LIMA: A PQC Encryption Scheme

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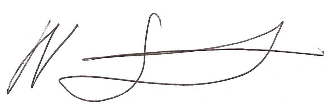

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Chapter 1

Discussion and Design Rationale

We introduce LIMA (LattIce MAthematics), a set of lattice-based public-key encryption and key encapsulation mechanisms, offering chosen plaintext security and chosen ciphertext security options. LIMA mixes conservative, standard and boring design choices with some efficiency improvements and flexibility. These factors are exhibited in its genesis: it is based on the ring variant [LPR10] of the LWE problem [Reg05] and on the encryption construction in [LP11]. We use the Fujisaki-Okamoto transform [FO99] to obtain an IND-CCA secure public-key encryption scheme. Our IND-CCA key encapsulation mechanism (KEM) is obtained via a transform of Dent [Den03]. This provides improved communication efficiency over using our IND-CCA public-key encryption scheme directly as a KEM; we also give a tight security proof for our IND-CCA KEM.

Thus our basic building blocks use highly respected and well studied cryptographic components. Our preference for "boring and simple" is illustrated by the fact that while our construction is efficient, other constructions such as [BDK$^+$17] achieve higher encryption and decryption speeds. However, run times for lattice-based schemes are already generally faster than current public-key schemes and thus we view optimizing run times as being less important compared to simplicity.

Mathematical Structure

For efficiency we use a ring variant. Encryption based on Ring-LWE has been extensively studied in the literature [BG14, LPR10, LPR13, CMV$^+$15]. The core component is essentially the Lyubashevsky *et al.* scheme [LPR10], or (equivalently) the BGV [BGV14] homomorphic encryption scheme, with the message being in the upper bits as opposed to the lower bits. This is sometimes referred to as the FV scheme [FV12].[1] The scheme also bears some resemblance to the scheme in [Pei14], although we adopt a completely different approach to ciphertext compression and producing IND-CCA variants. A lot of prior research has been conducted not only into the basic scheme, but also into implementation aspects [LSR$^+$15, RVM$^+$14], including protecting against side-channel attacks [RRVV15].

We opted for the ring variant of LWE to reduce the ciphertext size and to increase performance compared to LWE. In between Ring-LWE and LWE sits the Module-LWE [LS15] problem. As recently shown in [AD17], this problem is polynomial-time equivalent to Ring-LWE with a large modulus. Such large modulus Ring-LWE instances are distinguished from our Ring-LWE instances by the module-rank required to solve them. That is, while the Ring-LWE problem considered here requires one to find unusually short vectors in a module of rank three, this dimension is bigger in constructions such as [BDK$^+$17]. At present, it is unclear if any loss of security is implied by using smaller module rank for ranks > 1.

We offer two forms of ring to use within the LIMA system:

- Power-of-two Cyclotomics: These are relatively well studied and admit very efficient implementations.

[1] We do not use any homomorphic properties in this document, we just mention this to show the theoretical pedigree and prior analysis of our approach.

Name	Ring Variant	Functionality	Security
LIMA-2p-Enc-CPA	Two-Power	Encryption	IND-CPA
LIMA-sp-Enc-CPA	Safe-Prime	Encryption	IND-CPA
LIMA-2p-KEM-CPA	Two-Power	Key Encapsulation	IND-CPA
LIMA-sp-KEM-CPA	Safe-Prime	Key Encapsulation	IND-CPA
LIMA-2p-Enc-CCA	Two-Power	Encryption	IND-CCA
LIMA-sp-Enc-CCA	Safe-Prime	Encryption	IND-CCA
LIMA-2p-KEM-CCA	Two-Power	Key Encapsulation	IND-CCA
LIMA-sp-KEM-CCA	Safe-Prime	Key Encapsulation	IND-CCA

Table 1.1: The different LIMA schemes.

- Safe-Prime Cyclotomics: These are almost as good as power-of-two cyclotomics in terms of the ring geometry, but are slightly less efficient.

The reasons for offering the second option for the ring are twofold. Firstly, using safe-prime cyclotomics allows us to create a more flexible parameter space in terms of the ring dimension. Secondly, recall that a prime p is called "safe" if $p = 2 \cdot p' + 1$ for another prime p'. This implies that there are only two non-trivial subfields of the cyclotomic field $\mathbb{Q}[X]/\Phi_p(X)$. By contrast, the presence of subfield towers in the power-of-two case raises the question of whether this could leave open routes for attack such as those in [CJL16a, ABD16]. While it was shown soon after that towers of subfields are not required for these attacks to succeed [KF17], it appears prudent to hedge against possible future attacks regardless. For example, [BBdV$^+$17] shows that, for some fields, the presence of subfields can lead to a much easier lattice problem. We stress, however, that none of these subfield attacks are currently applicable to Ring-LWE. In summary, for added flexibility and to allow a more conservative choice of field, we give the *option* of using safe-prime cyclotomics.

The use of safe-primes appears to mitigate the possibility of attacks via subfields, but it also avoids the complex ring geometry associated with picking fields with large Galois group, such as the choice made in [BCLvV16]. Thus, we believe that using safe-prime fields offers a nice compromise between efficiency and protection against potential future attacks. In addition, using safe-prime cyclotomics allows us, with a little extra work, to be able to still use FFTs to evaluate the main polynomial products involved in the operation of our schemes. Thus, we think safe prime cyclotomic fields offer a good compromise between the benefits of the use of two power cyclotomics and fields defined by polynomials with large Galois groups.

We stress that we consider our schemes instantiated with the power-of-two rings to be secure, and they are more efficient in this case. We only offer the safe-prime variant if subfield attacks are a concern, and to offer additional flexibility in terms of parameter choices.

Cryptographic Structure

We offer IND-CPA and IND-CCA variants of a public-key encryption scheme. The IND-CCA variant is based on the Fujisaki-Okamoto transform [FO99]. Our choice of this methodology for achieving IND-CCA security is that it permits a tight security proof. We also offer IND-CPA and IND-CCA variants of a KEM construction, with the IND-CCA variant being built using a construction of Dent [Den03, Table 5] applied to our IND-CPA public-key encryption scheme. Dent's transformation did not have a tight security proof, so in [AOP$^+$17a] we also provide a new proof that is tight in our specific context. We note that a related but more generic tight reduction was recently given in [HHK17].

Ignoring parameter sizes for the moment, this document therefore defines eight possible configurations for LIMA, as listed in Table 1.1.

The first two schemes in Table 1.1 should not be used (without care) in any application. The third and fourth schemes should also be used with care, but are included here as they are mentioned in the NIST call as being potentially desirable in some key exchange applications.

Scheme	N	q
LIMA-2p	1024	133121
LIMA-2p	2048	184321
LIMA-sp	1018	12521473
LIMA-sp	1306	48181249
LIMA-sp	1822	44802049
LIMA-sp	2062	16900097

Table 1.2: Parameter sets for use with LIMA.

All the LIMA schemes are derived from a base encryption algorithm Enc-CPA-Sub which can result in decryption failures. In particular Enc-CPA-Sub could make a choice of randomness that produces in a ciphertext which results in a decryption failure (or even decrypts to the wrong message). This is a standard problem in lattice based schemes and can cause problems in operation and in security proofs. To avoid this issue, we then modify the base encryption algorithm Enc-CPA-Sub so that it rejects any choice of randomness which *could* result in an eventual decryption failure. Thus, we apply rejection sampling at the encryption stage so as to remove the problem of decryption failures. By choosing our parameters carefully, we can make the probability that repeated sampling is needed during encryption relatively small, thus avoiding in almost all cases the need to repeat the encryption procedure. This results in a non-constant time encryption algorithm, though it will be constant time with high probability and will not leak any information about the message. We discuss this further below.

We define six parameter sets to be used in conjunction with the eight different LIMA schemes. Two of these parameter sets are in the "power-of-two" setting and four in the "safe-prime" setting. This yields a total of 24 different schemes, with this flexibility enabling scaling to larger messages spaces and/or increased security levels. The six parameter sets are shown in Table 1.2

Note that for LIMA-sp parameter sets, the larger value of q is to enable the FFT to be efficiently computed via Bluestein's algorithm. This places some requirements on the value of q. Using the FFT also means that ring multiplication can be more easy implemented on highly parallel processors such as GPUs. Another option in the LIMA-sp case would have been to abandon the usage of the FFT methods and use smaller values of q. This, however, would have come at the expense of an estimated slowdown by a factor of about two.[2]

Random Seeds

All random seeds/coins for our algorithms are passed through a NIST approved XOF, from NIST SP 800 185 [NIS16]. This means *we do not* pass the seeds/coins through a DRBG first, as recommended by NIST (in the FAQ related to this call).

The XOF output is used to generate random finite field elements, symmetric keys, and (importantly for us) samples from a distribution somewhat close to a Gaussian distribution. For this latter task we adopt a method suggested in [ADPS16], which is both efficient (in software and hardware) and constant-time.

Ciphertext Compression

All ciphertexts are compressed in the sense of ignoring *coefficients* of the "message carrying component" which contains no information about the message. Further compression can be obtained by Huffman encoding the ring elements.

In the case of our IND-CPA KEM, a further form of compression is possible by utilizing the reconciliation

[2] In this case we would recommend using Karatsuba multiplication to multiply the ring elements, as opposed to coordinate wise multiplication in the FFT domain.

approach of [Pei14]. We did not use this for two reasons. Firstly, it is only applicable to the IND-CPA KEM.[3] Secondly, there is some concern in the community over patenting of reconciliation mechanisms, and we wanted to avoid uncertainties arising from unresolved issues relating to the licensing of patents.

Intellectual Property

To our knowledge, the algorithms contained in this proposal are not covered by any intellectual property constraints.

The implementations provided in this proposal were written by Valery Osheter, Guy Peer and Nigel P. Smart.

[3] In [Pei14] a method of obtaining an IND-CCA secure KEM is given, but it is less efficient than ours.

Chapter 2

Specification

In this chapter, we slowly develop our specification from the bottom up.

2.1 Random Numbers, Hashing, XOFs and KDFs

To ensure efficient implementation, and to avoid multiple low level components (e.g. block ciphers and hash functions) we build all of our symmetric key components out of the SHA-3 hash function. Our constructions makes use of the following two functions, which are modelled as random oracles in our proofs of security:

- A function which takes a string and produces an essentially unbounded "random" string of bytes from this input string. This function will be used as the basis of all of the sampling needed in our algorithms, i.e. all the "Gaussian samples" and other random values, in ways which we will describe in detail below. Technically this function is an XOF (Extendable Output Function), and for our purposes we will use KMAC256, which is based on the SHA-3 hash function and defined in NIST SP 800 185 [NIS16].
- A function similar to that above, but which will only produce an output of fixed length. This fixed length output is intended to be used as a key in a higher level application. Technically, this function is a KDF (Key Derivation Function), and again we will use KMAC256 for this purpose.

2.1.1 KMAC256

We will model the KMAC256 algorithm via the following API for the purposes of this document, for details see [NIS16]. When called in the form $\mathsf{XOF} \leftarrow \mathsf{KMAC}(\mathsf{key}, \mathsf{data}, 0)$ the output is an XOF object, and when called in the form $K \leftarrow \mathsf{KMAC}(\mathsf{key}, \mathsf{data}, L)$ the output is a string of L bits in length. In both cases the input is a key key (of length at least 256 bits), a (one-byte) data string data, and a length field L in bits. The data string data will be used as a diversifier in our application, and it will correspond to the *domain separation* field in the KMAC standard. We use the empty string for the *input* field in the KMAC standard. Thus different values of data will specify different uses of the KMAC construction within our algorithms. In particular we will use single byte values of data only, as follows:

- $\mathsf{data} = 0x00$: Use a KDF.
- $\mathsf{data} = 0x01$: Use in LIMA key generation as an XOF.
- $\mathsf{data} = 0x02$: Use in LIMA Enc-CPA as an XOF.
- $\mathsf{data} = 0x03$: Use in LIMA Enc-CCA as an XOF.
- $\mathsf{data} = 0x04$: Use in LIMA Encap-CPA as an XOF.
- $\mathsf{data} = 0x05$: Use in LIMA Encap-CCA as an XOF.

In the case when $L = 0$ we shall let $a \leftarrow \mathsf{XOF}[n]$ denote the process of obtaining n bytes from the XOF object returned by the call to KMAC.

$\underline{\mathsf{KDF}^{[n]}(k)}$

In particular, our KDF can be derived as follows. Given a key k and desired output length n (in bytes) the KDF is then defined by:

1. $K \leftarrow \mathsf{KMAC}(k, 0x00, n)$
2. Output K.

2.1.2 Random Field Elements

At various points we will need to select an element uniformly at random from a finite field $\mathbb{F}_q$, where q is a prime, or we will need to produce vectors of such elements uniformly at random. These are defined in the following two operations, which assume a XOF has already been set up as above.

If $\mathbf{x}$ and $\mathbf{y}$ are two vectors of the same length over $\mathbb{F}_q$, we let $\mathbf{x} \otimes \mathbf{y}$ denote their Schur product, i.e. the coordinatewise product, over $\mathbb{F}_q$. Similarly we let $\mathbf{x} \oplus \mathbf{y}$ denote the coordinate-wise sum mod q.[1]

$\underline{a \underset{\mathsf{XOF}}{\leftarrow} \mathbb{F}_q}$

We consume twice as many random bits as q has from our XOF in order to obtain a value with a distribution that is close to uniform.

1. $s \leftarrow \mathsf{XOF}[\ 2 \cdot \lceil \log_{256} q \rceil\ ]$.
2. Convert s to an integer (msb is the left most bit).
3. $a \leftarrow s \pmod q$.
4. Output a.

$\underline{\mathbf{a} \underset{\mathsf{XOF}}{\leftarrow} \mathbb{F}_q^n}$

1. For i from 1 to n do
   (a) $a_i \underset{\mathsf{XOF}}{\leftarrow} \mathbb{F}_q$.
2. Output $\mathbf{a}$.

2.1.3 Random Approximate Discrete Gaussians

We define a distribution χ_σ from which the coefficients of an element in our ring R (to be defined later) will be drawn. This distribution is an approximation to the discrete Gaussian distribution with standard deviation σ and mean $\mu = 0$. There are various methods to approximately sample from such a distribution, for example Knuth-Yao or Box-Muller. We instead use a coarse approximation which can be realized by a constant time algorithm, and which is also suitable for implementation in hardware. In particular, we use the method of approximating a Discrete Gaussian via a centred binomial distribution, as suggested in [ADPS16], parametrizd by a value B. Starting with $2 \cdot B + 2$ random bits (b_i, b'_i) for $i = 0, \ldots, B$, one samples from χ_σ by computing

$$\sum_{i=0}^{B} (b_i - b'_i)$$

[1] We also use $\oplus$ to define exclusive-or, the difference in the two uses of this symbol should be clear from the context.

which is a binomial distribution centred on zero and with standard deviation $\sqrt{(B+1)/2} \approx \sigma$. Thus this distribution will approximate the discrete Gaussian distribution with standard deviation σ. In our scheme we select $B = 19$, and so we obtain $\sigma \approx 3.16$, and we require 40 bits (i.e. 5 bytes) of randomness per sample.

$\underline{\mathsf{GenerateGaussianNoise}_{\mathsf{XOF}}(\sigma)\textbf{:}}$

1. $B \leftarrow \lceil 2 \cdot \sigma^2 \rfloor - 1 = 19$. (Here and throughout $\lceil \cdot \rfloor$ denotes rounding of the argument to the nearest integer.)
2. $t = \mathsf{XOF}[5]$; interpret t as a bit string of length 40 in the natural way.
3. $s \leftarrow 0$.
4. For $i = 0$ to B do
   (a) $s \leftarrow s - t[2 \cdot i] + t[2 \cdot i + 1]$.
5. Return s.

2.1.4 Checking

To aid the implementor we provide in the `KAT` directory a file `XOF-KAT.txt` which will allow the implementor to check the above methods for generating random field elements and elements from an approximate discrete Gaussian. The KAT values are given in terms of the input string to the XOF, the associated data item, and then the corresponding outputs.

2.2 Cyclotomic Rings

As discussed at the beginning, our cryptographic system will come in two variants, depending on which type of cyclotomic field one chooses to use.

