HW5

Q1. Solution is in the code.

Result = 3.6797 when $h=0.1$ and
Result = 4.5581 when $h=0.05$

Error: Romberg integration

$$\frac{(\text{Result of } h=0.05)-(\text{Result of } h=0.1)}{2^2-1} = \frac{4.5581-3.6797}{3} = 0.2928$$

Q2.

$$\int_{t_n}^{t_{n+1}} f(t)\,dt \approx c_0 f_{n-3} + c_1 f_{n-2} + c_2 f_{n-1} + c_3 f_n$$

$$\begin{array}{l} f(t)=1 \\ f(t)=t \\ f(t)=t^2 \\ f(t)=t^3 \end{array} \quad \begin{bmatrix} 1 & 1 & 1 & 1 \\ -3h & -2h & -h & 0 \\ 9h^2 & 4h^2 & h^2 & 0 \\ -27h^3 & -8h^3 & -h^3 & 0 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} h \\ h^2/2 \\ h^3/3 \\ h^4/4 \end{bmatrix} \Rightarrow \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \frac{h}{24} \begin{bmatrix} -9 \\ 37 \\ -59 \\ 55 \end{bmatrix}$$

$$\Rightarrow \int_{t_n}^{t_{n+1}} f(t)\,dt \approx \frac{h}{24}[-9 f_{n-3} + 37 f_{n-2} - 59 f_{n-1} + 55 f_n]$$

$$\int_{t_n}^{t_{n+1}} f(t)\,dt \approx c_0 f_{n-4} + c_1 f_{n-3} + c_2 f_{n-2} + c_3 f_{n-1} + c_4 f_n$$

error $= \frac{251}{270}$

$$\begin{array}{l} f(t)=1 \\ f(t)=t \\ f(t)=t^2 \\ f(t)=t^3 \\ f(t)=t^4 \end{array} \quad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -4h & -3h & -2h & -h & 0 \\ 16h^2 & 9h^2 & 4h^2 & h^2 & 0 \\ -64h^3 & -27h^3 & -8h^3 & -h^3 & 0 \\ 256h^4 & 81h^4 & 16h^4 & h^4 & 0 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} h \\ h^2/2 \\ h^3/3 \\ h^4/4 \\ h^5/5 \end{bmatrix} \Rightarrow \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \frac{h}{270} \begin{bmatrix} 251 \\ -1274 \\ 2616 \\ -2774 \\ 1901 \end{bmatrix}$$

$$\Rightarrow x_{n+1} = x_n + \frac{h}{24}[-9 f_{n-3} + 37 f_{n-2} - 59 f_{n-1} + 55 f_n] + \frac{251}{270} h^5 x^{(5)}(\xi)$$

$$\int_{t_n}^{t_{n+1}} f(t)\,dt \approx c_0 f_{n-2} + c_1 f_{n-1} + c_2 f_n + c_3 f_{n+1}$$

$$\begin{array}{l} f(t)=1 \\ f(t)=t \\ f(t)=t^2 \\ f(t)=t^3 \end{array} \quad \begin{bmatrix} 1 & 1 & 1 & 1 \\ -2h & -h & 0 & h \\ 4h^2 & h^2 & 0 & h^2 \\ -8h^3 & -h^3 & 0 & h^3 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} h \\ h^2/2 \\ h^3/3 \\ h^4/4 \end{bmatrix} \Rightarrow \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \frac{h}{24}\begin{bmatrix} 1 \\ -5 \\ 19 \\ 9 \end{bmatrix}$$

$$\Rightarrow \int_{t_n}^{t_{n+1}} f(t)\,dt \approx \frac{h}{24}\left[f_{n-2} - 5f_{n-1} + 19f_n + 9f_{n+1}\right]$$

$$\int_{t_n}^{t_{n+1}} f(t)\,dt \approx c_0 f_{n-3} + c_1 f_{n-2} + c_2 f_{n-1} + c_3 f_n + c_4 f_{n+1}$$

