HW4

Q1.

$f(x) = 1 + \log_{10} x$

i	x_i	f_i	$f[x_i, x_{i+1}]$	$f[x_i, \cdots, x_{i+2}]$
0	0.15	0.1761	2.4355	-5.7505
1	0.21	0.3222	1.9754	-3.9088
2	0.23	0.3617	1.7409	-2.9464
3	0.27	0.4314	1.4757	-2.2307
4	0.32	0.5051	1.2973	
5	0.35	0.5441		

$$f'(0.268) \approx P'(0.268) = f[x_0, x_1] + f[x_0, x_1, x_2]\left[(0.268 - x_0) + (0.268 - x_1)\right]$$
$$= 2.4355 - 5.7505(0.118 + 0.058) = 1.4234$$

$$f'(0.268) \approx P'(0.268) = f[x_1, x_2] + f[x_1, x_2, x_3]\left[(0.268 - x_1) + (0.268 - x_2)\right]$$
$$= 1.9754 - 3.9088(0.058 + 0.038) = 1.6002$$

$$f'(0.268) \approx P'(0.268) = f[x_2, x_3] + f[x_2, x_3, x_4]\left[(0.268 - x_2) + (0.268 - x_3)\right]$$
$$= 1.7409 - 2.9464(0.038 - 0.002) = 1.6348$$

$$f'(0.268) \approx P'(0.268) = f[x_3, x_4] + f[x_3, x_4, x_5]\left[(0.268 - x_3) + (0.268 - x_4)\right]$$
$$= 1.4757 - 2.2307(-0.002 - 0.052) = 1.5962$$

According to a calculator, $f'(0.268) = 1.6205$ where $f(x) = 1 + \log_{10} x$. So the points with the least error are x_2, x_3, and x_4. We should be able to guess this answer because $x_3 = 0.27$ is the closest to $x = 0.268$.

Q2. $f(x) = x + \frac{\sin(x)}{3}$ $\quad h = 0.2 \quad s = \frac{x - x_0}{h}$

i	x_i	f_i	Δf_i	$\Delta^2 f_i$	$\Delta^3 f_i$	$\Delta^4 f_i$
0	0.3	0.3985	0.2613	-0.0064	-0.0022	0.0004
1	0.5	0.6598	0.2549	-0.0086	-0.0018	0.0004
2	0.7	0.9147	0.2464	-0.0104	-0.0014	0.0004
3	0.9	1.1611	0.2360	-0.0118	-0.0010	
4	1.1	1.3971	0.2241	-0.0128		
5	1.3	1.6212	0.2113			
6	1.5	1.8325				

$$f(x) = f_i + s\Delta f_i + \frac{s(s-1)}{2!}\Delta^2 f_i + \frac{s(s-1)(s-2)}{3!}\Delta^3 f_i + \frac{s(s-1)(s-2)(s-3)}{4!}\Delta^4 f_i \cdots$$

$$f'(x) = \frac{1}{h}\Big(\Delta f_i + \frac{s+(s-1)}{2!}\Delta^2 f_i + \frac{s(s-1)+(s-1)(s-2)+s(s-2)}{3!}\Delta^3 f_i$$
$$+ \frac{s(s-1)(s-2)+(s-1)(s-2)(s-3)+s(s-2)(s-3)+s(s-1)(s-3)}{4!}\Delta^4 f_i \cdots\Big)$$

(a) $s = \frac{0.72-0.7}{0.2} = 0.1$

$$f'(0.72) = \frac{1}{0.2}\left(0.2464 + \frac{0.1-0.9}{2}\times(-0.0104) + \frac{0.1\times(-0.9)+(-0.9)(-1.9)+0.1\times(-1.9)}{6}\times(-0.0014)\right)$$

$$= 1.2511$$

(b) $s = \frac{1.33-1.1}{0.2} = 1.15$

$$f'(1.33) = \frac{1}{0.2}\left(0.2241 + \frac{1.15+0.15}{2}\times(-0.0128)\right) = 1.0789$$

(c) $s = \frac{0.5-0.5}{0.2} = 0$

$$f'(0.5) = \frac{1}{0.2}\Big(0.2549 + \frac{0-1}{2}\times(-0.0086) + \frac{0\times(-1)+(-1)(-2)+0\times(-2)}{6}\times(-0.0018)$$
$$+ \frac{0\times(-1)(-2)+(-1)(-2)(-3)+0\times(-2)(-3)+0\times(-1)(-3)}{24}\times(0.0004)\Big)$$

$$= 1.2925$$

Q3.

$$f''(x_0) = C_{-2}f_{-2} + C_{-1}f_{-1} + C_0f_0 + C_1f_1 + C_2f_2$$

$$P(u) = au^4 + bu^3 + cu^2 + du + e$$

$P(u) = 1$:

$$f_{-2} = f_{-1} = f_0 = f_1 = f_2 = P(u) = 1$$
$$f'(x_0) = C_{-2} + C_{-1} + C_0 + C_1 + C_2 = P''(u) = 0$$