- **LIMA-2p: Power-of-Two:** In this variant N is a power of two and q is a prime such that $q \equiv 1 \pmod{2 \cdot N}$. We have $M = 2 \cdot N$ and $N = \phi(M)$, thus $q \equiv 1 \pmod{M}$. In this case we define the rings as
$$R = \mathbb{Z}[X]/(X^N + 1), \quad R_2 = \mathbb{Z}_2[X]/(X^N + 1), \text{ and } R_q = \mathbb{Z}_q[X]/(X^N + 1).$$
Note that $\Phi_M(X) = X^N + 1$ in this case.
- **LIMA-sp: Safe-Prime:** In this variant we select p to be a "safe prime", i.e. a prime such that $p = 2 \cdot p' + 1$ for another prime p'. We let e denote the smallest integer such that $2^e > 2 \cdot p$ and we let q be a prime such that $q \equiv 1 \pmod{2^e \cdot p}$. In this case we define the rings as
$$R = \mathbb{Z}[X]/(X^N + X^{N-1} + \cdots + X + 1), \quad R_2 = \mathbb{Z}_2[X]/(X^N + X^{N-1} + \cdots + X + 1),$$
and
$$R_q = \mathbb{Z}_q[X]/(X^N + X^{N-1} + \cdots + X + 1),$$
where $N = p - 1 = 2 \cdot p'$. Note that $\Phi_p(X) = X^{p-1} + X^{p-2} + \cdots + X + 1 = X^N + X^{N-1} + \cdots + X + 1$ in this case.

Elements of these rings are degree $(N - 1)$ polynomials with coefficients from $\mathbb{Z}, \mathbb{Z}_2, \mathbb{Z}_q$, respectively. Equivalently, these are represented as vectors of length N, with elements in the $\mathbb{Z}, \mathbb{Z}_2, \mathbb{Z}_q$, respectively.

2.2.1 Roots of Unity

Our schemes use a fixed set of roots of unity. We define these using the following function:

RootOfUnity(m, q):

1. $a \leftarrow 1$.
2. Do
   (a) $a \leftarrow a + 1$.
   (b) $\alpha \leftarrow a^{(q-1)/m} \pmod q$.
   (c) $\beta \leftarrow \Phi_m(\alpha) \pmod q$.
3. While $(\beta \neq 0)$.
4. Output α.

Note that if m is a power of two then $\Phi_m(X) = X^{m/2} + 1$ and if $m = 2 \cdot p$ is twice a prime p then $\Phi_m(X) = X^{2 \cdot p - 2} + X^{2 \cdot p - 4} + \cdots + X^2 + 1 = \Phi_p(X^2)$.

We use the above procedure to derive the following roots of unity:

LIMA-2p: By construction $\mathbb{F}_q$ contains a $(2 \cdot N)$-th root of unity, so we let

$$\alpha_0 \leftarrow \mathsf{RootOfUnity}(2 \cdot N, q).$$

Given α_0, we set $\alpha_1 = 1/\alpha_0 \pmod q$. We also set $\beta_0 = 1/N \pmod q$. For our two parameter sets in this setting, we have the following values for $\alpha_0, \alpha_1, \beta_0$:

N	q	α_0	α_1	β_0
1024	133121	32141	100666	132991
2048	184321	88992	152704	184231

LIMA-sp: By construction $\mathbb{F}_q$ contains a 2^e-th root of unity, and a $2 \cdot p$-th root of unity. We define

$$\begin{aligned} \alpha_0 &\leftarrow \mathsf{RootOfUnity}(2 \cdot p, q), \\ \alpha_1 &\leftarrow 1/\alpha_0 \pmod q, \\ \beta_0 &\leftarrow \mathsf{RootOfUnity}(2^e, q), \\ \beta_1 &\leftarrow 1/\beta_0 \pmod q. \end{aligned}$$

For our four parameter sets in this setting we have the following values of $\alpha_0, \alpha, \beta_0, \beta_1$:

N	q	e	α_0	α_1	β_0	β_1
1018	12521473	11	1561269	8501297	9597006	10910567
1306	48181249	12	30019814	39013233	5599915	28280508
1822	44802049	12	43213195	19941338	8284672	1121361
2062	16900097	13	12381941	15641966	213248	7202243

2.2.2 Representation

For each ring elment $\mathbf{a} \in R_q$ we represent it as a polynomial of degree $N - 1$, which we also think of as a vector of N elements in $\mathbb{F}_q$. To store/transmit such elements on-the-wire we use an ordering of the coefficients in which the coefficient corresponding to the smallest coefficient is to the left (i.e. transmitted first). Thus we encode

$$a_0 + a_1 X + \cdots + a_{N-1} \cdot X^{N-1}$$

as the array of integers $(a_0, a_1, \ldots, a_{N-1})$. See Section 2.8 for how we encode data in more detail.

BV-2-RE($\mathbf{b}$):

Given a byte array $\mathbf{b}$ we produce a ring element, where each coefficient corresponds to a *bit* of $\mathbf{b}$, as follows, We let $\mathbf{b} = (b_0, \ldots, b_{t-1})$ where $8 \cdot t < N$.

1. $\mathbf{a} \leftarrow 0$.
2. $d \leftarrow 0$.
3. For $i = 0, \ldots, t-1$
   (a) $t \leftarrow b_i$
   (b) For $i = 0, \ldots, 7$
      i. $\mathbf{a} \leftarrow \mathbf{a} + (t \wedge 1) \cdot X^d$.
      ii. $t \leftarrow t \gg 1$.
      iii. $d \leftarrow d + 1$.
4. Return $\mathbf{a}$.

The inverse operation we denote by $\mathbf{b} \leftarrow \text{RE-2-BV}(\mathbf{a})$, where in this case a is assumed to have coefficients in $\{0, 1\}$.

2.2.3 Truncation

To aid bandwidth efficiency, we sometimes truncate a ring element to a vector of integers modulo q, of smaller size. Given a ring element $\mathbf{a} \in R_q$ of the form

$$\mathbf{a} = a_0 + a_1 \cdot X + \cdots + a_{N-1} \cdot X^{N-1},$$

we define, for $1 \leq T \leq N$,

$$\mathsf{Trunc}(\mathbf{a}, T) = a_0 + a_1 \cdot X + \cdots + a_{T-1} \cdot X^{T-1}.$$

2.3 Fourier Transforms Over Cyclotomic Rings

The reason for defining the restrictions on q in the previous section, and for fixing specified roots of unity, is so that we can define the following Fourier Transform algorithms. These enable us to perform arithmetic in the above rings via coordinate-wise additions and multiplications.

$\mathsf{FFT}_{sub-1}(\mathbf{x}, n, \tau)$:

This algorithm takes as input a vector $\mathbf{x} = (x_0, x_1, x_2, \ldots, x_{n-1})$ of length n (a power of two), and a root of unity τ and performs the following recursive steps:

1. If $n = 1$ then return $\mathbf{x}$.
2. $\mathbf{y} \leftarrow (x_0, x_2, x_4, \ldots)$.
3. $\mathbf{z} \leftarrow (x_1, x_3, x_5, \ldots)$.
4. $\beta \leftarrow \tau^2 \pmod q$.
5. $\mathbf{y} \leftarrow \mathsf{FFT}_{sub-1}(\mathbf{y}, n/2, \beta)$,
6. $\mathbf{z} \leftarrow \mathsf{FFT}_{sub-1}(\mathbf{z}, n/2, \beta)$.
7. $\omega \leftarrow \tau$.

8. For i from 0 to $n/2-1$
   (a) $s \leftarrow \omega \cdot z_i \pmod q$.
   (b) $t \leftarrow y_i$.
   (c) $x_i \leftarrow t + s \pmod q$.
   (d) $x_{i+n/2} \leftarrow t - s \pmod q$.
   (e) $\omega \leftarrow \omega \cdot \beta \pmod q$.
9. Return $\mathbf{x}$.

$\mathsf{FFT}_{sub-2}(\mathbf{x}, n, \tau)$:

This algorithm takes as input a vector $\mathbf{x} = (x_0, x_1, x_2, \ldots, x_{n-1})$ of length n (a power of two), and a root of unity τ and performs the following recursive steps, which are identical to those above, except for the way the final output is produced:

1. If $n = 1$ then return $\mathbf{x}$.
2. $\mathbf{y} \leftarrow (x_0, x_2, x_4, \ldots)$.
3. $\mathbf{z} \leftarrow (x_1, x_3, x_5, \ldots)$.
4. $\beta \leftarrow \tau^2 \pmod q$.
5. $\mathbf{y} \leftarrow \mathsf{FFT}_{sub-2}(\mathbf{y}, n/2, \beta)$,
6. $\mathbf{z} \leftarrow \mathsf{FFT}_{sub-2}(\mathbf{z}, n/2, \beta)$.
7. $\omega \leftarrow 1$.
8. For i from 0 to $n/2-1$
   (a) $s \leftarrow \omega \cdot z_i \pmod q$.
   (b) $t \leftarrow y_i$.
   (c) $x_i \leftarrow t + s \pmod q$.
   (d) $x_{i+n/2} \leftarrow t - s \pmod q$.
   (e) $\omega \leftarrow \omega \cdot \tau \pmod q$.
9. Return $\mathbf{x}$.

$\mathsf{BFFTData}()$:

For the LIMA-sp fields we will be using Bluestein's FFT algorithm and as such we will require a number of items of precomputed data. We first define three arrays with the following ranges

$$\begin{aligned}&\mathsf{powers}[0\ldots1][0\ldots p-1],\\ &\mathsf{powersi}[0\ldots1][0\ldots p-1],\\ &\mathbf{bd}[0\ldots1][0\ldots 2^e-1].\end{aligned}$$

We will refer to the (i,j)-th element of powers by $\mathsf{powers}_{i,j}$ etc, and the i-th row of $\mathbf{bd}$ by $\mathbf{bd}_i$. We initialize these data structures as follows:

1. $\mathsf{powersi}_{0,0} \leftarrow 1/2^e \pmod q$.
2. $\mathsf{powersi}_{1,0} \leftarrow 1/(2^e \cdot p) \pmod q$.

3. For r from 0 to 1 do
   (a) $\mathsf{powers}_{r,0} \leftarrow 1$.
   (b) $\mathbf{bd}_{r,p-1} = 1$.
   (c) For i from 1 to $p-1$ do
      i. $s \leftarrow i^2 \pmod{2 \cdot p}$.
      ii. $\mathsf{powers}_{r,i} \leftarrow (\alpha_r)^s \pmod{q}$.
      iii. $\mathsf{powersi}_{r,i} \leftarrow \mathsf{powers}_{r,i} \cdot \mathsf{powersi}_{r,0} \pmod{q}$.
      iv. $b \leftarrow (\alpha_{1-r})^s \pmod{q}$.
      v. $\mathbf{bd}_{r,p-1+i} \leftarrow b$.
      vi. $\mathbf{bd}_{r,p-1-i} \leftarrow b$.
   (d) $\mathbf{bd}_{r,i} \leftarrow 0$ for $i = 2 \cdot p - 1$ to $2^e - 1$.
   (e) $\mathbf{bd}_r \leftarrow \mathsf{FFT}_{sub-2}(\mathbf{bd}_r, 2^e, \beta_0)$.

$\mathsf{BFFT}(\mathbf{a}, r)$**:**

This algorithm performs both the forward and the backward directions for the FFT in the LIMA-sp field variant. The input in both cases is a vector $\mathbf{a}$ of length p, indexed from zero, the output is a vector of length p, again indexed from zero. The forward direction is implemented by setting $r = 0$, and the backward direction is implemented by setting $r = 1$. In the forward direction the $(p-1)$-th element in the input array $\mathbf{a}$ is equal to zero, whereas in the backward directtion the first element in the array $\mathbf{a}$ is equal to zero. Thus (essentially) the input has size $p-1$, and not p; but for ease of exposition we think of it as being of size p.

1. Define $\mathbf{x}$ as a vector indexed by 0 to $2^e - 1$, initialized to zero, and $\mathbf{b}$ as a vector indexed by 0 to $p-1$.
2. $x_i \leftarrow \mathsf{powers}_{r,i} \cdot a_i \pmod{q}$ for $i = 0, \ldots, p-1$.
3. $\mathbf{x} \leftarrow \mathsf{FFT}_{sub-2}(\mathbf{x}, 2^e, \beta_0)$
4. $x_i \leftarrow x_i \cdot \mathbf{bd}_{r,i} \pmod{q}$ for $i = 0, \ldots, 2^e - 1$.
5. $\mathbf{x} \leftarrow \mathsf{FFT}_{sub-2}(\mathbf{x}, 2^e, \beta_1)$
6. $b_i \leftarrow x_{i+p-1} \cdot \mathsf{powersi}_{r,i} \pmod{q}$ for $i = 0, \ldots, p-1$.
7. Return $\mathbf{b}$.

2.3.1 Fourier Transforms

We can now define the Fourier Transform algorithms themselves.

$\mathsf{FFT}(f)$**:**

We can think of the polynomial $f \in R_q$ given by $f = f_0 + f_1 \cdot X + f_2 \cdot X^2 + \cdots + f_{N-1} \cdot X^{N-1}$ as the vector $\mathbf{f} = (f_0, f_1, \ldots, f_{N-1}) \in \mathbb{F}_q^N$. To compute the FFT of f we then perform the steps:

LIMA-2p:

1. $\mathbf{x} \leftarrow \mathsf{FFT}_{sub-1}(\mathbf{f}, N, \alpha_0)$.
2. Return $\mathbf{x}$.

LIMA-sp:

1. $\mathbf{y} \leftarrow \mathsf{BFFT}(\mathbf{f}, 0)$.
2. $x_i \leftarrow y_{i+1}$ for $i = 0, \ldots, p-2$.
3. Return $\mathbf{x}$.

$\mathsf{FFT}^{-1}(\mathbf{x})$:

To invert the above operation we perform the following steps:

LIMA-2p:

1. $\gamma \leftarrow \alpha_1^2 \pmod q$.
2. $\mathbf{f} \leftarrow \mathsf{FFT}_{sub-2}(\mathbf{x}, N, \gamma)$.
3. $\delta \leftarrow \beta_0$.
4. For i from 0 to $N-1$ do
   (a) $f_i \leftarrow f_i \cdot \delta \pmod q$.
   (b) $\delta \leftarrow \delta \cdot \alpha_1 \pmod q$.
5. Return $\mathbf{f}$.

LIMA-sp:

1. $y_0 \leftarrow 0$.
2. $y_{i+1} \leftarrow x_i$ for $i = 0, \ldots, p-2$.
3. $\mathbf{f} \leftarrow \mathsf{BFFT}(\mathbf{y}, 1)$.
4. $f_i \leftarrow f_i - f_{p-1} \pmod q$ for $i = 0, \ldots, p-1$.
5. Return $\mathbf{f}$.

Note that step 4 immediately above performs reduction modulo $\Phi_p(X)$.

2.3.2 Checking

Again, since this is a relatively complex set of operations, we provide a KAT file in the KAT directory called `FFT-KAT.txt` to enable implementors to check they are getting the same output from the FFT routines for specific inputs.

2.4 IND-CPA Public-Key Encryption

In this section, we define an IND-CPA public-key encryption scheme which will encrypt messages of size at most $\lfloor N/8 \rfloor$ bytes in length. In the next sub-sections, we will use elements of this IND-CPA encryption scheme to build our IND-CCA encryption scheme (which will encrypt messages of $\lfloor N/8 \rfloor - 32$ bytes in length) and our IND-CPA and IND-CCA key encapsulation mechanisms. We define

$$\Delta_q = \left\lfloor \frac{q}{2} \right\rfloor.$$

$\mathsf{KeyGen}(\mathsf{s})$:

Key generation proceeds as follows, where the input $\mathbf{seed}_0$ is a string containing *at least* 256 bits of entropy.

1. $\mathsf{XOF} \leftarrow \mathsf{KMAC}(\mathbf{seed}_0, 0x01, 0)$
2. $a = (a_0, \ldots, a_{N-1}) \underset{\mathsf{XOF}}{\leftarrow} \mathbb{F}_q^N$. Interpret a as an element of R_q.
3. For $i = 0$ to $N-1$ do $s_i \leftarrow \mathsf{GenerateGaussianNoise}_{\mathsf{XOF}}(\sigma)$.

