{-2} \\ 0 \\ e^{-1} \end{bmatrix}$$

$$= (2e^{-1} - 5e^{-2})u^3 + (-3e^{-1} + 7e^{-2})u^2 + e^{-1}$$

$$f(\tfrac{3}{2}) = P_1(\tfrac{1}{2}) = 0.3362$$

$$P_2(u) = [u^3\ u^2\ u\ 1] \cdot \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2e^{-2} \\ 3e^{-3} \\ -e^{-2} \\ -2e^{-3} \end{bmatrix}$$

$$= [u^3\ u^2\ u\ 1] \begin{bmatrix} 3e^{-2} - 8e^{-3} \\ -4e^{-2} + 11e^{-3} \\ -e^{-2} \\ 2e^{-2} \end{bmatrix}$$

$$= (3e^{-2} - 8e^{-3})u^3 + (-4e^{-2} + 11e^{-3})u^2 - e^{-2}u + 2e^{-2}$$