$P(u) = u$:

$$f_{-2} = P(-2h) = -2h$$
$$f_{-1} = P(-h) = -h$$
$$f_0 = P(0) = 0$$
$$f_1 = P(h) = h$$
$$f_2 = P(2h) = 2h$$
$$f'(x_0) = C_{-2}(-2h) + C_{-1}(-h) + C_0(0) + C_1(h) + C_2(2h) = P''(0) = 0$$

$P(u) = u^2$:

$$f_{-2} = P(-2h) = 4h^2$$
$$f_{-1} = P(-h) = h^2$$
$$f_0 = P(0) = 0$$
$$f_1 = P(h) = h^2$$
$$f_2 = P(2h) = 4h^2$$
$$f'(x_0) = C_{-2}(4h^2) + C_{-1}(h^2) + C_0(0) + C_1(h) + C_2(4h^2) = P''(0) = 2$$

$P(u) = u^3$:

$$f_{-2} = P(-2h) = -8h^3$$
$$f_{-1} = P(-h) = -h^3$$
$$f_0 = P(0) = 0$$
$$f_1 = P(h) = h^3$$
$$f_2 = P(2h) = 8h^3$$
$$f'(x_0) = C_{-2}(-8h^3) + C_{-1}(-h^3) + C_0(0) + C_1(h^3) + C_2(8h^3) = P''(0) = 0$$

$P(u) = u^4$:

$$f_{-2} = P(-2h) = 16h^4$$
$$f_{-1} = P(-h) = h^4$$
$$f_0 = P(0) = 0$$
$$f_1 = P(h) = h^4$$
$$f_2 = P(2h) = 16h^4$$
$$f'(x_0) = C_{-2}(16h^4) + C_{-1}(h^4) + C_0(0) + C_1(h^4) + C_2(16h^4) = P''(0) = 0$$

$$\begin{array}{l} P(u)=1 \rightarrow \\ P(u)=u \rightarrow \\ P(u)=u^2 \rightarrow \\ P(u)=u^3 \rightarrow \\ P(u)=u^4 \rightarrow \end{array} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2h & -h & 0 & h & 2h \\ 4h^2 & h^2 & 0 & h^2 & 4h^2 \\ -8h^3 & -h^3 & 0 & h^3 & 8h^3 \\ 16h^4 & h^4 & 0 & h^4 & 16h^4 \end{bmatrix} \begin{bmatrix} C_{-2} \\ C_{-1} \\ C_0 \\ C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 0 \end{bmatrix} \Rightarrow C = \begin{bmatrix} -1/12 \\ 4/3 \\ -5/2 \\ 4/3 \\ -1/12 \end{bmatrix}$$

$$f''(x_0) = -\frac{1}{12} f_{-2} + \frac{4}{3} f_{-1} - \frac{5}{2} f_0 + \frac{4}{3} f_1 - \frac{1}{12} f_2$$

$$f'''(x_0) = C_{-2} f_{-2} + C_{-1} f_{-1} + C_0 f_0 + C_1 f_1 + C_2 f_2$$

$$\begin{array}{l} P(u)=1 \rightarrow P'''(u)=0 \rightarrow \\ P(u)=u \rightarrow P'''(u)=0 \rightarrow \\ P(u)=u^2 \rightarrow P'''(u)=0 \rightarrow \\ P(u)=u^3 \rightarrow P'''(u)=6 \rightarrow \\ P(u)=u^4 \rightarrow P'''(u)=0 \rightarrow \end{array} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2h & -h & 0 & h & 2h \\ 4h^2 & h^2 & 0 & h^2 & 4h^2 \\ -8h^3 & -h^3 & 0 & h^3 & 8h^3 \\ 16h^4 & h^4 & 0 & h^4 & 16h^4 \end{bmatrix} \begin{bmatrix} C_{-2} \\ C_{-1} \\ C_0 \\ C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 6 \\ 0 \end{bmatrix} \Rightarrow C = \begin{bmatrix} -1/2 \\ 1 \\ 0 \\ -1 \\ 1/2 \end{bmatrix}$$

$$f'''(x_0) = -\frac{1}{2} f_{-2} + f_{-1} - f_1 + \frac{1}{2} f_2$$

$$\text{error} = (x - x_{-2})(x - x_{-1})(x - x_0)(x - x_1)(x - x_2) \frac{f^6(\xi)}{6!}$$

$$\frac{d}{dx} \text{error} \Big|_{x=x_0} = (x - x_{-2})(x - x_{-1})(x - x_1)(x - x_2) \frac{f^6(\xi)}{6!}$$

Q4.