4. For $i = 0$ to $N-1$ do $e'_i \leftarrow \mathsf{GenerateGaussianNoise}_{\mathsf{XOF}}(\sigma)$.
5. $\mathbf{a} \leftarrow \mathsf{FFT}(a)$.
6. $\mathbf{s} \leftarrow \mathsf{FFT}(s)$.
7. $\mathbf{e}' \leftarrow \mathsf{FFT}(e')$.
8. $\mathbf{b} \leftarrow (\mathbf{a} \otimes \mathbf{s}) \oplus \mathbf{e}'$, with pointwise multiplication and addition $\pmod q$.
9. $\mathfrak{sk} \leftarrow (\mathbf{s}, \mathbf{a}, \mathbf{b})$.
10. $\mathfrak{pk} \leftarrow (\mathbf{a}, \mathbf{b})$.
11. Return $(\mathfrak{pk}, \mathfrak{sk})$

Observe that s is a polynomial with small (Gaussian) coefficients, a is a random polynomial (with large, truly random coefficients), and $\mathbf{b}$ is the Fourier transform of $b = a \cdot s + e'$ where e' is also a polynomial with small (Gaussian) coefficients.

An alternative form of key generation which produces a "compressed" public/secret key is as follows:

1. $\mathsf{XOF} \leftarrow \mathsf{KMAC}(\mathbf{seed}_0, 0x01, 0)$
2. $\mathbf{seed}_1 \leftarrow \mathsf{KMAC}^{[384]}(\mathbf{seed}_0)$.
3. $\mathsf{XOF}' \leftarrow \mathsf{KMAC}(\mathbf{seed}_1, 0x01, 0)$
4. $a = (a_0, \ldots, a_{N-1}) \underset{\mathsf{XOF}'}{\leftarrow} \mathbb{F}_q^N$. Interpret a as an element of R_q.
5. For $i = 0$ to $N-1$ do $s_i \leftarrow \mathsf{GenerateGaussianNoise}_{\mathsf{XOF}}(\sigma)$.
6. For $i = 0$ to $N-1$ do $e'_i \leftarrow \mathsf{GenerateGaussianNoise}_{\mathsf{XOF}}(\sigma)$.
7. $\mathbf{a} \leftarrow \mathsf{FFT}(a)$.
8. $\mathbf{s} \leftarrow \mathsf{FFT}(s)$.
9. $\mathbf{e}' \leftarrow \mathsf{FFT}(e')$.
10. $\mathbf{b} \leftarrow (\mathbf{a} \otimes \mathbf{s}) \oplus \mathbf{e}'$, with pointwise multiplication and addition $\pmod q$.
11. $\mathfrak{sk} \leftarrow (\mathbf{seed}_0, \mathbf{seed}_1, \mathbf{b})$.
12. $\mathfrak{pk} \leftarrow (\mathbf{seed}_1, \mathbf{b})$.
13. Return $(\mathfrak{pk}, \mathfrak{sk})$

Then to "uncompress" the public key one computes:

1. $\mathsf{XOF}' \leftarrow \mathsf{KMAC}(\mathbf{seed}_1, 0x01, 0)$
2. $a = (a_0, \ldots, a_{N-1}) \underset{\mathsf{XOF}'}{\leftarrow} \mathbb{F}_q^N$
3. $\mathbf{a} \leftarrow \mathsf{FFT}(a)$.

A similar procedure can be performed to uncompress the components of the private key.

Note that, either in the regular or compressed case, the public key $\mathfrak{pk}$ is implicitly contained in the private key $\mathfrak{sk}$. We will exploit this property without further comment in the remainder of this chapter.

RandCheck(v, e)

To reduce parameter sizes, we reject certain pairs sampled from $(\chi_\sigma^N)^2$. Later we shall analyse the probability of this rejection. In particular we reject any pair (v, e) for which the following conditions are not satisfied. We let $t_i = v_i + e_i$ and, for the case of LIMA-2p we accept (v, e) if

$$\Big|\sum_{i=0}^{N-1} t_i\Big| \leq 11 \cdot \sqrt{2 \cdot N} \cdot \sigma.$$

For the case of LIMA-sp we accept (v, e) if for $k = 0, \ldots, N-1$ we have

$$\Big|\left(\sum_{i=0}^{k} t_i\right) + \left(\sum_{i=1}^{N-1} t_i\right) + \left(\sum_{i=k+2}^{N-1} t_i\right)\Big| \leq 11 \cdot \sqrt{4 \cdot N} \cdot \sigma.$$

Thus, we have the following algorithm

1. $a \leftarrow 1$.
2. For i from 0 to $N-1$ do
   (a) $t_i \leftarrow v_i + e_i$.
3. If LIMA-2p
   (a) $e \leftarrow \sum_{i=0}^{N-1} t_i$.
   (b) If $|e| > 11 \cdot \sqrt{2 \cdot N} \cdot \sigma$ then $a \leftarrow 0$.
4. Else
   (a) For $k = 0$ to $N-1$ do
      i. $e_k \leftarrow \left(\sum_{i=0}^{k} t_i\right) + \left(\sum_{i=1}^{N-1} t_i\right) + \left(\sum_{i=k+2}^{N-1} t_i\right)$.
      ii. If $|e_k| > 11 \cdot \sqrt{4 \cdot N} \cdot \sigma$ then $a \leftarrow 0$.
5. Return a.

Enc-CPA-Sub$(\mathbf{m}, \mathfrak{pk}, \mathsf{XOF})$:

We now define an algorithm Enc-CPA-Sub that we will use as a subroutine in our all of our encryption schemes and KEMs. Enc-CPA-Sub makes use of RandCheck and so may produce an invalid ciphertext symbol $\perp$ upon execution; we will analyse the probability of this event in Section 4.1 and show that it is small. On the other hand, whenever Enc-CPA-Sub executes successfully, its ciphertext output is guaranteed to be decryptable. We use rejection sampling on executions of Enc-CPA-Sub within our main encryption/encapsulation algorithms to guarantee that they have no decryption failures in normal operation. This comes at the cost of making encryption/encapsulation non-constant time, though only with very low probability.

The encryption mechanism takes as input the public key $\mathfrak{pk} = (\mathbf{a}, \mathbf{b})$, a message $\mathbf{m} \in \{0,1\}^{|\mathbf{m}|}$, where $|\mathbf{m}| < N$, and an already initialised XOF XOF. We assume that $|\mathbf{m}|$ is a multiple of eight, i.e. we transmit whole bytes only, thus we think of $\mathbf{m}$ as a byte vector of length $|\mathbf{m}|/8$. The encryption algorithm will return $\perp$ if the current XOF state does not produce a valid tuple of randomness.

1. $\ell = |\mathbf{m}|$.
2. If $\ell > N$ then return $\perp$.
3. $\mu \leftarrow$ BV-2-RE$(\mathbf{m})$,

4. For $i = 0$ to $N-1$ do $v_i \leftarrow \mathsf{GenerateGaussianNoise}_{\mathsf{XOF}}(\sigma)$.

5. For $i = 0$ to $N-1$ do $e_i \leftarrow \mathsf{GenerateGaussianNoise}_{\mathsf{XOF}}(\sigma)$.

6. For $i = 0$ to $N-1$ do $d_i \leftarrow \mathsf{GenerateGaussianNoise}_{\mathsf{XOF}}(\sigma)$.

7. $\mathbf{v} \leftarrow \mathsf{FFT}(v)$, $\mathbf{e} \leftarrow \mathsf{FFT}(e)$.

8. $x \leftarrow d + \Delta_q \cdot \mu \pmod q$.

9. $s \leftarrow \mathsf{FFT}^{-1}(\mathbf{b} \otimes \mathbf{v})$.

10. $t \leftarrow s + x$.

11. $c_0 \leftarrow \mathsf{Trunc}(t, \ell)$.

12. $\mathbf{c}_1 \leftarrow (\mathbf{a} \otimes \mathbf{v}) \oplus \mathbf{e}$, with pointwise multiplication and addition $\pmod q$.

13. If $\mathsf{RandCheck}(v, e) = 0$ then return $\perp$, else return $(c_0, \mathbf{c}_1)$.

The $\mathsf{RandCheck}$ routine is performed at the end so as to have a single exit point. It could be carried out after step 5. However, as $\mathsf{RandCheck}$ passes with high probability, exactly where it is performed will make little difference to performance.

Note that c_0 is the ring element $b \cdot v + d + \Delta_q \cdot \mu$ truncated to ℓ coefficients, whilst $\mathbf{c}_1$ is the Fourier transform of the element $a \cdot v + e$. We implicitly assume that the internal representation of c_0 holds the number of coefficients ℓ; see Section 2.8 for details of on the wire encodings of c_0 and $\mathbf{c}_1$. Also note that the size of a ciphertext (in bits) is equal to approximately

$$(N + \ell) \cdot \log_2 q \approx (N + |\mathbf{m}|) \cdot \log_2 q.$$

We are now ready to define an IND-CPA public key encryption scheme. It uses the same KeyGen algorithm as above, and an encryption algorithm $\mathsf{Enc\text{-}CPA}$ and a decryption algorithm $\mathsf{Dec\text{-}CPA}$ that are defined immediately below.

$\mathsf{Enc\text{-}CPA}(\mathbf{m}, \mathfrak{pk}, \mathbf{r})$**:**

The encryption algorithm is guaranteed never to return an invalid ciphertext. This is achieved by repeatedly calling $\mathsf{Enc\text{-}CPA\text{-}Sub}$ until a valid ciphertext is produced. Because, as we show later, $\mathsf{Enc\text{-}CPA\text{-}Sub}$ fails only with low probability, we expect that only one call to $\mathsf{Enc\text{-}CPA\text{-}Sub}$ is needed with high probability.

The encryption algorithm again takes as input the public key $\mathfrak{pk} = (\mathbf{a}, \mathbf{b})$, a message $\mathbf{m} \in \{0,1\}^{|\mathbf{m}|}$, where $|\mathbf{m}| < N$, but this time it also takes as input a string $\mathbf{r}$ containing at least 256-bits of entropy (although we recommend $\mathbf{r}$ contains at least 384 bits of entropy).

1. If $|\mathbf{r}| < 256$ or $|\mathbf{m}| > N$ then return $\perp$.

2. $\mathsf{XOF} \leftarrow \mathsf{KMAC}(\mathbf{r}, 0x02, 0)$.

3. Do

   (a) $\mathbf{c} \leftarrow \mathsf{Enc\text{-}CPA\text{-}Sub}(\mathbf{m}, \mathfrak{pk}, \mathsf{XOF})$

4. While $\mathbf{c} = \perp$.

5. Return $\mathbf{c}$.

Dec-CPA($\mathbf{c}, \mathfrak{sk}$):

On input a ciphertext $\mathbf{c} = (c_0, \mathbf{c}_1)$ and a secret key $\mathfrak{sk} = (\mathbf{s}, \mathbf{a}, \mathbf{b})$, decryption is performed as follows.

1. Define ℓ to be the length of c_0, i.e. the number of field elements used to represent c_0.
2. If $\ell \neq 0 \pmod 8$ then return $\perp$.
3. $v \leftarrow \mathsf{FFT}^{-1}(\mathbf{s} \otimes \mathbf{c}_1)$.
4. $t \leftarrow \mathsf{Trunc}(v, \ell)$.
5. $f \leftarrow c_0 - t$.
6. Convert f into centered-representation. That is, let $f = (f_0, \ldots, f_{\ell-1})$ where each $f_i \in \mathbb{Z}_q$. If $0 \leq f_i \leq \frac{q-1}{2}$, then leave it unchanged. Else if $\frac{q}{2} < f_i \leq q-1$, then set $f_i \leftarrow f_i - q$ (over the integers).
7. $\mu \leftarrow \left| \left\lfloor \frac{2}{q} f \right\rceil \right|$ (i.e., round the scaled coefficients of f to the nearest integer and take the absolute value; the resulting coefficents will be 0 or 1).
8. $\mathbf{m} \leftarrow \mathsf{RE\text{-}2\text{-}BV}(\mu)$.
9. Return $\mathbf{m}$.

2.5 IND-CCA Public-Key Encryption

The scheme above is not secure under chosen-ciphertext attacks. In this section, we show how to achieve IND-CCA security by utilizing the highly efficient (first) transform of Fujisaki and Okamoto [FO99]. The key generation and basic primitives all remain the same; the modification is only with respect to encryption and decryption. If the original encryption scheme ($\mathsf{KeyGen}, \mathsf{Enc\text{-}CPA}, \mathsf{Dec\text{-}CPA}$) can encrypt messages of $\lfloor N/8 \rfloor$ bytes in length, this IND-CCA scheme encrypts messages of $\lfloor N/8 \rfloor - 32$ bytes in length; again we assume messages are a whole number of bytes long. We select the constant 32 here so as to enable the encryption of as large a message as possible, whilst achieving the desired security levels.

This algorithm will always terminate with a valid encryption (assuming the input random string is of the correct length), and the probability of requiring two or more executions of Enc-CPA-Sub is small.

In what follows we let $\mathbf{r} \leftarrow \mathbf{r} + 1$ denote the operation of converting the bit string to an integer in a little-endian manner, adding one to the value, and then converting back to a bit string of the same length as the input string.

Enc-CCA($\mathbf{m}, \mathfrak{pk}, \mathbf{r}$):

1. If $|\mathbf{r}| \neq 256$ or $|\mathbf{m}| \geq N - 256$ then return $\perp$.
2. Do
   (a) $\mu \leftarrow \mathbf{m} \| \mathbf{r}$.
   (b) $\mathsf{XOF} \leftarrow \mathsf{KMAC}(\mu, 0x03, 0)$.
   (c) $\mathbf{c} \leftarrow \mathsf{Enc\text{-}CPA\text{-}Sub}(\mu, \mathfrak{pk}, \mathsf{XOF})$.
   (d) $\mathbf{r} \leftarrow \mathbf{r} + 1$.
3. While $\mathbf{c} = \perp$.
4. Return $\mathbf{c}$.

Dec-CCA($\mathbf{c}, \mathfrak{sk}$):

1. $\mu \leftarrow$ Dec-CPA($\mathbf{c}, \mathfrak{sk}$).
2. If $|\mu| < 256$ then return $\perp$.
3. XOF $\leftarrow$ KMAC($\mu, 0x03, 0$).
4. $\mathbf{c}' \leftarrow$ Enc-CPA-Sub($\mu, \mathfrak{pk}$, XOF).
5. If $\mathbf{c} \neq \mathbf{c}'$ then return $\perp$.
6. $\mathbf{m} \| \mathbf{r} \leftarrow \mu$, where $\mathbf{r}$ is 256 bits long.
7. Return $\mathbf{m}$.

Note for this IND-CCA encryption algorithm the bit-size of a ciphertext is equal to approximately

$$(N + |\mathbf{m}| + 256) \cdot \log_2 q$$

and messages are limited to being at most $N - 256$ bits in size.

2.6 IND-CPA Key Encapsulation Mechanism

In this section, and following the NIST call, we present an IND-CPA KEM. This is useful in contexts where an *ephemeral* public key is generated and then used to transmit a symmetric key to another party, with the ephemeral public key being signed by a long term static key. The idea is that the KEM should only be used to transport a single symmetric key.

For key encapsulation in the post-quantum setting, we aim to transmit a key with at least $\ell \geq 256$ bits of entropy, where ℓ is divisible by eight and $\ell \leq N$. To do this we require sufficient entropy to perform the encapsulation; we recommend at least 384 bits of entropy for this. Thus, in total we assume an input entropy pool of at least $\ell + 384 \geq 256 + 384 = 640 = 8 \cdot 80$ bits.

The key generation algorithm KeyGen is the same as in that of our basic CPA encryption scheme, detailed in Section 2.4. The encapsulation algorithm Encap-CPA and decapsulation algorithm Decap-CPA are defined immediately below.

Encap-CPA($\ell, \mathfrak{pk}, \mathbf{r}$):

This algorithm takes as input a public key $\mathfrak{pk}$, a bit length ℓ and an input string of random bits $\mathbf{r}$ such that $|\mathbf{r}| \geq \ell + 384$. The procedure outputs an encapsulation $\mathbf{c} = (c_0, \mathbf{c}_1)$ and the key $\mathbf{k}$, of length ℓ bits, it encapsulates. Again we expect, with high probability, that the while loop in the pseudo-code is only executed once.

1. If $|\mathbf{r}| < \ell + 384$ or $\ell > N$ or $\ell < 256$ then return $\perp$.
2. Write $\mathbf{r}$ as $\mathbf{t} \| \mathbf{k}$ where $|\mathbf{k}| = \ell$.
3. XOF $\leftarrow$ KMAC($\mathbf{t}, 0x04, 0$).
4. Do
   (a) $\mathbf{c} \leftarrow$ Enc-CPA-Sub($\mathbf{k}, \mathfrak{pk}$, XOF).
5. While $\mathbf{c} = \perp$.
6. Return $(\mathbf{c}, \mathbf{k})$.