error $= \frac{-19}{270}$

$$\begin{array}{l} f(t)=1 \\ f(t)=t \\ f(t)=t^2 \\ f(t)=t^3 \\ f(t)=t^4 \end{array} \quad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -3h & -2h & -h & 0 & h \\ 9h^2 & 4h^2 & h^2 & 0 & h^2 \\ -27h^3 & -8h^3 & -h^3 & 0 & h^3 \\ 81h^4 & 16h^4 & h^4 & 0 & h^4 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} h \\ h^2/2 \\ h^3/3 \\ h^4/4 \\ h^5/5 \end{bmatrix} \Rightarrow \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \frac{h}{270}\begin{bmatrix} -19 \\ 106 \\ -264 \\ 646 \\ 251 \end{bmatrix}$$

$$\Rightarrow x_{n+1} = x_n + \frac{h}{24}\left[f_{n-2} - 5f_{n-1} + 19f_n + 9f_{n+1}\right] - \frac{19}{720} h^5 x^{(5)}(\xi)$$

Q3.

$$y'' + ty' - 2y = t \qquad y(0) = y''(0) = 0 \quad y'(0) = 1 \qquad t = [0, 0.2, 0.4, 0.6, 0.8, 1]$$

$$\begin{array}{l} x_1(t) = y \\ x_2(t) = y' \\ x_3(t) = y'' \end{array} \quad \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ f(t, y, y', y'') \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ t - tx_2 + 2x_1 \end{bmatrix}$$

$$x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} y(0) \\ y'(0) \\ y''(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

(a) Result is in code.

$\Rightarrow y(0.2) = 0.2001 \qquad y(0.4) = 0.4021 \qquad y(0.6) = 0.6108$

(b) Result is in code.

$\Rightarrow y(0.8)$ predictor $= 0.834004$

$y(0.8)$ corrector $= 0.833975$

$\Rightarrow y(1)$ predictor $= 1.082738$

$y(1)$ corrector $= 1.082645$

(c) Error $= |\tilde{y}(1) - y(1)| \cdot \left(\frac{19}{251+19}\right) = |1.082645 - 1.082738| \cdot \frac{19}{270}$

$= 0.00000654 = 6.54 \times 10^{-6}$

Q4.

$y'' + \frac{y}{4} = 0 \quad y(0) = 0 \quad y(\pi) = 2 \quad y = 2\sin\left(\frac{x}{2}\right)$

(a) $h = \frac{\pi}{4} \quad y''_i = \frac{1}{h^2}(y_{i+1} - 2y_i + y_{i-1})$ points : $0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$

0 1 2 3 4

$\Rightarrow \frac{1}{h^2}(y_{i+1} - 2y_i + y_{i-1}) + \frac{y_i}{4} = 0$

$\Rightarrow \frac{1}{h^2} y_{i+1} + \left(\frac{1}{4} - \frac{2}{h^2}\right) y_i + \frac{1}{h^2} y_{i-1} = 0$

$\Rightarrow \frac{16}{\pi^2} y_{i+1} + \left(\frac{\pi^2 - 128}{4\pi^2}\right) y_i + \frac{16}{\pi^2} y_{i-1} = 0$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{16}{\pi^2} & \frac{\pi^2-128}{4\pi^2} & \frac{16}{\pi^2} & 0 & 0 \\ 0 & \frac{16}{\pi^2} & \frac{\pi^2-128}{4\pi^2} & \frac{16}{\pi^2} & 0 \\ 0 & 0 & \frac{16}{\pi^2} & \frac{\pi^2-128}{4\pi^2} & \frac{16}{\pi^2} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2 \end{bmatrix} \Rightarrow y = \begin{bmatrix} 0 \\ 0.7702 \\ 1.4215 \\ 1.8537 \\ 2 \end{bmatrix}$$

$y\left(\frac{\pi}{4}\right) = 2\sin\left(\frac{\pi}{8}\right) = 0.7654 \qquad 0.7702 - 0.7654 = 0.0048$

$y\left(\frac{\pi}{2}\right) = 2\sin\left(\frac{\pi}{4}\right) = 1.4142 \quad \overset{\text{error}}{\Longrightarrow} \quad 1.4215 - 1.4142 = 0.0073$

$y\left(\frac{3\pi}{4}\right) = 2\sin\left(\frac{3\pi}{8}\right) = 1.8478 \qquad 1.8537 - 1.8478 = 0.0059$