$\frac{1}{3}$ rule : $\int_a^b f(x)dx = \frac{h}{3}[f(a)+4f(a+h)+f(b)]$ $h=\frac{b-a}{2}$

$\frac{8}{3}$ rule : $\int_a^b f(x)dx = \frac{3}{8}h[f(a)+3f(a+h)+3f(a+2h)+f(b)]$ $h=\frac{b-a}{3}$

① $\underbrace{(1.0, 1.1, 1.2)}_{1/3} + \underbrace{(1.2, 1.3, 1.4, 1.5)}_{8/3} + \underbrace{(1.5, 1.6, 1.7, 1.8)}_{8/3}$

$$\int_1^{1.8} f(x)dx = \frac{0.1}{3}(1.543+4\times1.669+1.811)$$
$$+\frac{3\times0.1}{8}(1.811+3\times1.971+3\times2.151+2.352)$$
$$+\frac{3\times0.1}{8}(2.352+3\times2.577+3\times2.828+3.107)$$
$$=0.3343+0.6198+0.8128 = 1.7669$$

② $\underbrace{(1.0, 1.1, 1.2, 1.3)}_{8/3} + \underbrace{(1.3, 1.4, 1.5)}_{1/3} + \underbrace{(1.5, 1.6, 1.7, 1.8)}_{8/3}$

$$\int_1^{1.8} f(x)dx = \frac{3\times0.1}{8}(1.543+3\times1.669+3\times1.811+1.971)$$
$$+\frac{0.1}{3}(1.971+4\times2.151+2.352)$$
$$+\frac{3\times0.1}{8}(2.352+3\times2.577+3\times2.828+3.107)$$
$$=0.5233+0.4309+0.8128 = 1.7670$$

③ $\underbrace{(1.0, 1.1, 1.2, 1.3)}_{8/3} + \underbrace{(1.3, 1.4, 1.5, 1.6)}_{8/3} + \underbrace{(1.6, 1.7, 1.8)}_{1/3}$

$$\int_1^{1.8} f(x)dx = \frac{3\times0.1}{8}(1.543+3\times1.669+3\times1.811+1.971)$$
$$+\frac{3\times0.1}{8}(1.971+3\times2.151+3\times2.352+2.577)$$
$$+\frac{0.1}{3}(2.577+4\times2.828+3.107)$$

$= 0.5233 + 0.6771 + 0.5665 = 1.7669$

To calculate the error, we use Simpson's 1/3 rule to be the "correct" resut

$$(1.0, 1.1, 1.2) + (1.2, 1.3, 1.4) + (1.4, 1.5, 1.6) + (1.6, 1.7, 1.8)$$

1/3 1/3 1/3 1/3

$$\int_1^{1.8} f(x)\,dx = \frac{0.1}{3}(1.543 + 4\times 1.669 + 1.811)$$

$$+ \frac{0.1}{3}(1.811 + 4\times 1.971 + 2.151)$$

$$+ \frac{0.1}{3}(2.151 + 4\times 2.352 + 2.577)$$

$$+ \frac{0.1}{3}(2.577 + 4\times 2.828 + 3.107)$$

$$= 0.3343 + 0.3949 + 0.4712 + 0.5665 = 1.7669$$

According to Simpson's 1/3 rule, ① and ③ has equally acurate answers while ② is off by 0.0001.

Q5. Code is provided.

$f(x) = \frac{1}{x^2}$ on $[0.2, 1]$

$$\int_{x_i}^{x_{i+1}} f(x)\,dx \approx \frac{f_i + f_{i+1}}{2}(x_{i+1} - x_i)$$

$$\Rightarrow \int_a^b f(x)\,dx \approx \sum_{i=0}^{n-1} \frac{h}{2}\left(f(a+ih) + f(a+(i+1)h)\right)$$

h terminates at $h = 0.0125$ and integral $= 4.0032$.

Q6.

Gauss Quadrature 3 terms:

$$\int_a^b f(x)\,dx = \frac{b-a}{2}\left[w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3)\right]$$

$$x_1 = \frac{b-a}{2} z_1 + \frac{b+a}{2} \qquad w_1 = \frac{5}{9} \qquad w_2 = \frac{8}{9} \qquad w_3 = \frac{5}{9}$$

$$x_2 = \frac{b-a}{2} z_2 + \frac{b+a}{2} \qquad z_1 = -\sqrt{\frac{3}{5}} \qquad z_2 = 0 \qquad z_3 = \sqrt{\frac{3}{5}}$$

$$x_3 = \frac{b-a}{2} z_3 + \frac{b+a}{2}$$

$$\int_{-0.2}^{1.4}\int_{0.4}^{2.6} e^x \sin(2y)\,dy\,dx = \int_{-0.2}^{1.4} e^x\,dx \int_{0.4}^{2.6} \sin(2y)\,dy$$

let $f(x) = e^x$ and $g(y) = \sin(2y)$

calculate $\int_{-0.2}^{1.4} f(x)\,dx$ and $\int_{0.4}^{2.6} g(y)\,dy$ separately.

$\int_{-0.2}^{1.4} f(x)\,dx = 3.23646921 \qquad \int_{0.4}^{2.6} g(y)\,dy = 0.11409502$

$\int_{-0.2}^{1.4}\int_{0.4}^{2.6} f(x)\,g(y)\,dy\,dx = 3.23646921 \times 0.11409502 = 0.3692650192 4$

Calculation is in the code.