$\underline{\textsf{Decap-CPA}(\mathbf{c}, \mathfrak{sk})\textbf{:}}$

This algorithm takes as input a secret key $\mathfrak{sk}$ and an encapsulation $\mathbf{c} = (c_0, \mathbf{c}_1)$, and outputs the key $\mathbf{k}$ it encapsulates. It simply runs the decryption algorithm of our IND-CPA public-key encryption scheme.

1. $\mathbf{k} \leftarrow \textsf{Dec-CPA}(\mathbf{c}, \mathfrak{sk})$.
2. Output $\mathbf{k}$.

Note that for this IND-CPA KEM, the size of an encapsulation is equal to approximately

$$(N + \ell) \cdot \log_2 q$$

bits.

2.7 IND-CCA Key Encapsulation Mechanism

In this section, we present an IND-CCA KEM. We adopt the method of Dent [Den03, Theorem 5], which builds an IND-CCA secure KEM from an IND-CPA secure encryption scheme. We essentially reuse the encryption scheme from Section 2.4 with algorithms ($\textsf{KeyGen}, \textsf{Enc-CPA}, \textsf{Dec-CPA}$), but we replace Enc-CPA with an algorithm derived from Enc-CPA-Sub. The "hashing of random coins" approach that Dent's transform relies on is reflected in our construction by using a random value $\mathbf{r}$ as an input to an XOF to generate the randomness consumed by Enc-CPA-Sub, and by also treating $\mathbf{r}$ as the message input to Enc-CPA-Sub. Our security analysis will treat the XOF as a random oracle.

The key generation algorithm KeyGen of our IND-CCA KEM is the same as in that detailed in Section 2.4. The encapsulation algorithm Encap-CCA and decapsulation algorithm Decap-CCA are defined immediately below.

$\underline{\textsf{Encap-CCA}(\ell, \mathfrak{pk}, \mathbf{s})\textbf{:}}$

This algorithm takes as input a public key $\mathfrak{pk}$, a bit length ℓ (divisible by eight), and a string of random bits $\mathbf{r}$ such that $256 \leq |\mathbf{r}| \leq N$. The output is an encapsulation $\mathbf{c} = (c_0, \mathbf{c}_1)$ and the key $\mathbf{k} \in \{0,1\}^{\ell}$ it encapsulates.

1. If $|\mathbf{r}| < 384$ or $|\mathbf{r}| > N$ then return $\perp$.
2. Do
   (a) $\textsf{XOF} \leftarrow \textsf{KMAC}(\mathbf{r}, 0x05, 0)$.
   (b) $\mathbf{c} \leftarrow \textsf{Enc-CPA-Sub}(\mathbf{r}, \mathfrak{pk}, \textsf{XOF})$.
   (c) $\mathbf{r} \leftarrow \mathbf{r} + 1$.
3. While $\mathbf{c} = \perp$.
4. $\mathbf{k} \leftarrow \textsf{KDF}^{[\ell]}(\mathbf{r})$.
5. Return $(\mathbf{c} = (c_0, \mathbf{c}_1), \mathbf{k})$.

$\underline{\textsf{Decap-CCA}(\ell, \mathbf{c}, \mathfrak{sk})\textbf{:}}$

This algorithm takes as input a secret key key $\mathfrak{sk}$ and an encapsulation $\mathbf{c} = (c_0, \mathbf{c}_1)$, and outputs the key $\mathbf{k}$ it encapsulates, or an error symbol $\perp$.

1. $\mathbf{r} \leftarrow \textsf{Dec-CPA}(\mathbf{c}, \mathfrak{sk})$.
2. If $|\mathbf{r}| < 384$ then return $\perp$.

3. $\mathsf{XOF} \leftarrow \mathsf{KMAC}(\mathbf{r}, 0x05, 0)$.

4. $\mathbf{c}' \leftarrow \mathsf{Enc\text{-}CPA\text{-}Sub}(\mathbf{r}, \mathfrak{pk}, \mathsf{XOF})$.

5. If $\mathbf{c} \neq \mathbf{c}'$ then return $\perp$.

6. $\mathbf{k} \leftarrow \mathsf{KDF}^{[\ell]}(\mathbf{r})$.

Note for this IND-CCA KEM, the size of a ciphertext is equal to approximately

$$(N + |\mathbf{r}|) \cdot \log_2 q$$

bits.

2.8 Encoding of Ciphertexts and Public Keys

In this section we detail how data items are encoded as strings of bytes. We first encode the parameter set as a single byte for ease of reference, using the following table:

Scheme	N	q	pCode
LIMA-2p	1024	133121	0
LIMA-2p	2048	184321	1
LIMA-sp	1018	12521473	2
LIMA-sp	1306	48181249	3
LIMA-sp	1822	44802049	4
LIMA-sp	2062	16900097	5

We encode integers, such as length fields and elements of $\mathbb{F}_q$, given by

$$a = a_0 + 256 \cdot a_1 + \cdots + a_t \cdot 256^t$$

as the sequence of bytes

$$a_t \| \ldots \| a_1 \| \ldots \| a_0.$$

Thus, the number 12345 gets encoded as the byte string $0x30 \| 0x39$. We write this byte string as $\mathsf{Val}(a)$. In many situations the value of t is implicit, i.e. $t = \lceil \log_{256} q \rceil$. If we want to pad the representation (to the left), to exactly t bytes then we write $\mathsf{Val}_t(a)$.

To encode a public key $\mathfrak{pk} = (\mathbf{a}, \mathbf{b})$ where

$$\mathbf{a} = a_0 + a_1 \cdot X + \cdots + a_{N-1} \cdot X^{N-1} \text{ and } \mathbf{b} = a_0 + a_1 \cdot X + \cdots + a_{N-1} \cdot X^{N-1},$$

as a byte string, which we express as $B \leftarrow \mathsf{Encode}(\mathfrak{pk})$, we use the byte string

$$B = \mathsf{pCode} \| \mathsf{Val}(a_0) \| \mathsf{Val}(a_1) \| \ldots \| \mathsf{Val}(a_{N-1}) \| \mathsf{Val}(b_0) \| \mathsf{Val}(b_1) \| \ldots \| \mathsf{Val}(b_{N-1}),$$

where we take $a_i, b_i \in [0, q-1)$. The inverse operation is denoted by $\mathsf{Decode}(B)$. Note that the value of pCode allows us to infer the value N and the byte length of the integers q

As the IND-CCA versions of our public-key encryption scheme and KEM require the public key for decryption, we store the public-key with the secret key, in one data item, we encode a secret key $\mathfrak{sk} = (\mathbf{s}, \mathbf{a}, \mathbf{b})$ in a similar way by writing $B \leftarrow \mathsf{Encode}(\mathfrak{sk})$ where:

$$B = \mathsf{pCode} \| \mathsf{Val}(a_0) \| \mathsf{Val}(a_1) \| \ldots \| \mathsf{Val}(a_{N-1}) \| \mathsf{Val}(b_0) \| \mathsf{Val}(b_1) \| \ldots \| \mathsf{Val}(b_{N-1}) \| \mathsf{Val}(s_0) \| \mathsf{Val}(s_1) \| \ldots \| \mathsf{Val}(s_{N-1}).$$

Note that the first bytes in such a byte string B correspond to the encoding of a public key, so one can extract the public key by calling $\mathfrak{pk} \leftarrow \mathsf{Decode}(B)$, using a suitable overloading of the function Decode.

In the case of our compressed public keys we replace the byte string

$$\mathsf{Val}(a_0)\|\mathsf{Val}(a_1)\|\ldots\|\mathsf{Val}(a_{N-1})$$

in the above representations by the byte string representing the 384-bits of $\mathbf{p}$.

To encode a ciphertext $\mathbf{c} = (c_0, \mathbf{c}_1)$ where

$$c_0 = c_0 + c_1 \cdot X + \cdots + c_{\ell-1} \cdot X^{\ell-1}$$

and

$$\mathbf{c}_1 = (c'_0, c'_1, \ldots, c'_{N-1}),$$

we write $B \leftarrow \mathsf{Encode}(\mathbf{c})$ where:

$$B = \mathsf{pCode}\|\mathsf{Val}_2(\ell)\|\mathsf{Val}(c_0)\|\mathsf{Val}(c_1)\|\ldots\|\mathsf{Val}(c_{\ell-1})\|\mathsf{Val}(c'_0)\|\mathsf{Val}(c'_1)\|\ldots\|\mathsf{Val}(c'_{N-1})$$

The inverse operation is denoted by $\mathbf{c} \leftarrow \mathsf{Decode}(B)$. Notice that the ciphertext also contains a reference to the parameter set to which it is related.

Chapter 3

Known Answer Test Values

The KAT subdirectory contains a number of files containing Known Answer Test values. The names are *not the same* as the ones NIST test files produce. This is because we would get a clash in names as we have two encryption and two encapsulation algorithms (one CPA and one CCA in each case). Thus we have renamed the files so that our naming is more consistent with what the contents represent.

We overview these here:

- The first two files give test data on intermediate routines, in particular the XOF and FFT algorithms.
- The second set of files gives the input and output data from the encryption and encapsulation programs.

XOF-KAT.txt	This file specifies KAT values for the XOF functionalities. Given an input string and a diversifier it gives the first 32 bytes of output of the associated XOF object. From this 32 bytes of output it also gives the finite field elements and Gaussian samples that would be generated from such an output string. This file thereby enables testing of the XOF output, and how the XOF output is processed into random elements consumed by our schemes.
FFT-KAT.txt	For each ring used in LIMA this gives the output of applying the FFT algorithm to the all ones vector. Thus this file enables testing of FFT algorithms.

lima_2p_1024_EncCPA-KAT.txt lima_2p_2048_EncCPA-KAT.txt lima_sp_1018_EncCPA-KAT.txt lima_sp_1306_EncCPA-KAT.txt lima_sp_1822_EncCPA-KAT.txt lima_sp_2062_EncCPA-KAT.txt	For each of the six parameter sets we give input and output of key generation, encryption and decryption for Enc-CPA. For key generation we give the associated input seed, and then the output public key and secret key (using the encoding methods in Section 2.8). For encryption we give a message, seed and then the associated ciphertext. This is then decrypted to give the original message back.
lima_2p_1024_EncCCA-KAT.txt lima_2p_2048_EncCCA-KAT.txt lima_sp_1018_EncCCA-KAT.txt lima_sp_1306_EncCCA-KAT.txt lima_sp_1822_EncCCA-KAT.txt lima_sp_2062_EncCCA-KAT.txt	For each of the six parameter sets we give input and output of key generation, encryption and decryption for Enc-CCA. For key generation we give the associated input seed, and then the output public key and secret key (using the encoding methods in Section 2.8); these are the same keys are earlier as we use the same initial seed to generate them. For encryption we give a message, seed and then the associated ciphertext. This is then decrypted to give the original message back.
lima_2p_1024_EncapCPA-KAT.txt lima_2p_2048_EncapCPA-KAT.txt lima_sp_1018_EncapCPA-KAT.txt lima_sp_1306_EncapCPA-KAT.txt lima_sp_1822_EncapCPA-KAT.txt lima_sp_2062_EncapCPA-KAT.txt	For each of the six parameter sets we give input and output of key generation, encryption and decryption for Encap-CPA. For key generation we give the associated input seed, and then the output public key and secret key (using the encoding methods in Section 2.8); these are the same keys are earlier as we use the same initial seed to generate them. For encapsulation we give a seed and then the associated ciphertext, and the key which it encapsulates. This is then decapsulated to check the same key is produced.
lima_2p_1024_EncapCCA-KAT.txt lima_2p_2048_EncapCCA-KAT.txt lima_sp_1018_EncapCCA-KAT.txt lima_sp_1306_EncapCCA-KAT.txt lima_sp_1822_EncapCCA-KAT.txt lima_sp_2062_EncapCCA-KAT.txt	For each of the six parameter sets we give input and output of key generation, encryption and decryption for Encap-CCA. For key generation we give the associated input seed, and then the output public key and secret key (using the encoding methods in Section 2.8); these are the same keys are earlier as we use the same initial seed to generate them. For encapsulation we give a seed and then the associated ciphertext, and the key which it encapsulates. This is then decapsulated to check the same key is produced.

Chapter 4

Security Analysis

In this chapter, we formally analyse the security of the schemes, i.e. we present provable security results. We also discuss the probability of decryption errors, i.e. correctness. Since correctness is simpler, we start there. In the next chapter, we will then discuss known attacks on the schemes and how we make use of these attacks to derive the parameter sets presented in Chapter 1.

4.1 Correctness

Recall that all our schemes share the same KeyGen algorithm. Furthermore, all our schemes make use of Enc-CPA-Sub as a sub-route by calling it repeatedly during encryption resp. encapsulation until an output other than $\perp$ is returned. Thus, we first estimate the probability that the Enc-CPA-Sub algorithm outputs $\perp$. This will bound the probability of needing to perform additional loops in our higher-level encryption resp. encapsulation algorithms so as to obtain a valid ciphertext.

Recall that our conditions are as follows. For the randomness $(v, e, d) \underset{\mathsf{XOF}}{\leftarrow} \chi_\sigma^N$ we accept any tuple (v, e) for which the following conditions are satisfied. We let $t_i = v_i + e_i$ and, for the case of LIMA-2p, we accept (v, e) if

$$\Big|\sum_{i=0}^{N-1} t_i\Big| \leq 11 \cdot \sqrt{2 \cdot N} \cdot \sigma. \tag{4.1}$$

Note that the sum on the left-hand side will behave like a Gaussian with standard deviation $\sqrt{2 \cdot N} \cdot \sigma$. Thus, the probability that the absolute value of the sum is greater than the term on the right-hand side is bounded by 2^{-91} since $\mathsf{erfc}(11/\sqrt{2}) \approx 2^{-91}$.

For the case of LIMA-sp, we accept (v, e) if for $k = 0, \ldots, N-1$ we have

$$\Big|\left(\sum_{i=0}^{k} t_i\right) + \left(\sum_{i=1}^{N-1} t_i\right) + \left(\sum_{i=k+2}^{N-1} t_i\right)\Big| \approx \Big|2 \cdot \sum_{i=0}^{N-1} t_i\Big| \leq 11 \cdot \sqrt{4 \cdot N} \cdot \sigma. \tag{4.2}$$

This sum of three sums will act like a Gaussian with standard deviation bounded by $\sqrt{4 \cdot N} \cdot \sigma$. Thus the probability that the sum is greater than the term on the right-hand side will also be bounded by a value close to 2^{-91}.

4.1.1 Correctness of Dec-CPA

We first look at correctness of decryption of our algorithm Dec-CPA. Note that f in this routine is the truncation of the following element to degree at most ℓ:

$$\begin{aligned} c_0 - s \cdot c_1 &= (b \cdot v + d + \Delta_q \cdot \mu) - s \cdot (a \cdot v + e) \\ &= ((a \cdot s + e') \cdot v + d + \Delta_q \cdot \mu) - s \cdot (a \cdot v \cdot e) \\ &= (e' \cdot v + d + \Delta_q \cdot \mu) - s \cdot e \\ &= (e' \cdot v + d - s \cdot e) + \Delta_q \cdot \mu, \end{aligned}$$

which is equal to "small" plus $\Delta_q \cdot \mu$ since all of e', v, d, s, e are polynomials with small Gaussian coefficients. Thus, when multiplying by $\frac{2}{q}$, these all disappear and the only value that remains after rounding is $\frac{2}{q} \cdot \Delta_q \cdot \mu = \mu$.

Observe that decryption works as long as $e' \cdot v + d - s \cdot e$ remains small. We first note that by construction, i.e. by our choice of the method for generating an approximation to a discrete Gaussian in the key generation algorithm, we have that

$$\|e'\|_\infty, \quad \|s\|_\infty \leq 20,$$

where $\| \cdot \|_\infty$ is the infinity (a.k.a. max) norm on the coefficients of the polynomial. To ensure correctness we want to obtain a bound B on each coefficient E_k of the expression

$$d + v \cdot e' - s \cdot e,$$

such that we can ensure correctness of decryption.