(b) try $h = \frac{\pi}{5}$ points : $\begin{array}{cccccc} 0, & \frac{\pi}{5}, & \frac{2\pi}{5}, & \frac{3\pi}{5}, & \frac{4\pi}{5}, & \pi \\ 0 & 1 & 2 & 3 & 4 & 5 \end{array}$

$$y''_i = \frac{1}{h^2}(y_{i+1} - 2y_i + y_{i-1})$$

$$\Rightarrow \frac{1}{h^2} y_{i+1} + \left(\frac{1}{4} - \frac{2}{h^2}\right) y_i + \frac{1}{h^2} y_{i-1} = 0$$

$$\Rightarrow \frac{25}{\pi^2} y_{i+1} + \left(\frac{\pi^2 - 200}{4\pi^2}\right) y_i + \frac{25}{\pi^2} y_{i-1} = 0$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{25}{\pi^2} & \frac{\pi^2-200}{4\pi^2} & \frac{25}{\pi^2} & 0 & 0 & 0 \\ 0 & \frac{25}{\pi^2} & \frac{\pi^2-200}{4\pi^2} & \frac{25}{\pi^2} & 0 & 0 \\ 0 & 0 & \frac{25}{\pi^2} & \frac{\pi^2-200}{4\pi^2} & \frac{25}{\pi^2} & 0 \\ 0 & 0 & 0 & \frac{25}{\pi^2} & \frac{\pi^2-200}{4\pi^2} & \frac{25}{\pi^2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 2 \end{bmatrix} \Rightarrow y = \begin{bmatrix} 0 \\ 0.6205 \\ 1.1998 \\ 1.6229 \\ 1.9054 \\ 2 \end{bmatrix}$$

$$\begin{aligned} y(\tfrac{\pi}{5}) &= 2\sin(\tfrac{\pi}{10}) = 0.6180 \\ y(\tfrac{2\pi}{5}) &= 2\sin(\tfrac{\pi}{5}) = 1.1756 \\ y(\tfrac{3\pi}{5}) &= 2\sin(\tfrac{3\pi}{10}) = 1.6180 \\ y(\tfrac{2\pi}{5}) &= 2\sin(\tfrac{2\pi}{5}) = 1.9021 \end{aligned} \quad \xRightarrow{\text{error}} \quad \begin{aligned} 0.6205 - 0.6180 &= 0.0025 \\ 1.1998 - 1.1756 &= 0.0042 \\ 1.6227 - 1.6180 &= 0.0047 \\ 1.9054 - 1.9021 &= 0.0033 \end{aligned}$$

$\Rightarrow$ max error $= 0.0047 = 0.47\% < 0.5\%$ with $h = \frac{\pi}{5}$

(c) $y' = \cos(\frac{\pi}{2})$ $\quad y'(0) = \cos(\frac{0}{2}) = 1$

Q5.

$x'' - tx' + t^2 x = t^3$ $\quad h = \frac{1}{4}$ on $[0, 1]$ points : $\begin{array}{ccccc} [0, & \frac{1}{4}, & \frac{1}{2}, & \frac{3}{4}, & 1] \\ 0 & 1 & 2 & 3 & 4 \end{array}$

$x(0) + x'(0) - x(1) + x'(1) = 3 \quad \cdots ①$

$x(0) - x'(0) + x(1) - x'(1) = 2 \quad \cdots ②$

① + ② : $2x(0) = 5$ $\qquad x(0) = \frac{5}{2}$

① − ② : $2(x'(0) - x(1) + x'(1)) = 1$ $\qquad x'(0) - x(1) + x'(1) = \frac{1}{2}$

Central difference :

$$x'(t_i) \approx \frac{x_{i+1} - x_{i-1}}{2h} \qquad x''(t_i) \approx \frac{x_{i+1} - 2x_i + x_{i-1}}{h^2}$$

