Numerical Methods, 2023 Spring

Assignment 2
Due on April 6

Note: you should explain how you obtain your solution in your submission. If you use MATLAB or any other software to compute your results, you should provide your code and describe your solving process. This is good practice for you to explain things in a logical, organized, and concise way!

1. (25%) Solve the system

$$2.51\,x + 1.48\,y + 4.53\,z = 0.05,$$
$$1.48\,x + 0.93\,y - 1.30\,z = 1.03,$$
$$2.68\,x + 3.04\,y - 1.48\,z = -0.53.$$

   (a) Use Gaussian elimination, but use only three significant digits and do no interchanges. Observe the small divisor in reducing the third column. The correct solution is $x = 1.45310$, $y = -1.58919$, $z = -0.27489$.
   (b) Repeat part (a) but now do partial pivoting.
   (c) Repeat part (b) but now chop the numbers rather than rounding.
   (d) Substitute the solutions found in (a), (b), and (c) into the equations. How well do these match the original right-hand sides?

2. (25%) The system of Exercise 34 is an example of a symmetric matrix. Because the elements at opposite positions across the diagonal are exactly the same, it can be stored as a matrix with n rows but only three columns. (Please submit your code to E3)

▶34. Given this tridiagonal system:

$$\begin{bmatrix} 4 & -1 & 0 & 0 & 0 & 0 & 100 \\ -1 & 4 & -1 & 0 & 0 & 0 & 200 \\ 0 & -1 & 4 & -1 & 0 & 0 & 200 \\ 0 & 0 & -1 & 4 & -1 & 0 & 200 \\ 0 & 0 & 0 & -1 & 4 & -1 & 200 \\ 0 & 0 & 0 & 0 & -1 & 4 & 100 \end{bmatrix}.$$

   (a) Write an algorithm for solving a symmetric tridiagonal system that takes advantage of such compacting.
   (b) Use the algorithm from part (a) to solve the system in Exercise 34.
   (c) How many arithmetic operations are needed with this algorithm for a system of n equations?

3. (25%) Solve this system of equations, starting with the initial vector of [0, 0, 0]:

$$4.63\,x_1 - 1.21\,x_2 + 3.22\,x_3 = 2.22,$$
$$-3.07\,x_1 + 5.48\,x_2 + 2.11\,x_3 = -3.17,$$
$$1.26\,x_1 + 3.11\,x_2 + 4.57\,x_3 = 5.11.$$

   (a) Solve using the Jacobi method.
   (b) Solve using the Gauss-Seidel method.

4. (10%) Find a good value of the overrelaxation factor that speeds the convergence of 3. (b) mostly by trying different values.

5. (15%) Compute the condition number of the following matrices and classify each of them as well-conditioned or ill-conditioned.

   (a) $\begin{bmatrix} 10^{10} & 0 \\ 0 & 10^{-10} \end{bmatrix}$ (b) $\begin{bmatrix} 10^{10} & 0 \\ 0 & 10^{10} \end{bmatrix}$ (c) $\begin{bmatrix} 10^{-10} & 0 \\ 0 & 10^{-10} \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

6. (15%) (Bonus) Derive an algorithm and write a program to solve a linear system **Ax**=**b,** where **A** is an *N*x*N* band matrix with bandwidth *W*. You can generalize the algorithm you develop in 2. to handle a band matrix. (Please submit your code to E3)