LIMA-2p: Writing v_i for the coefficients of v, e'_i for the coefficients of e' and so on, we find that the k-th coefficient of this term is equal to

$$E_k = d_k + \sum_{\substack{0 \leq i,j < N, \\ i+j=k}} (v_i \cdot e'_j + e_i \cdot s_j) - \sum_{\substack{0 \leq i,j < N, \\ i+j=N+k}} (v_i \cdot e'_j + e_i \cdot s_j).$$

We thus have that

$$|E_k| \leq |d_k| + 20 \cdot \Big| \sum_{j=0}^{N-1} v_j + e_j \Big|.$$

We know that by construction $|d_k| \leq 20$, and by our analysis above we know that the sum is bounded by $11 \cdot \sigma \cdot \sqrt{2 \cdot N}$, as we reject any pair (v, e) which does not satisfy this bound. Thus, we are guaranteed that

$$|E_k| \leq 20 + 20 \cdot 11 \cdot \sigma \cdot \sqrt{2 \cdot N} = 20 \left(1 + 11 \cdot \sigma \cdot \sqrt{2 \cdot N}\right).$$

So as our bound B on $|E_k|$ we take

$$B = 20 \cdot (1 + 11 \cdot \sigma \cdot \sqrt{2 \cdot N}). \tag{4.3}$$

LIMA-sp: The expression for E_k is now more complicated, due to the more complex form of reduction modulo $\Phi_p(X)$. We have

$$\begin{aligned} E_k = d_k &+ \left(\sum_{i+j=k} s_i \cdot e_j + e'_i \cdot v_j \right) - \left(\sum_{i+j=N} s_i \cdot e_j + e'_i \cdot v_j \right) \\ &+ \left(\sum_{i+j=N+1+k} s_i \cdot e_j + e'_i \cdot v_j \right). \end{aligned}$$

Thus, we have that

$$|E_k| \le |d_k| + 20 \cdot \left| \left(\sum_{j=0}^{k} t_j\right) + \left(\sum_{j=1}^{N-1} t_j\right) + \left(\sum_{j=k+2}^{N-1} t_j\right) \right|.$$

Hence, by a similar argument, we have guaranteed that $|E_k| \le B$ where

$$B = 20 \cdot (1 + 11 \cdot \sigma \cdot \sqrt{4 \cdot N}). \tag{4.4}$$

Thus, in both cases, if $q > 4 \cdot B$, decryption will be correct.

4.1.2 Correctness of Dec-CCA

Correctness of Dec-CCA follows analogously to the correctness of Dec-CPA.

4.1.3 Correctness of Decap-CPA

Correctness of Decap-CPA follows analogously to the correctness of Dec-CPA.

4.1.4 Correctness of Decap-CCA

Correctness of Decap-CCA follows analogously to the correctness of Dec-CPA.

4.2 Security of Gaussian Sampling

Formally, the usual worst-case to average-case security reductions for LWE will not apply to our scheme as we are not using a true rounded Gaussian as in [Reg05, LPR10] but only an approximation. However, we are well outside the range for the worst-case to average-case analysis to hold in any case, since $\sigma < \sqrt{n}$. However, there is no known attack which exploits these differences of distributions in LWE encryption resp. encapsulation. Thus we expect that our approximation of a Discrete Gaussian will pose no security problem in practice. Indeed, we could have selected errors from a suitably chosen uniform distribution, but we use the Gaussian to enable a finer grained correctness analysis above and to provide some design validation, having based the general design on a closely related problem which does have worst-case to average-case security.

Another concern might be the fact that in the safe-prime variant we do not define errors in the canonical embedding but in the polynomial embedding. An often claimed advantage of power-of-two cyclotomics is that choosing small errors in the polynomial embedding is equivalent to selecting small errors in the canonical embedding (where the actual Ring-LWE problem lies). However, for safe-prime cyclotomics the two embeddings are quite close geometrically (for example the ring constant defined in [DPSZ12] is very close to one for prime cyclotomics). Thus, we contend that this difference can be ignored, much like the difference between our approximate Gaussians and true Gaussians mentioned above.

4.3 Security Reductions

We are now ready to examine the formal security assurances of our two public-key encryption schemes and two key encapsulation mechanisms. Throughout this section, to aid the reader, we assume the public key and second ciphertext components are not located in the FFT domain. This is purely a notational convenience to aid exposition; converting between the standard and FFT domains is a keyless operation.

4.3.1 Hard Problems

We recall the Ring-LWE problem:

Definition 1 (Ring-LWE)**.** *Consider the following experiment: a challenger picks $s \in R_q$ and a bit $\beta \in \{0,1\}$ uniformly at random. The adversary is given an oracle which on empty input returns a pair $(a,b) \in R_q^2$, where if $\beta = 0$ the two elements are chosen uniformly at random, and if $\beta = 1$ the value a is chosen uniformly at random and b is selected such that $b = a \cdot s + e$ where e is selected according to a Gaussian distribution in the canonical embedding. The adversary's goal is to output a guess as to the bit β.*

In our situation, we slightly modify the above Ring-LWE problem. In particular, the secret values s are selected from χ_σ^N. It is well known that we may pick s from a "small" distribution without affecting security, see for example [MR09, ACPS09]. We therefore define the LIMA-LWE problem as follows.

Definition 2 (LIMA Ring-LWE)**.** *Let χ_σ denote the distribution defined earlier. Consider the following experiment: a challenger picks $s \in \chi_\sigma^N \subset R_q$ and a bit $\beta \in \{0,1\}$. The adversary $\mathcal{A}$ is given an oracle which on empty input returns a pair $(a,b) \in R_q^2$, where if $\beta = 0$ the two elements are chosen uniformly at random, and if $\beta = 1$ the value a is chosen uniformly at random and b is selected such that $b = a \cdot s + e$ where $e \in \chi_\sigma^N \subset R_q$. At the end of the experiment the adversary outputs its guess β' as to the hidden bit β. For an adversary which makes n_Q calls to its oracle and running in time t, we define*

$$\mathsf{Adv}^{\mathsf{LWE}}(\mathcal{A}, n_Q, t) = 2 \cdot \left| \Pr[\beta = \beta'] - \frac{1}{2} \right|.$$

We conjecture that $\mathsf{Adv}^{\mathsf{LWE}}(\mathcal{A}, n_Q, t)$ is negligible for all adversaries and all LIMA parameter sets. The only known "attack" on this problem is essentially via lattice reduction.

Conjecture 1. *For suitable choices of σ, N and q depending on the security parameter λ, $\epsilon = \mathsf{Adv}^{\mathsf{LWE}}(A, n_Q, t)$ is a negligible function in the security parameter λ. In particular, for all adversaries running in time t we have $t/\epsilon^2 \geq 2^\lambda$.*

We note that in the conjecture above we normalize the running time by success probability as $1/\epsilon^2$ — instead of the more customary $1/\epsilon$ — because we are considering a decision problem. In Section 5.1, we estimate the expected probabilities of an adversary solving this problem in a given time period via lattice reduction. However, due to our rejection of certain tuples (v, e, d) in the encryption routine, we need to define some modified problems.

Definition 3 (LIMA-LWE†)**.** *Let χ_σ denote the distribution defined earlier. Consider the following experiment: A challenger picks $v \in \chi_\sigma^N \subset R_q$ and a bit $\beta \in \{0,1\}$. The adversary $\mathcal{A}$ is given a pair of LWE challenges where, if $\beta = 1$,*

$$a' = a \cdot v + e \text{ and } b' = b \cdot v + d$$

where $a, b \in R_q$ are selected uniformly at random, and $e, d \in \chi_\sigma^N$, subject to the tuple (v, e, d) satisfying condition (4.1) or (4.2). But if $\beta = 0$ the adversary is given a uniformly random tuple $(a, b, a', b;)$. At the end of the experiment the adversary outputs its guess β' as to the hidden bit β. For an adversary running in time t, we can define

$$\mathsf{Adv}^{\mathsf{LWE}^\dagger}(\mathcal{A}, t) = 2 \cdot \left| \Pr[\beta = \beta'] - \frac{1}{2} \right|.$$

We have the following theorem.

Theorem 1. *For all adversaries $\mathcal{A}$, there is an adversary $\mathcal{B}$ such that*

$$\mathsf{Adv}^{\mathsf{LWE}^\dagger}(\mathcal{A}, t) \leq \mathsf{Adv}^{\mathsf{LWE}}(\mathcal{B}, 2, t) + 2^{-91}.$$

Proof. We construct adversary $\mathcal{B}$, by simply first making two oracle calls and then running adversary $\mathcal{A}$. When $\beta = 0$, i.e. the input is uniformly random, the distribution passed to adversary $\mathcal{A}$ is "valid" under its game. However, when $\beta = 1$ the distribution can be invalid. We let E denote the event that adversary $\mathcal{B}$ passes an invalid challenge to adversary $\mathcal{A}$ in the case of $\beta = 1$. Such an invalid challenge will violate one of the conditions (4.1) or (4.2). When E does not occur two adversaries are identical. We know, from above, that we have $\Pr[E] \leq 2^{-91}$, thus we have

$$\begin{aligned}
\mathsf{Adv}^{\mathsf{LWE}^\dagger}(\mathcal{A}, t) &= \Pr[\beta'_{\mathcal{A}} = 1 | \beta = 1] - \Pr[\beta'_{\mathcal{A}} = 1 | \beta = 0] \\
&= \Pr[\beta'_{\mathcal{A}} = 1 | \beta = 1 \wedge E] \cdot \Pr[E] + \Pr[\beta'_{\mathcal{A}} = 1 | \beta = 1 \wedge \neg E] \cdot \Pr[\neg E] - \Pr[\beta'_{\mathcal{A}} = 1 | \beta = 0] \\
&\leq \Pr[E] + \Pr[\beta'_{\mathcal{A}} = 1 | \beta = 1 \wedge \neg E] - \Pr[\beta'_{\mathcal{A}} = 1 | \beta = 0] \\
&\leq 2^{-91} + \Pr[\beta'_{\mathcal{B}} = 1 | \beta = 1] - \Pr[\beta'_{\mathcal{B}} = 1 | \beta = 0] \\
&= \mathsf{Adv}^{\mathsf{LWE}}(\mathcal{B}, 2, t) + 2^{-91}.
\end{aligned}$$

□

Our security proofs will reduce the security of our schemes down to the LIMA-LWE† problem. Note that it is reasonable to assume that the LIMA-LWE† is as hard as the standard LIMA-LWE problem, as there is no known attack on LIMA-LWE† which would not also break the standard LIMA-LWE problem. Thus, we treat the 2^{-91} term in the above reduction simply as an artefact of the security proof, as opposed to something which needs to worried about in practice.

4.3.2 Security Reduction for the Basic Encryption Scheme

The IND-CPA security of our basic encryption scheme (KeyGen, Enc-CPA, Dec-CPA) is established in the following theorem, whose proof is given in the Appendix of the full version [AOP+17b] of [AOP+17a].

Theorem 2. *If the LWE/LWE† problem is hard, then the scheme* (KeyGen, Enc-CPA, Dec-CPA) *is IND-CPA secure in the random oracle model. In particular, if there is an adversary $\mathcal{A}$ against the IND-CPA security of* (KeyGen, Enc-CPA, Dec-CPA) *in the random oracle model, then there are adversaries $\mathcal{B}$ and $\mathcal{D}$ such that*

$$\mathsf{Adv}^{\mathsf{IND\text{-}CPA}}(\mathcal{A}) \leq 2 \cdot \mathsf{Adv}^{\mathsf{LWE}}(\mathcal{B}, 1, t) + 2 \cdot \mathsf{Adv}^{\mathsf{LWE}^\dagger}(\mathcal{D}, t).$$

4.3.3 Security Reduction for our IND-CCA Secure PKE scheme

Our construction of an IND-CCA secure encryption scheme uses the Fujisaki-Okamoto transform [FO99] applied to our basic scheme. Before we can apply this transform, we first need to establish its γ-uniformity.

Definition 4 (γ-Uniformity)**.** *Consider an IND-CPA encryption scheme given by the tuple of algorithms* (KeyGen*,* Enc-CPA*,* Dec-CPA) *with* $\mathsf{Enc\text{-}CPA} : \mathcal{M} \times \mathcal{R} \longrightarrow \mathcal{C}$ *being the encryption function mapping messages and randomness to ciphertexts. Such a scheme is said to be γ uniform if for all public keys $\mathfrak{pk}$ output by* KeyGen*, all $m \in \mathcal{M}$ and all $c \in \mathcal{C}$ we have $\gamma(\mathfrak{pk}, m, c) \leq \gamma$, where*

$$\gamma(\mathfrak{pk}, m, c) = \Pr[r \in \mathcal{R} : c = \mathsf{Enc\text{-}CPA}(m, \mathfrak{pk}, r)].$$

The lemma below establishes that Ring-LWE-based encryption has low γ-uniformity. The proof can be found in [AOP+17a].

Lemma 1. *Let* (KeyGen, Enc-CPA, Dec-CPA) *with parameters N, χ_σ, q be the basic PKE scheme described in Section 2.4 and let σ such that $\Pr[X = x \mid X \leftarrow_r \chi_\sigma] \leq 1/2$ for any x. Then this scheme is γ-uniform with $\gamma \leq 2^{-N}$.*

Note that in our construction the condition $\forall x, \Pr[X = x \mid X \leftarrow_r \chi_\sigma] \leq 1/2$ is always satisfied by picking $\sigma > 1$. Also note that if we truncate c_0 to ℓ components then the above bound becomes $2^{-(N+\ell)}$ by considering d truncated to ℓ components directly as being sampled from χ_σ^ℓ. Applying the main result (Theorem 3) of Fujisaki and Okamoto [FO99], we obtain the following:[1]

[1]Using $k = N$ and $k_0 = 256$ in Theorem 3 of [FO99].

Theorem 3. *Suppose that* (KeyGen, Enc-CPA, Dec-CPA) *is* (t', ϵ') *IND-CPA secure and* γ*-uniform. For any* q_H, q_D*, the scheme* (KeyGen, Enc-CCA, Dec-CCA)*, derived from* (KeyGen, Enc-CPA, Dec-CPA) *as in Section 2.5, is* (t, ϵ) *IND-CCA secure for any adversary making at most* q_H *queries to* XOF *(modelled as a random oracle) and at most* q_D *queries to the decryption oracle, where*

$$t = t' - q_H \cdot (T_{\mathsf{Enc}} + v \cdot N),$$
$$\epsilon = \epsilon' \cdot (1 - \gamma)^{-q_D} + q_H \cdot 2^{-255}.$$

Here T_{Enc} *is the running time of the encryption function and* v *is a constant.*

We write "(KeyGen, Enc-CCA, Dec-CCA), derived from (KeyGen, Enc-CPA, Dec-CPA)" in the theorem statement above to express that both Enc-CPA and Enc-CCA make use of Enc-CPA-Sub as a sub-routine in a while loop which terminates as soon as a valid ciphertext is returned by Enc-CPA-Sub. While the two while-loops differ in how they construct the random seed per loop iteration, they are equivalent in the random oracle model. Thus, we may think of Enc-CCA as essentially making use of Enc-CPA as in the FO transform [FO99].

4.3.4 Security Reduction for our IND-CPA KEM

Theorem 4. *If the LIMA-LWE problem is hard then the KEM* (KeyGen, Encap-CPA, Decap-CPA) *is IND-CPA in the random oracle model.*

Proof. Given an adversay $\mathcal{A}$ against the IND-CPA security of (KeyGen, Encap-CPA, Decap-CPA), we can trivially construct an adversary $\mathcal{B}$ against the IND-CPA security of (KeyGen, Enc-CPA, Dec-CPA) such that

$$\mathsf{Adv}^{\mathsf{IND\text{-}CPA}}(\mathcal{A}) = \mathsf{Adv}^{\mathsf{IND\text{-}CPA}}(\mathcal{B}).$$

The result follows immediately. □

4.3.5 Security Reduction for our IND-CCA KEM

As remarked earlier, our IND-CCA KEM construction is obtained by applying the construction of Dent [Den03, Table 5]. This builds an IND-CCA secure KEM from a OW-CPA secure PKE scheme. By Theorem 2, we know that our encryption scheme is IND-CPA secure. It also has large message space. It follows that it is OW-CPA secure.