Substitute :

$$\frac{x_{i+1} - 2x_i + x_{i-1}}{h^2} - t\,\frac{x_{i+1} - x_{i-1}}{2h} + t^2 x_i = t^3$$

$$\Rightarrow 2x_{i+1} - 4x_i + 2x_{i-1} - th\,x_{i+1} + th\,x_{i-1} + 2t^2h^2x_i = 2t^3h^2$$

$$\Rightarrow (2-th)x_{i+1} + (-4+2t^2h^2)x_i + (2+th)x_{i-1} = 2t^3h^2$$

For $i = 1, 2, 3, 4$, $t = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$

$$(2-(\tfrac{1}{4})(\tfrac{1}{4}))x_2 + (-4+2(\tfrac{1}{4})^2(\tfrac{1}{4})^2)x_1 + (2+(\tfrac{1}{4})(\tfrac{1}{4}))x_0 = 2(\tfrac{1}{4})^3(\tfrac{1}{4})^2$$
$$\frac{31}{16}x_2 - \frac{511}{128}x_1 + \frac{33}{16}x_0 = \frac{1}{512}$$

$$(2-(\tfrac{1}{2})(\tfrac{1}{4}))x_3 + (-4+2(\tfrac{1}{2})^2(\tfrac{1}{4})^2)x_2 + (2+(\tfrac{1}{2})(\tfrac{1}{4}))x_1 = 2(\tfrac{1}{2})^3(\tfrac{1}{4})^2$$
$$\frac{15}{8}x_3 - \frac{127}{32}x_2 + \frac{17}{8}x_1 = \frac{1}{64}$$

$$(2-(\tfrac{3}{4})(\tfrac{1}{4}))x_4 + (-4+2(\tfrac{3}{4})^2(\tfrac{1}{4})^2)x_3 + (2+(\tfrac{3}{4})(\tfrac{1}{4}))x_2 = 2(\tfrac{3}{4})^3(\tfrac{1}{4})^2$$
$$\frac{29}{16}x_4 - \frac{503}{128}x_3 + \frac{35}{16}x_2 = \frac{27}{512}$$

$$(2-(1)(\tfrac{1}{4}))x_5 + (-4+2(1)^2(\tfrac{1}{4})^2)x_4 + (2+(1)(\tfrac{1}{4}))x_3 = 2(1)^3(\tfrac{1}{4})^2$$
$$\frac{7}{4}x_5 - \frac{31}{8}x_4 + \frac{9}{4}x_3 = \frac{1}{8}$$

We don't have x_5 , so use $x'(0) - x(1) + x'(1) = \frac{1}{2}$

$$\Rightarrow -x_4 + \frac{x_5 - x_4}{2h} = \frac{1}{2} - \frac{x_1 - x_0}{2h} \qquad \Rightarrow -2hx_4 + x_5 - x_4 = h - x_1 + x_0$$

$$\Rightarrow x_5 = h - x_1 + x_0 + (2h+1)x_4 \qquad \overset{h=\frac{1}{4}}{\Rightarrow} x_5 = \frac{1}{4} - x_1 + x_0 + \frac{3}{2}x_4$$

Substitute x_5

$$\frac{7}{4}\left(\frac{1}{4}-x_1+x_0+\frac{3}{2}x_4\right)-\frac{31}{8}x_4+\frac{9}{4}x_3=\frac{1}{8}$$

$$\Rightarrow \frac{7}{16}-\frac{7}{4}x_1+\frac{7}{4}x_0+\frac{21}{8}x_4-\frac{31}{8}x_4+\frac{9}{4}x_3=\frac{1}{8}$$

$$\Rightarrow -\frac{5}{4}x_4+\frac{9}{4}x_3-\frac{7}{4}x_1+\frac{7}{4}x_0=-\frac{5}{16}$$

Solve linear systems

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{33}{16} & -\frac{511}{128} & \frac{31}{16} & 0 & 0 \\ 0 & \frac{17}{8} & -\frac{127}{32} & \frac{15}{8} & 0 \\ 0 & 0 & \frac{35}{16} & -\frac{503}{128} & \frac{27}{16} \\ \frac{7}{4} & -\frac{7}{4} & 0 & \frac{9}{4} & -\frac{5}{4} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 1/512 \\ 1/64 \\ 27/512 \\ -5/16 \end{bmatrix} \Rightarrow \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 25.4289 \\ 49.7356 \\ 76.4626 \\ 105.7822 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} x(0) &= 2.5 \\ x\left(\tfrac{1}{4}\right) &= 25.4289 \\ x\left(\tfrac{1}{2}\right) &= 49.7356 \\ x\left(\tfrac{3}{4}\right) &= 76.4626 \\ x(1) &= 105.7822 \end{aligned}$$