Both Enc-CPA and Encap-CCA make use of Enc-CPA-Sub as a sub-routine in a while loop which terminates as soon as a valid ciphertext is returned by Enc-CPA-Sub. As above, we we note that while the two while-loops differ in how they construct the random seed per loop iteration, they are equivalent in the random oracle model. Thus, we may think of Encap-CCA as essentially making use of Enc-CPA, as in Dent's construction. In what follows, we will thus refer to (KeyGen, Enc-CPA, Dec-CPA) as the "underlying encryption scheme". With this in mind, directly applying the generic result [Den03, Theorem 5], we would obtain the following security theorem for our IND-CCA KEM:

Theorem 5. *Suppose there is an adversary* $\mathcal{A}$ *which breaks the IND-CCA security of* (KeyGen, Encap-CCA, Decap-CCA) *in the random oracle model, with advantage* ϵ*, running in time* t*, making at most* q_D *decapsulation queries,* q_H *queries to the random oracle implementing the XOF function and* q_K *queries to the random oracle implementing the KDF. Then there is an adversary* $\mathcal{B}$ *breaking the OW-CPA security of the underlying encryption scheme* (KeyGen, Enc-CPA, Dec-CPA) *running in time essentially* t*, with advantage* ϵ'' *such that*

$$\varepsilon \le (q_D + q_H + q_K) \cdot \varepsilon' + \frac{q_D}{2^{\ell'}} + \gamma \cdot q_D$$

where $\ell' \ge 384$ *is the bit-size of* $\mathbf{r}$ *in our construction.*

The problem with this result is that it does not give a very tight reduction. We thus present a new tight proof of our construction, which is *not generic*, i.e. we make explicit use of the Ring-LWE based construction of the underlying encryption scheme. The proof is again given in [AOP+17a].[2] We note that a related but more generic tight reduction was recently given in [HHK17].

Theorem 6. *If the LIMA-LWE problem is hard then* $(\mathsf{KeyGen}, \mathsf{Encap\text{-}CCA}, \mathsf{Decap\text{-}CCA})$ *is an IND-CCA secure KEM In the random oracle model. In particular if $\mathcal{A}$ is an adversary against the IND-CCA security of* $(\mathsf{KeyGen}, \mathsf{Encap\text{-}CCA}, \mathsf{Decap\text{-}CCA})$ *running in time t, making at most q_H queries to the random oracle implementing the XOF function and at most q_K queries to the random oracle implementing the KDF, then there are adversaries $\mathcal{B}$ and $\mathcal{D}$ such that*

$$\epsilon \leq 2 \cdot \left(\epsilon' + \epsilon'' + \frac{q_H + q_K}{2^{\ell'}} + \gamma \cdot q_D \right),$$

where $\epsilon = \mathsf{Adv}^{\mathsf{IND\text{-}CCA}}(\mathcal{A}, t)$, $\epsilon' = \mathsf{Adv}^{\mathsf{LWE}}(\mathcal{B}, 1, t)$, $\epsilon'' = \mathsf{Adv}^{\mathsf{LWE}^\dagger}(\mathcal{D}, t)$, *and* $\ell' \geq 384$ *is the bit-size of* $\mathbf{r}$ *in our construction.*

We end this section in noting that a recent result in [JZC+17] reports that the above KEM is also IND-CCA secure in the QROM.

[2]Note that the roles of ℓ and ℓ' are reversed here compared to the original presentation in [AOP+17a].

Chapter 5

Known Attacks

Due to our provable security results the main *mathematical* attacks against our schemes will those on the underlying LWE problem. All current best known attacks against the LWE problem are based on lattice reduction. Thus, we spend the first part of this section discussing lattice reduction. We then go on to discuss how lattice reduction can be applied to attack the LWE problem and hence our scheme. Using estimates from this analysis, we are able to estimate the security offered by our selected parameter sets. We then go on to consider other potential attack strategies, including *implementation* attacks.

5.1 Lattice Reduction

Lattice reduction algorithms have been studied for many years in [LLL82, Sch87, GN08, HPS11, CN11, MW16]. From a theoretical perspective, one of the best lattice reduction algorithm is the slide reduction algorithm from [GN08]. Alternatively, we may call the BKZ algorithm [Sch87] and its variants [HPS11, CN11], performing best in practice. The BKZ algorithm is parameterized by a block size β and consists of repeated calls to an oracle solving the Shortest Vector Problem (SVP) in dimension β combined with calls to the famous LLL algorithm. After performing BKZ-β reduction the first vector in the transformed lattice basis will have norm $\delta_0^m \cdot \det(\Lambda)^{1/m}$ where $\det(\Lambda)$ is the determinant of the lattice under consideration, m its dimension and the root-Hermite factor δ_0 is a constant based on the block size parameter β. Increasing the parameter β leads to a smaller δ_0 but also leads to an increase in run-time; the run-time grows at least exponentially in β.

In estimating costs for running an attack via BKZ we have a number of different costs which we need to estimate.

- An attack strategy has a probability of success (see for example the ϵ for the dual attack on LWE mentioned later) depending on the block size β. It is often more efficient to repeatedly perform lattice reduction with a block size $\beta' < \beta$ which leads to a low probability of success for solving LWE than to target a high success probability directly using a bigger block size β. Thus, an entire attack strategy may need to be repeated a number of times to amplify the chance of success.

- Each call to BKZ will itself make a certain number of calls to an internal SVP oracle. The precise number depends on how one optimizes BKZ (e.g. using early abort).

- Each SVP oracle will itself require a certain amount of time to execute. It is at this point that potential quantum speed-ups could come to fruition.

In what follows, we make the following assumptions about the BKZ algorithm and lattice reduction in general.

Block Size. To establish the required block size β we solve

$$\log \delta_0 = \log \left(\frac{\beta}{2\pi e} (\pi\beta)^{\frac{1}{\beta}} \right) \cdot \frac{1}{2(\beta - 1)}$$

for β, see the PhD Thesis of Yuanmi Chen [Che13] for a justification of this.

Cost of SVP. Several algorithms can be used to realize the SVP oracle inside BKZ. Asymptotically, the fastest known algorithms are sieving algorithms. The fastest, known *classical* algorithm runs in time $2^{0.292\,\beta+o(\beta)}$ [BDGL16]. The fastest, known *quantum* algorithm runs in time $2^{0.265\,\beta+o(\beta)}$ [Laa15]. We note that this algorithm applies Grover's algorithm assuming an optimal speed-up. In [ADPS16] a third category — "paranoid" — was introduced, which gives a plausible lower bound for currently known techniques as $2^{0.2075\,\beta+o(\beta)}$, but we do not make use of this bound here. Experiments in [BDGL16] suggest $o(\beta) \approx 16$. All times are expressed in elementary operations mod q. For estimating time on a classical computer, we assume that such an operation can be performed using 1 clock cycle on a modern 64-bit CPU. To obtain a rough binary gate count, these numbers would need to be multiplied by $\log_2 q$. However, this will affect the final result only marginally.

Calls to SVP. The BKZ algorithm proceeds by repeatedly calling an oracle for computing a shortest vector on a smaller lattice of dimension β. In each "tour", n such calls are made and the algorithm is typically terminated once it stops making sufficient progress in reducing the basis. Experimentally, it has been established that only the first few tours make significant progress [Che13], so we can assume that one BKZ call costs as much as $4n$ calls to the SVP oracle. However, it seems plausible that the cost of these calls can be amortized across different calls, which is why [ADPS16] assumes the cost of BKZ to be the same as *one* SVP oracle call.

BKZ Cost. In summary, we assume a call to BKZ-β costs $2^{0.292\,\beta+16}$ operations classically and $2^{0.265\,\beta+16}$ operations quantumly. The additive constant 16 accounts for the $o(\beta)$ overhead of sieving and the fact that the SVP oracle needs to be called several times in BKZ.

Calls to BKZ. To pick parameters, we normalize running times to a fixed success probability. That is, all our expected costs are for an adversary winning with probability 51%. However, as mentioned above, it is often more efficient to run some algorithm many times with parameters that have a low probability of success instead of running the same algorithm under parameter choices which ensure a high probability of success. Thus, in general, we would expect several calls to BKZ to be required: $\approx 1/\varepsilon$ for search problems resp. $\approx 1/\varepsilon^2$ for decision problems.

5.2 Solving LWE

Having discussed lattice reduction, we now show how this can be applied to solve the LWE problem. Since, according to current knowledge, there is no difference between the attacks on Ring-LWE and standard LWE, to simplify the presentation we look at standard LWE.

Standard LWE is the problem of determining given $(A, \vec{c}) \in \mathbb{Z}_q^{m \times N} \times \mathbb{Z}_q^m$ whether $\vec{c} = A \cdot \vec{s} + \vec{e}$ or whether $\vec{c}$ is chosen uniformly at random. In what follows, $\alpha = \sqrt{2\pi}\sigma/q$. Here, we use the matrix form of writing LWE, where m is the number of samples made available to an attacker, N is the dimension of the secret, and q the modulus as before. In what follows, we will assume the attacker has access to as many samples as it requires, which is optimistic for the attacker, i.e. it gets to choose m.

We may pursue one of the two following strategies for solving LWE, called the primal and dual strategies. There are other possible strategies but we will point out that they are not competitive.

Primal. Find some $\vec{s}'$ such that $\|\vec{w} - \vec{c}\|$ with $\vec{w} = \vec{A} \cdot \vec{s}'$ is minimized, under the guarantee that $\vec{w}$ is not too far from $\vec{c}$. This is known as the Bounded Distance Decoding problem (BDD). To solve BDD, we may embed the BDD instance into a unique SVP (uSVP) instance and apply lattice reduction to solve it. To predict the required root-Hermite factor, [ADPS16] gives

$$\sqrt{\beta}\sigma \leq \delta_0^{2\beta-d} \cdot \mathrm{Vol}(L)^{1/d},$$

where $d = m + 1$ is the dimension of the lattice L. For the primal attack, only one BKZ call suffices, i.e. the attack either succeeds with high probability or it fails. To solve BDD, we may also perform lattice reduction followed by lattice point enumeration [LP11, LN13]. However, since, according to our estimates, this approach does not improve performance considerably compared to the uSVP approach, we do not consider it further here.

Dual. Find a short $\vec{y}$ in the integral row span of A. This problem is known as the Short Integer Solution problem (SIS). Given such a $\vec{y}$, we can then compute $\langle \vec{y}, \vec{c} \rangle$. On the one hand, if $\vec{c} = A \cdot \vec{s} + \vec{e}$, then $\langle \vec{y}, \vec{c} \rangle = \langle \vec{y} \cdot A, \vec{s} \rangle + \langle \vec{y}, \vec{e} \rangle \equiv \langle \vec{y}, \vec{e} \rangle \pmod q$. If $\vec{y}$ is short then $\langle \vec{y}, \vec{e} \rangle$ is also short. On the other hand, if $\vec{c}$ is uniformly random, so is $\langle \vec{y}, \vec{c} \rangle$, cf., for example, [LP11]. Following [APS15], the required log root-Hermite factor is

$$\log \delta_0 = \frac{\log^2 \left(\alpha \left(\sqrt{\ln(\frac{1}{\epsilon})/\pi} \right)^{-1} \right)}{4\, N \log q},$$

where ϵ is the advantage of distinguishing $\langle \vec{c}, \vec{e} \rangle$ from uniform mod q.

Note that the dual attack solves the decision version of LWE. Thus, applying the Chernoff bound to amplify an advantage ϵ to a constant advantage, we need to perform $\approx 1/\epsilon^2$ experiments and pick by majority vote. However, for the dual attack, too, we are going to assume that *one* BKZ call is sufficient regardless of ϵ. This BKZ-β is then followed by $\approx 1/\epsilon^2$ calls to LLL. This assumption is justified heuristically in that we can rerandomise an already reduced basis followed by some light lattice reduction such as LLL to achieve a different basis which is almost as reduced as the input [Alb17]. In contrast to [Alb17], however, we are going to assume that the LLL reduced bases have the same quality as the initial BKZ-β reduced basis, which is optimistic for the attacker. As a consequence, we assume that amplifying success probability is relatively cheap compared to previous work.

Other Methods. The dual strategy can also be realized using variants of the BKW algorithm [GJS15, KF15]. However, for the parameter choices considered here, these algorithms are not competitive with lattice-reduction based algorithms.

Another potential vector of attack is to combine combinatorial methods with lattice reduction leading to the "hybrid attack". Recently, this attack was revisited in the quantum setting [GvVW17] and the authors found that it offers not advantage over the attacks considered here if (a) the secret is not unusually short and (b) quantum sieving is assumed to realise the SVP oracle inside BKZ. Both conditions are satisfied in our security analysis.

Finally, Arora and Ge proposed an asymptotically efficient algorithm for solving LWE [AG11], which was later improved in [ACF+15]. However, these algorithms involve large constants in the exponent, ruling them out for parameters typically considered in cryptography such as here.

5.3 Parameter Analysis

We now analyze the parameters of our scheme to determine the estimated classical and quantum hardness, given the various options for lattice based attacks presented above. In selecting our parameters, we implemented the following strategy (which is implemented in Sage code in Appendix A). We first recall that the choice on q is restricted in the following ways:

LIMA-2p $q \equiv 1 \pmod{2 \cdot N}$.

LIMA-sp $q \equiv 1 \pmod{2^e \cdot p}$, where $e = \lceil \log_2(2 \cdot p) \rceil$ and $p = N + 1$.

Secondly, note that for a given dimension N, there is a minimal q of the right form such that

$$q > 4 \cdot B, \tag{5.1}$$

where the bound B comes from Equation 4.3 or 4.4 depending on which type (LIMA-2p or LIMA-sp) we are in. Thus, given N and a type, we pick the smallest q satisfying Equation 5.1 subject to the constraints above. We always use the same σ. Hence, given N, we can derive the remaining parameters.

Then, given N, q, σ we estimate the cost of solving an LWE instance with such parameters as outlined above, i.e. applying the primal or the dual attack and considering different classical or quantum implementations for the SVP oracle required by the BKZ algorithm. Using this strategy and calling the code in the appendix, we can estimate the hardness of the parameter sets. In Table 5.1 we present for each parameter set the best classical and quantum attack given our model above. For more precise details on these estimates we give all the data in Table 5.2.

		LIMA-2p				
N	q	$\lceil (N \cdot \lceil \log_2 q \rceil)/8 \rceil$	classical	strategy	quantum	strategy
1024	133121	2304	227.5	dual	208.8	dual
2048	184321	4608	481.7	dual	444.5	dual
		LIMA-sp				
$N = 2p'$	q	$\lceil (N \cdot \lceil \log_2 q \rceil)/8 \rceil$	classical	strategy	quantum	strategy
1018	12521473	3054	151.8	primal	139.2	primal
1306	48181249	4245	183.3	primal	167.8	primal
1822	44802049	5922	271.5	primal	247.9	primal
2062	16900097	6444	329.6	dual	303.5	dual

Table 5.1: Parameter choices for LIMA-2p and LIMA-sp: N is the ring dimension, q the modulus, $\lceil (N \cdot \lceil \log_2 q \rceil)/8 \rceil$ is the size of one ring element in bytes. We also list the best known running times ($\log_2$ of number of elementary operations) for attacking the underlying LWE problem in the classical and in the quantum setting with success probability at least 51%. The strategy columns state which of the strategies in Section 5.2 is most efficient and hence listed here.

5.4 Mapping to NIST Security Levels

NIST asks for security levels to be expressed in relation to the cost of classically and quantumly breaking AES and SHA3. We reproduce NIST's estimates for these tasks in Table 5.3. In particular, note that NIST parameterizes its quantum predictions by MAXDEPTH: the maximum depth a quantum circuit is expected to support.

Looking at Table 5.3, we observe that quantum computers will not provide any advantage if MAXDEPTH $< 2^{27}$ ($md < 27$ in our notation), if we furthermore assume that a single quantum gate costs at least as much as a classical gate. Thus, in what follows, we will assume MAXDEPTH is at least 2^{27}.

In contrast, our quantum estimates assume perfect Grover speed-up, i.e. we potentially assume a more powerful quantum adversary than NIST depending on the value of MAXDEPTH. Thus, to map our cost estimates to NIST, we say that a LIMA set of parameters offers security comparable to AES-x resp. SHA3-x whenever our log-cost estimates is bigger than NIST's quantum log-cost estimate minus 27 resp. NIST's classical log-cost estimate. Thus, we obtain Table 5.4.

strategy	adversary	$\log_2$ cost	δ_0	dim	β	repeat
		LIMA-2p, $n = 1024$, $q = 133121$				
primal	classical	231.8	1.002563	2152	739	$2^{0.0}$
dual	classical	227.5	1.002623	2148	717	$2^{185.8}$
primal	quantum	211.8	1.002563	2152	739	$2^{0.0}$
dual	quantum	208.8	1.002594	2159	727	$2^{163.8}$
		LIMA-2p, $n = 2048$, $q = 184321$				
primal	classical	488.5	1.001411	4116	1618	$2^{0.0}$
dual	classical	481.7	1.001426	4174	1595	$2^{356.4}$
primal	quantum	444.8	1.001411	4116	1618	$2^{0.0}$
dual	quantum	444.5	1.001411	4196	1617	$2^{314.4}$
		LIMA-sp, $n = 1018$, $q = 12521473$				
primal	classical	151.8	1.003584	2080	465	$2^{0.0}$
dual	classical	151.8	1.003589	2155	465	$2^{89.4}$
primal	quantum	139.2	1.003584	2080	465	$2^{0.0}$
dual	quantum	140.4	1.003610	2149	461	$2^{97.8}$
		LIMA-sp, $n = 1306$, $q = 48181249$				
primal	classical	183.3	1.003091	2713	573	$2^{0.0}$
dual	classical	183.3	1.003091	2736	573	$2^{109.8}$
primal	quantum	167.8	1.003091	2713	573	$2^{0.0}$
dual	quantum	167.8	1.003091	2736	573	$2^{109.8}$
		LIMA-sp, $n = 1822$, $q = 44802049$				
primal	classical	271.5	1.002260	3734	875	$2^{0.0}$
dual	classical	272.4	1.002254	3776	878	$2^{162.4}$
primal	quantum	247.9	1.002260	3734	875	$2^{0.0}$
dual	quantum	248.7	1.002254	3776	878	$2^{161.8}$
		LIMA-sp, $n = 2062$, $q = 16900097$				
primal	classical	336.3	1.001904	4190	1097	$2^{0.0}$
dual	classical	329.6	1.001935	4213	1074	$2^{277.4}$
primal	quantum	306.7	1.001904	4190	1097	$2^{0.0}$
dual	quantum	303.5	1.001920	4229	1085	$2^{244.4}$

Table 5.2: Cost of primal and dual attacks for best known quantum [Laa15] and classical [BDGL16] adversaries. The column "dim" holds the dimension of the lattice considered. The column "repeat" gives the number of times such a reduction would have to be repeated to amplify the success probability to $> 50\%$, depending on whether we are solving a search or decision problem. This cost is already taken into account in the "$\log_2$ cost" given. The primal attack is predicted to always succeed with probability close to one.

Algorithm	Quantum Gates	Classical Gates
AES 128	$170 - \log_2 \text{MAXDEPTH}$	143
SHA3-256	—	146
AES 192	$233 - \log_2 \text{MAXDEPTH}$	207
SHA3-384	—	210
AES 256	$298 - \log_2 \text{MAXDEPTH}$	272
SHA3-512	—	274

Table 5.3: NIST Cost Estimates for AES and SHA3.

n	q	NIST Level	Comment
		LIMA-2p	
1024	133121	AES-192	classical/quantum estimates > 207
2048	184321	SHA-512	classical/quantum estimates > 274
		LIMA-sp	
1018	12521473	AES-128	classical/quantum estimates ≈ 143
1306	48181249	SHA-256	classical/quantum estimates > 146
1822	44802049	AES-192	classical/quantum estimates > 207
2062	16900097	SHA-512	classical/quantum estimates > 274

Table 5.4: NIST Mapping

5.5 (Quantum) Algebraic Attacks

A potential concern for Ring-LWE based schemes are attacks exploiting the additional structure implied by the ring setting. That is, while the best known attacks against Ring-LWE work by treating it as a normal LWE instance, there could potentially exist additional structural weaknesses.

For example, polynomial-time quantum attacks finding short generators of principal ideals in cyclotomic rings were introduced in [CGS14, CDPR16]. This result was then extended in [CDW17] to short elements in general ideal lattices. While the latter will only recover vectors which are sub-exponentally longer than a shortest vector, this improves on what can be achieved classically, i.e. exponential approximation factors [LLL82]. On the other hand, we note that these approximation factors are too big to apply to most Ring-LWE schemes. Furthermore, it is currently not clear if these results to extend to Ring-LWE, i.e. it is possible that Ring-LWE is strictly harder than Ideal-SVP.

Another line of attack was first sketched in [GS02] and recently revived in [ABD16, CJL16b]. In this attack on NTRU with very large moduli, an attacker exploits the presence of subfields to map the challenge instance to a smaller dimensional instance. Soon after, it was shown that the presence of subfields is not required [KF17]. On the other hand, [BBdV+17] shows that in some rings subfields can be exploited to find shorter vectors. However, it is not clear how to extend these attacks to Ring-LWE and to the rings considered here. Thus, at present, they do not seem to pose a threat to Ring-LWE based public-key encryption.

5.6 Implementation Attacks

Whilst we use constant time methods for sampling from an approximate Gaussian, we do have an obvious timing side-channel in that our encryption algorithms use a form of rejection sampling of the underlying random coins. We contend that such a non-constant time implementation does not cause a security concern. Firstly, this occurs in the public-key encryption and key encapsulation operations only, and so involves no side-channel on the secret key. Secondly, any message-dependent leakage (for example in our IND CCA encryption scheme) is masked by first applying SHA-3 to the message and randomness, before deciding whether to reject. Thirdly, and probably most importantly, the probability of rejection is so small that the probability that it occurs can actually be ignored in practice.

Chapter 6

Seed-Sizes, Bandwidth, Performance, Advantages and Disadvantages

Here we examine various aspects of our constructions.

6.1 Seed Size Discussion

In general, and in the test routines attached to this submission, we use 48 bytes for the seed sizes for all routines (key generation and public-key encryption/key encapsulation), except for the IND-CCA encryption algorithm, where we use only 32 bytes of random seed, and the IND-CPA encapsulation algorithm, where we use 80 bytes of random seed. These are our recommendations for use with the LIMA system. The reason for the restriction in the case of IND-CCA encryption is that we want to maximize the possible message size which could be encrypted in the IND-CCA case. We believe 32 bytes of randomness to be sufficient for most purposes, but err on the side of caution in recommending 48 bytes in most instances. For the case of IND-CPA encapsulation we use 80 bytes of randomness so as to enable a simple implementation to encapsulate a 32 byte key. If 80 bytes of randomness is considered too much then expansion can be done via the XOF. Table 6.1 summarises our choices.

Operation	Recommended Seed Size (Bytes)
IND-CPA Encryption	
KeyGen	48
Enc-CPA	48
IND-CCA Encryption	
KeyGen	48
Enc-CCA	32
IND-CPA Encapsulation	
KeyGen	48
Encap-CPA	80
IND-CCA Encapsulation	
KeyGen	48
Encap-CCA	48

Table 6.1: Recommended Seed Sizes

In the NIST subdirectories we present test programs using the NIST test harness to produce KAT vectors for the various algorithms and parameters. For the case of key encapsulation we always encapsulate a 32 byte key, and for the case of encryption we encrypt message lengths up to the maximum supported by the parameter set/primitive combination.

6.2 Bandwidth

A major concern in relation to lattice based schemes is the consumption in bandwidth of public keys, secret keys and ciphertexts. In what follows we assume that finite field elements are held in an uncompressed byte format, i.e. an element in $\mathbb{F}_q$ takes $\lceil \log_{256} q \rceil$ bytes to represent it, and not (the possibly lower) $\lceil \log_2 q \rceil$ bits.

In the Table 6.2 we present various sizes (in bytes) for the different parameter sets when stored as byte strings using our Encode routines. For encryption operations we assume first a message of 32-bytes (256 bits), and then a message of the maximum possible length enabled by the scheme. For key encapsulation operations we assume a key of length 32-bytes (256 bits). In all cases we assume the seed lengths given in the previous subsection. We present two sizes for the public key size, one the standard size (which is the one used in the Reference and Optimized implementations), and one for the compressed representation given in the text above. A similar compression can be obtained for the secret key, as also alluded to above. Further bandwidth savings are possible by compressing the representation of the finite field elements, using (say) Huffman coding. We then obtain the (approximate) sizes (again in whole bytes) in Table 6.3, giving savings of between 20 and 30 percent. Here we just give the public and secret key sizes, with similar savings for the other measures.

	LIMA-2p	**LIMA-2p**	**LIMA-sp**	**LIMA-sp**	**LIMA-sp**	**LIMA-sp**
N	1024	2048	1018	1306	1822	2062
q	133121	184321	12521473	48181249	44802049	16900097
$b = \lceil \log_{256} q \rceil$	3	3	3	4	4	4
$\lvert \mathfrak{sk} \rvert = 3 \cdot N \cdot b + 1$	9217	18433	9163	15673	21865	24745
$\lvert \mathfrak{pk} \rvert = 2 \cdot N \cdot b + 1$	6145	12289	6109	10449	14577	16497
$\lvert \mathfrak{pk} \rvert = N \cdot b + 385$	3457	6529	3439	5609	7673	8633
$\lvert c_{\mathsf{Enc\text{-}CPA}} \rvert = (N + 256) \cdot b + 3$	3843	6915	3825	6251	8315	9275
$\lvert c_{\mathsf{Enc\text{-}CCA}} \rvert = (N + 512) \cdot b + 3$	4611	7683	4593	7275	9339	10299
$\lvert c_{\mathsf{Enc\text{-}CPA}} \rvert = 2 \cdot N \cdot b + 3$	6147	12291	6111	10451	14579	16499
$\lvert c_{\mathsf{Enc\text{-}CCA}} \rvert = 2 \cdot N \cdot b + 3$	6147	12291	6111	10451	14579	16499
$\lvert c_{\mathsf{Encap\text{-}CPA}} \rvert = (N + 256) \cdot b + 3$	3843	6915	3825	6251	8315	9275
$\lvert c_{\mathsf{Encap\text{-}CCA}} \rvert = (N + 384) \cdot b + 3$	4227	7299	4209	6763	8827	9787

Table 6.2: Bandwidth Estimates for Various Parameters

6.3 Performance

We give some performance numbers in milliseconds for various architectures and our six different parameter sets in Tables 6.4–6.9. We picked two relatively modern processors, as well as historical ones, so as to get some idea as to the performance in different situations. For these timings we utilize the seed sizes defined above, and give times for encrypting (resp. encapsulating) a 32 byte message (resp. secret key). The underlying Keccak routines are compiled using the `asmX86-64` target for the Keccak library for the two modern processors, and `generic32` for the legacy ones. All performance numbers were computed using the *Optimized* variant supplied in our submission. This is an ANSI-C compliant implementation.

	LIMA-2p	**LIMA**-2p	**LIMA**-sp	**LIMA**-sp	**LIMA**-sp	**LIMA**-sp
N	1024	2048	1018	1306	1822	2062
q	133121	184321	12521473	48181249	44802049	16900097
$b' = \lceil \log_2 q \rceil$	18	18	24	26	26	25
$\lvert\mathfrak{sk}\rvert \approx \lceil 3 \cdot N \cdot b'/8 \rceil$	6912	13824	9162	12734	17765	19332
$\lvert\mathfrak{pk}\rvert \approx \lceil 2 \cdot N \cdot b'/8 \rceil$	4608	9216	6108	8489	11843	12888

Table 6.3: Bandwidth Estimates Under Huffman Encoding

The seed for each routine is generated by called `dev/urandom` and then using this "as is"; since we pass the output to an XOF, we deemed it unnecessary to carry our further processing via the NIST DRBG (as used in the NIST test harness).

Note that on some processors the encryption/encapsulation time is sometimes much slower than the associated decryption/decapsulation time. Upon investigation we found that this is because on this test system the time to access `dev/urandom` is an order of magnitude different compared to other systems. Since encryption requires entropy (although only a small amount), this equates to making encryption slower than decryption. This is despite our IND-CCA decryption algorithms involving a re-encryption step, albeit with a known random seed.

	Modern Processors		Legacy Processors	
Processor	Intel i7-3770	Intel Xeon X5460	Atom N270	Pentium 4
Clock Speed	3.10 GHz	3.16 GHz	1.60 GHz	3.6 GHz
Cache Size	8192 KB	6144 KB	512 KB	2048 KB
Key Generation	0.21 ms	0.38 ms	2.94 ms	2.65 ms
IND-CPA Encryption	0.15 ms	0.24 ms	2.14 ms	1.86 ms
IND-CPA Decryption	0.05 ms	0.08 ms	0.65 ms	0.58 ms
IND-CCA Encryption	0.15 ms	0.25 ms	2.15 ms	1.91 ms
IND-CCA Decryption	0.19 ms	0.32 ms	2.69 ms	2.39 ms
IND-CPA Encapsulation	0.15 ms	0.25 ms	2.15 ms	1.85 ms
IND-CPA Decapsulation	0.05 ms	0.08 ms	0.65 ms	0.58 ms
IND-CCA Encapsulation	0.15 ms	0.25 ms	2.18 ms	1.96 ms
IND-CCA Decapsulation	0.19 ms	0.31 ms	2.69 ms	2.30 ms

Table 6.4: Timings for **LIMA-2p-1024**

	Modern Processors		Legacy Processors	
Processor	Intel i7-3770	Intel Xeon X5460	Atom N270	Pentium 4
Clock Speed	3.10 GHz	3.16 GHz	1.60 GHz	3.6 GHz
Cache Size	8192 KB	6144 KB	512 KB	2048 KB
Key Generation	0.42 ms	0.78 ms	5.96 ms	5.33 ms
IND-CPA Encryption	0.31 ms	0.50 ms	4.34 ms	3.77 ms
IND-CPA Decryption	0.10 ms	0.17 ms	1.28 ms	1.15 ms
IND-CCA Encryption	0.31 ms	0.50 ms	4.35 ms	3.86 ms
IND-CCA Decryption	0.39 ms	0.64 ms	5.38 ms	4.75 ms
IND-CPA Encapsulation	0.31 ms	0.50 ms	4.34 ms	3.74 ms
IND-CPA Decapsulation	0.10 ms	0.17 ms	1.29 ms	1.14 ms
IND-CCA Encapsulation	0.31 ms	0.50 ms	4.37 ms	3.92 ms
IND-CCA Decapsulation	0.39 ms	0.63 ms	5.37 ms	4.57 ms

Table 6.5: Timings for **LIMA-2p-2048**

	Modern Processors		Legacy Processors	
Processor	Intel i7-3770	Intel Xeon X5460	Atom N270	Pentium 4
Clock Speed	3.10 GHz	3.16 GHz	1.60 GHz	3.6 GHz
Cache Size	8192 KB	6144 KB	512 KB	2048 KB
Key Generation	0.45 ms	0.80 ms	6.15 ms	5.46 ms
IND-CPA Encryption	0.39 ms	0.65 ms	5.33 ms	4.66 ms
IND-CPA Decryption	0.13 ms	0.21 ms	1.67 ms	1.48 ms
IND-CCA Encryption	0.39 ms	0.65 ms	5.36 ms	4.75 ms
IND-CCA Decryption	0.51 ms	0.86 ms	6.89 ms	6.10 ms
IND-CPA Encapsulation	0.39 ms	0.66 ms	5.34 ms	4.68 ms
IND-CPA Decapsulation	0.12 ms	0.21 ms	1.66 ms	1.48 ms
IND-CCA Encapsulation	0.39 ms	0.66 ms	5.36 ms	4.78 ms
IND-CCA Decapsulation	0.51 ms	0.85 ms	6.89 ms	6.02 ms

Table 6.6: Timings for **LIMA-sp-1018**

	Modern Processors		Legacy Processors	
Processor	Intel i7-3770	Intel Xeon X5460	Atom N270	Pentium 4
Clock Speed	3.10 GHz	3.16 GHz	1.60 GHz	3.6 GHz
Cache Size	8192 KB	6144 KB	512 KB	2048 KB
Key Generation	0.83 ms	1.46 ms	11.19 ms	9.97 ms
IND-CPA Encryption	0.75 ms	1.25 ms	10.03 ms	8.86 ms
IND-CPA Decryption	0.24 ms	0.41 ms	3.20 ms	2.85 ms
IND-CCA Encryption	0.75 ms	1.25 ms	10.02 ms	8.94 ms
IND-CCA Decryption	0.99 ms	1.64 ms	13.06 ms	11.61 ms
IND-CPA Encapsulation	0.75 ms	1.25 ms	10.03 ms	8.84 ms
IND-CPA Decapsulation	0.24 ms	0.41 ms	3.20 ms	2.85 ms
IND-CCA Encapsulation	0.75 ms	1.25 ms	10.06 ms	8.93 ms
IND-CCA Decapsulation	0.98 ms	1.64 ms	13.06 ms	11.48 ms

Table 6.7: Timings for **LIMA-sp-1306**

	Modern Processors		Legacy Processors	
Processor	Intel i7-3770	Intel Xeon X5460	Atom N270	Pentium 4
Clock Speed	3.10 GHz	3.16 GHz	1.60 GHz	3.6 GHz
Cache Size	8192 KB	6144 KB	512 KB	2048 KB
Key Generation	0.91 ms	1.62 ms	12.40 ms	11.11 ms
IND-CPA Encryption	0.80 ms	1.32 ms	10.75 ms	9.49 ms
IND-CPA Decryption	0.26 ms	0.43 ms	3.37 ms	3.03 ms
IND-CCA Encryption	0.80 ms	1.32 ms	10.86 ms	9.63 ms
IND-CCA Decryption	1.04 ms	1.73 ms	13.87 ms	12.40 ms
IND-CPA Encapsulation	0.80 ms	1.32 ms	10.75 ms	9.52 ms
IND-CPA Decapsulation	0.26 ms	0.44 ms	3.37 ms	3.05 ms
IND-CCA Encapsulation	0.80 ms	1.32 ms	10.78 ms	9.70 ms
IND-CCA Decapsulation	1.04 ms	1.73 ms	13.87 ms	12.27 ms

Table 6.8: Timings for **LIMA-sp-1822**

	Modern Processors		Legacy Processors	
Processor	Intel i7-3770	Intel Xeon X5460	Atom N270	Pentium 4
Clock Speed	3.10 GHz	3.16 GHz	1.60 GHz	3.6 GHz
Cache Size	8192 KB	6144 KB	512 KB	2048 KB
Key Generation	1.63 ms	2.85 ms	22.23 ms	19.98 ms
IND-CPA Encryption	1.51 ms	2.51 ms	20.35 ms	18.16 ms
IND-CPA Decryption	0.49 ms	0.83 ms	6.57 ms	5.85 ms
IND-CCA Encryption	1.51 ms	2.51 ms	20.36 ms	18.16 ms
IND-CCA Decryption	1.98 ms	3.31 ms	26.63 ms	23.75 ms
IND-CPA Encapsulation	1.51 ms	2.51 ms	20.95 ms	18.08 ms
IND-CPA Decapsulation	0.49 ms	0.83 ms	6.57 ms	5.83 ms
IND-CCA Encapsulation	1.51 ms	2.51 ms	20.38 ms	18.13 ms
IND-CCA Decapsulation	1.98 ms	3.31 ms	26.62 ms	23.45 ms

Table 6.9: Timings for **LIMA-sp-2062**

We also give in Table 6.10 cycle counts on an Intel i7-3770 machine running at 3.1 GHz with a cache of 8192 KB.

LIMA-2p-1024	
Key Generation	654921
IND-CPA Encryption	480700
IND-CPA Decryption	156880
IND-CCA Encryption	482597
IND-CCA Decryption	615220
IND-CPA Encapsulation	480913
IND-CPA Decapsulation	156297
IND-CCA Encapsulation	486938
IND-CCA Decapsulation	611232
LIMA-2p-2048	
Key Generation	1325909
IND-CPA Encryption	971660
IND-CPA Decryption	310484
IND-CCA Encryption	977969
IND-CCA Decryption	1242139
IND-CPA Encapsulation	1117377
IND-CPA Decapsulation	481230
IND-CCA Encapsulation	1262893
IND-CCA Decapsulation	1229593
LIMA-sp-1018	
Key Generation	1429742
IND-CPA Encryption	1236494
IND-CPA Decryption	395944
IND-CCA Encryption	1239122
IND-CCA Decryption	1610046
IND-CPA Encapsulation	1233953
IND-CPA Decapsulation	396764
IND-CCA Encapsulation	1241867
IND-CCA Decapsulation	1612433

LIMA-sp-1306	
Key Generation	2600237
IND-CPA Encryption	2354094
IND-CPA Decryption	770324
IND-CCA Encryption	2360157
IND-CCA Decryption	3091742
IND-CPA Encapsulation	2355666
IND-CPA Decapsulation	770710
IND-CCA Encapsulation	2361683
IND-CCA Decapsulation	3085679
LIMA-sp-1822	
Key Generation	2853857
IND-CPA Encryption	2506669
IND-CPA Decryption	813067
IND-CCA Encryption	2513027
IND-CCA Decryption	3264289
IND-CPA Encapsulation	2505937
IND-CPA Decapsulation	812501
IND-CCA Encapsulation	2512619
IND-CCA Decapsulation	3263201
LIMA-sp-2062	
Key Generation	5114770
IND-CPA Encryption	4726471
IND-CPA Decryption	1549057
IND-CCA Encryption	4728991
IND-CCA Decryption	6235608
IND-CPA Encapsulation	4729707
IND-CPA Decapsulation	1553638
IND-CCA Encapsulation	4738128
IND-CCA Decapsulation	6237127

Table 6.10: Cycle Counts on an Intel i7-2600

The encryption and decryption algorithms could be further optimized to run the FFT in parallel, using AVX2 and AVX-512 instructions. We have not implemented these optimizations since recent research[1] shows that this results in severe performance penalties on real systems. In particular, when using AVX2 and AVX-512 instructions, the clock speed is reduced due to power issues. If a piece of code only uses AVX2 and AVX-512 instructions, then the overall speedup that can be achieved may be very good because the high parallelization offsets the lower clock speed. However, if the code uses additional, non-AVX2/AVX-512, instructions, then those instructions run at the lower clock speed and the overall performance is impaired. This means that using AVX2 and AVX-512 instructions for a task like encryption, which is typically run in conjunction with arbitrary other tasks, may be problematic.

6.4 Advantages and Disadvantages

We end with a summary of the known advantages and disadvantages of our LIMA proposal.

6.4.1 Advantages

- We provide the choice of LIMA-2p and LIMA-sp parameter sets, depending on confidence in subfield tower protections.

[1] See `https://blog.cloudflare.com/on-the-dangers-of-intels-frequency-scaling/`.

- We provide the Enc-CCA scheme for public-key encryption of short messages.
- The Encap-CCA scheme, when combined with a DEM such as AES-GCM, gives public-key encryption for long messages.
- We use the well-studied underlying primitive of Ring-LWE. a relatively boring, but safe, pedigree/design choice.
- We employ Gaussian-like distributions to provide design validation against worst-case/average-case results.
- We use rejection sampling in encryption to remove the issue of decryption failures.
- All the schemes have tight security proofs.

6.4.2 Disadvantages

- It is reported in [JZC+17] that our IND-CCA KEM is secure in the Quantum-ROM model, but at the present time we make no claim as to the security in the QROM model of our IND-CCA public-key encryption scheme.
- Our Reference and Optimized implementations are not currently constant time. But operations dependent on secret data (keys and messages) could be made constant time with little cost in performance.

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Appendix A

Parameter Search Code

We used the script in this section in conjunction with the LWE estimator from [APS15] to estimate the cost of solving LWE instances with our parameter choices.

In contrast to [Alb17], in our estimates we conservatively assume that producing many short bases from a short basis does not increase the norm of the output vectors. Thus, we apply the following patch to commit 6f2534014b7c4a4798ebe2df829c45bf01e47cbc of https://bitbucket.org/malb/lwe-estimator:

```
--- estimator.orig.py   2017-06-23 14:55:33.229724436 +0100
+++ estimator.py        2017-06-23 14:55:27.281917030 +0100
@@ -2147,7 +2147,7 @@
     delta_0 = best["delta_0"]

     if use_lll:
-        scale = 2
+        scale = 1  # TODO conservative compared to upstream
     else:
         scale = 1
```

Our parameter search is then implemented as:

```
# -*- coding: utf-8 -*-

import estimator as est
from sage.all import RR, log, ceil, floor, next_prime, sqrt, is_prime, pi, Infinity, get_verbose
from sage.all import ZZ
from functools import partial
from collections import OrderedDict


sigma = sqrt((20)/2)


def adps16_primal(n, alpha, q, secret_distribution=True,
                  m=est.oo, reduction_cost_model=est.reduction_default_cost,
                  success_probability=0.99, **kwds):
    u"""
    Primal attack according to [USENIX:ADPS16]_

    [USENIX:ADPS16] Alkim, E., Léo Ducas, Thomas Pöppelmann, & Schwabe, P.  (2015).
    Post-quantum key exchange - a new hope.

    :param n:
    :param alpha:
    :param q:
    :param optimisation_target:
    """

    kwds = {"n": n, "alpha": alpha, "q": q,
            "secret_distribution": secret_distribution,
            "reduction_cost_model": reduction_cost_model,
            "m": m}

    cost = est.binary_search(_adps16_primal_cost, start=40, stop=2*n, param="block_size",
                             predicate=lambda x, best: x["red"]<=best["red"], **kwds)
```

```
    for block_size in range(32, cost["beta"]+1)[::-1]:
        t = _adps16_primal_cost(block_size=block_size, **kwds)
        if t["red"] == Infinity:
            break
        cost = t

    return cost


def _adps16_primal_cost(block_size, n, alpha, q, secret_distribution=True, m=est.oo,
                        reduction_cost_model=est.reduction_default_cost,
                        success_probability=0.99):

    n, alpha, q = est.Param.preprocess(n, alpha, q)
    RR = alpha.parent()
    q = RR(q)
    delta_0 = est.delta_0f(block_size)
    sigma = est.stddevf(alpha*q)
    block_size = RR(block_size)

    if secret_distribution:
        m += n

    m = min(2*ceil(sqrt(n*log(q)/log(delta_0))), m)

    log_b_star = lambda d: delta_0.log()*(2*block_size-d) + ((d-n-1)*q.log())/d

    C = sigma.log() + block_size.log()/2

    for d in xrange(n, m):
        if log_b_star(d) - C >= 0:
            break

    ineq = lambda d: sigma * RR(sqrt(block_size)) <= delta_0**(2*block_size-d) * (q**ZZ(d-n-1))**(ZZ(1)/d)

    ret = est.lattice_reduction_cost(reduction_cost_model, delta_0, d)
    if not ineq(d):
        ret["red"] = Infinity

    ret["d"] = d
    ret["m"] = m
    if secret_distribution:
        ret["m"] -= n

    ret["delta_0"] = delta_0

    return ret


def ADPS16(beta, d, B=None, mode="classical"):
    u"""
     Runtime estimation from [USENIX:ADPS16]_ but with additive constant 16.

     :param beta: block size
     :param n: LWE dimension ‘n > 0‘
     :param B: bit-size of entries
     """

    if mode not in ("classical", "quantum", "paranoid"):
        raise ValueError("Mode ’%s’ not understood"%mode)

    c = {"classical": 0.2920,
         "quantum":   0.2650,  # paper writes 0.262 but this isn’t right, see above
         "paranoid":  0.2075}

    c = c[mode]

    return ZZ(2)**RR(c*beta + 16)


cost_models = OrderedDict([(partial(ADPS16, mode="classical"), "classical"),
                           (partial(ADPS16, mode="quantum"),   "quantum")])


def _compose(strategy, n, alpha, q, reduction_cost_model, *args, **kwds):
    """
```

```
    Run `strategy` using `reduction_cost_model`.

    :param strategy: dual, primal, etc.
    :param n: LWE dimension
    :param alpha: noise rate
    :param q: modulus
    :param reduction_cost_model: cost model for BKZ

    """
    if get_verbose() >= 1:
        print "%20s"%cost_models[reduction_cost_model],

    r = strategy(n=n, alpha=alpha, q=q, *args,
                 reduction_cost_model=reduction_cost_model,
                 success_probability=0.51, **kwds)

    if get_verbose() >= 1:
        print r
    return r


dual   = partial(_compose,
                 strategy=est.dual_scale,
                 secret_distribution=True,
                 use_lll=True)

primal = partial(_compose, strategy=adps16_primal,
                 secret_distribution=True)


def complete_parameters(N, type=1):
    """

    Extend dimension ``N`` to complete parameter set.

    :param N: dimension
    :param type: 1 or 2

    """

    if type == 1:
        B = 20 * (1 + 11 * sigma * sqrt(2*N))  # noise bound
        q = ((ceil(4*B))//(2*N) + 1) * (2*N) + 1  # q % (2*N ) âĹą 1

        while not is_prime(q):
            q += 2*N

        assert(q>4*B)
        assert(q%(2*N) == 1)

    elif type == 2:
        pd = N
        p = 2*pd + 1
        N = p - 1
        B = 20 * (1 + 11 * sigma * sqrt(4*N))
        e = ceil(log(2*p, 2))
        l = floor(4*B/(2**e * p))
        q = l * 2**e * p + 1

        while not is_prime(q):
            l += 1
            q = l * 2**e * p + 1

        assert(q>4*B)
        assert(q%(2**e * p) == 1)

    else:
        raise ValueError("Only Type-I or Type-II are defined.")

    alpha = RR(sqrt(2*pi)*sigma/q)
    return N, alpha, q


def security_level(n, alpha, q, reduction_cost_model):
    cost_1 = primal(n=n, alpha=alpha, q=q, reduction_cost_model=reduction_cost_model)
    cost_2 =   dual(n=n, alpha=alpha, q=q, reduction_cost_model=reduction_cost_model)
```

```
    best = sorted([cost_1, cost_2], key=lambda x: x["red"])[0]

    return best, (cost_1, cost_2)


def all_candidates(up_to=128, type=1, start=400):

    if type not in (1, 2):
        raise ValueError("Type must be either 1 or 2")

    candidates = []

    def log_cost(cost):
        return log(cost.values()[0], 2.).n()

    if type == 1:
        i = 2**ceil(log(start, 2))

        def next_i(i):
            return 2*i

    elif type == 2:
        i = next_prime(start)

        def next_i(pd):
            pd = next_prime(pd)
            while not is_prime(2*pd+1):
                pd = next_prime(pd+1)
            return pd

    while True:
        n, alpha, q = complete_parameters(i, type=type)

        best_c, costs_c = security_level(n, alpha, q, reduction_cost_model=cost_models.keys()[0])
        alg_c = costs_c.index(best_c) + 1

        best_q, costs_q = security_level(n, alpha, q, reduction_cost_model=cost_models.keys()[1])
        alg_q = costs_q.index(best_q) + 1

        alg_c = "primal" if alg_c == 1 else "dual"
        alg_q = "primal" if alg_q == 1 else "dual"

        candidates.append((n, q, ceil(ceil(log(q, 2))*n/8.0),
                          log_cost(best_c), alg_c, log_cost(best_q), alg_q))

        print "%4d & %10d & %4d & %5.1f & %6s & %5.1f & %6s\\\\"%candidates[-1]

        if log(best_q.values()[0], 2) > up_to:
            break
        else:
            i = next_i(i)

    return candidates


def print_candidate_security(i, type=1):
    n, alpha, q = complete_parameters(i, type=type)
    _, costs_c = security_level(n, alpha, q, reduction_cost_model=cost_models.keys()[0])
    _, costs_q = security_level(n, alpha, q, reduction_cost_model=cost_models.keys()[1])

    fmt = "%6s & %10s & %5.1f & %8.6f & %4d & %3d & $2^{%5.1f}$\\\\"

    print "\\multicolumn{7}{c}{Type %d, $n=%d$, $q=%d$}\\\\"%(type, n, q)
    print "\\midrule"

    for i, c_ in enumerate(costs_c):
        if i == 1:
            attack = "dual"
        elif i == 0:
            attack = "primal"
        else:
            continue

        r_ = log(c_.data.get("repeat", 1), 2)
        print(fmt%(attack, "classical", log(c_.values()[0], 2), c_["delta_0"], c_["d"], c_["beta"], r_))

    for i, c_ in enumerate(costs_q):
```

```
        if i == 1:
            attack = "dual"
        elif i == 0:
            attack = "primal"
        else:
            continue

        r_ = log(c_.data.get("repeat", 1), 2)
        print(fmt%(attack, "quantum", log(c_.values()[0], 2), c_["delta_0"], c_["d"], c_["beta"], r_))

    print "\\midrule"


def print_all_candidate_securities():
    """
    """
    print_candidate_security(1024, 1)
    print_candidate_security(2048, 1)
    print_candidate_security(1018/2, 2)
    print_candidate_security(1306/2, 2)
    print_candidate_security(1822/2, 2)
    print_candidate_security(2062/2, 2)
```

In particular, we used the following calls to produce Tables 5.1 and 5.2.

```
sage: attach("security_estimates.py")
sage: _ = all_candidates(up_to=320, type=1)
sage: _ = all_candidates(up_to=320, type=2)
sage: print_all_candidate_securities()
```